The Instability of Differential Operators

Nils Dencker

Lund University

19 September 2023

Introduction

Let $x \in \mathbf{R}^n$, the results are local and generalize to manifolds. The complex derivative $D = \frac{1}{i}\partial$ gives

$$P(x, D)u(x) = (2\pi)^{-n} \int e^{ix\cdot\xi} P(x, \xi)\hat{u}(\xi)\, d\xi \qquad u \in C_0^\infty(\mathbf{R}^n)$$

Here $P(x, \xi)$ is the **symbol** of the operator.

$P(x, D)$ is PDO if $\xi \mapsto P(x, \xi)$ is polynomial in ξ, ΨDO if

$$P(x, \xi) = p_m(x, \xi) + p_{m-1}(x, \xi) + \dots$$

with p_j *homogeneous* of degree j in ξ, m is the *order*, $p = p_m$ is the **principal symbol**. The principal symbol is invariant as a function on the cotangent space $T^*\mathbf{R}^n$.

One can localize in cones in phase space $(x, \xi) \in T^*\mathbf{R}^n$ with ΨDO, so called *microlocal analysis*.

Introduction

Let $x \in \mathbf{R}^n$, the results are local and generalize to manifolds. The complex derivative $D = \frac{1}{i}\partial$ gives

$$P(x, D)u(x) = (2\pi)^{-n} \int e^{ix\cdot\xi} P(x,\xi)\hat{u}(\xi)\, d\xi \qquad u \in C_0^\infty(\mathbf{R}^n)$$

Here $P(x,\xi)$ is the **symbol** of the operator.

$P(x, D)$ is PDO if $\xi \mapsto P(x,\xi)$ is polynomial in ξ, ΨDO if

$$P(x,\xi) = p_m(x,\xi) + p_{m-1}(x,\xi) + \dots$$

with p_j *homogeneous* of degree j in ξ, m is the *order*, $p = p_m$ is the **principal symbol**. The principal symbol is invariant as a function on the cotangent space $T^*\mathbf{R}^n$.

One can localize in cones in phase space $(x,\xi) \in T^*\mathbf{R}^n$ with ΨDO, so called *microlocal analysis*.

Introduction

Let $x \in \mathbf{R}^n$, the results are local and generalize to manifolds. The complex derivative $D = \frac{1}{i}\partial$ gives

$$P(x,D)u(x) = (2\pi)^{-n} \int e^{ix\cdot\xi} P(x,\xi)\hat{u}(\xi)\, d\xi \qquad u \in C_0^\infty(\mathbf{R}^n)$$

Here $P(x,\xi)$ is the **symbol** of the operator.

$P(x,D)$ is PDO if $\xi \mapsto P(x,\xi)$ is polynomial in ξ, ΨDO if

$$P(x,\xi) = p_m(x,\xi) + p_{m-1}(x,\xi) + \dots$$

with p_j *homogeneous* of degree j in ξ, m is the *order*, $p = p_m$ is the **principal symbol**. The principal symbol is invariant as a function on the cotangent space $T^*\mathbf{R}^n$.

One can localize in cones in phase space $(x,\xi) \in T^*\mathbf{R}^n$ with ΨDO, so called *microlocal analysis*.

Local solvability

P is locally solvable near x_0 if

$$Pu = f$$

has a local weak (distribution) solution u near x_0 for all $f \in C^\infty$ in a set of finite codimension, thus P has locally a finite cokernel.

This is equivalent to *a priori* estimates for the L^2 adjoint P^*.

One can prove local nonsolvability by constructing local *approximate* solutions to $P^* u = 0$, which are called *pseudomodes*.

Observe that in the **analytic category** all nondegenerate PDO are locally solvable by the Cauchy-Kovalevsky theorem.

Constant coefficient PDO are locally solvable. (Malgrange 1955)

Elliptic case: $p(x, \xi) \neq 0$ for $\xi \neq 0$ are solvable. (Lax-Milgram 1954)

Local solvability

P is locally solvable near x_0 if

$$Pu = f$$

has a local weak (distribution) solution u near x_0 for all $f \in C^\infty$ in a set of finite codimension, thus P has locally a finite cokernel.

This is equivalent to *a priori* estimates for the L^2 adjoint P^*.

One can prove local nonsolvability by constructing local *approximate* solutions to $P^*u = 0$, which are called *pseudomodes*.

Observe that in the **analytic category** all nondegenerate PDO are locally solvable by the Cauchy-Kovalevsky theorem.

Constant coefficient PDO are locally solvable. (Malgrange 1955)

Elliptic case: $p(x, \xi) \neq 0$ for $\xi \neq 0$ are solvable. (Lax-Milgram 1954)

Local solvability

P is locally solvable near x_0 if

$$Pu = f$$

has a local weak (distribution) solution u near x_0 for all $f \in C^\infty$ in a set of finite codimension, thus P has locally a finite cokernel.

This is equivalent to *a priori* estimates for the L^2 adjoint P^*.

One can prove local nonsolvability by constructing local *approximate* solutions to $P^* u = 0$, which are called *pseudomodes*.

Observe that in the **analytic category** all nondegenerate PDO are locally solvable by the Cauchy-Kovalevsky theorem.

Constant coefficient PDO are locally solvable. (Malgrange 1955)

Elliptic case: $p(x, \xi) \neq 0$ for $\xi \neq 0$ are solvable. (Lax-Milgram 1954)

Local solvability

P is locally solvable near x_0 if

$$Pu = f$$

has a local weak (distribution) solution u near x_0 for all $f \in C^\infty$ in a set of finite codimension, thus P has locally a finite cokernel.

This is equivalent to *a priori* estimates for the L^2 adjoint P^*.

One can prove local nonsolvability by constructing local *approximate* solutions to $P^*u = 0$, which are called *pseudomodes*.

Observe that in the **analytic category** all nondegenerate PDO are locally solvable by the Cauchy-Kovalevsky theorem.

Constant coefficient PDO are locally solvable. (Malgrange 1955)

Elliptic case: $p(x, \xi) \neq 0$ for $\xi \neq 0$ are solvable. (Lax-Milgram 1954)

Local solvability

P is locally solvable near x_0 if

$$Pu = f$$

has a local weak (distribution) solution u near x_0 for all $f \in C^\infty$ in a set of finite codimension, thus P has locally a finite cokernel.

This is equivalent to *a priori* estimates for the L^2 adjoint P^*.

One can prove local nonsolvability by constructing local *approximate* solutions to $P^* u = 0$, which are called *pseudomodes*.

Observe that in the **analytic category** all nondegenerate PDO are locally solvable by the Cauchy-Kovalevsky theorem.

Constant coefficient PDO are locally solvable. (Malgrange 1955)

Elliptic case: $p(x, \xi) \neq 0$ for $\xi \neq 0$ are solvable. (Lax-Milgram 1954)

Principal type

Definition

The operator P is of **principal type** if $|dp| \neq 0$ when $p = 0$.

Then P has simple characteristics, which is a *generic* condition for nonelliptic operators.

The Hamilton field of p: $H_p = \partial_\xi p \partial_x - \partial_x p \partial_\xi$ should be *nonradial*, observe that $H_p p = 0$.

Theorem (Hörmander's thesis 1955)

PDO of principal type with principal symbol p such that $H_p \overline{p} = \{ p, \overline{p} \} = -2i \{ \operatorname{Re} p, \operatorname{Im} p \} = 0$ are solvable.

For example PDO of principal type with **real** principal symbol are solvable, since then $\overline{p} = p$. Observe that the condition is *not invariant*.

Principal type

Definition

The operator P is of **principal type** if $|dp| \neq 0$ when $p = 0$.

Then P has simple characteristics, which is a *generic* condition for nonelliptic operators.

The Hamilton field of p: $H_p = \partial_\xi p \partial_x - \partial_x p \partial_\xi$ should be *nonradial*, observe that $H_p p \equiv 0$.

Theorem (Hörmander's thesis 1955)

PDO of principal type with principal symbol p such that $H_p \overline{p} = \{ p, \overline{p} \} = -2i \{ \operatorname{Re} p, \operatorname{Im} p \} \equiv 0$ are solvable.

For example PDO of principal type with **real** principal symbol are solvable, since then $\overline{p} = p$. Observe that the condition is *not invariant*.

Principal type

Definition

The operator P is of **principal type** if $|dp| \neq 0$ when $p = 0$.

Then P has simple characteristics, which is a *generic* condition for nonelliptic operators.

The Hamilton field of p: $H_p = \partial_\xi p \partial_x - \partial_x p \partial_\xi$ should be *nonradial*, observe that $H_p p \equiv 0$.

Theorem (Hörmander's thesis 1955)

PDO of principal type with principal symbol p such that $H_p \overline{p} = \{ p, \overline{p} \} = -2i \{ \operatorname{Re} p, \operatorname{Im} p \} \equiv 0$ *are solvable.*

For example PDO of principal type with **real** principal symbol are solvable, since then $\overline{p} = p$. Observe that the condition is *not invariant*.

Principal type

Definition

The operator P is of **principal type** if $|dp| \neq 0$ when $p = 0$.

Then P has simple characteristics, which is a *generic* condition for nonelliptic operators.

The Hamilton field of p: $H_p = \partial_\xi p \partial_x - \partial_x p \partial_\xi$ should be *nonradial*, observe that $H_p p \equiv 0$.

Theorem (Hörmander's thesis 1955)

PDO of principal type with principal symbol p such that $H_p \overline{p} = \{ p, \overline{p} \} = -2i \{ \operatorname{Re} p, \operatorname{Im} p \} \equiv 0$ *are solvable.*

For example PDO of principal type with **real** principal symbol are solvable, since then $\overline{p} = p$. Observe that the condition is *not invariant*.

The Lewy counterexample

Hans Lewy's counterexample (1957) The vector field

$$P = D_{x_1} + iD_{x_2} + i(x_1 + ix_2)D_{x_3} = D_{x_1} - x_2 D_{x_3} + i(D_{x_2} + x_1 D_{x_3})$$

is **not locally solvable anywhere** in $\mathbf{R}^3$. In fact, the range of P is a set of the *first category* in C^∞, a meagre set.

By the Cauchy-Kovalevsky theorem P is solvable for *analytic* functions.

P is the tangential Cauchy-Riemann operator on the boundary of the **strictly pseudoconvex set**

$$\Omega = \{ (z_1, z_2) : |z_1|^2 + 2\,\mathrm{Im}\, z_2 < 0 \} \subset \mathbf{C}^2$$

Observe that $Pf\big|_{\partial\Omega} = 0$ for any analytic function $f(z_1, z_2)$.

But almost all vector fields on $\mathbf{R}^3$ have **trivial kernels:** the constants (Treves et al 1983). Thus there exist arbitrarily small perturbations of P that are not tangential Cauchy-Riemann operators.

The Lewy counterexample

Hans Lewy's counterexample (1957) The vector field

$$P = D_{x_1} + iD_{x_2} + i(x_1 + ix_2)D_{x_3} = D_{x_1} - x_2 D_{x_3} + i(D_{x_2} + x_1 D_{x_3})$$

is **not locally solvable anywhere** in $\mathbf{R}^3$. In fact, the range of P is a set of the *first category* in C^∞, a meagre set.

By the Cauchy-Kovalevsky theorem P is solvable for *analytic* functions.

P is the tangential Cauchy-Riemann operator on the boundary of the **strictly pseudoconvex set**

$$\Omega = \{ (z_1, z_2) : |z_1|^2 + 2 \operatorname{Im} z_2 < 0 \} \subset \mathbf{C}^2$$

Observe that $Pf\big|_{\partial\Omega} = 0$ for any analytic function $f(z_1, z_2)$.

But almost all vector fields on $\mathbf{R}^3$ have **trivial kernels:** the constants (Treves et al 1983). Thus there exist arbitrarily small perturbations of P that are not tangential Cauchy-Riemann operators.

The Lewy counterexample

Hans Lewy's counterexample (1957) The vector field

$$P = D_{x_1} + iD_{x_2} + i(x_1 + ix_2)D_{x_3} = D_{x_1} - x_2 D_{x_3} + i(D_{x_2} + x_1 D_{x_3})$$

is **not locally solvable anywhere** in $\mathbf{R}^3$. In fact, the range of P is a set of the *first category* in C^∞, a meagre set.

By the Cauchy-Kovalevsky theorem P is solvable for *analytic* functions.

P is the tangential Cauchy-Riemann operator on the boundary of the **strictly pseudoconvex set**

$$\Omega = \{ (z_1, z_2) : \ |z_1|^2 + 2 \operatorname{Im} z_2 < 0 \} \subset \mathbf{C}^2$$

Observe that $Pf\big|_{\partial\Omega} = 0$ for any analytic function $f(z_1, z_2)$.

But almost all vector fields on $\mathbf{R}^3$ have **trivial kernels:** the constants (Treves et al 1983). Thus there exist arbitrarily small perturbations of P that are not tangential Cauchy-Riemann operators.

The Lewy counterexample

Hans Lewy's counterexample (1957) The vector field

$$P = D_{x_1} + iD_{x_2} + i(x_1 + ix_2)D_{x_3} = D_{x_1} - x_2 D_{x_3} + i(D_{x_2} + x_1 D_{x_3})$$

is **not locally solvable anywhere** in $\mathbf{R}^3$. In fact, the range of P is a set of the *first category* in C^∞, a meagre set.

By the Cauchy-Kovalevsky theorem P is solvable for *analytic* functions.

P is the tangential Cauchy-Riemann operator on the boundary of the **strictly pseudoconvex set**

$$\Omega = \{ (z_1, z_2) : \; |z_1|^2 + 2 \operatorname{Im} z_2 < 0 \} \subset \mathbf{C}^2$$

Observe that $Pf\big|_{\partial\Omega} = 0$ for any analytic function $f(z_1, z_2)$.

But almost all vector fields on $\mathbf{R}^3$ have **trivial kernels:** the constants (Treves et al 1983). Thus there exist arbitrarily small perturbations of P that are not tangential Cauchy-Riemann operators.

The Lewy counterexample

Hans Lewy's counterexample (1957) The vector field

$$P = D_{x_1} + iD_{x_2} + i(x_1 + ix_2)D_{x_3} = D_{x_1} - x_2 D_{x_3} + i(D_{x_2} + x_1 D_{x_3})$$

is **not locally solvable anywhere** in $\mathbf{R}^3$. In fact, the range of P is a set of the *first category* in C^∞, a meagre set.

By the Cauchy-Kovalevsky theorem P is solvable for *analytic* functions.

P is the tangential Cauchy-Riemann operator on the boundary of the **strictly pseudoconvex set**

$$\Omega = \{\, (z_1, z_2) : \; |z_1|^2 + 2\,\mathrm{Im}\, z_2 < 0 \,\} \subset \mathbf{C}^2$$

Observe that $Pf\big|_{\partial\Omega} = 0$ for any analytic function $f(z_1, z_2)$.

But almost all vector fields on $\mathbf{R}^3$ have **trivial kernels:** the constants (Treves et al 1983). Thus there exist arbitrarily small perturbations of P that are not tangential Cauchy-Riemann operators.

The Lewy counterexample

Hans Lewy's counterexample (1957) The vector field

$$P = D_{x_1} + iD_{x_2} + i(x_1 + ix_2)D_{x_3} = D_{x_1} - x_2 D_{x_3} + i(D_{x_2} + x_1 D_{x_3})$$

is **not locally solvable anywhere** in $\mathbf{R}^3$. In fact, the range of P is a set of the *first category* in C^∞, a meagre set.

By the Cauchy-Kovalevsky theorem P is solvable for *analytic* functions.

P is the tangential Cauchy-Riemann operator on the boundary of the **strictly pseudoconvex set**

$$\Omega = \{\, (z_1, z_2) : \ |z_1|^2 + 2\,\mathrm{Im}\, z_2 < 0 \,\} \subset \mathbf{C}^2$$

Observe that $Pf\big|_{\partial\Omega} = 0$ for any analytic function $f(z_1, z_2)$.

But almost all vector fields on $\mathbf{R}^3$ have **trivial kernels:** the constants (Treves et al 1983). Thus there exist arbitrarily small perturbations of P that are not tangential Cauchy-Riemann operators.

The bracket condition

Theorem (Hörmander 1960)

Local solvability of PDO P implies that $\{\operatorname{Re} p, \operatorname{Im} p\} = H_{\operatorname{Re} p} \operatorname{Im} p = 0$ on $p^{-1}(0)$, where p is the principal symbol of P.

This is an invariant and nongeneric condition so **almost all** nonelliptic linear PDO are **not** locally solvable.

The principal symbol of $[P, P^*]$ is $\frac{1}{i}\{p, \overline{p}\} = -2\{\operatorname{Re} p, \operatorname{Im} p\}$, so a vanishing bracket means that the operator is **approximately normal**.

For the Lewy vector field we have $\{\operatorname{Re} p, \operatorname{Im} p\} = 2\xi_3$ and $p = 0$ if and only if $\xi_1 = x_2\xi_3$ and $\xi_2 = -x_1\xi_3$, so $\{\operatorname{Re} p, \operatorname{Im} p\}$ does not vanish at the characteristics.

Observe that for **ΨDO**, the bracket condition is that $H_{\operatorname{Re} p} \operatorname{Im} p \le 0$ on $p^{-1}(0)$.

The bracket condition

Theorem (Hörmander 1960)

Local solvability of PDO P implies that $\{\,\mathrm{Re}\, p, \mathrm{Im}\, p\,\} = H_{\mathrm{Re}\, p} \mathrm{Im}\, p = 0$ on $p^{-1}(0)$, where p is the principal symbol of P.

This is an invariant and nongeneric condition so **almost all** nonelliptic linear PDO are **not** locally solvable.

The principal symbol of $[P, P^*]$ is $\frac{1}{i}\{\,p, \overline{p}\,\} = -2\{\,\mathrm{Re}\, p, \mathrm{Im}\, p\,\}$, so a vanishing bracket means that the operator is **approximately normal**.

For the Lewy vector field we have $\{\,\mathrm{Re}\, p, \mathrm{Im}\, p\,\} = 2\xi_3$ and $p = 0$ if and only if $\xi_1 = x_2\xi_3$ and $\xi_2 = -x_1\xi_3$, so $\{\,\mathrm{Re}\, p, \mathrm{Im}\, p\,\}$ does not vanish at the characteristics.

Observe that for **ΨDO**, the bracket condition is that $H_{\mathrm{Re}\, p} \mathrm{Im}\, p \le 0$ on $p^{-1}(0)$.

The bracket condition

Theorem (Hörmander 1960)

Local solvability of PDO P implies that $\{\operatorname{Re} p, \operatorname{Im} p\} = H_{\operatorname{Re} p} \operatorname{Im} p = 0$ on $p^{-1}(0)$, where p is the principal symbol of P.

This is an invariant and nongeneric condition so **almost all** nonelliptic linear PDO are **not** locally solvable.

The principal symbol of $[P, P^*]$ is $\frac{1}{i}\{p, \overline{p}\} = -2\{\operatorname{Re} p, \operatorname{Im} p\}$, so a vanishing bracket means that the operator is **approximately normal**.

For the Lewy vector field we have $\{\operatorname{Re} p, \operatorname{Im} p\} = 2\xi_3$ and $p = 0$ if and only if $\xi_1 = x_2\xi_3$ and $\xi_2 = -x_1\xi_3$, so $\{\operatorname{Re} p, \operatorname{Im} p\}$ does not vanish at the characteristics.

Observe that for **ΨDO**, the bracket condition is that $H_{\operatorname{Re} p} \operatorname{Im} p \le 0$ on $p^{-1}(0)$.

The bracket condition

Theorem (Hörmander 1960)

Local solvability of PDO P implies that $\{\operatorname{Re} p, \operatorname{Im} p\} = H_{\operatorname{Re} p} \operatorname{Im} p = 0$ on $p^{-1}(0)$, where p is the principal symbol of P.

This is an invariant and nongeneric condition so **almost all** nonelliptic linear PDO are **not** locally solvable.

The principal symbol of $[P, P^*]$ is $\frac{1}{i}\{p, \overline{p}\} = -2\{\operatorname{Re} p, \operatorname{Im} p\}$, so a vanishing bracket means that the operator is **approximately normal**.

For the Lewy vector field we have $\{\operatorname{Re} p, \operatorname{Im} p\} = 2\xi_3$ and $p = 0$ if and only if $\xi_1 = x_2\xi_3$ and $\xi_2 = -x_1\xi_3$, so $\{\operatorname{Re} p, \operatorname{Im} p\}$ does not vanish at the characteristics.

Observe that for **ΨDO**, the bracket condition is that $H_{\operatorname{Re} p} \operatorname{Im} p \le 0$ on $p^{-1}(0)$.

The bracket condition

Theorem (Hörmander 1960)

Local solvability of PDO P implies that $\{\operatorname{Re} p, \operatorname{Im} p\} = H_{\operatorname{Re} p} \operatorname{Im} p = 0$ on $p^{-1}(0)$, where p is the principal symbol of P.

This is an invariant and nongeneric condition so **almost all** nonelliptic linear PDO are **not** locally solvable.

The principal symbol of $[P, P^*]$ is $\frac{1}{i}\{p, \overline{p}\} = -2\{\operatorname{Re} p, \operatorname{Im} p\}$, so a vanishing bracket means that the operator is **approximately normal**.

For the Lewy vector field we have $\{\operatorname{Re} p, \operatorname{Im} p\} = 2\xi_3$ and $p = 0$ if and only if $\xi_1 = x_2\xi_3$ and $\xi_2 = -x_1\xi_3$, so $\{\operatorname{Re} p, \operatorname{Im} p\}$ does not vanish at the characteristics.

Observe that for **ΨDO**, the bracket condition is that $H_{\operatorname{Re} p} \operatorname{Im} p \le 0$ on $p^{-1}(0)$.

Range of vector fields

Actually, complex vector fields with nonvanishing brackets are **determined** by their range, up to *right multiplication* by functions.

Theorem (Hörmander 1963)

Let P and Q be two complex vector fields such that $p = \sigma(P)$ satisfies

$$\{\operatorname{Re} p, \operatorname{Im} p\} \neq 0 \quad \text{on } p^{-1}(0)$$

so that P is not solvable. If locally $\operatorname{Ran} Q \subset \operatorname{Ran} P$ then $Q = PE$ for some $E \in C^\infty$ locally. If $Q \neq 0$ they generate the same right module.

The condition is that for any $f \in C^\infty$ there exists a distribution u such that $Qf = Pu$ locally.

This has been generalized to unsolvable ΨDO but the conclusions are then microlocal and weaker (D. and Wittsten 2013).

Range of vector fields

Actually, complex vector fields with nonvanishing brackets are **determined** by their range, up to *right multiplication* by functions.

Theorem (Hörmander 1963)

Let P and Q be two complex vector fields such that $p = \sigma(P)$ satisfies

$$\{\operatorname{Re} p, \operatorname{Im} p\} \neq 0 \quad \text{on } p^{-1}(0)$$

so that P is not solvable. If locally $\operatorname{Ran} Q \subset \operatorname{Ran} P$ then $Q = PE$ for some $E \in C^\infty$ locally. If $Q \neq 0$ they generate the same right module.

The condition is that for any $f \in C^\infty$ there exists a distribution u such that $Qf = Pu$ locally.

This has been generalized to unsolvable ΨDO but the conclusions are then microlocal and weaker (D. and Wittsten 2013).

Range of vector fields

Actually, complex vector fields with nonvanishing brackets are **determined** by their range, up to *right multiplication* by functions.

Theorem (Hörmander 1963)

Let P and Q be two complex vector fields such that $p = \sigma(P)$ satisfies

$$\{\operatorname{Re} p, \operatorname{Im} p\} \neq 0 \quad \text{on } p^{-1}(0)$$

so that P is not solvable. If locally $\operatorname{Ran} Q \subset \operatorname{Ran} P$ *then $Q = PE$ for some $E \in C^\infty$ locally. If $Q \neq 0$ they generate the same* right module.

The condition is that for any $f \in C^\infty$ there exists a distribution u such that $Qf = Pu$ locally.

This has been generalized to unsolvable ΨDO but the conclusions are then microlocal and weaker (D. and Wittsten 2013).

Range of vector fields

Actually, complex vector fields with nonvanishing brackets are **determined** by their range, up to *right multiplication* by functions.

Theorem (Hörmander 1963)

Let P and Q be two complex vector fields such that $p = \sigma(P)$ satisfies

$$\{\operatorname{Re} p, \operatorname{Im} p\} \neq 0 \quad \textit{on } p^{-1}(0)$$

so that P is not solvable. If locally $\operatorname{Ran} Q \subset \operatorname{Ran} P$ then $Q = PE$ for some $E \in C^\infty$ locally. If $Q \neq 0$ they generate the same right module.

The condition is that for any $f \in C^\infty$ there exists a distribution u such that $Qf = Pu$ locally.

This has been generalized to unsolvable ΨDO but the conclusions are then microlocal and weaker (D. and Wittsten 2013).

Instability for the Cauchy problem

The Cauchy-Kovalevsky theorem gives local solvability for the Cauchy problem for **quasilinear analytic vector fields** with *analytic* data on a noncharacteristic *analytic* initial surface.

But **almost all** data has an *arbitrarily small* C^∞ perturbation so that the Cauchy problem has **no** C^2 solution — for example when the bracket does not vanish. (Lerner, Morimoto and Xu 2010)

Example: the nonhomogeneous Burger's equation

$$\partial_t u + u\partial_{x_1} u = f(t,x,u) \qquad (t,x) \in \mathbf{R} \times \mathbf{R}^n$$

with analytic f has *no* C^2 solution for **almost all nonanalytic** f and Cauchy data $u(0,x)$.

For example when the bracket $[\partial_t + \operatorname{Re} u\,\partial_x, \operatorname{Im} u\,\partial_x] \neq 0$, that is, $2 \operatorname{Im} u\,\partial_{x_1} \operatorname{Re} u(t,x) \neq \operatorname{Im} f(t,x,u)$ when $t = 0$.

Instability for the Cauchy problem

The Cauchy-Kovalevsky theorem gives local solvability for the Cauchy problem for **quasilinear analytic vector fields** with *analytic* data on a noncharacteristic *analytic* initial surface.

But **almost all** data has an *arbitrarily small* C^∞ perturbation so that the Cauchy problem has **no** C^2 solution — for example when the bracket does not vanish. (Lerner, Morimoto and Xu 2010)

Example: the nonhomogeneous Burger's equation

$$\partial_t u + u\partial_{x_1} u = f(t,x,u) \qquad (t,x) \in \mathbf{R} \times \mathbf{R}^n$$

with analytic f has *no* C^2 solution for **almost all nonanalytic** f and Cauchy data $u(0,x)$.

For example when the bracket $[\partial_t + \operatorname{Re} u\,\partial_x, \operatorname{Im} u\,\partial_x] \neq 0$, that is, $2\operatorname{Im} u\,\partial_{x_1} \operatorname{Re} u(t,x) \neq \operatorname{Im} f(t,x,u)$ when $t = 0$.

Instability for the Cauchy problem

The Cauchy-Kovalevsky theorem gives local solvability for the Cauchy problem for **quasilinear analytic vector fields** with *analytic* data on a noncharacteristic *analytic* initial surface.

But **almost all** data has an *arbitrarily small* C^∞ perturbation so that the Cauchy problem has **no** C^2 solution — for example when the bracket does not vanish. (Lerner, Morimoto and Xu 2010)

Example: the nonhomogeneous Burger's equation

$$\partial_t u + u\partial_{x_1} u = f(t,x,u) \qquad (t,x) \in \mathbf{R} \times \mathbf{R}^n$$

with analytic f has *no* C^2 solution for **almost all nonanalytic** f and Cauchy data $u(0,x)$.

For example when the bracket $[\partial_t + \operatorname{Re} u\, \partial_x, \operatorname{Im} u\, \partial_x] \neq 0$, that is, $2 \operatorname{Im} u\, \partial_{x_1} \operatorname{Re} u(t,x) \neq \operatorname{Im} f(t,x,u)$ when $t = 0$.

Instability for the Cauchy problem

The Cauchy-Kovalevsky theorem gives local solvability for the Cauchy problem for **quasilinear analytic vector fields** with *analytic* data on a noncharacteristic *analytic* initial surface.

But **almost all** data has an *arbitrarily small* C^∞ perturbation so that the Cauchy problem has **no** C^2 solution — for example when the bracket does not vanish. (Lerner, Morimoto and Xu 2010)

Example: the nonhomogeneous Burger's equation

$$\partial_t u + u\partial_{x_1} u = f(t, x, u) \qquad (t, x) \in \mathbf{R} \times \mathbf{R}^n$$

with analytic f has *no* C^2 solution for **almost all nonanalytic** f and Cauchy data $u(0, x)$.

For example when the bracket $[\partial_t + \operatorname{Re} u\,\partial_x, \operatorname{Im} u\,\partial_x] \neq 0$, that is, $2\operatorname{Im} u\,\partial_{x_1} \operatorname{Re} u(t, x) \neq \operatorname{Im} f(t, x, u)$ when $t = 0$.

Pseudospectrum

The semiclassical Schrödinger operator with potential $V \in C^\infty(\mathbf{R}^n)$:

$$P(h) = -h^2\Delta + V(x) = p(x, hD) \qquad 0 < h \le 1$$

$z \in \mathbf{C}$ is in the *spectrum* $\sigma(P(h))$ if $P(h)u = zu$ has a solution $u \neq 0$ so that $P(h) - z$ is *not invertible*.

If $z \in \{|\xi|^2 + V(x) : \ \{\operatorname{Re} p, \operatorname{Im} p\} = 2\langle \xi, \nabla \operatorname{Im} V(x)\rangle \neq 0\}$ then

$$\|(P(h) - z)^{-1}\| \ge C_N h^{-N} \qquad \forall\, N$$

By Sard's theorem, this holds for **almost all values** when $\operatorname{Im} V \not\equiv 0$. It also holds for any semiclassical ΨDO. (D., Sjöstrand and Zworski 2004) and has been generalized to systems (D. 2008).

The "almost eigenvalues" are called **pseudospectrum** and the "almost eigenfunctions" are called *pseudomodes*.

Pseudospectrum

The semiclassical Schrödinger operator with potential $V \in C^\infty(\mathbf{R}^n)$:

$$P(h) = -h^2\Delta + V(x) = p(x, hD) \qquad 0 < h \le 1$$

$z \in \mathbf{C}$ is in the *spectrum* $\sigma(P(h))$ if $P(h)u = zu$ has a solution $u \neq 0$ so that $P(h) - z$ is *not invertible*.

If $z \in \{|\xi|^2 + V(x) : \ \{\operatorname{Re} p, \operatorname{Im} p\} = 2\langle \xi, \nabla \operatorname{Im} V(x)\rangle \neq 0\}$ then

$$\|(P(h) - z)^{-1}\| \ge C_N h^{-N} \qquad \forall\, N$$

By Sard's theorem, this holds for **almost all values** when $\operatorname{Im} V \neq 0$. It also holds for any semiclassical ΨDO. (D., Sjöstrand and Zworski 2004) and has been generalized to systems (D. 2008).

The "almost eigenvalues" are called **pseudospectrum** and the "almost eigenfunctions" are called *pseudomodes*.

Pseudospectrum

The semiclassical Schrödinger operator with potential $V \in C^\infty(\mathbf{R}^n)$:

$$P(h) = -h^2\Delta + V(x) = p(x, hD) \qquad 0 < h \leq 1$$

$z \in \mathbf{C}$ is in the *spectrum* $\sigma(P(h))$ if $P(h)u = zu$ has a solution $u \neq 0$ so that $P(h) - z$ is *not invertible*.

If $z \in \{|\xi|^2 + V(x) : \ \{\,\mathrm{Re}\, p, \mathrm{Im}\, p\,\} = 2\langle \xi, \nabla \,\mathrm{Im}\, V(x)\rangle \neq 0\}$ then

$$\|(P(h) - z)^{-1}\| \geq C_N h^{-N} \qquad \forall\, N$$

By Sard's theorem, this holds for **almost all values** when $\mathrm{Im}\, V \neq 0$. It also holds for any semiclassical ΨDO. (D., Sjöstrand and Zworski 2004) and has been generalized to systems (D. 2008).

The "almost eigenvalues" are called **pseudospectrum** and the "almost eigenfunctions" are called *pseudomodes*.

Pseudospectrum

The semiclassical Schrödinger operator with potential $V \in C^\infty(\mathbf{R}^n)$:

$$P(h) = -h^2\Delta + V(x) = p(x, hD) \qquad 0 < h \le 1$$

$z \in \mathbf{C}$ is in the *spectrum* $\sigma(P(h))$ if $P(h)u = zu$ has a solution $u \neq 0$ so that $P(h) - z$ is *not invertible*.

If $z \in \{|\xi|^2 + V(x) : \{\operatorname{Re} p, \operatorname{Im} p\} = 2\langle \xi, \nabla \operatorname{Im} V(x)\rangle \neq 0\}$ then

$$\|(P(h) - z)^{-1}\| \ge C_N h^{-N} \qquad \forall N$$

By Sard's theorem, this holds for **almost all values** when $\operatorname{Im} V \not\equiv 0$. It also holds for any semiclassical ΨDO. (D., Sjöstrand and Zworski 2004) and has been generalized to systems (D. 2008).

The "almost eigenvalues" are called **pseudospectrum** and the "almost eigenfunctions" are called *pseudomodes*.

Pseudospectrum

The semiclassical Schrödinger operator with potential $V \in C^\infty(\mathbf{R}^n)$:

$$P(h) = -h^2\Delta + V(x) = p(x, hD) \qquad 0 < h \le 1$$

$z \in \mathbf{C}$ is in the *spectrum* $\sigma(P(h))$ if $P(h)u = zu$ has a solution $u \ne 0$ so that $P(h) - z$ is *not invertible*.

If $z \in \{|\xi|^2 + V(x) :\ \{\operatorname{Re} p, \operatorname{Im} p\} = 2\langle \xi, \nabla \operatorname{Im} V(x)\rangle \ne 0\}$ then

$$\|(P(h) - z)^{-1}\| \ge C_N h^{-N} \qquad \forall\, N$$

By Sard's theorem, this holds for **almost all values** when $\operatorname{Im} V \not\equiv 0$. It also holds for any semiclassical ΨDO. (D., Sjöstrand and Zworski 2004) and has been generalized to systems (D. 2008).

The "almost eigenvalues" are called **pseudospectrum** and the "almost eigenfunctions" are called *pseudomodes*.

Generalization: the Nirenberg-Treves conjecture

Condition (Ψ)**:** $\operatorname{Im} p$ does **not** change sign from $-$ to $+$ on the oriented bicharacteristics of $\operatorname{Re} p$.

Condition (P)**: no change of sign**, i.e., both (Ψ) and $(\overline{\Psi})$ hold.

These conditions are *invariant* and gives that $H_{\operatorname{Re} p} \operatorname{Im} p \le 0$ at $p^{-1}(0)$.

Conditions (P) and (Ψ) are *equivalent* for PDO, but not for ΨDO. (Switch $\xi \leftrightarrow -\xi$, if the degree of p is even/odd, then H_p odd/even.)

Example For $P = D_t + if(t, x, D_x)$ with real first order operator f, $H_{\operatorname{Re} p} \operatorname{Im} p = \partial_t f(t, x, \xi) \le 0$ is the *bracket condition* and (Ψ) means that $t \mapsto f(t, x, \xi)$ cannot change sign from $-$ to $+$ for increasing t.

The Nirenberg-Treves conjecture (1969)

A principal type ΨDO is locally solvable if and only if the principal symbol satisfies condition (Ψ). Nirenberg and Treves proved this for ΨDOs with analytic symbols, which must vanish of finite order.

Generalization: the Nirenberg-Treves conjecture

Condition (Ψ)**:** $\operatorname{Im} p$ does **not** change sign from $-$ to $+$ on the oriented bicharacteristics of $\operatorname{Re} p$.

Condition (P)**: no change of sign**, i.e., both (Ψ) and $(\overline{\Psi})$ hold.

These conditions are *invariant* and gives that $H_{\operatorname{Re} p} \operatorname{Im} p \le 0$ at $p^{-1}(0)$.

Conditions (P) and (Ψ) are *equivalent* for PDO, but not for ΨDO. (Switch $\xi \leftrightarrow -\xi$, if the degree of p is even/odd, then H_p odd/even.)

Example For $P = D_t + if(t,x,D_x)$ with real first order operator f, $H_{\operatorname{Re} p} \operatorname{Im} p = \partial_t f(t,x,\xi) \le 0$ is the *bracket condition* and (Ψ) means that $t \mapsto f(t,x,\xi)$ cannot change sign from $-$ to $+$ for increasing t.

The Nirenberg-Treves conjecture (1969)

A principal type ΨDO is locally solvable if and only if the principal symbol satisfies condition (Ψ). Nirenberg and Treves proved this for ΨDOs with analytic symbols, which must vanish of finite order.

Generalization: the Nirenberg-Treves conjecture

Condition (Ψ)**:** $\operatorname{Im} p$ does **not** change sign from $-$ to $+$ on the oriented bicharacteristics of $\operatorname{Re} p$.

Condition (P)**: no change of sign**, i.e., both (Ψ) and $(\overline{\Psi})$ hold.

These conditions are *invariant* and gives that $H_{\operatorname{Re} p} \operatorname{Im} p \le 0$ at $p^{-1}(0)$.

Conditions (P) and (Ψ) are *equivalent* for PDO, but not for ΨDO. (Switch $\xi \leftrightarrow -\xi$, if the degree of p is even/odd, then H_p odd/even.)

Example For $P = D_t + if(t, x, D_x)$ with real first order operator f, $H_{\operatorname{Re} p} \operatorname{Im} p = \partial_t f(t, x, \xi) \le 0$ is the *bracket condition* and (Ψ) means that $t \mapsto f(t, x, \xi)$ cannot change sign from $-$ to $+$ for increasing t.

The Nirenberg-Treves conjecture (1969)

A principal type ΨDO is locally solvable if and only if the principal symbol satisfies condition (Ψ). Nirenberg and Treves proved this for ΨDOs with analytic symbols, which must vanish of finite order.

Generalization: the Nirenberg-Treves conjecture

Condition (Ψ)**:** $\operatorname{Im} p$ does **not** change sign from $-$ to $+$ on the oriented bicharacteristics of $\operatorname{Re} p$.

Condition (P)**: no change of sign**, i.e., both (Ψ) and $(\overline{\Psi})$ hold.

These conditions are *invariant* and gives that $H_{\operatorname{Re} p} \operatorname{Im} p \le 0$ at $p^{-1}(0)$.

Conditions (P) and (Ψ) are *equivalent* for PDO, but not for ΨDO. (Switch $\xi \leftrightarrow -\xi$, if the degree of p is even/odd, then H_p odd/even.)

Example For $P = D_t + if(t, x, D_x)$ with real first order operator f, $H_{\operatorname{Re} p} \operatorname{Im} p = \partial_t f(t, x, \xi) \le 0$ is the *bracket condition* and (Ψ) means that $t \mapsto f(t, x, \xi)$ cannot change sign from $-$ to $+$ for increasing t.

The Nirenberg-Treves conjecture (1969)

A principal type ΨDO is locally solvable if and only if the principal symbol satisfies condition (Ψ). Nirenberg and Treves proved this for ΨDOs with analytic symbols, which must vanish of finite order.

Generalization: the Nirenberg-Treves conjecture

Condition (Ψ)**:** $\operatorname{Im} p$ does **not** change sign from $-$ to $+$ on the oriented bicharacteristics of $\operatorname{Re} p$.

Condition (P)**: no change of sign**, i.e., both (Ψ) and $(\overline{\Psi})$ hold.

These conditions are *invariant* and gives that $H_{\operatorname{Re} p} \operatorname{Im} p \le 0$ at $p^{-1}(0)$.

Conditions (P) and (Ψ) are *equivalent* for PDO, but not for ΨDO. (Switch $\xi \leftrightarrow -\xi$, if the degree of p is even/odd, then H_p odd/even.)

Example For $P = D_t + if(t, x, D_x)$ with real first order operator f, $H_{\operatorname{Re} p} \operatorname{Im} p = \partial_t f(t, x, \xi) \le 0$ is the *bracket condition* and (Ψ) means that $t \mapsto f(t, x, \xi)$ cannot change sign from $-$ to $+$ for increasing t.

The Nirenberg-Treves conjecture (1969)

A principal type ΨDO is locally solvable if and only if the principal symbol satisfies condition (Ψ). Nirenberg and Treves proved this for ΨDOs with analytic symbols, which must vanish of finite order.

Resolution of the Nirenberg-Treves conjecture

Condition (P) is **sufficient** for local solvability for PDO. (Beals-Fefferman 1973)

Condition (Ψ) is **necessary** for solvability for ΨDO. (Hörmander 1980)

Condition (Ψ) is **sufficient** for solvability for ΨDO in **two** variables. (Lerner 1988)

Theorem (D. 2006)

If P is a principal type ΨDO with principal symbol satisfying condition (Ψ) then P is locally solvable.

This finally resolved the NT conjecture, that condition (Ψ) is *equivalent* to local solvability for ΨDO of principal type.

This has been generalized to *systems* of principal type with constant characteristics (D. 2011), but not much is known in general for *nonprincipal type operators* .

Resolution of the Nirenberg-Treves conjecture

Condition (P) is **sufficient** for local solvability for PDO. (Beals-Fefferman 1973)

Condition (Ψ) is **necessary** for solvability for ΨDO. (Hörmander 1980)

Condition (Ψ) is **sufficient** for solvability for ΨDO in **two** variables. (Lerner 1988)

Theorem (D. 2006)

If P is a principal type ΨDO with principal symbol satisfying condition (Ψ) then P is locally solvable.

This finally resolved the NT conjecture, that condition (Ψ) is *equivalent* to local solvability for ΨDO of principal type.

This has been generalized to *systems* of principal type with constant characteristics (D. 2011), but not much is known in general for *nonprincipal type operators* .

Resolution of the Nirenberg-Treves conjecture

Condition (P) is **sufficient** for local solvability for PDO. (Beals-Fefferman 1973)

Condition (Ψ) is **necessary** for solvability for ΨDO. (Hörmander 1980)

Condition (Ψ) is **sufficient** for solvability for ΨDO in **two** variables. (Lerner 1988)

Theorem (D. 2006)

If P is a principal type ΨDO with principal symbol satisfying condition (Ψ) then P is locally solvable.

This finally resolved the NT conjecture, that condition (Ψ) is *equivalent* to local solvability for ΨDO of principal type.

This has been generalized to *systems* of principal type with constant characteristics (D. 2011), but not much is known in general for *nonprincipal type operators* .

Resolution of the Nirenberg-Treves conjecture

Condition (P) is **sufficient** for local solvability for PDO. (Beals-Fefferman 1973)

Condition (Ψ) is **necessary** for solvability for ΨDO. (Hörmander 1980)

Condition (Ψ) is **sufficient** for solvability for ΨDO in **two** variables. (Lerner 1988)

Theorem (D. 2006)

If P is a principal type ΨDO with principal symbol satisfying condition (Ψ) then P is locally solvable.

This finally resolved the NT conjecture, that condition (Ψ) is *equivalent* to local solvability for ΨDO of principal type.

This has been generalized to *systems* of principal type with constant characteristics (D. 2011), but not much is known in general for *nonprincipal type operators* .

Resolution of the Nirenberg-Treves conjecture

Condition (P) is **sufficient** for local solvability for PDO. (Beals-Fefferman 1973)

Condition (Ψ) is **necessary** for solvability for ΨDO. (Hörmander 1980)

Condition (Ψ) is **sufficient** for solvability for ΨDO in **two** variables. (Lerner 1988)

Theorem (D. 2006)

If P is a principal type ΨDO with principal symbol satisfying condition (Ψ) then P is locally solvable.

This finally resolved the NT conjecture, that condition (Ψ) is *equivalent* to local solvability for ΨDO of principal type.

This has been generalized to *systems* of principal type with constant characteristics (D. 2011), but not much is known in general for *nonprincipal type operators* .

Resolution of the Nirenberg-Treves conjecture

Condition (P) is **sufficient** for local solvability for PDO. (Beals-Fefferman 1973)

Condition (Ψ) is **necessary** for solvability for ΨDO. (Hörmander 1980)

Condition (Ψ) is **sufficient** for solvability for ΨDO in **two** variables. (Lerner 1988)

Theorem (D. 2006)

If P is a principal type ΨDO with principal symbol satisfying condition (Ψ) then P is locally solvable.

This finally resolved the NT conjecture, that condition (Ψ) is *equivalent* to local solvability for ΨDO of principal type.

This has been generalized to *systems* of principal type with constant characteristics (D. 2011), but not much is known in general for *nonprincipal type operators* .

Why condition (Ψ)?

P is solvable $\Leftrightarrow$ P^* has a finite-dimensional kernel

Example Let $(t, x) \in \mathbf{R} \times \mathbf{R}^n$, $|D_x| = \Delta_x^{1/2} \geq 0$ and

$$P = D_t + it|D_x| \qquad \text{with} \qquad p = \tau + it|\xi|$$

Thus, p does **not** satisfy condition (Ψ) and

$$P^* = D_t - it|D_x| = \frac{1}{i}(\partial_t + t|D_x|)$$

We have that $P^* u = 0$ if

$$u(t, x) = \int e^{i\langle x, \xi \rangle - t^2|\xi|/2} \phi(\xi)\, d\xi \in C^0 \qquad \forall\ \phi \in L^1$$

so P^* has an **infinite-dimensional kernel** and P is **not** solvable. The solutions u are called *pseudomodes*.

Why condition (Ψ)?

P is solvable $\Leftrightarrow$ P^* has a finite-dimensional kernel

Example Let $(t, x) \in \mathbf{R} \times \mathbf{R}^n$, $|D_x| = \Delta_x^{1/2} \geq 0$ and

$$P = D_t + it|D_x| \qquad \text{with} \qquad p = \tau + it|\xi|$$

Thus, p does **not** satisfy condition (Ψ) and

$$P^* = D_t - it|D_x| = \frac{1}{i}(\partial_t + t|D_x|)$$

We have that $P^*u = 0$ if

$$u(t, x) = \int e^{i\langle x, \xi\rangle - t^2|\xi|/2}\phi(\xi)\, d\xi \in C^0 \qquad \forall\ \phi \in L^1$$

so P^* has an **infinite-dimensional kernel** and P is **not** solvable. The solutions u are called *pseudomodes*.

Why condition (Ψ)?

P is solvable $\Leftrightarrow$ P^* has a finite-dimensional kernel

Example Let $(t, x) \in \mathbf{R} \times \mathbf{R}^n$, $|D_x| = \Delta_x^{1/2} \geq 0$ and

$$P = D_t + it|D_x| \qquad \text{with} \qquad p = \tau + it|\xi|$$

Thus, p does **not** satisfy condition (Ψ) and

$$P^* = D_t - it|D_x| = \frac{1}{i}(\partial_t + t|D_x|)$$

We have that $P^* u = 0$ if

$$u(t, x) = \int e^{i\langle x, \xi\rangle - t^2|\xi|/2} \phi(\xi)\, d\xi \in C^0 \qquad \forall\ \phi \in L^1$$

so P^* has an **infinite-dimensional kernel** and P is **not** solvable. The solutions u are called *pseudomodes*.

Why condition (Ψ)?

P is solvable $\Leftrightarrow$ P^* has a finite-dimensional kernel

Example Let $(t,x) \in \mathbf{R} \times \mathbf{R}^n$, $|D_x| = \Delta_x^{1/2} \geq 0$ and

$$P = D_t + it|D_x| \qquad \text{with} \qquad p = \tau + it|\xi|$$

Thus, p does **not** satisfy condition (Ψ) and

$$P^* = D_t - it|D_x| = \frac{1}{i}(\partial_t + t|D_x|)$$

We have that $P^*u = 0$ if

$$u(t,x) = \int e^{i\langle x,\xi\rangle - t^2|\xi|/2}\phi(\xi)\, d\xi \in C^0 \qquad \forall\ \phi \in L^1$$

so P^* has an **infinite-dimensional kernel** and P is **not** solvable. The solutions u are called *pseudomodes*.

Why condition (Ψ)?

P is solvable $\Leftrightarrow$ P^* has a finite-dimensional kernel

Example Let $(t, x) \in \mathbf{R} \times \mathbf{R}^n$, $|D_x| = \Delta_x^{1/2} \geq 0$ and

$$P = D_t + it|D_x| \qquad \text{with} \qquad p = \tau + it|\xi|$$

Thus, p does **not** satisfy condition (Ψ) and

$$P^* = D_t - it|D_x| = \frac{1}{i}(\partial_t + t|D_x|)$$

We have that $P^* u = 0$ if

$$u(t, x) = \int e^{i\langle x,\xi\rangle - t^2|\xi|/2}\phi(\xi)\, d\xi \in C^0 \qquad \forall\ \phi \in L^1$$

so P^* has an **infinite-dimensional kernel** and P is **not** solvable. The solutions u are called *pseudomodes*.

Nonprincipal type operators

If the principal symbol p_m vanishes of **at least second order**, the *subprincipal symbol* $p_s \cong p_{m-1}$ is an important invariant.

Example Let

$$P = D_1 D_2 + B(x, D) \qquad \text{on } \mathbf{R}^n \qquad n \geq 3$$

where B is first order. The principal symbol $\xi_1 \xi_2$ vanishes of second order at the **double characteristic set** $\Sigma_2 = \{\, \xi_1 = \xi_2 = 0 \,\}$ which is *involutive*, i.e., if p and q vanish on Σ_2 then $\{p, q\}$ vanishes.

The invariant **subprincipal symbol** is the principal symbol b of B on Σ_2.

If $\operatorname{Im} b$ *changes sign* on x_1 or x_2 lines, then P is *not solvable*. If $\operatorname{Im} b \neq 0$ then P is solvable. (Mendoza–Uhlmann 1983-84).

Nonprincipal type operators

If the principal symbol p_m vanishes of **at least second order**, the *subprincipal symbol* $p_s \cong p_{m-1}$ is an important invariant.

Example Let

$$P = D_1 D_2 + B(x, D) \qquad \text{on } \mathbf{R}^n \quad n \geq 3$$

where B is first order. The principal symbol $\xi_1 \xi_2$ vanishes of second order at the **double characteristic set** $\Sigma_2 = \{\, \xi_1 = \xi_2 = 0 \,\}$ which is *involutive*, i.e., if p and q vanish on Σ_2 then $\{p, q\}$ vanishes.

The invariant **subprincipal symbol** is the principal symbol b of B on Σ_2.

If $\operatorname{Im} b$ *changes sign* on x_1 or x_2 lines, then P is *not solvable*. If $\operatorname{Im} b \neq 0$ then P is solvable. (Mendoza–Uhlmann 1983-84).

Nonprincipal type operators

If the principal symbol p_m vanishes of **at least second order**, the *subprincipal symbol* $p_s \cong p_{m-1}$ is an important invariant.

Example Let

$$P = D_1 D_2 + B(x, D) \qquad \text{on } \mathbf{R}^n \qquad n \geq 3$$

where B is first order. The principal symbol $\xi_1 \xi_2$ vanishes of second order at the **double characteristic set** $\Sigma_2 = \{\, \xi_1 = \xi_2 = 0 \,\}$ which is *involutive*, i.e., if p and q vanish on Σ_2 then $\{p, q\}$ vanishes.

The invariant **subprincipal symbol** is the principal symbol b of B on Σ_2.

If $\operatorname{Im} b$ *changes sign* on x_1 or x_2 lines, then P is *not solvable*. If $\operatorname{Im} b \neq 0$ then P is solvable. (Mendoza–Uhlmann 1983-84).

Nonprincipal type operators

If the principal symbol p_m vanishes of **at least second order**, the *subprincipal symbol* $p_s \cong p_{m-1}$ is an important invariant.

Example Let

$$P = D_1 D_2 + B(x, D) \qquad \text{on } \mathbf{R}^n \qquad n \geq 3$$

where B is first order. The principal symbol $\xi_1 \xi_2$ vanishes of second order at the **double characteristic set** $\Sigma_2 = \{\, \xi_1 = \xi_2 = 0 \,\}$ which is *involutive*, i.e., if p and q vanish on Σ_2 then $\{p, q\}$ vanishes.

The invariant **subprincipal symbol** is the principal symbol b of B on Σ_2.

If $\operatorname{Im} b$ *changes sign* on x_1 or x_2 lines, then P is *not solvable*. If $\operatorname{Im} b \neq 0$ then P is solvable. (Mendoza–Uhlmann 1983-84).

Nonprincipal type operators

If the principal symbol p_m vanishes of **at least second order**, the *subprincipal symbol* $p_s \cong p_{m-1}$ is an important invariant.

Example Let

$$P = D_1 D_2 + B(x, D) \qquad \text{on } \mathbf{R}^n \qquad n \geq 3$$

where B is first order. The principal symbol $\xi_1 \xi_2$ vanishes of second order at the **double characteristic set** $\Sigma_2 = \{ \xi_1 = \xi_2 = 0 \}$ which is *involutive*, i.e., if p and q vanish on Σ_2 then $\{p, q\}$ vanishes.

The invariant **subprincipal symbol** is the principal symbol b of B on Σ_2.

If $\operatorname{Im} b$ *changes sign* on x_1 or x_2 lines, then P is *not solvable*. If $\operatorname{Im} b \neq 0$ then P is solvable. (Mendoza–Uhlmann 1983-84).

Necessary conditions: limit characteristics on Σ_2

Definition

$\Gamma \subset \Sigma_2$ is a **limit bicharacteristic** (possibly a point) of real p if there exist bicharacteristics Γ_j of p that converge to Γ as smooth curves.

Example $p = \prod_j p_j$, p_j of real principal type, $p_j = \{ p_j, p_k \} = 0$ on the double characteristic set Σ_2 and $p_j \neq p_k$ on $p^{-1}(0) \setminus \Sigma_2$. Then the limit bicharacteristics of p are the bicharacteristics of p_j on Σ_2, $\forall\, j$.

Assuming a curvature condition on the limit bicharacteristics we get the following result for nonprincipal type operators.

Theorem (D. 2016)

If P has real principal symbol and the imaginary part of the subprincipal symbol $\operatorname{Im} p_s$ changes sign from $-$ to $+$ on a limit bicharacteristic on Σ_2, then P is not locally solvable.

Necessary conditions: limit characteristics on Σ_2

Definition

$\Gamma \subset \Sigma_2$ is a **limit bicharacteristic** (possibly a point) of real p if there exist bicharacteristics Γ_j of p that converge to Γ as smooth curves.

Example $p = \prod_j p_j$, p_j of real principal type, $p_j = \{ p_j, p_k \} = 0$ on the double characteristic set Σ_2 and $p_j \neq p_k$ on $p^{-1}(0) \setminus \Sigma_2$. Then the limit bicharacteristics of p are the bicharacteristics of p_j on Σ_2, $\forall j$.

Assuming a curvature condition on the limit bicharacteristics we get the following result for nonprincipal type operators.

Theorem (D. 2016)

If P has real principal symbol and the imaginary part of the subprincipal symbol $\operatorname{Im} p_s$ changes sign from $-$ to $+$ on a limit bicharacteristic on Σ_2, then P is not locally solvable.

Necessary conditions: limit characteristics on Σ_2

Definition

$\Gamma \subset \Sigma_2$ is a **limit bicharacteristic** (possibly a point) of real p if there exist bicharacteristics Γ_j of p that converge to Γ as smooth curves.

Example $p = \prod_j p_j$, p_j of real principal type, $p_j = \{ p_j, p_k \} = 0$ on the double characteristic set Σ_2 and $p_j \neq p_k$ on $p^{-1}(0) \setminus \Sigma_2$. Then the limit bicharacteristics of p are the bicharacteristics of p_j on Σ_2, $\forall j$.

Assuming a curvature condition on the limit bicharacteristics we get the following result for nonprincipal type operators.

Theorem (D. 2016)

If P has real principal symbol and the imaginary part of the subprincipal symbol $\operatorname{Im} p_s$ changes sign from $-$ to $+$ on a limit bicharacteristic on Σ_2, then P is not locally solvable.

Necessary conditions: limit characteristics on Σ_2

Definition

$\Gamma \subset \Sigma_2$ is a **limit bicharacteristic** (possibly a point) of real p if there exist bicharacteristics Γ_j of p that converge to Γ as smooth curves.

Example $p = \prod_j p_j$, p_j of real principal type, $p_j = \{ p_j, p_k \} = 0$ on the double characteristic set Σ_2 and $p_j \neq p_k$ on $p^{-1}(0) \setminus \Sigma_2$. Then the limit bicharacteristics of p are the bicharacteristics of p_j on Σ_2, $\forall j$.

Assuming a curvature condition on the limit bicharacteristics we get the following result for nonprincipal type operators.

Theorem (D. 2016)

If P has real principal symbol and the imaginary part of the subprincipal symbol $\operatorname{Im} p_s$ changes sign from $-$ to $+$ on a limit bicharacteristic on Σ_2, then P is not locally solvable.

Necessary conditions: limit characteristics on Σ_2

Definition

$\Gamma \subset \Sigma_2$ is a **limit bicharacteristic** (possibly a point) of real p if there exist bicharacteristics Γ_j of p that converge to Γ as smooth curves.

Example $p = \prod_j p_j$, p_j of real principal type, $p_j = \{ p_j, p_k \} = 0$ on the double characteristic set Σ_2 and $p_j \neq p_k$ on $p^{-1}(0) \setminus \Sigma_2$. Then the limit bicharacteristics of p are the bicharacteristics of p_j on Σ_2, $\forall j$.

Assuming a curvature condition on the limit bicharacteristics we get the following result for nonprincipal type operators.

Theorem (D. 2016)

If P has real principal symbol and the imaginary part of the subprincipal symbol $\operatorname{Im} p_s$ changes sign from $-$ to $+$ on a limit bicharacteristic on Σ_2, then P is not locally solvable.

Necessary conditions: subprincipal characteristics

Assume Σ_2 is involutive with symplectic **foliation** given by the Hamilton fields, it is **nonradial** if *all* its Hamilton fields are.

Example $\Sigma_2 = \{\, \xi = 0 \,\}$ with leaves $L = \{\, (x, y_0; 0, \eta_0) : x \in \mathbf{R}^k \,\}$.

Definition

P is of **subprincipal type** if if p vanishes of second order at Σ_2 and $H_{p_s}\big|_{\Sigma_2} \subseteq T\Sigma_2$ is transversal to the leaves L when $p_s = 0$.

Examples $\partial_x p_s = 0$ and $\partial_{y,\eta} p_s \neq 0$ when $p_s = 0$ on $\Sigma_2 = \{\, \xi = 0 \,\}$.

$\Delta_x + D_t$ is of *subprincipal type*, but $\Delta_x + D_x$ is *not*.

H_{p_s} is only defined modulo TL. To define condition (Ψ) for p_s we need

$$\left| dp_s\big|_{TL} \right| \le C|p_s| \qquad \text{for leaves } L \text{ of } \Sigma_2$$

Then p_s is **constant on the leaves** after multiplication with a nonvanishing factor, so $\operatorname{Im} p_s$ does *not* changes sign on the leaves.

Necessary conditions: subprincipal characteristics

Assume Σ_2 is involutive with symplectic **foliation** given by the Hamilton fields, it is **nonradial** if *all* its Hamilton fields are.

Example $\Sigma_2 = \{\, \xi = 0 \,\}$ with leaves $L = \{\, (x, y_0; 0, \eta_0) : x \in \mathbf{R}^k \,\}$.

Definition

P is of **subprincipal type** if if p vanishes of second order at Σ_2 and $H_{p_s}\big|_{\Sigma_2} \subseteq T\Sigma_2$ is transversal to the leaves L when $p_s = 0$.

Examples $\partial_x p_s = 0$ and $\partial_{y,\eta} p_s \neq 0$ when $p_s = 0$ on $\Sigma_2 = \{\, \xi = 0 \,\}$. $\Delta_x + D_t$ is of *subprincipal type*, but $\Delta_x + D_x$ is *not*.

H_{p_s} is only defined modulo TL. To define condition (Ψ) for p_s we need

$$\left| dp_s\big|_{TL} \right| \leq C|p_s| \qquad \text{for leaves } L \text{ of } \Sigma_2$$

Then p_s is **constant on the leaves** after multiplication with a nonvanishing factor, so $\operatorname{Im} p_s$ does *not* changes sign on the leaves.

Necessary conditions: subprincipal characteristics

Assume Σ_2 is involutive with symplectic **foliation** given by the Hamilton fields, it is **nonradial** if *all* its Hamilton fields are.

Example $\Sigma_2 = \{\, \xi = 0 \,\}$ with leaves $L = \{\, (x, y_0; 0, \eta_0) : x \in \mathbf{R}^k \,\}$.

Definition

P is of **subprincipal type** if if p vanishes of second order at Σ_2 and $H_{p_s}\big|_{\Sigma_2} \subseteq T\Sigma_2$ is transversal to the leaves L when $p_s = 0$.

Examples $\partial_x p_s = 0$ and $\partial_{y,\eta} p_s \neq 0$ when $p_s = 0$ on $\Sigma_2 = \{\, \xi = 0 \,\}$.

$\Delta_x + D_t$ is of *subprincipal type*, but $\Delta_x + D_x$ is *not*.

H_{p_s} is only defined modulo TL. To define condition (Ψ) for p_s we need

$$\left| dp_s \big|_{TL} \right| \le C |p_s| \qquad \text{for leaves } L \text{ of } \Sigma_2$$

Then p_s is **constant on the leaves** after multiplication with a nonvanishing factor, so $\operatorname{Im} p_s$ does *not* changes sign on the leaves.

Necessary conditions: subprincipal characteristics

Assume Σ_2 is involutive with symplectic **foliation** given by the Hamilton fields, it is **nonradial** if *all* its Hamilton fields are.

Example $\Sigma_2 = \{\, \xi = 0 \,\}$ with leaves $L = \{\, (x, y_0; 0, \eta_0) : x \in \mathbf{R}^k \,\}$.

Definition

P is of **subprincipal type** if if p vanishes of second order at Σ_2 and $H_{p_s}\big|_{\Sigma_2} \subseteq T\Sigma_2$ is transversal to the leaves L when $p_s = 0$.

Examples $\partial_x p_s = 0$ and $\partial_{y,\eta} p_s \neq 0$ when $p_s = 0$ on $\Sigma_2 = \{\, \xi = 0 \,\}$.

$\Delta_x + D_t$ is of *subprincipal type*, but $\Delta_x + D_x$ is *not*.

H_{p_s} is only defined modulo TL. To define condition (Ψ) for p_s we need

$$\left| dp_s \big|_{TL} \right| \le C |p_s| \qquad \text{for leaves } L \text{ of } \Sigma_2$$

Then p_s is **constant on the leaves** after multiplication with a nonvanishing factor, so $\operatorname{Im} p_s$ does *not* changes sign on the leaves.

Necessary conditions: subprincipal characteristics

Assume Σ_2 is involutive with symplectic **foliation** given by the Hamilton fields, it is **nonradial** if *all* its Hamilton fields are.

Example $\Sigma_2 = \{\, \xi = 0 \,\}$ with leaves $L = \left\{\, (x, y_0; 0, \eta_0) : x \in \mathbf{R}^k \,\right\}$.

Definition

P is of **subprincipal type** if if p vanishes of second order at Σ_2 and $H_{p_s}\big|_{\Sigma_2} \subseteq T\Sigma_2$ is transversal to the leaves L when $p_s = 0$.

Examples $\partial_x p_s = 0$ and $\partial_{y,\eta} p_s \neq 0$ when $p_s = 0$ on $\Sigma_2 = \{\, \xi = 0 \,\}$.

$\Delta_x + D_t$ is of *subprincipal type*, but $\Delta_x + D_x$ is *not*.

H_{p_s} is only defined modulo TL. To define condition (Ψ) for p_s we need

$$\left| dp_s \big|_{TL} \right| \le C|p_s| \qquad \text{for leaves } L \text{ of } \Sigma_2$$

Then p_s is **constant on the leaves** after multiplication with a nonvanishing factor, so $\operatorname{Im} p_s$ does *not* changes sign on the leaves.

Necessary conditions: subprincipal characteristics

Assume Σ_2 is involutive with symplectic **foliation** given by the Hamilton fields, it is **nonradial** if *all* its Hamilton fields are.

Example $\Sigma_2 = \{\, \xi = 0 \,\}$ with leaves $L = \left\{\, (x, y_0; 0, \eta_0) : x \in \mathbf{R}^k \,\right\}$.

Definition

P is of **subprincipal type** if if p vanishes of second order at Σ_2 and $H_{p_s}\big|_{\Sigma_2} \subseteq T\Sigma_2$ is transversal to the leaves L when $p_s = 0$.

Examples $\partial_x p_s = 0$ and $\partial_{y,\eta} p_s \neq 0$ when $p_s = 0$ on $\Sigma_2 = \{\, \xi = 0 \,\}$.

$\Delta_x + D_t$ is of *subprincipal type*, but $\Delta_x + D_x$ is *not*.

H_{p_s} is only defined modulo TL. To define condition (Ψ) for p_s we need

$$\left| dp_s \big|_{TL} \right| \leq C |p_s| \qquad \text{for leaves } L \text{ of } \Sigma_2$$

Then p_s is **constant on the leaves** after multiplication with a nonvanishing factor, so $\operatorname{Im} p_s$ does *not* changes sign on the leaves.

Definition

Condition $Sub(\Psi)$: $\operatorname{Im} p_s$ has constant sign on the leaves L of Σ_2 and p_s satisfies condition (Ψ) on $T^\sigma\Sigma_2 = T\Sigma_2/TL$, which is symplectic.

For PDE we have condition (P), thus *no* sign changes.

Example $T^\sigma\Sigma_2 \cong \{\, (w_0; y, \eta) : w_0 \in \Sigma_2 \,\}$ for $\Sigma_2 = \{\, \xi = 0 \,\}$.

When the sign change is of *infinite order* we also need assumtions that make the pseudomodes stay *local* and not get dispersed.

Theorem (D. 2017)

If P has principal symbol vanishing of at least second order at a nonradial involutive manifold Σ_2, is of subprincipal type and does not satisfy condition $Sub(\Psi)$, then P is not locally solvable.

Also have counterexamples when the Taylor expansion of the **refined principal symbol** $p_r = p + p_s$ does not satisfy (Ψ) (D. 2018).

Definition

Condition $Sub(\Psi)$: $\operatorname{Im} p_s$ has constant sign on the leaves L of Σ_2 and p_s satisfies condition (Ψ) on $T^\sigma\Sigma_2 = T\Sigma_2/TL$, which is symplectic.

For PDE we have condition (P), thus *no* sign changes.

Example $T^\sigma\Sigma_2 \cong \{\,(w_0; y, \eta) : w_0 \in \Sigma_2\,\}$ for $\Sigma_2 = \{\,\xi = 0\,\}$.

When the sign change is of *infinite order* we also need assumtions that make the pseudomodes stay *local* and not get dispersed.

Theorem (D. 2017)

If P has principal symbol vanishing of at least second order at a nonradial involutive manifold Σ_2, is of subprincipal type and does not satisfy condition $Sub(\Psi)$, then P is not locally solvable.

Also have counterexamples when the Taylor expansion of the **refined principal symbol** $p_r = p + p_s$ does not satisfy (Ψ) (D. 2018).

Definition

Condition $Sub(\Psi)$: $\operatorname{Im} p_s$ has constant sign on the leaves L of Σ_2 and p_s satisfies condition (Ψ) on $T^\sigma\Sigma_2 = T\Sigma_2/TL$, which is symplectic.

For PDE we have condition (P), thus *no* sign changes.

Example $T^\sigma\Sigma_2 \cong \{\, (w_0; y, \eta) : w_0 \in \Sigma_2 \,\}$ for $\Sigma_2 = \{\, \xi = 0 \,\}$.

When the sign change is of *infinite order* we also need assumtions that make the pseudomodes stay *local* and not get dispersed.

Theorem (D. 2017)

If P has principal symbol vanishing of at least second order at a nonradial involutive manifold Σ_2, is of subprincipal type and does not satisfy condition $Sub(\Psi)$, then P is not locally solvable.

Also have counterexamples when the Taylor expansion of the **refined principal symbol** $p_r = p + p_s$ does not satisfy (Ψ) (D. 2018).

Definition

Condition $Sub(\Psi)$: $\operatorname{Im} p_s$ has constant sign on the leaves L of Σ_2 and p_s satisfies condition (Ψ) on $T^\sigma\Sigma_2 = T\Sigma_2/TL$, which is symplectic.

For PDE we have condition (P), thus *no* sign changes.

Example $T^\sigma\Sigma_2 \cong \{\,(w_0; y, \eta) : w_0 \in \Sigma_2\,\}$ for $\Sigma_2 = \{\,\xi = 0\,\}$.

When the sign change is of *infinite order* we also need assumtions that make the pseudomodes stay *local* and not get dispersed.

Theorem (D. 2017)

If P has principal symbol vanishing of at least second order at a nonradial involutive manifold Σ_2, is of subprincipal type and does not satisfy condition $Sub(\Psi)$, then P is not locally solvable.

Also have counterexamples when the Taylor expansion of the **refined principal symbol** $p_r = p + p_s$ does not satisfy (Ψ) (D. 2018).

Definition

Condition $Sub(\Psi)$: $\operatorname{Im} p_s$ has constant sign on the leaves L of Σ_2 and p_s satisfies condition (Ψ) on $T^\sigma\Sigma_2 = T\Sigma_2/TL$, which is symplectic.

For PDE we have condition (P), thus *no* sign changes.

Example $T^\sigma\Sigma_2 \cong \{\, (w_0; y, \eta) : w_0 \in \Sigma_2 \,\}$ for $\Sigma_2 = \{\, \xi = 0 \,\}$.

When the sign change is of *infinite order* we also need assumtions that make the pseudomodes stay *local* and not get dispersed.

Theorem (D. 2017)

If P has principal symbol vanishing of at least second order at a nonradial involutive manifold Σ_2, is of subprincipal type and does not satisfy condition $Sub(\Psi)$, then P is not locally solvable.

Also have counterexamples when the Taylor expansion of the **refined principal symbol** $p_r = p + p_s$ does not satisfy (Ψ) (D. 2018).

Definition

Condition $Sub(\Psi)$: $\operatorname{Im} p_s$ has constant sign on the leaves L of Σ_2 and p_s satisfies condition (Ψ) on $T^\sigma\Sigma_2 = T\Sigma_2/TL$, which is symplectic.

For PDE we have condition (P), thus *no* sign changes.

Example $T^\sigma\Sigma_2 \cong \{ (w_0; y, \eta) : w_0 \in \Sigma_2 \}$ for $\Sigma_2 = \{ \xi = 0 \}$.

When the sign change is of *infinite order* we also need assumtions that make the pseudomodes stay *local* and not get dispersed.

Theorem (D. 2017)

If P has principal symbol vanishing of at least second order at a nonradial involutive manifold Σ_2, is of subprincipal type and does not satisfy condition $Sub(\Psi)$, then P is not locally solvable.

Also have counterexamples when the Taylor expansion of the **refined principal symbol** $p_r = p + p_s$ does not satisfy (Ψ) (D. 2018).

Sufficient conditions: the refined principal symbol

Definition

P is of *real subprincipal type* if the principal symbol is real, vanish of exactly second order at Σ_2 and $H_{\mathrm{Re}\, p_s}\big|_{\Sigma_2} \subseteq T\Sigma_2$ is transversal to the leaves L. Condition $Sub_r(\Psi)$: p_r satisfies condition (Ψ) at the limit Σ_2.

Example If $\operatorname{Hess} p \neq 0$ then $Sub_r(\Psi)$ holds iff $\operatorname{Im} p_s$ has constant sign on the leaves of Σ_2 and (Ψ) is holds for **any bicharacteristic** of $\operatorname{Re} p_s$.

Theorem (D. 2023)

If P is of real subprincipal type and satisfies condition $Sub_r(\Psi)$ then P is locally solvable.

Thus, condition $Sub_r(\Psi)$ is equivalent to solvability of ΨDO of **real subprincipal type**.

Sufficient conditions: the refined principal symbol

Definition

P is of *real subprincipal type* if the principal symbol is real, vanish of exactly second order at Σ_2 and $H_{\mathrm{Re}\, p_s}\big|_{\Sigma_2} \subseteq T\Sigma_2$ is transversal to the leaves L. Condition $Sub_r(\Psi)$: p_r satisfies condition (Ψ) at the limit Σ_2.

Example If $\mathrm{Hess}\, p \neq 0$ then $Sub_r(\Psi)$ holds iff $\mathrm{Im}\, p_s$ has constant sign on the leaves of Σ_2 and (Ψ) is holds for **any bicharacteristic** of $\mathrm{Re}\, p_s$.

Theorem (D. 2023)

If P is of real subprincipal type and satisfies condition $Sub_r(\Psi)$ then P is locally solvable.

Thus, condition $Sub_r(\Psi)$ is equivalent to solvability of ΨDO of **real subprincipal type**.

Sufficient conditions: the refined principal symbol

Definition

P is of *real subprincipal type* if the principal symbol is real, vanish of exactly second order at Σ_2 and $H_{\mathrm{Re}\, p_s}\big|_{\Sigma_2} \subseteq T\Sigma_2$ is transversal to the leaves L. Condition $Sub_r(\Psi)$: p_r satisfies condition (Ψ) at the limit Σ_2.

Example If $\mathrm{Hess}\, p \neq 0$ then $Sub_r(\Psi)$ holds iff $\mathrm{Im}\, p_s$ has constant sign on the leaves of Σ_2 and (Ψ) is holds for **any bicharacteristic** of $\mathrm{Re}\, p_s$.

Theorem (D. 2023)

If P is of real subprincipal type and satisfies condition $Sub_r(\Psi)$ then P is locally solvable.

Thus, condition $Sub_r(\Psi)$ is equivalent to solvability of ΨDO of **real subprincipal type**.

Sufficient conditions: the refined principal symbol

Definition

P is of *real subprincipal type* if the principal symbol is real, vanish of exactly second order at Σ_2 and $H_{\operatorname{Re} p_s}\big|_{\Sigma_2} \subseteq T\Sigma_2$ is transversal to the leaves L. Condition $Sub_r(\Psi)$: p_r satisfies condition (Ψ) at the limit Σ_2.

Example If $\operatorname{Hess} p \neq 0$ then $Sub_r(\Psi)$ holds iff $\operatorname{Im} p_s$ has constant sign on the leaves of Σ_2 and (Ψ) is holds for **any bicharacteristic** of $\operatorname{Re} p_s$.

Theorem (D. 2023)

If P is of real subprincipal type and satisfies condition $Sub_r(\Psi)$ then P is locally solvable.

Thus, condition $Sub_r(\Psi)$ is equivalent to solvability of ΨDO of **real subprincipal type**.

Example

Let

$$P = D_{x_1} D_{x_2} + D_t + if(t, x, y, D_y)$$

with $f(t, x, y, \eta)$ real and of first order.

If $x_j \mapsto f(t, x, y, \eta)$ *changes sign* for $j = 1$ or 2 on $\Sigma_2 = \{\, \xi_1 = \xi_2 = 0 \,\}$, then P is *not solvable.* (Mendoza–Uhlmann)

If $|\partial_x f| \le C|f|$ then f has no sign changes in the x variables.

P is of **real subprincipal type**, so P is *solvable* $\Leftrightarrow$

f has no sign changes in the (x_1, x_2) variables and

$t \mapsto f(t, x, \xi)$ has *no* sign change from $-$ to $+$ on Σ_2 as t *increases.*

Example

Let

$$P = D_{x_1} D_{x_2} + D_t + if(t, x, y, D_y)$$

with $f(t, x, y, \eta)$ real and of first order.

If $x_j \mapsto f(t, x, y, \eta)$ *changes sign* for $j = 1$ or 2 on $\Sigma_2 = \{ \xi_1 = \xi_2 = 0 \}$, then P is *not solvable*. (Mendoza–Uhlmann)

If $|\partial_x f| \le C|f|$ then f has no sign changes in the x variables.

P is of **real subprincipal type**, so P is *solvable* $\Leftrightarrow$

f has no sign changes in the (x_1, x_2) variables and

$t \mapsto f(t, x, \xi)$ has *no* sign change from $-$ to $+$ on Σ_2 as t *increases*.

Example

Let

$$P = D_{x_1} D_{x_2} + D_t + if(t, x, y, D_y)$$

with $f(t, x, y, \eta)$ real and of first order.

If $x_j \mapsto f(t, x, y, \eta)$ *changes sign* for $j = 1$ or 2 on $\Sigma_2 = \{ \xi_1 = \xi_2 = 0 \}$, then P is *not solvable*. (Mendoza–Uhlmann)

If $|\partial_x f| \le C|f|$ then f has no sign changes in the x variables.

P is of **real subprincipal type**, so P is *solvable* $\Leftrightarrow$
f has no sign changes in the (x_1, x_2) variables and
$t \mapsto f(t, x, \xi)$ has *no* sign change from $-$ to $+$ on Σ_2 as t *increases*.

Example

Let

$$P = D_{x_1} D_{x_2} + D_t + if(t, x, y, D_y)$$

with $f(t, x, y, \eta)$ real and of first order.

If $x_j \mapsto f(t, x, y, \eta)$ *changes sign* for $j = 1$ or 2 on $\Sigma_2 = \{\, \xi_1 = \xi_2 = 0 \,\}$, then P is *not solvable*. (Mendoza–Uhlmann)

If $|\partial_x f| \le C|f|$ then f has no sign changes in the x variables.

P is of **real subprincipal type**, so P is *solvable* $\Leftrightarrow$

f has no sign changes in the (x_1, x_2) variables and

$t \mapsto f(t, x, \xi)$ has *no* sign change from $-$ to $+$ on Σ_2 as t *increases*.

Open problems

Case when limit characteristics converge as continuous curves.

Solvability of weakly hyperbolic operators.

Complex limit bicharacteristics of complex principal symbols.

Condition $Sub_r(\Psi)$ for complex principal symbol.

Higher order vanishing of the principal symbol.

Systems of nonprincipal type.

Open problems

Case when limit characteristics converge as continuous curves.

Solvability of weakly hyperbolic operators.

Complex limit bicharacteristics of complex principal symbols.

Condition $Sub_r(\Psi)$ for complex principal symbol.

Higher order vanishing of the principal symbol.

Systems of nonprincipal type.

Open problems

Case when limit characteristics converge as continuous curves.

Solvability of weakly hyperbolic operators.

Complex limit bicharacteristics of complex principal symbols.

Condition $Sub_r(\Psi)$ for complex principal symbol.

Higher order vanishing of the principal symbol.

Systems of nonprincipal type.

Open problems

Case when limit characteristics converge as continuous curves.

Solvability of weakly hyperbolic operators.

Complex limit bicharacteristics of complex principal symbols.

Condition $Sub_r(\Psi)$ for complex principal symbol.

Higher order vanishing of the principal symbol.

Systems of nonprincipal type.

Open problems

Case when limit characteristics converge as continuous curves.

Solvability of weakly hyperbolic operators.

Complex limit bicharacteristics of complex principal symbols.

Condition $Sub_r(\Psi)$ for complex principal symbol.

Higher order vanishing of the principal symbol.

Systems of nonprincipal type.

Open problems

Case when limit characteristics converge as continuous curves.

Solvability of weakly hyperbolic operators.

Complex limit bicharacteristics of complex principal symbols.

Condition $Sub_r(\Psi)$ for complex principal symbol.

Higher order vanishing of the principal symbol.

Systems of nonprincipal type.