Positivity, Toeplitz operators, and Berger-Coburn Conjecture

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Workshop on Microlocal Analysis and Mathematical Physics, Lund, September 19-21, 2023

In honor of ANDERS MELIN

Joint work with L. Coburn and J. Sjöstrand

Introduction. Positive complex Lagrangians

Main theme of the talk : explore/develop links between Toeplitz operators and Fourier integral operators in the complex domain. An important role throughout the talk will be played by complex canonical transformations, enjoying certain positivity properties. For motivation, let us start by considering a couple of examples.

Example I. Let $q : \mathbb{R}^n_x \times \mathbb{R}^n_\xi \to \mathbb{C}$ be a quadratic form such that

$$\mathrm{Re}\, q > 0.$$

Associated to q is the Hamilton map $F : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ defined by

$$q(X, Y) = \sigma(X, FY), \quad X, Y \in \mathbb{C}^{2n}.$$

Here

$$\sigma = \sum_{j=1}^{n} d\xi_j \wedge dx_j$$

is the complex symplectic form on $\mathbb{C}^{2n} = \mathbb{C}^n_x \times \mathbb{C}^n_\xi$.

We have $\mathrm{Spec}(F) \cap \mathbb{R} = \emptyset$ and

$$\lambda \in \mathrm{Spec}(F) \Longleftrightarrow -\lambda \in \mathrm{Spec}(F).$$

Let us set

$$\Lambda^+ = \bigoplus_{\substack{\lambda \in \mathrm{Spec}(F) \\ \mathrm{Im}\,\lambda > 0}} \mathrm{Ker}\,(F-\lambda)^{2n} \subset \mathbb{C}^{2n}.$$

The complex linear subspace Λ^+ is a complex Lagrangian plane in the sense that $\dim_{\mathbb{C}} \Lambda^+ = n$ and

$$\sigma(X, Y) = 0, \quad X, Y \in \Lambda^+.$$

Fundamental observation (J. Sjöstrand, 1974) : the $\mathbb{C}$–Lagrangian Λ^+ is strictly positive :

$$\frac{1}{i}\sigma(X, \overline{X}) > 0, \quad 0 \neq X \in \Lambda^+.$$

Example. Let $q(x, \xi) = x^2 + \xi^2$. We have

$$\Lambda^+ = \left\{X = (x, \xi) \in \mathbb{C}^{2n};\ \xi = ix\right\}.$$

L. Hörmander (1971), ... A. Melin – J. Sjöstrand (1974–1977).

Let

$$\varphi(x,y) = \frac{i(x-y)^2}{2}, \quad x, y \in \mathbb{C}^n,$$

and consider the complex linear map

$$\kappa : \mathbb{C}^{2n} \ni (y, -\varphi'_y(x,y)) \mapsto (x, \varphi'_x(x,y)) \in \mathbb{C}^{2n}.$$

We have : κ is a complex canonical transformation in the sense that

$$\kappa^* \sigma = \sigma,$$

and

$$\kappa(\mathbb{R}^{2n}) = \Lambda_{\Phi_0} := \left\{ \left(x, \frac{2}{i}\frac{\partial \Phi_0}{\partial x}(x) \right), x \in \mathbb{C}^n \right\} \subset \mathbb{C}^{2n} = \mathbb{C}^n_x \times \mathbb{C}^n_\xi.$$

Here

$$\Phi_0(x) = \sup_{y\in\mathbb{R}^n} (-\mathrm{Im}\, \varphi(x,y)) = \frac{|\mathrm{Im}\, x|^2}{2}.$$

The $\mathbb{C}$-Lagrangian plane $\kappa(\Lambda^+) \subset \mathbb{C}^{2n}$ enjoys the following positivity property,

$$\frac{1}{i}\sigma(X, \iota_{\Phi_0}(X)) \geq 0, \quad X \in \kappa(\Lambda^+).$$

Here

$$\iota_{\Phi_0} : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$$

is the unique anti-linear involution such that $\iota_{\Phi_0} = 1$ along Λ_{Φ_0} (the complex conjugation with respect to Λ_{Φ_0}).

Let now Φ_0 be a strictly plurisubharmonic quadratic form on $\mathbb{C}^n$,

$$\sum_{j,k=1}^{n} \frac{\partial^2 \Phi_0(x)}{\partial x_j \partial \overline{x}_k} \zeta_j \overline{\zeta}_k > 0, \qquad 0 \neq \zeta \in \mathbb{C}^n.$$

The associated manifold

$$\Lambda_{\Phi_0} = \left\{ \left(x, \frac{2}{i} \frac{\partial \Phi_0}{\partial x}(x) \right), \, x \in \mathbb{C}^n \right\} \subset \mathbb{C}^{2n} = \mathbb{C}^n_x \times \mathbb{C}^n_\xi$$

is maximally totally real $\Longrightarrow \exists$ a corresponding anti-linear involution ι_{Φ_0}.
Let $\Lambda \subset \mathbb{C}^{2n}$ be a $\mathbb{C}$–Lagrangian plane which is positive relative to Λ_{Φ_0}, i.e.

$$\frac{1}{i}\sigma(X, \iota_{\Phi_0}(X)) \geq 0, \quad X \in \Lambda.$$

Proposition (J. Sjöstrand, 1982)

A $\mathbb{C}$–Lagrangian plane Λ is positive relative to Λ_{Φ_0} precisely when we have

$$\Lambda = \Lambda_\Psi,$$

where Ψ is pluriharmonic quadratic with $\Psi \leq \Phi_0$.

The proposition plays a crucial role in the work by J. Sjöstrand – J. Viola – M.H. (2013) on sharp resolvent estimates for the Weyl quantization of q.

Example II. Let $g = (g_{jk})$ be a real analytic Riemannian metric on $\mathbb{R}^n$. Associated to g is the Laplace-Beltrami operator $-\Delta_g$ with the principal symbol

$$p(y,\eta) = \sum_{j,k=1}^{n} g^{jk}(y)\eta_j\eta_k, \quad (g^{jk}) = (g_{jk})^{-1}, \quad (y,\eta) \in \mathbb{R}^{2n}.$$

Let $\rho_0 \in p^{-1}(1) \subset \mathbb{R}^{2n}$ and for $\rho \in \mathrm{neigh}(\rho_0, p^{-1}(1))$, consider the Hamiltonian trajectory emanating from ρ,

$$\Gamma_\rho = \{\exp(tH_p)(\rho); t \in \mathrm{neigh}(0,\mathbb{R})\} \subset p^{-1}(1).$$

Here

$$H_p = \partial_\eta p \cdot \partial_y - \partial_y p \cdot \partial_\eta$$

is the Hamilton vector field of p.

Let

$$\Lambda = \bigcup_{\rho} \Gamma_{\rho}^{\mathbb{C}} \subset \mathbb{C}^{2n},$$

where $\Gamma_{\rho}^{\mathbb{C}}$ is the complexification of Γ_{ρ},

$$\Gamma_{\rho}^{\mathbb{C}} = \{\exp(tH_p)(\rho); t \in \mathrm{neigh}(0, \mathbb{C})\} \subset (p^{-1}(1))^{\mathbb{C}}.$$

What can we say about the positivity properties of Λ? To understand this, we apply the canonical transformation of Example I,

$$\kappa : \mathbb{C}^{2n} \ni (y, -\varphi_y'(x,y)) \mapsto (x, \varphi_x'(x,y)) \in \mathbb{C}^{2n}.$$

Proposition (J. Sjöstrand, 1982, G. Lebeau, 1985)

We have $\kappa(\Lambda) = \Lambda_{\Phi}$, where Φ is plurisubharmonic with $\Phi \leq \Phi_0$.

This result is instrumental in the work in progress by J. Sjöstrand – M.H, devoted to a heat evolution approach to second microlocalization with respect to the hypersurface $p^{-1}(1) \subset \mathbb{R}^{2n}$.

Positive canonical transformations

L. Hörmander (1983, 1994).

Let Φ_0 be a strictly plurisubharmonic quadratic form on $\mathbb{C}^n$, with the anti-linear involution ι_{Φ_0}. A complex linear canonical transformation

$$\kappa : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$$

is said to be positive relative to Λ_{Φ_0} provided that

$$\frac{1}{i}\Big(\sigma(\kappa(\rho), \iota_{\Phi_0}\kappa(\rho)) - \sigma(\rho, \iota_{\Phi_0}(\rho))\Big) \geq 0, \quad \rho \in \mathbb{C}^{2n}.$$

Example. Let q be a holomorphic quadratic form on $\mathbb{C}^{2n}_{x,\xi}$ such that

$$\operatorname{Re} q|_{\Lambda_{\Phi_0}} \geq 0.$$

Then the canonical transformation

$$\kappa = \exp\left(\frac{1}{i} H_q\right)$$

is positive relative to Λ_{Φ_0}, $H_q = \partial_\xi q \cdot \partial_x - \partial_x q \cdot \partial_\xi$.

Characterizing positive canonical transformations

Theorem (L. Coburn – J. Sjöstrand — M. H., 2019)

Let Φ_0 be a strictly plurisubharmonic quadratic form on $\mathbb{C}^n$. A complex linear canonical transformation $\kappa : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ is positive relative to Λ_{Φ_0} precisely when we have

$$\kappa(\Lambda_{\Phi_0}) = \Lambda_{\Phi},$$

where Φ is a strictly plurisubharmonic quadratic form such that $\Phi \leq \Phi_0$.

A couple of words about the proof

Let $\varphi(x, y, \theta)$ be a holomorphic quadratic form on $\mathbb{C}^n_x \times \mathbb{C}^n_y \times \mathbb{C}^N_\theta$ which is a non-degenerate phase function in the sense of Hörmander,

$$\mathrm{rank}\,\left(\varphi''_{\theta x}\;\; \varphi''_{\theta y}\;\; \varphi''_{\theta\theta}\right) = N,$$

and such that

$$\kappa : \mathbb{C}^{2n} \ni (y, -\varphi'_y(x, y, \theta)) \mapsto (x, \varphi'_x(x, y, \theta)) \in \mathbb{C}^{2n}, \quad \varphi'_\theta(x, y, \theta) = 0.$$

The proof of the necessity part is based on direct geometric arguments, using that

$$\Phi(x) = \mathrm{vc}_{y,\theta}\left(-\mathrm{Im}\,\varphi(x, y, \theta) + \Phi_0(y)\right).$$

Here $\mathrm{vc}_{y,\theta}$ = the critical value with respect to (y, θ).

Fourier integral operators

When establishing the sufficiency part, we consider a (formal) Fourier integral operator quantizing κ,

$$Au(x) = \iint e^{i\varphi(x,y,\theta)} a\, u(y)\, dy \wedge d\theta. \tag{1}$$

Here $a \in \mathbb{C}$ and $u \in \mathrm{Hol}(\mathbb{C}^n) \Longrightarrow$ the integrand is an $(n+N, 0)$ differential form on $\mathbb{C}^{n+N}_{y,\theta}$ which is closed. More specifically, we introduce the Bargmann space

$$H_{\Phi_0}(\mathbb{C}^n) = \mathrm{Hol}(\mathbb{C}^n) \cap L^2(\mathbb{C}^n, e^{-2\Phi_0} L(dx)),$$

and let $u \in H_{\Phi_0}(\mathbb{C}^n)$.

How do we choose the contour of integration in (1) ?

Good contours I

Main point (J. Sjöstrand, 1982) : the mapping property

$$\kappa(\Lambda_{\Phi_0}) = \Lambda_{\Phi},$$

with Φ_0, Φ strictly plurisubharmonic implies that the plurisubharmonic quadratic form

$$\mathbb{C}^{n+N}_{y,\theta} \ni (y,\theta) \mapsto -\mathrm{Im}\, \varphi(0,y,\theta) + \Phi_0(y)$$

is non-degenerate of signature $(n+N, n+N)$. It follows that for each $x \in \mathbb{C}^n$, there exists an affine subspace $\Gamma(x) \subset \mathbb{C}^{n+N}_{y,\theta}$ of real dimension $n+N$, passing through the critical point $(y_c(x), \theta_c(x))$ of the function

$$(y,\theta) \mapsto -\mathrm{Im}\, \varphi(x,y,\theta) + \Phi_0(y),$$

such that

$$-\mathrm{Im}\, \varphi(x,y,\theta) + \Phi_0(y) \leq \Phi(x) - \frac{1}{C} \mathrm{dist}\left((y,\theta), (y_c(x), \theta_c(x))\right)^2,$$

along $\Gamma(x)$.

Good contours II

We say that $\Gamma(x) \subset \mathbb{C}^{n+N}_{y,\theta}$ is a good contour for the plurisubharmonic function

$$(y, \theta) \mapsto -\mathrm{Im}\,\varphi(x, y, \theta) + \Phi_0(y).$$

Example. Let $N = n$, $\varphi(x, y, \theta) = (x - y) \cdot \theta$, so that $\kappa = 1$. The contour

$$\theta = \frac{2}{i}\frac{\partial \Phi_0}{\partial x}\left(\frac{x+y}{2}\right) + \frac{i}{C}\overline{(x-y)}, \quad C > 1,$$

is good.

Good contours III

Consider the realization of A,

$$A_\Gamma u(x) = \iint_{\Gamma(x)} e^{i\varphi(x,y,\theta)} a\, u(y)\, dy \wedge d\theta, \quad u \in H_{\Phi_0}(\mathbb{C}^n),$$

where $\Gamma(x)$ is a good contour.

It follows that

$$A_\Gamma = A : H_{\Phi_0}(\mathbb{C}^n) \to H_\Phi(\mathbb{C}^n)$$

is bounded. Here

$$H_\Phi(\mathbb{C}^n) = \mathrm{Hol}(\mathbb{C}^n) \cap L^2(\mathbb{C}^n, e^{-2\Phi} L(dx)).$$

Bergman representation of FIO

Using a version of the Schwartz kernel theorem (J. Peetre (1990),...), we obtain a Bergman type representation for the realization $A_\Gamma = A$: the operator may uniquely be written in the form

$$Au(x) = \widehat{a} \int e^{2\Psi(x,\overline{y})} u(y) e^{-2\Phi_0(y)} \, L(dy), \quad u \in H_{\Phi_0}(\mathbb{C}^n).$$

Here $\widehat{a} \in \mathbb{C}$ and $\Psi(x, z)$ is a holomorphic quadratic form on $\mathbb{C}^{2n}_{x,z}$.

Inferring the positivity of κ

Main point : We have

$$2\mathrm{Re}\,\Psi(x,\overline{y}) \le \Phi(x) + \Phi_0(y), \quad (x,y) \in \mathbb{C}^{2n},$$

implying that

$$2\mathrm{Re}\,\Psi(x,\overline{y}) \le \Phi_0(x) + \Phi_0(y), \quad (x,y) \in \mathbb{C}^{2n}.$$

It follows that the $\mathbb{C}$–Lagrangian plane

$$\Lambda_{2\mathrm{Re}\,\Psi} = \left\{ \left(x, \frac{2}{i}\partial_x \Psi(x,y); y, \frac{2}{i}\partial_y \Psi(x,y) \right) \right\} \subset \mathbb{C}^{4n}$$

is positive relative to $\Lambda_{\Phi_0(x)+\Phi_0(\overline{y})}$. This implies the positivity of κ.

Corollary

Let Φ_0 *be a strictly plurisubharmonic quadratic form on* $\mathbb{C}^n$ *and let*

$$\kappa : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$$

be a complex linear canonical transformation which is positive relative to Λ_{Φ_0}*. Then, if* U *is a Fourier integral operator quantizing* κ*, we have*

$$U : H_{\Phi_0}(\mathbb{C}^n) \to H_{\Phi_0}(\mathbb{C}^n)$$

is bounded.

Example. Let q be a holomorphic quadratic form on $\mathbb{C}^{2n}_{x,\xi}$ such that

$$\mathrm{Re}\, q|_{\Lambda_{\Phi_0}} \geq 0,$$

and let q^w be the Weyl quantization of q. Then the heat evolution semigroup e^{-tq^w}, $t \geq 0$, is an FIO quantizing the positive canonical transformation

$$\exp\left(\frac{t}{i} H_q\right), \quad t \geq 0,$$

and therefore for all $t \geq 0$ we have

$$\exp\left(\frac{t}{i} H_q\right)(\Lambda_{\Phi_0}) = \Lambda_{\Phi_t},$$

and

$$e^{-tq^w} : H_{\Phi_0}(\mathbb{C}^n) \rightarrow H_{\Phi_t}(\mathbb{C}^n)$$

is bounded.

Here the family $t \mapsto \Phi_t$ is decreasing. We have the Bergman representation

$$e^{-tq^w}u(x) = \widehat{a}(t) \int e^{2\Psi_t(x,\overline{y})}u(y)e^{-2\Phi_0(y)}\, L(dy),$$

valid globally for all $t \geq 0$.

F. Hérau – J. Sjöstrand – C. Stolk (2004), ... F. White (2022) : propagation of global analytic singularities for quadratic non-selfadjoint evolution equations, L^p bounds for the semigroup.

Applications to Toeplitz operators

C. Berger – L. Coburn (1986, 1987, 1994), L. Coburn (1992, 2001, 2005), G. Rozenblum (2012), G. Rozenblum – A. Borichev (2015), G. Rozenblum – N. Vasilevski (2014, 2020), J. Toft (2002, 2004, 2007, 2012), C. Pfeuffer – J. Toft (2019), L. Amour – J. Nourrigat (2019).

There are also numerous developments in complex geometry/Toeplitz quantization, in the context of holomorphic sections of high powers of a positive holomorphic line bundle over a complex manifold : F. Berezin (1975), . . . , J. Peetre (1990), . . . , D. Catlin (1999), S. Zelditch (1998), . . . , R. Berman – B. Berndtsson – J. Sjöstrand (2008), . . . , O. Rouby – J. Sjöstrand – S. Vũ Ngoc (2021), A. Deleporte (2021), H. Hezari – H. Xu (2021), A. Deleporte – J. Sjöstrand – M. H. (2023), M. Stone – M. H. (2022), R. Melrose (2004), C. Epstein – R. Melrose (1998).

C. Fefferman (1974), L. Boutet de Monvel – J. Sjöstrand (1975) (asymptotics of the Bergman and Szegő kernels for strictly pseudoconvex smooth domains $\Subset \mathbb{C}^n$), . . . , M. Kashiwara (1977), A. Deleporte (2023) (the Szegő kernel for domains with analytic boundary).

Toeplitz quantization on Bargmann space

Let Φ_0 be a strictly plurisubharmonic quadratic form on $\mathbb{C}^n$. Given a measurable function $p : \mathbb{C}^n \to \mathbb{C}$, let us consider the Toeplitz operator with symbol p,

$$\mathrm{Top}(p) = \Pi_{\Phi_0} \circ p \circ \Pi_{\Phi_0} : H_{\Phi_0}(\mathbb{C}^n) \to H_{\Phi_0}(\mathbb{C}^n),$$

equipped with the natural (maximal) domain

$$\mathcal{D}(\mathrm{Top}(p)) = \{u \in H_{\Phi_0}(\mathbb{C}^n);\ pu \in L^2(\mathbb{C}^n, e^{-2\Phi_0} L(dx))\}.$$

Here

$$\Pi_{\Phi_0} : L^2(\mathbb{C}^n, e^{-2\Phi_0} L(dx)) \to H_{\Phi_0}(\mathbb{C}^n)$$

is the orthogonal (Bergman) projection.

Toeplitz vs Weyl quantization I

Let $p \in L^\infty(\mathbb{C}^n)$, say. We have

$$\mathrm{Top}(p) = a^w(x, D_x).$$

Here the Weyl quantization $a^w(x, D_x)$ of $a \in C^\infty(\Lambda_{\Phi_0})$ is given by

$$a^w(x, D_x)\, u(x) = \frac{1}{(2\pi)^n} \iint_{\Gamma_{\Phi_0(x)}} e^{i(x-y)\cdot\theta} a\left(\frac{x+y}{2}, \theta\right) u(y)\, dy \wedge d\theta,$$

with $\Gamma_{\Phi_0}(x) \subset \mathbb{C}^{2n}_{y,\theta}$ being the natural contour of integration given by

$$\theta = \frac{2}{i} \frac{\partial \Phi_0}{\partial x}\left(\frac{x+y}{2}\right).$$

Toeplitz vs Weyl quantization II

The Weyl symbol $a \in C^\infty(\Lambda_{\Phi_0})$ is given by

$$a\left(x, \frac{2}{i}\frac{\partial \Phi_0}{\partial x}(x)\right) = \left(\exp\left(\frac{1}{4}(\Phi''_{0,x\overline{x}})^{-1}\partial_x \cdot \partial_{\overline{x}}\right) p\right)(x), \quad x \in \mathbb{C}^n.$$

The symbol of $(\Phi''_{0,x\overline{x}})^{-1}\partial_x \cdot \partial_{\overline{x}}$ is

$$-\frac{1}{4}(\Phi''_{0,x\overline{x}})^{-1}\overline{\zeta} \cdot \zeta < 0, \quad 0 \neq \zeta \in \mathbb{C}^n \simeq \mathbb{R}^{2n} \Longrightarrow$$

the Weyl symbol a is given by the forward heat flow acting on p.

V. Guillemin (1985), . . . , C. Berger – L. Coburn (1994), J. Sjöstrand (1994), . . . , M. Zworski (2012).

When is a Toeplitz operator bounded on $H_{\Phi_0}(\mathbb{C}^n)$?

Example (C. Berger – L. Coburn, 1994). Let $\Phi_0(x) = \frac{|x|^2}{2}$ and let

$$p(x) = \exp(\lambda |x|^2), \quad \lambda \in \mathbb{C}, \quad \mathrm{Re}\,\lambda < 1/2.$$

Explicit computations show that

$$\mathrm{Top}(p) \in \mathcal{L}(H_{\Phi_0}(\mathbb{C}^n), H_{\Phi_0}(\mathbb{C}^n)) \iff |1-\lambda| \geq 1.$$

Furthermore,

- $|1-\lambda| > 1 \Longrightarrow \mathrm{Top}(p)$ is of trace class
- $|1-\lambda| = 1 \Longrightarrow \mathrm{Top}(p)$ is unitary

The Weyl symbol a can be computed by exact stationary phase and we see that

$$|1-\lambda| \geq 1 \iff a \in L^\infty(\Lambda_{\Phi_0}).$$

The Berger-Coburn Conjecture

Conjecture (C. Berger – L. Coburn, 1994) For any "reasonable" Toeplitz symbol p [1], we have

$$\mathrm{Top}(p) \in \mathcal{L}(H_{\Phi_0}(\mathbb{C}^n), H_{\Phi_0}(\mathbb{C}^n)) \Longleftrightarrow \text{the Weyl symbol } a \in L^\infty(\Lambda_{\Phi_0}).$$

The conjecture still stands.

C. Berger – L. Coburn, 1994 : some partial results towards the conjecture.

1. such that $p\, e^{2\Psi_0(\cdot,\overline{y})} \in L^2_{\Phi_0}$, for all $y \in \mathbb{C}^n$. Here Ψ_0 is the polarization of Φ_0.

Theorem (L. Coburn – J. Sjöstrand – M. H., 2019, 2023)

Let Φ_0 be a strictly plurisubharmonic quadratic form on $\mathbb{C}^n$ and let q be a complex valued quadratic form on $\mathbb{C}^n$. Assume that

$$\mathrm{Re}\, q(x) < \Phi_{\mathrm{herm}}(x) := \frac{1}{2}\left(\Phi_0(x) + \Phi_0(ix)\right), \quad 0 \neq x \in \mathbb{C}^n,$$

and that

$$\det \partial_{\overline{x}}\partial_x \left(2\Phi_0 - q\right) \neq 0.$$

The Toeplitz operator

$$\mathrm{Top}(e^q) : H_{\Phi_0}(\mathbb{C}^n) \to H_{\Phi_0}(\mathbb{C}^n)$$

is bounded if and only if the Weyl symbol $a \in C^\infty(\Lambda_{\Phi_0})$ of $\mathrm{Top}(e^q)$ satisfies $a \in L^\infty(\Lambda_{\Phi_0})$. Furthermore, $\mathrm{Top}(e^q)$ is compact precisely when the Weyl symbol a vanishes at infinity.

An extension to the inhomogeneous case

Remark. The result is still valid when the quadratic form q is replaced by a complex inhomogeneous quadratic polynomial on $\mathbb{C}^n$:

L. Coburn – J. Sjöstrand – F. White – M. H. (2021) : sufficiency
H. Xiong (2023) : necessity

Some ideas of the proof : Toeplitz operators as FIOs

Let Ψ_0 be the holomorphic quadratic form on $\mathbb{C}^{2n}$ such that $\Psi_0(x,\overline{x}) = \Phi_0(x)$ (the polarization of Φ_0) and let us write for $u \in \mathcal{D}$,

$$\mathrm{Top}(e^q)u(x) = C\int e^{2\Psi_0(x,\overline{y})}e^{q(y,\overline{y})}u(y)e^{-2\Phi_0(y)}dyd\overline{y}$$
$$= C\iint_\Gamma e^{2(\Psi_0(x,\theta)-\Psi_0(y,\theta))+q(y,\theta)}u(y)\,dy\,d\theta.$$

Here Γ is the contour in $\mathbb{C}^{2n}_{y,\theta}$ given by $\theta = \overline{y}$.

The holomorphic quadratic form

$$F(x,y,\theta) = \frac{2}{i}\left(\Psi_0(x,\theta) - \Psi_0(y,\theta)\right) + \frac{1}{i}q(y,\theta)$$

is a non-degenerate phase function in the sense of Hörmander $\Longrightarrow$ the operator $\mathrm{Top}(e^q)$ can be regarded as a Fourier integral operator associated to the (complex linear) canonical transformation

$$\kappa : (y, -F'_y(x,y,\theta)) \mapsto (x, F'_x(x,y,\theta)), \quad F'_\theta(x,y,\theta) = 0.$$

Passing to the Weyl quantization

We write

$$\mathrm{Top}(e^q) = a^w(x, D_x),$$

where $a \in C^\infty(\Lambda_{\Phi_0})$ is given by

$$a\left(x, \frac{2}{i}\frac{\partial \Phi_0}{\partial x}(x)\right) = \left(\exp\left(\frac{1}{4}(\Phi''_{0,x\overline{x}})^{-1}\partial_x \cdot \partial_{\overline{x}}\right) e^q\right)(x), \quad x \in \mathbb{C}^n.$$

By exact stationary phase, we conclude that

$$a(x,\xi) = C \exp\left(iF(x,\xi)\right), \quad (x,\xi) \in \mathbb{C}^{2n}, \quad C \neq 0,$$

where F is a holomorphic quadratic form on $\mathbb{C}^{2n}$.

Positivity : from bounded Weyl symbols to bounded Toeplitz operators

It follows that

$$\kappa : \rho + \frac{1}{2} H_F(\rho) \mapsto \rho - \frac{1}{2} H_F(\rho), \quad \rho \in \mathbb{C}^{2n}.$$

where $H_F = \partial_\xi F \cdot \partial_x - \partial_x F \cdot \partial_\xi$ is the Hamilton vector field of F.

Main observation : We have :

the Weyl symbol $a = Ce^{iF} \in L^\infty(\Lambda_{\Phi_0}) \Longleftrightarrow \kappa$ is positive relative to Λ_{Φ_0}.

The implication

$$a \in L^\infty(\Lambda_{\Phi_0}) \Longrightarrow \mathrm{Top}(e^q) \in \mathcal{L}(H_{\Phi_0}(\mathbb{C}^n), H_{\Phi_0}(\mathbb{C}^n))$$

follows.

Compactness of Toeplitz operators

Remark. We have : the Weyl symbol

$$a(x,\xi) \to 0, \quad \Lambda_{\Phi_0} \ni (x,\xi) \to \infty$$

precisely when the canonical transformation κ is strictly positive,

$$\frac{1}{i}\Big(\sigma(\kappa(\rho), \iota_{\Phi_0}\kappa(\rho)) - \sigma(\rho, \iota_{\Phi_0}(\rho))\Big) > 0, \quad 0 \neq \rho \in \mathbb{C}^{2n}.$$

It follows that

$$\mathrm{Top}(e^q) : H_{\Phi_0}(\mathbb{C}^n) \to H_{\Phi_0}(\mathbb{C}^n)$$

is compact (in fact, of trace class), cf. with the Berger-Coburn example.

From bounded Toeplitz operators to bounded Weyl symbols

Assume that

$$\mathrm{Top}(e^q) \in \mathcal{L}(H_{\Phi_0}(\mathbb{C}^n), H_{\Phi_0}(\mathbb{C}^n)).$$

Main idea : consider the operator $\mathrm{Top}(e^q)$ acting on the space of coherent states,

$$k_y(x) = C_{\Phi_0} e^{2\Psi_0(x,\overline{y}) - \Phi_0(y)}, \quad y \in \mathbb{C}^n.$$

We have $|| \, k_y \, ||_{H_{\Phi_0}} = 1$ and

$$-\Phi_0(x) + 2\mathrm{Re}\, \Psi_0(x, \overline{y}) - \Phi_0(y) \asymp - |x-y|^2 ,$$

so that k_y is centered at $y \in \mathbb{C}^n$.

Remark. We have, essentially,

$$k_y(x) e^{-\Phi_0(y)} = (\Pi_{\Phi_0} \delta_y)(x),$$

where Π_{Φ_0} is the Bergman projection.

Bergman form for Toeplitz operators and positivity

Exact stationary phase shows that

$$\left(\mathrm{Top}(e^q)k_y\right)(x) = C\, e^{2f(x,\overline{y})-\Phi_0(y)}, \quad y \in \mathbb{C}^n,$$

where $f(x,z)$ is a holomorphic quadratic form on $\mathbb{C}^{2n}_{x,z}$, giving a Bergman representation for the bounded operator $\mathrm{Top}(e^q)$,

$$\mathrm{Top}(e^q)u(x) = C \iint e^{2f(x,\overline{y})} u(y) e^{-2\Phi_0(y)}\, dy\, d\overline{y}.$$

Main observation : $\mathrm{Top}(e^q) \in \mathcal{L}(H_{\Phi_0}(\mathbb{C}^n), H_{\Phi_0}(\mathbb{C}^n)) \Longrightarrow$

$$2\mathrm{Re}\, f(x,\overline{y}) \leq \Phi_0(x) + \Phi_0(y), \quad (x,y) \in \mathbb{C}^{2n},$$

implying the positivity of κ relative to Λ_{Φ_0},

$$\kappa : \left(y, \frac{2}{i}\partial_y \Psi_0(y,z)\right) \mapsto \left(x, \frac{2}{i}\partial_x f(x,z)\right), \quad \partial_z f(x,z) = \partial_z \Psi_0(y,z),$$

and hence the boundedness of the Weyl symbol.

More Toeplitz surprises

The composition problem : For which Toeplitz symbols f, g is there an h such that

$$\mathrm{Top}(f)\,\mathrm{Top}(g) = \mathrm{Top}(h)?$$

Example. L. Coburn (2001) : let $\Phi_0(x) = \frac{|x|^2}{2}$. There exists $\lambda_0 \in \mathbb{C}$ with $0 < \mathrm{Re}\,\lambda_0 < 1/2$, $|\lambda_0 - 1| = 1$, such that the unitary operator

$$\mathrm{Top}(e^{\lambda_0 |x|^2}) : H_{\Phi_0}(\mathbb{C}^n) \to H_{\Phi_0}(\mathbb{C}^n)$$

satisfies : $\left(\mathrm{Top}(e^{\lambda_0 |x|^2})\right)^2$ is not a Toeplitz operator.

Composing Toeplitz FIOs I

Let Φ_0 be a strictly plurisubharmonic quadratic form on $\mathbb{C}^n$ and let q, $\widetilde{q}$ be complex valued quadratic forms on $\mathbb{C}^n$ such that

$$\mathrm{Re}\, q(x) < \Phi_{\mathrm{herm}}(x) := \frac{1}{2}\left(\Phi_0(x) + \Phi_0(ix)\right), \quad x \neq 0,$$

$$\mathrm{Re}\, \widetilde{q}(x) < \Phi_{\mathrm{herm}}(x), \quad x \neq 0,$$

satisfying

$$\det \partial_{\overline{x}} \partial_x \left(2\Phi_0 - q\right) \neq 0, \quad \det \partial_{\overline{x}} \partial_x \left(2\Phi_0 - \widetilde{q}\right) \neq 0.$$

Assume that the Weyl symbols a, $\widetilde{a}$ of $\mathrm{Top}(e^q)$, $\mathrm{Top}(e^{\widetilde{q}})$ satisfy

$$a \in L^\infty(\Lambda_{\Phi_0}), \quad \widetilde{a} \in L^\infty(\Lambda_{\Phi_0}),$$

so that the Toeplitz operators

$$\mathrm{Top}(e^q) : H_{\Phi_0}(\mathbb{C}^n) \to H_{\Phi_0}(\mathbb{C}^n), \quad \mathrm{Top}(e^{\widetilde{q}}) : H_{\Phi_0}(\mathbb{C}^n) \to H_{\Phi_0}(\mathbb{C}^n)$$

are bounded.

Composing Toeplitz FIOs II

L. Coburn – J. Sjöstrand – M. H. (2023) : we have (under an additional mild assumption),

$$\mathrm{Top}(e^{\widetilde{q}}) \circ \mathrm{Top}(e^{q}) = C\, \mathrm{Op}^{w}(e^{i\widehat{F}}) : H_{\Phi_0}(\mathbb{C}^n) \to H_{\Phi_0}(\mathbb{C}^n),$$

for some $0 \neq C \in \mathbb{C}$. Here $\widehat{F}$ is a holomorphic quadratic form on $\mathbb{C}^{2n}$ such that

$$\mathrm{Im}\, \widehat{F}|_{\Lambda_{\Phi_0}} \geq 0.$$

We also give a general criterion for when the Weyl quantization $\mathrm{Op}^{w}(e^{iG})$, where G is a holomorphic quadratic form on $\mathbb{C}^{2n}$, is of the form $C\, \mathrm{Top}(e^{Q})$ where Q is a quadratic form on $\mathbb{C}^n$.

Theorem (L. Coburn – J. Sjöstrand — M. H., 2023)

Let G be a holomorphic quadratic form on $\mathbb{C}^{2n}$ such that $\mathrm{Im}\, G|_{\Lambda_{\Phi_0}} \geq 0$. Assume that the holomorphic quadratic form

$$iG\left(x, \frac{2}{i}\frac{\partial \Psi_0}{\partial x}(x,z)\right) + 4\Psi_{\mathrm{herm}}(x,z), \quad (x,z) \in \mathbb{C}^{2n}$$

is non-degenerate, where Ψ_{herm} is the polarization of Φ_{herm}, and let us set

$$Q^{\pi}(y,\theta) = \mathrm{vc}_{x,z}\left(4\Psi_{\mathrm{herm}}(x-y, z-\theta) + iG\left(x, \frac{2}{i}\frac{\partial \Psi_0}{\partial x}(x,z)\right)\right).$$

Assume that the restriction $Q(y) = Q^{\pi}(y, \overline{y})$ of the holomorphic quadratic form Q^{π} on $\mathbb{C}^{2n}$ to the anti-diagonal satisfies

$$\mathrm{Re}\, Q(y) < \Phi_{\mathrm{herm}}(y), \quad 0 \neq y \in \mathbb{C}^n.$$

Then the Weyl quantization $\mathrm{Op}^w(e^{iG})$ is a bounded Toeplitz operator, with the Toeplitz symbol of the form $C\, e^{Q}$, for some $C \neq 0$.

TACK SÅ MYCKET, ANDERS, FÖR DEN FANTASTISKA HANDLEDNINGEN !

HA DEN ÄRAN PÅ FÖDELSEDAGEN !