Radon transforms supported in hypersurfaces

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Workshop on Microlocal Analysis and Mathematical Physics

in honor of Anders Melin's 80th birthday

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The Radon transform

Define

$$Rf(L) = \int_L f\, ds, \quad f \in C_c(\mathbb{R}^n), \quad L \text{ hyperplane in } \mathbb{R}^n.$$

Assume $Rf(L)$ known for all hyperplanes L. Find f.

Application ($n = 2$): Computerized Tomography (CT).

$f(x)$ attenuation of X-rays at x.

$Rf(L)$ total attenuation along line L.

Coordinates: $L(\omega, p)$ is hyperplane $\{x \in \mathbb{R}^n;\ x \cdot \omega = p\}$, ω unit vector. Thus

$$Rf(\omega, p) = Rf(L(\omega, p)), \quad \omega \in S^{n-1}, \quad p \in \mathbb{R}.$$

Rf is even, $\quad Rf(\omega, p) = Rf(-\omega, -p)$.

The formula $\widehat{Rf}(\omega, \tau) = \widehat{f}(\tau\omega)$ solves the inversion problem.

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Define

$$R^*\phi(x) = \text{mean}\{\phi(L);\ L \ni x\} = \int_{S^{n-1}} \phi(\omega, x \cdot \omega) d\omega.$$

If f is a compactly supported distribution, Rf is defined by

$\langle Rf, \varphi \rangle = \langle f, R^*\varphi \rangle$ for test functions φ on the mfd of hyperplanes.

Moreover

$$R^*Rf(x) = \frac{c_n}{|x|} * f(x),$$

so

$$f = c_n'(-\Delta)^{(n+1)/2} R^*Rf.$$

Johan Radon (1887-1956) published inversion formulas for R in 1917. Fourier transform was not used in Radon's paper.

In Linz, Austria, there is a RICAM institute (Radon Institute of Computational and Applied Mathematics).

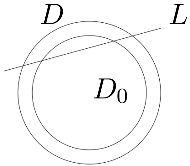
Theorem 1 (JB 2020, 2021). Let $D \subset \mathbb{R}^n$ be a bounded, convex domain. Assume that there exists a distribution $f \neq 0$, supported in $\overline{D}$, such that Rf is supported in the set of supporting planes to ∂D. Then the boundary of D is an ellipsoid.

If ∂D is C^1 smooth, the supporting planes for D are of course tangent planes to ∂D.

The Interior Problem for the Radon transform, $n = 2$

Let D_0, the region of interest, be a proper subset of D. One would like to reconstruct the restriction to D_0 of a function supported in $\overline{D}$ from measurements of $Rf(L)$ only for lines that intersect D_0.

But this is in general not possible.



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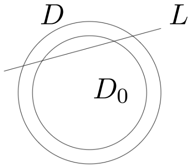

In fact, given two disks D and $\overline{D_0} \subset D$ there exist functions f with support *equal* to $\overline{D}$ such that

$$Rf(L) = 0 \quad \text{for all lines } L \text{ that meet } D_0.$$

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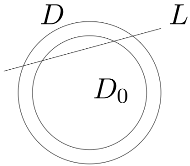

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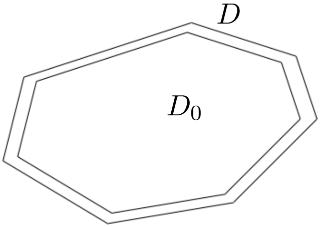
$$Rf(L) = 0 \quad \text{for all lines } L \text{ that meet } D_0.$$

If D and D_0 are concentric and centered at the origin, one can take f radial, that is, $f(x) = f(r)$ with $r = |x|$, which makes the problem 1-dimensional.

The Interior Problem, cont.

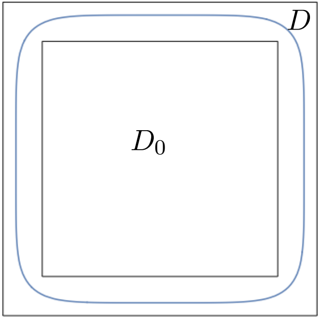
It is natural to replace the disks by arbitrary convex sets.

Conjecture. Let D and D_0 be bounded convex domains in the plane with $\overline{D_0} \subset D$. Then there exists a smooth function f with $\operatorname{supp} f \subset \overline{D}$ and $\operatorname{supp} f \cap D_0 \neq \emptyset$, such that its Radon transform $Rf(L)$ vanishes for every line L that intersects D_0.



Note: not true in odd dimensions!

Proof idea: find a compactly supported distribution f whose Radon transform is supported on the set of tangents to the blue curve.



Then a regularization of f, $f_1 = f * \phi$, will solve our problem, because $Rf_1 = g_1$ will be a smooth function (on the manifold of lines) that is supported in a neighborhood of the set of tangents to the curve.

Newton's lemma

A bounded domain in the plane is called *algebraically integrable*, if the area of a segment cut off by a secant line is an algebraic function of the parameters defining the line.

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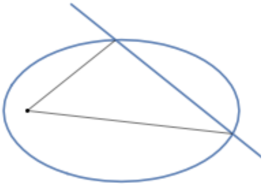
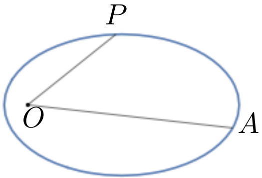
Lemma 28 in *Principia* reads according to Arnold and Vassiliev in *Newton's Principia read 300 years later* (Notices of the AMS 1989):

Theorem. There exists no algebraically integrable convex non-singular algebraic curve.

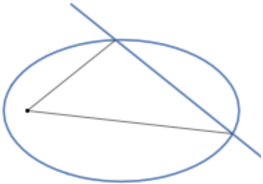
Newton's lemma, cont.

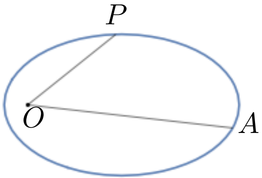




A segment is equal to a sector minus a triangle, and the area of the triangle depends algebraically on the coordinates of the corners.

Newton's lemma, cont.





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Newton's proof. Let A be fixed, and let $f(P)$ be the area of the sector defined by the lines OA and OP. This function is multivalued, and as P comes back to A after a full cycle, its value will be the area of the region bounded by the oval. After two full cycles $f(P)$ will be equal to twice the area. And so on.

So the function $f(P)$ must have infinitely many values, which is impossible if it is algebraic.

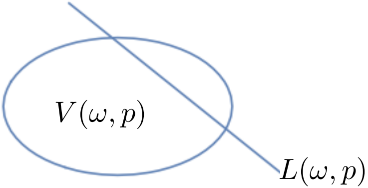
Arnold's Problem

Problem 1987-14 in *Arnold's Problems* asks:

Is it true that

$V(\omega, p)$ algebraic $\Longrightarrow$

n odd and ∂D ellipsoid.



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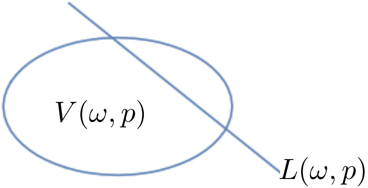

Vassiliev 1988: There exist no convex algebraically integrable bounded domains in even dimensions.

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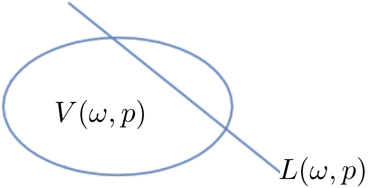

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Case of odd dimension still unsolved.

Arnold's Problem, cont.

Special case: assume n is odd and the volume function $p \mapsto V(\omega, p)$ is *polynomial* for all ω. Prove that the boundary of D is an ellipsoid.

Solved by Koldobsky, Merkurjev, and Yaskin 2017.

Arnold's Problem, cont.

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Theorem 1 implies the result of Koldobsky, Merkurjev, and Yaskin.

Because if $p \mapsto V(\omega, p)$ is a polynomial of degree $\leq N$ for all ω, then the Radon transform, $p \mapsto R\chi_D(\omega, p)$, of the characteristic function for the domain D is a polynomial of degree $\leq N$ (for p in some interval that depends on ω).

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Because if $p \mapsto V(\omega, p)$ is a polynomial of degree $\leq N$ for all ω, then the Radon transform, $p \mapsto R\chi_D(\omega, p)$, of the characteristic function for the domain D is a polynomial of degree $\leq N$ (for p in some interval that depends on ω). Hence

$$\partial_p^{2m} R\chi_D(\omega, p) = R(\Delta^m \chi_D)(\omega, p)$$

is supported on the set of tangent planes, if $2m > N$. By Theorem 1 the boundary of D must then be an ellipsoid.

Just a reminder:

Theorem 1. Assume that there exists a distribution $f \neq 0$, supported in $\overline{D}$, D convex and bounded, such that Rf is supported in the set of supporting planes to ∂D. Then the boundary of D is an ellipsoid.

On the proof of Theorem 1

Strategy of proof ($n = 2$):

1. Write down an expression for an arbitrary distribution $g(\omega, p)$ on the manifold of lines in $\mathbb{R}^2$ that is supported on the set of tangents to the boundary of D.

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Strategy of proof ($n = 2$):

1. Write down an expression for an arbitrary distribution $g(\omega, p)$ on the manifold of lines in $\mathbb{R}^2$ that is supported on the set of tangents to the boundary of D.

2. Write down the condition on $g(\omega, p)$ for g (compactly supported) to be the Radon transform of a distribution f on $\mathbb{R}^2$.

The condition is that

$$\omega = (\omega_1, \omega_2) \mapsto \int_{\mathbb{R}} g(\omega, p) p^k dp \quad \text{is a homogeneous polynomial}$$

of degree k for every k.

3. Prove that those conditions imply that the boundary curve is an ellipse.

On the proof of Theorem 1, case $D = -D$

Let $\rho_D(\omega) = \rho(\omega)$ be the supporting function for D

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The line $L(\omega, p)$ is tangent to ∂D iff

$$p = \rho(\omega) \quad \text{or} \quad p = \inf\{x \cdot \omega;\ x \in D\} = -\rho(-\omega) = -\rho(\omega).$$

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We may assume that g is even with respect to ω and p separately. If g is of order 0, then for some density $q(\omega)$

$$g(\omega, p) = q(\omega)\big(\delta(p - \rho(\omega)) + \delta(p + \rho(\omega))\big).$$

Here $\delta(\cdot)$ denotes the Dirac measure.

Use range conditions to deduce information on $\rho(\omega)$.

Case $D = -D$ and $Rf = g$ is a distribution of order 0, cont.

$$g(\omega, p) = q(\omega)\big(\delta(p - \rho(\omega)) + \delta(p + \rho(\omega))\big).$$

$k = 0$:

$\int_{\mathbb{R}} g(\omega, p) p^0 dp = 2\, q(\omega)$ must be constant, $q(\omega) = q \neq 0$.

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$k = 2$:

$\int_{\mathbb{R}} g(\omega, p)p^2 dp = 2q\,\rho(\omega)^2$ must be polynomial of degree 2, hence

$\rho(\omega)^2 = \rho(\omega_1, \omega_2)^2$ is a homogeneous polynomial of degree 2.

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If $D = -D$, then ∂D is an ellipsoid iff $\rho(\omega)^2$ is a (quadratic) polynomial.

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It follows that ∂D is an ellipse.

Assume next that $Rf = g$ is a distribution of order 1 of the form

$$g(\omega, p) = q(\omega)\big(\delta'(p - \rho(\omega)) - \delta'(p + \rho(\omega))\big).$$

Then $\int g(\omega, p)dp = 0$.

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Moreover

$$\int g(\omega, p)p^2 dp = -4q(\omega)\rho(\omega) = p_2(\omega)$$
$$\int g(\omega, p)p^4 dp = -24q(\omega)\rho(\omega)^3 = p_4(\omega).$$

Hence

$$\rho(\omega)^2 = 6\,\frac{p_4(\omega)}{p_2(\omega)}$$

must be a rational function.

But

$$\int g(\omega, p)p^6 dp = c_1\, q(\omega)\rho(\omega)^5 = p_6(\omega)$$

so

$$\rho(\omega)^4 = c_2 \frac{p_6(\omega)}{p_2(\omega)},$$

and similarly

$$\rho(\omega)^{2k} = c_k\, \frac{p_{2k+2}(\omega)}{p_2(\omega)}.$$

That is, an arbitrarily high power of $\rho(\omega)^2$ is a rational function with the same denominator, hence $\rho(\omega)^2$ must be a polynomial.

The same argument applies if $g(\omega, p)$ is assumed to be a distribution of arbitrarily high order, for instance if k is even

$$g(\omega, p) = q(\omega)\big(\delta^{(k)}(p - \rho(\omega)) + \delta^{(k)}(p + \rho(\omega))\big).$$

without lower order terms.

D not necessarily symmetric, $g(\omega, p)$ of order 0

Then $\rho(\omega)$ and $\rho(-\omega)$ may be different, same with $q(\omega)$ and $q(-\omega)$. An arbitrary $g(\omega, p)$ of order zero can then be written

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The first few moments of $g(\omega, p)$ will be

$$\int g(\omega, p)dp = q(\omega) + q(-\omega)$$

$$\int g(\omega, p)p\,dp = q(\omega)\rho(\omega) - q(-\omega)\rho(-\omega)$$

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Write $\rho(\omega) = \rho,\ q(\omega) = q$, and $\rho(-\omega) = \check{\rho},\ q(-\omega) = \check{q}$.

Then the range conditions will read

$$\begin{aligned} q + \check{q} &= p_0 \\ q\rho - \check{q}\check{\rho} &= p_1 \\ q\rho^2 + \check{q}\check{\rho}^2 &= p_2 \\ q\rho^3 - \check{q}\check{\rho}^3 &= p_3 \\ \text{etc.} \end{aligned} \qquad \text{or} \qquad \begin{pmatrix} 1 & 1 \\ \rho & -\check{\rho} \\ \rho^2 & \check{\rho}^2 \\ \rho^3 & -\check{\rho}^3 \\ \dots & \dots \end{pmatrix} \begin{pmatrix} q \\ \check{q} \end{pmatrix} = \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \\ \dots \end{pmatrix}$$

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Write the system of the first four equations as a set of three matrix equations:

$$
\begin{pmatrix} 1 & 1 \\ \rho & -\check{\rho} \end{pmatrix} \begin{pmatrix} q \\ \check{q} \end{pmatrix} = \begin{pmatrix} p_0 \\ p_1 \end{pmatrix}, \quad \begin{pmatrix} \rho & -\check{\rho} \\ \rho^2 & \check{\rho}^2 \end{pmatrix} \begin{pmatrix} q \\ \check{q} \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix},
$$

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\begin{pmatrix} \rho^2 & \check{\rho}^2 \\ \rho^3 & -\check{\rho}^3 \end{pmatrix} \begin{pmatrix} q \\ \check{q} \end{pmatrix} = \begin{pmatrix} p_2 \\ p_3 \end{pmatrix}.
$$

The important point is that the three square matrices form a geometric series:

$$\begin{pmatrix} \rho & -\check{\rho} \\ \rho^2 & \check{\rho}^2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\rho\check{\rho} & \rho + \check{\rho} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \rho & -\check{\rho} \end{pmatrix}$$

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Introduce a name for the important matrix

$$S = \begin{pmatrix} 0 & 1 \\ -\rho\check{\rho} & \rho+\check{\rho} \end{pmatrix}.$$

We can now easily eliminate $\begin{pmatrix} q \\ \check{q} \end{pmatrix}$. Indeed, we have shown that

$$S\begin{pmatrix} p_0 \\ p_1 \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \quad S\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} p_2 \\ p_3 \end{pmatrix}, \quad \text{and so on.}$$

Recall that $\det S = \rho\check{\rho} = \rho(\omega)\rho(-\omega)$. To make use of this fact we form matrix equations by combining the previous equations in pairs:

$$S \begin{pmatrix} p_0 & p_1 \\ p_1 & p_2 \end{pmatrix} = \begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix}, \quad S \begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix} = \begin{pmatrix} p_2 & p_3 \\ p_3 & p_4 \end{pmatrix}, \text{ etc.}$$

The product rule for determinants now shows that
$\det S = \rho(\omega)\rho(-\omega)$ must be a rational function.

(Provided $\det \begin{pmatrix} p_0 & p_1 \\ p_1 & p_2 \end{pmatrix}$ is not identically zero; I will come back to this question.)

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(Provided $\det \begin{pmatrix} p_0 & p_1 \\ p_1 & p_2 \end{pmatrix}$ is not identically zero; I will come back to this question.)

To show that $\det S$ must in fact be a polynomial we argue as above, showing that an arbitrary power of $\det S$, $(\rho\check{\rho})^k$, must be a rational function with the same denominator. Just use the formula

$$S^k\begin{pmatrix} p_0 & p_1 \\ p_1 & p_2 \end{pmatrix} = \begin{pmatrix} p_k & p_{k+1} \\ p_{k+1} & p_{k+2} \end{pmatrix}$$

for arbitrarily large k.

The condition

$$\rho(\omega)^2 \text{ is polynomial}$$

is not translation invariant. Because if $D_a = D + (a_1, a_2)$, then $\rho_{D_a}(\omega) = \rho_D(\omega) + a \cdot \omega$, and if D is the unit disk, $\rho_D(\omega) = \sqrt{\omega_1^2 + \omega_2^2}$, then

$$\rho_{D_a}(\omega)^2 = \left(\sqrt{\omega_1^2 + \omega_2^2} + a \cdot \omega)\right)^2$$

$$= \omega_1^2 + \omega_2^2 + 2(a \cdot \omega)\sqrt{\omega_1^2 + \omega_2^2} + (a \cdot \omega)^2,$$

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$$\begin{aligned}\rho_{D_a}(\omega)^2 &= \left(\sqrt{\omega_1^2 + \omega_2^2} + a \cdot \omega)\right)^2 \\ &= \omega_1^2 + \omega_2^2 + 2(a \cdot \omega)\sqrt{\omega_1^2 + \omega_2^2} + (a \cdot \omega)^2,\end{aligned}$$

which is not polynomial. On the other hand, for *symmetric* D (with respect to some point) the condition that $\rho(\omega)\rho(-\omega)$ is a polynomial is translation invariant, because

$$\begin{aligned}&\big(\rho(\omega) + a \cdot \omega\big)\big(\rho(-\omega) - a \cdot \omega\big) \\ &= \rho(\omega)\rho(-\omega) - (a \cdot \omega)^2 - (a \cdot \omega)\big(\rho(\omega) - \rho(-\omega)\big) \\ &= \rho(\omega)\rho(-\omega) - (a \cdot \omega)^2.\end{aligned}$$

Lemma. Assume that $\rho_{D_a}(\omega)\rho_{D_a}(-\omega)$ is polynomial in (ω_1, ω_2) for two distinct $a = (a_1, a_2)$. Then the boundary of D is an ellipse.

Proof. We may assume that the two points are $(0,0)$ and $(a_1, a_2) \neq (0,0)$. The formula

$$\begin{aligned}&\big(\rho(\omega) + a\cdot\omega\big)\big(\rho(-\omega) - a\cdot\omega\big)\\ &= \rho(\omega)\rho(-\omega) - (a\cdot\omega)^2 - (a\cdot\omega)\big(\rho(\omega) - \rho(-\omega)\big)\\ &= \rho(\omega)\rho(-\omega) - (a\cdot\omega)^2.\end{aligned}$$

then shows that

$$(a\cdot\omega)\big(\rho(\omega) - \rho(-\omega)\big)$$

must be a quadratic polynomial, hence $\rho(\omega) - \rho(-\omega)$ is linear, say

$$\rho(\omega) - \rho(-\omega) = -2b\cdot\omega$$

for some $b = (b_1, b_2)$. But this means that

$$\rho(\omega) + b\cdot\omega = \rho(-\omega) - b\cdot\omega.$$

Hence $(\rho(\omega) + b\cdot\omega)^2$ is a quadratic polynomial, so ∂D_b is a quadric.

On the determinant $p_0 p_2 - p_1^2$

Using the expressions

$$\begin{aligned} p_0 &= q + \check{q} \\ p_1 &= q\rho - \check{q}\check{\rho} \\ p_2 &= q\rho^2 + \check{q}\check{\rho}^2 \end{aligned}$$

we find that

$$p_0 p_2 - p_1^2 = q\check{q}(\rho + \check{\rho})^2.$$

Since the left hand side is a polynomial, it is enough to prove that the right hand side is different from zero at some point. If we choose the origin inside D, then $\rho(\omega)$ and $\rho(-\omega)$ will be positive for all ω. So it is enough to prove that $q(\omega)q(-\omega)$ cannot be identically zero.

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Solving q and $\check{q}$ from the first two equations we obtain

$$q\check{q} = \frac{p_0^2\rho\check{\rho} - p_1^2 + p_1p_0(\rho - \check{\rho})}{(\rho + \check{\rho})^2}.$$

It is easy to see that this expression cannot be identically zero.

$Rf = g$ contains terms of different order

If the distribution $g(\omega, p)$ is of order 3 and $D = -D$, then

$$\begin{aligned} g(\omega, p) = {} & q_0(\omega)\big(\delta(p - \rho(\omega)) + \delta(p + \rho(\omega))\big) \\ & + q_1(\omega)\big(\delta'(p - \rho(\omega)) - \delta'(p + \rho(\omega))\big) \\ & + q_2(\omega)\big(\delta''(p - \rho(\omega)) + \delta''(p + \rho(\omega))\big) \\ & + q_3(\omega)\big(\delta^{(3)}(p - \rho(\omega)) - \delta^{(3)}(p + \rho(\omega))\big). \end{aligned}$$

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The range conditions can then be written

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ \rho & 1 & 0 & 0 \\ \rho^2 & 2\rho & 2 & 0 \\ \rho^3 & 3\rho^2 & 6\rho & 6 \\ \rho^4 & 4\rho^3 & 12\rho^2 & 24\rho \\ \rho^5 & 5\rho^4 & 20\rho^3 & 60\rho^2 \\ \dots & \dots & \dots & \dots \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ \dots \end{pmatrix}.$$

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Let me denote the sequence of 4×4 submatrices of the infinite matrix by M_0, M_1, M_2, etc.

The important fact is that this sequence is a geometric series in the sense that

$$M_1 = SM_0, \quad M_2 = SM_1, \quad M_1 = M_0T, \quad M_2 = M_1T, \quad \text{etc.},$$

in other words

$$M_k = S^k M_0 \quad M_k = M_0 T^k \quad \text{for all } k, \quad \text{where}$$

$$S = M_1 M_0^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\rho^4 & 4\rho^3 & -12\rho^2 & 24\rho \end{pmatrix}$$

and

$$T = M_0^{-1} M_1 = \begin{pmatrix} \rho^2 & 1 & 0 & 0 \\ 0 & \rho^2 & 2 & 0 \\ 0 & 0 & \rho^2 & 3 \\ 0 & 0 & 0 & \rho^2 \end{pmatrix}.$$

So

$$\det S = \det T = (\rho^2)^4 = \rho^8.$$

The three equations and one more

$$S\begin{pmatrix}p_0\\p_1\\p_2\\p_3\end{pmatrix}=\begin{pmatrix}p_1\\p_2\\p_3\\p_4\end{pmatrix},\quad S\begin{pmatrix}p_1\\p_2\\p_3\\p_4\end{pmatrix}=\begin{pmatrix}p_2\\p_3\\p_4\\p_5\end{pmatrix},\quad S\begin{pmatrix}p_2\\p_3\\p_4\\p_5\end{pmatrix}=\begin{pmatrix}p_3\\p_4\\p_5\\p_6\end{pmatrix},$$

can then be combined to the matrix equation

$$S\begin{pmatrix}p_0 & p_1 & p_2 & p_3\\p_1 & p_2 & p_3 & p_4\\p_2 & p_3 & p_4 & p_5\\p_3 & p_4 & p_5 & p_6\end{pmatrix}=\begin{pmatrix}p_1 & p_2 & p_3 & p_4\\p_2 & p_3 & p_4 & p_5\\p_3 & p_4 & p_5 & p_6\\p_4 & p_5 & p_6 & p_7\end{pmatrix}.$$

Similarly for arbitrary k

$$S^k \begin{pmatrix} p_0 & p_1 & p_2 & p_3 \\ p_1 & p_2 & p_3 & p_4 \\ p_2 & p_3 & p_4 & p_5 \\ p_3 & p_4 & p_5 & p_6 \end{pmatrix} = \begin{pmatrix} p_k & p_{k+1} & p_{k+2} & p_{k+3} \\ p_{k+1} & p_{k+2} & p_{k+3} & p_{k+4} \\ p_{k+2} & p_{k+3} & p_{k+4} & p_{k+5} \\ p_{k+3} & p_{k+4} & p_{k+5} & p_{k+6} \end{pmatrix}.$$

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Provided the matrix in the left hand side is nonsingular. And it must be, because its determinant is equal to

$$c\big(\rho(\omega)^2\big)^{3\cdot 3} q_3(\omega)^2$$

with $c \neq 0$. And the same for $g(\omega, p)$ of arbitrary order.

This point — to prove that the determinant in the denominator is not identically zero — gave me very big difficulties in the case when D is not assumed symmetric. Because then the expression for the determinant contains the factor

$$q_m(\omega)q_m(-\omega) \quad \text{instead of} \quad q_m(\omega)^2.$$

And it is not obvious that

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Local questions I

Assume that there exists a distribution f with support in $\overline{D}$ (convex), a tangent plane L_0, a point $x^0 \in L_0 \cap \operatorname{supp} f$, and a neighborhood V of L_0 in the manifold of hyperplanes, such that *the restriction of* Rf *to* V *is supported on the set of supporting planes to* ∂D *in* V. Does it follow that ∂D is a quadric in some neighborhood of x^0?

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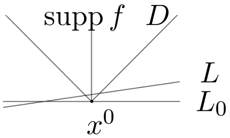
NO, if x^0 is a corner point of D.

Example:
$f(x_1, x_2) = \delta'(x_1)\chi_{[0,1]}(x_2)$.



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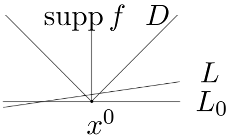
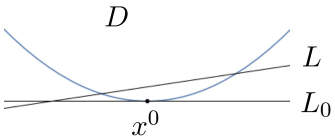


If ∂D is C^1 near x^0, we don't know.



Local questions II. Singularities of a distribution and the geometry of its support

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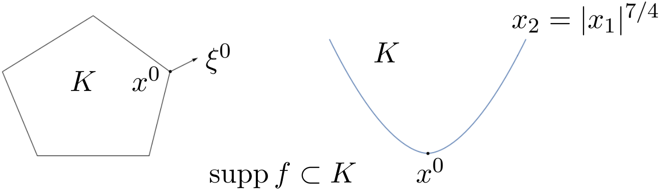
Assume that ξ^0 is an outer conormal to supp f at $x^0 \in \partial(\operatorname{supp} f)$.
Then $(x^0, \pm\xi^0) \in WF_A(f)$.



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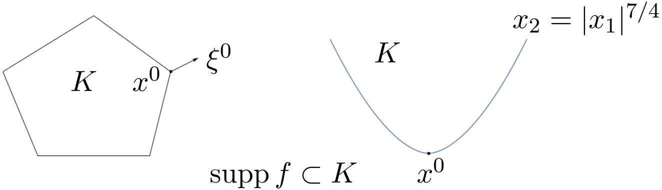

Stronger theorems connecting the geometry of $\operatorname{supp} f$ at boundary points of $\operatorname{supp} f$ with analytic singularities of f were later given by Hörmander, Sjöstrand, Kashiwara. In the figures $\operatorname{supp} f \subset K$ and $x^0 \in \operatorname{supp} f$.

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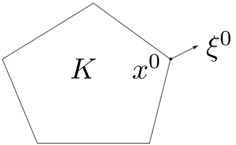

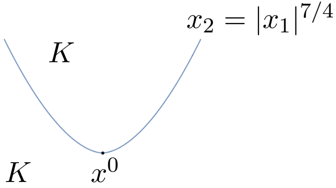

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The following is an easy consequence of the definition of $WF(f)$:

If f is a C^∞ density on a C^∞ hypersurface Σ, then $WF(f)$ is contained in the set $N^*(\Sigma)$ of conormals to Σ,

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And if f is the characteristic function for a domain D with real analytic boundary, then

$$WF_A(f) = N^*(\partial D).$$

Similarly, for distributions of higher order:

Let Σ be a hypersurface in R^{n+1} defined by $y = \Psi(x)$ and f be the distribution

$$\begin{aligned}\langle f, \varphi \rangle &= \sum_{j=0}^{m-1} \int_{\Sigma} q_j \, \partial_y^j \varphi \, dx \\ &= \sum_{j=0}^{m-1} \int_{\mathbb{R}^n} q_j(x) \, (\partial_y^j \varphi)(x, \Psi(x)) dx, \quad \varphi \in C_c^\infty(U).\end{aligned}$$

If Ψ and all q_j are real analytic, then $WF_A(f) \subset N^*(\Sigma)$.

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It turned out that the arguments in the proof of Theorem 1 could prove a theorem of this kind.

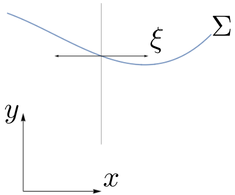
Theorem 2. Let f be the distribution above, supported on the C^1 surface $\Sigma: \ y = \Psi(x)$, $x \in U \subset \mathbb{R}^n$, q_j continuous, that is

$$\langle f, \varphi \rangle = \sum_{j=0}^{m-1} \int_\Sigma q_j \, \partial_y^j \varphi \, dx.$$

Assume that $WF_A(f)$ contains no horisontal cotangent vectors $(\xi, \eta) = (\xi, 0)$, i.e. that

$$N^*(\gamma_x) \cap WF_A(f) = \emptyset,$$

for every line $\gamma_x : \ y \mapsto (x, y)$ for $x \in \mathbb{R}^n$. Then the surface Σ and all densities q_j are real analytic.



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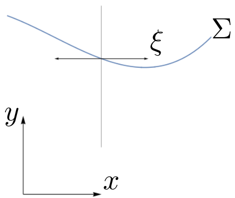
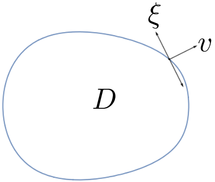


In particular, if $WF_A(f) \subset N^*(\Sigma)$, then the surface Σ and all densities q_j are real analytic.

Corollary. Let f be the characteristic function $\chi_D(x)$ for a domain D with C^1 boundary, or the product of $\chi_D(x)$ with a real analytic function, and let $x^0 \in \partial D$. Let v be a tangent vector that is transversal to the boundary at x^0. Assume that $(x^0, \xi) \notin WF_A(f)$ for all ξ that are conormal to v. Then the boundary of D is real analytic in a neighborhood of x^0.



There is in fact a coordinate free formulation of the theorem.

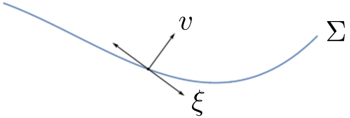
Theorem 2′. Let Σ be a C^1 hypersurface in a real analytic manifold M, let $f \in \mathcal{D}'(M)$ be supported in Σ, and let $z \in \operatorname{supp} f$. Assume that $v \in T_z(M)$ is a tangent vector to M at z that is transversal to Σ and that

$$(z, \xi) \notin WF_A(f) \text{ for every } \xi \text{ that is conormal to } v.$$

Then there exists a neighborhood U of z such that the surface Σ is real analytic in U and the distribution f has the form

$$\langle f, \varphi \rangle = \sum_{j=0}^{m-1} \int_{\mathbb{R}^n} q_j(x)\,(\partial_y^j \varphi)(x, \Psi(x))dx, \quad \varphi \in C_c^\infty(U).$$

in suitable local coordinates in U with all q_j real analytic.



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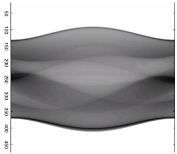
$$S^{n-1} \ni \omega \mapsto \int g(\omega, p) p^k \, dp \quad \text{is a polynomial for every } k.$$

This is a microlocal regularity assumption on g, because it implies that the conormal of $p \mapsto (\omega, p)$ is disjoint from $WF_A(g)$ for every ω.



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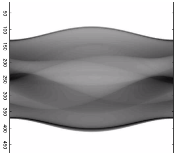

The *conclusion* of Theorem 1 is a very strong regularity property of the supporting hypersurface of g; indeed, it says that the surface is an ellipsoid.

Theorem 3. Let D and D_0 be bounded convex domains in the plane with $\overline{D_0} \subset D$. Then there exists a smooth function f, supported in D, such that $Rf(L) = 0$ for every line L that intersects D_0.

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Sketch of proof.

Take D_1 such that $\overline{D}_0 \subset D_1 \subset \overline{D}_1 \subset D$.

Denote by $\widehat{D}_0$ the set of lines that meet D_0.

Set $\quad \mu = \inf\{\|Rf\|_{L^2(\widehat{D}_0)};\ \|f\|_{L^2(\overline{D})} \le M,\ f = 1 \text{ in } D \setminus D_1\}$.

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Hence $Rf_0 = 0$ in $\widehat{D}_0$. $\square$

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