The backscattering transformation

Ingrid Belțiță (IMAR), joint work with Anders Melin

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Scattering operator

$H_v = -\Delta + v, \quad v \in C_0^\infty(\mathbb{R}^n)$, $n \geq 3$ odd.

- *Wave operators*: $W_\pm = \lim_{t\to\pm\infty} e^{itH_v} e^{-itH_0}$ exist and are complete $(\operatorname{Ran} W_\pm = P_{ac}(H)L^2(\mathbb{R}^n))$
- *Scattering operator:* $S = W_+^* W_-$, $SH_0 = H_0 S$, $\hat{S} = \mathcal{F} S \mathcal{F}^*$.

- $S - I = W_+^*(W_- - W_+) = -i \int_{-\infty}^{\infty} e^{itH_0}(W_+^* v)e^{-itH_0} dt.$

$$\Longrightarrow (\mathcal{F}(S-I)\mathcal{F}^*)\varphi(\xi) = -2\pi i \int \delta(|\xi|^2 - |\eta|^2)\alpha_+(\xi,\eta)\varphi(\eta)\, d\eta,$$

when $\operatorname{supp}\varphi \subseteq \mathbb{R}^n \setminus \{0\}$ and $\xi \neq 0$, where

$$\alpha_+(\xi,\eta) = (\mathcal{F} W_+^* v \mathcal{F}^*)(\xi,\eta) = (2\pi)^{-n}\widehat{W_+^* v}(\xi,-\eta).$$

Note that $(i + H_0)^{-N} W_+^* v \in \mathcal{B}_2(L^2)$ for N large enough, hence $\widehat{W_+^* v}(\xi,\eta)$ is L^2_{loc} continuous in η, and

$$(1 + |\xi|^2)^{-N} |\widehat{(W_+^* v)}(\xi,\eta)| \leq g(\xi)$$

a.e. ξ and for every η, where $g \in L^2(\mathbb{R}^n)$.

*** $T\colon \mathcal{S}' \to \mathcal{S}$ linear cont., $T(x,y)$ is its distribution kernel.

Backscattering data

- *Backscattering data* the function

$$\xi \mapsto \overline{\alpha_+(-\xi,\xi)} = \widehat{vW_+}(\xi,-\xi)$$

or its inverse Fourier transform, that is,

$$(B_{\mathsf{class}}v)(x) = 2^n \int (vW_+)(x-y, x+y)\,dy.$$

- *Real backscattering data* the function

$$\begin{aligned}\xi \mapsto 2^{-1}(\overline{\alpha_+(-\xi,\xi)} + \alpha(\xi,-\xi)) &= 2^{-1}(\widehat{vW_+}(\xi,-\xi) + \overline{\widehat{vW_+}(-\xi,\xi)}) \\ &= 2^{-1}(\widehat{vW_+} + \widehat{vW_-})(\xi,-\xi),\end{aligned}$$

or

$$(B_{\mathsf{class,re}}v)(x) = 2^{n-1} \int v(x-y)(W_+ + W_-)(x-y, x+y)\,dy.$$

Link to the wave equation

Some hand-waving computations:

- Assume for the moment that $H_v \geq 0$ and $0 \notin \sigma_p(H_v)$.
- Birman invariance principle: $W_\pm(H_v, H_0) = W_\pm(\sqrt{H_v}, \sqrt{H_0})$, and

$$``W_+ = I - \int_0^\infty \frac{\sin(t\sqrt{H_v})}{\sqrt{H_v}} v e^{-it\sqrt{H_0}} dt''$$

Link to the wave equation

The wave "group" [Melin (2003)]

$v \in L^q_{\mathrm{cpt}}(\mathbb{R}^n)$, $q > n$, There is a a unique
$K_v \in C^2([0,\infty), \mathcal{B}(H^2, L^2)) \cap C([0,\infty), \mathcal{B}(H^2, H^2))$ such that $K_v(t)$ solves
$(\partial_t^2 + H_v)K_v(t) = 0$, $K_v(0) = 0$, $\dot{K}_v(0) = I$,

- $K_0(t) = \frac{\sin t|D|}{|D|}$, $t \geq 0$.

Conv. kernel $k_0(\cdot, t)$ of $K_0(t)$,
$k_0(x,t) = -i\pi(2\pi i)^{-n} \int_{S^{n-1}} \delta^{(n-2)}(x\theta - t)\, d\theta$ supported on $|x| = t$.

- $K_N(t) := \int\limits_0^t K_{N-1}(t-s) v K_0(s)\, ds$

  $\|K_N(t)\| \leq C^N t^{1+N\delta} \|v\|_{L^q} / \Gamma(2 + N\delta)$, where $\delta = 2 - n/q$,

Then

$$K_v(t) = \sum_N (-1)^N K_N(t),$$

and $L^q_{\mathrm{cpt}} \ni v \to K_v(t) \in \mathcal{B}(L^2(\mathbb{R}^n))$ is entire analytic.

- $|x - y| \leq t$ in the support of $K_v(t, x, y)$

Link to the wave equation

- $W_\pm = P_{\mathsf{ac}} - \int\limits_0^\infty P_{\mathsf{ac}} K_v(t) v e^{\mp it\sqrt{H_0}} dt$, where $P_{\mathsf{ac}} := P^H_{\mathsf{ac}}$.
- $G := -\int\limits_0^\infty K_v(t) v \dot{K}_0(t) dt$
- $W_+ + W_- = 2(I + G + S)$

with S a sum of projections defined in terms of eigenvalues ≤ 0 and their and eigenfunctions, such that $S(\cdot,\cdot)$ is smooth in the second variable.
Thus

$$(B_{\mathsf{class,re}} v)(x) = 2^{n-1} \int v(x-y)(W_+ + W_-)(x-y, x+y)\, dy$$
$$= v(x) + 2^n \int v(x-y) G(x-y, x+y)\, dy + \int v(y) S(y,x)\, dy$$

(A. Melin (2004)).

The backscattering transformation and the problem

Backscattering transform of v

$$(Bv)(x) = v(x) + 2^n \int v(x-y)G(x-y, x+y)\, dy.$$

Problem. Find v (singularities of v) from Bv.

- When v is real $Bv - B_{\text{class,re}}v$ is a smooth function. If there are no bound states $Bv = B_{\text{class,re}}v$.
- $C_0^\infty(\mathbb{R}^n) \ni v \mapsto Bv \in C^\infty(\mathbb{R}^n)$ is entire analytic in v:

$$Bv = \sum_1^\infty B_N v, \quad B_1 v = v.$$

- $B_N v$ is the value at $(v, \ldots, v)$ of an N-linear singular integral operator.

First result

- The regularity of $B_N v$ increases with N in the sense of the next theorem.

• [A. Melin (03)] The larger degree terms are as regular as we want: *Let $q > n$ and k be a nonnegative integer. Then there is a positive integer $N_0 = N_0(n, q, k)$ such that if $R_1, R_2 > 0$ there is a $C = C(n, k, R_1, R_2, q)$ such that*

$$\|\Delta^k B_N v\|_{L^2(B(0,R_1))} \leq C^N \|v\|_{L^q}^N / N!, \qquad N \geq N_0,$$

whenever $v \in L^q$ has support in the ball $B(0, R_2)$.

Formula for $B_N v$

Recall:
$(Bv)(x) = v(x) + 2^n \int v(x-y) G(x-y, x+y)\, dy$, where

$$G = -\int_0^\infty K_v(t) v \dot{K}_0(t) dt.$$

$$K_v(t) = \sum_N (-1)^N K_N(t),$$

$$K_0(t) = \frac{\sin t|D|}{|D|},\ K_N(t) = \int_0^t K_{N-1}(t-s) v K_0(s)\, ds,$$

$$G = \sum_1^\infty G_N;\ G_N = (-1)^N \int_0^\infty K_{N-1}(t) v \dot{K}_0(t) dt$$

Formula for $B_N v$

Thus

$$Bv = \sum B_N v$$

with

$$B_1 v = v$$

and

$$B_N = 2^n \int v(x-y) G_{N-1}(x-y, x+y)\, dy, \quad N \geq 2,$$

with

$$G_{N-1} = (-1)^N \int_0^\infty K_{N-2}(t) v \dot{K}_0(t) dt, \quad G_0 = K_0(t),$$

$$K_{N-2}(t) = \int_0^t K_{N-3}(t-s) v K_0(s)\, ds, \quad K_0(t) = \frac{\sin t|D|}{|D|},$$

Formula for $B_N v$

- Define
  $Q_1(x,t) = k_0(x,t)$,
  $Q_N(x_1, \cdot x_N; t) = \int_0^t Q_{N-1}(x_1, \ldots, x_{N-1}; t-s) k_0(x_N; s)\, ds.$
- Then

$$K_N(x,y) = \int v(x_1) \ldots v(x_N) Q_{N+1}(x - x_1, x_1 - x_2, \ldots, x_N - y; t) d\bar{x}$$

and

$$G_N(x,y) = (-1)^N \int v(x_1) \cdots v(x_N) \times$$
$$\times E_{N+1}(x - x_1, x_1 - x_2, \ldots, x_{N-2} - x_{N-1}, x_N - y) d\bar{x}$$

where

$$E_{N+1}(x_1, \ldots, x_{N+1}) = (-1)^N \int_0^\infty Q_N(x_1, \ldots, x_N; t) \dot{k}_0(x_{N+1}; t) dt.$$

Formula for $B_N v$

- $E_N \in \mathcal{D}'((\mathbb{R}^n)^N)$, $N \geq 2$

$$E_N(x_1, \ldots, x_N) := (-1)^{N-1} \int_0^\infty Q_{N-1}(x_1, \ldots, x_{N-1}; t) \dot{k}_0(x_N; t) \mathrm{d}t$$

▲ $Bv = v + \sum_2^\infty B_N v$, with

$$(B_N v)(x) = \int_{(\mathbb{R}^n)^N} E_N(y_1, \ldots, y_N) v\Big(x + \frac{y_N}{2} + \frac{y_1}{2} + \cdots + \frac{y_{N-1}}{2}\Big)$$

$$v\Big(x + \frac{y_N}{2} - \frac{y_1}{2} + \frac{y_2}{2} + \cdots + \frac{y_{N-1}}{2}\Big) \cdots v\Big(x + \frac{y_N}{2} - \frac{y_1}{2} - \cdots - \frac{y_{N-1}}{2}\Big) \mathrm{d}\vec{y}$$

when $v \in C_0^\infty(\mathbb{R}^n)$.

The distribution E_N

Set

$$P_N = (\Delta_1 - \Delta_N) \cdots (\Delta_{N-1} - \Delta_N),$$

Δ_j in the Laplacian in the variables x_j.

Properties of E_N

E_N is a fundamental solution of P_N. It has the following properties:

(i) $E_N(x_1, \ldots, x_N)$ is rotation invariant in each x_j;

(ii) $|x_1| + \cdots + |x_{N-1}| = |x_N|$ in the support of E_N;

(iii) E_N is homogeneous of degree $2(N-1) - nN$.

If E is a fundamental solution of P_N that satisfies *(i)-(iii)*, then $E = E_N$.

- $E_2(x, y) = 4^{-1}(i\pi)^{1-n}\delta^{(n-2)}(x^2 - y^2)$.

A result for $n = 3$

- R. Langergren (2011) There is a constant $C > 0$ such that

$$\|B_N v\|_{(1)} \leq C N^3 (8\pi)^{-N} \|v\|_{(1)}^N$$

Here, if $n \geq 3$,

$$\|f\|_{(n-2)} = \|\nabla^{n-2} f\|.$$

- For $n \geq 3$ odd
  - $\|B_2 v\|_{(n-2)} \leq C \|v\|_{(n-2)}^2$ (A. Melin (2004))
  - $\|B_3 v\|_{(n-2)} \leq C \|v\|_{(n-2)}^3$ (A. Melin, I.B. (2013))

Smoothning properties of B_N

Theorem

$s \geq (n-3)/2$, N_0 integer such that

$$a < N_0 - 1, \quad a \leq (N_0 - 1)(s - (n-3)/2).$$

- There is $C = C(n, s, a) > 0$ such that

$$\|B_N v\|_{H^{s+a}(B(0,R))} \leq C^N R^{(N-1)/2} N^{-N/2} \|v\|_{H^s}^N$$

  when $N \geq N_0$, $R > 0$ and $v \in C_0^\infty(B(0,R))$.

- There is $C_1 = C_1(n, a, s) > 0$ such that

$$\|B_N v\|_{H^{s+a}(\mathbb{R}^n)} \leq C_1^N R^{N-1} \|v\|_{H^s}^N,$$

  for every $N \geq N_0$, $R > 0$ and $v \in C_0^\infty(B(0,R))$.

- $0 \leq a < 1$, $a \leq s - (n-3)/2$ is good for all $N \geq 2$,
  $\Rightarrow$ $Bv - v$ is in $H_{(s+a)}$.

Proof

$N = 3$: $\phi_a(t) = H(t)(\sin(ta)/a$,

- "$\hat{E}_3(\xi_1, \xi_2, \xi_3) = \int_0^\infty (\phi_{|\xi_1|} * \phi_{|\xi_2|})(t) \cos(|\xi_3| t)\, dt$"
- $\sigma > 0$:

$$F(\xi_1, \xi_2, \xi_3, \sigma) := \int_0^\infty (\phi_{|\xi_1|} * \phi_{|\xi_2|})(t) \cos(|\xi_3| t) e^{-t\sigma}\, dt,$$

$$= \mathrm{Re}\Big(\frac{1}{|\xi_1|^2 - (|\xi_3| - i\sigma)^2} \cdot \frac{1}{|\xi_2|^2 - (|\xi_3| - i\sigma)^2}\Big).$$

- Good cut-off: $\chi \in C_0^\infty(\mathbb{R})$,

$$E_{3,\chi}(x_1, x_2, x_3) = \int_0^\infty Q_2(x_1, x_2; t) \dot{k}_0(x_3; t) \chi(t)\, dt,$$

with good estimates for the Fourier transform.

- If $\chi = 1$ on $|t| < R$, then $E_{3,\chi} = E_3$ on $|y_3| < R$.
- $\widehat{B_N v}(\xi) = C \int \int \hat{E}_{3,\chi}(\xi_1, \xi_2, \xi) \hat{v}(\xi/2 + \xi_1) \hat{v}(\xi_1 - \xi_2) \hat{v}(\xi/2 - \xi_2)\, d\xi_1\, d\xi_2$, for appropriate χ.

Generic local uniqueness

Corollary

$s > (n-3)/2$, $R > 0$ fixed.
There is a closed subset G_s of $H^s(B(0,R)$ such that

- $\mathbb{C}f \cap G_{s,R}$ discrete for every $f \in H_0^s(B(0,R))$
- B is a locally isomorphism at every $v \in H_0^s(B(0,R)) \setminus G_s$.

In particular, $Bv = 0 \Rightarrow v = 0$ for v small.

The quadratic term

$N = 2$:

$$B_2(v_1, v_2)(x) = \iint E_2(y,z) v_1(x + \frac{y+z}{2}) v_2(x - \frac{y-z}{2}) \mathrm{d}y \, \mathrm{d}z.$$

$$H_a^b(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n); \langle x \rangle^a \langle D \rangle^b u \in L^2(\mathbb{R}^n)\}$$

where $a, b \in \mathbb{R}$.

Theorem

Assume that

$$0 \le \bar{a} \le a,\ 0 \le \bar{b} \le b,\ \bar{a} + \bar{b} < 1/2.$$

Then B_2 is continuous from $H^{m+b}_{1/2+a}(\mathbb{R}^n) \times H^{m+b}_{1/2+a}(\mathbb{R}^n)$ to $H^{m+b+\bar{b}}_{1/2+a+\bar{a}}(\mathbb{R}^n)$. In particular, B_2 is a continuous bilinear operator on H_a^b when $a \ge 1/2$ and $b \ge m$.

Proof

$n = 3$

- $B_2(f,g)(x) = \iint E(y,z) f(x + \frac{y+z}{2}) g(x - \frac{y-z}{2}) \mathrm{d}y \, \mathrm{d}z,.$

Then

$$\langle B_2(f,g), h \rangle = -4 \iint\limits_{\mathbb{R}^3 \times \mathbb{R}^+} A(f,g)(x,t)(\cos(t|D|)h)(x) \mathrm{d}x \mathrm{d}t$$

where

$$A(f,g)(x,t) = Ct \int\limits_{\mathbb{S}^2} f(x - t\omega) g(x + t\omega) \, d\omega.$$