Discrete spectrum of polynomially compact pseudodifferential operators and applications to the Neumann-Poincare operator in 3D Elasticity.

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The project is inspired by the Neumann-Poincare operator in 3D elasticity: the 'double-layer potential.' A renewed interest - wave propagation in auxetics -materials with negative Poisson constant. It was proved 28 years ago that it is a pseudodiff. operator of order 0, with 3 points of essential spectrum. What can one say about eigenvalues converging to these points?

Pseudodifferential operators.

Zero order matrix operators on a manifold : $a_0(x, \xi)$ is the principal symbol of order 0, $a_{-1}(x, \xi)$ is the lower order symbol. $(x, \xi) \in T^*\Gamma$. Vector case: $a(x, \xi)$ is a $N \times N$ matrix, the operator acts on N component vector-functions. The essential spectrum: the set of EIGENVALUES of $a_0(x, \xi)$.

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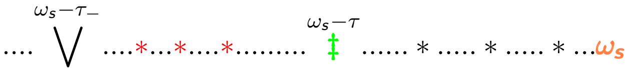
Polynomially compact

$p(A)$ is compact. Ess. spectrum consists of several points ω_l, the zeros of the polynomial p; the discrete eigenvalues may converge to each of them from above and from below.

What are we going to count??? Let ω_s be one of the zeros of $p(t)$. We fix $\tau_- > 0$ and for $\tau \in (0, \tau_-)$ we denote $n_-(A; \omega_s, \tau)$ the number of eigenvalues of A in the interval $(\omega_s - \tau_-, \omega_s - \tau)$.



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Similarly, for eigenvalues above ω_s of $\boldsymbol{\Sigma}$ and $0 < t < \tau_+$, denote $n_+(A; \omega_s, \tau)$ the number of eigenvalues of A in $(\omega_s + \tau, \omega_s + \tau_+)$ (A is can be omitted in this notation).

$$\boldsymbol{\omega_s} * \dots * \dots * \dots\dots * \dots\ \overset{\omega+\tau}{\ddagger}\ . * \dots * \dots * \dots\ \overset{\omega+\tau_+}{\bigvee}\ \dots\dots\dots$$

We are interested in estimates and asymptotics of $n_\pm(\omega, \tau)$ as $\tau \to 0$. $n(\omega, \tau) = n_+(\omega, \tau) + n_-(\omega, \tau)$.

Birman-Solmyak: B is compact,order -1. Let $b_{-1}(x,\xi)$ be the principal symbol, of order -1.

$$n_{\pm}(0,\tau) \sim \tau^{-d} d^{-1} (2\pi)^{-d} \int_{\Gamma} \int_{|\xi|=1} \mathrm{Tr}\left([b_{-1}(x,\xi)]_{\pm}^{d}\right) dx d\xi$$

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Important: if at some (x,ξ) one of eigenvalues of $b_{-1}(x,\xi)$ is positive (negative) then the corresponding integral is nonzero!!!

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Important: d an even integer–the integrand is the trace of a power of the matrix b_{-1}, the polynomial function of the entries. Can be controlled.

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$\mathbf{\Sigma}(A)$ consists of several discrete points ω_l, $l = 1, \ldots, L$. We study the eigenvalues converging to ONE of these points, ω_s. Approach: We GUESS a polynomial, which distinguishes the eigenvalues of A near the given ω_s but almost 'destroys' eigenvalues near other $\omega_l, l \neq s$.

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Fix $s \in [1, N]$. $\mathbf{p}_s(t) = \prod_{l \neq s}(t - \omega_l)^2(\tau - \omega_s)$, $B_s = \mathbf{p}_s(A)$. The leading term (order τ^{-d}) in $n_{\pm}(A, \omega_s, \tau)$ is determined by the asymptotics of $n_{\pm}(B_s, 0, \tau)$. The symbol of order -1 of B_s is calculated explicitly.

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The case $N = 3$ is of major interest and we will write formulas for this case, $d = 2$, Γ is a smooth compact surface, $a_0(x, \xi)$, is the principal symbol (in a fixed co-ordinate system and a frame), $a_{-1}(x, \xi)$ symbol of order -1.

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For the eigenvalue ω_1, the polynomial is $\mathbf{p}_1(t) = (t - \omega_1)(t - \omega_2)^2(t - \omega_3)^2$. We want to calculate the PRINCIPAL SYMBOL of $p_1(A)$, of order -1 (it is invariant, although contains non-invariant terms.)

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$$\mathrm{sym}(\mathbf{p}_1(A))_{-1} \sim a_{-1}(a_0-\omega_2)^2(a_0-\omega_3)^2 + (a_0-\omega_1)a_{-1}(a_0-\omega_2)(a_0-\omega_3)^2 +$$
$$(a_0-\omega_1)(a_0-\omega_2)a_{-1}(a_0-\omega_3)^2 + (a_0-\omega_1)(a_0-\omega_2)^2 a_{-1}(a_0-\omega_3) +$$
$$b_1c_1(a_0-\omega_2)(a_0-\omega_3)^2 + b_1(a_0-\omega_2)c_1(a_0-\omega_3)^2$$
$$+b_1(a_0-\omega_2)^2c_1(a_0-\omega_3) + b_1(a_0-\omega_2)^2(a_0-\omega_3)c_1 +$$
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24 terms. For a concrete operator: calculate the symbol$\mathbf{p}_s(A)_{-1}$ apply the B-S formula for spectrum.

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24 terms. For a concrete operator: calculate the symbol$\mathbf{p}_s(A)_{-1}$ apply the B-S formula for spectrum.Important: the algebraic structure is explicit. .

Part 2. Elasticity NP operator

Lamé equations, homogeneous isotropic material.

$$\mathcal{L}u \equiv \mathcal{L}_{\mu,\lambda} = \mathrm{div}\,(\mu \mathrm{grad}\, u) + \mathrm{grad}\,((\lambda+\mu)\mathrm{div}\, u) = 0, x = (x_1, x_2, x_3) \in \Omega,$$

μ, λ : Lamé parameters. $\varkappa = \frac{\lambda+\mu}{\mu} = (1-2\sigma)^{-1}$, σ - the Poisson constant, $u = (u_1, u_2, u_3)^{\perp}$

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Fundamental solution, 'Kelvin matrix':

$$\Phi_{jk}(x) = -\frac{\alpha}{4\pi}\delta_{jk}|x|^{-1} - \frac{\beta}{4\pi}x_j x_k |x|^{-3},\ d = 3.$$

$\alpha = \frac{1}{2}(\mu^{-1} + (2\mu+\lambda)^{-1})$, $\beta = \frac{1}{2}(\mu^{-1} - (2\mu+\lambda)^{-1})$.
Conormal derivative (traction): matrix 3×3 operator $\mathbf{T}$

$$\mathbf{T}_{jk} \equiv \mathbf{T}_{jk}(\partial_x, \nu_x) = \lambda\nu_x^j \partial_k(x) + \lambda\nu_x^k \partial_j + (\lambda+\mu)\delta_{jk}\partial_{\nu_x}.$$

$\mathcal{K}f(x) = \int_\Gamma \mathbf{T}_y \Phi(x-y) f(y) dS_y$. Even for a smooth boundary, the NP operator is not compact (off-diagonal terms).

Dim 3, M.Agranovich, B.Amosov, M.Levitin 1999; K. Ando, H.Kang, Y.Miyanishi 2017; Y.Miyanishi, G.Rozenblum 2021,

NP operator is a pseudodifferential operator on Γ with principal symbol

$$a_0(\xi') = \frac{i\kappa_0}{|\xi'|}\begin{pmatrix} 0 & 0 & -\xi_1 \\ 0 & 0 & -\xi_2 \\ \xi_1 & \xi_2 & 0 \end{pmatrix}.$$

$\kappa_0 = \frac{\mu}{2(2\mu+\lambda)}$, $\xi' = (\xi_1, \xi_2) \in T_x^*\Gamma$. Eigenvalues of the principal symbol: $0, \pm\kappa_0$.

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Denote: $\mathfrak{m}(x,\xi) = \mathfrak{m}^{\sigma}(x,\xi)_{-1}$ is the principal symbol of $\mathbf{p}_s(\mathcal{K})$ (here $d = 2$, $\mathbf{N} = 3$, $k = -1$). B-S formula:

$$n_{\pm}(\omega,\tau) \sim \tau^{-2} 2^{-1} (2\pi)^{-2} \int_G \int_{|\xi|=1} \operatorname{Tr}\left([\mathfrak{m}(x,\xi)]_{\pm}^2\right) dx d\xi,$$

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important: the integrand is terrible, but, if all eigenvalues of the symbol $\mathfrak{m}(x,\xi$ are positive (negative) than we need not calculate the eigenvalues of the symbol but only the polynomial operations with entries of the symbol.
The formulas for the eigenvalue asymptotics are pseudo-local.

The only previously known case

A sphere, Deng, Li, Liu, 2019 (J. Spectr. Theory)

$$\Lambda_j^0(\mathcal{K}) = \frac{3}{2(2j+1)} \sim \frac{3}{4j}, \tag{0.1}$$

$$\Lambda_j^-(\mathcal{K}) = \frac{3\lambda - 2\mu(2j^2 - 2j - 3)}{2(\lambda + 2\mu)(4j^2 - 1)} \sim -\kappa_0 + \frac{2\mu}{(\lambda + 2\mu)j},$$

$$\Lambda_j^+(\mathcal{K}) = \frac{-3\lambda + 2\mu(2j^2 + 2j - 3)}{2(\lambda + 2\mu)(4j^2 - 1)}, \sim \kappa_0 + \frac{2\mu}{(\lambda + 2\mu)j},$$

all of them of multiplicity j, where λ, μ are Lamé constants. pause all three sequences approach their limit points from above; there are no eigenvalues that approach these points from below. Moreover, the eigenvalues tending to 0 do not depend on the material. Even the ball case is not easy, since variables do not separate.

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all of them of multiplicity j, where λ, μ are Lamé constants. pause all three sequences approach their limit points from above; there are no eigenvalues that approach these points from below. Moreover, the eigenvalues tending to 0 do not depend on the material. Even the ball case is not easy, since variables do not separate.We investigate whether these properties remain for a general surface.

The main theorem.

1. Let χ, $\mathbf{W} = \frac{1}{4}\int_\Gamma (k_1 + k_2)^2 dS$ be the Euler characteristic and the Willmore energy of Γ. Then

   $$n(\omega_s, \tau) = n_+(\omega_s, \tau) + n_-(\omega_s, \tau) \sim \tau^{-2}(\mathcal{A}(\omega_s, \lambda, \mu)\chi + \mathcal{B}(\omega_s, \lambda, \mu)\mathbf{W})$$

   where $\mathcal{A}, \mathcal{B}$ are universal, depending only on κ_0 quadratically, coefficients.
2. Corollary. There are *always* infinitely many eigenvalues near each of the points ω_s;
3. The asymptotic formulas hold

   $$n_\pm(\omega_s, \tau) \sim \mathcal{C}_\pm(\omega_s, \lambda, \mu, \Gamma)\tau^{-2}.$$

   The coefficients are given by explicit terrible formulas.
4. Coefficients $\mathcal{C}_+(\omega_s, \lambda, \mu, \Gamma)$ are always positive; this means that there are infinitely many eigenvalues converging to ω_s *from above*.
5. If there exists a point where the surface Γ is strictly concave, then $\mathcal{C}_-(\omega_s, \lambda, \mu, \Gamma) > 0$, i.e., there are infinitely many eigenvalues of the NP operator converging to ω_s from below.
6. If the body has a cavity inside, there are infinitely many

How is this proved!!

We calculate the principal and subprincipal symbols of the NP operator $\mathcal{K}$ in a specially chosen co-ordinate system and frame. It turns out that

1. the principal symbol $a_0(x, \xi')$,

$$a_0(\xi') = \frac{i\kappa_0}{|\xi'|} \begin{pmatrix} 0 & 0 & -\xi_1 \\ 0 & 0 & -\xi_2 \\ \xi_1 & \xi_2 & 0 \end{pmatrix}.$$

   does not depend on $\xi' = (\xi_1, \xi_2)$ but depends on x via the change frame on Γ.

2. the x-derivatives of a_0 are linear functions of the principal curvatures $k_1(x), k_2(x)$ with UNIVERSAL coefficients
3. the subprincipal symbol $a_{-1}(x)$ is a linear function of $k_1(x), k_2(x)$ with universal coefficients.

We repeat: $b_\alpha = -i\partial_{\xi_\alpha} a_0, c_\alpha = \partial_{x_\alpha} a_0;$

$$
\begin{aligned}
&\mathrm{symb}(\mathbf{p}_1(A))_{-1} \\
&\sim a_{-1}(a_0-\omega_2)^2(a_0-\omega_3)^2 + (a_0-\omega_1)a_{-1}(a_0-\omega_2)(a_0-\omega_3)^2 + \\
&(a_0-\omega_1)(a_0-\omega_2)a_{-1}(a_0-\omega_3)^2 + (a_0-\omega_1)(a_0-\omega_2)^2 a_{-1}(a_0-\omega_3) + \\
&b_1c_1(a_0-\omega_2)(a_0-\omega_3)^2 + b_1(a_0-\omega_2)c_1(a_0-\omega_3)^2 \\
&+ b_1(a_0-\omega_2)^2c_1(a_0-\omega_3) + b_1(a_0-\omega_2)^2(a_0-\omega_3)c_1 + \\
&(a_0-\omega_1)b_1c_1(a_0-\omega_3)^2 + (a_0-\omega_1)b_1(a_0-\omega_2)c_1(a_0-\omega_3) + \\
&(a_0-\omega_1)b_1(a_0-\omega_2)(a_0-\omega_3)c_1 + (a_0-\omega_1)(a_0-\omega_2)b_1c_1(a_0-\omega_3) + \\
&(a_0-\omega_1)(a_0-\omega_2)b_1(a_0-\omega_3)c_1 + (a_0-\omega_1)(a_0-\omega_2)^2 b_1c_1 + \\
&b_2c_2(a_0-\omega_2)(a_0-\omega_3)^2 + b_2(a_0-\omega_2)c_2(a_0-\omega_3)^2 + \\
&b_2(a_0-\omega_2)^2c_2(a_0-\omega_3) + b_2(a_0-\omega_2)^2(a_0-\omega_3)c_2 + \\
&(a_0-\omega_1)b_2c_2(a_0-\omega_3)^2 + (a_0-\omega_1)b_2(a_0-\omega_2)c_2(a_0-\omega_3) + \\
&(a_0-\omega_1)b_2(a_0-\omega_2)(a_0-\omega_3)c_2 + (a_0-\omega_1)(a_0-\omega_2)b_2c_2(a_0-\omega_3) + \\
&(a_0-\omega_1)(a_0-\omega_2)b_2(a_0-\omega_3)c_2 + (a_0-\omega_1)(a_0-\omega_2)^2 b_2c_2.
\end{aligned}
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(expressed in the special co-ordinates system, determined by the curvature lines and the corresponding frame) has the form

$$\mathfrak{m}_s(x,\xi) = k_1(x)M_1(\xi) + k_2(x)M_2(\xi);\, M_\iota = M_\iota^{(s)}$$

Where $M_1(\xi), M_2(\xi)$ are matrices determined only by the Lamé constants λ, μ. They also depend on the point ω_s of the essential spectrum.

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Remark: since the normal is exterior, k_1, k_2 are negative at the points where the surface Γ is convex.

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1. M_1, M_2 do not depend on Γ, but we know their integrals for a sphere, $k_1 = k_2 = -1$: Use the symmetricity of the sphere: therefore $M_1(\xi), M_2(\xi) \leq 0$. 2. Since there exists at least one point on Γ where both of curvatures are negative (the body is convex), there are infinitely many eigenvalues converging to ω_s from above. If there exists at least one point where both curvatures are positive (a point of concavity,) there are infinitely many eigenvalues converging to the ω_s from below.

Calculation of eigenvalue asymptotics

Calculation of eigenvalue asymptotics

The dependence of $M_\iota(\xi)$ on λ, μ, ξ is hard to find, but we still can do a lot. We might have used by means of a symbolic manipulation program, but a lot can be still done by hand. They do not depend on Γ, so we choose Γ where they are easy to calculate. It is a cylinder $k_2 \equiv 0$, and instead of 24 terms we need to calculate only 14. And they are rather simple. Lots of other entries equal zero. But even here to find the separate asymptotics of eigenvalues is, probably, impossible, since one needs to find explicitly eigenvalues of a matrix 3×3 depending on parameters, and then integrate in x, ξ.

But: in the convex case, the symbol $\mathfrak{m}(x, \xi)$ is nonnegative, and here, fortunately, the Birman-Solomyak coefficient is in dimension 2 a rational expression:
$\operatorname{tr}(\mathfrak{m}(x, \xi)^2)$ is the sum of squares of absolute values of the entries, $\operatorname{tr}(\mathfrak{m}(x, \xi)^2) = \sum_{p,q} \mathfrak{m}(x, \xi)^2_{p,q}$, it is a quadratic function of curvatures, and this gives the asymptotics of eigenvalues.

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Since the matrix $\mathfrak{m}(x,\xi)$ is linear in curvatures, its square is a quadratic form in curvatures, with universal coefficients. Integration of this form gives a linear expression in the Euler characteristics and the Willmore energy.

Dependence on the material

The results of the calculation for a cylinder:

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The symbol

$$\mathfrak{m}(x,\xi) = k_1(x)(\kappa_0 X_1(\xi)+(\frac{1}{2}-\kappa_0)Y_1(\xi))+k_2(x)(\kappa_0 X_2(\xi)+(\frac{1}{2}-\kappa_0)Y_2(\xi))$$

$\kappa_0 = \frac{\mu}{2(2\mu+\lambda)}$,, X_j, Y_j are uinversal matrices.
Integrate: a separate integration in x and in ξ.

$$n(\omega_s,\tau) \sim \tau^{-2}(\mathbf{C}_1\chi + \mathbf{C}_2 W$$

where $\mathbf{C}_1, \mathbf{C}_2$ are universal quadratic functions in $\kappa_0, \frac{1}{2}-\kappa_0$- this is the dependence of the material! Moreover, for the eigenvalues close to 0, these quadratic functions are constants.

Just a desert. The pseudodifferential approach can handle the case of a body made of a nonhomogeneous material. TheLame constants are not constants now, but $\lambda(x)$, $\mu(x)$. I consider a version of Lamé-Navier equations, for variable coefficients and the corresponding NP operator (no responsibility for correctness of the mechanical sense.) The coefficient $\kappa_0(x) = \frac{\mu(x)}{2(\lambda(x)+2\mu(x))}$. The essential spectrum is 0 plus the union of intervals of values of $\pm\kappa_0(x), x \in \Gamma$. There may be eigenvalues converging to the tips of the essential spectrum. In [Y.Miyanishi, G.R] estimates for the rate of this convergence were found. In the recent paper by me the asymptotics of these eigenvalues is found.

Suppose that $\kappa_0(x)$, $x \in \Gamma$, has nondegenerate maximum at $x_0 \in \Gamma$. Transformation $\widetilde{\mathcal{K}} = \kappa_0(x_0) - \mathcal{K}$.

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Near x_0 the symbol of $\widetilde{\mathcal{K}}$ in proper co-ordinates is

$$a(x,\xi) = P(|x - x_0|^2 + r_0|\xi'|^{-1})P + (1 - P)R(x,\xi')(1 - P)$$

where P is a 1-dim projection and $R(x,\xi')$ is a positive symbol.

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After making Fourier transform, we obtain the PsDo with symbol

$$\mathbf{a}(y, \eta') = |\eta'|^2 - r_0|y|^{-1} + (\textit{something not influensing the negative spectrum}$$

Schrödinger operator with Coulomb potential. $n(\lambda) \sim C\lambda^{-1}$.

Many questions on the NP operator. Even for the Laplacian: when are there negative eigenvalues? The ball: *no*; some oblate ellipsoid: *yes*.

Sources

Y. Miyanishi, G. Rozenblum, *Spectral properties of the Neumann–Poincaré operator in 3D elasticity.* **Int. Math. Res. Not. IMRN** 2021, no. 11, 8715–8740. G. Rozenblum. *Eigenvalue asymptotics for polynomially compact pseudodifferential operators.* **Algebra i Analiz** 33 (2021), no. 2, 215–232.(**St.Petersburg Math.Journal**, **33** (2) 2022); `arXiv:2006.10568`
G. Rozenblum. *Asymptotics of eigenvalues of the Neumann–Poincaré operator in 3D elasticity.*
J. Pseudo-Differ. Oper. Appl. 14 (2023), no. 2, Paper No. 26.
G. Rozenblum. *Discrete spectrum of zero order pseudodifferential operators.* **Opuscula Math.** 43, no. 2 (2023), 247–268

THANK YOU FOR YOUR ATTENTION!!!