Left definite Sturm-Liouville equations.
A sketch of Ch 5 of Bennewitz, Brown and Weikard: Spectral theory for ordinary differential equations

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1. Spectral theory

- Spectral theory is concerned with the simultaneous diagonalization of Hermitian forms or, equivalently, a corresponding eigenvalue equation $Su = \lambda Tu$.
- For this to be possible one needs that some linear combination of the Hermitian forms is positive. This is then used as a scalar product.
- If S and T are differential operators the convention is that S has higher order than T.
- In the simplest cases T is multiplication by a positive function w, and then one may consider the problem in the L^2-space with weight w. This would be a *right definite equation*.
- If one requires positivity of a Hermitian form associated with the operator S one talks of a *left definite equation*.

2. Sturm-Liouville equations

- We now restrict ourselves to consider a Sturm–Liouville equation

$$-(pu')' + qu = \lambda wu$$

  on an open interval (a, b).
  Standard assumptions are that 1/p, q and w are realvalued, locally integrable functions.

- A Hermitian form associated with the left hand side is the *Dirichlet integral*

$$\int_a^b (pu'\overline{v'} + qu\overline{v}).$$

  We call the problem *left-definite* if $p > 0$ a.e. and $q \geq 0$ but not $\equiv 0$.

3. Brief history

- In 1906 Hilbert created a theory of *polar* integral equations and applied it to left definite Sturm-Liouville equations with bounded base interval and integrable $1/p$, q and w. This implies discrete spectrum. As scalar product he used an expression which in a sense is dual to the Dirichlet integral.
- In 1910 Herman Weyl proved the first eigenfunction expansion theorem for general right definite Sturm-Liouville equations. Such equations do not necessarily have discrete spectrum. In a second note 1910 he treated the left definite case, using the scalar product introduced by Hilbert.
- The first to use the Dirichlet integral as scalar product seems to have been Max Mason who treated a regular case in 1907.
- Later results are mainly from the period 1960–1980, but interest was revived when the Camassa-Holm equation appeared in the 1990s. This is an integrable system in a similar sense as the KdV equation, but is associated with a left definite spectral problem.

4. Normal form

- Positivity of the Dirichlet integral requires $p > 0$ a.e. If we pick $c \in (a,b)$ and change variable by setting $t(x) = \int_c^x 1/p$ the equation is transformed into one where $p \equiv 1$ and q, w still locally integrable. So, we shall only deal with equations of the form $-u'' + qu = \lambda wu$. For the moment we assume $q \geq 0$ but not a.e. 0, which is sufficient but not necessary for positivity of the form $\int_a^b (u'\overline{v'} + qu\overline{v})$.
- We study the equation in a Hilbert space $\mathcal{H}_1$ with scalar product $\langle u, v \rangle = \int_a^b (u'\overline{v'} + qu\overline{v})$ and norm $\|u\| = \langle u, u \rangle^{1/2}$.
  The space $\mathcal{H}_1$ is the set of locally absolutely continuous functions u defined in (a,b) which have finite norm.
- It is easily proved that if $K \subset (a,b)$ is compact there is a constant C_K such that if $u \in \mathcal{H}_1$, then $|u(x)| \leq C_K \|u\|$ for $x \in K$.
  Thus point evaluations are bounded linear forms on $\mathcal{H}_1$.

5. Reproducing kernel

- Let L_c be the compactly supported elements of $L^1(a,b)$.
  If $v \in L_c$ it follows that the linear form $\mathcal{H}_1 \ni u \mapsto \int_a^b u\overline{v}$ is bounded.
  By use of Riesz' representation theorem we obtain a linear operator $G_0 : L_c \to \mathcal{H}_1$ such that $\langle u, G_0 v\rangle = \int_a^b u\overline{v}$.
- The operator G_0 is central to our approach to left definite problems.

Another simple consequence of the Riesz representation theorem is the following proposition.

Proposition

There exists a unique real-valued reproducing kernel *function g_0 defined in $(a,b)\times(a,b)$ such that $g_0(x,\cdot) \in \mathcal{H}_1$ for every fixed $x \in (a,b)$ and $u(x) = \langle u, g_0(x,\cdot)\rangle$ for every $u \in \mathcal{H}_1$ and $x \in (a,b)$.*

The reproducing kernel has the properties $g_0(x,y) = g_0(y,x)$, $g_0(x,x) > 0$ and the norm of the linear form $\mathcal{H}_1 \ni u \mapsto u(x)$ is $\sqrt{g_0(x,x)}$.

6. Max and min relations I

- A linear relation on $\mathcal{H}_1$ is a subspace of the orthogonal sum $\mathcal{H}_1 \oplus \mathcal{H}_1$. This is a trivial generalization of the concept of linear operator.
- To study $-u'' + qu = wf$ we use the theory of relations, since the same u will solve the equation for many f if $w = 0$ on an open set.
- We define a relation $T_c = \{(G_0(wv), v) : v \in L_c \cap \mathcal{H}_1\}$. T_c is a symmetric relation, for if $u, v \in L_c \cap \mathcal{H}_1$ we have

$$\langle G_0(wu), v\rangle = \overline{\langle v, G_0(wu)\rangle} = \int_a^b wu\bar{v} = \langle u, G_0(wv)\rangle.$$

- Define the minimal relation T_0 as the closure of T_c, and the maximal relation T_1 as its adjoint, i.e.,

$$T_1 = \{(u, f) \in \mathcal{H}_1 \oplus \mathcal{H}_1 : \langle u, v\rangle = \langle f, G_0(wv)\rangle \text{ for all } v \in L_c \cap \mathcal{H}_1\}.$$

7. Max and min relations II

Proposition

T_1 is the maximal realization of our equation in $\mathcal{H}_1$, i.e., $(u, f) \in T_1$ if and only if u and $f \in \mathcal{H}_1$, u' is locally absolutely continuous, and $-u'' + qu = wf$.

Proof.

If u and $f \in \mathcal{H}_1$ the definition of G_0 shows that for any $v \in L_c \cap \mathcal{H}_1$

$$\langle u, v \rangle - \langle f, G_0(wv) \rangle = \int_a^b (u' \overline{v'} + qu\overline{v} - wf\overline{v}) \, .$$

In particular, this vanishes for all $v \in C_0^\infty(a, b) \subset L_c \cap \mathcal{H}_1$ if and only if $-u'' + qu = wf$ in the sense of distributions. Since $qu - wf$ is locally integrable so is u'', and u' is locally absolutely continuous. □

8. Abstract spectral theory

- Abstract spectral theory now shows that any self-adjoint realization of our equation in $\mathcal{H}_1$ is a restriction of T_1 and an extension of T_0.
- Let D_λ be the *defect space at* λ of T_1. This is the space of elements $(u, \lambda u) \in T_1$. In our case, the set of solutions of $-u'' + qu = \lambda wu$ which are in $\mathcal{H}_1$, so $\dim D_\lambda \le 2$.
- In general, $\dim D_\lambda$ is constant in the open upper and lower half-planes. For $\operatorname{Im} \lambda > 0$ we denote the dimension by d_+, for $\operatorname{Im} \lambda < 0$ by d_-, and $d_\pm$ are called the *defect indices* of T_1. Self-adjoint restrictions of T_1 exist if and only if $d_+ = d_-$.
- If T is a self-adjoint restriction of T_1 and $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then $T = T_0 \dot{+} D$ where $D \subset D_\lambda \dot{+} D_{\overline{\lambda}}$. Defect indices are equal and we have $\dim D = d_+ = d_-$.
- In our case we always have $d_+ = d_- \le 2$, so there are self-adjoint realizations T of our equation, and they are obtained by imposing $d_+ = d_-$ linear, homogeneous conditions on T_1.

9. Boundary conditions I

- If (u, f) and (v, g) are in T_1 integration by parts shows that

$$\langle u, g\rangle - \langle f, v\rangle = \lim_{K\uparrow(a,b)} (u'\overline{g} - f\overline{v'})\big|_K,$$

the limit taken over compact intervals $K \subset (a, b)$.

- For a self-adjoint restriction T of T_1 this *boundary form* must vanish, so in general we must impose *boundary conditions* on T_1.
- An interval endpoint may be *limit point* (LP), at which $u'\overline{g} - f\overline{v'}$ vanishes on T_1, or *limit circle* (LC) at which it does not.
- T_1 is already self-adjoint if both endpoints are LP; this is the case of $d_\pm = 0$. If just one endpoint is LP, then $d_\pm = 1$ and a boundary condition is needed at the other endpoint. If both endpoints are LC, then $d_\pm = 2$ and two boundary conditions involving both endpoints are needed.

10. Boundary conditions II

- u' may not be well-defined at a LC endpoint, but if W is a real-valued primitive of w, then f and $u' + Wf$ always are.
- Note that $u'\overline{g} - f\overline{v'} = (u' + Wf)\overline{g} - f(\overline{v' + Wg})$. This means that f and $u' + Wf$ play the same role in the boundary form as u and u' do in the right definite (regular) case.
- Thus, if both endpoints are LC, one type of self-adjoint boundary conditions is

$$\begin{pmatrix} f(b) \\ (u' + Wf)(b) \end{pmatrix} = S \begin{pmatrix} f(a) \\ (u' + Wf)(a) \end{pmatrix}$$

  where S is a *symplectic* 2×2 matrix. This means that S is a real matrix of determinant 1 multiplied by a complex number of absolute value 1.
- Special cases are 'periodic' conditions, when S is the unit matrix, and 'semi-periodic' conditions, when S is the negative of the unit matrix.

11. Limit point criterion

- The other type of self-adjoint realization when both endpoints are LC is obtained by giving *separated* boundary conditions at the endpoints.
- At a a separated condition is $f(a)\cos\alpha + (u' + Wf)(a)\sin\alpha = 0$ for some fixed $\alpha \in [0, \pi)$. A similar condition is applied at b.
- If just one endpoint is LC we just need a separated boundary condition at this endpoint, and if both endpoints are LP no boundary conditions at all are applied.
- In contrast to the right definite case one may give explicit necessary and sufficient conditions for LP in the left definite case.

Theorem

Suppose W is a primitive of w. If an endpoint is finite, and q and W^2 are integrable near the endpoint, then the endpoint is LC.
Otherwise the endpoint is LP.

12. More abstract spectral theory

- Let $\mathcal{D}_T = \{u : \text{ there exists } (u, f) \in T\}$, $\mathcal{H}_\infty = \{f : (0, f) \in T\}$ and let $\mathcal{H}$ be the closure of $\mathcal{D}_T$ in $\mathcal{H}_1$.
  If T is self-adjoint, then $\mathcal{H}_1 = \mathcal{H} \oplus \mathcal{H}_\infty$ and $T \cap \mathcal{H}^2$ is the graph of a self-adjoint *operator* in $\mathcal{H}$. We denote also this operator by T.
- We shall finally turn to eigenfunction expansions for T, and start with the simplest case.
- If a self-adjoint operator T on a Hilbert space $\mathcal{H}$ has discrete spectrum (compact resolvent) with eigenvalues $\lambda_1, \lambda_2, \ldots$ and corresponding orthonormal eigenfunctions $e_1, e_2, \ldots$, then every element u of the Hilbert space has an expansion $u = \sum_{k=1}^{\infty} \hat{u}_k e_k$ where $\hat{u}_k = \langle u, e_k \rangle$. The series is norm convergent in $\mathcal{H}$, but note that in our case, because of the existence of a reproducing kernel, this implies locally uniform convergence.

13. Compact resolvent

We have the following sufficient conditions for compact resolvent.

Theorem

Suppose that near each endpoint of (a, b) *either* $x \mapsto w(x)g_0(x, x)$ *is integrable or else there is a primitive* W *of* w *such that* $x \mapsto W^2(x)g_0(x, x)$ *is integrable near the endpoint. Then* T *has compact resolvent.*
In particular, if both endpoints of (a, b) *are LC, then* T *has compact resolvent.*

In general neither condition implies the other, and we are not aware of any explicit necessary and sufficient conditions for compact resolvent.

14. Green's function

- To obtain an expansion theorem in case of a non-compact resolvent we need detailed information on the resolvent $R_\lambda = (T - \lambda I)^{-1}$. Note that R_λ has adjoint $R_{\overline{\lambda}}$. Using the reproducing kernel we find that $R_\lambda u(x) = \langle R_\lambda u, g_0(x,\cdot)\rangle = \langle u, R_{\overline{\lambda}} g_0(x,\cdot)\rangle$, so we may say that $G(x,y;\lambda) = \overline{R_{\overline{\lambda}} g_0(x,\cdot)}(y)$ is Green's function at λ.
- However, it is convenient to introduce the kernel $g(x,y;\lambda) = G(x,y;\lambda) + g_0(x,y)/\lambda$, so that $R_\lambda u(x) = \langle u, \overline{g(x,\cdot\,;\lambda)}\rangle - u(x)/\lambda$.
- We now need an explicit expression for $g(x,y;\lambda)$. It is convenient to first consider the case when at least one endpoint is LC. The reason is that if both endpoints are LP one may have continuous spectrum of multiplicity two complicating the eigenfunction expansion.
- So, from now on assume a is LC while b can be either LP or LC. Since spectrum is discrete if b is LC, the interesting case is when b is LP.

15. The m-function

Assume that the boundary condition at a is

$$f(a)\cos\alpha + (u' + Wf)(a)\sin\alpha = 0.$$

Let φ and θ be solutions of $-u'' + qu = \lambda wu$ for $\lambda \neq 0$ with initial data

$$\begin{cases} \lambda\varphi(a,\lambda) = -\sin\alpha \\ (\varphi' + W\lambda\varphi)(a,\lambda) = \cos\alpha \end{cases}, \quad \begin{cases} \lambda\theta(a,\lambda) = \cos\alpha \\ (\theta' + W\lambda\theta)(a,\lambda) = \sin\alpha \end{cases}.$$

Then φ and θ are analytic in $\lambda \neq 0$, locally uniformly in x.

Theorem

There exists a unique function $\lambda \mapsto m(\lambda)$ defined in the resolvent set of T except at 0 and such that $\psi(x,\lambda) = \theta(x,\lambda) + m(\lambda)\varphi(x,\lambda)$ is in $\mathcal{H}_1$, and satisfies the boundary condition at b if b is LC.
We then have $g(x,y;\lambda) = \varphi(\min(x,y),\lambda)\psi(\max(x,y),\lambda)$.

The function m is the (left definite) Titchmarsh-Weyl m-function for our problem.

16. The spectral measure

- The function m is a *Nevanlinna function*, i.e., it satisfies $m(\overline{\lambda}) = \overline{m(\lambda)}$, is analytic in $\mathbb{C} \setminus \mathbb{R}$, and maps the open upper half plane into itself.
  Nevanlinna functions have a unique representation $m(\lambda) = A + B\lambda + \int_{\mathbb{R}} \left(\frac{1}{t-\lambda} - \frac{t}{t^2+1} \right) d\eta(t)$, where $A \in \mathbb{R}$, $B \geq 0$ and the positive measure η satisfies $\int_{\mathbb{R}} \frac{d\eta(t)}{t^2+1} < \infty$.
- The measure η is called the *spectral measure* for our self-adjoint realization.
  It gives rise to a Hilbert space L^2_η with scalar product $\langle \hat{u}, \hat{v} \rangle_\eta = \int_{\mathbb{R}} \hat{u}\overline{\hat{v}}\, d\eta$.
- By a *finite function* we shall mean a function in $\mathcal{H}_1$ which is identically zero near b.
- Because of the special role played by $\lambda = 0$ in the left-definite equation we may need to remove a 1-dimensional space from $\mathcal{H}_1$ to obtain a simple expansion theorem.

17. The expansion theorem

So, let $\mathcal{H}_0 = \mathcal{H}_1$ unless b is LC with the boundary condition $\lambda u(b,\lambda) = 0$. In the exceptional case, let $\mathcal{H}_0$ be the closure of the finite functions in $\mathcal{H}_1$. Then $\mathcal{H} \subset \mathcal{H}_0$ and the expansion theorem is:

Theorem

1. *The map* $u \mapsto \hat{u}(t) = \langle u, \varphi(\cdot, t)\rangle$*, defined for finite* $u \in \mathcal{H}_1$*, extends to a map* $\mathcal{F} : \mathcal{H}_0 \to L^2_\eta$*, the Fourier transform associated with* T*.*
2. $\mathcal{F}$ *has nullspace* $\mathcal{H}_\infty$ *and is unitary from* $\mathcal{H}$ *to* L^2_ρ*, so that Parsevals formula* $\langle u, v\rangle = \langle \hat{u}, \hat{v}\rangle_\eta$ *holds for functions in* $\mathcal{H}$*.*
3. $\varphi(x, \cdot) \in L^2_\eta$ *for each* x *and* $\langle \hat{u}, \varphi(x, \cdot)\rangle_\eta$ *converges in* $\mathcal{H}$*. The map* $\hat{u} \mapsto \langle \hat{u}, \varphi(x, \cdot)\rangle_\eta$ *is the inverse of* $\mathcal{F} : \mathcal{H} \to L^2_\eta$*.*
4. $\hat{f}(t) = t\hat{u}(t)$ *if and only if* $(u, f) \in T$*, so* $\mathcal{F}$ *diagonalizes* T*.*

The proof is by contour integration of $\langle R_\lambda u, v\rangle$. This yields Parseval's formula and then the rest follows easily.

18. Two LP endpoints

- To deal with two LP endpoints one may use a device introduced by Weyl in 1910 and also used by Titchmarsh.
  Pick $c \in (a, b)$ and solutions φ and θ as before, but with initial data in c. As before we obtain solutions ψ_+ for the interval (c, b) and ψ_- for the interval (a, c) satisfying the boundary conditions at b respectively a.
- Introducing $\Phi = \left(\begin{smallmatrix} \varphi \\ \theta \end{smallmatrix}\right)$ and $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ we may now write

  $$g(x, y; \lambda) = \Phi^*(\min(x, y), \lambda)(M(\lambda) - \tfrac{1}{2}J)\Phi(\max(x, y), \lambda$$

  where M is a 2×2-valued Nevanlinna function, with a corresponding 2×2 spectral measure.
- The corresponding vector-valued L^2-space now serves as the transform space in which T is diagonalized.
  We omit the details.

19. Generalized left definite equations I

Assuming $\int_a^b(|u'|^2 + q|u|^2) > 0$ for $u \neq 0$ does not imply $q \geq 0$, but it does imply that solutions of $-u'' + qu = 0$ can have at most one zero. It also implies that there is a positive solution with reciprocal in L^2. This gives a hint of how to generalize the concept of a left definite Sturm-Liouville equation.

Definition

Let $F \in H^1_{\text{loc}}(a,b)$ be strictly positive and satisfy $1/F \in L^2(a,b)$.

Here $H^1_{\text{loc}}(a,b)$ is the set of functions with distributional derivatives in $L^2_{\text{loc}}(a,b)$. Similarly, $H^{-1}_{\text{loc}}(a,b)$ is the set of distributional derivatives of functions in $L^2_{\text{loc}}(a,b)$.
Thus $-F'' + qF = 0$ where $q = F''/F \in H^{-1}_{\text{loc}}(a,b)$.
Let $h = F'/F \in L^2_{\text{loc}}(a,b)$, note that $h' + h^2 = q$, and put

$$\langle u,v\rangle = \int_a^b (u/F)'\overline{(v/F)'}F^2 = \int_a^b (u' - hu)\overline{(v' - hv)}.$$

If u or v is compactly supported $\langle u,v\rangle = \int_a^b (u'\overline{v'} + qu\overline{v})$ and $\langle u,u\rangle > 0$ if u is not a multiple of F. However, $\langle F,F\rangle = 0$.
To obtain a positive definite scalar product we note that if $\langle u,u\rangle < \infty$, then $\int_a^b |(u/F)'| = \int_a^b |(u/F)'F|/F \le \sqrt{\langle u,u\rangle \int_a^b F^{-2}}$ so that $\tilde{u} = u/F$ has finite limits at a and b.
We may therefore define $\langle u,v\rangle_Q = \langle u,v\rangle + Q(u,v)$ where

$$Q(u,v) = \alpha\tilde{u}(a)\overline{\tilde{v}(a)} + \beta\tilde{u}(b)\overline{\tilde{v}(b)} + \gamma\tilde{u}(a)\overline{\tilde{v}(b)} + \overline{\gamma}\tilde{u}(b)\overline{\tilde{v}(a)}.$$

This gives a positive definite scalar product if $\alpha + 1 > 0$ and $(\alpha+1)(\beta+1) > |\gamma - 1|^2$.
One may now create a complete spectral theory very similar to the one described, concerning the equation $-u'' + qu = wf$, where q and $w \in H^{-1}_{\mathrm{loc}}(a,b)$, provided $q = F''/F$ with F as in our definition.

21. Examples

- $F(x) = \cosh x$. Here $q = F''/F = 1$ so the equation is $-u'' + u = wf$. This is covered by our earlier theory (if $w \in L^1_{\text{loc}}$).
- $F(x) = 2 + |x|$ on $\mathbb{R}$. We obtain the equation $-u'' + \delta u = wf$, where δ is the Dirac measure at 0. δ is a positive measure and it would be easy to extend our previous theory to cover such a case.
- $F(x) = x^\alpha + x^{1-\alpha}$ on $(0, \infty)$. Here $q = \alpha(\alpha - 1)x^{-2}$, which is < 0 in $(0, \infty)$ if $0 < \alpha < 1$, while $1/F \in L^2(0, \infty)$ unless $\alpha = 1/2$.
  Thus we may find F satisfying all requirements in spite of q being strictly negative.
  Since this means that F is positive and concave this is only possible on bounded or semi-bounded intervals.
- Even on $\mathbb{R}$ one may have q negative on a large set.
  For example, with $F(x) = (1 + x^2)^{1/3}$ we have $1/F \in L^2(\mathbb{R})$, but $q = \frac{1}{9}(6 - 2x^2)(1 + x^2)^{-2}$.
  Thus q is negative except on the compact interval $[-\sqrt{3}, \sqrt{3}]$.

22. A final example

Consider, finally, $F(x) = 1 + |x|^\alpha$ on $\mathbb{R}$.
Then $1/F \in L^2(\mathbb{R})$ if $\alpha > 1/2$, and $F'(x) = \alpha|x|^{\alpha-2}x$.
Differentiating for $x \neq 0$ we obtain $F''(x) = \alpha(\alpha-1)|x|^{\alpha-2}$, which is negative if $1/2 < \alpha < 1$.
F' has an infinite jump at 0 so it is not of locally bounded variation, but it is locally square integrable. Thus $F'' \in H^{-1}_{\text{loc}}(\mathbb{R})$ but is not a measure.
We have $q(x) = \alpha(\alpha-1)|x|^{\alpha-2}/(1+|x|^\alpha) < 0$ for $x \neq 0$, but in any neighborhood of 0 we have $q \in H^{-1}_{\text{loc}}$, but it is not a measure.
Thus the formula does not represent q there and it is meaningless to speak of its sign in a neighborhood of 0.

Thanks for listening.

The End.