Tunnel effect and symmetries for non-selfadjoint operators

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Abstract. We study low lying eigenvalues for non-selfadjoint semiclassical differential operators, where symmetries play an important role. In the case of the Kramers-Fokker-Planck operator, we show how the presence of certain supersymmetric and $\mathcal{PT}$-symmetric structures leads to precise results concerning the reality and the size of the exponentially small eigenvalues in the semiclassical (here the low temperature) limit. This analysis also applies sometimes to chains of oscillators coupled to two heat baths, but when the temperatures of the baths are different, we show that the supersymmetric approach may break down. We also discuss $\mathcal{PT}$–symmetric quadratic differential operators with real spectrum and characterize those that are similar to selfadjoint operators. This talk is based on joint works with Emanuela Caliceti, Sandro Graffi, Frédéric Hérau, and Johannes Sjöstrand.

1. Introduction

The purpose of this talk is to describe some recent results [13], [14], [4], concerning the spectral analysis of some classes of non-selfadjoint operators in the presence of symmetries.

Let us consider a differential operator

$$P = P(x, hD_x; h) \tag{1.1}$$

on M, where $M = \mathbf{R}^n$ or a compact n-dimensional manifold, equipped with a smooth strictly positive density. Here $D_x = -i\partial_x$ and $h \to 0$ is a semiclassical parameter, which can be Planck's constant or the temperature. Let us assume that P has a natural closed realization in $L^2(M)$, such that its spectrum $\mathrm{Spec}\,(P) \subset \{z \in \mathbf{C}; \mathrm{Re}\, z \geq 0\}$, and such that $0 \in \mathrm{Spec}\,(P)$ is a simple isolated eigenvalue. Let $e_0 \in L^2(M)$ be a corresponding eigenfunction.

We are interested in the following two problems, that are closely related and in fact equivalent when the operator P is selfadjoint:

- Problem of return to equilibrium: study how fast $e^{-tP/h}u$ converges to a multiple of e_0 as $t \to +\infty$, when $u \in L^2(M)$.
- Related spectral problem: study the spectral gap between 0 and $\mathrm{Spec}\,(P)\backslash\{0\}$.

Traditionally such problems arise when P is the Schrödinger operator, in particular when we are in the presence of multiple potential wells, [19], [20], [8], [9],[10], but here we shall be concerned with second order operators coming from kinetic and transport theory, such as the Kramers-Fokker-Planck operator [18] and the operator modeling the chain of two oscillators coupled to two heat baths [5]. A new difficulty compared with the Schrödinger case is that our operators are non-selfadjoint and also lack ellipticity, both locally and at infinity. A simplifying feature for the Kramers-Fokker-Planck operator, essential for the analysis of the

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problems above, is the presence of a supersymmetric structure, showing that we have a non-selfadjoint Witten Laplacian in degree zero, observed in [1] and [22]. An important role is also played by a certain reflection (generalized $\mathcal{PT}$) symmetry.

The plan of the talk is as follows. In Section 2, we shall follow [14] and discuss some general conditions for a second order differential operator to possess a supersymmetric structure. In Section 3, we describe the results of [13], giving complete asymptotic expansions for the exponentially small eigenvalues of the Kramers-Fokker-Planck operator. Section 4 is concerned with the case of the operator describing the short chain of two oscillators coupled to two heat baths, where it turns out that the supersymmetry may break down in the case when the temperatures of the baths are distinct, [14]. Finally, in Section 5, we describe the result of [4], concerning $\mathcal{PT}$–symmetric quadratic differential operators with real spectrum, that are similar to selfadjoint operators.

2. Generalities on supersymmetric structures

Let M be $\mathbf{R}^n$ or a smooth compact manifold of dimension n, equipped with a smooth strictly positive volume density $\omega(dx)$. Let

$$d : C^\infty(M;\Lambda^k T^*M) \to C^\infty(M;\Lambda^{k+1}T^*M) \tag{2.1}$$

be the de Rham exterior differentiation, and let

$$\delta : C^\infty(M;\Lambda^{k+1} TM) \to C^\infty(M;\Lambda^k TM) \tag{2.2}$$

be the adjoint of d with respect to the natural pointwise duality between $\Lambda^k T^*M$ and $\Lambda^k TM$, integrated against $\omega(dx)$. In what follows, we shall restrict the attention to real-valued sections and operators.

Let $A(x) : T_x^*M \to T_xM$ be a linear map depending smoothly on $x \in M$. We have the bilinear non-symmetric product,

$$(u,v)_A = \int \langle \Lambda^k A(x)u(x), v(x)\rangle \omega(dx), \quad u,v \in C_0^\infty(M;\Lambda^k T^*M).$$

When the map A is pointwise bijective, there is natural way of defining the formal adjoint of a linear operator Q taking k–forms to ℓ–forms. This adjoint, denoted by $Q^{A,*}$, maps ℓ-forms to k-forms, and is given by

$$(Qu,v)_A = (u, Q^{A,*}v)_A.$$

Example. For the restriction of the de Rham operator to 0–forms, we get

$$d^{A,*} = \delta A^t.$$

Let now $\varphi \in C^\infty(M;\mathbf{R})$ and consider the Witten-de Rham differential

$$d_\varphi = e^{-\varphi/h} \circ hd \circ e^{\varphi/h} = hd + d\varphi\wedge, \quad h \in (0,h_0], \quad h_0 > 0.$$

We then have the associated twisted Witten Laplacian,

$$\Box_{A,\varphi} = d_\varphi^{A,*} d_\varphi + d_\varphi d_\varphi^{A,*}. \tag{2.3}$$

The operator $\Box_{A,\varphi}$ preserves the degree of a differential form and its restriction to 0-forms is given by $\Box_{A,\varphi}^{(0)} = d_\varphi^{A,*} d_\varphi$.

Remark. The ordinary Witten Laplacian [24], [11] is defined as in (2.3), when M is a compact Riemannian manifold, say, with $A = \mathcal{G}^{-1}$, where $\mathcal{G} = (g_{jk})$ is the Riemannian metric.

Let $P = P(x, hD_x; h)$ be a second order scalar real semiclassical differential operator on M. Viewing P as acting on 0-forms, we would like to determine when there exists a linear h-dependent map $A = A(x; h)$ as above, such that for all $h \in (0, h_0]$,

$$P = d_\psi^{A,*} d_\varphi, \tag{2.4}$$

for some real-valued h–independent functions φ, $\psi \in C^\infty$, either locally or globally on M. If the representation (2.4) holds, we say that P has a supersymmetric structure on M, in the semiclassical sense.

The following result, established in [14], gives a necessary and sufficient condition for (2.4) to hold.

Proposition 2.1. *In order for the operator P to have a supersymmetric structure on M in the semiclassical sense it is necessary that there should exist h-independent functions φ, $\psi \in C^\infty(M; \mathbf{R})$, such that*

$$P(e^{-\varphi/h}) = 0, \quad P^*(e^{-\psi/h}) = 0, \tag{2.5}$$

for all $h \in (0, h_0]$, $h_0 > 0$. If (2.5) holds and the δ–complex

$$C^\infty(M; \Lambda^2 TM) \xrightarrow{\delta} C^\infty(M; TM) \xrightarrow{\delta} C^\infty(M)$$

is exact in degree 1 for smooth sections, then P has a semiclassical supersymmetric structure on M.

Remark. Let $h = 1$ and assume that M is a compact Riemannian manifold. Let $P = -\Delta + X$, where X is a smooth real vector field. Then (2.5) holds with $\varphi = \psi = 0$ precisely when the vector field X is divergence free.

3. The Kramers-Fokker-Planck operator

In this section, we shall be concerned with the low lying eigenvalues of the Kramers-Fokker-Planck operator P, given by

$$P = y \cdot h\partial_x - V'(x) \cdot h\partial_y + \frac{\gamma}{2}(y - h\partial_y) \cdot (y + h\partial_y) \quad \text{on} \quad \mathbf{R}^{2n}_{x,y}. \tag{3.1}$$

Here $h > 0$ is the temperature and we shall work in the low temperature limit $h \to 0$. The constant $\gamma > 0$ is the friction.

Let us assume that the potential $V \in C^\infty(\mathbf{R}^n; \mathbf{R})$ is such that

$$\partial^\alpha V \in L^\infty(\mathbf{R}^n), \quad |\alpha| \geq 2, \quad |V'(x)| \geq 1/C, \quad |x| \geq C > 0,$$

and that

$$V(x) \to +\infty, \quad |x| \to \infty.$$

According to [15], [7], the graph closure of P on $\mathcal{S}(\mathbf{R}^{2n})$ is maximally accretive on $L^2(\mathbf{R}^{2n})$. Furthermore, $\operatorname{Re} P \geq 0$ and $\operatorname{Spec}(P)$ is contained in the right half plane $\operatorname{Re} z \geq 0$.

Let

$$\varphi(x, y) = \frac{y^2}{2} + V(x).$$

The Maxwellian $e_0 = e^{-\varphi/h} \in L^2$ satisfies $P(e_0) = 0$ so that $0 \in \operatorname{Spec}(P)$, and it is the only eigenvalue of P on the imaginary axis. Furthermore, (3.1) also gives that

$$P^*(e^{-\varphi/h}) = 0,$$

and an application of Proposition 2.1 allows us to conclude that the operator P has a semi-classical supersymmetric structure on $\mathbf{R}^{2n}_{x,y}$. The existence of such a structure for the Kramers-Fokker-Planck operator was established in [1], [22].

Example. We have explicitly $P = d_\varphi^{A,*} d_\varphi$, where

$$A = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & \gamma \end{pmatrix}.$$

A global hypoellipticity result for P was established in [15], and it was also shown that the spectrum of P avoids a parabolic neighborhood of $i\mathbf{R}$ away from a disc around the origin, where the spectrum is discrete.

We shall now turn the attention to the analysis of the low lying eigenvalues of the operator P in the semiclassical limit $h \to 0$, where a fundamental study has been carried out in the work [16]. Assume that V is a Morse function. According to [16], the spectrum of P in any strip $0 \leq \operatorname{Re} z \leq Ch$ is discrete, and the eigenvalues have complete asymptotic expansions, with the leading terms of the form

$$\lambda_{j,k}(h) = \mu_{j,k} h + o(h).$$

Here the $\mu_{j,k}$ are the eigenvalues of the quadratic approximation

$$Q_j = y \cdot \partial_x - V''(x_j)x \cdot \partial_y + \frac{\gamma}{2}\left(-\partial_y + y\right) \cdot \left(\partial_y + y\right),$$

at the points $(x_j, 0)$, where x_j are the finitely many critical points of V. The eigenvalues $\mu_{j,k}$ can be computed explicitly and are confined to an angle $|\arg z| \leq \theta_0 < \pi/2$. Furthermore, when x_j is not a local minimum, then $\operatorname{Re} \lambda_{j,k}(h) \geq h/C$. When x_j is a local minimum, then precisely one of the $\lambda_{j,k}(h)$ is $\mathcal{O}(h^\infty)$, while the others have real part $\geq h/C$.

Remark. The eigenvalues of the quadratic operator Q_j are given by the familiar formulas, established in [21], [2] in the case when the quadratic operator in question is elliptic. Let $q(x, y, \xi, \eta)$ be the (non-elliptic) Weyl symbol of $Q = Q_j + \gamma/2$, given by

$$q(x, y, \xi, \eta) = iy \cdot \xi - iV''(x_j)x \cdot \eta + \frac{\gamma}{2}\left(y^2 + \eta^2\right).$$

Then $\operatorname{Re} q \geq 0$ and if we put

$$\langle \operatorname{Re} q \rangle_T = \frac{1}{T}\int_0^T \operatorname{Re} q \circ \exp\left(tH_{\operatorname{Im} q}\right) dt, \quad T > 0 \quad \text{fixed},$$

then the following averaging property holds,

$$\langle \operatorname{Re} q \rangle_T > 0. \tag{3.2}$$

The averaging property alone implies that the spectrum of Q is discrete, given by the formulas of [21], and that the heat semigroup e^{-tQ} is strongly regularizing, $t > 0$. See [17].

Remark. The spectrum of the operator Q_j is real precisely when all the eigenvalues of $V''(x_j)$ are $\leq \gamma^2/4$.

3.1. Exponentially small eigenvalues of the Kramers-Fokker-Planck operator

We now come to describe the main result of [13]. Let us recall the potential

$$V \text{ is a Morse function with } n_0 \text{ local minima.}$$

We take $\varphi(x,y) = y^2/2 + V(x)$ and write $d = 2n$. The critical points of φ of index 1 will be called the saddle points. An application of the Morse lemma shows that if $s \in \mathbf{R}^d$ is a saddle point, then for $r > 0$ small enough, the set

$$\{(x,y) \in B(s,r); \varphi(x,y) < \varphi(s)\}$$

has two connected components. Here $B(s,r)$ is the open ball in $\mathbf{R}^d$ of radius $r > 0$ centered at s. We shall say that s is a separating saddle point if these components belong to different components in the sublevel set

$$\{(x,y) \in \mathbf{R}^d; \varphi(x,y) < \varphi(s)\}.$$

A connected component E of the set $\varphi^{-1}((-\infty,\sigma)) = \{(x,y) \in \mathbf{R}^d; \varphi(x,y) < \sigma\}$ is called a critical component if there exists a separating saddle point $\in \partial E$ or if $E = \mathbf{R}^d$.

There exists a natural map from the set of the local minima to the set of the critical components, which can be described as follows. Let us consider the sublevel sets $\{\varphi < \sigma\}$ for σ large and decreasing. For $\sigma = +\infty$, we get $\mathbf{R}^d$, which is connected. Let m_1 be a point of global minimum of φ, and let us write $E_{m_1} = \mathbf{R}^d$. When decreasing σ, the set $\{\varphi < \sigma\}$ remains connected and non-empty until one of the following happens:

(i) We reach a critical value $\sigma = \varphi(s)$, where s is one or several separating saddle points. Then $\{\varphi < \sigma\}$ splits into several connected components.

(ii) We reach $\sigma = \varphi(m_1)$ and the connected component disappears.

In the case (i), one of the components contains m_1. For each of the other components, E_k, we choose a global minimum $m_k \in E_k$ of the restriction $\varphi|_{E_k}$ of φ to E_k, and write $E_{m_k} = E_k$, $\sigma = \sigma(m_k)$. We then continue the procedure with each of the connected components, including the one containing m_1, until all the local minima have been recovered.

Let us set

$$S_k = \sigma(m_k) - \varphi(m_k) > 0, \quad S_1 = +\infty.$$

The following result was established in [13].

Theorem 3.1. *We have*

- *There exists $C > 0$ such that for all $h > 0$ small enough,* $\operatorname{Spec}(P) \cap D(0, h/C)$ *consists of n_0 eigenvalues that are real and exponentially small:*

  $$\mu_k \asymp h e^{-2S_k/h}, \quad 1 \le k \le n_0.$$

  Here $\mu_1 = 0$.

- *If we assume, after relabeling, that $S_2 > \max_{j \ge 3} S_j$ and that ∂E_{m_2} contains only one separating saddle point, then the smallest non-vanishing eigenvalue is of the form*

  $$\mu_2 = h b_2(h) e^{-2S_2/h}, \quad b_2 \sim b_{2,0} + h b_{2,1} + \dots \quad b_{2,0} > 0.$$

Remark. Theorem 3.1 is analogous to the results of [3], [6] in the case of the Witten Laplacian. The main ingredient in the proof of Theorem 3.1 is the development of tunneling techniques, including exponential decay estimates, in the present non-elliptic non-selfadjoint situation. The presence of additional symmetries, including the supersymmetric structure, allows for simplifications in the analysis.

Remark. Under an even stronger generic assumption in Theorem 3.1, we get complete asymptotic expansions for all the eigenvalues μ_k, $2 \leq k \leq n_0$,

$$\mu_k = hb_k(h)e^{-2S_k/h}, \quad b_k(h) \sim \sum_{j=0}^{\infty} b_{k,j}h^j, \quad b_{k,0} > 0.$$

3.2. A $\mathcal{PT}$–type symmetry

The reality of the exponentially small eigenvalues of P is a consequence of the following generalized $\mathcal{PT}$–symmetry. Let $\kappa(x,y) = (x,-y)$ and let us define

$$U_\kappa : L^2(\mathbf{R}^d) \to L^2(\mathbf{R}^d)$$

by $U_\kappa u = u \circ \kappa$. We have

$$U_\kappa^2 = 1, \quad U_\kappa^* = U_\kappa,$$

and

$$P^* U_\kappa = U_\kappa P.$$

Let us introduce the non-degenerate Hermitian form,

$$(u,v)_\kappa := (U_\kappa u, v)_{L^2}, \quad u, v \in L^2.$$

Then

$$(Pu,v)_\kappa = (U_\kappa Pu, v)_{L^2} = (P^* U_\kappa u, v)_{L^2} = (U_\kappa u, Pv)_{L^2} = (u, Pv)_\kappa,$$

so that P is formally selfadjoint with respect to $(\cdot,\cdot)_\kappa$.

Proposition 3.2. *Let $E_0 \subset L^2(\mathbf{R}^d)$ be the spectral subspace corresponding to $\mu_1, \ldots, \mu_{n_0}$. Then the Hermitian form $(\cdot,\cdot)_\kappa$ is positive definite on $E_0 \times E_0$, for $h > 0$ small enough, and hence, is a scalar product there. The operator $P : E_0 \to E_0$ is selfadjoint and so the eigenvalues $\mu_1, \ldots, \mu_{n_0}$ are real.*

As explained in [13], the supersymmetric structure of the Kramers-Fokker-Planck operator allows one to work with a Witten-like complex in the tunneling analysis of P, and the additional symmetry given by κ can also be used in higher degrees, allowing for a selfadjoint situation when restricting the attention to the exponentially small eigenvalues.

4. Chain of oscillators coupled to heat baths

The purpose of this section is to describe, following [14], an example of a physically significant second order semiclassical operator, for which the supersymmetric structure may break down. Specifically, we shall be concerned with the semigroup generator for the stochastic process describing a chain of two oscillators, coupled to two heat baths, [5],

$$\widetilde{P}_W = \frac{\gamma}{2}\sum_{j=1}^{2} \alpha_j(-h\partial_{z_j})\left(h\partial_{z_j} + \frac{2}{\alpha_j}(z_j - x_j)\right) + y \cdot h\partial_x - (W'(x) + x - z)\cdot h\partial_y.$$

Here $(x_j, y_j) \in \mathbf{R}^{2n}$ are the position and velocity of the jth particle, $j = 1, 2$, $x = (x_1, x_2)$, $y = (y_1, y_2)$, and the classical Hamiltonian is of the form $y^2/2 + W(x) + x^2/2$. The variable $z_j \in \mathbf{R}^n$ is the effective heat bath variable, $j = 1, 2$, and $\gamma > 0$ is the friction. Finally, the temperature in the jth bath is given by $\alpha_j h/2 > 0$, $j = 1, 2$.

The operator $\widetilde{P}_W$ possesses a supersymmetric structure in the following two cases:

- The equilibrium case: $\alpha_1 = \alpha_2$.
- The decoupled case: $W = W_0(x) = W_1(x_1) + W_2(x_2)$.

In each of the two cases, there exists an explicit function $\varphi_0(x, y, z) \in C^\infty(\mathbf{R}^{6n}; \mathbf{R})$ and a constant matrix A such that if $P_W = e^{\varphi_0/h} \widetilde{P}_W e^{-\varphi_0/h}$ then we have

$$P_W = d_{\varphi_0}^{A,*} d_{\varphi_0}, \quad P_W \left(e^{-\varphi_0/h}\right) = P_W^* \left(e^{-\varphi_0/h}\right) = 0.$$

In the equilibrium case, a result analogous to Theorem 3.1 has been obtained in [12] when W is a Morse function with two local minima and one saddle point.

In the decoupled case, we have

$$\varphi_0(x, y, z) = \sum_{j=1}^{2} \frac{1}{\alpha_j} \left(\frac{y_j^2}{2} + W_j(x_j) + \frac{(x_j - z_j)^2}{2} \right),$$

and the conjugated operator $P_{W_0} = e^{\varphi_0/h} \widetilde{P}_{W_0} e^{-\varphi_0/h}$ is given by

$$P_{W_0} = \frac{\gamma}{2} \sum_{j=1}^{2} \alpha_j \left(-h\partial_{z_j} + \frac{1}{\alpha_j}(z_j - x_j) \right) \cdot \left(h\partial_{z_j} + \frac{1}{\alpha_j}(z_j - x_j) \right) \\ + y \cdot h\partial_x - (\partial_x W_0(x) + x - z) \cdot h\partial_y$$

Associated to the operator P_{W_0} is the real symbol,

$$q_{W_0}(x, y, z, \xi, \eta, \zeta) = \frac{\gamma}{2} \sum_{j=1}^{2} \alpha_j \left(\zeta_j^2 - \frac{1}{\alpha_j^2}(x_j - z_j)^2 \right) + y \cdot \xi - (\partial_x W_0(x) + x - z) \cdot \eta, \quad (4.1)$$

so that to the leading order, we have

$$P_{W_0} = -q_{W_0}(x, y, z, -h\partial_x, -h\partial_y, -h\partial_z).$$

The phase function φ_0 satisfies the eikonal equation,

$$q_{W_0}(x, y, z, \partial_x \varphi_0, \partial_y \varphi_0, \partial_z \varphi_0) = 0,$$

reflecting the fact that $P_{W_0}(e^{-\varphi_0/h}) = 0$. Let us now perturb the operator $\widetilde{P}_{W_0}$ by replacing W_0 by $W = W_0 + \delta W$, so that

$$\widetilde{P}_W = \widetilde{P}_{W_0} - \partial_x(\delta W(x)) \cdot h\partial_y.$$

After a conjugation, we obtain an operator

$$P_W = P_{W_0} - \partial_x (\delta W(x)) \cdot (h\partial_y - \partial_y \varphi_0).$$

We are interested in the question whether the perturbed conjugated operator P_W still possesses a smooth supersymmetric structure on $\mathbf{R}^{6n}$, in the semiclassical sense. According to

Proposition 2.1, a necessary condition for that is the existence of a smooth solution φ of the eikonal equation

$$q_W(x, y, z, \partial_x\varphi, \partial_y\varphi, \partial_z\varphi) = 0. \tag{4.2}$$

The following result of [14] shows the non-existence of a supersymmetric structure for P_W, for some perturbations, in the non-equilibrium case.

Theorem 4.1. *Let us take $\gamma = 1$ and assume that $\alpha_1 \neq \alpha_2$. Let $W_1(x_1)$ be a Morse function with two local minima m_1, m_2, and one saddle point s_0, tending to $+\infty$ when $x_1 \to \infty$. Let $W_2(x_2)$ be a positive definite quadratic form. Let $3 \leq m \in \mathbf{N}$. There exists $C^\infty(\mathbf{R}^{2n}) \ni \delta W = \mathcal{O}(|x_2|^m)$ arbitrarily small and vanishing near M_j and S_0, such that the eikonal equation (4.2) has no smooth solution on $\mathbf{R}^{6n}$, with $\varphi(\widetilde{M_1}) = 0$, $\varphi'(\widetilde{M_1}) = 0$, $\varphi''(\widetilde{M_1}) > 0$. Here $M_j = (m_j, 0)$, $S_0 = (s_0, 0)$, and $\widetilde{M_1} = (M_1, 0, M_1)$.*

Remark. In the proof of Theorem 4.1 one shows that the eikonal equation (4.2) has no C^∞-solutions on any open set $\Omega \subset \mathbf{R}^{6n}$ such that $\Omega_+ \Subset \Omega$. Here Ω_+ is the connected component of the set $\varphi_0^{-1}((-\infty, \varphi_0(\widetilde{S}_0)))$, which contains $\widetilde{M_1}$, and $\widetilde{S}_0 = (S_0, 0, S_0)$. In other words, when trying to extend the local solution defined near the local minimum $\widetilde{M_1}$, we see that it develops singularities when approaching the saddle point $\widetilde{S}_0$.

Theorem 4.1 implies that for the operator P_W, it is in general not possible to write $P_W = d_\psi^{A,*} d_\varphi$, with a smooth h-independent function φ. This indicates that the tunneling analysis of P_W will necessarily be more challenging and should be carried out directly on the level of our second order differential operator.

5. Quadratic $\mathcal{PT}$–symmetric operators

Let us finally describe the results of [4], concerning the case of quadratic differential operators enjoying the property of $\mathcal{PT}$–symmetry.

Let $\kappa : \mathbf{R}^n \to \mathbf{R}^n$ be linear with $\kappa^2 = 1$. Associated with the linear involution κ, we have the parity operator $\mathcal{P}$, given by

$$\mathcal{P}u = u \circ \kappa, \quad u \in L^2(\mathbf{R}^n).$$

Let us also introduce the time reversal symmetry $\mathcal{T}$ defined by $\mathcal{T}u = \overline{u}$.

Let $q(x, \xi)$ be a complex-valued quadratic form on $\mathbf{R}^{2n} = \mathbf{R}^n_x \times \mathbf{R}^n_\xi$, and assume that q is elliptic on $\mathbf{R}^{2n}$ in the sense that $q(X) = 0$, $X \in \mathbf{R}^{2n}$, precisely when $X = 0$. From [21] we may then recall that unless $q(\mathbf{R}^{2n}) = \mathbf{C}$ (which can only occur when $n = 1$), the range of q on $\mathbf{R}^{2n}$ is a closed sector of aperture $< \pi$, and without loss of generality we may assume that $\mathrm{Re}\, q > 0$. Associated to the quadratic form q is the Hamilton map

$$F : \mathbf{C}^{2n} \to \mathbf{C}^{2n}$$

defined by

$$q(X, Y) = \sigma(X, FY), \quad X, Y \in \mathbf{C}^{2n},$$

where σ is the canonical complex symplectic form on $\mathbf{C}^{2n}$.

The following result has been established in [4].

Theorem 5.1. *Let $q : \mathbf{R}^{2n} \to \mathbf{C}$ be a quadratic form such that $\mathrm{Re}\, q > 0$. Assume that the Weyl quantization of q, the operator $q^w(x, D_x)$, is $\mathcal{P}\mathcal{T}$–symmetric in the sense that*

$$[\mathcal{P}\mathcal{T}, q^w(x, D_x)] = 0.$$

Assume furthermore that $\mathrm{Spec}\,(q^w) \subset \mathbf{R}$. *Then the operator $q^w(x, D_x)$ is similar to a selfadjoint operator precisely when the Hamilton map F of q has no Jordan blocks.*

Remark. The commutation property $[\mathcal{P}\mathcal{T}, q^w(x, D_x)] = 0$ is equivalent to the fact that

$$\overline{q(\kappa(x), -\kappa^t(\xi))} = q(x, \xi), \quad (x, \xi) \in \mathbf{R}^{2n}.$$

Remark. Let $\mathcal{H}_j$, $j = 1, 2$ be complex separable Hilbert spaces. In the context of Theorem 5.1, we say that two closed densely defined operators $\mathcal{A}_j : \mathcal{H}_j \to \mathcal{H}_j$, $j = 1, 2$, with discrete spectra, are similar if there exist linear subspaces $\mathcal{S}_j \subset \mathcal{D}(\mathcal{A}_j)$, with $\mathcal{A}_j(\mathcal{S}_j) \subset \mathcal{S}_j$, containing the spectral subspaces of $\mathcal{A}_j$, and a linear bijection $S : \mathcal{S}_1 \to \mathcal{S}_2$ such that

$$S\mathcal{A}_1 = \mathcal{A}_2 S \quad \text{on } \mathcal{S}_1.$$

It follows that the similar operators $\mathcal{A}_1$ and $\mathcal{A}_2$ are isospectral.

Example. Let

$$q(x, y, \xi, \eta) = iy \cdot \xi - iV''(x_0)x \cdot \eta + \frac{\gamma}{2}\left(y^2 + \eta^2\right)$$

be the Weyl symbol of the quadratic Kramers-Fokker-Planck operator. If we set $\kappa(x, y) = (-x, -y)$, then it follows that the property of the $\mathcal{P}\mathcal{T}$–symmetry holds,

$$\overline{q(\kappa(x, y), -\kappa^t(\xi, \eta))} = q(x, y, \xi, \eta).$$

Assume that the eigenvalues of $V''(x_0)$ do not exceed $\gamma^2/4$, so that the spectrum of q^w is purely real. To be able to apply Theorem 5.1 to the non-elliptic quadratic operator q^w, we should observe that the result of Theorem 5.1 continues to be valid in the case when $\mathrm{Re}\, q \geq 0$ and

$$\frac{1}{T}\int_0^T \mathrm{Re}\, q \circ \exp\left(tH_{\mathrm{Im}\, q}\right) dt > 0, \quad T > 0.$$

This follows by inspection of the proof of Theorem 5.1 in [4], together with some arguments of [17], [23].

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