The Iwasawa μ–Invariant of p–Adic Hecke L–functions

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February 12, 2010

Abstract

For an odd prime p, we compute the μ-invariant of the anticyclotomic Katz p-adic L-function of a p-ordinary CM field if the conductor of the branch character is a product of primes split over the maximal real subfield. Except for rare cases where the root number of the p-adic functional equation is congruent to -1 modulo p, the invariant vanishes.

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*The author is partially supported by the following NSF grants: DMS 9988043, DMS 0244401, DMS 0456252 and DMS 0753991.

1 Introduction

We fix a rational prime p. In our book [PAF] 4.2.4 and 8.4, we computed the monodromy group at p inside the automorphism group $\mathcal{G}$ of the arithmetic automorphic function field of the Shimura variety of symplectic and unitary type. In this paper, we shall carry out a similar computation of the monodromy group of the Serre–Tate deformation space realized as a formal completion of the tower of the Hilbert modular varieties at an ordinary abelian variety with real multiplication (see Corollary 3.5). This combined with the q–expansion principle enables us to compute the μ–invariant of the anti-cyclotomic Katz p–adic L–function in an explicit manner. In other words, under mild assumptions, we shall prove the vanishing of the μ–invariant of the p–adic Hecke L–functions constructed by Katz in [K3] (see also [HT]). Let $\mathbb{F}$ be an algebraic closure of $\mathbb{F}_p$ and $W(\mathbb{F})$ be the ring of Witt vectors with coefficients in $\mathbb{F}$. We fix a p–adic valuation ring W finite flat over $W(\mathbb{F})$. Fix an algebraic closure $\overline{\mathbb{Q}}_p$ (resp. $\overline{\mathbb{Q}}$) of $\mathbb{Q}_p$ (resp. $\mathbb{Q}$) and write $\widehat{\mathbb{Q}}_p$ for the p–adic completion of $\overline{\mathbb{Q}}_p$. We regard $W \subset \widehat{\mathbb{Q}}_p$.

To state the result precisely, we first recall nice properties of the p–adic Katz measure $\varphi = \varphi_{\mathfrak{C}}$ (of prime-to–p conductor $\mathfrak{C}$) interpolating Hecke L–values. Let F be a totally real number field and M be a totally imaginary quadratic extension of F (hereafter such fields will be called CM fields). We write D_F for the discriminant of F. We write $\mathfrak{O}$ (resp. O) for the integer ring of M (resp. F). We fix two embeddings throughout the paper: $i_\infty : \overline{\mathbb{Q}} \to \mathbb{C}$ and $i_p : \overline{\mathbb{Q}} \to \widehat{\mathbb{Q}}_p$. We suppose throughout the paper the following ordinarity hypothesis:

(ord) *Every prime factor of p in F splits in M.*

Then, writing c both for complex conjugation of $\mathbb{C}$ and of $\overline{\mathbb{Q}}$ induced under i_∞, we can choose a set of embeddings Σ of M into $\overline{\mathbb{Q}}$ such that

(cm1) *$\Sigma \sqcup \Sigma c$ is the set of all embeddings of M into $\overline{\mathbb{Q}}$;*

(cm2) *the p–adic place induced by any element of Σ composed with i_p is distinct from any of those induced by elements in Σc.*

The set Σ satisfying (cm1-2) is called a p–adic CM-type. Under (ord), we can find a p–adic CM-type, and we fix one such Σ. We write Σ_p for the set of p–adic places (hence of prime ideals of M over p) induced by the embedding $i_p \circ \sigma$ for $\sigma \in \Sigma$. We fix a finite idele $d \in M_{\mathbb{A}}^{(\infty)}$ (resp. $d_F \in F_{\mathbb{A}}^{(\infty)}$) such that the ideal

corresponding to d (resp. d_F) is the different $\mathfrak{d}_M$ (resp. $\mathfrak{d}$) of $M/\mathbb{Q}$ (resp. $F/\mathbb{Q}$). Let $\lambda : M_{\mathbb{A}}^\times/M^\times \to \mathbb{C}^\times$ be a Hecke character such that

$$\lambda(x_\infty) = \prod_{\sigma\in\Sigma} x_\infty^{(k+\kappa_\sigma(1-c))\sigma},$$

where k and κ_σ are integers. Then λ has values in $\overline{\mathbb{Q}}$ on the finite part $M_{\mathbb{A}^{(\infty)}}^\times$ of $M_{\mathbb{A}}^\times$. Moreover, the map $\widehat{\lambda} : M_{\mathbb{A}^{(\infty)}}^\times/M^\times \to \overline{\mathbb{Q}}_p^\times$ defined by $\widehat{\lambda}(x) = \lambda(x)\prod_{\sigma\in\Sigma} x_p^{(k+\kappa_\sigma(1-c))\sigma}$ is a well defined continuous character, which is called the p–adic avatar of λ. By class field theory, we may regard $\widehat{\lambda}$ as a Galois character $\widehat{\lambda} : \mathrm{Gal}(\overline{\mathbb{Q}}/M) \to \overline{\mathbb{Q}}_p^\times$. We can associate λ its dual λ^* given by $\lambda^*(x) = \lambda(x^c)^{-1}|x|_{\mathbb{A}}$. Then the p–adic avatar of λ^* is given by $\widehat{\lambda}^*(x) = \widehat{\lambda}(x^c)^{-1}N(x)^{-1}$ for the p-adic cyclotomic character N. Let $\mathfrak{Q}$ be a prime of M dividing the conductor of λ and let $d_{\mathfrak{Q}}$ be a generator of the different of $M_{\mathfrak{Q}}$. We define the local Gauss sum of λ at prime ideals $\mathfrak{Q}$ dividing the conductor of λ by

$$G(d_{\mathfrak{Q}}, \lambda_{\mathfrak{Q}}) = \lambda(\varpi_{\mathfrak{Q}}^{-e}) \sum_{u\in(\mathfrak{O}_{\mathfrak{Q}}/\mathfrak{Q}^e)^\times} \lambda_{\mathfrak{Q}}(u)\mathbf{e}_M(\varpi_{\mathfrak{Q}}^{-e}d_{\mathfrak{Q}}^{-1}u), \tag{1.1}$$

where $\varpi_{\mathfrak{Q}}$ is a prime element of the $\mathfrak{Q}$–adic completion $M_{\mathfrak{Q}}$, $\mathfrak{O}_{\mathfrak{Q}}$ is the $\mathfrak{Q}$–adic integer ring of $M_{\mathfrak{Q}}$, $\lambda_{\mathfrak{Q}}$ is the restriction of λ to $M_{\mathfrak{Q}}^\times$, e is the exponent of $\mathfrak{Q}$ in the conductor of λ and $\mathbf{e}_M : M_{\mathbb{A}}/M \to \mathbb{C}^\times$ is the standard additive character normalized as $\mathbf{e}_M(x_\infty) = \exp(2\pi\sqrt{-1}\mathrm{Tr}(x_\infty))$. Outside the conductor of λ, we simply put $G(d_{\mathfrak{Q}}, \lambda_{\mathfrak{Q}}) = 1$. We can define the complex and the p–adic period $\Omega_\infty \in (F\otimes_{\mathbb{Q}}\mathbb{C})^\times \cong (\mathbb{C}^\Sigma)^\times$ and $\Omega_p \in (O\otimes_{\mathbb{Z}} W)^\times$ as in [K3] (see Section 4.4 in the text for more details). In fact, these numbers are defined uniquely only modulo $\overline{\mathbb{Q}}^\times$ but the ratio "Ω_∞/Ω_p" is uniquely determined. Finally, we fix an $\mathfrak{O}$-ideal $\mathfrak{C}$ prime to p and choose an element $\delta \in M$ such that

(d1) $\delta^c = -\delta$ and $i_\infty(\mathrm{Im}(\delta^\sigma)) > 0$ for all $\sigma \in \Sigma$,

(d2) The alternating form $\langle x, y\rangle = \mathrm{Tr}_{M/F}(\frac{xy^c}{2\delta})$ induces an isomorphism $\mathfrak{O} \wedge \mathfrak{O} \cong \mathfrak{c}^{-1}\mathfrak{d}^{-1}$ for an ideal $\mathfrak{c}$ prime to $p\mathfrak{C}\mathfrak{C}^c$,

where $\mathfrak{d}$ is the different of $F/\mathbb{Q}$. By (d2) above, if $\mathfrak{Q}$ is prime to $\mathfrak{c}$, one can choose $d_{\mathfrak{Q}}$ in (1.1) to be 2δ or $(2\delta)^c$. Then we define root numbers:

$$W_p(\lambda) = \prod_{P\in\Sigma_p} N_{M/\mathbb{Q}}(P^{-e(P)})G(2\delta;\lambda_P),$$
$$W'(\lambda) = \prod_{\mathfrak{L}|\mathfrak{F}} G((2\delta)^c, \lambda_{\mathfrak{L}}^{-1}) \prod_{\mathfrak{L}|\mathfrak{F}_c} G(2\delta, \lambda_{\mathfrak{L}^c}^{-1}) \prod_{\mathfrak{l}|\mathfrak{J}} G((2\delta)^c, \lambda_{\mathfrak{l}}^{-1}), \tag{1.2}$$

where we decomposed $\mathfrak{C} = \mathfrak{F}\mathfrak{F}_c\mathfrak{J}$ so that $\mathfrak{F}\mathfrak{F}_c$ consists of split primes over F, $\mathfrak{J}$ consists of inert or ramified primes over F, $\mathfrak{F} + \mathfrak{F}_c = \mathfrak{O}$ and $\mathfrak{F}_c^c \supset \mathfrak{F}$. We constructed in [HT] (following [K3] where the case $\mathfrak{C} = 1$ is treated) a unique measure φ on the ray class group $Z = Z(\mathfrak{C})$ modulo $\mathfrak{C}p^\infty$ of M characterized

by the following formula:

$$\frac{\int_{Z(\mathfrak{C})}\widehat{\lambda}d\varphi}{\Omega_p^{k\Sigma+2\kappa}}=(\mathfrak{O}^\times:O^\times)W_p(\lambda)\frac{(-1)^{k\Sigma}\pi^\kappa\Gamma_\Sigma(k\Sigma+\kappa)}{\sqrt{|D_F|}\,\mathrm{Im}(\delta)^\kappa\Omega_\infty^{k\Sigma+2\kappa}}$$
$$\times\prod_{\mathfrak{L}|\mathfrak{C}}(1-\lambda(\mathfrak{L}))\{\prod_{P\in\Sigma_p}(1-\lambda(P^c))\prod_{P\in\Sigma_p}(1-\lambda^*(P^c))\}L(0,\lambda)\quad(1.3)$$

for all Hecke characters λ modulo $\mathfrak{C}p^\infty$ such that (i) its conductor is divisible by all prime factors of $\mathfrak{F}$, (ii) the infinity type of λ is $k\Sigma+\kappa(1-c)$ for an integer k and $\kappa=\sum_{\sigma\in\Sigma}\kappa_\sigma\sigma$ with integers κ_σ satisfying either $k>0$ and $\kappa_\sigma\geq 0$ or $k\leq 1$ and $\kappa_\sigma\geq 1-k$. Moreover denoting the measure φ for $Z(\mathfrak{C}^c)$ by φ_c, we have the following functional equation

$$\int_{Z(\mathfrak{C})}\widehat{\lambda}d\varphi=\lambda N(\mathfrak{c}^{-1})W'(\lambda)\int_{Z(\mathfrak{C}^c)}\widehat{\lambda}^*d\varphi_c$$

as long as the conductor of λ is divisible by all prime factors of $\mathfrak{F}$. Here, we used the following convention for an element ξ of the formal free module generated by Σ and for $x\in\mathbb{C}^\Sigma$ and $x\in W^\Sigma$:

$$x^\xi=\prod_{\sigma\in\Sigma}x_\sigma^{\xi_\sigma}\quad\text{and}\quad\Gamma_\Sigma(\xi)=\prod_{\sigma\in\Sigma}\Gamma(\xi_\sigma).$$

The set Σ is also identified with the formal sum $\sum_{\sigma\in\Sigma}\sigma$, and $a\in M$ (including -1) is considered to be an element of $\mathbb{C}^\Sigma$ via diagonal embedding $a\mapsto(a^\sigma)_{\sigma\in\Sigma}$. By abusing this convention, π is considered to be the diagonal element $(\pi)_{\sigma\in\Sigma}$ in $\mathbb{C}^\Sigma$. We have written Σ_p for the set of prime ideals corresponding to p–adic places induced by $i_p\circ\sigma$ for $\sigma\in\Sigma$. The L–functions in (1.3) is always the primitive one associated with a primitive Hecke character. We also tacitly agree to put $\lambda(\mathfrak{Q})=0$ if $\mathfrak{Q}$ divides the conductor of λ.

Let $\Delta=\Delta(\mathfrak{C})$ be the maximal torsion subgroup of $Z(\mathfrak{C})$. A character $\psi:\Delta\to W^\times$ is called a *branch* character. We fix a splitting $Z(\mathfrak{C})=\Delta\times\Gamma$ for a $\mathbb{Z}_p$-free subgroup Γ so that ψ and any function ϕ on Γ can be considered to be functions on $Z(\mathfrak{C})$ via pullback by the projections: $Z(\mathfrak{C})\twoheadrightarrow\Delta$ and $Z(\mathfrak{C})\twoheadrightarrow\Gamma$. The ψ–branch φ_ψ of the measure φ is defined on Γ and is given by

$$\int_\Gamma\phi d\varphi_\psi=\int_{Z(\mathfrak{C})}\psi\phi d\varphi.$$

Since Γ is isomorphic to $Z(1)/\Delta(1)$, $\mathrm{Gal}(M/F)$ acts on Γ naturally. We write π^- for the projection of Γ onto $\Gamma^-=\Gamma/\Gamma^{\mathrm{Gal}(M/F)}$, on which the generator $c\in\mathrm{Gal}(M/F)$ acts by $-1:x\mapsto x^c=x^{-1}$. We write $\varphi_\psi^-=\pi_*^-\varphi_\psi$:

$$\int_{\Gamma^-}\phi d\varphi_\psi^-=\int_\Gamma\phi\circ\pi^-d\varphi_\psi.$$

Take a Hecke character λ of infinity type $\kappa(1-c)$ for sufficiently large κ (so, $k=0$) which is trivial on $\Delta_\mathfrak{C}$ factoring through π^- (such a character always exists

for a well chosen κ). Then the characters $\{\widehat{\lambda}\psi\chi\}_\chi$ for finite order characters $\chi : \Gamma^- \to \overline{\mathbb{Q}}_p^\times$ span a dense subspace of continuous functions on Γ^-, because finite order characters span the dense subspace of locally constant functions. The constant $\lambda\psi\chi(\mathfrak{c})^{-1}W'(\psi\widehat{\lambda}\chi)$ only depends on ψ and is equal to $\psi(\mathfrak{c})^{-1}W'(\psi)$. Indeed, because of $\widehat{\lambda}\chi(x^c) = \widehat{\lambda}\chi(x)^{-1}$ (and $\widehat{\lambda}\chi$ factoring through Γ^-), we have $\psi\widehat{\lambda}\chi(\mathfrak{c}) = \psi(\mathfrak{c})$ (by $\mathfrak{c}^c = \mathfrak{c}$), $\psi\widehat{\lambda}\chi|_{\mathfrak{O}_\mathfrak{Q}^\times} = \psi|_{\mathfrak{O}_\mathfrak{Q}^\times}$ for $\mathfrak{Q} \nmid p$, and $\widehat{\lambda}\chi(\varpi_\mathfrak{Q}\varpi_{\mathfrak{Q}^c}) = 1$ (taking $\varpi_{\mathfrak{Q}^c} = \varpi_\mathfrak{Q}^c$). This implies $G(d_\mathfrak{Q}, \psi\widehat{\lambda}\chi) = G(d_\mathfrak{Q}, \psi)$ and

$$\psi\widehat{\lambda}\chi(\mathfrak{c})^{-1}W'(\psi\widehat{\lambda}\chi) = \psi(\mathfrak{c})^{-1}W'(\psi)$$

as desired. Thus the above functional equation stated for characters is actually valid for all continuous functions ϕ on Γ^-:

$$\int_{\Gamma^-} \phi d\varphi_\psi^- = \psi N(\mathfrak{c}^{-1})W'(\psi)\int_{\Gamma^-} \phi^* d\varphi_{\psi^*}^- = \psi N(\mathfrak{c}^{-1})W'(\psi)\int_{\Gamma^-} \phi d\varphi_{\psi^*}^-,$$

where $\phi^*(x) = \phi(x^{-c})N(x)^{-1} = \phi(x)$ because $N(\Gamma^-) = 1$ and ϕ factors through Γ^- on which $x \mapsto x^{-c}$ is the identity map. From this, the functional equation for φ_ψ^- can be stated as an identity of the two measures on Γ^-:

$$d\varphi_\psi^- = \psi N(\mathfrak{c}^{-1})W'(\psi)d\varphi_{\psi^*}^-.$$

Thus the measure φ_ψ^- vanishes modulo $\mathfrak{m}_W$ if the following condition is satisfied:

$$\psi^* \equiv \psi \mod \mathfrak{m}_W \text{ and } \psi N(\mathfrak{c}^{-1})W'(\psi) \equiv -1 \mod \mathfrak{m}_W. \tag{V}$$

If (V) is satisfied, the μ–invariant of the measure φ_ψ^- is positive. Our main result of this paper is as follows:

Theorem I. *Suppose that $p > 2$ and that p is unramified in $F/\mathbb{Q}$. Further suppose that $\mathfrak{I} = 1$. Then the μ–invariant of φ_ψ^- vanishes, unless* (V) *is satisfied. When* (V) *is satisfied, $\mu(\varphi_\psi^-)$ is finite and positive.*

Actually, we prove a stronger result: Theorem 5.1, computing $\mu(\varphi_\psi^-)$ explicitly in terms of the branch character ψ, and $\mu(\varphi_\psi^-)$ is given by $\mu(\psi)$ in (5.27).

The above theorem of course implies the vanishing of $\mu(\varphi_\psi)$ unless (V) is satisfied. Even if (V) is satisfied, $\mu(\varphi_\psi)$ might well vanish, but we only study the anticyclotomic measure φ_ψ^- in this paper. The condition (V) is rarely satisfied because it is equivalent to the following three conditions (see Lemma 5.2):

(M1) M/F is unramified at every finite place;

(M2) The strict ideal class of the polarization ideal $\mathfrak{c}$ in F is not a norm class of an ideal class of M ($\Leftrightarrow \left(\frac{M/F}{\mathfrak{c}}\right) = -1$);

(M3) $\mathfrak{a} \mapsto (\psi(\mathfrak{a})N_{F/\mathbb{Q}}(\mathfrak{a}) \mod \mathfrak{m}_W)$ is the character $\left(\frac{M/F}{\cdot}\right)$ of M/F.

The last condition (M3) is equivalent to $\psi^* \equiv \psi \mod \mathfrak{m}_W$; (M2) depends on our choice of the CM-type Σ, and even if (M1) and (M3) are satisfied, (M2) could fail (see the example after the proof of the theorem in Subsection 5.4).

Basically at the same time when this paper was first written, for imaginary quadratic field $M = \mathbb{Q}(\sqrt{-D})$, the μ–invariant of the anti-cyclotomic part was determined by Finis [F2] without assuming $\mathfrak{I} = 1$, by a different method directly studying the associated CM elliptic curve (and perhaps, his method can be generalized to general CM fields). Our method does not yield a proof of the vanishing of μ of the restriction to the Galois group of the Coates–Wiles $\mathbb{Z}_p$–extension of an imaginary quadratic field (which has been proven in [G1]). We will recall the definition of the μ–invariant at the end of this introduction.

Recently, Vatsal in [V1], [V2] and [V3] has proposed an idea proving the vanishing of the μ–invariant for many p–adic L–functions of elliptic modular forms over an imaginary quadratic field (that is, the p–adic Rankin product of an elliptic modular form with an elliptic cusp form with complex multiplication by the imaginary quadratic field). His result also concerns with the anticyclotomic restriction of the p–adic L–function and is a modular generalization of the classical method of Ferrero-Washington [FW].

By this theorem, as long as $\mathfrak{I} = 1$ and p is unramified in $F/\mathbb{Q}$ and (M1-3) are not satisfied, the main divisibility result in [HT] Theorem I holds in the Iwasawa algebra Λ there in place of the weaker divisibility in $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$ proven in [HT]. We can prove this stronger divisibility even under (M1-3) (see [H07]), which results a proof of the anticyclotomic main conjecture under some mild assumptions (see [H06]).

In [Si], Sinnott gave an algebro-geometric proof of the theorem of Ferrero-Washington, relying on the analysis of rational functions on $\mathbb{G}_{m/\mathbb{F}_p}$ (under transcendental automorphisms of the formal group $\widehat{\mathbb{G}}_m$). Our idea is the use of Hilbert modular Shimura varieties and Eisenstein series in place of $\mathbb{G}_m$ and rational functions. Though the origin of our idea goes back to [Si], in order to make it work for the Shimura variety in place of geometrically easy $\mathbb{G}_m$, we are forced to go through an extensive study of the q–expansion of Eisenstein series and the geometry of the moduli space of abelian varieties with real multiplication by O (abbreviated as AVRM). The q–expansion principle is equivalent to geometric irreducibility of the mod p fiber of the variety, which was shown by Ribet [DR] Section 4 (the study of $\mathcal{G}$ also yields the irreducibility; see [PAF] 4.2.4 and [H08]). The datum of an ordinary CM type gives rise to an abelian scheme A of the given CM type over W. We will construct an Eisenstein series E_a indexed by $a \in \Omega$ for an appropriate finite subset Ω of automorphisms of the deformation space of A with the following properties:

1. E_a is congruent to an arithmetic Eisenstein series modulo p.
2. Elements in Ω are disjoint modulo the stabilizer of A inside the automorphism group of the moduli space (that is, the Hilbert modular Shimura variety).

3. The functions $a(E_a) = E_a \circ a$ for $a \in \Omega$ with $E_a \not\equiv 0 \mod p$ are linearly independent modulo p.

4. The expansion of a non-zero linear combination of $\{a(E_a)\}_{a\in\Omega}$ with respect to the canonical variable t of the Serre–Tate deformation space of A coincides with the power series expansion of a given branch of the (anticyclotomic) Katz measure in the theorem.

Some technical reasons aside, the assumption of unramifiedness of p in F is made to guarantee the smoothness over $\mathbb{Z}_p$ of Hilbert modular varieties of level prime to p. The smoothness might not be necessary, anyway; so, we might be able to dispose of this condition by applying our method more carefully.

After proving the theorem in Section 5, we discuss what happens when $\mathfrak{I} \neq 1$. In this case, Gillard showed that the anticyclotomic μ–invariant is positive for some order p branch characters for infinitely many choices of $\mathfrak{I}$ ([G2] Proposition 2). We will reprove this result of Gillard in Subsection 5.5, employing our technique. This is included in order to show that the q–expansion of our Eisenstein series fully reflects divisibility by p of the Katz measure (and also as a good evidence for the reliability of our method). The computation of the μ–invariant, when the branch character is ramified and primitive at a nonsplit prime of M over F, seems far more demanding than in the case of split-prime level. We hope to come back to this question in future.

We recall in the rest of the introduction the notion of the μ–invariant of p–adic measures and a brief history of proofs of vanishing of the μ–invariant of some other p–adic L–functions. The space Λ of p–adic measures on Γ^- with values in W is a p–adic Banach algebra under the convolution product induced from the group structure on Γ^-. Then Λ is isomorphic canonically to the continuous group algebra $W[[\Gamma^-]]$ via the isomorphism which takes the Dirac measure at $\gamma \in \Gamma^-$ to the element $\gamma \in W[[\Gamma^-]]$. Choosing a base of Γ^-, this non-canonical identification with $\mathbb{Z}_p^{[F:\mathbb{Q}]}$ induces in turn an isomorphism of $W[[\Gamma^-]]$ onto the formal power series ring over W of $[F:\mathbb{Q}]$ variables. Especially, Λ is regular and a unique factorization domain. The uniformizer ϖ of W is a prime element in Λ. The μ–invariant of a measure $\varphi^- \in \Lambda$ is the exponent μ such that ϖ^μ divides exactly φ^-. In other words,

$$|\varpi|_p^{\mu(\varphi^-)} = \mathrm{Sup}_\phi \left| \int_{\Gamma^-} \phi d\varphi^- \right|_p / |\phi|_p \quad (|\phi|_p = \mathrm{Sup}_x |\phi(x)|_p), \tag{1.4}$$

where $|\ \ |_p$ is the normalized absolute value of $\overline{\mathbb{Q}}_p$ (extended uniquely to $\widehat{\mathbb{Q}}_p$), and ϕ runs over all continuous functions on Γ^- with values in W.

In the case of Kubota-Leopoldt p–adic L–functions, the vanishing of the μ–invariant was predicted by Iwasawa from the point of view of his theory of cyclotomic $\mathbb{Z}_p$–extensions, and the conjecture was proven by Ferrero and Washington [FW] later, and more recently a new and simpler proof was given by Sinnott [Si]. The idea of Sinnott paved the way of treating the problem even for the elliptic $\mathbb{Z}_p^2$–extensions of imaginary quadratic fields, and a proof of the vanishing of the Katz-Yager p–adic L–functions for imaginary quadratic fields

was then given by Gillard [G1] and Schneps, independently, according to this line. I collaborated with R. Gillard in the early 1990's and proved a result similar to the one presented here for partial Hecke L–functions directly related to Katz's Eisenstein measure (see [G2]). A new input here is Shimura's determination ([Sh2] II) of the automorphism group $\mathcal{G}$ of the arithmetic Hilbert modular function field and the study of the action of $\mathcal{G}$ on the Serre–Tate canonical coordinate of the universal deformation space of a CM abelian variety. This new input combined with a Zariski density result of a positive dimensional subset stable under the action of an algebraic torus in $\mathcal{G}$ enabled us to prove the linear independence of $\{a(E_a)\}_{a\in\Omega}$ modulo p (see Corollary 3.21).

The density result (see Proposition 3.8 and its slight generalization: Proposition 3.11) is an adaptation of Chai's density result of a Hecke orbit (see [C2] Section 5) to our setting. In earlier versions of this paper, the proof of this density result relied on a lifting argument of the mod p subvariety to a characteristic 0 formal scheme. Although lifting works well over the ordinary locus, C.-L. Chai pointed out to me a flaw in the proof. He suggested to use the techniques in his three papers from [C2] to [C4] to recover the result. Also in the earlier versions, the condition (V) was not presented as it is now, and we claimed the vanishing rather unconditionally. Actually, the author found a discrepancy in the computation of the q–expansion of the Eisenstein series, which resulted a better understanding of the circumstances with non-triviality of μ only when (M1-3) ($\Leftrightarrow$ (V)) are satisfied.

The author would like to thank Ching-Li Chai for his remarks and assistance. The author would like to also thank Roland Gillard and Jacques Tilouine and the referees of this paper who read carefully the drafts of this paper and pointed out several mistakes.

2 Serre–Tate Deformation Space

In this section, we describe deformation theory of abelian schemes over local W_m–algebras for $W_m = W/p^mW$. We follow principally Katz's exposition [K2].

2.1 A Theorem of Drinfeld

Let R be a local W_m–algebra, and R–LR is the category of local R–algebras. Let $G : R\text{–}LR \to AB$ be a covariant functor into the category AB of abelian groups. When $m = \infty$ (that is, $W_\infty = W$), the category R–LR is made up of p–adically complete local R–algebras $B = \varprojlim_n B/p^nB$ and morphisms are supposed to be p–adically continuous. For simplicity, we always assume that rings we consider are noetherian. If we regard G as a functor from the category of affine R schemes (or formal schemes), it is contravariant. Suppose that, for any faithfully flat extension of finite type $B \hookrightarrow C$ of R–algebras,

1. The group $G(B)$ injects into $G(C)$, that is, $G(B) \hookrightarrow G(C)$;
2. Let $C' = C \otimes_B C$ and $C'' = C \otimes_B C \otimes_B C$. Write $\iota_i : C \hookrightarrow C'$ $(i = 1, 2)$ for the two natural inclusions (with $\iota_1(r) = r \otimes 1$ and $\iota_2(r) = 1 \otimes r$) and

$\iota_{ij} : C' \hookrightarrow C''$ for the three natural inclusions (i.e. $\iota_{12}(r \otimes s) = r \otimes s \otimes 1$ and so on). If $x \in G(C)$ satisfies $y = G(\iota_1)(x) = G(\iota_2)(x)$ and $G(\iota_{12})(y) = G(\iota_{23})(y) = G(\iota_{13})(y)$, then x is in the image of $G(B)$.

Such a G is called an abelian sheaf on R–LR under the $fppf$–topology (or simply *abelian $fppf$–sheaf*). We denote by R–Gp the category of abelian fppf sheaves over R. If $A_{/R}$ is an abelian scheme, then $G(B) = A(B) = \mathrm{Hom}_S(\mathrm{Spec}(B), A)$ ($S = \mathrm{Spec}(R)$ or $\mathrm{Spf}(R)$) is an $fppf$–sheaf.

The following definition of p-divisibility is in a naive sense *weaker* than Tate's notion of p-divisible groups. We call an abelian fppf sheaf G a *p–divisible* fppf sheaf if for any $x \in G(B)$, there exists a finite faithfully flat extension C of B and a point $y \in G(C)$ such that $x = py$. If G is an abelian scheme A (including non-p-torsion points), it is a p–divisible fppf sheaf.

We call a p-divisible fppf sheaf $G_{/S}$ a *p–divisible* group or a *Barsotti-Tate* group if $G = \varinjlim_n G[p^n]$ for finite flat group schemes $G[p^n] = \mathrm{Ker}(p^n : G \to G)$ over S with closed immersions $G[p^n] \hookrightarrow G[p^m]$ for $m > n$ and the multiplication $[p^{m-n}] : G[p^m] \to G[p^n]$ is an epimorphism in the category of finite flat group schemes. Thus $A[p^\infty] = \bigcup_n A[p^n]$ for $A[p^n] = \mathrm{Ker}(p^n : A \to A)$ is a Barsotti-Tate p-divisible group if $A_{/R}$ is an abelian scheme.

Let R be a local W–algebra and I be an ideal of R such that $I^{\nu+1} = 0$ and $NI = 0$ for an integer N equal to a power of p. Define functors G_I and $\widehat{G}$ by

$$G_I(B) = \mathrm{Ker}(G(B) \to G(B/I)) \quad \text{and} \quad \widehat{G}(B) = \mathrm{Ker}(G(B) \to G(B/\mathfrak{m}_B)),$$

where $\mathfrak{m}_B$ is the maximal ideal of B. When $G(B) = \mathrm{Hom}_{R\text{-}LR}(\mathcal{R}, B) (= \widehat{G}(B))$ for $\mathcal{R} = R[[T_1, \ldots, T_n]]$ (that is $G_{/R} = \mathrm{Spf}(\mathcal{R})_{/R}$) and the identity element $\mathbf{0}$ corresponding to the ideal $(T_1, \ldots, T_n)$, we call G a formal group. If G is formal, then the map $\mathrm{Hom}_{R-LR}(\mathcal{R}, B) \ni \phi \mapsto (\phi(T_1), \ldots, \phi(T_n))$ identifies $G_I(B)$ with the set $I \times I \times \cdots \times I$ (n times) endowed with a formal group law.

Suppose that $G_{/R}$ is formal. Then for any integer m, the endomorphism $[m]$ of multiplication by m on G induces a continuous algebra endomorphism $[m]^* : \mathcal{R} \to \mathcal{R}$; it induces multiplication by m on $\Omega_{G/R} = (T_1, \ldots, T_n)/(T_1, \ldots, T_n)^2$, hence on the tangent space $T_{G/R}$ too. Thus $[N](T_i) \equiv NT_i \mod (T_1, \ldots, T_n)^2$, and $[N](G_I(B)) = G_{I^2}(B)$ because $NI = 0$. Similarly, we have inductively, $[N](G_{I^a}(B)) = G_{I^{a+1}}(B)$. Thus $[N^\nu]G_I = G_0 = \{\mathbf{0}\}$. We get

$$G_I \subset G[N^\nu] \quad \text{if } G \text{ is formal}, \tag{2.1}$$

where $G[m] = \mathrm{Ker}([m] : G \to G)$ is the kernel of $[m]$.

Theorem 2.1 (Drinfeld)**.** *Let G and H be abelian $fppf$–sheaves over R–LR and I be as above. Let G_0 and H_0 be the restriction of G and H to R/I–LR. Suppose*

(i) *G is a p–divisible fppf sheaf;*

(ii) *H is formal (so, $H(B) \to H(B/J)$ is surjective for any nilpotent ideal J).*

Then

(1) *The modules* $\mathrm{Hom}_{R\text{-}Gp}(G,H)$ *and* $\mathrm{Hom}_{R/I\text{-}Gp}(G_0,H_0)$ *are* p*–torsion-free, where the symbol "*$\mathrm{Hom}_{X\text{-}Gp}$*" stands for the homomorphisms of abelian* $fppf$*–sheaves over* X*–LR;*

(2) *The natural map, so-called*

$$\text{"reduction} \mod I\text{"}: \mathrm{Hom}_{R\text{-}Gp}(G,H) \to \mathrm{Hom}_{R/I\text{-}Gp}(G_0,H_0)$$

is injective;

(3) *For any* $f_0 \in \mathrm{Hom}_{R/I\text{-}Gp}(G_0,H_0)$*, there exists a unique homomorphism* $\Phi \in \mathrm{Hom}_{R\text{-}Gp}(G,H)$ *such that* $\Phi \mod I = N^\nu f_0$*. We write as in* [K2] *"*$N^\nu f$*" for* Φ *even if* f *exists only in* $\mathrm{Hom}_{R\text{-}Gp}(G,H)\otimes_{\mathbb{Z}}\mathbb{Q}$*;*

(4) *In order that* $f \in \mathrm{Hom}_{R\text{-}Gp}(G,H)$*, it is necessary and sufficient that "*$N^\nu f$*" kills* $G[N^\nu]$*.*

Proof. The first assertion follows from p–divisibility, because if $pf(x)=0$ for all x, taking y with $py=x$, we find $f(x)=pf(y)=0$ and hence $f=0$.

We have an exact sequence: $0 \to H_I \to H \to H_0 \to 0$; so, we have another exact sequence:

$$0 \to \mathrm{Hom}(G,H_I) \to \mathrm{Hom}(G,H) \xrightarrow{\mod I} \mathrm{Hom}(G,H_0) = \mathrm{Hom}(G_0,H_0),$$

which tells us the injectivity since H_I is killed by N^ν and $\mathrm{Hom}(G,H)$ is p–torsion-free.

To show (3), take $f_0 \in \mathrm{Hom}(G_0,H_0)$. By surjectivity of $H(B) \to H_0(B/I)$, we can lift $f_0(x \mod I)$ to $y \in H(B)$. The class $y \mod \mathrm{Ker}(H \to H_0)$ is uniquely determined. Since $\mathrm{Ker}(H \to H_0)$ is killed by N^ν, for any $x \in G(B)$, therefore $N^\nu y$ is uniquely determined; so, $x \mapsto N^\nu y$ induces functorial map: "$N^\nu f$": $G(B) \to H(B)$. This shows (3).

The assertion (4) is then obvious from p–divisibility of G. The uniqueness of f follows from the p–torsion-freeness of $\mathrm{Hom}(G,H)$. □

2.2 A Theorem of Serre–Tate

Let $\mathcal{A}_{/R}$ be the category of abelian schemes defined over R. We consider a category $Def(R,R/I)$ of triples (A_0,D,ϵ), where A_0 is an abelian scheme over R/I, D is p–divisible, and $\epsilon : D_0 \cong A_0[p^\infty]$. We have a natural functor $\mathcal{A}_{/R} \to Def(R,R/I)$ given by $A \mapsto (A_0 = A \mod I, A[p^\infty], \mathrm{id})$.

Theorem 2.2 (Serre-Tate)**.** *The above functor:* $\mathcal{A}_{/R} \to Def(R,R/I)$ *is a canonical equivalence of categories.*

Proof. By Drinfeld's theorem applied to $A[p^\infty]$ and A (both abelian $fppf$–sheaf), the functor is fully faithful (see [K2] for details). It is known that we can lift A_0 to an abelian scheme B over R. This follows from the deformation theory of Grothendieck ([GIT] Section 6.3 and [CBT] 2.8.1). Assume that A_0 is ordinary. When R/I is a finite field, by a theorem of Tate, A_0 has complex

multiplication. By the theory of abelian varieties with complex multiplication, A_0 can be lifted to a unique abelian scheme B over R with complex multiplication (the canonical lift), because isomorphism classes of such abelian varieties of CM type corresponds bijectively to lattices (up to scalar multiplication) in a CM field. Thus we have an isomorphism $\alpha_0^{(p)} : B_0[p^\infty] \to A_0[p^\infty]$. Then we have a unique lifting (by the Drinfeld theorem) that $f : B[p^\infty] \to D$ of $N^\nu \alpha_0^{(p)}$. Clearly, f is an isogeny, whose (quasi) inverse is the lift of $N^\nu(\alpha_0^{(p)})^{-1}$. Thus $\mathrm{Ker}(f)$ is a finite flat group subscheme of B. The geometric quotient of B by a finite flat group subscheme exists (see [ABV] Section 12) and is an abelian scheme over R. Then dividing B by $\mathrm{Ker}(f)$, we get the desired $A_{/R} \in \mathcal{A}_{/R}$. □

2.3 Deformation of an Ordinary Abelian Variety

Let $S = \mathrm{Spec}(\mathcal{O}_S)$ be an affine scheme over $\mathbb{F}_p$ and (A, ω) be a pair of an abelian variety over S of relative dimension g and a basis $\omega = \omega_1, \ldots, \omega_g$ of $H^0(A, \Omega_{A/S})$ over $\mathcal{O}_S$. Write $\pi : A \to S$ for the structure morphism. We have the absolute Frobenius endomorphism $F_{abs} : S \to S$. Let $\mathcal{T}_{A/S}$ be the relative tangent bundle, and consider the direct image $\pi_*\mathcal{T}_{A/S}$ over S; so, $H^0(S, \pi_*\mathcal{T}_{A/S})$ is spanned by the dual base $\eta = \eta(\omega)$. For each invariant derivation D of $\mathcal{O}_A$, by the Leibnitz formula, we have

$$D^p(xy) = \sum_{j=0}^{p} \binom{p}{j} D^{p-j}xD^j y = xD^p y + yD^p x.$$

Thus D^p is again a derivation. The association: $D \mapsto D^p$ induces an F_{abs}–linear endomorphism F^* of $\mathcal{T}_{A/S}$. Then we define $H(A, \omega) \in \mathcal{O}_S$ by $F^* \bigwedge^g \eta = H(A, \omega) \bigwedge^g \eta$. Since $\eta(\lambda\omega) = {}^t\lambda^{-1}\eta(\omega)$ for $\lambda \in GL_g(\mathcal{O}_S)$, we see

$$\begin{aligned} H(A, \lambda\omega) \bigwedge^g \eta(\lambda\omega) &= F^* \bigwedge^g \eta(\lambda\omega) = F^*(\det(\lambda)^{-1} \bigwedge^g \eta(\omega)) \\ &= \det(\lambda)^{-p} F^* \bigwedge^g \eta(\omega) = \det(\lambda)^{-p} H(A, \omega) \bigwedge^g \eta(\omega) \\ &= \det(\lambda)^{-p} H(A, \omega) \det(\lambda) \bigwedge^g \eta(\lambda\omega) = \det(\lambda)^{1-p} H(A, \omega) \bigwedge^g \eta(\lambda\omega). \end{aligned}$$

Thus we get

$$H(A, \lambda\omega) = \det(\lambda)^{1-p} H(A, \omega).$$

We call A ordinary if we can embed μ_p^g into $A[p]$ after a faithfully flat étale base-change. As in the elliptic curve case (cf. [GME] 2.9.1), we know

$$H(A, \omega) = 0 \iff A \text{ is not ordinary.}$$

Let κ be an algebraically closed field over $\mathbb{F}_p$. Let R be a pro-artinian local ring with residue field κ. Write $CL_{/R}$ for the category of complete local R–algebras with residue field κ. We fix an *ordinary* abelian variety $A_{0/\kappa}$. Write $A^t_{/R}$ for the dual abelian scheme (representing $\mathrm{Pic}^0_{A/R}$) of an abelian scheme

$A_{/R}$. We write $TA[p^\infty]^{et}$ for the Tate module of the maximal étale quotient of $A[p^\infty]$. We consider the following deformation functor $\widehat{\mathcal{P}} : CL_{/R} \to SETS$:

$$\widehat{\mathcal{P}}_{A_0}(\mathcal{O}_S) = \left[(A_{/S}, \iota_A)\middle|\ A_{/S} \text{ is an abelian scheme and } \iota_A : A \otimes_{\mathcal{O}_S} \kappa \cong A_0\right].$$

Here "[ ]" indicates the set "{ }/ $\cong$" of isomorphism classes of the objects inside the straight brackets, and $f : (A, \iota_A)_{/S} \cong (A', \iota_{A'})_{/S}$ if $f : A \to A'$ is an isomorphism of abelian schemes with the following commutative diagram:

$$\begin{array}{ccc} A \otimes_{\mathcal{O}_S} \kappa & \xrightarrow{f_0} & A' \otimes_{\mathcal{O}_S} \kappa \\ {\scriptstyle \iota_A}\big\downarrow\wr & & {\scriptstyle \iota_{A'}}\big\downarrow\wr \\ A_0 & = & A_0. \end{array}$$

The functor $\widehat{\mathcal{P}}_{A_0}$ is representable by the formal torus

$$\mathrm{Hom}_{\mathbb{Z}_p}(TA_0[p^\infty]^{et} \times TA_0^t[p^\infty]^{et}, \widehat{\mathbb{G}}_m(S)),$$

and each deformation $(A_{/S}, i_A) \in \widehat{\mathcal{P}}_{A_0}(\mathcal{O}_S)$ gives rise to the Serre–Tate coordinate $q_{A/S} : TA_0[p^\infty]^{et} \times TA_0^t[p^\infty]^{et} \to \widehat{\mathbb{G}}_m(S)$. We give a sketch of the construction of $q_{A/S}$. We prepare some facts. Let $f : A \to B$ be an isogeny; so, $\mathrm{Ker}(f)$ is a finite flat group scheme over S. Pick $x \in \mathrm{Ker}(f)$, and let $\mathcal{L} \in \mathrm{Ker}(f^t) \subset B^t$ be the line bundle on B with $\mathbf{0}_B^*\mathcal{L} = \mathcal{O}_S$ ($S = \mathrm{Spec}(\mathcal{O}_S)$ for an artinian R–algebra $\mathcal{O}_S$). Thus $f^*\mathcal{L} = \mathcal{O}_A$. Cover B by open affine subschemes U_i so that $\mathcal{L}|_{U_i} = \phi_i^{-1}\mathcal{O}_{U_i}$. Since $\mathbf{0}_B^*\mathcal{L} = \mathcal{O}_S$, we may assume that $(\phi_i/\phi_j) \circ \mathbf{0}_B = 1$. Since $f : A \to B$ is finite, it is affine. Write $V_i = f^{-1}(U_i) = \mathrm{Spec}(\mathcal{O}_{U_i})$. Then $f^*\mathcal{L}|_{V_i} = \varphi_i^{-1}\mathcal{O}_{V_i}$ with $\varphi_i = \phi_i \circ f$, and we have, regarding $x : S \to \mathrm{Ker}(f)$,

$$\frac{\varphi_i \circ x}{\varphi_j \circ x} = \frac{\phi_i \circ f \circ x}{\phi_j \circ f \circ x} = \frac{\phi_i \circ \mathbf{0}_B}{\phi_j \circ \mathbf{0}_B} = 1.$$

Thus $\varphi_i \circ x$ glue into a morphism $[x, \mathcal{L}] : S \to \mathbb{G}_m$, and we get a pairing

$$e_f : \mathrm{Ker}(f) \times \mathrm{Ker}(f^t) \to \mathbb{G}_m.$$

Since A is a $\mathrm{Ker}(f)$–torsor over B, we have $A \times_B A \cong \mathrm{Ker}(f) \times_S B$. Thus for any homomorphism $\zeta : \mathrm{Ker}(f) \to \mathbb{G}_m$, we can find a function $\phi : \mathrm{Ker}(f) \times_S B \to \mathbf{P}^1$ such that $\phi(y + t) = \zeta(t)\phi(y)$ for $t \in \mathrm{Ker}(f)$. This function ϕ gives rise to a divisor D on $B_A = B \times_S A$. By definition, $f_A^*\mathcal{L}(D) = \mathcal{O}_{A \times_S A}$ for $f_A = f \times 1 : A \times_S A \to B \times_S A$, and $e_f(x, \mathcal{L}(D)) = \zeta(x)$. Thus, over A, $e_{f/A} : \mathrm{Ker}(f)_{/A} \times \mathrm{Ker}(f^t)_{/A} \to \mathbb{G}_m$ is a perfect pairing. Since $A \to S$ is faithfully flat, we find that the original e_f is perfect. Write A° for the formal completion at the origin of the mod p fiber of A.

We apply the above argument to $f = [p^n] : A \to A$, write the pairing as e_n and verify the following points:

(P1) $e_n(\alpha(x), y) = e_n(x, \alpha^t(y))$ for $\alpha \in \mathrm{End}(A_{/B})$;

(P2) Write $A_0[p^n]^\circ = \mu_{p^n}^g \subset A_0[p^n]$. Then e_n induces an isomorphism of group schemes: $A_0[p^n]^\circ \cong \mathrm{Hom}(A_0^t[p^n]^{et}, \mu_{p^n})$;

(P3) Taking limit of the above isomorphisms with respect to n, we find

$$A^\circ \cong \mathrm{Hom}(TA^t[p^\infty]^{et}, \widehat{\mathbb{G}}_m) \cong \mathrm{Hom}(TA_0^t[p^\infty]^{et}, \widehat{\mathbb{G}}_m)$$

as formal groups. In particular $A^\circ \cong \widehat{\mathbb{G}}_m^g$.

We are now ready to describe the Serre–Tate coordinate $q_{A/S}$. Since $\mathcal{O}_S \in CL_{/R}$ is a projective limit of local R–algebras with nilpotent maximal ideal, we may assume that $\mathcal{O}_S$ is a local artinian R–algebra with nilpotent maximal ideal $\mathfrak{m}_S$. Then $A^\circ(S)$ is killed by p^{n_0} for sufficiently large n_0 (applying Drinfeld's theorem to $I = \mathfrak{m}_S$). Taking a lift $\widetilde{x} \in A(S)$ of $x \in A(\mathbb{F})$ (such that $\widetilde{x} \mod \mathfrak{m}_S = x$), $\widetilde{x}$ is determined modulo $\mathrm{Ker}(A(S) \to A(\mathbb{F})) = A^\circ(S)$ which is a subgroup of $A[p^n]$ if $n \geq n_0$. By the smoothness of $A_{/S}$, a lift $\widetilde{x} \in A(S)$ of $x \in A(\mathbb{F})$ always exists. Thus $p^n\widetilde{x} \in A(S)$ is uniquely determined by $x \in A(\mathbb{F})$. If $x \in A[p^n]$, $p^n\widetilde{x} =$"p^n"$x \in A^\circ(S)$ by definition, getting a homomorphism "p^n":$A[p^n](\mathbb{F}) \to A^\circ(S)$. We have an obvious commutative diagram (if $n \geq n_0$)

$$\begin{array}{ccccc} A[p^{n+1}]^{et}(S) & \xrightarrow{\sim} & A_0[p^{n+1}](\mathbb{F}) & \xrightarrow{\text{“}p^{n+1}\text{”}} & A^\circ(S) \\ {\scriptstyle p}\downarrow & & {\scriptstyle p}\downarrow & & \downarrow\| \\ A[p^n]^{et}(S) & \xrightarrow{\sim} & A_0[p^n](\mathbb{F}) & \xrightarrow{\text{“}p^n\text{”}} & A^\circ(S), \end{array}$$

which gives rise to a morphism $TA_0[p^\infty]^{et} \to A^\circ(S)$. Thus the structure of the Barsotti–Tate group $A[p^\infty]$ is uniquely determined by the extension class of the exact sequence of fppf sheaves:

$$0 \to A^\circ[p^\infty]_{/S} \to A[p^\infty]_{/S} \xrightarrow{\pi} A[p^\infty]^{et}_{/S} \to 0. \tag{2.2}$$

Take $x = \varprojlim_n x_n \in TA[p^\infty]^{et}$ with $x_n \in A[p^n]^{et}$. Lift x_n to $v_n \in A(S)$ so that $\pi(v_n) = x_n$. Then, for "p^n": $A[p^n] \to A^\circ$,

$$q_n(x) = \text{“}p^n\text{”}v_n \in A^\circ(S).$$

The value $q_n(x_n)$ becomes stationary if $n \geq n_0$, and taking limit of $q_n(x_n)$ as $n \to \infty$, we get $q(x) \in A^\circ(S) \cong \mathrm{Hom}(TA_0^t[p^\infty]^{et}, \widehat{\mathbb{G}}_m(S))$. Then we define $q_{A/S}(x, y) = q(x)(y)$ (see [K2]).

Theorem 2.3 (Serre-Tate)**.** *We have*

(1) *A canonical isomorphism*

$$\widehat{\mathcal{P}}(\mathcal{O}_S) \cong \mathrm{Hom}_{\mathbb{Z}_p}(TA_0[p^\infty]^{et} \times TA_0^t[p^\infty]^{et}, \widehat{\mathbb{G}}_m(S))$$

taking $(A_{/S}, \iota_A)$ *to* $q_{A/S}(\cdot, \cdot)$.

(2) *The functor* $\widehat{\mathcal{P}}$ *is represented by the formal scheme*

$$\mathrm{Hom}_{\mathbb{Z}_p}(TA_0[p^\infty]^{et} \times TA_0^t[p^\infty]^{et}, \widehat{\mathbb{G}}_m) \cong \widehat{\mathbb{G}}_m^{g^2}.$$

(3) $q_{A/S}(x,y) = q_{A^t/S}(y,x)$ *under the canonical identification:* $(A^t)^t = A$.

(4) *Let* $f_0 : A_{0/\kappa} \to B_{0/\kappa}$ *be a homomorphism of two ordinary abelian varieties with the dual map:* $f_0^t : B_0^t \to A_0^t$. *Then* f_0 *is induced by a homomorphism* $f : A_{/S} \to B_{/S}$ *for* $A \in \widehat{\mathcal{P}}_{A_0}(\mathcal{O}_S)$ *and* $B \in \widehat{\mathcal{P}}_{B_0}(\mathcal{O}_S)$ *if and only if* $q_{A/S}(x, f_0^t(y)) = q_{B/S}(f_0(x), y)$.

Proof. Here is a brief outline of the proof. Let $T_{/S}$ and $E_{/S}$ be a multiplicative and an étale p-divisible group over a scheme S, respectively. Consider the sheafification $\underline{\mathrm{Hom}}_{S_{\mathrm{fppf}}}(E[p^n], T[p^n])$ (resp. $\underline{\mathrm{Ext}}^1_{S_{\mathrm{fppf}}}(E[p^n], T[p^n])$) of presheaf $U \mapsto \mathrm{Hom}_U(E[p^n]_{/U}, T[p^n]_{/U})$ (resp. $U \mapsto \mathrm{Ext}^1_U(E[p^n]_{/U}, T[p^n]_{/U})$) over the small fppf site S_{fppf} over S. Any connected-étale extension $T[p^n] \to X \twoheadrightarrow E[p^n]$ in the category of finite flat $\mathbb{Z}/p^n\mathbb{Z}$-modules over S split over an fppf extension S'/S; so, we have $\underline{\mathrm{Ext}}^1_{S_{\mathrm{fppf}}}(E[p^n], T[p^n]) = 0$ and a splitting $X = T[p^n] \oplus E[p^n]$. Taking a module section $i : E[p^n] \to X[p^n]$ and projecting down to $T[p^n]$ over S', we get a homomorphism $\phi_{S'} \in \mathrm{Hom}_{S'}(E[p^n], T[p^n])$. Since $S' \mapsto \phi_{S'}$ satisfies the descent datum, it is a Čech 1-cocycle with values in $\underline{\mathrm{Hom}}_{S_{\mathrm{fppf}}}(E[p^n], T[p^n])$. Thus we have a morphism $\mathrm{Ext}^1_{S_{\mathrm{fppf}}}(E[p^n], T[p^n]) \to H^1(S_{\mathrm{fppf}}, \underline{\mathrm{Hom}}_{S_{\mathrm{fppf}}}(E[p^n], T[p^n]))$. By fppf descent, this is an isomorphism. Applying this to $S = \mathrm{Spec}(\mathcal{O}_S)$, $T[p^n] = A[p^n]^\circ$ and $E[p^n] = A[p^n]^{et}$, we get

$$\mathrm{Ext}^1_{S_{\mathrm{fppf}}}(A[p^n]^{et}, A[p^n]^\circ) \cong H^1(S_{\mathrm{fppf}}, \underline{\mathrm{Hom}}_{S_{\mathrm{fppf}}}(A[p^n]^{et}, A[p^n]^\circ)).$$

When S is affine, [C4] Proposition 2.4 (iii) and (iv) combined tells us

$$\begin{aligned}\mathrm{Ext}^1_{S_{\mathrm{fppf}}}(A[p^\infty]^{et}, A[p^\infty]^\circ) &\cong \varprojlim_n \mathrm{Ext}^1_{S_{\mathrm{fppf}}}(A[p^n]^{et}, A[p^n]^\circ) \\ &\cong \mathrm{Hom}_{\mathbb{Z}_p}(TA^t[p^\infty]^{et} \otimes TA^t[p^\infty]^{et}, \widehat{\mathbb{G}}_m(S)))\end{aligned}$$

since $\widehat{\mathbb{G}}_m \cong \varprojlim_n R^1\pi_*\mu_{p^n} \cong \varprojlim_n \mu_{p^n}$ (for $\pi : S_{\mathrm{fppf}} \to S_{\mathrm{et}}$) as sheaves over the small étale site S_{et} (see [C4] Section 2 for more details). Since the residue field $\mathbb{F}$ of $\mathcal{O}_S$ is algebraically closed, $A[p^n]^{et}$ and $A^t[p^n]^{et}$ are constant over $\mathcal{O}_S$; so, we may replace $TA^t[p^\infty]^{et}$ and $TA^t[p^\infty]^{et}$ by their special fibers $TA_0[p^\infty]^{et}$ and $A_0^t[p^\infty]^{et}$, and $q_{A/S}$ completely determines the extension class of the p-divisible group in (2.2). Therefore, $q_{A/S}$ determines the isomorphism class of $A[p^\infty]_{/S}$. Then by the Serre–Tate theorem in the previous subsection, the deformation $A_{/S}$ is determined by $(A_0, A[p^\infty])$ and hence by $q_{A/S}$. This shows the assertions (1) and (2). All other assertions follows from (P1-3) easily. □

2.4 Abelian Variety with Real Multiplication

Let $F/\mathbb{Q}$ be a totally real finite extension unramified at the fixed prime p. Write O for the integer ring of F, and put $d = [F : \mathbb{Q}]$. Consider an abelian scheme A

over a scheme S of relative dimension d with an embedding $i : O \hookrightarrow \mathrm{End}(A_{/S})$ sending the identity to the identity automorphism of $A_{/S}$.

An abelian scheme $A_{/S}$ can be considered as an fppf sheaf on $SCH_{/S}$ with coefficients in abelian groups. For any O–module M, the fppf sheaf $A \otimes M$ which is the fppf sheafification of the presheaf taking an S–scheme T to $A(T) \otimes_O M$ gives rise to another abelian scheme, written as $A \otimes M$. If $M = \mathfrak{c}^{-1}$ for an integral ideal $\mathfrak{c} \subset O$, tensoring A with the exact sequence:

$$0 \to O \to \mathfrak{c}^{-1} \to \mathfrak{c}^{-1}/O \to 0,$$

we get another exact sequence:

$$0 \to \mathrm{Tor}_1(A, O/\mathfrak{c}) \to A \to A \otimes \mathfrak{c}^{-1} \to 0.$$

Thus $A \otimes \mathfrak{c}^{-1}$ is represented by $A/A[\mathfrak{c}]$, because $\mathrm{Tor}_1^O(A, O/\mathfrak{c}) \cong A[\mathfrak{c}]$ canonically (since O is a Dedekind domain).

Here is a brief description of polarization on an abelian scheme $A_{/S}$ satisfying the four conditions (rm1–4) below (called an AVRM). See [R] Section 1 for more details on polarizations on an AVRM. An ample line bundle L on A gives rise to an isogeny $\lambda_L : A \to A^t$ as follows (cf. [ABV] Section 6 and [DAV] page 3). Pick a T-point $a \in A(T)$ for an S-scheme T. Then by addition, a induces a morphism $T_a : A_{/T} \to A_{/T}$ sending x to $x + a$. Then $A(T) \ni a \mapsto T_a^*(L) \otimes L^{-1} \otimes a^*(L)^{-1} \otimes \mathbf{0}^*(L) \in \mathrm{Pic}^0_{A/S}(T) = A^t(T)$ is a morphism of group functors, which gives rise to the homomorphism $\lambda_L : A_{/S} \to A^t_{/S}$. A line bundle is called symmetric if $(-1)^*L = L$. If L is symmetric, $\lambda_L^t = \lambda_L$. A polarization is an O–linear isogeny $\lambda : A \to A^t$ induced by a symmetric line bundle $L_{/A_s}$ fiber by fiber over geometric points $s \in S$ (cf. [GIT] 6.3). If $\lambda : A \to A^t$ is a polarization, $\mathrm{Ker}(\lambda)$ is given by $A[\mathfrak{c}^{-1}]$ for an integral ideal $\mathfrak{c}^{-1} \neq 0$, because $\mathrm{Ker}(\lambda)$ is self dual under Cartier duality. Then λ induces $A^t \cong A \otimes \mathfrak{c}$. Such a polarization is called a $\mathfrak{c}$–*polarization*. By definition, $\lambda_{L \otimes L'} = \lambda_L + \lambda_{L'}$. For $a \in O$, we see easily that $a \circ T_x = T_{a(x)} \circ a$ and that $\lambda_{a^*L} = a^2 \lambda_L$. The set of totally positive elements in a square ideal $\mathfrak{a}^2$ is generated over $\mathbb{N}$ by square elements of $\mathfrak{a}$. Thus the subset of $\mathrm{Hom}(A, A^t)$ generated by polarizations forms a positive cone $P(A)$. If S is a $\mathbb{Q}$–scheme, the module $Lie(A)$ is a faithful module over $\mathrm{End}_O^{\mathbb{Q}}(A) = \mathrm{End}_O(A) \otimes_{\mathbb{Z}} \mathbb{Q}$. In particular, F–linear symmetric endomorphisms $\mathrm{End}^{\mathbb{Q}}_{O-sym}(A)$ (those fixed by the Rosati involution) is isomorphic to F. Thus we have $\mathrm{End}_{O-sym}(A) = O$. Therefore if λ is a $\mathfrak{c}$–polarization, $\mathrm{Hom}_{O-sym}(A, A^t) = \mathrm{Hom}_{O-sym}(A, A) \otimes \mathfrak{c} = \mathfrak{c}$, and hence $P(A) \cong \mathfrak{c}_+$ canonically, where $\mathfrak{c}_+$ is the cone inside $\mathfrak{c}$ made up of totally positive elements.

We consider the following fiber category $\mathcal{A}_F$ of abelian schemes over the category of $\mathbb{Z}_{(p)}$–schemes. Here $\mathbb{Z}_{(p)} \subset \mathbb{Q}$ is the valuation ring of the p–adic valuation. An object of $\mathcal{A}_F$ is the triple $(A_{/S}, i : O \hookrightarrow \mathrm{End}(A_{/S}), \lambda)$, where

(rm1) $i = i_A$ is an embedding of algebras taking identity to identity;

(rm2) λ is an O–linear symmetric polarization $\lambda : A \to A^t$ with $p \nmid \deg(\lambda)$;

(rm3) The image of i_A is stable under the Rosati involution induced by λ;

(rm4) As $O\otimes_{\mathbb{Z}}\mathcal{O}_S$–modules, we have $Lie(A)\cong O\otimes_{\mathbb{Z}}\mathcal{O}_S$ locally under the Zariski topology of S, where the O–module structure of $Lie(A)$ is induced by i.

A morphism $\phi:(A,i,\lambda)_{/S}\to(A',i',\lambda')_{/S}$ in the category $\mathcal{A}_F$ is an O–linear morphism $\phi:A_{/S}\to A'_{/S}$ of abelian schemes over S with $\lambda=\phi^t\circ\lambda'\circ\phi$.

Fix an algebraic closure $\mathbb{F}$ of $\mathbb{F}_p$. Take an ordinary abelian scheme (A_0,i_0,λ_0) defined over $\mathbb{F}$. We fix a polarization $\lambda_0:A_0\to A_0^t$ of degree prime to p. We consider the following functor defined from $CL_{/W(\mathbb{F})}$ into $SETS$:

$$\widehat{\mathcal{P}}_{A_0,i_0,\lambda_0}(R)$$
$$=\left[(A_{/R},\iota_A,i,\lambda)\in\mathcal{A}_F\middle|(A,\iota_A)\in\widehat{\mathcal{P}}_{A_0}(R),\quad\lambda\text{ and }i\text{ induce }\lambda_0\text{ and }i_0\right].$$

Here we call $f:(A,\lambda_A,\iota_A)\to(B,\lambda_B,\iota_B)$ an isomorphism if $f:(A,\iota_A)\cong(B,\iota_B)$ and $f^t\circ\lambda_B\circ f=\lambda_A$. Note that by Theorem 2.1 (1) (Drinfeld's theorem), $\mathrm{End}(A_{/R})$ is torsion-free, and hence, $\mathrm{End}(A_{/R})\hookrightarrow\mathrm{End}^{\mathbb{Q}}(A_{/R})=\mathrm{End}(A_{/R})\otimes_{\mathbb{Z}}\mathbb{Q}$. We write $\alpha^*=\lambda_0^{-1}\circ\alpha^t\circ\lambda_0$ for $\alpha\in\mathrm{End}(A_0)\otimes_{\mathbb{Z}}\mathbb{Q}$. Since $\mathrm{End}(A_{/R})\subset\mathrm{End}(A_0)$ again by Theorem 2.1 (2), the involution keeps $\mathrm{End}^{\mathbb{Q}}(A_{/R})$ stable (because on $\mathrm{End}^{\mathbb{Q}}(A_{/R})$, it is given by $\alpha^*=\lambda^{-1}\circ\alpha^t\circ\lambda$). The Rosati involution $\alpha\mapsto\alpha^*$ is known to be positive (see [ABV] Section 21). The polarization λ_0 induces an isomorphism $\lambda_0:A[p^\infty]^{et}\cong A^t[p^\infty]^{et}$. We identify $TA_0[p^\infty]^{et}$ and $TA_0^t[p^\infty]^{et}$ by λ_0. Then the involution $\alpha\mapsto\alpha^t$ in the Serre–Tate theorem (4) is replaced by the positive involution "$*$"; in particular, "$*$" is the identity map on $i(O)$ (which is the unique positive involution of the totally real field). Then it is clear from the previous theorem that, for $O_p=O\otimes_{\mathbb{Z}}\mathbb{Z}_p$,

$$\widehat{\mathcal{P}}_{A_0,\lambda_0}(R)\cong\mathrm{Hom}_{\mathbb{Z}_p}(TA_0[p^\infty]^{et}\otimes_{O_p}TA_0[p^\infty]^{et},\widehat{\mathbb{G}}_m(R)).$$

Proposition 2.4. *We have $TA_0[p^\infty]^{et}\cong O_p$ as O–modules.*

Proof. Since A_0 and the connected component $A_0[p]^\circ$ of the finite flat group scheme $A_0[p]$ share the tangent space $Lie(A_0)$ at the origin, as O–modules, they are free of rank 1 over $O\otimes_{\mathbb{Z}}\mathbb{F}$. Write $A_0[p]^\circ=\mathrm{Spec}(R)$ for an $\mathbb{F}$–bialgebra R. Then for its unique maximal ideal $\mathfrak{m}\subset R$, we have $Lie(A_0)=\mathrm{Hom}_{\mathbb{F}}(\mathfrak{m}/\mathfrak{m}^2,\mathbb{F})$. By Cartier duality (e.g. [GME] 1.7), we have

$$A_0^t[p]^{et}\cong\mathrm{Hom}_{Gp-sch}(A_0[p]^\circ,\mu_p)\hookrightarrow\mathrm{Hom}_{SCH}(A_0[p]^\circ,\mu_p)$$
$$\cong\mathrm{Hom}_{\mathbb{F}-alg}(\mathbb{F}[t]/(t^p),R)\twoheadrightarrow\mathrm{Hom}_{\mathbb{F}-alg}(\mathbb{F}[t]/(t^p),R/\mathfrak{m}^2)\cong\mathfrak{m}/\mathfrak{m}^2.$$

Since $A[p]^\circ\cong\mu_p^d$ over $\mathbb{F}$ for $d=\dim A_0$, it is easy to see that the above morphism induces $A_0^t[p]^{et}\otimes_{\mathbb{F}_p}\mathbb{F}\cong H^0(A_0,\Omega_{A_0/\mathbb{F}})$. Then by duality and polarization, we get $A_0[p]^{et}\otimes_{\mathbb{F}_p}\mathbb{F}\cong Lie(A_0)$ as $O\otimes_{\mathbb{Z}}\mathbb{F}$–modules. This shows that

$$Lie(A_0)\cong TA_0[p^\infty]^{et}\otimes_{\mathbb{Z}}\mathbb{F}\ \text{ as } O\otimes_{\mathbb{Z}}\mathbb{F}\text{–modules.}\tag{2.3}$$

Then by Nakayama's lemma, we conclude from (rm4) the desired assertion. □

Corollary 2.5. *Suppose that O is unramified at p. Let $S = \mathbb{G}_m \otimes_{\mathbb{Z}} \mathfrak{d}^{-1} = \mathrm{Spec}(\mathbb{Z}[O])$ for the group algebra $\mathbb{Z}[O]$. Then identifying $TA_0[p^\infty]^{et}$ with O_p, the functor $\widehat{\mathcal{P}}_{A_0,i,\lambda_0}$ is represented by the formal scheme $\widehat{S}_{/W}$, where $\widehat{S}$ is the formal completion of S along the identity section of $\mathbb{G}_m \otimes_{\mathbb{Z}} \mathfrak{d}^{-1}(\mathbb{F})$.*

Proof. We have seen that the deformation space $\widehat{S}$ is given by

$$\begin{aligned}&\mathrm{Hom}_{\mathbb{Z}_p}(TA_0[p^\infty]^{et} \otimes_{O_p} TA_0[p^\infty]^{et}, \widehat{\mathbb{G}}_m(R))\\ &\quad \cong \mathrm{Hom}_{\mathbb{Z}_p}(O_p, \widehat{\mathbb{G}}_m(R)) \cong \widehat{\mathbb{G}}_m(R) \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(O, \mathbb{Z}) \cong \widehat{S}(R) \quad (R \in CL_{/W}).\end{aligned}$$

Here $t \otimes a \in \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}} \mathfrak{d}^{-1}$ corresponds to $q : O_p \to \widehat{\mathbb{G}}_m \in \mathrm{Hom}_{\mathbb{Z}_p}(O_p, \widehat{\mathbb{G}}_m(R))$ with $q(b) = t^{\mathrm{Tr}(ab)}$. This supplies us with the desired identity. □

3 Hilbert Modular Shimura Varieties

Let $G = \mathrm{Res}_{F/\mathbb{Q}}(GL(2))$. We write $h_{\mathbf{0}} : \mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to G_{/\mathbb{R}}$ for the homomorphism of real algebraic groups sending $a + b\sqrt{-1}$ to the matrix $\left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$. We write $\mathfrak{X}$ for the conjugacy class of $h_{\mathbf{0}}$ under $G(\mathbb{R})$. The group $G(\mathbb{R})$ acts on $\mathfrak{X}$ from the left by conjugation. Since the centralizer of $h_{\mathbf{0}}$ is the product of the maximal compact subgroup of the identity connected component $G(\mathbb{R})_+$ of the real Lie group $G(\mathbb{R})$ and its center $Z(\mathbb{R})$, the identity connected component $\mathfrak{X}^+$ containing $\mathbf{0} = h_{\mathbf{0}}$ is isomorphic to the product $\mathfrak{Z} = \mathfrak{H}^I$ of copies of the upper half complex plane $\mathfrak{H}$ indexed by embeddings I of F into $\mathbb{R}$ by $g(\mathbf{0}) \mapsto g(\mathbf{i})$ for $\mathbf{i} = (\sqrt{-1}, \dots, \sqrt{-1})$. Here the action of $(g_\sigma)_{\sigma\in I} \in G(\mathbb{R})$ with $g_\sigma = \left(\begin{smallmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{smallmatrix}\right)$ on $\mathfrak{Z}$ is given by $z = (z_\sigma) \mapsto \left(\frac{a_\sigma z_\sigma + b_\sigma}{c_\sigma z_\sigma + d_\sigma}\right)$. Thus $\mathfrak{X}$ is a finite union of the hermitian symmetric domain isomorphic to $\mathfrak{Z}$, and for an arithmetic subgroup $\Gamma \subset G(\mathbb{Q})$, $\Gamma\backslash\mathfrak{X}$ is a finite union of Hilbert modular varieties.

The pair $(G, \mathfrak{X})$ satisfies Deligne's axiom for Shimura varieties in [D3] 2.1.1. The Shimura variety over $\mathbb{C}$ is given by

$$\begin{aligned}Sh_{\mathbb{C}}(\mathbb{C}) = Sh_{\mathbb{C}}(G, \mathfrak{X})(\mathbb{C}) = \varprojlim_K G(\mathbb{Q})\backslash \left(\mathfrak{X} \times G(\mathbb{A}^{(\infty)})\right)/K\\ = G(\mathbb{Q})\backslash \left(\mathfrak{X} \times G(\mathbb{A}^{(\infty)})\right)/\overline{Z(\mathbb{Q})}, \quad (3.1)\end{aligned}$$

where $(\gamma, u) \in G(\mathbb{Q}) \times K$ acts on $(z, g) \in \mathfrak{X} \times G(\mathbb{A}^{(\infty)})$ by $\gamma(z, g)u = (\gamma(z), \gamma g u)$, $\overline{Z(\mathbb{Q})}$ is the closure of the center $Z(\mathbb{Q})$ in $G(\mathbb{A}^{(\infty)})$. See [M] page 324. We write $[z, g]$ for the point of $Sh_{\mathbb{C}}(\mathbb{C})$ represented by $(z, g) \in \mathfrak{X} \times G(\mathbb{A}^{(\infty)})$. This pro-algebraic variety has a unique canonical model $Sh(G, \mathfrak{X})$ defined over $\mathbb{Q}$, which we recall later. In this section, we review the construction of the model, emphasizing its automorphism group $\mathcal{G}$. Strictly speaking, the group $\mathcal{G}$ we will study is a subgroup of finite index in the full automorphism group, and the full automorphism group is a semi-direct product of $\mathcal{G}$ with the field automorphism group $\mathrm{Aut}(F/\mathbb{Q})$. As is clear from Shimura's original construction of canonical models [Sh2], full knowledge of $\mathcal{G}$ is almost equivalent to the existence of the canonical model itself (see [AAF] Chapter II).

3.1 Abelian Varieties up to Isogenies

Let $V = F^2$ be a column vector space, and put $V(\mathbb{A}^{(\infty)}) = V_{\mathbb{A}^{(\infty)}} := V \otimes_{\mathbb{Q}} \mathbb{A}^{(\infty)}$. We often write $F_{\mathbb{A}^{(\infty)}}$ for $F \otimes_{\mathbb{Q}} \mathbb{A}^{(\infty)}$, which is the finite part of the adele ring $F_{\mathbb{A}} = F \otimes_{\mathbb{Q}} \mathbb{A}$. Then $V(\mathbb{A}^{(\infty)})$ is an $F_{\mathbb{A}^{(\infty)}}$-free module of rank 2. We consider the fibered category $\mathcal{A}_F^{\mathbb{Q}}$ over $\mathbb{Q}$-SCH defined by the following data:

(Object) abelian schemes with real multiplication by O;

(Morphism) $\mathrm{Hom}_F^{\mathbb{Q}}(A, A') = \mathrm{Hom}_O(A, A') \otimes_{\mathbb{Z}} \mathbb{Q}$.

For an object $A_{/S}$, we take a geometric point $s \in S$, consider the Tate module $T(A) = T_s(A) = \varprojlim_N A[N](k(s))$, and define $V(A) = V_s(A) = T(A) \otimes_{\mathbb{Z}} \mathbb{A}^{(\infty)}$. The module $V(A)$ is an $F_{\mathbb{A}^{(\infty)}}$-free module of rank 2 and has an $\widehat{O}$-stable lattice $T(A)$, where $\widehat{O} = O \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}} = \prod_{\ell:\mathrm{prime}} O_\ell$.

Picking a geometric point s in each connected component of S, a full level structure on A is an isomorphism $\eta : V(\mathbb{A}^{(\infty)}) \cong V_s(A)$ of $F_{\mathbb{A}^{(\infty)}}$-modules. For a closed subgroup $K \subset G(\mathbb{A}^{(\infty)})$, a level K-structure is the K-orbit $\overline{\eta} = \eta K$ of η for the right action $\eta \mapsto \eta \circ u$ ($u \in K$). Strictly speaking, we consider the étale (set theoretic) sheaf $\mathcal{L}(S') = \mathrm{Isom}_F(V_{\mathbb{A}^{(\infty)}}, V_s(A_{/S'}))$ (over the small étale site over S) of level structures of A on which K acts, and $\overline{\eta}$ is supposed to be an element of the sheaf quotient $\mathcal{L}/K$. For many instances, we assume K to be open compact. Since $A[N]_{/S}$ is an étale finite group scheme, the algebraic fundamental group $\pi_1(S, s)$ with base point s acts on $A[N](k(s))$ for any integer N and hence on the full Tate module $V_s(A) = \varprojlim_N A[N](k(s)) \otimes \mathbb{Q}$. The level K-structure is defined over S if $\sigma \circ \overline{\eta} = \overline{\eta}$ for each $\sigma \in \pi_1(S, s)$. If the compatibility $\sigma \circ \overline{\eta} = \overline{\eta}$ is valid at one geometric point s for each connected component of S, it is valid for all $s \in S$ (see [PAF] 6.4.1).

Two polarizations $\lambda, \lambda' : A \to A^t$ are said to be equivalent (written as $\lambda \sim \lambda'$) if $\lambda = a\lambda' = \lambda' \circ a$ for a totally positive $a \in F$. Here a is any fraction in $F_+^\times$, writing $F_+^\times$ for the set of all totally positive elements in F. Without introducing the category $\mathcal{A}_F^{\mathbb{Q}}$ up to isogeny, our notion of polarization classes does not make sense. The equivalence class of a polarization λ defined over S is written as $\overline{\lambda}$. If the class $\overline{\lambda}$ is defined over S, we can find a polarization $\lambda \in \overline{\lambda}$ really defined over S (e.g., [PAF] pages 100–101). Our requirement (rm4) in Section 2.4 is often stated as the condition on characteristic polynomials satisfied by the action of $\alpha \in O$ on the $\mathcal{O}_S$-module $Lie(A)$ in papers and books dealing with Shimura varieties of PEL type (for example, [Ko] Section 5 and the condition (det) of [PAF] 4.2.1). For an open compact subgroup K, we consider the following functor from $SCH_{/\mathbb{Q}}$ into $SETS$,

$$\mathcal{P}_K^{\mathbb{Q}}(S) = \left[(A, \overline{\lambda}, \overline{\eta})_{/S} \text{ with (rm1–4)}\right],$$

where $\overline{\eta}$ is a level K-structure as defined above, and $[\ ] = \{\ \}/\cong$ indicates the set of isomorphism classes in $\mathcal{A}_F^{\mathbb{Q}}$ of the objects defined over S in the brackets. For a compact subgroup K, $\mathcal{P}_K^{\mathbb{Q}}(S)$ is defined by the natural projective limit $\varprojlim_U \mathcal{P}_U^{\mathbb{Q}}(S)$ for U running over open compact subgroups containing K. An F-linear morphism $\phi \in \mathrm{Hom}_F^{\mathbb{Q}}(A, A')$ is an isomorphism between triples $(A, \overline{\lambda}, \overline{\eta})_{/S}$

and $(A', \overline{\lambda}', \overline{\eta}')_{/S}$ if it is compatible with all data; that is,

$$\phi \circ \overline{\eta} = \overline{\eta}' \text{ and } \phi^t \circ \overline{\lambda} = \overline{\lambda}' \circ \phi.$$

Equip $V = F^2$ with an alternating form $\Lambda : V \wedge_F V \cong F$ given by $(x, y) = {}^t x J_1 y$ for $J_1 = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. We define a $\mathbb{Q}$-alternating pairing $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{Q}$ by $\mathrm{Tr}_{F/\mathbb{Q}} \circ \Lambda$. Suppose that the point $s \in S$ is a complex point $s \in S(\mathbb{C})$; so, we have the Betti homology group $H_1(A, \mathbb{Q}) := H_1(A(k(s)), \mathbb{Q})$. Then the polarization $\lambda : A \to A^t$ induces a nondegenerate F-Hermitian alternating pairing $E_\lambda : \bigwedge^2 H_1(A, \mathbb{Q}) \to \mathbb{Q}$ (the Riemann form; see [ABV] Sections 1 and 20). Here the word: "F-Hermitian" means $E_\lambda(\alpha x, y) = E_\lambda(x, \alpha y)$ for all $\alpha \in F$. We write $e_\lambda : H_1(A, \mathbb{Q}) \wedge_F H_1(A, \mathbb{Q}) \cong F$ for a unique alternating form satisfying $\mathrm{Tr}_{F/\mathbb{Q}} \circ e_\lambda = E_\lambda$. The Hodge decomposition: $H^1(A, \mathbb{C}) = H^0(A(k(s)), \Omega^{an}_{A/\mathbb{C}}) \oplus H^0(A(k(s)), \overline{\Omega}^{an}_{A/\mathbb{C}})$ induces, by Poincaré duality, an embedding $h = h_A : \mathbb{C}^\times = \mathbb{S}(\mathbb{R}) \to \mathrm{Aut}_F(H_1(A, \mathbb{R}))$ such that

1. $h(z)\omega = z\omega$ for all $\omega \in \mathrm{Hom}_{\mathbb{C}}(H^0(A(k(s)), \Omega_{A/\mathbb{C}}), \mathbb{C})$ (and $h(z)\overline{\omega} = \overline{z\omega}$);
2. $E_\lambda(x, h(\sqrt{-1})y)$ is a positive definite Hermitian form on $H_1(A, \mathbb{R})$ ($\cong V_{\mathbb{R}} := V \otimes_{\mathbb{Q}} \mathbb{R}$) under the complex structure given by h.

In the above definition of $\mathcal{P}^{\mathbb{Q}}_K$ for an open compact K, missing is a condition usually required in papers dealing with Shimura varieties:

(pol) There exists an F-linear isomorphism $f : V \cong H_1(A, \mathbb{Q})$ such that $f^{-1} \circ h_A \circ f$ is a conjugate of $h_{\mathbf{0}}$ under $G(\mathbb{R})$, $f \equiv \eta \mod K$ under the canonical isomorphism $V_s(A) \cong H_1(A, \mathbb{A}^{(\infty)}) = H_1(A, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{A}^{(\infty)}$ and $e_\lambda(f(x), f(y)) = \alpha \cdot \Lambda(x \wedge y)$ for some $\alpha \in F^\times$.

Since V and $H_1(A, \mathbb{Q})$ both have a non-degenerate F-bilinear alternating form, we can find an F-linear isomorphism $f_0 : V \xrightarrow{\sim} H_1(A, \mathbb{Q})$ with $e_\lambda(f_0(x), f_0(y)) = \Lambda(x \wedge y)$. After tensoring $\mathbb{A}^{(\infty)}$ and scaling by an element in $F^\times_{\mathbb{A}^{(\infty)}}$, we may assume that $g := \eta^{-1} \circ f_0$ belongs in $SL_2(F^{(\infty)}_{\mathbb{A}})$. By the strong approximation theorem, we have $\gamma \in SL_2(F)$ such that $g = u\gamma^{-1}$ for $u \in K$; in other words, putting $f = f_0 \circ \gamma$, we have $f \in \eta \circ K$ as in (pol). Since $G(\mathbb{R})$ is the full group of $F_{\mathbb{R}}$-linear automorphisms of $V_{\mathbb{R}}$, $f^{-1} \circ h_A \circ f$ is always conjugate to $h_{\mathbf{0}}$. Thus this condition (pol) is redundant; so, we ignore it.

By [Sh1] and [D2] 4.16–21,

(rep) the canonical model $Sh(G, \mathfrak{X})_{/\mathbb{Q}}$ represents the functor $\mathcal{P}^{\mathbb{Q}}_1$ over $\mathbb{Q}$ for the trivial subgroup 1 made of the identity element of $G(\mathbb{A}^{(\infty)})$.

This fact will be confirmed over $\mathbb{C}$ by a straight calculation (see the paragraph following (3.2)). Through the action of $G(\mathbb{A}^{(\infty)})$ on $F^2_{\mathbb{A}^{(\infty)}}$, $g \in G(\mathbb{A}^{(\infty)})$ acts on the level structure by $\eta \mapsto \eta \circ g$ and hence on the variety $Sh(G, \mathfrak{X})$ from the right. If K is open and sufficiently small (so that $\mathrm{Aut}((A, \overline{\lambda}, \overline{\eta})_{/S}) = \{1\}$ for all test objects $(A, \overline{\lambda}, \overline{\eta})_{/S}$), $Sh_K(G, \mathfrak{X}) := (Sh(G, \mathfrak{X})/K)_{/\mathbb{Q}}$ (whose complex points are given by the manifold $G(\mathbb{Q})\backslash(\mathfrak{X} \times G(\mathbb{A}^{(\infty)}))/K$) represents $\mathcal{P}^{\mathbb{Q}}_K$ over $\mathbb{Q}$.

Over $\mathbb{C}$, by (3.1), we have

$$[z,g]=[\gamma(z),\gamma g]\ (\Leftrightarrow [\gamma^{-1}(z),g]=[z,\gamma g]) \tag{3.2}$$

for $(z,g)\in\mathfrak{Z}\times G(\mathbb{A}^{(\infty)})$ and $\gamma\in G(\mathbb{Q})_+$, taking the expression

$$Sh(G,X)(\mathbb{C})=G(\mathbb{Q})_+\backslash\left(\mathfrak{Z}\times G(\mathbb{A}^{(\infty)})\right)/\overline{Z(\mathbb{Q})}$$

and noting $\mathfrak{X}^+=\mathfrak{Z}$. In the complex uniformization, each point $[z,g]$ corresponds to the test triple $(A_z,\lambda_z,\eta_z\circ g)$, where $A_z(\mathbb{C})=\mathbb{C}^I/(O^*+Oz)$ and $\eta_z\left(\begin{smallmatrix}a\\b\end{smallmatrix}\right)=bz-a$ identifying $T(A_z)=\widehat{O}^*+\widehat{O}z$. To see this, we note that the map: $[z,g]\mapsto(A_z,\lambda_z,\eta_z\circ g)$ sends $\left(\mathfrak{Z}\times G(\mathbb{A}^{(\infty)})\right)$ surjectively onto $\mathcal{P}^{\mathbb{Q}}_K(\mathbb{C})$ for each open compact subgroup K. Thus we need to check

$$(A_{\gamma^{-1}(z)},\overline{\lambda}_{\gamma^{-1}(z)},\eta_{\gamma^{-1}(z)}\circ g)\cong(A_z,\overline{\lambda}_z,\eta_z\circ\gamma^{(\infty)}g)\ \text{ in } \mathcal{A}^{\mathbb{Q}}_F \text{ for } \gamma\in G(\mathbb{Q})_+$$

which is equivalent to $[\gamma^{-1}(z),g]=[z,\gamma^{(\infty)}g]$. This is because $\alpha_\gamma\circ\eta_{\gamma^{-1}(z)}=\eta_z\circ\gamma^{(\infty)}$ for the isogeny $\alpha_\gamma: A_{\gamma^{-1}(z)}\to A_z$ given via the multiplication by $(-cz+a)$ on $\mathbb{C}^I$ (writing $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$).

We now give a very brief outline of the proof of the representability (assuming that K is open-compact), reducing it to the representability of a functor classifying abelian schemes up to isomorphisms not up to isogenies. Let G_1 be the derived group $\mathrm{Res}_{O/\mathbb{Z}}SL(2)$ of G. By shrinking K, we may assume that $\det(K)\cap O_+^\times\subset(K\cap Z(\mathbb{Z}))^2$. This is to guarantee that the images of $gKg^{-1}\cap G_1(\mathbb{Q})$ and $gKg^{-1}\cap G(\mathbb{Q})^+$ in $PG(\mathbb{Q})$ ($PG=G/Z$) are equal; so, $Sh_K(\mathbb{C})$ can be embedded into $Sh_{K_1}(\mathbb{C})$ for $K_1=G_1(\mathbb{A}^{(\infty)})\cap K$, because the moduli problem with respect to K_1 is neat without having any nontrivial automorphisms. Let $L\subset V$ be an O-lattice. We may assume that $L=\mathfrak{a}^*\oplus\mathfrak{b}$ for a pair $(\mathfrak{a},\mathfrak{b})$ of two fractional ideals, where $\mathfrak{a}^*$ is the dual ideal given by $\{\xi\in F|\mathrm{Tr}(\xi\mathfrak{a})\subset\mathbb{Z}\}=\mathfrak{a}^{-1}\mathfrak{d}^{-1}$. We define the polarization ideal $\mathfrak{c}$ by $\mathfrak{c}^*=\Lambda(L\wedge L)\subset F$. For each point $h_z\in\mathfrak{X}$, we have a unique point $z\in(\mathbb{C}-\mathbb{R})^I$ fixed by $h_z(\mathbb{C}^\times)$ (in this way, we identify $\mathfrak{Z}$ with the connected component $\mathfrak{X}^+$ of $\mathfrak{X}$ containing $h_{\mathbf{0}}$). By changing the $F\otimes\mathbb{R}$-linear identification $V\otimes_{\mathbb{Q}}\mathbb{C}=F^2\otimes_{\mathbb{Q}}\mathbb{C}$, we may assume that $z\in\mathfrak{X}^+=\mathfrak{Z}$. The action of $h_z(\mathbb{C}^\times)$ on $V_{\mathbb{R}}=V\otimes_{\mathbb{Q}}\mathbb{R}$ gives a structure of a complex vector space of dimension $g=[F:\mathbb{Q}]$ on $V_{\mathbb{R}}$; that is, $V_{\mathbb{R}}=\mathbb{C}^I$ via $(a,b)\mapsto -a+bz=(a,b)J_1\cdot{}^t(z,1)$ for $J_1=\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$. Then $L\subset V_{\mathbb{R}}$ gives rise to the lattice $\mathcal{L}_z$, and Λ induces the $\mathfrak{c}$-polarization λ_z. Set $\widehat{L}=L\otimes_{\mathbb{Z}}\widehat{\mathbb{Z}}\subset V_{\mathbb{A}}^{(\infty)}=V\otimes_{\mathbb{Q}}\mathbb{A}^{(\infty)}$, and define an abelian variety $A_{z/\mathbb{C}}$ by $A_z(\mathbb{C})=\mathbb{C}^I/\mathcal{L}_z$. Then we have $T(A_z)=\widehat{L}$, which induces $\eta_z: V_{\mathbb{A}^{(\infty)}}\cong V(A_z)$ and gives rise to a level N-structure $\phi_N: N^{-1}L/L\cong A_z[N]$ for any $N>0$.

Let $Cl^+(K)=F^\times_{\mathbb{A}^{(\infty)}}/\det(K)F^\times_+$, which is a finite group (by the open property of K). We fix a complete representative set $\{c\in F^\times_{\mathbb{A}^{(\infty)}}\}$ for $Cl^+(K)$ so that $c\widehat{O}\cap F=\mathfrak{c}$. We define an O–lattice $L_{\mathfrak{c}}=\mathfrak{c}^*\oplus O\subset V$ as above with $\Lambda(L_{\mathfrak{c}}\wedge L_{\mathfrak{c}})=\mathfrak{c}^*$, and put $L=L_O$. Note that $L=L_{\mathfrak{c}}\cdot\left(\begin{smallmatrix}c&0\\0&1\end{smallmatrix}\right)$ in $F^2=V$.

For each isogeny class of $(A,\overline{\lambda},\overline{\eta})_{/S}\in\mathcal{P}^{\mathbb{Q}}_K(S)$, we can functorially find a unique triple $(A',\overline{\lambda'},\overline{\eta}')_{/S}$ and a polarization ideal $\mathfrak{c}$ (representing a unique class

in $Cl^+(K)$) such that $\eta'(\widehat{L}_{\mathfrak{c}}) = T(A)$. Once this is done, as explained after (pol), we can find a polarization λ' in $\overline{\lambda'}$ so that the alternating pairing induced on $T(A)$ by the polarization coincides with Λ under η. See [PAF] pages 135–6 for the details of this process of finding a unique triple $(A', \lambda', \overline{\eta}')_{/S}$ in the isogeny class of $(A, \overline{\lambda}, \overline{\eta})_{/S}$. Thus once we have adjusted the $\mathfrak{c}$-polarization λ' in $\overline{\lambda}$ to Λ for each member $(A, \overline{\lambda}, \overline{\eta}) \in \mathcal{P}_K^{\mathbb{Q}}(S)$, we have a unique triple $(A', \lambda', \overline{\eta}')_{/S}$ with $\mathfrak{c}$-polarization λ'. If two such choices are isogenous, the isogeny between them has to be an isomorphism keeping the polarization. Thus we get an isomorphism of functors: $\mathcal{P}_K^{\mathbb{Q}}(S) \cong \mathcal{P}'_K(S) := \bigsqcup_{\mathfrak{c}\in Cl^+(K)} \mathcal{P}'_{K,\mathfrak{c}}(S)$, where $\mathfrak{c}$ runs over the ideal classes in $Cl^+(K) = {F_{\mathbb{A}}^{(\infty)}}^{\times}/F_+^{\times} \det(K)$, and

$$\mathcal{P}'_{K,\mathfrak{c}}(S) = \left\{(A', \lambda', \overline{\eta}')_{/S} \text{ with (rm1-4)} \;\middle|\; \eta'(\widehat{L}_{\mathfrak{c}}) = T(A') \text{ and } \mathfrak{c}(\lambda') = \mathfrak{c}\right\} / \cong .$$

Here $\cong$ means an isomorphism (not an isogeny) for a chosen polarization integral over the fixed lattice $L_{\mathfrak{c}}$ in the class of $\overline{\lambda}$ (in other words, λ induces a fixed alternating form on the space V integral over $L_{\mathfrak{c}}$ (up to units in $F \cap \det(K)$). As we now see, this functor $\mathcal{P}'_{K,\mathfrak{c}}$ is represented by a scheme $\mathfrak{M}(\mathfrak{c}, K)$ over a specific abelian extension k_K of $\mathbb{Q}$ dependent on K (see below for a description of k_K for some specific K's). See [PAF] Section 4.2 for details of this process.

Recall $L_{\mathfrak{c}} = \mathfrak{c}^* \oplus O \subset V$, $\widehat{L}_{\mathfrak{c}} = \varprojlim_N L_{\mathfrak{c}}/NL_{\mathfrak{c}} = L_{\mathfrak{c}} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ and $L_{\mathfrak{c}} \wedge L_{\mathfrak{c}} \cong \mathfrak{c}^*$ by $(a, b) \wedge (a', b') \mapsto a'b - ab'$. Take the principal congruence subgroup $\Gamma_{\mathfrak{c}}(N) = \mathrm{Ker}(GL(\widehat{L}_{\mathfrak{c}}) \twoheadrightarrow GL(L_{\mathfrak{c}}/NL_{\mathfrak{c}}))$ of $G(\mathbb{A}^{(\infty)})$ for an integer $N > 0$. We write $\Gamma(N)$ for $\Gamma_O(N)$. We identify μ_N with $\mathbb{Z}/N\mathbb{Z}$ by choosing a primitive Nth root $\zeta = \zeta_N$ of unity in $\mathbb{Q}[\mu_N]$. Then, having a level $\Gamma(N)$-structure $\overline{\eta}$ is equivalent to having a level $\Gamma_{\mathfrak{c}}(N)$-structure $\overline{\eta}'$, because we can identify $\widehat{L}_{\mathfrak{c}}$ and $\widehat{L}$ via the left multiplication by $\left(\begin{smallmatrix} c & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Giving $\overline{\eta}'$ is equivalent to giving an isomorphism of locally free group schemes

$$\phi_N : (\mathfrak{c}^* \otimes \mu_N) \times (O \otimes \mathbb{Z}/N\mathbb{Z}) \cong N^{-1}L_{\mathfrak{c}}/L_{\mathfrak{c}} \overset{\overline{\eta}'}{\cong} A'[N].$$

Thus $\mathcal{P}'_{\Gamma(N),\mathfrak{c}}$ is the standard moduli functor classifying the level structure for the principal congruence subgroup $\Gamma_{\mathfrak{c}}(N)$:

$$\mathcal{P}'_{\Gamma(N),\mathfrak{c}}(S) = \left\{(A, \lambda, \phi_N)_{/S} \middle| \begin{matrix} \phi_N : (\mathfrak{c}^* \otimes \mu_N) \times (O \otimes \mathbb{Z}/N\mathbb{Z}) \cong A[N] \\ \text{and } \ \mathfrak{c}(\lambda) = \mathfrak{c} \end{matrix}\right\} / \cong .$$

By a standard argument (see [R], [K3] and [PAF] 4.1), this functor is represented by a geometrically irreducible quasi projective variety $\mathfrak{M}(\mathfrak{c}, \Gamma(N))_{/\mathbb{Q}[\mu_N]}$.

Over $k_{\Gamma(N)} = \mathbb{Q}[\mu_N]$, the component $\mathfrak{M}(\mathfrak{c}, \Gamma(N))$ of $Sh_{\Gamma(N)}(G, \mathfrak{X})$ represents the functor $\mathcal{P}'_{\Gamma(N),\mathfrak{c}}$. This irreducible component in turn corresponds to the component $G(\mathbb{Q})\backslash(\mathfrak{X} \times G(\mathbb{Q})\left(\begin{smallmatrix} c & 0 \\ 0 & 1 \end{smallmatrix}\right)\Gamma(N))/\Gamma(N) \subset Sh_{\Gamma(N)}(\mathbb{C})$ in (3.1). The choice ζ gives rise to the identification $\mathbb{Q}[\mu_N] = \mathbb{Q}[T]/(\Phi_N(T))$ with $\mathbb{Q}[\zeta]$ for the cyclotomic polynomial $\Phi_N(T) \in \mathbb{Z}[T]$, and an automorphism $\sigma \in \mathrm{Gal}(\mathbb{Q}[\zeta]/\mathbb{Q})$ changes the identification by $\zeta \mapsto \zeta^\sigma$, whose action is induced by $\phi_N \mapsto \phi_N \circ \left(\begin{smallmatrix} c & 0 \\ 0 & 1 \end{smallmatrix}\right)$ for a unit $c \in \widehat{\mathbb{Z}}^\times$ such that $\zeta^\sigma = \zeta^c$. In other words, the action of

$\left(\begin{smallmatrix} c & 0 \\ 0 & 1 \end{smallmatrix}\right) \in G(\mathbb{A}^{(\infty)})$ on $Sh(G,\mathfrak{X})_{/\mathbb{Q}}$ brings $\mathfrak{M}(O,\Gamma(N))_{/\mathbb{Q}[\zeta]} \subset Sh_{\Gamma(N)}(G,\mathfrak{X})$ to its σ-conjugate $\mathfrak{M}(O,\Gamma(N))^{\sigma}_{/\mathbb{Q}[\zeta]}$ in $Sh_{\Gamma(N)}(G,\mathfrak{X})_{/\mathbb{Q}}$.

Summing up all these, we have

$$\mathcal{P}'_{\Gamma(N)} = \bigsqcup_{\mathfrak{c}\in Cl^+_F(N)} \mathcal{P}'_{\Gamma(N),\mathfrak{c}} \ \text{ over } \mathbb{Q}[\mu_N]\text{-}SCH, \tag{3.3}$$

which implies

$$Sh_{\Gamma(N)}(G,\mathfrak{X})_{/\mathbb{Q}[\mu_N]} = \bigsqcup_{\mathfrak{c}\in Cl^+_F(N)} \mathfrak{M}(\mathfrak{c},\Gamma(N)) \ \text{ over } \mathbb{Q}[\mu_N]. \tag{3.4}$$

Since $L_{\mathfrak{c}} = \left(\widehat{L}\cdot \left(\begin{smallmatrix} c & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1}\right) \cap V$, this corresponds to the decomposition

$$G(\mathbb{A}^{(\infty)}) = \bigsqcup_{\mathfrak{c}\in Cl^+_F(N)} G(\mathbb{Q}) \left(\begin{smallmatrix} c & 0 \\ 0 & 1 \end{smallmatrix}\right) \Gamma(N).$$

By the Galois action on $\mathfrak{M}(\mathfrak{c},\Gamma(N))_{/\mathbb{Q}[\mu_N]}$, we can descend the right-hand side of (3.4) to the base field $\mathbb{Q}$ to obtain the model $Sh_{\Gamma(N)}(G,\mathfrak{X})$ over $\mathbb{Q}$, because $\mathfrak{M}(\mathfrak{c},\Gamma(N))$ is quasi-projective as we already mentioned.

To construct p-integral models of Shimura varieties, we use the following variant (due to Kottwitz [Ko]) of the functor $\mathcal{P}^{\mathbb{Q}}_K$. We fix a rational prime p unramified in $F/\mathbb{Q}$. This concerns an open-compact subgroup K *maximal at* p (i.e., $K = G(\mathbb{Z}_p)\times K^{(p)}$), where $O_p = O\otimes_{\mathbb{Z}}\mathbb{Z}_p$. We have written $K^{(p)} = \{x\in K | x_p = 1\}$. Recall $\mathbb{A}^{(p\infty)} = \{x\in\mathbb{A} | x_p = x_\infty = 0\}$. We identify the multiplicative group $\mathbb{A}^{(p\infty)^\times}$ with $\{x\in\mathbb{A}^\times | x_p = x_\infty = 1\}$.

We consider the following fibered category $\mathcal{A}^{(p)}_F$ over $\mathbb{Z}_{(p)}$-schemes:

(Object) abelian schemes with real multiplication by O;

(Morphism) We define $\mathrm{Hom}_{\mathcal{A}^{(p)}_F}(A,A') = \mathrm{Hom}_{\mathcal{A}_F}(A,A')\otimes_{\mathbb{Z}}\mathbb{Z}_{(p)}$, where

$$\mathbb{Z}_{(p)} = \left\{ \frac{a}{b} \middle| b\mathbb{Z} + p\mathbb{Z} = \mathbb{Z} \right\}.$$

This means that to classify test objects, we now allow only isogenies with degree prime to p (called "prime-to-p isogenies"), and the degree of the polarization λ is supposed to be also prime to p. Two polarizations are equivalent if $\lambda = a\lambda' = \lambda'\circ a$ for a totally positive $a\in F$ prime to p.

Fix an O-lattice $L\subset V = F^2$ with $\Lambda(L\wedge L) = \mathfrak{c}^*$, and assume self O_p-duality of $L_p = L\otimes_{\mathbb{Z}}\mathbb{Z}_p$ under the alternating pairing $\Lambda : V\wedge V\cong F$. Consider test objects $(A,\overline{\lambda},\overline{\eta}^{(p)})_{/S}$. Here $\eta^{(p)} : V(\mathbb{A}^{(p\infty)}) = V\otimes_{\mathbb{Q}}\mathbb{A}^{(p\infty)} \cong V^{(p)}(A) = T(A)\otimes_{\mathbb{Z}}\mathbb{A}^{(p\infty)}$ and $\lambda\in\overline{\lambda}$ are supposed to satisfy the following requirement, $V^{(p)}(A)\wedge V^{(p)}(A) \xrightarrow{e_\lambda} F^{(p\infty)}_{\mathbb{A}}$ is proportional to $\Lambda : V\wedge V\cong F$ up to scalars in $(F\otimes\mathbb{A}^{(p\infty)})^\times$. Here e_λ is the alternating form induced by the polarization λ.

We write the $K^{(p)}$-orbit of $\eta^{(p)}$ as $\overline{\eta}^{(p)}$. Then we consider the following functor from $\mathbb{Z}_{(p)}$-schemes into $SETS$.

$$\mathcal{P}_K^{(p)}(S) = \left[(A, \overline{\lambda}, \overline{\eta}^{(p)})_{/S} \text{ with (rm1-4)}\right]. \tag{3.5}$$

Let $O_{(p)+}^{\times} = O_p^{\times} \cap F_+^{\times}$. As long as K is maximal at p, we can identify $Cl^+(K) = {F_{\mathbb{A}}^{(\infty)}}^{\times}/F_+^{\times} \det(K)$ with ${F_{\mathbb{A}}^{(p\infty)}}^{\times}/O_{(p)+}^{\times} \det(K^{(p)})$. Thus we may choose the representatives $\{\mathfrak{c}\}$ prime to p (and we may assume the self-duality of L at p). By the same process as bringing $\mathcal{P}_K^{\mathbb{Q}}$ isomorphically to $\mathcal{P}'_{K/\mathbb{Q}}$, the functor is equivalent to $\mathcal{P}'_{K/\mathbb{Z}_{(p)}}$ defined over $\mathbb{Z}_{(p)}$-SCH; so, it is representable over $\mathbb{Z}_{(p)}$, giving a canonical model $Sh_K^{(p)}(G, \mathfrak{X})_{/\mathbb{Z}_{(p)}}$ over $\mathbb{Z}_{(p)}$. The functor $\mathcal{P}'_{K/\mathbb{Z}_{(p)}}$ is a disjoint union of the functors $\mathcal{P}'_{K,\mathfrak{c}}$ indexed by $\mathfrak{c} \in Cl^+(K)$, where

$$\mathcal{P}'_{K,\mathfrak{c}}(S) = \left[(A, \lambda, \overline{\eta}^{(p)})_{/S} \text{ with (rm1-4) } \left|\eta^{(p)}(\widehat{L}_{\mathfrak{c}}) = T^{(p)}(A),\ \mathfrak{c}(\lambda) = \mathfrak{c}\right.\right]. \tag{3.6}$$

A subtle point is to relate $Sh^{(p)}_{/\mathbb{Z}_{(p)}}$ to $Sh_{/\mathbb{Q}}$. The equivalence of functors $\mathcal{P}^{(p)}_{\Gamma(N)} \cong \mathcal{P}'_{\Gamma(N)/\mathbb{Z}_{(p)}}$ are compatible when N varies over integers prime to p; similarly, for $\mathcal{P}^{\mathbb{Q}}_{\Gamma(N)} \cong \mathcal{P}'_{\Gamma(N)/\mathbb{Q}}$; therefore,

$$Sh^{(p)} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q} \cong Sh/G(\mathbb{Z}_p).$$

The functor $\mathcal{P}'_{\Gamma(N),O/\mathbb{Z}_{(p)}[\mu_N]}$ for N prime to p is represented by a scheme $\mathfrak{M}(O, \Gamma(N))_{/\mathbb{Z}_{(p)}[\mu_N]}$ and gives rise to a closed subscheme of $Sh^{(p)}_{\Gamma(N)/\mathbb{Z}_{(p)}[\mu_N]}$. The characteristic 0 fiber $\mathfrak{M}(O, \Gamma(N)) \otimes_{\mathbb{Z}_{(p)}[\mu_N]} \mathbb{Q}[\mu_N]$ gives $\mathfrak{M}(O, \Gamma(N))_{/\mathbb{Q}[\mu_N]}$ in (3.4). We define a closed subscheme $\mathfrak{M}^{(p)}$ of $Sh^{(p)}$ over the integer ring $\mathbb{Z}_{(p)}^{(p)-ab} = \bigcup_{p\nmid N} \mathbb{Z}_{(p)}[\mu_N]$ by

$$\mathfrak{M}^{(p)} := \varprojlim_{p\nmid N} \mathfrak{M}(O, \Gamma(N))_{/\mathbb{Z}_{(p)}[\mu_N]} \subset Sh^{(p)} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_{(p)}^{(p)-ab}. \tag{3.7}$$

Similarly, we define a closed subscheme $\mathfrak{M}$ of $Sh_{\mathbb{Q}^{ab}}$ over the maximal abelian extension $\mathbb{Q}^{ab} = \bigcup_{N>0} \mathbb{Q}[\mu_N]$ by

$$\mathfrak{M}_{/\mathbb{Q}^{ab}} := \varprojlim_{N>0} \mathfrak{M}(O, \Gamma(N))_{/\mathbb{Q}[\mu_N]} \subset Sh \otimes_{\mathbb{Q}} \mathbb{Q}^{ab}. \tag{3.8}$$

Since $\widehat{L}' \cong \widehat{L}$ for any O-lattice $L' \subset V$, it is essential to allow all O-isomorphism classes of O-lattices L' to define $\mathcal{P}'_{\Gamma(N)}$, because in the definition of $\mathcal{P}^{(p)}$, only $\widehat{L}^{(p)}$ is specified (which does not determine the isomorphism class of L if the class group of F is nontrivial). This problem is more acute at p because over $\mathbb{Z}_{(p)}$, $T_p(A)$ does not determine L_p. Indeed the p-adic Tate module of an abelian scheme of characteristic p has less rank than its characteristic 0 counterpart. The self-duality at p of L has to be imposed to overcome this point

(see the argument just above Remark 7.4 of [PAF]). Also we need the density of the derived group $G_1(\mathbb{Q})$ in $G_1(\mathbb{A}^{(\infty)})$ (the strong approximation theorem) in order to know that geometrically irreducible components of $Sh_K^{(p)}$ are indexed by the class group $Cl^+(K)$: $\pi_0(Sh^{(p)}_{K/\overline{\mathbb{Q}}}) \cong Cl^+(K)$.

Since p is unramified in $F/\mathbb{Q}$ (and $K^{(p)}$ is sufficiently small), $Sh_K^{(p)}$ is smooth over $\mathbb{Z}_{(p)}$ by the infinitesimal criterion of smoothness (e.g., [NMD] Proposition 2.2.6); that is, we can show that any characteristic p test object lifts to characteristic 0 infinitesimally. To explain this, let R be a $\mathbb{Z}_{(p)}$-algebra with a nilpotent ideal $I \subset R$ containing a power of p. Put $R_0 = R/I$. We want to show the existence of a lifting of a test object $(A_0, \overline{\lambda}_0, \overline{\eta}_0^{(p)})_{/R_0}$ to R. The abelian variety A_0 lifts to an abelian scheme $A_{/R}$ (with $A \otimes_R R_0 \cong A_0$) by the deformation theory of Grothendieck–Messing–Mumford (cf. [CBT] V.1.6, [GIT] Section 6.3, [R] 1.5–10, [DAV] I.3, and also [PAF] Theorem 8.8 and the remark after the theorem). Since the degree of the polarization is prime to p (here we use the fact that we can choose a representative $\mathfrak{c}$ prime to p in a given class in $Cl^+(K)$), λ also lifts because we may assume that $\lambda_0 : A_0 \to A_0^t$ is étale (and hence $A^t = A/E$ for an étale subgroup $E \subset A$ lifting $\mathrm{Ker}(\lambda_0)$; see [ECH] I.3.12). As for the level structure $\eta_0^{(p)}$, it is prime to p and hence étale over R_0. Then it extends uniquely to a level structure $\eta^{(p)} : V_{\mathbb{A}}^{(p)} \cong V^{(p)}(A)$ over R. By the deformation theory of Barsotti–Tate groups (see [CBT] V.1.6 and [R] 1.5–10), using (rm4), we can find a deformation $A_{/R}$ of A_{0/R_0} with an embedding $O \hookrightarrow \mathrm{End}(A_{/R})$ compatible with $O \hookrightarrow \mathrm{End}(A_{0/R_0})$.

We can let $g \in G(\mathbb{A}^{(\infty)})$ act on $Sh(G, \mathfrak{X})_{/\mathbb{Q}}$ by

$$(A, \overline{\lambda}, i, \eta) \mapsto (A, \overline{\lambda}, i, \eta \circ g), \tag{3.9}$$

which gives a right action of $G(\mathbb{A}^{(\infty)})$ on $Sh(G, \mathfrak{X})$. Define

$$\mathcal{G} = \mathcal{G}(G, \mathfrak{X}) = \left\{ g \in G(\mathbb{A}) \middle| \det(g) \in \mathbb{A}^\times \overline{F^\times F_{\infty+}^\times} / \overline{F^\times F_{\infty+}^\times} \right\},$$

and write $\overline{\mathcal{E}} = \overline{\mathcal{E}}(G, \mathfrak{X}) = \mathcal{G}(G, \mathfrak{X}) / \overline{Z(\mathbb{Q}) G(\mathbb{R})_+}$ (see [Sh2] II, [Sh3] and [AAF] Section 8). Here $F_{\infty+}^\times$ is the subgroup of totally positive elements in $F_\infty = F \otimes_{\mathbb{Q}} \mathbb{R}$. By (3.1) (and by our construction), we have $\pi_0(Sh(G, \mathfrak{X})(\mathbb{C})) \cong F_{\mathbb{A}^{(\infty)}}^\times / \overline{F_+^\times} \cong F_{\mathbb{A}}^\times / \overline{F^\times F_{\infty+}^\times} = \varprojlim_N Cl_F^+(N)$. The action of $g \in G(\mathbb{A}^{(\infty)})$ permutes transitively connected components of $Sh(G, \mathfrak{X})(\mathbb{C})$.

The neutral irreducible component of $Sh(G, \mathfrak{X})(\mathbb{C})$ is the image of $\mathfrak{Z} \times 1$ in $Sh(G, \mathfrak{X})(\mathbb{C})$ under the projection in (3.1) and is given by the complex points $\mathfrak{M}(\mathbb{C})$ of the closed subscheme $\mathfrak{M}_{/\mathbb{Q}^{ab}}$ of $Sh_{/\mathbb{Q}^{ab}}$ defined in (3.8). Since $\mathfrak{M}(O, \Gamma(N))(\mathbb{C})$ is a connected complex manifold for $N \gg 0$, $\mathfrak{M}_{/\mathbb{Q}^{ab}}$ is geometrically irreducible. Composing the structure morphism $\mathfrak{M} \to \mathrm{Spec}(\mathbb{Q}^{ab})$ with the unique morphism $\mathrm{Spec}(\mathbb{Q}^{ab}) \to \mathrm{Spec}(\mathbb{Q})$, we regard $\mathfrak{M}$ as an irreducible (but geometrically reducible) $\mathbb{Q}$-scheme. Thus we can think of the rational function field $\mathbb{Q}(\mathfrak{M}_{/\mathbb{Q}})$. The field of definition of $\mathfrak{M}$ (that is, the algebraic closure of $\mathbb{Q}$ in the function field $\mathbb{Q}(\mathfrak{M}_{/\mathbb{Q}})$) is the maximal abelian extension $\mathbb{Q}^{ab}/\mathbb{Q}$ (so,

$\mathbb{Q}(\mathfrak{M}_{/\mathbb{Q}}) = \mathbb{Q}^{ab}(\mathfrak{M}_{/\mathbb{Q}^{ab}}))$, because the values of the Weil pairing on all the torsion points of the universal abelian scheme over $\mathfrak{M}$ generate $\mathbb{Q}^{ab}$. Then we can think of the scheme $\mathfrak{M} \times_{\mathbb{Q}} \mathbb{Q}^{ab}$ (over $\mathbb{Q}^{ab}$) which is no longer connected:

$$\mathfrak{M} \times_{\mathbb{Q}} \mathbb{Q}^{ab} = \bigsqcup_{\sigma \in \mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})} \mathfrak{M} \times_{\mathbb{Q}^{ab},\sigma} \mathbb{Q}^{ab}.$$

Since Sh is defined over $\mathbb{Q}$ and $\mathfrak{M}_{/\mathbb{Q}^{ab}} \subset Sh_{/\mathbb{Q}^{ab}}$, $\mathfrak{M}^{\sigma}_{/\mathbb{Q}^{ab}} := \mathfrak{M} \times_{\mathbb{Q}^{ab},\sigma} \mathbb{Q}^{ab}$ gives another connected components of $Sh \otimes_{\mathbb{Q}} \mathbb{Q}^{ab}$; in other words, the nonconnected scheme $\mathfrak{M} \times_{\mathbb{Q}} \mathbb{Q}^{ab}$ has an open immersion into $Sh(G,\mathfrak{X})_{/\mathbb{Q}^{ab}}$, and the action of $g \in G(\mathbb{A}^{(\infty)})$ preserves $\pi_0(\mathfrak{M} \times_{\mathbb{Q}} \mathbb{Q}^{ab})$ if and only if $\det(g) \in \mathbb{A}^{\times}\overline{F^{\times}F^{\times}_{\infty+}}$. The action of g with $\det(g) \in \mathbb{A}^{\times}$ permutes transitively geometrically irreducible components of Sh through the action of the Artin symbol $[\det(g), \mathbb{Q}]$ on $\mathbb{Q}^{ab}$ (see [PAF] Proof of Theorem 4.14). Thus we may regard $\mathcal{G}$ as the stabilizer inside $G(\mathbb{A}^{(\infty)})$ of the neutral component $\mathfrak{M}_{/\mathbb{Q}}$. Since $G(\mathbb{A}^{(\infty)})$ acts transitively on the set $\pi_0(Sh(G,\mathfrak{X})_{/\overline{\mathbb{Q}}})$, the stabilizer of another component $\mathfrak{M}\cdot g$ in $G(\mathbb{A}^{(\infty)})$ is given by $g^{-1}\mathcal{G}g$. Since $\mathcal{G}$ is a normal subgroup of $G(\mathbb{A}^{(\infty)})$, $\mathcal{G}$ is the stabilizer of any other geometrically irreducible component of $Sh(G,\mathfrak{X})$.

We shall give another description of $\overline{\mathcal{E}}$ due to Deligne. We recall it, because recently Shimura's reciprocity is often written down using Deligne's formulation and it is also easier to describe the action of $G(\mathbb{A}^{(\infty)})$ (up to isogeny) in group theoretic terms if we use his definition. Write $\overline{G} = G(\mathbb{A}^{(\infty)})/\overline{Z(\mathbb{Q})}$, $\Gamma = G(\mathbb{Q})_+$, and $\Delta = G^{ad}(\mathbb{Q}) = G(\mathbb{Q})/Z(\mathbb{Q})$. We have the projection $\Gamma \ni \gamma \mapsto \overline{\gamma}$ onto a subgroup $\overline{\Gamma}$ of $\overline{G}$ and the following commutative diagram of group homomorphisms:

$$\begin{array}{ccc} \overline{\Gamma} & \xrightarrow{\varphi} & \Delta \\ \cap\downarrow & & \downarrow r \\ \overline{G} & \xrightarrow[ad]{} & \mathrm{Aut}(\overline{G}). \end{array} \tag{3.10}$$

Here r is the inclusion, φ is induced by the projection $G(\mathbb{Q}) \to G^{ad}(\mathbb{Q})$, $\mathrm{Aut}(\overline{G})$ is the automorphism group of the group $\overline{G}$, and $ad(g)(x) = gxg^{-1}$ for $g \in \overline{G}$. We often write $\varphi(\gamma)$ for $\varphi(\overline{\gamma})$ and by definition, $r(\delta)$ ($\delta \in \Delta$) preserves $\overline{\Gamma}$ as a whole. Plainly, we have the following two compatibility conditions,

(a) $r(\varphi(\gamma)) = ad(\overline{\gamma})$ for all $\gamma \in \Gamma$ (commutativity of (3.10));

(b) $\varphi(r(\delta)(\overline{\gamma})) = ad(\delta)(\varphi(\gamma))$ for all $\delta \in \Delta$ and $\gamma \in \Gamma$.

We consider the semi-direct product: $\overline{G} \rtimes \Delta$ whose multiplication law is given by $(g,\delta)(h,\epsilon) = (g \cdot (r(\delta)(h)), \delta\epsilon)$, and we have $(g,\delta)^{-1} = (r(\delta^{-1})(g^{-1}), \delta^{-1})$. By computation, we have

$$(g,\delta)(\overline{\gamma}^{-1}, \varphi(\gamma))(g,\delta)^{-1} = (g \cdot r(\delta)(\overline{\gamma}^{-1})r(\delta \cdot \varphi(\gamma))(r(\delta^{-1})(g^{-1})), ad(\delta)(\varphi(\gamma))).$$

Then again by computation,

$$\begin{aligned} & g \cdot r(\delta)(\overline{\gamma}^{-1}) r(\delta \cdot \varphi(\gamma))(r(\delta^{-1})(g^{-1})) \\ & \quad = g \cdot r(\delta)(\overline{\gamma}^{-1}) r(\delta)(\overline{\gamma}(r(\delta^{-1})(g^{-1}))\overline{\gamma}^{-1}) = g \cdot r(\delta)((r(\delta^{-1})(g^{-1}))\overline{\gamma}^{-1}) \\ & \qquad = g \cdot r(\delta)(r(\delta^{-1})(g^{-1}))(r(\delta)(\overline{\gamma}^{-1})) = r(\delta)(\overline{\gamma})^{-1}. \end{aligned}$$

This shows that $\widetilde{\Gamma} = \{(\overline{\gamma}^{-1}, \varphi(\gamma)) | \gamma \in \Gamma\}$ is a normal subgroup of the semi-direct product $\overline{G} \rtimes \Delta$. We then define

$$\overline{G} *_\Gamma \Delta = (\overline{G} \rtimes \Delta)/\widetilde{\Gamma}. \tag{3.11}$$

We have the following commutative diagram with exact rows.

$$\begin{array}{ccccccc} \mathrm{Ker}(\varphi) & \xrightarrow{\subset} & \overline{\Gamma} & \xrightarrow{\varphi} & \Delta & \xrightarrow{\text{onto}} & \mathrm{Coker}(\varphi) \\ \| \downarrow & & \cap \downarrow & & \cap \downarrow{\scriptstyle \delta \mapsto [1,\delta]} & & \downarrow \| \\ \mathrm{Ker}(\varphi) & \xrightarrow[\subset]{} & \overline{G} & \xrightarrow[g \mapsto [g,1]]{} & \overline{G} *_\Gamma \Delta & \xrightarrow[\text{onto}]{} & \mathrm{Coker}(\varphi). \end{array}$$

Then by the (suitably applied) snake lemma, we get a canonical isomorphism

$$\overline{\Gamma} \backslash \overline{G} \cong \Delta \backslash (\overline{G} *_\Gamma \Delta). \tag{3.12}$$

Note that $\overline{\Gamma} \backslash (\mathfrak{X} \times \overline{G}) = Sh(G, \mathfrak{X})(\mathbb{C})$ by (3.1). By this isomorphism, the amalgamated product $\overline{G} *_\Gamma \Delta$ acts on $\overline{\Gamma} \backslash (\mathfrak{X} \times \overline{G})$, and the action of $[g, \delta] \in \overline{G} *_\Gamma \Delta$ on the class $[z]$ in $\overline{\Gamma} \backslash (\mathfrak{X} \times \overline{G}) = Sh(G, \mathfrak{X})(\mathbb{C})$ (which is sent to $[z, 1] \in \Delta \backslash (\overline{G} *_\Gamma \Delta)$) is given by

$$[z] \cdot [g, \delta] = [z, 1][g, \delta] = [zg, \delta] = [1, \delta][r(\delta)^{-1}(zg), 1] = [r(\delta)^{-1}(zg)]. \tag{3.13}$$

Thus $\overline{G} *_\Gamma \Delta$ acts on the Shimura variety $Sh(G, \mathfrak{X})$, and by (3.13) combined with (3.2), the action coincides with the one in (3.9) (see [D3] and [PAF] 4.2.2). In particular, $\overline{\mathcal{E}}(G, \mathfrak{X})$ is identified with the stabilizer of $\mathfrak{M}$ (and of any other geometrically irreducible component of $Sh(G, \mathfrak{X})$) in $\overline{G} *_\Gamma \Delta$. The map

$$\overline{G} *_\Gamma \Delta \ni (g, ad(\gamma)) \mapsto \det(g) \in F_{\mathbb{A}}^\times / \overline{F^\times F_{\infty+}^\times}$$

is a well defined homomorphism, and $\overline{\mathcal{E}}(G, \mathfrak{X})$ is identified with the inverse image of $\mathbb{A}^{\times} \overline{F^\times F_{\infty+}^\times} / \overline{F^\times F_{\infty+}^\times}$ in $\overline{G} *_\Gamma \Delta$. The following fact (whose proof we have sketched) has been shown in [Sh2] II 6.5 and [Mt] Theorem 2 (see also [MS] 4.6 and 4.13 and [PAF] Theorem 4.14):

Theorem 3.1. *The stabilizer in $G(\mathbb{A}^{(\infty)})$ of the geometrically irreducible component of $Sh(G, \mathfrak{X})$ which contains the image of $\mathfrak{X}^+ \times 1$ is given by $\overline{\mathcal{E}}(G, \mathfrak{X})$. The right action of $(g, ad(\gamma)) \in \overline{\mathcal{E}}(G, \mathfrak{X})$ ($\gamma \in G(\mathbb{Q})$) on $[z, g']$ is given by*

$$[z, g'] \mapsto [\gamma^{-1}(z), (g'g)^{ad(\gamma)}],$$

where $(g'g)^{ad(\gamma)} = \gamma^{-1}(g'g)\gamma$.

Since Shimura does not formulate his result in the language of scheme, it is hard to say which part of Sh is Shimura's canonical model, though we can probably say that the (projective) system $\{\mathfrak{M}_K := \mathfrak{M}/K\}_{K\subset\overline{\mathcal{E}}(G,\mathfrak{X})}$ of quasi-projective varieties irreducible over $\mathbb{Q}$ (indexed by open compact subgroups K) each regarded as defined over its field of definition k_K (that is, the algebraic closure of $\mathbb{Q}$ in its function field $\mathbb{Q}(\mathfrak{M}_{K/\mathbb{Q}})$) is essentially his canonical model. Since other geometrically connected components V of $Sh_{/\mathbb{Q}}$ is isomorphic to $\mathfrak{M}$ by an action of $g\in G(\mathbb{A}^{(\infty)})$, more precisely, Shimura's canonical models give a system of geometrically irreducible varieties of the form $g(\mathfrak{M}_K)_{/k_K}$ with a specific isomorphism onto $\mathfrak{M}_{g^{-1}Kg/k_K}$ given by each element of $g\in G(\mathbb{A}^{(\infty)})$. His theory includes an explicit determination of k_K as an abelian extension of $\mathbb{Q}$ via class field theory, the local reciprocity low at each CM point on $\mathfrak{M}_K$ and an explicit description of the action of $\overline{\mathcal{E}}(G,\mathfrak{X})$ on each member $\mathfrak{M}_K$ (the global reciprocity law). The above result is an interpretation in Deligne's language of the result of Shimura in [Sh2] II 6.5. When we regard $g\in\overline{\mathcal{E}}(G,\mathfrak{X})$ as an automorphism of $\mathcal{O}_{Sh}$ or $Sh(G,\mathfrak{X})_{/\mathbb{Q}}$, we write it as $\tau(g)$.

3.2 Shimura's Reciprocity Law

Since $Sh(G,\mathfrak{X})(\mathbb{C}) = G(\mathbb{Q})\backslash\left(\mathfrak{X}\times G(\mathbb{A}^{(\infty)})\right)/\overline{Z(\mathbb{Q})}$, we write $[z,g]\in Sh(\mathbb{C})$ for the image of $(z,g)\in\mathfrak{X}\times G(\mathbb{A}^{(\infty)})$. A point $x=[z,g]$ is called a *CM point* if $z=(z_\sigma)_{\sigma\in I}\in\mathfrak{X}=(\mathbb{C}-\mathbb{R})^I\subset F\otimes_{\mathbb{Q}}\mathbb{C}$ generates a totally imaginary quadratic extension $M_x=F[z]\subset F\otimes_{\mathbb{Q}}\mathbb{C}$ of F (a CM field over F). We write $\mathfrak{O}=\mathfrak{O}_x$ for the integer ring of M_x and $O_x=\{\alpha\in\mathfrak{O}_x|\alpha\mathcal{L}_z\subset\mathcal{L}_z\}$ (the order of $\mathcal{L}_z = O^*+Oz$). Let $T_x=T_z$ be the (abstract) group scheme $\mathrm{Res}_{\mathfrak{O}_x/\mathbb{Z}}\mathbb{G}_m$ (which is an abstract torus over $\mathbb{Z}[\frac{1}{D}]$ for the discriminant D of O_x). We assume $p\nmid D$ for the prime p (so, we assume that $\mathcal{L}_z\otimes_{\mathbb{Z}}\mathbb{Z}_p=\mathfrak{O}_x\otimes_{\mathbb{Z}}\mathbb{Z}_p$ and $O_x\otimes_{\mathbb{Z}}\mathbb{Z}_p=\mathfrak{O}_x\otimes_{\mathbb{Z}}\mathbb{Z}_p$). The regular representation $\rho_z: T_x(\mathbb{Q})=M_x^\times\to G(\mathbb{Q})$ given by $\left(\begin{smallmatrix}\alpha z\\ \alpha\end{smallmatrix}\right)=\rho_z(\alpha)\left(\begin{smallmatrix}z\\1\end{smallmatrix}\right)$ gives rise to a representation $T_{x/\mathbb{Z}[1/D]}\to G_{/\mathbb{Z}[1/D]}$ because $(1,z)$ gives rise to a basis of $\mathcal{L}_z\otimes_{\mathbb{Z}}\mathbb{Z}[\frac{1}{D}]$. Since $(1,z)$ gives a basis of $\mathcal{L}_z\otimes_{\mathbb{Z}}\widehat{\mathbb{Z}}[\frac{1}{D}]$ over $\widehat{\mathbb{Z}}[\frac{1}{D}]$ for the discriminant D of O_x, we may regard ρ_z as a representation $\widehat{\rho}_z: T_z\to G$ defined over $\widehat{\mathbb{Z}}[\frac{1}{D}]$. Now conjugating by g, we get $\widehat{\rho}_x: T_{x/\mathbb{A}^{(\infty)}}\to G_{/\mathbb{A}^{(\infty)}}$ defined over $\mathbb{A}^{(\infty)}$ given by $\widehat{\rho}_x(\alpha)=g^{-1}\widehat{\rho}_z(\alpha)g$. Here we used the fact that $\mathbb{A}^{(\infty)}=\widehat{\mathbb{Z}}[\frac{1}{D}]\otimes_{\mathbb{Z}}\mathbb{Q}$. We assume that A_x has complex multiplication by $\mathfrak{O}_x$; that is, under the action of $T_x(\widehat{\mathbb{Z}})=\widehat{\mathfrak{O}}_x^\times$ via $\widehat{\rho}_x$, $\widehat{L}\cdot g\cap F^2$ is identified with a fractional ideal of M_x prime to p. On the other hand, the level structure $\eta_x=\eta_z\circ g$ identifies $T(A_x)$ with $\widehat{L}\cdot g=\widehat{L}_{\mathfrak{c}}$ for a polarization ideal $\mathfrak{c}$ prime to p.

We let $G(\mathbb{Q})$ act on the column vector space $V=F^2$ through the matrix multiplication. The action of T_x via ρ_z on V makes V a vector space over M_x of dimension 1. Then the subspace $V_x=V\otimes_{\mathbb{Q}}\mathbb{C}$ on which h_z acts by its restriction $\mu_x=h_z|_{\mathbb{G}_m\times 1}$ is preserved by multiplication by M_x, yielding an isomorphism class Σ_x of representations of M_x. Since the isomorphism class Σ_x is determined by its diagonal entries $\sigma_i: M_x\hookrightarrow\mathbb{C}$, we may identify Σ_x with a formal sum $\sum_i\sigma_i$. Since $\mu_x\times\overline{\mu}_x=h_z$, we find that $\{\sigma_i,c\sigma_i\}_{i=1,\dots,d}$ ($d=[F:\mathbb{Q}]$) is the total set I_x of complex embeddings of M_x into $\mathbb{C}$. Taking the fiber $A=A_x$ at

$x \in Sh(\mathbb{C})$ of the universal abelian scheme over Sh, we find that A has complex multiplication by M_x with CM type (M_x, Σ_x). Let (M'_x, Σ'_x) be the reflex of (M_x, Σ_x) as defined in [ACM] Chapter IV. Then $a \mapsto \prod_{\sigma \in \Sigma'_x} \sigma(a)$ induces a morphism $r_x : T' = \mathrm{Res}_{M'_x/\mathbb{Q}}\mathbb{G}_m \to T_x \subset G$. The field M'_x is by definition the field of definition of $\mu_x : \mathbb{G}_m \to G$. The map r_x can be realized as

$$r_x : T'_x = \mathrm{Res}_{M'_x/\mathbb{Q}}\mathbb{G}_m \xrightarrow{\mu_x} \mathrm{Res}_{M'_x/\mathbb{Q}}T_x \xrightarrow{Norm} T_x.$$

For each $b \in T'_x(\mathbb{A}^{(\infty)}) = (M'_{x,\mathbb{A}^{(\infty)}})^\times$, we have the Artin reciprocity image $[b, M'_x] \in \mathrm{Gal}(M'^{ab}_x/M'_x)$, where M'^{ab}_x is the maximal abelian extension of M'_x. Since $T_x(\mathbb{R})$ is the stabilizer of z, $[z, \gamma g] = [\gamma^{-1}(z), g] = [z, g]$ for $\gamma \in T_x(\mathbb{Q})$, and hence $[z, g] \mapsto [z, r_x(b)g]$ only depends on $[b, M'_x]$ by class field theory. Also we find that elements of $\widehat{\rho}_x(T_x(\mathbb{Q})) \subset G(\mathbb{A}^{(\infty)})$ stabilize the CM point $[z, g]$ under right multiplication. Now we are ready to state Shimura's reciprocity law for the CM point $[z, g]$ (see [ACM] 18.6, 18.8 and [M] II.5.1):

Proposition 3.2. *Let $x = [z, g]$ be a CM point in $Sh(G, \mathfrak{X})_{/\mathbb{Q}}$. Then the point x is M'^{ab}_x-rational, and for any $b \in T'_x(\mathbb{A}^{(\infty)})$, we have*

$$[b^{-1}, M'_x]([z, g]) = [z, g]\widehat{\rho}_x(r_x(b)) = [z, \widehat{\rho}_z(r_x(b))g]$$

and $[z, g]\widehat{\rho}_x(\gamma) = [z, \widehat{\rho}_z(\gamma)g] = [\gamma^{-1}(z), g] = [z, g]$ for any $\gamma \in T_x(\mathbb{Q})$.

3.3 Reciprocity Law for Deformation Spaces

We suppose that p is unramified in $F/\mathbb{Q}$. We start with a fixed CM point $x = [z, g]$ and the associated abelian variety $(A_x, \overline{\lambda}, i, \eta)$ of CM type (M_x, Σ_x). Unless confusion seems likely, we write (M, Σ) for (M_x, Σ_x). We suppose that $i : O \hookrightarrow \mathrm{End}(A_x)$ extends to $i : \mathfrak{O} \hookrightarrow \mathrm{End}(A_x)$ for the integer ring $\mathfrak{O}$ of M. Take $W = W(\overline{\mathbb{F}}_p)$ and consider the reduction A_0 modulo (p) of A_x. Suppose that A_0 is ordinary. Diagonalizing the action of M on $Lie(A_x)_{/W}$, we may assume that $\sigma \in \Sigma$ embeds $\mathfrak{O}$ into W. We write v_i $(i = 1, 2, \cdots)$ for the p–adic place of M associated to $\sigma \in \Sigma$. We write $\Sigma_p = \Sigma_{x,p}$ for the set of places v_i. This condition of A_0 being ordinary is equivalent to

(ord) *Each $v \in \Sigma_p$ is not equivalent to $v \circ c$ for $1 \neq c \in \mathrm{Gal}(M/F)$.*

This implies that all prime factors of p in F split in M. We pick a base of $M_{\mathbb{A}^{(\infty)}}$ over $F_{\mathbb{A}^{(\infty)}}$ and identify $M_{\mathbb{A}^{(\infty)}}$ with $V(\mathbb{A}^{(\infty)}) = F^2_{\mathbb{A}^{(\infty)}}$ so that the fixed lattice in the definition of $\mathcal{P}^{(p)}_K$ is a fractional ideal of M. If $x = [z, g]$, the choice of g is tantamount to the choice of the base of $M_{\mathbb{A}^{(\infty)}}$ over $F_{\mathbb{A}^{(\infty)}}$. Then the polarization λ induces an alternating pairing $\langle \alpha, \beta \rangle = \mathrm{Tr}_{M/\mathbb{Q}}(\delta\alpha c(\beta))$ for the unique non-trivial automorphism c of M/F. Here $\delta \in M$ is a purely imaginary element $\delta = \sqrt{-\Delta}$ for a totally positive element $\Delta \in F$ with $\mathrm{Im}(\sigma(\delta)) > 0$ for all $\sigma \in \Sigma$. We then have $A_x(\mathbb{C}) = L\backslash(M \otimes_{\mathbb{Q}} \mathbb{R})$ for a fractional ideal $L \subset M$ identifying $M \otimes_{\mathbb{Q}} \mathbb{R}$ with $\mathbb{C}^\Sigma$ through $a \otimes t \mapsto (\sigma(a)t)_{\sigma \in \Sigma}$. This induces $\eta^{(p)} = \eta^{(p)}_z \circ g^{(p)} : M \otimes_{\mathbb{Q}} \mathbb{A}^{(p\infty)} \cong V^{(p)}(A_x)$. Since K is maximal at p, we may

assume that $L_p = \mathfrak{O}_p = \mathfrak{O} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ (so, $g_p \in G(\mathbb{Z}_p)$ because $\mathcal{L}_z \otimes_{\mathbb{Z}} \mathbb{Z}_p = \mathfrak{O}_p$ inside M_p). We are dealing with Kottwitz's moduli problem (as in (3.5) in 3.1). By reduction mod p, $\eta^{(p)}$ induces a prime-to-p level structure $\eta_0^{(p)}$ on A_0. Let $(A, \iota_A, \lambda)_{/R}$ be any deformation of $(A_0, \iota_0, \lambda_0)_{/\mathbb{F}}$ ($\mathbb{F} = \overline{\mathbb{F}}_p$) over $\mathrm{Spec}(R)$ for an artinian W–algebra R. Since $A[N]$ for N prime to p is étale over $\mathrm{Spec}(R)$, the level structure $\eta_0^{(p)}$ at the special fiber extends uniquely to a level structure $\eta_A^{(p)}$ on $A_{/R}$. Thus the level structure is insensitive to the deformation of the underlying triple $(A_0, \iota_0, \lambda_0)$. Therefore, for the deformation functor:

$$\widehat{\mathcal{P}}(R) = \left[(A, \iota_A, i, \lambda, \eta_A^{(p)})_{/R} \middle| (A, \iota_A, i, \lambda, \eta_A^{(p)}) \mod \mathfrak{m}_R = (A_0, \iota_0, i_0, \lambda_0, \eta_0^{(p)})\right],$$

the forgetful morphism: $(A, \iota_A, i, \lambda, \eta_A^{(p)})_{/R} \mapsto (A, \iota_A, i, \lambda)_{/R}$ of $\widehat{\mathcal{P}}$ into the original deformation functor $\widehat{\mathcal{P}}_{A_0, i_0, \lambda_0}$ induces an isomorphism of functors; so, they have identical deformation spaces.

We consider the Serre–Tate deformation space $\widehat{S}$ representing $\widehat{\mathcal{P}}$. We take the Kottwitz model $Sh^{(p)}(G, \mathfrak{X})_{/W}$ over W and consider $x = [z, g]$ as a point of $Sh^{(p)}(G, \mathfrak{X})(W)$. Let $Sh^{ord} = Sh^{ord}(G, \mathfrak{X}) = Sh^{(p)}(G, \mathfrak{X})[\frac{1}{E}]$, that is, we invert over $Sh^{(p)}$ a lift E of a power of the Hasse invariant H. The formal completion Sh^{ord}_{∞} of Sh^{ord} along $Sh^{ord}_1 = Sh^{ord} \otimes_W \mathbb{F}$ is uniquely determined independently of the choice of E and gives the ordinary locus of $\mathbf{A}$. Writing Sh^{ord} for $Sh^{(p)}[\frac{1}{E}]$ is therefore a slight abuse of notation. We assume that x gives rise to a closed point of Sh^{ord}_{∞}.

Since $\widehat{S}$ carries the universal deformation $\underline{\mathcal{A}} = (\mathcal{A}, i, \lambda, \eta^{(p)})_{/\widehat{S}}$ which is an element of $\mathcal{P}^{(p)}(\widehat{S})$, by the universality of the Shimura variety, we have an inclusion

$$\varphi : \widehat{S} \hookrightarrow Sh^{ord}_{\infty}(G, \mathfrak{X}) \text{ such that } \varphi^* \underline{\mathbf{A}}^{ord} = \underline{\mathcal{A}} \tag{3.14}$$

for the universal quadruple $\underline{\mathbf{A}}^{ord}$ over $Sh^{ord}_{\infty}(G, \mathfrak{X})$. Since $\eta^{(p)}$ lacks the information about $A_0[p^{\infty}]$, the identification of $\widehat{S}$ with $\widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}} \mathfrak{d}^{-1}$ is not yet specified.

Since $\widehat{S}$ is connected, we have the connected component $V_{/W} \subset Sh^{(p)}_{/W}$ containing the image of φ. Then $V_{/\mathbb{F}} = V \otimes_W \mathbb{F}$ is the connected component containing the point x carrying $(A_0, \iota_0, i_0, \lambda_0, \eta_0^{(p)})$. We can lift the morphism φ to the Igusa tower over the formal completion V^{ord}_{∞} of $V[\frac{1}{E}]_{/W}$ along $V^{ord}_{/\mathbb{F}} = V[\frac{1}{E}]_{/\mathbb{F}}$. The Igusa tower $Ig_{/V^{ord}_{\infty}}$ studied in [PAF] Chapter 8 is given by $\mathrm{Isom}_O(F_p/O_{p/V^{ord}_{\infty}}, \mathbf{A}^{ord}[p^{\infty}]^{et}_{/V^{ord}_{\infty}})$ for the universal abelian scheme $\mathbf{A}^{ord}$ over V^{ord}_{∞}. Strictly speaking, in [PAF], we studied principally the Igusa tower on the neutral component of Sh^{ord}_{∞}, but here we study it over V^{ord}_{∞}, because we need to study it over the component containing the fixed CM point $x \in Sh^{ord}_{\infty}$. We can also write $Ig_{/V^{ord}_{\infty}} \cong \mathrm{Isom}_O(\mu_{p^{\infty}} \otimes_{\mathbb{Z}} \mathfrak{d}^{-1}_{/V^{ord}_{\infty}}, \mathbf{A}^{ord}[p^{\infty}]^{\circ}_{/V^{ord}_{\infty}})$ for the connected component $\mathbf{A}^{ord}[p^{\infty}]^{\circ}$ of $\mathbf{A}^{ord}[p^{\infty}]$ (Cartier duality). Let $\mathbf{p} = \prod_{v \in \Sigma_p} \mathfrak{p}_v$ for the prime $\mathfrak{p}_v$ associated to the valuation $v \in \Sigma_p$. Then $\bigcup_j (\mathbf{p}^c)^{-j}/\mathfrak{O} \cong A_x[(\mathbf{p}^c)^{\infty}]$, which induces

$$\eta_p^{ord} : O_p \cong \mathfrak{O}_{\mathbf{p}^c} \cong \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p/\mathbb{Z}_p, A_x[(\mathbf{p}^c)^{\infty}]) = TA_x[p^{\infty}]^{et}.$$

We can therefore extend $\eta^{(p)}$ to

$$\eta^{ord} : O_p \times (M_x \otimes_{\mathbb{Q}} \mathbb{A}^{(p\infty)}) \cong TA_x[p^\infty]^{et} \times V^{(p)}(A_x).$$

Let $\widehat{\mathcal{K}}$ be the field of fractions of W. Over the field $\widehat{\mathcal{K}}[\mu_{p^\infty}]$, we can further extend η_p^{ord} to $\eta_p : O_p \times O_p = \mathfrak{O}_p \cong TA_x[p^\infty]$ by identifying $\bigcup_j \mathbf{p}^{-j}/\mathfrak{O} \cong A_x[\mathbf{p}^\infty]$. This choice is tantamount to the choice of g_p which brings the base of L_p to the base given by the two idempotent $1_{\mathbf{p}} := (1,0)$ of $\mathfrak{O}_{\mathbf{p}}$ and $1_{\mathbf{p}^c} := (0,1)$ of $\mathfrak{O}_{\mathbf{p}^c}$ in $\mathfrak{O}_{\mathbf{p}} \times \mathfrak{O}_{\mathbf{p}^c} = O_p \times O_p$. We write $\eta = \eta_p \times \eta^{(p)}$ and $\eta^{ord} = \eta_p^{ord} \times \eta^{(p)}$.

We can think of the deformation of $(A_0, i, \iota_0, \lambda_0, \eta_0^{ord})_{/\overline{\mathbb{F}}_p}$ for $\eta_0^{ord} = \eta^{ord}$ mod p. The p-part of the level p–structure η_0^{ord} provides the canonical identification of the deformation space $\widehat{S}$ with $\widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}} \mathfrak{d}^{-1}$. For any complete local W-algebra C and any deformation $A_{/C}$ of A_0, $A[p^\infty]^{et}$ is étale over $\mathrm{Spec}(C)$; so, again the deformation is insensitive to the ordinary level structure. Thus we get a canonical immersion:

$$\varphi_{\eta_0^{ord}} : \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}} \mathfrak{d}^{-1} \hookrightarrow Ig \ \text{ such that } \varphi^* \underline{\mathbf{A}}^{ord} = \underline{\mathcal{A}}^{ord}. \tag{3.15}$$

Here $\underline{\mathbf{A}}^{ord}$ (resp. $\underline{\mathcal{A}}^{ord}$) denotes the universal ordinary quadruple over Ig (resp. the universal quintuple over $\widehat{S}$).

The abelian variety $\underline{A}_x = (A_x, i, \lambda, \eta^{ord})$ of CM type (M, Σ) is the fiber of $\underline{\mathcal{A}}^{ord}$ at a point $q_0 \in \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}} \mathfrak{d}^{-1}(W)$. Here q_0 is an O_p–bilinear form on $TA_0[p^\infty]^{et}$. Since any element $a \in i_0(\mathfrak{O}) \subset \mathrm{End}(A_{0/\overline{\mathbb{F}}_p})$ can be lifted to A_x, by the Serre–Tate theorem, we have

$$q_0(i(\alpha)y, y') = q_0(y, i(\overline{\alpha})y') \ (\alpha \in \mathfrak{O}), \tag{3.16}$$

where $\overline{\alpha} = c(\alpha)$ for $1 \neq c \in \mathrm{Gal}(M/F)$. This forces q_0 to be 0, that is, $q_0(y, y')$ is the constant 1 of the group $\widehat{\mathbb{G}}_m \otimes \mathfrak{d}^{-1}$ identically, because q_0 is also O_p–linear and $O_p = \mathfrak{O}_{\mathbf{p}}$. Indeed, the connected étale exact sequence of $A_x[p^\infty]$ does split by complex multiplication, and hence $q_0 = 1$ by definition.

We now compute the effect of the isogeny $i_0(\alpha) : A_0 \to A_0$ ($A_0 = A_{x/\mathbb{F}}$ and $\alpha \in \mathfrak{O}$) on the deformation space $\widehat{S}$. Pick a deformation $A_{/R}$ of $A_0 = A_{x/\mathbb{F}}$ for an artinian $R \in CL_{/W}$, and we look into the following diagram with exact rows:

$$\begin{array}{ccccc} \mathrm{Hom}(TA_0[p^\infty]^{et}, \widehat{\mathbb{G}}_m(R)) & \hookrightarrow & A[p^n](R) & \xrightarrow{\pi} & A_0[p^n]^{et}(R) \\ \alpha\uparrow & & & & \downarrow\alpha^{-c} \\ \mathrm{Hom}(TA_0[p^\infty]^{et}, \widehat{\mathbb{G}}_m(R)) & \hookrightarrow & A[p^n](R) & \xrightarrow{\pi} & A_0[p^n]^{et}(R). \end{array} \tag{3.17}$$

Take $u = \varprojlim_n u_n \in TA_0[p^\infty]^{et}$, and lift it to $v = \varprojlim_n v_n$ for $v_n \in A(R)$ (but $v_n \notin A[p^n]$). Then

$$q(u) = \varprojlim_n q_n(u_n) \in \mathrm{Hom}(TA_0[p^\infty]^{et}, \widehat{\mathbb{G}}_m) \ \text{ for } q_n(u_n) = \text{“}p^n\text{”} v_n.$$

Note that the identification of $\mathrm{Hom}(TA_0[p^\infty]^{et}, \widehat{\mathbb{G}}_m)$ with the formal group A° of A is given by the Cartier duality composed with the polarization; so, if α is

prime to p, α sends q to $q^{\alpha^{1-c}} = \varprojlim_n \alpha(\text{“}p^n\text{”}\alpha^{-c}(v_n))$.Thus the effect of α on q is given by $q \mapsto q^{\alpha^{1-c}}$. Once the identification of $\widehat{S}$ with $\widehat{\mathbb{G}}_m \otimes \mathfrak{d}^{-1}$ is given (that is, a level p^∞–structure $\eta_0^{ord} : F_p/O_p \cong A_0[p^\infty]^{et}$ is chosen), $\alpha \in \mathfrak{O}$ prime to p acts on the coordinate t (of $\widehat{\mathbb{G}}_m \otimes \mathfrak{d}^{-1}$) by $t \mapsto t^{\alpha^{1-c}}$.

Write $\widehat{\mathbb{Z}_p[O]}$ for the formal completion of $\mathbb{Z}_p[O]$ at the origin $1 \in S(\mathbb{F})$ for $S = \mathbb{G}_m \otimes \mathfrak{d}^{-1}$. Identify $\widehat{\mathbb{Z}_p[O]}$ with the ring made up of series: $\sum_{\xi \in O} a(\xi) t^\xi$ for $a(\xi) \in \mathbb{Z}_p$ (here $\widehat{\mathbb{Z}_p[O]} \cong \mathbb{Z}_p[[(t^{\xi_1} - 1), \dots, (t^{\xi_d} - 1)]]$ for a base $\xi_1, \dots, \xi_d$ of O over $\mathbb{Z}$). Let $T = \mathrm{Res}_{O/\mathbb{Z}}\mathbb{G}_m$. Since $\widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}} \mathfrak{d}^{-1} = \mathrm{Spf}(\widehat{\mathbb{Z}_p[O]})$ for the completion at the origin $1 \in S(\mathbb{F})$ for $S = \mathbb{G}_m \otimes_{\mathbb{Z}} \mathfrak{d}^{-1}$, $O_p^\times = T(\mathbb{Z}_p)$ acts on $\widehat{S}$ as follows: We have a character $O \to \widehat{\mathbb{Z}_p[O]}^\times$ with $s \mapsto t^s$. Then the variable change $t \mapsto t^s$ induces an automorphism of the formal group $\widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}} \mathfrak{d}^{-1}$, and all O–linear automorphisms are obtained in this way. On the points q of the formal scheme $\widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}} \mathfrak{d}^{-1}(W)$, the action induces $q \mapsto q^s$.

The inclusion $O \hookrightarrow \mathfrak{O}$ induces an identification of p-adic rings $\mathfrak{O}_{\mathbf{p}}$ with O_p which we fix in this paper and use always in the sequel. Note that $\mathfrak{O}_p = \mathfrak{O}_{\mathbf{p}} \times \mathfrak{O}_{\mathbf{p}^c}$. This same inclusion: $O \hookrightarrow \mathfrak{O}$ induces an inclusion of $\mathbb{Z}_{(p)}$-tori $T \hookrightarrow T_x$. Let $\mathcal{T} := T_x/T$. By the identification above, the map $\mathfrak{O}_{(p)}^\times \to \mathfrak{O}_{\mathbf{p}}^\times$ given by $\alpha \mapsto \alpha^{1-c}$ induces an injective homomorphism

$$\mathcal{T}(\mathbb{Z}_{(p)}) \to O_p^\times = T(\mathbb{Z}_p). \tag{3.18}$$

Thereby, the action of $T(\mathbb{Z}_p)$ on $\widehat{S}$ and that of $\mathcal{T}(\mathbb{Z}_{(p)})$ are compatible. The torus $\mathcal{T}(\mathbb{Z}_{(p)})$ is isomorphic to the image (under $\widehat{\rho}_x$) of $T_x(\mathbb{Z}_{(p)})$ in $\overline{\mathcal{E}}(G, \mathfrak{X})$, and its action on $\widehat{S}$ factors through the action of $\overline{\mathcal{E}}(G, \mathfrak{X})$ on $Ig(G, \mathfrak{X})$ via (3.15). The O_p–module structure of $\widehat{S}$ given by $t \mapsto t^s$ therefore commutes with the isogeny action of T_x on $\widehat{S}$.

By the level structure η_p^{ord} (and its dual), we identify A_x° with $\widehat{\mathbb{G}}_m \otimes \mathfrak{d}^{-1}$ and $A_x[p^\infty]^{et}$ with F_p/O_p. In this way, we may identify the torus T with the diagonal torus T^δ of SL_2. The action of $t \in T(\mathbb{Z}_p) = O_p^\times$ on the quotient $A_x[p^\infty]^{et}$ is given by the multiplication by $t \in O_p^\times$, and hence the two tori are identified by $T(\mathbb{Z}_p) \ni t \mapsto \left(\begin{smallmatrix} t^{-1} & 0 \\ 0 & t \end{smallmatrix}\right) \in T^\delta(\mathbb{Z}_p) \subset SL_2(O_p)$ taking the lower diagonal entry of T^δ as the coordinate of the quotient.

Change of level structure $\eta^{ord} \mapsto \eta^{ord} \circ a$ for $a \in T(\mathbb{Z}_p)$ is given by the action of an element $\left(\begin{smallmatrix} a^{-1} & 0 \\ 0 & a \end{smallmatrix}\right)$ of the diagonal torus in $T^\delta(\widehat{\mathbb{Z}}) \subset SL_2(F_{\mathbb{A}}^{(\infty)})$, which moves the point $x \in Ig$ to a different point $y = a(x)$ étale over the image of x in Sh_K (for $K = G(\widehat{\mathbb{Z}})$) and brings the canonical coordinate at x to that of the image y. In other words, the action of a by the change of η^{ord} to $\eta^{ord} \circ a$:

$$(A, i, \iota, \lambda, \eta^{ord}) \mapsto (A, i, \iota, \lambda, \eta^{ord} \circ a)$$

sends the deformation space $\widehat{S}$ centered at $(A_0, i_0, \iota_0, \lambda_0, \eta_0^{ord})$ on x to the different deformation space $\widehat{S}_a$ centered at $(A_0, i_0, \iota_0, \lambda_0, \eta_0^{ord} \circ a)$ on $y = a(x)$ (as long as the two quintuples are not isogenous). The action of $\widehat{\rho}_x : T_x(\mathbb{Q}) \hookrightarrow G(\mathbb{A}^{(\infty)})$

and the action of $T^\delta(\mathbb{Z}_p)$ via change of level structure are compatible, since the intersection of the images of the two groups in $\overline{\mathcal{E}}(G,\mathfrak{X})$ is trivial (and the p-component $T^\delta(\mathbb{Z}_p)$ and the projection of $\widehat{\rho}_x(T_x(\mathbb{Q}))$ to the p-component $G(\mathbb{Q}_p)$ are both diagonal).

By the definition of $\widehat{\rho}_x$ given above, we have $\alpha\circ\eta^{ord}=\eta^{ord}\circ\widehat{\rho}_x(\alpha)$ automatically. If $\widehat{\rho}_x(\alpha)\in T_x(\mathbb{Z}_{(p)})$, it acts on $Ig(G,\mathfrak{X})$ as an automorphism, while $\widehat{\rho}_x(\alpha)\in T_x(\mathbb{Q})$ may expand or shrink $\mathcal{O}_{Ig}$ because it would induce a morphism like the Frobenius map on the special fiber. The action of $\tau(\widehat{\rho}_x(\alpha))$ sends the canonical coordinate t into $t^{\alpha^{1-c}}$ (identifying α with its image in $\mathfrak{O}_{\mathfrak{p}}=\prod_{\mathfrak{p}\in\Sigma_p}\mathfrak{O}_{\mathfrak{p}}=O_p$).

Lemma 3.3. *If $h\in\overline{\mathcal{E}}(G,\mathfrak{X})$ fixes x and is an image of $\widetilde{h}\in G(\mathbb{A}^{(\infty)})$ with $\widetilde{h}_p\in G(\mathbb{Z}_p)$, then it is induced by an endomorphism $\alpha\in\mathrm{End}_F^{\mathbb{Q}}(A_x)=M$, and h induces $t\mapsto t^{\alpha^{1-c}}$.*

Proof. Since h fixes x, it has to preserve Ig and $\widehat{S}$ by the irreducibility of Ig (a theorem of Ribet; see [PAF] Theorem 4.21). Take $\widetilde{h}\in G(\mathbb{A}^{(\infty)})$ with $\widetilde{h}_p\in G(\mathbb{Z}_p)$ projecting down to h. Thus $\widetilde{h}_p$ is in the upper triangular Borel subgroup $B(\mathbb{Z}_p)$ by [PAF] Corollary 4.22. The Borel subgroup B is upper triangular with respect to the coordinate given by η^{ord} (and its dual) under which we identified T and T^δ. By the universality of $Sh_{/\mathbb{Q}}$, there exists an isogeny $\alpha:A_x\to A_x$ such that $\eta^{(p)}\circ\widetilde{h}=\alpha\circ\eta^{(p)}$ and $\eta^{ord}\circ\widetilde{h}=\alpha\circ\eta^{ord}$. Since $\alpha\in\mathrm{End}(A_x)=\mathfrak{O}$, we have $\widetilde{h}=\widehat{\rho}_x(\alpha)$ modulo $Z(\mathbb{Q})$, and therefore, h is the image of $\widehat{\rho}_x(\alpha)$ in $\overline{\mathcal{E}}(G,\mathfrak{X})$. The assertion follows from the above discussion. □

Summing up the above discussion, we have the following fact:

Proposition 3.4. *Let $a\in T(\mathbb{Z}_p)$ for $T=\mathrm{Res}_{O/\mathbb{Z}}\mathbb{G}_m$. Then the action:*

$$(A,i,\iota,\lambda,\eta^{ord})\mapsto(A,i,\iota,\lambda,\eta^{ord}\circ a)$$

induces an isomorphism: $\widehat{S}_a\cong\widehat{S}$ sending $f=\sum_\xi c(\xi)t'^\xi\in\Gamma(\widehat{S}_a,\mathcal{O}_{\widehat{S}_a})$ to $f\circ a(t)=\sum_\xi c(\xi)t^\xi\in\Gamma(\widehat{S},\mathcal{O}_{\widehat{S}})$, where t (resp. t') is the canonical coordinate of $\widehat{S}$ (resp. $\widehat{S}_a$). For an isogeny $\alpha\in\mathrm{End}_F^{\mathbb{Q}}(A_x)$ regarded as an element of $T_x(\mathbb{Q})$ by $\widehat{\rho}_x$, we have $t\circ\tau(\widehat{\rho}_x(\alpha))=t^{\alpha^{1-c}}$.

Here is how to relate the characteristic 0 Shimura variety Sh_{U_n} of level $\Gamma_1(p^n)$ with the characteristic p Igusa tower of level p^n. A more localized argument can be found in [H09]. Let $\mathcal{W}=i_p^{-1}(W(\mathbb{F}))\subset\overline{\mathbb{Q}}$ (a strict henselization of $\mathbb{Z}_{(p)}$ inside $\overline{\mathbb{Q}}$). We regard $Sh^{(p)}$ as a (pro-)scheme over $\mathcal{W}$. Let $\mathcal{K}$ (resp. $\widehat{\mathcal{K}}$) be the field of fractions of $\mathcal{W}$ (resp. W). Consider the quotient $Sh_{U_\infty/\mathcal{K}}=Sh(G,\mathfrak{X})/U_\infty$ for the stabilizer $U_\infty=U_{p,\infty}\subset G(\mathbb{Z}_p)$ of the infinity cusp. Thus $U_\infty=\bigcap_n U_n$ and U_n consists of elements $g\in G(\mathbb{Z}_p)$ with $g\equiv\left(\begin{smallmatrix}1&*\\0&*\end{smallmatrix}\right)\mod p^n$. Thus $Sh_{U_\infty/\mathcal{K}}=\varprojlim_n Sh_{U_n/\mathcal{K}}$, and $Sh_{U_n/\mathcal{K}}$ $(n=1,2,\ldots,\infty)$ can be written as the scheme representing the functor

$$\mathrm{Isom}_O(\mu_{p^n}\otimes\mathfrak{d}^{-1}_{/Sh^{(p)}_{/\mathcal{K}}},\mathbf{A}[p^n]_{/Sh^{(p)}_{/\mathcal{K}}}),$$

because the level p-structure $\eta_p \mod U_n$ for a test object $(A,\lambda,\eta \mod U_\infty)_{/S}$ can be given by an O-linear closed immersion: $\mu_{p^\infty}\otimes\mathfrak{d}^{-1}\hookrightarrow A[p^\infty]$ ($\Leftrightarrow O_p(1)=T_p(\mu_{p^\infty}\otimes\mathfrak{d}^{-1})\hookrightarrow T_pA$) if $n=\infty$ and an O-linear closed immersion: $\mu_{p^n}\otimes\mathfrak{d}^{-1}_{/S}\hookrightarrow A[p^n]_{/S}$ if $n<\infty$. Here $\mathrm{Isom}_O(\mathcal{G}_{/B},\mathcal{H}_{/B})$ for finite flat O-modules (or Barsotti-Tate O-modules) $\mathcal{G}$ and $\mathcal{H}$ over a base B is a contravariant functor from B-SCH to $SETS$ which assigns a B-scheme R the set of O-linear closed immersions $\mathcal{G}\times_B R\hookrightarrow\mathcal{H}\times_B R$ defined over R. By the theory of the Hilbert scheme (e.g. [PAF] 6.1.5–6), the above functor $\mathrm{Isom}_O(\mathcal{G}_{/B},\mathcal{H}_{/B})$ is representable by a scheme quasi-finite affine over B if $\mathcal{G}$ and $\mathcal{H}$ is finite flat over B, because flatness and projectivity of $\mathcal{G}_{/B}$ and $\mathcal{H}_{/B}$ (following from finiteness) is the requirement of representability by the Hilbert scheme. Put $I_n=\mathrm{Isom}_O(\mu_{p^n}\otimes\mathfrak{d}^{-1}_{/Sh^{(p)}_{/\mathcal{W}}},\mathbf{A}[p^n]_{/Sh^{(p)}_{/\mathcal{W}}})$.

We perform the same construction over the category of (p-adic) formal schemes over Sh^{ord}_∞. We then get the formal completion $Ig_n(G,\mathfrak{X})$ of I_n along its special fiber over $\mathbb{F}$:

$$\begin{aligned}Ig_n(G,\mathfrak{X})&=\mathrm{Isom}_O(\mu_{p^n}\otimes\mathfrak{d}^{-1}_{/Sh^{ord}_\infty},\mathbf{A}[p^n]_{/Sh^{ord}_\infty})\\&\cong\mathrm{Isom}_O(\mu_{p^n}\otimes\mathfrak{d}^{-1}_{/Sh^{ord}_\infty},\mathbf{A}[p^n]^{mult}_{/Sh^{ord}_\infty})\\&\cong\mathrm{Isom}_O(\mathbf{A}[p^n]^{et}_{/Sh^{ord}_\infty},p^{-n}O/O_{/Sh^{ord}_\infty})\quad\text{(Cartier duality)}\end{aligned}$$

for the multiplicative part $\mathbf{A}[p^n]^{mult}_{/Sh^{ord}_\infty}$ of $\mathbf{A}^{ord}[p^n]_{/Sh^{ord}_\infty}$ (which is only well defined over the formal scheme Sh^{ord}_∞). The Igusa tower Ig/V^{ord}_∞ we discussed earlier is the pull back of the full Igusa tower $Ig=\varprojlim_n Ig_n(G,\mathfrak{X})$ to the integral formal subscheme $V^{ord}_\infty\subset Sh^{ord}_\infty$. Though $Ig_n(G,\mathfrak{X})$ is étale finite over Sh^{ord}_∞, $I_n/Sh^{(p)}$ is étale quasi-finite over $Sh^{(p)}$ (because elements I_n over non-ordinary locus in characteristic 0 fiber of I_n does not extend to characteristic p fiber). In any case, by definition $I_n\otimes_{\mathcal{W}}\mathcal{K}=Sh_{U_n/\mathcal{K}}$.

We look at the normalization $\mathcal{I}_n=\mathcal{I}_n(G,\mathfrak{X})$ of the scheme $Sh^{ord}_{/\mathcal{W}}$ in $Sh_{U_\infty/\mathcal{K}}$. Since $I_n/Sh^{(p)}$ is étale quasi-finite, I_n is normal. Thus $\mathcal{O}_{I_n}\supset\mathcal{O}_{\mathcal{I}_n}$ and $\mathcal{O}_{I_n,x}\supset\mathcal{O}_{\mathcal{I}_n,x}$ at all closed points $x\in I_n$; in other words, $\mathcal{O}_{I_n}$ is a localization of $\mathcal{O}_{\mathcal{I}_n}$ over the topological space of I_n; so, we have an open immersion $I_n\hookrightarrow\mathcal{I}_n$ because of $I_n\otimes_{\mathcal{W}}\mathcal{K}=Sh_{U_n/\mathcal{K}}$.

Since V^{ord}_∞ is a connected component of Sh^{ord}_∞, Ig/V^{ord}_∞ (defined earlier) is a closed subscheme of $Ig(G,\mathfrak{X})$ (and actually a connected component of $Ig(G,\mathfrak{X})$ by a result of Ribet). The n-th layer

$$\begin{aligned}Ig_n(G,\mathfrak{X}):=\mathrm{Isom}_O(\mu_{p^n}\otimes\mathfrak{d}^{-1}_{/Sh^{ord}_\infty},\mathbf{A}^{ord}[p^n]^{mult}_{/Sh^{ord}_\infty})\\\overset{(*)}{\cong}\mathrm{Isom}_O(O/p^nO_{/Sh^{ord}_\infty},\mathbf{A}^{ord}[p^n]^{et}_{/Sh^{ord}_\infty})\end{aligned}$$

is finite étale over the formal scheme Sh^{ord}_∞, and $Ig(G,\mathfrak{X})=\varprojlim_n Ig_n(G,\mathfrak{X})$. The isomorphism $(*)$ is given by sending an isomorphism of the left-hand-side to its Cartier dual inverse. Each layer $Ig_n(G,\mathfrak{X})$ is finite over Sh^{ord}_∞. As we have seen, $Ig_n(G,\mathfrak{X})$ is the formal completion $\widehat{I}_n$ of I_n along its special fiber $I_{n/\mathbb{F}}=I_n\otimes_{\mathcal{W}}\mathbb{F}=Ig_n(G,\mathfrak{X})_{/\mathbb{F}}\subset\mathcal{I}_{n/\mathbb{F}}$ and hence is an open formal subscheme

of the formal completion $\widehat{\mathcal{I}}_n$ of $\mathcal{I}_n$ along its special fiber. In summary, the special fiber $\mathcal{I}_{n/\mathbb{F}}$ over $\mathbb{F}$ has $Ig_n(G,\mathfrak{X})_{/\mathbb{F}}$ as an open subscheme of maximal dimension, the formal scheme $Ig(G,\mathfrak{X})_{/W}$ is the formal completion of $I_{\infty/\mathcal{W}}$ along $Ig(G,\mathfrak{X})_{/\mathbb{F}}$, and $\widehat{I}_\infty$ is an open formal subscheme of the formal completion $\widehat{\mathcal{I}}_\infty$ along its special fiber.

The quadruple $\underline{A}_x = (A_x, i, \overline{\lambda}, \eta)$ of CM type (M,Σ) gives a unique point $x \in Sh(G,\mathfrak{X})(\mathcal{K}[\mu_{p^\infty}])$ and the ordinary quadruple $\underline{A}_x = (A_x, i, \overline{\lambda}, \eta^{ord})$ gives a unique integral point $x \in \mathcal{I}_\infty(\mathcal{W})$. Consider the $\mathcal{W}$–point $x \in \mathcal{I}_\infty$. Then writing $\mathcal{O}_{x/\mathcal{W}}$ for the stalk at the closed point $\widetilde{x} = (x \mod \mathfrak{m}_{\mathcal{W}}) \in \mathcal{I}_\infty(\mathbb{F})$ (for the maximal ideal $\mathfrak{m}_{\mathcal{W}} \subset \mathcal{W}$), we have an isomorphism $\widehat{S} = \mathrm{Spf}(\widehat{\mathcal{O}}_{x/W})$, where $\widehat{\mathcal{O}}_x = \varprojlim_n \mathcal{O}_x/\mathfrak{m}_x^n$ for the maximal ideal $\mathfrak{m}_x$ of $\mathcal{O}_{x/\mathcal{W}}$. Since $\widehat{S} = \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}} \mathfrak{d}^{-1}$, the endomorphism ring $\mathrm{End}(\widehat{S})$ as a formal group is isomorphic to $M_d(\mathbb{Z}_p)$. By $t \mapsto t^a$, $a \in O_p$ acts on $\widehat{S}$; so, we write $\mathrm{End}_O(\widehat{S})$ for the commutant of $\mathrm{Res}_{O_p/\mathbb{Z}_p}\mathbb{G}_a(\mathbb{Z}_p)$ in $\mathrm{End}(\widehat{S})$. Then $\mathrm{End}_O(\widehat{S}) \cong O_p$. For each $f \in \mathcal{O}_{\widehat{S}}$ and $a \in O_p^\times$, we write $a(f) = f \circ a$. Recall the torus $\mathcal{T}$ defined by T_x/T. We may consider the reversed exact sequence of tori over $\mathbb{Z}_{(p)}$: $1 \to \mathcal{T} \to T_x \xrightarrow{norm} T \to 1$, where the map "*norm*" is induced by the norm map: $\mathfrak{O}_{(p)}^\times \to O_{(p)}^\times$. The character $T_x \ni \alpha \mapsto \alpha^{1-c}$ factors through $\mathcal{T}$ with kernel T. The inclusion $\widehat{\rho}_x : T_x \hookrightarrow G$ (over $\mathbb{A}^{(\infty)}$) induces $\rho : M_x^\times \to \overline{\mathcal{E}}(G,\mathfrak{X})$ (and by abusing the symbol, we have $\rho : \mathcal{T}(\mathbb{Q}) \cong \widehat{\rho}_x(T_x(\mathbb{Q}))/Z(\mathbb{Q}) \to \overline{\mathcal{E}}(G,\mathfrak{X})$). Let $\mathcal{D}$ be the stabilizer in $\overline{\mathcal{E}}(G,\mathfrak{X})$ of the generic point of the irreducible component of $Ig(G,\mathfrak{X})_{/\mathbb{F}}$ containing x. As seen in [PAF] Corollary 4.22, we may identify $\mathcal{D}$ with

$$\mathcal{D} = \frac{\left\{h \in \mathcal{G}(G,\mathfrak{X}) \middle| h_p \text{ is upper triangular and } \det(h) \in \overline{\mathbb{Q}_p^\times Z(\mathbb{Q})Z(\mathbb{R})_+}\right\}}{\overline{Z(\mathbb{Q})}G(\mathbb{R})_+}.$$

Here $\mathcal{D}$ contains $\widehat{\rho}_x(T_x(\mathbb{Q}))$, $\widehat{\rho}_x(h)_p$ ($h \in M_{x,p}^\times$) is in the diagonal torus in $\mathcal{D}$ and $\widehat{\rho}_x(\mathcal{T}(\mathbb{Q}))$ is a discrete subgroup of $\mathcal{D}$.

Corollary 3.5. *If $\alpha \in \mathfrak{O}_{(p)}^\times$ ($\cong T_x(\mathbb{Z}_{(p)})$), then $\tau(\widehat{\rho}_x(\alpha))$ fixes x and preserves $\mathcal{O}_{x/\mathcal{W}}$. If $\tau(h)$ for $h \in \overline{\mathcal{E}}(G,\mathfrak{X})$ fixes x, then h is in the image of $M_x^\times$. Moreover writing ι for the embedding $\mathcal{O}_{x/\mathcal{W}} \hookrightarrow \mathcal{O}_{\widehat{S}}$ associated to $(A_x, \overline{\lambda}, i, \eta^{ord})$, we have $\alpha^{1-c}(\iota(f)) = \iota({}^{\tau(\widehat{\rho}_x(\alpha))}f)$. The effect of $\widehat{\rho}_x(\alpha) \in \overline{\mathcal{E}}(G,\mathfrak{X})$ (for $\alpha \in M^\times$) on the canonical coordinate $t \in \widehat{S}$ is given by $t \mapsto t^{\alpha^{1-c}}$.*

Since the action of $T_x(\mathbb{Q})$ ($x = [z,g]$) on $\widehat{S}$ factors through $T_x(\mathbb{Q})/T(\mathbb{Q}) = \mathcal{T}(\mathbb{Q})$ by $\alpha \mapsto \alpha^{1-c}$, we regard $\rho(\mathcal{T}(\mathbb{Z}_{(p)}))$ as the isotropy group in $\overline{\mathcal{E}}(G,\mathfrak{X})$ of $\widehat{S} \hookrightarrow Ig(G,\mathfrak{X})$ (by Lemma 3.3). However we need to keep in mind the fact that the image of $\alpha \in T_x(\mathbb{Z}_{(p)})$ in $\mathcal{T}(\mathbb{Z}_{(p)})$ acts on $\widehat{S}$ through the action of $\tau(\rho(\alpha))$ whose action on $q \in \widehat{S}$ is given by $q \mapsto q^{\alpha^{1-c}}$.

3.4 Rigidity for Formal p-Divisible Groups

We set up some notation to quote a result of Chai (Theorem 4.2 in [C3] and Theorem 6.6 in [C4]). Let k be an algebraically closed field of characteristic

$p > 0$. Let $\widehat{\mathbf{T}}$ be a finite dimensional p-divisible smooth formal group over k. Let $E_{\mathbb{Z}_p} = \mathrm{End}(\widehat{\mathbf{T}})$, and let $E = E_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Denote by $\underline{E}^\times$ the linear algebraic group over $\mathbb{Q}_p$ whose $\mathbb{Q}_p$-rational points is $E^\times$. Let $\mathbf{G}$ be a connected linear algebraic group over $\mathbb{Q}_p$, and let $\boldsymbol{\rho} : \mathbf{G} \to \underline{E}^\times$ be a homomorphism of algebraic groups over $\mathbb{Q}_p$. Let $\mathbf{G}(\mathbb{Z}_p) = \boldsymbol{\rho}^{-1}(E^\times_{\mathbb{Z}_p})$. The compact p-adic group $\mathbf{G}(\mathbb{Z}_p)$ operates on the p-divisible formal group $\widehat{\mathbf{T}}$ via $\boldsymbol{\rho}$.

Theorem 3.6 (C.-L. Chai)**.** *Assume that the trivial representation is not a subquotient of the linear representation $(\boldsymbol{\rho}, E)$. Suppose that $\widehat{\mathcal{Z}}$ is an integral closed formal subscheme of the p-divisible formal group $\widehat{\mathbf{T}}$ which is closed under the action of an open subgroup U of $\mathbf{G}(\mathbb{Z}_p)$. Then $\widehat{\mathcal{Z}}$ is stable under the group law of $\widehat{\mathbf{T}}$ and hence is a p-divisible smooth formal subgroup of $\widehat{\mathbf{T}}$.*

A proof of this fact is given as [C4] Theorem 6.6 (see also [C3] Theorem 4.2). We now interpret this result in the following setting. In the sequel, $k = \mathbb{F} = \overline{\mathbb{F}}_p$. We keep the notation introduced in the previous subsection. In particular, we recall the torus $\mathcal{T}$ fixing the CM point x. Let L be a $\mathbb{Z}_p$–free module of finite rank on which $\mathcal{T}(\mathbb{Z}_p)$ acts by a $\mathbb{Q}_p$–rational linear representation. We take $(\mathbf{G}_{/\mathbb{Q}_p}, \widehat{\mathbf{T}}_{/k})$ in the theorem to be $(\mathcal{T}_{/\mathbb{Q}_p}, \widehat{\mathbf{T}}_{L/\mathbb{F}} = \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} L)$. Then $\widehat{\mathbf{T}}_{L/\mathbb{F}}$ inherits the action of $\mathcal{T}$ from L; so, we get $\boldsymbol{\rho} : \mathcal{T}_{/\mathbb{Q}_p} \to E$ for $E = \mathrm{End}(\widehat{\mathbf{T}}_L) = \mathrm{End}_{\mathbb{Z}_p}(L)$. Then we get from the theorem the following lemma:

Lemma 3.7. *Suppose that the trivial representation of $\mathcal{T}(\mathbb{Z}_p)$ is not a subquotient of $L \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. If $\widehat{\mathcal{Z}}_{/\mathbb{F}}$ is an integral closed formal subscheme of $\widehat{\mathbf{T}}_L$ stable under the action of an open subgroup U of $\mathcal{T}(\mathbb{Z}_p)$. Then there exists a $\mathbb{Z}_p$–direct summand $L_{\widehat{\mathcal{Z}}} \subset L$ stable under $\mathcal{T}(\mathbb{Z}_p)$ such that $\widehat{\mathcal{Z}} = \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} L_{\widehat{\mathcal{Z}}}$; in particular, $\widehat{\mathcal{Z}}$ is a smooth formal subtorus of $\widehat{\mathbf{T}}_L$.*

We recall the definition of Tate-linear subvarieties in the Hilbert modular variety given in [C4] Section 5. Fix a closed point $x \in Sh^{ord}_{/\mathbb{F}}$ $(\mathbb{F} = \overline{\mathbb{F}}_p)$ carrying a triple $(A_x, \overline{\lambda}, \eta^{(p)})$ (thus A_x is of CM type (M, Σ) and satisfies (ord) in 3.3). Let V be the irreducible component of $Sh^{(p)}_{/\mathbb{F}}$ containing x, and put $V^{ord} = V \cap Sh^{ord}$. Let $m \geq 1$ be a positive integer. Suppose that $\mathcal{Z}$ is an irreducible closed subvariety of $(V^{ord})^m = V^{ord} \times V^{ord} \times \cdots \times V^{ord}$ defined over $\mathbb{F}$.

(T1) Let $z = (z_1, \ldots, z_m)$ $(z_j \in V^{ord})$ be any closed point of $\mathcal{Z}$. We say that $\mathcal{Z}$ is *Tate-linear at* z if the formal completion of $\mathcal{Z}$ at z is a formal subtorus of the Serre–Tate formal torus $\prod_{j=1}^m \widehat{V^{ord}_{z_j}} \cong (\widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}} O)^m$,

(T2) We say that $\mathcal{Z}$ is Tate-linear if it is *Tate-linear* at every closed point of $\mathcal{Z}$.

(T3) Denote by $f : Y \to \mathcal{Z}$ the normalization of $\mathcal{Z}$. We say that $\mathcal{Z}$ is *weakly Tate-linear* if for every closed point y of Y, the morphism induced by f on the formal completion $\widehat{Y}_y$ of Y along y is an isomorphism of $\widehat{Y}_y$ to a formal subtorus of the Serre–Tate formal torus $(\widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}} O)^m$ (at $f(y) \in V^m$).

Obviously, we can modify the above definition to define Tate O-linearity insisting O-linearity in (T1–3). In [C4], the definition of Tate linear subvarieties is given

for a closed subvariety of the ordinary locus of the Siegel modular variety. Since $(V^{ord})^m$ has a canonical closed immersion into a Siegel modular variety (e.g., [PAF] Corollary 7.2 and 8.4.2), this definition is equivalent to Chai's definition for closed subvarieties of the Hilbert modular variety. It is conjectured by Chai that a weakly Tate linear subvariety is actually Tate linear (see [C4] 5.3.1)), which has been shown to hold for our $V \subset Sh^{(p)}_{/\mathbb{F}}$ (see [C4] Theorem 8.6).

If $\mathcal{Z}$ is a variety with a morphism $\pi : \mathcal{Z} \to (V^{ord})^m$ and if for a closed point $z \in \mathcal{Z}$, π induces an embedding of $\widehat{\mathcal{Z}}_z$ into the formal completion of $(V^{ord})^m$ at $\pi(z)$, we can still speak of Tate-linearity (and weak Tate-linearity) at z of $\mathcal{Z}$ (we shall make this abuse often later).

3.5 Linear Independence

We prove a key result on linear independence of arithmetic modular functions (Theorem 3.20 below), respectively, forms (Corollary 3.21 below) and their image under a transcendental automorphism of the deformation space $\widehat{S}$ over W. We keep the assumption of unramifiedness of p in $F/\mathbb{Q}$ and the notation introduced in 3.3. Thus $\mathfrak{d}_p^{-1} = O_p$, and we have

$$\widehat{S} = \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}} \mathfrak{d}^{-1} = \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \mathfrak{d}_p^{-1} = \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} O_p = \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}} O,$$

where $\widehat{\mathbb{G}}_m$ denotes the completion of $\mathbb{G}_m$ over W along the origin 0 in the special fiber at p. Thus the definition of the Tate linearity of the previous section applies to this case. Recall that M/F is the fixed CM quadratic extension of F with integer ring $\mathfrak{O}$, and $x \in Ig(\mathbb{F})$ is the CM point corresponding to an ordinary abelian variety with complex multiplication of type (M, Σ). We may assume that the point x has expression $x = [z, g] = [z, 1] \cdot g$ for $g \in G(\mathbb{Z}_p \times \mathbb{A}^{(p\infty)})$. Indeed, we can choose the CM abelian variety A_x so that its lattice $L = \mathcal{L}_z \cdot g$ (which is a fractional M-ideal prime to p) is given by $\mathfrak{c}^* + Oz$ for a fractional F-ideal $\mathfrak{c}$ prime to p. By our choice, $\mathcal{L}_z \otimes_{\mathbb{Z}} \mathbb{Z}_p = \mathfrak{O}_p = \mathfrak{O}_{\mathbf{p}} \oplus \mathfrak{O}_{\mathbf{p}^c}$; so, we may choose g with $\eta_x = \eta_z \circ g$ so that $g_p \in G(\mathbb{Z}_p)$ is the matrix of change of base from $(1, z) \in F_p^2$ to the basis $(1_{\mathbf{p}}, 1_{\mathbf{p}^c}) \in F_p^2$ for the idempotents $1_{\mathbf{p}} \in \mathfrak{O}_{\mathbf{p}}$ and $1_{\mathbf{p}^c} \in \mathfrak{O}_{\mathbf{p}^c}$. The level p^∞-structure η_p^{ord} (of A_x) sends $a \in O_p$ to $a \cdot 1_{\mathbf{p}^c}$ identifying $A_x[p^\infty]^{et}$ with $M_{\mathbf{p}}/\mathfrak{O}_{\mathbf{p}^c}$ by $\mathcal{L}_z \otimes_{\mathbb{Z}} \mathbb{Z}_p = \mathfrak{O}_{\mathbf{p}} \oplus \mathfrak{O}_{\mathbf{p}^c}$. For this choice $x = [z, g]$, we recall the representation $\widehat{\rho}_x : T_x \to G$ defined over $\mathbb{A}^{(\infty)}$ given at the beginning of Section 3.2 and the quotient torus $\mathcal{T}(\mathbb{Z}_{(p)}) = T_x(\mathbb{Z}_{(p)})/T(\mathbb{Z}_{(p)})$ defined just above (3.18). As studied in Lemma 3.3, $\widehat{\rho}_x(\mathcal{T}(\mathbb{Z}_{(p)}))$ gives the stabilizer of $x = [z, g]$ in $G(\mathbb{Z}_p \times \mathbb{A}^{(p\infty)})/\overline{Z(\mathbb{Z}_{(p)})}$. We simply write ρ for $\widehat{\rho}_x$ hereafter.

For each open compact subgroup K of $G(\mathbb{A}^{(\infty)})$ such that $K = K_p \times K^{(p)}$ with $K_p = GL_2(O_p)$, let V_K be the geometrically irreducible component containing x in the reduction $(Sh^{(p)}/K)_{/\mathbb{F}}$ modulo p of the Kottwitz model. Let $V = \varprojlim_K V_K$ for K running through open compact subgroups of $G(\mathbb{A}^{(\infty)})$ maximal at p. Strictly speaking, the point x gives rise to a projective system of points $x_K \in V_K(\mathbb{F})$ (the image of x in V_K), but we write this point as $x \in V_K(\mathbb{F})$.

The formal completion $\widehat{S}$ of V along x is isomorphic to $\widehat{\mathbb{G}}_m \otimes O$ whose automorphism group is isomorphic to $O_p^\times$. Through the injective homomorphism (3.18): $\alpha \mapsto \alpha^{1-c}$, we regard $\mathcal{T}(\mathbb{Z}_{(p)})$ as a subgroup of $O_p^\times$, identifying $\mathfrak{O}_{\mathbf{p}}$ with O_p by the inclusion $O \hookrightarrow \mathfrak{O}$.

Let $\mathcal{O}_{V,x} = \varinjlim_K \mathcal{O}_{V_K,x}$ be the stalk of V at x, and let $S = \mathrm{Spec}(\mathcal{O}_{V,x})$. The local ring $\mathcal{O}_{V,x}$ is a dense subring of the affine ring $\widehat{\mathcal{O}}_S$ of $\widehat{S}$.

Take $a_1, \dots, a_m \in O_p^\times$. By the action of a_j on $\widehat{S}$ (and hence on $\widehat{\mathcal{O}}_S$), we have an algebra homomorphism

$$\phi : \overbrace{\mathcal{O}_{V,x} \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} \mathcal{O}_{V,x}} \to \widehat{\mathcal{O}}_S \text{ sending } f_1 \otimes \cdots \otimes f_m \text{ to } \prod_{j=1}^{m} a_j(f) \in \widehat{\mathcal{O}}_S. \tag{3.19}$$

If a_j's $(j = 1, \dots, m)$ are pairwise distinct modulo $\mathcal{T}(\mathbb{Z}_{(p)})$, we would like to prove that ϕ is injective. Thus for a nonconstant modular function $f \in \mathcal{O}_{V,x}$, $\{a_1(f), \dots, a_m(f)\}$ are linearly independent over $\mathbb{F}$. Since f is a ratio of two modular forms, this is not too far from the claim (made in the introduction) that $\{a_1(E), \dots, a_m(E)\}$ are linearly independent over $\mathbb{F}$ for a suitable Eisenstein series E. Thus we study $\mathrm{Ker}(\phi)$ for $a_1, \dots, a_m \in O_p^\times$.

Since $\rho(\mathcal{T}(\mathbb{Z}_{(p)}))$ fixes x (Lemma 3.3), $\mathcal{T}(\mathbb{Z}_{(p)})$ acts on $\mathcal{O}_S$ by ring automorphisms, and by Corollary 3.5, this action is compatible with action of $O_p^\times$ via the embedding $\mathcal{T}(\mathbb{Z}_{(p)}) \hookrightarrow O_p^\times$. Thus we have $\phi(\alpha(f_1) \otimes \cdots \otimes \alpha(f_m)) = \alpha^{1-c}(\phi(f_1 \otimes \cdots \otimes f_m))$ for all $\alpha \in \mathcal{T}(\mathbb{Z}_{(p)})$. In other words, the closed subscheme $\mathrm{Spec}(\mathrm{Im}(\phi)) \subset S^m = S \times \cdots \times S$ is stable under the diagonal action of $\mathcal{T}(\mathbb{Z}_{(p)})$ on S^m. Thus we study in the following couple of propositions the (local) structure of a closed subscheme of S^m stable under the diagonal action of $\mathcal{T}(\mathbb{Z}_{(p)})$. After determining the structure of such formal subschemes, we will globalize the result to reach our desired conclusion of the injectivity of ϕ if a_js are independent.

Since $\mathrm{Ker}(\phi)$ is a prime ideal of $\mathcal{O}_{V,x} \otimes \cdots \otimes \mathcal{O}_{V,x}$ stable under the diagonal action of $\mathcal{T}(\mathbb{Z}_{(p)})$, it is induced by an irreducible closed (pro-)subscheme $X \subset V$ passing through $x^m = (x, x, \dots, x)$. In other words, X is the Zariski closure in V^m of $\mathrm{Spec}(\mathcal{O}_{V,x} \otimes \cdots \otimes \mathcal{O}_{V,x}/\mathfrak{b})$ for the prime ideal $\mathfrak{b} = \mathrm{Ker}(\phi)$. We take a more general setting specified as follows (we use the following notation throughout).

(N0) Let $S = \mathrm{Spec}(\mathcal{O}_{V,x})_{/\mathbb{F}}$ and $S_K = \mathrm{Spec}(\mathcal{O}_{V_K,x})_{/\mathbb{F}}$ with their formal completion $\widehat{S}$ and $\widehat{S}_K$ along x isomorphic to $\widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}} \mathfrak{d}^{-1} = \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}} O$;

(N1) For a prime ideal $\mathfrak{b} \subset (\mathcal{O}_{V,x} \otimes \cdots \otimes \mathcal{O}_{V,x})$ (the m-fold tensor product) stable under a p-adically open subgroup $\mathbb{T}$ of $\mathcal{T}(\mathbb{Z}_{(p)})$, we write $\mathcal{X}_{/\mathbb{F}} = \mathrm{Spec}(\mathcal{O}_{V,x} \otimes \cdots \otimes \mathcal{O}_{V,x}/\mathfrak{b}) \subset S^m$ and let $\mathcal{Y} \twoheadrightarrow \mathcal{X}$ be the normalization;

(N2) $\widehat{X} \subset \widehat{S}^m$ is a formal completion of $\mathcal{X}$ along its closed point $x^m = (x, \dots, x)$;

(N3) $X_{/\mathbb{F}}$ is the Zariski closure of $\mathcal{X}$ in V^m (so, X is stable under $\mathbb{T}$ and $\widehat{X}$ is the formal completion of X along x^m). Let $Y \twoheadrightarrow X$ be the normalization. Write $X = \varprojlim_K X_K \subset V^m$ with irreducible closed subschemes $X_K \subset V_K^m$

(the image of X in V_K^m) and $Y_K \twoheadrightarrow X_K$ for the normalization of X_K (so, $Y = \varprojlim_K Y_K$), where K runs over open compact subgroups with $K = K^{(p)} \times G(\mathbb{Z}_p)$.

We first deal with the simplest case of $m = 1$. We start with an irreducible closed (pro-)subscheme $X \subset V$ passing through x stable under the action of a subgroup $\mathbb{T}$ of $\mathcal{T}(\mathbb{Z}_{(p)})$ as above. Define $X^{ord} = X \cap V^{ord}$ and $X_K^{ord} = X_K \cap V_K^{ord}$. We want to prove that $X = V$ if $\dim X > 0$, and as we will see after the following proposition, this implies injectivity of ϕ. By the étaleness of $\mathcal{O}_{V,x}/\mathcal{O}_{V_K,x}$, $\widehat{X}$ is canonically isomorphic to the formal completion $\widehat{X}_K$ of X_K at x.

Proposition 3.8. *Let the notation and the assumption be as in* (N0–3) *with* $m = 1$. *If* $\dim X_{/\mathbb{F}} > 0$, *then we have* $X_{/\mathbb{F}} = V_{/\mathbb{F}}$ *and* $X_{K/\mathbb{F}} = V_{K/\mathbb{F}}$.

Proof. We first follow the argument in [C2] Sections 4 and 5. By the Serre–Tate deformation theory and unramifiedness of p in $F/\mathbb{Q}$ (which implies $\mathfrak{d}_p = O_p$), we have a canonical identification:

$$\widehat{V}_{/\mathbb{F}} \cong \widehat{S} = \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}} \mathfrak{d}^{-1} = \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \mathfrak{d}_p^{-1} = \prod_{\mathfrak{p} \in \Sigma_p} \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} O_{\mathfrak{p}},$$

where $\widehat{V}$ is the formal completion of V along x. For an open compact subgroup K maximal at p (so that Ig/V_K^{ord} is étale at x), the above identity induces

$$\widehat{V}_{K/\mathbb{F}} \cong \widehat{S} = \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}} \mathfrak{d}^{-1} = \prod_{\mathfrak{p} \in \Sigma_p} \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} O_{\mathfrak{p}}.$$

Since $\mathcal{O}_{X_K,x}$ is a localization of an $\mathbb{F}$-algebra of finite type, it is an excellent ring (see [EGA] IV.7.8.3 (ii) and (iii)). Since $\mathcal{O}_{X_K,x}$ is an excellent integral domain, $\widehat{X} \cong \widehat{X}_K = \mathrm{Spf}(\widehat{\mathcal{O}_{X_K,x}})$ is reduced (see [EGA] IV.7.8.3 (vii)).

As we have seen, an element $\alpha \in M^\times$ prime to p acts on the Serre–Tate canonical coordinate by $t \mapsto t^{\alpha^{1-c}}$ for the generator c of $\mathrm{Gal}(M/F)$. By the stability of X under $\mathbb{T}$, the formal completion $\widehat{X}$ along the point x is stable under the closure $\overline{\mathbb{T}}$ of $\mathbb{T}$. Since $\widehat{X} \cong \widehat{X}_K$ is a noetherian reduced formal scheme, it has only finitely many irreducible components. Thus the stabilizer of each irreducible component of $\widehat{X}$ is an open subgroup of $\overline{\mathbb{T}}$ and hence is an open subgroup of $\mathcal{T}(\mathbb{Z}_p)$. Applying Lemma 3.7 to $\widehat{S} = \widehat{\mathbf{T}}_L$ ($L = O_p$) and an irreducible component I of $\widehat{X} = \widehat{X}_K$ (which is reduced as we already remarked), we find

$$I = \prod_{\mathfrak{p} \in \Xi_I} \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} O_{\mathfrak{p}}$$

for a subset $\Xi_I \subset \Sigma_p$. The formal scheme I is a smooth formal subgroup of $\widehat{S}$.

The group $\mathcal{T}(\mathbb{Z}_{(p)})$ acts naturally on the normalization $Y = \varprojlim_K Y_K$ of X. For a closed point $y \in Y_K$ over x, $\mathcal{O}_{Y_K,y}$ (which is finite type over $\mathcal{O}_{X_K,x}$) is excellent, and the formal completion $\widehat{Y}_K$ is integral ([EGA] IV.7.8.3 (vii)). Thus there is a unique point $y_I \in Y$ over $x \in X$ such that the projection $Y_K \to X_K$ induces an isomorphism of the formal completion $\widehat{Y}_{y_I}$ along y_I onto I.

Let U be any of the (pro-)varieties Ig, V and V_K. On U, the tangent bundle Θ_U is decomposed into the direct sum of eigenspaces under the O–action:

$$\Theta_U \cong \mathcal{O}_U \otimes_{\mathbb{Z}} O \text{ locally, and } \Theta_U = \bigoplus_{\mathfrak{p}\in\Sigma_p} \Theta_{U,\mathfrak{p}}, \tag{3.20}$$

where $\Theta_{U,\mathfrak{p}}$ is a locally free $O_U \otimes_{\mathbb{Z}_p} O_{\mathfrak{p}}$-module of rank 1 (see [C2] page 473). To see this, let $f : A \to Z$ be an AVRM over a scheme $Z_{/\mathbb{F}}$. Then we have the Kodaira-Spencer map $\kappa : f_*\Omega_{A/Z} \otimes_{O\otimes_{\mathbb{Z}}\mathcal{O}_Z} f_*\Omega_{A/Z} \to \Omega_{Z/\mathbb{F}}$ (see [K3] 1.0). The Kodaira-Spencer map κ is an isomorphism if A is the universal abelian scheme over $Z = U$ (see [K3] (1.0.21)); hence, $f_*\Omega_{Z/\mathbb{F}} \cong O \otimes_{\mathbb{Z}} \mathcal{O}_Z$ and, taking the dual, $\Theta_U \cong O \otimes_{\mathbb{Z}} \mathcal{O}_Z$. Therefore $\Theta_U = \bigoplus_{\mathfrak{p}\in\Sigma_p} \Theta_{U,\mathfrak{p}}$ for the $O_{\mathfrak{p}} \otimes_{\mathbb{Z}} \mathcal{O}_Z$-eigen sub-bundles $\Theta_{U,\mathfrak{p}}$, and we obtain the expression (3.20).

Let $\mathbf{A}$ be the universal abelian scheme over V_K, and write $A = \mathbf{A} \times_{V_K} Y_K$. We again have the Kodaira-Spencer map $\kappa_Y : f_*\Omega_{A/Y_K} \otimes_{O\otimes_{\mathbb{Z}}\mathcal{O}_{Y_K}} f_*\Omega_{A/Y_K} \to \Omega_{Y_K/\mathbb{F}}$. Since $I = \prod_{\mathfrak{p}\in\Xi_I} \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} O_{\mathfrak{p}}$, after taking the formal completion along y_I, this map induces an isomorphism

$$\left(\widehat{f_*\Omega_{A/Y_K}} \otimes_{O\otimes_{\mathbb{Z}}\mathcal{O}_{Y_K}} \widehat{f_*\Omega_{A/Y_K}}\right) \otimes_O \prod_{\mathfrak{p}\in\Xi_I} O_{\mathfrak{p}} \cong \widehat{\Omega}_{Y_{K,y_I}/\mathbb{F}}.$$

By this expression, via the normalization map: $Y \to X$, the tangent space $\Theta_{\widehat{Y}}$ of $\widehat{Y}$ at y_I is identified with $\oplus_{\mathfrak{p}\in\Xi_I}(\Theta_{\mathfrak{p}} \otimes_{\mathcal{O}_{V,x}} \widehat{\mathcal{O}}_{Y,y_I})$, where $\Theta_{\mathfrak{p}} = \Theta_{V,\mathfrak{p}}$. By faithfully flat descent, we have $\Theta_Y \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y,y_I} = \oplus_{\mathfrak{p}\in\Xi_I}(\Theta_{\mathfrak{p}} \otimes_{\mathcal{O}_V} \mathcal{O}_{Y,y_I})$. Thus on an open dense subscheme $Y_I \subset Y$ with $y_I \in Y_I$, we have

$$\Theta_{Y_I} = \oplus_{\mathfrak{p}\in\Xi_I}(\Theta_{\mathfrak{p}} \otimes_{\mathcal{O}_V} \mathcal{O}_{Y_I}).$$

Since Y is irreducible, $\cap_I Y_I$ for I running over all irreducible components of $\widehat{X}$ is still open dense in Y. This implies that Ξ_I is independent of I; hence, $\widehat{X}$ is integral and smooth, and we have

$$\widehat{Y} = \widehat{X} = \prod_{\mathfrak{p}\in\Xi} \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} O_{\mathfrak{p}} \tag{3.21}$$

for a subset Ξ of Σ. Therefore, X is smooth at x.

Suppose that $\Xi \neq \Sigma_p$ and let $\mathfrak{p} \in \Sigma_p - \Xi$. We only need to prove that $V_K = X_K$ for a choice of an open subgroup K maximal at p. Choosing K sufficiently small, we may assume that V_K is smooth over $\mathbb{F} = \overline{\mathbb{F}}_p$. Recall the universal abelian scheme $\mathbf{A}_{/V_K}$. Define $A = A_K = \mathbf{A} \times_{V_K} X_K$. Write $\mathfrak{p}_F = F \cap \mathfrak{p}$, and consider the $\mathfrak{p}_F$–divisible group $A[\mathfrak{p}_F^\infty]$.

We need here a lemma (Lemma 3.10 below) about an ordinary AVRM: $A \to Z$. In our setting, $Z = X_K$, which is an irreducible excellent affine scheme. Since $\mathfrak{p} \notin \Xi$, the p-divisible group $A[\mathfrak{p}_F^\infty]_{/\widehat{X}_K}$ splits canonically into a direct sum $A[\mathfrak{p}_F^\infty]^\circ \oplus A[\mathfrak{p}_F^\infty]^{et}_{/\widehat{X}_K}$. By the lemma, on an open dense subscheme $U_K \subset X_K$ the p-divisible group $A[\mathfrak{p}_F^\infty]_{/U_K}$ splits canonically into a direct sum $A[\mathfrak{p}_F^\infty]^\circ \oplus A[\mathfrak{p}_F^\infty]^{et}_{/U_K}$ for the connected component $A[\mathfrak{p}_F^\infty]^\circ$ and the étale quotient

$A[\mathfrak{p}_F^\infty]^{et}$. We now follow the proof of Theorem 8.6 in [C4] to get a contradiction (and hence we conclude $\Xi = \Sigma_p$). Consider the decomposition $A[p^\infty]_{/U_K} = \prod_{\mathfrak{p}' \in \Sigma_p} A[\mathfrak{p}'^\infty_F]_{/U_K}$ of the Barsotti-Tate group $A[p^\infty]_{/U_K}$ over U_K. This étale-connected splitting of $A[\mathfrak{p}_F^\infty]$ over U_K gives two orthogonal idempotents e° and e^{et} in $\mathrm{End}_{U_K}(A[\mathfrak{p}_F^\infty])$, with the following properties.

- The idempotents e° and e^{et} commute with the action of O on $A[\mathfrak{p}_F^\infty]$,
- $e^\circ + e^{et} = \mathrm{id} \in \mathrm{End}_{U_K}(A[\mathfrak{p}_F^\infty])$,
- The image of e° is the multiplicative part of $A[\mathfrak{p}_F^\infty]$, and the image of e^{et} is naturally isomorphic to the maximal étale quotient of $A[\mathfrak{p}_F^\infty]$.

Thus, we have

$$\mathrm{End}_O(A[p^\infty]_{/U_K}) \supset \left(\bigoplus_{\mathfrak{p}' \neq \mathfrak{p}} O_{\mathfrak{p}}\right) \oplus \left(O_{\mathfrak{p}} e^\circ \oplus O_{\mathfrak{p}} e^{et}\right) \supsetneq O_p. \tag{$*$}$$

On the other hand, Theorem 2.6 of [J] tells us that

$$\mathrm{End}_O(A[p^\infty]_{/U_K}) = \mathrm{End}_O(A_{/U_K}) \otimes_{\mathbb{Z}} \mathbb{Z}_p.$$

The endomorphism algebra $\mathrm{End}^{\mathbb{Q}}(A_{/U_K}) = \mathrm{End}_O(A_{/U_K}) \otimes \mathbb{Q}$ is isomorphic to either a CM quadratic extension or F itself. Since $\Xi \neq \emptyset$ ($\Leftrightarrow \dim X > 0$), $\mathrm{End}_O(A[\mathfrak{p}'^{\infty}_F]_{/U_K}) = O_{\mathfrak{p}'}$ for $\mathfrak{p}' \in \Xi$, because $A[\mathfrak{p}'^{\infty}_F]_{/\widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}'}}$ is the universal Barsotti-Tate group over $\widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}'} \subset \widehat{X}$ deforming $A_x[\mathfrak{p}'^{\infty}_F]_{/\mathbb{F}}$ and hence $A[\mathfrak{p}'^{\infty}_F]^\circ \hookrightarrow A[\mathfrak{p}'^{\infty}_F] \twoheadrightarrow A[\mathfrak{p}'^{\infty}_F]^{et}$ is non-split over U_K. Thus $\mathrm{End}^{\mathbb{Q}}(A_{/U_K})$ cannot be a CM quadratic extension; so, $\mathrm{End}^{\mathbb{Q}}(A_{/U_K}) = F$. This is a contradiction (against $(*)$), hence $\Xi = \Sigma_p$, and $X_K = V_K$ as desired. □

Corollary 3.9. *Let the notation and the assumption be as in* (N0–3) *and as in* Proposition 3.8. *In particular, $m = 1$ and $\mathfrak{b} \subset \mathcal{O}_{V,x}$ is a non-maximal prime ideal stable under $\mathbb{T}$. Let $\widetilde{\mathfrak{b}}$ be the unique prime ideal of $\mathcal{O}_{Ig,x}$ above $\mathfrak{b}$, and write $\mathcal{X} = \mathrm{Spec}(\mathcal{O}_{V,x}/\mathfrak{b})$. Then $\mathfrak{b} = 0$, $\widetilde{\mathfrak{b}} = 0$ and $\mathcal{X} = \mathrm{Spec}(\mathcal{O}_{V,x})$; in particular, ϕ for $m = 1$ in* (3.19) *is injective.*

Proof. Let $\mathfrak{b}_K = \mathfrak{b} \cap \mathcal{O}_{V_K,x}$. Since $\mathcal{O}_{Ig,x}$ is étale over $\mathcal{O}_{V_K,x}$, we have a unique prime ideal $\widetilde{\mathfrak{b}} \subset \mathcal{O}_{Ig,x}$ which is over $\mathfrak{b}_K$. Thus $\widetilde{\mathfrak{b}}$ is also stable under $\mathbb{T}$, and we have $\widetilde{\mathfrak{b}} = 0 \Leftrightarrow \mathfrak{b} = 0 \Leftrightarrow \mathfrak{b}_K = 0$ for any open compact subgroup K maximal at p. We consider the Zariski closure X_K of $\mathrm{Spec}(\mathcal{O}_{V_K,x}/\mathfrak{b}_K)$ in $V_{K/\mathbb{F}}$.

For any Zariski open neighborhood $U \subset V_K$ of x, put $\mathcal{O}_U \cap \mathfrak{b}_K = \mathrm{Ker}(\mathrm{Res} : \mathcal{O}_U \to \mathcal{O}_{V_K,x}/\mathfrak{b}_K)$. Then $U \cap X_K$ is given by the spectrum relative to U: $\mathrm{Spec}_U(\mathcal{O}_U/\mathcal{O}_U \cap \mathfrak{b}_K)$. Since $\mathcal{O}_U \cap \mathfrak{b}_K$ is a (sheaf) prime ideal of $\mathcal{O}_U$, $U \cap X_K$ is irreducible reduced, and hence X_K is irreducible reduced. Thus

$$X = V \Leftrightarrow \mathfrak{b} = 0\ (\Leftrightarrow \widetilde{\mathfrak{b}} = 0) \Leftrightarrow \mathfrak{b}_K = 0 \Leftrightarrow X_K = V_K.$$

The irreducible reduced $\mathbb{T}$-invariant closed subscheme X is either a single point $\{x\}$ or V itself by the above proposition, and hence we conclude $\mathfrak{b} = 0$; in particular, $\mathrm{Ker}(\phi) = 0$, taking $\mathfrak{b}$ to be $\mathrm{Ker}(\phi)$. □

Lemma 3.10. *Let $A \to Z$ be an ordinary AVRM with real multiplication by O over a reduced excellent affine base scheme Z over $\overline{\mathbb{F}}_p$. For a closed point $s \in Z(\overline{\mathbb{F}}_p)$ and a formal completion $\widehat{Z}$ along s, if the p-divisible group $A[\mathfrak{p}_F^\infty]\times_Z \widehat{Z}$ splits into a product of its connected component $A[\mathfrak{p}_F^\infty]^\circ_{/\widehat{Z}}$ and étale quotient $A[\mathfrak{p}_F^\infty]^{et}_{/\widehat{Z}}$, then on an open subscheme $U \subset Z$ containing s, the p-divisible group $A[\mathfrak{p}_F^\infty]_{/U}$ canonically splits into $A[\mathfrak{p}_F^\infty]^\circ \oplus A[\mathfrak{p}_F^\infty]^{et}_{/U}$ for the connected component $A[\mathfrak{p}_F^\infty]^\circ_{/U}$ and the étale quotient $A[\mathfrak{p}_F^\infty]^{et}_{/U}$.*

Proof. The splitting, if it exists, is canonical, because Z is reduced. Indeed, such splitting is canonical over an algebraically closed field (cf. [ABV] Section 14, specifically, page 136), and if the base scheme is reduced, under the existence of the splitting over the base, it has to be unique at all geometric points (and hence unique over the reduced base scheme). Replacing Z by its irreducible component containing s, we may assume that Z is irreducible. The $\mathfrak{p}_F$-part of the Serre–Tate coordinate $t_\mathfrak{p}$ around s measures the degree of non-splitting of the exact sequence $A[\mathfrak{p}_F^\infty]^\circ \hookrightarrow A[\mathfrak{p}_F^\infty] \twoheadrightarrow A[\mathfrak{p}_F^\infty]^{et}_{/\widehat{Z}}$. Because of the splitting over $\widehat{Z}$, we find that $t_\mathfrak{p}(A[\mathfrak{p}_F^\infty]_{/\widehat{Z}}) = 0$. By assumption, $Z = \mathrm{Spec}(\mathcal{R})$ for an excellent integral domain $\mathcal{R}$. Then by [C4] Proposition 8.4 (ii), there exists an open neighborhood U of s in Z over which we have a splitting $A[\mathfrak{p}_F^\infty]^\circ \oplus A[\mathfrak{p}_F^\infty]^{et}_{/U}$. □

Our goal is to prove the injectivity of ϕ for general $m \geq 1$ in (3.19) under the assumption that the a_j's are pairwise distinct modulo $\mathcal{T}(\mathbb{Z}_{(p)})$. The injectivity is equivalent to $\mathrm{Spec}(\mathrm{Im}(\phi)) = S^m$; so, we study the local property of $\mathrm{Spec}(\mathrm{Im}(\phi))$ to show that $\dim(\mathrm{Im}(\phi)) < \dim S^m$ implies the equality of two of the a_j's modulo $\mathcal{T}(\mathbb{Z}_{(p)})$. The following result dealing with the local structure of $\mathrm{Spec}(\mathrm{Im}(\phi))$ when $m > 1$ is a key to prove the linear independence.

Proposition 3.11. *Let the notation be as in* (N0–3), Proposition 3.8 *and its proof. In particular, for a positive integer m, let $\mathcal{X}$ be a closed integral subscheme of S^m containing $x^m = (x, x, \ldots, x)$ for the closed point $x \in S$, and let $\Pi_\mathcal{Y} : \mathcal{Y} \to \mathcal{X}$ be the normalization of $\mathcal{X}$. Write $S^m = S' \times S''$ for the first $(m-1)$–factor $S' = S^{m-1} \subset S^m$ and the last factor $S'' = S$. Suppose that the projection to S' induces a dominant morphism $\pi_\mathcal{X} : \mathcal{X} \to S'_{/\mathbb{F}}$. Suppose further that $\mathcal{X}$ is stable under the diagonal action of a subgroup $\mathbb{T}$ of $\mathcal{T}(\mathbb{Z}_{(p)}) \subset \mathrm{Aut}(\mathcal{O}_{V,x})$ whose p–adic closure is open in $\mathcal{T}(\mathbb{Z}_p)$. Then,*

(1) *$\mathcal{Y}$ has finitely many points y over x^m, is Tate O-linear at every point y over x^m; so, $\widehat{\mathcal{Y}}_y = \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} L$ for an O_p-direct summand L of $X_*(\widehat{S}^m)$. Moreover the isomorphism class of L as O_p-module is independent of y.*

(2) *$\mathcal{Y}$ is smooth over $\mathbb{F}$ and is flat over S'.*

(3) *Either $\mathcal{X} = S^m$ or $\mathcal{X}$ is finite over S' via $\pi_\mathcal{X}$. If $\mathcal{X}$ is finite over S', $\mathcal{Y}$ is finite flat over S'.*

(4) *If $\pi_\mathcal{X} \circ \Pi_\mathcal{Y}$ induces a surjection of the tangent space at one $y \in \mathcal{Y}$ over x^m onto that of S' at x^{m-1} and $\mathcal{X}$ is a proper subscheme of S^m, then $\pi_\mathcal{X} \circ \Pi_\mathcal{Y} : \mathcal{Y} \to S'$ is étale.*

Comment: In Proposition 3.8 dealing with the case of $m = 1$, the factor S' is equal to $\mathrm{Spec}(\mathbb{F})$. Later in Corollaries 3.16 and 3.19, we prove that $\mathcal{X}$ is smooth; so, $\mathcal{Y} = \mathcal{X}$.

Proof. By Serre–Tate theory, we have $\widehat{S} \cong \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} O_p$. Since the case $m = 1$ has already been taken care of by Proposition 3.8, we may assume that $m \geq 2$. Since $\mathcal{X}$ is dominant over S', we have $\dim \mathcal{X} > 0$. Let K be an open compact subgroup of $G(\mathbb{A}^{(\infty)})$ maximal at p, and recall $S_K = \mathrm{Spec}(\mathcal{O}_{V_K,x})$. We assume that K is small so that S/S_K is étale. Consider the image of $\mathcal{X}_K$ of $\mathcal{X}$ in S_K^m. Then $\mathcal{X}_K = \mathrm{Spec}(\mathcal{R}_K)$ for an integral domain $\mathcal{R}_K$ and $\mathcal{X} = \mathrm{Spec}(\mathcal{R})$ for $\mathcal{R} = \varinjlim_K \mathcal{R}_K$. Since $\mathcal{R}_K$ is a localization of an integral domain of finite type over $\mathbb{F}$, $\mathcal{R}_K$ is excellent ([EGA] IV.7.8.3 (ii) and (iii)). Thus its formal completion $\widehat{\mathcal{X}}_K \cong \widehat{\mathcal{X}}$ along x^m is reduced ([EGA] IV.7.8.3 (vii)).

Since $\mathcal{X}$ is stable under the diagonal action of the subgroup $\mathbb{T}$ of $\mathcal{T}(\mathbb{Z}_{(p)}) \subset \mathrm{Aut}(\mathcal{O}_{V,x})$ and $\mathcal{X}$ is integral, by Lemma 3.7, $\widehat{\mathcal{X}}$ is a union of finitely many formal O_p–submodules of $\widehat{S}^m$ of the form $\widehat{\mathbb{G}}_m \otimes L$ for O_p–direct summands L of the cocharacter group $X_*(\widehat{S}^m) \cong O_p^m$: $\widehat{\mathcal{X}} = \bigcup_{L \in I} \widehat{\mathbb{G}}_m \otimes L$ for a finite index set I of O_p–direct summands L of $X_*(\widehat{S}^m)$. In particular, $\widehat{\mathcal{X}}$ is stable under the action of $\mathcal{T}(\mathbb{Z}_{(p)})$ (not just $\mathbb{T}$) diagonally embedded into $\mathrm{Aut}_O(\widehat{S})^m$, and $\mathcal{X}$ and X are stable under $\mathcal{T}(\mathbb{Z}_{(p)})$. The normalization $\mathcal{Y} \to \mathcal{X}$ is given by $\varprojlim_K \mathcal{Y}_K$ for the normalization $\mathcal{Y}_K$ of $\mathcal{X}_K$. Naturally the semigroup $\mathrm{End}_{SCH}(\mathcal{X})$ of endomorphisms of the scheme $\mathcal{X}$ acts on $\mathcal{Y}$; in particular, $\mathcal{T}(\mathbb{Z}_{(p)})$ acts on $\mathcal{Y}$. The formal completion $\widehat{\mathcal{Y}}_y$ along y for each point $y \in \mathcal{Y}$ over x^m is isomorphic to the formal completion $\widehat{\mathcal{Y}}_{K,y}$ of $\mathcal{Y}_K$ along the image of y in $\mathcal{Y}_K$. The scheme $\mathcal{Y}_K$ is excellent, because $\mathcal{Y}_K$ is finite over $\mathcal{X}_K$ ([EGA] IV.7.8.3 (ii), (vii)). Since $\widehat{\mathcal{Y}}_{K,y}$ is the normalization of $\widehat{\mathcal{X}}_K$ and $\widehat{\mathcal{Y}}_{K,y}$ is integral ([EGA] IV.7.8.3 (vii)), points y_L of $\mathcal{Y}$ over x^m are indexed by the irreducible components of $\widehat{\mathcal{X}}$ and hence by $L \in I$ so that $\widehat{\mathcal{Y}}_{K,y_L} = \widehat{\mathbb{G}}_m \otimes L$. Since $\mathcal{Y} \to S'$ is dominant, for at least one $L_0 \in I$, the projection $L_0 \otimes_O F \to X_*(\widehat{S}') \otimes_O F$ is surjective (i.e., the image of L_0 is of finite index in $X_*(\widehat{S}')$).

Recall that we denote by X the Zariski closure of $\mathcal{X}$ in V^m and the normalization $\Pi : Y \to X$ of X. Again $\mathrm{End}_{SCH}(X)$ acts on Y; in particular, $\mathcal{T}(\mathbb{Z}_{(p)})$ acts on Y. We have $Y = \varprojlim_K Y_K$ for the normalization Y_K of X_K. We look at the tangent bundle Θ_Z for $Z = V^m, Y, X, S^m$ and $\widehat{S}^m$. Since V^m carries the self product of the universal abelian scheme $\mathbf{A}^m$, by the Kodaira-Spencer map with respect to $\mathbf{A}^m/V^m$: $\Omega_{\mathbf{A}^m/V^m} \otimes_{(O^m \otimes_{\mathbb{Z}} \mathcal{O}_{V^m})} \Omega_{\mathbf{A}^m/V^m} \cong \Omega_{V^m}$, and taking dual, we have the diagonal action of O on Θ_{V^m} (actually O^m acts on Θ_{V^m}). We have the tangent bundle $\Theta_X \subset \Theta_{V^m}$ (which is stable under O), but Θ_X may not be locally free around x^m since X may have singularity at x^m. The action of O on Θ_X extends to Θ_Y compatibly. Let $y_L \in Y$ be a point above x^m with $\widehat{Y}_{y_L} = \widehat{\mathbb{G}}_m \otimes L$. Since Y is stable under $\mathcal{T}(\mathbb{Z}_{(p)})$ and $\widehat{Y}_{y_L} \cong \widehat{\mathbb{G}}_m \otimes L$ for each point $y_L \in Y$ above x^m, we have $\Theta_Y = \bigoplus_{\mathfrak{p}} \Theta_{Y,\mathfrak{p}}$ for $O_{\mathfrak{p}}$-eigen sub-bundle $\Theta_{Y,\mathfrak{p}}$. Since $\widehat{Y}_{y_L} = \widehat{\mathbb{G}}_m \otimes L$, $\Theta_{\widehat{Y}_{y_L}} = L \otimes_{\mathbb{Z}} \mathcal{O}_{\widehat{Y}_{y_L}}$, and $\mathrm{rank}_{\mathbb{Z}_p} L \otimes_O O_{\mathfrak{p}} = \mathrm{rank}_{\mathcal{O}_Y} \Theta_{Y,\mathfrak{p}} = \mathrm{rank}_{\mathbb{Z}_p} L_0 \otimes_O O_{\mathfrak{p}}$. Thus $L \cong L_0$ as O_p-modules

for all $L \in I$, and $\widehat{\mathcal{Y}}$ is equidimensional (the equidimensionality also follows from excellency of $\mathcal{X}_K$ by [EGA] IV.7.8.3 (x)). This proves (1).

If $L_0 \otimes_O F = X_*(\widehat{S}^m) \otimes_O F$, we get $\mathcal{X} = S^m$, and we are done. Hereafter we assume $\operatorname{rank}_{\mathbb{Z}_p} L_0 < \operatorname{rank}_{\mathbb{Z}_p} X_*(\widehat{S}^m)$ ($\Leftrightarrow L_0 \otimes_O F \neq X_*(\widehat{S}^m) \otimes_O F$); so, $\operatorname{rank}_{\mathbb{Z}_p} L < \operatorname{rank}_{\mathbb{Z}_p} X_*(\widehat{S}^m)$ for all $L \in I$. Recall the decomposition $S^m = S' \times S''$ with the last factor $S'' = S$. Similarly we decompose $V^m = V' \times V''$ for the product V' of the first $(m-1)$ copies of V and the last copy $V'' = V$. If $\pi : \widehat{\mathbb{G}}_m \otimes L \to \widehat{S}'$ is not dominant for $L \in I$, the image $\pi(\widehat{\mathbb{G}}_m \otimes L)$ is a proper closed formal subscheme of $\widehat{S}'$. In particular, $\widehat{\mathcal{Y}} \to \widehat{S}'$ is not flat over $\pi(\widehat{\mathbb{G}}_m \otimes L) \subsetneq \widehat{S}'$. Define the non-flat locus $Y^{nf} \subset Y$ as the Zariski-closure of the set of closed points $y \in Y$ such that $\mathcal{O}_{Y,y}$ is not flat over $\mathcal{O}_{V',v}$ for the image $v \in V'$ of y. The non-flat locus $y_L \in Y^{nf} \subset Y$ is a nonempty proper closed subscheme, because $\widehat{\mathbb{G}}_m \otimes L_0 \subset \widehat{\mathcal{Y}} = \widehat{Y}_{y_{L_0}}$ is flat over $\widehat{S}'$ and flatness is an open property. Since the formal completion of Y^{nf} at y_L contains $\widehat{\mathbb{G}}_m \otimes L$, $\dim Y^{nf} = \dim Y$; so, Y cannot be irreducible (because Y^{nf} is a proper closed subscheme of Y), a contradiction. Thus $\widehat{\mathbb{G}}_m \otimes L \to \widehat{S}'$ is flat for all $L \in I$, and hence $\widehat{\mathcal{Y}}$ is flat over $\widehat{S}'$. Since $\widehat{S}'/S'$ is faithfully flat, $\mathcal{Y}$ is flat over S'. In particular, for all $L \in I$, $\widehat{\mathbb{G}}_m \otimes L$ is dominant finite over $\widehat{S}'$. This proves (2).

Since $\pi : \widehat{\mathcal{X}} \to \widehat{S}'$ induces a surjection $\pi_* : L \otimes_O F \to X_*(\widehat{S}') \otimes_O F$, the intersection $\widehat{S}'' \cap (\widehat{\mathbb{G}}_m \otimes L)$ has dimension equal to $\operatorname{rank}_{\mathbb{Z}_p} L_0 + \operatorname{rank}_{\mathbb{Z}_p} X_*(\widehat{S}) - \operatorname{rank}_{\mathbb{Z}_p} X_*(\widehat{S}^m)$ for all $L \in I$. Then $\widehat{S}'' \cap (\widehat{\mathbb{G}}_m \otimes L)$ is a formal O_p-submodule isomorphic to $\prod_{\mathfrak{p} \in \Xi_L} \widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}}$ in $\widehat{S}'' = \widehat{S}$, where Ξ_L is the set of all primes $\mathfrak{p} \in \Sigma_p$ such that $\operatorname{rank}_{O_{\mathfrak{p}}} L_0 \otimes_O O_{\mathfrak{p}} > m-1$. By Proposition 3.8 (applied to an irreducible component of $\mathcal{X} \cap S''$), we have either $\Xi_L = \Sigma_p$ or $\Xi_L = \emptyset$. If $\Xi_L = \Sigma_p$, we have $\operatorname{rank}_{\mathbb{Z}_p} L_0 = \operatorname{rank} X_*(\widehat{S}^m)$, and hence $\mathcal{X} = S^m$; so, we are done. Since $L \cong L_0$ as O_p-modules for all $L \in I$, we actually knew that $\widehat{S}'' \cap (\widehat{\mathbb{G}}_m \otimes L)$ and that $\Xi = \Xi_L$ (indexed by L) are independent of $L \in I$, and $\widehat{\mathcal{X}} \cap \widehat{S}'' = \widehat{S}'' \cap (\widehat{\mathbb{G}}_m \otimes L)$ for all $L \in I$ (though we did not use this fact). We hereafter assume that $\Xi = \emptyset$. Since $\pi : \widehat{\mathbb{G}}_m \otimes L \to \widehat{S}'$ is dominant and $\dim \widehat{\mathbb{G}}_m \otimes L = \dim \widehat{S}' = \operatorname{rank}_{\mathbb{Z}_p} L$, $\pi : \widehat{\mathbb{G}}_m \otimes L \to \widehat{S}'$ is finite flat for all $L \in I$. Thus $\mathcal{X} \to S'$ is finite, and $\mathcal{Y}$ is finite flat over S'. This proves (3).

To prove (4), consider the differential sheaf $\Omega_{Y/S'}$. We may assume that $(\pi_{\mathcal{X}} \circ \Pi_{\mathcal{Y}})_* : \Theta_Y \to \Theta_{S'}$ is surjective at $y_0 = y_{L_0}$. Since $\widehat{Y}_{y_0}/\widehat{S}'$ is finite flat, $\widehat{Y}_{y_0}$ is étale over $\widehat{S}'$, and hence $\mathcal{Y}_{y_0}/S'$ is étale. Thus $\Omega_{Y/S'}|_{\mathcal{Y}_{y_0}} = 0$, and hence $\Omega_{Y/S'}$ vanishes on a nonempty open subscheme of Y. Thus the support $Y^{ram} \subset Y$ of $\Omega_{Y/S'}$ is a proper closed subscheme of Y. If $\mathcal{Y}_{y_L}/S'$ is not étale, $\mathcal{Y}_{y_L} \subset Y^{ram}$. Thus Y^{ram} is a closed subscheme of dimension equal to $\dim Y$; so, it is an irreducible component of Y, and hence Y is reducible, a contradiction. Thus $\mathcal{Y} \to S'$ is étale finite. □

Remark 3.12. Let the notation and assumption be as in Proposition 3.11. We suppose $m = 2$ and that $\mathcal{Y} \to \mathcal{X} \subsetneq S^2$ has two dominant projections onto the left and the right factor S of S^2. We write the Serre–Tate coordinate (induced by the ordinary level structure on A_x) of the left factor (resp. the right factor) of $\widehat{S}^2$

as t (resp. t'). Then by Lemma 3.7, the formal completion $\widehat{\mathcal{Y}}_y$ of $\mathcal{Y}$ along a point y above $x^2 = (x, x)$ is canonically isomorphic to a formal subtorus of $\widehat{S}^2$ given by $\widehat{\mathbb{G}}_m \otimes L$ for an O_p–free direct summand L of O_p^2. Thus if $\dim \mathcal{X} = \dim V$, $\widehat{\mathcal{Y}}_y$ is defined by the equation $t^u = t'^v$ for non-zero-divisors $u, v \in O_p$ with $uO_p + vO_p = O_p$, and $L \subset O_p^2$ is given by $L = \{(x, y) \in O_p^2 | ux = vy\}$. If two projections are étale, (u, v) can be chosen to be $(1, a)$ for a unit $a = v/u \in O_p^\times$.

Corollary 3.13. *Let $\mathfrak{b} = \mathrm{Ker}(\phi)$ for ϕ as in (3.19). Let $y \in Y$ be a point above x^m and $\widehat{Y}_y$ be the formal completion of Y along y. Then $\widehat{Y}_y$ for at least one $y \in Y$ contains $\widehat{\Delta} = \{(t^{a_1}, \dots, t^{a_m}) | t \in \widehat{S}\}$, and for $i = 1, 2, \dots, m$, writing $\widehat{S}_i$ for the i-th copy of $\widehat{S}$ in $\widehat{S}^m$, the projection $X_*(\widehat{Y}_y) \to X_*(\widehat{S}_i)$ is surjective, regarding $\widehat{Y}_y \subset \widehat{S}^m$. In particular, if $m = 2$ and $\pi_{\mathcal{X}}$ is finite, $\mathcal{Y} \to S'$ is étale finite.*

Proof. We have the following commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{V,x} \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} \mathcal{O}_{V,x} & \xrightarrow{\phi} & \widehat{\mathcal{O}}_S \\ \iota \downarrow & & \| \downarrow \\ \widehat{\mathcal{O}}_{V,x} \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} \widehat{\mathcal{O}}_{V,x} & \xrightarrow[\Phi]{} & \widehat{\mathcal{O}}_S, \end{array} \tag{3.22}$$

where $\widehat{\mathcal{O}}_{V,x} = \widehat{\mathcal{O}}_S$, $\Phi(f_1 \otimes \cdots \otimes f_m) = \prod_{j=1}^m a_j(f_j)$ and the map ι is the tensor product of the natural inclusion $\mathcal{O}_{V,x} \subset \widehat{\mathcal{O}}_{V,x}$. Thus $\mathrm{Ker}(\Phi) \supset \iota(\mathfrak{b})$. Note that $\mathrm{Spf}(\widehat{\mathcal{O}}_{V,x} \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} \widehat{\mathcal{O}}_{V,x}/\mathrm{Ker}(\Phi))$ is the skew diagonal image $\widehat{\Delta}$ in $\widehat{S}^m$. Taking the formal completion along x^m, the map ι brings $\widehat{\Delta}$ into $\widehat{\mathcal{X}}$ because $\mathrm{Ker}(\Phi) \supset \iota(\mathfrak{b})$. Thus an irreducible component $\widehat{\mathbb{G}}_m \otimes L$ of $\widehat{\mathcal{X}}$ contains $\widehat{\Delta}$. In particular, if $m = 2$, $\pi_{\mathcal{X},*}(L)$ contains $\pi_{\mathcal{X},*}(X_*(\widehat{\Delta})) = X_*(\widehat{S}')$. Then the rest follows from (4) of the above proposition. □

We keep the assumption and the notation in (N0–3) for $\mathfrak{b} = \mathrm{Ker}(\phi)$ (ϕ as in (3.19)). We have globalized $\mathcal{X}$ taking its Zariski closure X in V^m. We start with the simplest case where $m = 2$. There are two possibilities by the above result that $\dim \mathcal{X} = \dim V$ or $X = V^2$. The latter case implies ϕ is injective as desired; so, we are done. Assuming $\dim \mathcal{X} = \dim V$, we take the Zariski closure X of $\mathcal{X}$ in $V \times V$ and its normalization $\Pi : Y \to X$. We study X (resp. Y) as a global irreducible subvariety of the self product $V \times V$ (resp. as a correspondence $V \leftarrow Y \to V$). We are going to show in Corollary 3.16 after two preparatory propositions that $X = Y$ and that the variety X is the graph of an automorphism of V given by an action of an element in $G(\mathbb{A}^{(p\infty)})$ (in other words, a_1/a_2 has to be in $\mathcal{T}(\mathbb{Z}_{(p)})$). In this process of showing that X is a graph of an automorphism, we use repeatedly the fact that the diagonal action of $\mathcal{T}(\mathbb{Z}_{(p)})$ preserves X in $V \times V$ and extends to Y.

The subvariety X is a graph of an automorphism of V (as a correspondence in $V \times V$) if and only if the projections $\pi_j : X \to V$ ($j = 1, 2$) are isomorphisms. The only information we have is: (i) X is stable under the diagonal action of

$\mathcal{T}(\mathbb{Z}_{(p)})$ (or a finite index subgroup thereof) and (ii) the formal completion $\mathcal{Y}$ at any point $y \in Y$ over $(x, x) \in X$ has isomorphic projections to the formal completion of V at x (that is, we know that the two projections $\Pi_j = \Pi \circ \pi_j$ of Y to V are étale infinitesimally around y). Thus out of (i) and (ii), we need to show that π_j $(j = 1, 2)$ are isomorphisms. We shall do this by the following two steps:

Step 1. We show that Π_j is étale over a dense open subscheme of V (this is basically achieved by Propositions 3.14 and 3.15).

Step 2. We show that the two pullbacks $\mathbb{Y}_j := \Pi_j^*\mathbf{A}$ $(j = 1, 2)$ of the universal abelian scheme $\mathbf{A}_{/V}$ by Π_j are isogenous over Y. Writing the prime-to-p level structure of $\mathbb{Y}_j$ as $\boldsymbol{\eta}_j := \Pi_j^*\boldsymbol{\eta}$, for the isogeny $\varphi : \mathbb{Y}_1 \to \mathbb{Y}_2$, we have $\varphi \circ \boldsymbol{\eta}_1 = \boldsymbol{\eta}_2 \circ g$ for some $g \in G(\mathbb{A}^{(p\infty)})$, and we conclude $Y = X$ and X is the graph $\Delta_{1,g} \subset (V \times V)$ of $\tau(g)$ (Corollary 3.16).

After finishing off the case $m = 2$, we proceed by induction on m and show that $a_i/a_j \in \mathcal{T}(\mathbb{Z}_{(p)})$ (for some $i \neq j$) if $X \neq V^m$:

Step 3. Under a suitable assumption on $X \subset V^m$, by induction on m, we show that an irreducible subvariety $X \subset V^m$ (containing $(x, x, \dots, x)$) stable under the diagonal action of $\mathcal{T}(\mathbb{Z}_{(p)})$ (or its p-adically open subgroup) is contained in $V^{m-2} \times \Delta_{1,g}$ after permuting the components V. We get this result by applying Step 2 to the projected image of X to the product $V \times V$ of the last two factors in V^m (Corollary 3.19).

Then the linear independence of $\{a_1(E_{a_1}), \dots, a_m(E_{a_m})\}$ for elements a_j's mutually distinct modulo $\mathcal{T}(\mathbb{Z}_{(p)})$ in the introduction follows easily from this (Corollary 3.21).

In Step 2, the two AVRMs $\mathbb{Y}_{j/Y}$ are (O-linearly) isogenous if and only if $\mathrm{End}_O^{\mathbb{Q}}(\mathbb{Y}_{/Y}) = M_2(F)$ for $\mathbb{Y} := \mathbb{Y}_1 \times_Y \mathbb{Y}_2$. Our argument is by contradiction, supposing $\mathrm{End}_O^{\mathbb{Q}}(\mathbb{Y}_{/Y}) = F \times F$. Since V_K (for an open subgroup $K \subset G(\widehat{\mathbb{Z}})$) is actually defined over a finite field, the generic fiber of $\mathbb{Y}_{/Y}$ is an abelian scheme over a field of finite type over the prime field $\mathbb{F}_p$; so, generically, we can use finiteness theorem of Zarhin–Tate on the endomorphism ring of abelian scheme over a field of finite type over $\mathbb{F}_p$. Since π_j is étale over a big open subset, we can then study $\mathbb{Y}_1 \times_Y \mathbb{Y}_2$ specializing it to many CM points, and in such a way, we exhibit a contradiction against the assumption $\mathrm{End}(\mathbb{Y}_{/Y}) = F \times F$. This type of arguments is impossible just studying $\mathcal{X}$ because the scheme $\mathcal{X}$ has only one closed point and the function field of $\mathcal{X}$ is not finite type over $\mathbb{F}_p$.

The main tool in Step 1 is the local information from the Serre–Tate coordinates we have studied above and Zariski's main theorem (or equivalently, the Stein factorization of the projections $X \to V$), which requires us to have a smooth compactification (a toroidal compatification $\widetilde{V}$ of V). Though the minimal (Satake) compactification V^* of V is easy (and we still have the action of $\overline{\mathcal{E}}(G, \mathfrak{X})$ on the compactification), we lose smoothness which is vital in the use of the Zariski's main theorem. This point adds some technicalities to our arguments.

For a sufficiently small open compact subgroup K so that V_K is smooth over $\mathbb{F}$, we take a smooth toroidal compactification $\widetilde{V}_K$. The toroidal compactification depends on a choice of a simplicial cone decomposition $\mathcal{C}_K$ of the totally positive

cone $F_+^\times = \{\alpha \in F | \alpha \gg 0\}$ into a disjoint union $F_+^\times = \bigsqcup_{C \in \mathcal{C}_K} C$ stable under the multiplication by $O_+^\times$. Fix such a decomposition for K. For smaller $K' \triangleleft K$, we may take the smooth toroidal compactification $\widetilde{V}_{K'}$ associated to the same decomposition $\mathcal{C}_K$. Then K/K' acts faithfully on $\widetilde{V}_{K'}$ extending its action on $V_{K'}$, and $\widetilde{V}_{K'} \to \widetilde{V}_K$ is a finite morphism compatible with the action of K ([DAV] IV.6.7). Then we construct $\widetilde{V} = \varprojlim_K \widetilde{V}_K$ so that the starting maximal compact subgroup acts on $\widetilde{V}$ compatibly with the projection $\widetilde{V} \to V^*$ ([DAV] V.2.5).

Let $x \in V^{ord}(\mathbb{F}) = (V \cap Sh^{ord})(\mathbb{F})$. The fiber at x of $\mathbf{A}_{/V}$ is a test object $(A_x, \overline{\lambda}_x, \eta_x^{(p)})$. The abelian variety $A_{x/\mathbb{F}}$ has complex multiplication by a CM field M/F (by a theorem of Tate: [ABV] Section 22). Thus we have an embedding $\rho : T_x(\mathbb{Z}_{(p)}) \hookrightarrow G(\mathbb{A}^{(p\infty)})$ given by $\alpha\eta_x^{(p)} = \eta_x^{(p)}\rho(\alpha)$. Since the test object $\underline{A}_x$ is given in our application, the point of $Sh^{(p)}_{/\mathcal{W}}$ at which $\underline{A}_x$ is realized as a fiber of $\underline{\mathbf{A}}$ may not be in the neutral component, but it is the image of the neutral component under the right action by $g \in G(\mathbb{A}^{(p\infty)})$. The point x is therefore of the form $[z, g]$ whose level structure η_x is of the form $\eta_z \circ g$ ($g \neq 1$; otherwise, the image of $\mathbb{Q}$-anisotropic torus T_x under $\widehat{\rho}_z$ cannot be diagonal at p in $G(\mathbb{A}^{(\infty)})$). Recall $\rho = \widehat{\rho}_x : T_x(\mathbb{Z}_{(p)}) \hookrightarrow G(\mathbb{A}^{(p\infty)})$ in Section 3.2 with $\mathrm{Im}(\rho)$ in $G(\mathbb{A}^{(p\infty)})/\overline{Z(\mathbb{Z}_{(p)})}$ giving the stabilizer of the point $x = [z, g]$ (Lemma 3.3).

We start with the more general setting of (N0–3) with $m = 2$: Let

$$X_{/\mathbb{F}} \subset V \times V$$

be an irreducible subscheme with $(x, x') \in X^{ord}(\mathbb{F})$ ($X^{ord} = X \cap (V^{ord} \times V^{ord})$) stable under the diagonal action of a p-adically open subgroup $\mathbb{T}$ in $\mathcal{T}(\mathbb{Z}_{(p)})$. We write $\widetilde{V}$ for a smooth toroidal compactification of V, $\widetilde{X}$ for the Zariski closure of X in $\widetilde{V} \times \widetilde{V}$, and $\widetilde{\Pi} : \widetilde{Y} \to \widetilde{X}$ for the normalization $\widetilde{Y}$ of $\widetilde{X}$. The action of $\mathbb{T}$ on X extends to Y. For an open compact subgroups $K_1, K_2 \subset G(\mathbb{A}^{(p\infty)})$, we write V_{12} for $V_{K_1} \times V_{K_2}$, and we define X_{12} for the image of X in V_{12} and $\widetilde{X}_{12}$ for the image of $\widetilde{X}$ in $\widetilde{V}_{12} := \widetilde{V}_{K_1} \times \widetilde{V}_{K_2}$. We write $\widetilde{Y}_{12}$ for the normalization of $\widetilde{X}_{12}$. Thus $\widetilde{Y} = \varprojlim_{K_1 \times K_2} \widetilde{Y}_{12}$. We suppose (see Proposition 3.11 (3))

(DE) *The two projections $\Pi_1, \Pi_2 : Y \to V$ are finite at a point $y \in Y$ above a point (x, x') in $V(\mathbb{F}) \times V(\mathbb{F})$ fixed by the diagonal action of $T_x(\mathbb{Z}_{(p)})$.*

Since x and x' are fixed by $\rho(T_x(\mathbb{Z}_{(p)}))$, A_x and $A_{x'}$ are isogenous and have complex multiplication of the same type (M, Σ) (cf. [D1] Section 7). We also have $\dim_{\mathbb{F}} Y = \dim_{\mathbb{F}} X = \dim_{\mathbb{F}} V$ and $\dim_{\mathbb{F}} \widetilde{Y}_{12} = \dim \widetilde{X}_{12} = \dim \widetilde{V}_K$. If Π_j is not étale at (x, x'), $\Pi_{j,*}(X_*(\widehat{Y}_y)) \subset X_*(\widehat{S}) = O_p$ is an O_p-submodule of finite index, and we find $\alpha \in T_x(\mathbb{Z}_{(p)})$ such that $\alpha^{1-c} X_*(\widehat{S}) = \Pi_{j,*}(X_*(\widehat{Y}_y))$ in $X_*(\widehat{S}) = O_p$. Then $\rho(\alpha^{-1}) \circ \Pi_j$ is étale (by Proposition 3.11 (4)). The action of $\rho(\alpha)$ for a p-adic non-unit α^{1-c} is not an automorphism of V but a "radiciel" endomorphism of V. Indeed, $\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$ acts on $Sh^{(p)}$ as the relative Frobenius map of degree p, and hence if $\widehat{\rho}_x(\alpha)_p K = \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right) K$, the action of $\rho(\alpha)$ coincides with the Frobenius map composed with $\widehat{\rho}_x(\alpha^{(p)})$ on Sh_K. Since any statement

concerning the underlying topological space of our schemes is not affected by "radiciel" endomorphisms, we may assume (for such statements)

(A) Π_1 and Π_2 are étale finite at any point $y \in Y$ above (x, x'),

by modifying Π_j by $\rho(\alpha)$ for $\alpha \in M^\times$. This condition follows from Proposition 3.11 (4) and Corollary 3.13 in the case of our interest: $\mathfrak{b} = \mathrm{Ker}(\phi)$.

As already remarked, we will show that X is the graph of the action by an element $g \in G(\mathbb{A}^{(p\infty)})$. In this process, we may also assume the following condition without losing generality:

(B) $x = x'$.

By moving X under the automorphism $1 \times \rho(b^{(p)})$ of $V \times V$ for suitable $b \in M_{\mathbb{A}}^\times$, we can bring x' to x, and hence we may assume (B).

Let $V_j = \varprojlim_{K_j} V_{K_j}$, resp. $\widetilde{V}_j = \varprojlim_{K_j} \widetilde{V}_{K_j}$ $(j = 1, 2)$ be the j-th factor in $V \times V$, resp. in $\widetilde{V} \times \widetilde{V}$. We now study the finite locus ${}^{fi}V_j = \varprojlim_{K_j} {}^{fi}V_{K_j}$ in V_j of the projection $\Pi_j : Y \to V_j$ for $j = 1, 2$. By definition, the non-finite locus ${}^{nfi}\widetilde{V}_{K_j}$ is the closure of all closed points $v \in \widetilde{V}_{K_j}$ such that $\mathcal{O}_{\widetilde{Y}_{K_j}, y}$ is not finite over $\mathcal{O}_{\widetilde{V}_{K_j}, v}$ for at least one point $y \in \Pi_j^{-1}(v)$. We put ${}^{fi}\widetilde{V}_{K_j} = \widetilde{V}_{K_j} - {}^{nfi}\widetilde{V}_{K_j}$. Similarly, we define the non-flat locus ${}^{nfl}\widetilde{V}_{K_j}$ by the closure of all closed points $v \in \widetilde{V}_{K_j}$ such that $\mathcal{O}_{\widetilde{Y}_{K_j}, y}$ is not flat over $\mathcal{O}_{\widetilde{V}_{K_j}, v}$ for at least one point $y \in \Pi_j^{-1}(v)$. Since a flat morphism is an open map, ${}^{nfl}\widetilde{V}_{K_j}$ is a proper closed subscheme of $\widetilde{V}_{K_j}$. Let ${}^{fl}\widetilde{V}_{K_j} = \widetilde{V}_{K_j} - {}^{nfl}\widetilde{V}_{K_j}$. Thus ${}^{fl}\widetilde{\Pi}_j : \widetilde{\Pi}_j^{-1}({}^{fl}\widetilde{V}_{K_j}) \to {}^{fl}\widetilde{V}_{K_j}$ induced by $\widetilde{\Pi}_j$ is flat. Since ${}^{fl}\widetilde{\Pi}_{K_j}$ is proper flat and generically finite, each fiber of ${}^{fl}\widetilde{\Pi}_{K_j}$ is noetherian of dimension-zero (by [ALG] III.9.5); so, ${}^{fl}\widetilde{\Pi}_{K_j}$ is proper and quasi-finite; so, it is finite. Thus the non-flat locus ${}^{nfl}\widetilde{V}_{K_j}$ contains the non-finite locus ${}^{nfi}\widetilde{V}_{K_j}$; so,

$$ {}^{nfi}\widetilde{V}_{K_j} \text{ is a proper closed subscheme of } \widetilde{V}_{K_j} \text{ inside } {}^{nfl}\widetilde{V}_{K_j}. \tag{3.23} $$

Thus the finite-flat locus $\widetilde{V}_{K_j} - {}^{nfl}\widetilde{V}_{K_j}$ is a nonempty open subscheme. Similarly, we define ${}^{et}V_{K_j}$ by the maximal open subscheme of V_{K_j} over which Π_j is étale (the cuspidal divisors ramifies in $\widetilde{V}_j$ over $\widetilde{V}_{K_j}$; so, the étale locus is in V_j). Any of the properties $? = et, fi, nfl, nfi \ldots$, we write ${}^{?}V_{K_j} = V_{K_j} \cap {}^{?}\widetilde{V}_{K_j}$, ${}^{?}V_j = \varprojlim_{K_j} {}^{?}V_{K_j}$ and $V^{et} = {}^{et}V_1 \cap {}^{et}V_2$ (in other words, over V^{et}, Π_1 and Π_2 are *both* étale).

Proposition 3.14. *Suppose* (DE). *Then we have*

1. *The non-finite loci* ${}^{nfi}\widetilde{V}$ *and non-flat loci* ${}^{nfl}\widetilde{V}$ *of* $\widetilde{\Pi}_j : \widetilde{Y} \to \widetilde{V}$ $(j = 1, 2)$ *in* $\widetilde{V}$ *are of codimension at least* 2, *and* ${}^{nfi}\widetilde{V} \subset {}^{nfl}\widetilde{V}$.
2. *If* $\Pi_1 : \mathcal{Y} \to S$ *and* $\Pi_2 : \mathcal{Y} \to S$ *are étale,* $V^{et} = {}^{et}V_1 \cap {}^{et}V_2$ *is an open dense subscheme of* V *containing* (x, x) *stable under* $\mathbb{T}$.

Proof. As explained before, we may assume (A) and (B). We first show that we have a very big open subscheme ${}^{fi}V_j = \varprojlim_{K_j} {}^{fi}V_{K_j} \subset V_j$ so that each Y_{12} is finite over ${}^{fi}V_{K_j}$ $(j = 1, 2)$ under the two projections $\Pi_j : Y_{12} \to V_{K_j}$. The projection $\widetilde{\Pi}_j : \widetilde{Y}_{12} \to \widetilde{V}_{K_j}$ is a covering generically finite étale; so, over a dense open subscheme ${}^{fi}\widetilde{V}_{K_j} \subset \widetilde{V}_{K_j}$, $\widetilde{\Pi}_j$ is finite. Let $K'_j \subset K_j$ be open subgroups maximal at p. Let $\widetilde{X}'_{12}$ (resp. $\widetilde{Y}'_{12}$) be the image of $\widetilde{X}$ in $\widetilde{V}_{K'_1} \times \widetilde{V}_{K'_2}$ (resp. the normalization of $\widetilde{X}'_{12}$). We have the commutative diagram:

$$\begin{array}{ccccc} \widetilde{Y}'_{12} & \xrightarrow{\text{finite}} & \widetilde{X}'_{12} & \longrightarrow & \widetilde{V}_{K'_j} \\ \downarrow & & \downarrow{\scriptstyle \text{finite}} & & \pi \downarrow{\scriptstyle \text{finite}} \\ \widetilde{Y}_{12} & \xrightarrow[\text{finite}]{} & \widetilde{X}_{12} & \longrightarrow & \widetilde{V}_{K_j}. \end{array} \tag{3.24}$$

The middle down-pointed arrow is finite because $\widetilde{V}_{K'_1} \times \widetilde{V}_{K'_2} \to \widetilde{V}_{K_1} \times \widetilde{V}_{K_2}$ is finite. From this, the left-most down-pointed arrow is finite. Thus ${}^{fi}\widetilde{V}_{K'_j} = \pi^{-1}({}^{fi}\widetilde{V}_{K_j})$. Put ${}^{fi}V_{K_j} = {}^{fi}\widetilde{V}_{K_j} \cap V_{K_j}$. Then we have ${}^{fi}V_{K'_j} = \pi^{-1}({}^{fi}V_{K_j})$, and ${}^{fi}V_j = \varprojlim_{K_j} {}^{fi}V_{K_j} \subset V$ (for $j = 1, 2$) is a dense open subscheme of V whose image in V_{K_j} is ${}^{fi}V_{K_j}$. In other words, the projection $V_{K'_j} \twoheadrightarrow V_{K_j}$ induces surjective projection of the finite loci

$${}^{fi}V_{K'_j} = \pi^{-1}({}^{fi}V_{K_j}) \twoheadrightarrow {}^{fi}V_{K_j} \text{ for } K'_j \subset K_j \text{ for a fixed } K_j, \tag{3.25}$$

and the image of the non-finite locus ${}^{nfi}\widetilde{V}_{K'_j} := \widetilde{V}_{K'_j} - {}^{fi}\widetilde{V}_{K'_j}$ in $\widetilde{V}_{K_j}$ is a proper closed subscheme independent of $K'_j \subset K_j$ (for a fixed open compact subgroup $K_1 \times K_2 \subset G(\mathbb{A}^{(\infty)})$ maximal at p). The scheme ${}^{nfi}\widetilde{V}_j = \varprojlim_{K'_j} {}^{nfi}\widetilde{V}_{K'_j}$ is the non-finite locus of $\Pi_j : X \to V$ $(j = 1, 2)$. Put ${}^{fi}Y_j = \Pi_j^{-1}({}^{fi}V_j)$. Then $\Pi_j : {}^{fi}Y_j \to V_j$ is finite for $j = 1, 2$, and ${}^{fi}V_j$ is the maximal open subscheme of V with this property. By definition, $\widetilde{\Pi}_j : \widetilde{Y}_{12} \to \widetilde{V}_{K_j}$ is the normalization of $\widetilde{\pi}_j : \widetilde{X}_{12} \to \widetilde{V}_{K_j}$. Since $\widetilde{\pi}_j : \widetilde{V}_{12} \to \widetilde{V}_{K_j}$ is projective, $\widetilde{\pi}_j : \widetilde{X}_{12} \to \widetilde{V}_{K_j}$ is projective. Since the projection: $\widetilde{Y}_{12} \to \widetilde{X}_{12}$ is finite, it is projective; so, $\widetilde{\Pi}_j$ is projective, and we can take the Stein factorization of $\widetilde{Y}_{12} \to Y^{st}_{12} \xrightarrow{\Pi^{fi}_j} \widetilde{V}_{K_j}$ of $\widetilde{\Pi}_j$ (see [ALG] III.11.5). Thus we have

$$\widetilde{Y}_{12} \times_{\widetilde{V}_{K_j}} {}^{fi}\widetilde{V}_{K_j} \cong Y^{st}_{12} \times_{\widetilde{V}_{K_j}} {}^{fi}\widetilde{V}_{K_j}, \tag{3.26}$$

because over ${}^{fi}\widetilde{V}_{K_j}$, $\widetilde{Y}_{12} \to Y^{st}_{12}$ is birational with connected fiber.

We consider the non-finite locus ${}^{nfi}\widetilde{V}_j \subset \widetilde{V}$ and non-flat locus ${}^{nfl}\widetilde{V}_j \subset \widetilde{V}$ of $\widetilde{\Pi}_j : \widetilde{Y} \to \widetilde{V}$. As we already remarked before stating the proposition, ${}^{nfi}\widetilde{V}_j \subset {}^{nfl}\widetilde{V}_j$, and ${}^{nfi}\widetilde{V}_j$ is a proper closed subscheme of $\widetilde{V}_j$. Since $\Pi^{st}_j : \widetilde{Y} \to Y^{st}$ is birational and Y^{st} is projective and normal, ${\Pi^{st}_j}^{-1}$ is well defined outside the closed subscheme ${\Pi^{fi}_j}^{-1}({}^{nfl}\widetilde{V}_{K_j}) \subset Y^{st}_{12}$ of codimension ≥ 2 (see [ALG] V.5.1).

Since Π_1^{fi} is finite, ${}^{nfi}\widetilde{V}_{K_j}(\subset {}^{nfl}\widetilde{V}_{K_j})$ is at least of codimension 2 in $\widetilde{V}_{K_j}$. The projection $\widetilde{V}_{K'_j} \twoheadrightarrow \widetilde{V}_{K_j}$ again sends ${}^{nfi}\widetilde{V}_{K'_j}$ into ${}^{nfi}\widetilde{V}_{K_j}$, ${}^{nfi}\widetilde{V}_j = \varprojlim_{K_j} {}^{nfi}\widetilde{V}_{K_j}$ is a closed pro-subscheme of codimension 2 of the pro-variety $\widetilde{V}$ by the stability (3.25) of the non-finite locus with respect to K'_j (and by [EGA] IV.8.2.9).

The scheme Y_{12}^{st} is normal dominant finite over $\widetilde{V}_{K_j}$ (and is generically étale finite). Since $\widetilde{V}_{K_j}$ is smooth if K_j is sufficiently small, the ramified (non-étale) locus ${}^{ram}\widetilde{V}_{K_j} = \widetilde{V}_{K_j} - {}^{et}\widetilde{V}_{K_j}$ of Y_{12}^{st} over $\widetilde{V}_{K_j}$ is a divisor of V_{K_j}; so, it is of at least codimension 1 for a given K_j. Since $\widetilde{V}_{K_j} - V_{K_j}$ is the cuspidal divisor, the ramified (non-étale) locus ${}^{ram}V_{K_j} \subset V_{K_j}$ of Y_{12}^{st} over V_{K_j} is a divisor of V_{K_j}; so, it is of at least codimension 1 for a given K_j. To extend this result to the pro-variety V_j, we need to show that ${}^{ram}V_{K'_j}$ is sent into ${}^{ram}V_{K_j}$ under the projection map $V_{K'_j} \twoheadrightarrow V_{K_j}$. To show this, we look into the following commutative diagram similar to (3.24) (removing cuspidal divisors):

$$\begin{array}{ccccc} Y'_{12} & \xrightarrow{\text{finite}} & X'_{12} & \xrightarrow{\hookrightarrow} & V_{K'_1} \times V_{K'_2} \\ {\scriptstyle \pi_Y}\downarrow{\scriptstyle \text{finite}} & & {\scriptstyle \pi_X}\downarrow{\scriptstyle \text{finite}} & & {\scriptstyle \pi}\downarrow{\scriptstyle \text{étale}} \\ Y_{12} & \xrightarrow[\text{finite}]{} & X_{12} & \xrightarrow{\hookrightarrow} & V_{K_1} \times V_{K_2}. \end{array}$$

Taking fiber products, we get morphisms (from the commutativity of the above diagram): $Y'_{12} \to Y_{12} \times_{X_{12}} X'_{12}$, $X'_{12} \to X_{12} \times_{(V_{K_1} \times V_{K_2})} (V_{K'_1} \times V_{K'_2})$ and $Y_{12} \times_{(V_{K_1} \times V_{K_2})} (V_{K'_1} \times V_{K'_2}) \to X_{12} \times_{(V_{K_1} \times V_{K_2})} (V_{K'_1} \times V_{K'_2})$. Since the pull-back $X_{12} \times_{(V_{K_1} \times V_{K_2})} (V_{K'_1} \times V_{K'_2})$ of X_{12} to $V_{K'_1} \times V_{K'_2}$ is a closed subscheme of $V_{K'_1} \times V_{K'_2}$ étale over X_{12} containing (x, x). Thus by definition, X'_{12} is a closed irreducible subscheme of $X_{12} \times_{(V_{K_1} \times V_{K_2})} (V_{K'_1} \times V_{K'_2})$ of the same dimension. Thus X'_{12} is the irreducible component of $X_{12} \times_{(V_{K_1} \times V_{K_2})} (V_{K'_1} \times V_{K'_2})$ containing (x, x) and covering X_{12}, and π_X is étale finite. Thus we have a commutative diagram:

$$\begin{array}{ccc} Y_{12} \times_{X_{12}} X'_{12} & \xrightarrow{\text{étale}} & Y_{12} \\ \downarrow & & \downarrow{\scriptstyle \text{normalization}} \\ X'_{12} & \xrightarrow[\text{étale}]{} & X_{12} \end{array}$$

Since étale morphisms are isomorphisms at the level of completed local rings, they commute with the formation of normalization. Thus Y'_{12} is an irreducible component of $Y_{12} \times_{X_{12}} X'_{12}$, and therefore π_Y is étale finite. Hence, the projection $V_{K'_j} \twoheadrightarrow V_{K_j}$ sends ${}^{ram}V_{K'_j}$ into ${}^{ram}V_{K_j}$, and ${}^{ram}V_j = \varprojlim_{K_j} {}^{ram}V_{K_j}$ is a closed pro-subscheme of codimension 1 (by [EGA] IV.8.2.9) of the pro-variety V_j (whose image in V_{K_j} is contained in ${}^{ram}V_{K_j}$).

By (3.26), ${}^{et}V_j$ contains $V_j - ({}^{nfi}V_j \cup {}^{ram}V_j)$ which is an open dense subscheme of V_j, and hence $V^{et} = {}^{et}V_1 \cap {}^{et}V_2$ is open dense in V. By our assumption, $(x, x) \in V^{et}$, and V^{et} is stable under $\mathbb{T}$, since étaleness is preserved by the action of $\mathbb{T}$. □

Let v be a closed point of $Y^{ord}_{/\mathbb{F}} = Y \times_{V^2} (V^{ord})^2$ above $(v_1, v_2) \in V^2$. We consider the formal completion $\widehat{Y}_v$ (resp. $\widehat{V}_{\Pi_j(v)}$) along v (resp. $\Pi_j(v)$). If $\Pi_1 \times \Pi_2$ embeds $\widehat{Y}_v$ into $\widehat{V}_{v_1} \times \widehat{V}_{v_2}$ and the equation defining $\widehat{Y}_v$ in $\widehat{V}_{\Pi_1(v)} \times \widehat{V}_{\Pi_2(v)}$ is given by $t_1^a = t_2^b$ for the Serre–Tate coordinate t_j of $\widehat{V}_{v_j}$, we call Y *O-linear* at $\overline{v}$. Write $Y_{lin} \subset Y^{ord}(\mathbb{F})$ for the subset of all closed O-linear points.

Proposition 3.15. *Suppose* (DE)*, and let the notation be as in* (N0–3) *above* Proposition 3.8 *for $m = 2$. In particular, let X be a Zariski-closure of $\mathcal{X}$ in $V \times V$ and $Y \to X$ be the normalization of X. The subset $Y_{lin} \subset Y$ of O-linear points as defined above contains the set of all closed points of an open dense subscheme Y^{lin} in Y^{ord}. In other words, at each closed point $y' \in Y^{lin}(\mathbb{F})$, the formal completion of Y^{lin} along y' is defined by a linear equation.*

In the following corollary, we will find that the subvariety X is a graph of the action of an element in $G(\mathbb{A}^{(p\infty)})$; so, we conclude $Y^{lin} = X^{lin} = X^{ord}$. Since X^{ord} is Tate-linear at densely populated v in the image of Y^{lin}, by [C4] Proposition 5.3, X^{ord} is weakly Tate O-linear; so, Y^{ord} is Tate O-linear. We shall give here an argument (again suggested by Chai) sufficient to prove the weaker version as stated above.

Proof. Modifying Π_j by $\rho(\alpha)$ for $\alpha \in M^\times$ does not affect O-linearity at closed point of Y over (x, x'); so, by changing Π_j by $\rho(\alpha_j) \circ \Pi_j$ for $\alpha \in M^\times$ if necessary, we assume (A) and (B) (thus, Π_j is étale for $j = 1, 2$). An endomorphism $a_x \in \mathrm{End}(A_x)$ induces an endomorphism of the deformation space $\widehat{S} = \widehat{V}_x = \widehat{\mathbb{G}}_m \otimes O$. Modifying a_x by the central action of O_p, we may assume that a_x is the identity on the connected component of $A_x[p^\infty]$ without affecting the endomorphism of the deformation space $\widehat{S}$ induced by a_x. We fix an ordinary level p–structure η_p^{ord} on A_x. Identifying $\mathrm{End}_O(\widehat{S}) = O_p$ by η_p^{ord}, the action of a_x on $\widehat{S}$ is then given by the action of a_x over the étale quotient $A_x[p^\infty]^{et}$. By Proposition 3.11 (1), $\mathcal{Y}$ is O-linear at a point $y \in Y$ above (x, x), and hence, by Remark 3.12, we may assume that the formal completion $\widehat{Y}_y$ along y is defined by $t'^u = t^v$ $(u, v \in O_p = \mathrm{End}_O(\widehat{S}))$ for the Serre–Tate coordinate (t, t') of $\widehat{S} \times \widehat{S}$ for $\widehat{S} = \widehat{V}_x$, where t and t' are associated to the ordinary level structure η_p^{ord}. Since the two projections $\Pi_1, \Pi_2 : Y \to V$ are étale at y (by (A)), $a = v/u$ is a unit in O_p, and $\widehat{Y}_y$ is defined by $t' = t^a$. We write $a_x \in \mathrm{End}(A_x) \otimes_{\mathbb{Z}} \mathbb{Z}_p = \mathrm{End}(A_x[p^\infty])$ for the endomorphism inducing $a \in \mathrm{End}(\widehat{S})$ as normalized above. Then a_x is a unit in $\mathrm{End}_O(A_x[p^\infty])$.

We consider the universal abelian scheme $\mathbf{A}_{/V}$. We pull it back to $\Pi : Y \to X \subset V \times V$: $\mathbb{Y}_1 = \Pi_1^* \mathbf{A}$ and $\mathbb{Y}_2 = \Pi_2^* \mathbf{A}$. Identifying A_x with the fibers $\mathbb{Y}_{j,y}$ of $\mathbb{Y}_j$ $(j = 1, 2)$ at y, we regard the unit $a_x \in \mathrm{End}(A_x[p^\infty])$ as a homomorphism $a_x : \mathbf{A}_{1,x}[p^\infty] = A_x[p^\infty] \to A_x[p^\infty] = \mathbf{A}_{2,x}[p^\infty]$.

We now reduce the existence of the desired non-empty open subscheme $Y^{lin} \subset Y^{ord}$ to the existence of an étale irreducible covering $\widetilde{U}$ over an open dense subscheme $U \subset Y^{ord}$ containing the given point $y \in Y^{ord}$ such that a_x extends to an isomorphism $\widetilde{a} : \mathbb{Y}_1[p^\infty]_{/\widetilde{U}} \to \mathbb{Y}_2[p^\infty]_{/\widetilde{U}}$ of Barsotti-Tate groups over $\widetilde{U}$. Thus supposing the existence of the open subscheme U and

such an extension $\widetilde{a}$ over $\widetilde{U}$, we specify the open subscheme Y^{lin}. Shrinking $U \subset Y^{ord}$ (keeping y inside U), we may assume that the projections $\Pi_j : U \to V$ $(j = 1, 2)$ are both étale (cf. Proposition 3.14), because the projections are étale at the specific point y. Then the formal completion $\widehat{Y}_u$ along $u \in U$ with $\Pi(u) = (u_1, u_2) \in V \times V$ is isomorphic by Π_j $(j = 1, 2)$ to the universal deformation space of the p–divisible O–module $A_{u_j}[p^\infty]$ carrying the universal deformation $\mathbb{Y}_j[p^\infty]_{/\widehat{Y}_u} \cong \mathbf{A}_j[p^\infty]_{/\widehat{V}_{u_j}}$ via $\Pi_j : \widehat{Y}_u \cong \widehat{V}_{u_j}$. Choose an ordinary level structures $\eta^{ord}_{j,p} : F_p/O_p \cong A_{u_j}[p^\infty]$. Then the canonical coordinate t_j of $\mathbf{A}[p^\infty]_{/\widehat{V}_{u_j}} \cong \mathbb{Y}[p^\infty]_{/\widehat{Y}_u}$ is given by $t_j = \lim_{n\to\infty} \text{“}p^n\text{”}(\eta^{ord}_{j,p}(p^{-n}))$ for the Drinfeld's lift “p^n” in Theorem 2.1. Another level structure $\widetilde{a} \circ \eta^{ord}_{1,p}$ of $A_{u_2}[p^\infty]$ gives rise to the coordinate $t_2^{a_u}$ for a unit $a_u \in O_p^\times$ because $\mathrm{Aut}_O(\widehat{Y}_u) = O_p^\times$. Thus we get the relation $t_1 = t_2^{a_u}$ valid on $\widehat{Y}_u$, because $\widetilde{a}$ sends the $t_1 = \lim_n \text{“}p^n\text{”}(\eta^{ord}_{1,p}(p^{-n}))$ to $t_2^{a_u} = \lim_n \text{“}p^n\text{”}(\widetilde{a}(\eta^{ord}_{1,p}(p^{-n})))$. In other words, $\widehat{Y}_u$ is contained in the O-linear formal subscheme $\widehat{Y}'_u$ defined by $t_1 = t_2^{a_u}$. Since $\widehat{Y}_u \hookrightarrow \widehat{V}_{u_1} \times \widehat{V}_{u_2}$ is a smooth formal subscheme with two isomorphisms $\widehat{Y}_u \cong \widehat{V}_{u_j}$ induced by Π_j, we find $\widehat{Y}_u = \widehat{Y}'_u$, and hence $\widehat{Y}_u$ is defined by $t_1 = t_2^{a_u}$. Thus we may put $Y^{lin} = U$.

Next we shall show that a_x extends to $\widehat{a} : \mathbb{Y}_1[p^\infty] \to \mathbb{Y}_2[p^\infty]$ over $\widehat{Y}_y$. Identifying A_x with the fibers $\mathbb{Y}_{j,x}$ of $\mathbb{Y}_j$ $(j = 1, 2)$ at y, we regard the unit $a_x \in \mathrm{End}(A_x[p^\infty])$ as a homomorphism $a_x : \mathbb{Y}_{1,y}[p^\infty] = A_x[p^\infty] \to A_x[p^\infty] = \mathbb{Y}_{2,y}[p^\infty]$. As pointed out by one of the referees of this paper, the formal completion $\widehat{Y} = \widehat{Y}_y$ of Y along y is the maximal subscheme of $\widehat{S} \times \widehat{S}$ over which this a_x extends to a homomorphism $\widehat{a} : \mathbb{Y}_1[p^\infty]_{/\widehat{Y}} \to \mathbb{Y}_2[p^\infty]_{/\widehat{Y}}$ of $\widehat{Y}$–group schemes. To find $\widehat{a}$, as above, we identify a with an element of O_p by projecting a_x down to $\mathrm{End}_O(A_x[p^\infty]^{et}) = O_p$ for the maximal étale quotient $A_x[p^\infty]^{et}$ of the p–divisible group. The isogeny action $\rho(\alpha) : \mathbf{A}_{/V} \to \mathbf{A}_{/V}$ for $\alpha \in \mathrm{End}(A_x) = \mathfrak{O}$ induces $\rho(\alpha) : \mathbb{Y}_1[p^n]_{\widehat{Y}} \xrightarrow[\Pi_1]{\sim} \mathbf{A}[p^n]_{/\widehat{V}_x} \xrightarrow{\alpha} \mathbf{A}[p^n]_{/\widehat{V}_x} \xrightarrow[\Pi_2^{-1}]{\sim} \mathbb{Y}_2[p^n]_{/\widehat{Y}}$ if $\alpha \equiv a \mod p^n\mathfrak{O}_{\mathbf{p}}$ for $\mathbf{p} = \prod_{\mathfrak{p}\in\Sigma_p} \mathfrak{p}$ and if $\alpha \equiv 1 \mod p^n\mathfrak{O}_{\mathbf{p}^c}$. Choose α as above, and write it as α_n. Therefore the isogeny action of $\rho(\alpha_n)$ on $\mathbf{A}$ gives rise to $\alpha_n : \mathbb{Y}_1[p^n]_{/\widehat{Y}_n} \to \mathbb{Y}_2[p^n]_{/\widehat{Y}_n}$ well defined over an infinitesimal neighborhood $\widehat{Y}_n$ of y (isomorphic to the connected component of $A_x[p^n]$). Indeed, the embedding $\widehat{Y} \hookrightarrow \widehat{S} \times \widehat{S}$ is given by (t, t^a), $\rho(\alpha_n)$ sends $\mathbb{Y}_{1,v}[p^n]_{/\widehat{Y}_n} \xrightarrow[\Pi_1]{\sim} \mathbf{A}[p^n]_{\widehat{V}_n}$ to $\mathbb{Y}_{2,\rho(\alpha_n)_p(v)}[p^n]_{/\widehat{Y}_n} \xrightarrow[\Pi_2]{\sim} \mathbf{A}[p^n]_{/\widehat{V}_n}$, and $v^a = \rho(\alpha_n)_p(v)$ as long as $v \in \widehat{Y}_n$. Here $\widehat{V}_n$ is the infinitesimal neighborhood of x isomorphic to the connected component of $A_x[p^n]$), $\rho(\alpha_n)_p \in G(\mathbb{Q}_p)$ is the p–component of $\rho(\alpha_n)$, and α_n gives rise to $a|_{\mathbb{Y}_1[p^n]}$ after base-change to $\widehat{Y}_n$. By taking the limit $\widehat{a} = \lim_{n\to\infty} \alpha_n|_{\mathbb{Y}_2[p^n]_{/\widehat{Y}_n}}$ with respect to n, we find out that a_x gives rise to a unique homomorphism $\widehat{a} : \mathbb{Y}_1[p^\infty]_{/\widehat{Y}} \to \mathbb{Y}_2[p^\infty]_{/\widehat{Y}}$ of Barsotti-Tate groups over $\widehat{Y}$.

To show the existence of U, we follow an argument of Chai in the proof of [C4] Proposition 8.4. We choose a sufficiently small open compact subgroup $K_1 \times K_2 \subset G(\widehat{\mathbb{Z}})^2$ maximal at p so that $\widehat{Y}_{12,y} = \widehat{Y}_y$. Since Y^{ord}_{12} is irreducible, we only need to find a non-empty open subscheme $U_{12} \subset Y^{ord}_{12}$ with the required

property (then U is given by the pull back of U_{12} to Y^{ord}). We have $\widehat{a}$ well defined over $\widehat{Y} = \widehat{Y}_{12,y}$. For any reduced local $\mathcal{O}_{Y_{12},y}$–algebra $\mathcal{R}$, an O-linear isomorphism $\alpha : \mathbb{Y}_1[p^\infty]_{/\mathcal{R}} \to \mathbb{Y}_2[p^\infty]_{/\mathcal{R}}$ inducing a_x on $A_x[p^\infty]$ (if exists) is unique. Write simply $\widehat{\mathcal{O}}$ for $\widehat{\mathcal{O}}_{Y_{12},y} = \widehat{\mathcal{O}}_{Y,y}$ and $\mathcal{O}$ for $\mathcal{O}_{Y_{12},y}$. Since $\widehat{\mathcal{O}} \otimes_{\mathcal{O}} \widehat{\mathcal{O}}$ is reduced, the pullbacks of $\widehat{a}$ by the two projections $\pi_j : \widehat{Y} \times_{\mathcal{O}} \widehat{Y} \to \widehat{Y}$ coincide by the above uniqueness; that is, $\widehat{a}$ satisfies the descent datum (relative to $\widehat{\mathcal{O}}/\mathcal{O}$). By faithfully flat descent, $\widehat{a}$ descends to $\mathcal{O}$ and hence we can find an open Zariski neighborhood $U_{12} \subset Y_{12}^{ord}$ of y over which we have an extension $\widetilde{a}$ of a_x. Then $U = U_{12} \times_{Y_{12}} Y$ does the job.

There is an alternative argument: By M. Artin's approximation theorem in [A] (see also [NMD] Theorem 16 in 3.6) applied to the truncated $\widehat{a}_n = \widehat{a}|_{\mathbb{Y}_1[p^n]}$ defined over $\widehat{Y} = \widehat{Y}_{12,y}$, we can find an étale neighborhood $\widetilde{U}_n$ of y such that we have an extension $\widetilde{a}_n$ over $\widetilde{U}_n$. The two pullbacks of $\widetilde{a}_n$ to $\widetilde{U}_n \times_{Y_{12}^{ord}} \widetilde{U}_n$ coincide by the uniqueness of the extension of a_x because of the reducedness of $\widetilde{U}_n \times_{Y_{12}^{ord}} \widetilde{U}_n$; so, this time by étale descent, we get the desired open dense subscheme U_n. Since $\mathrm{Isom}_{Y_{12}^{ord}}(\mathbb{Y}_1[p^m], \mathbb{Y}_2[p^m])$ for $m > n$ is finite flat over $\mathrm{Isom}_{Y_{12}^{ord}}(\mathbb{Y}_1[p^n], \mathbb{Y}_2[p^n])$, the open set U_n stabilizes as n grows. Taking the intersection $U_{12} = \bigcap_n U_n$ with respect to n, we get the desired open dense subscheme $U_{12} \subset Y_{12}^{ord}$. □

Corollary 3.16. *Let the notation and the assumption be as in* Proposition 3.15. *Then X is everywhere smooth and $X = Y$. Moreover, if* (DE) *is satisfied, there exist non-zero $\alpha, \beta \in \mathfrak{O}_{(p)}$ such that X coincides with the skew diagonal*

$$\Delta_{\alpha,\beta} = \{(\rho(\alpha)(v), \rho(\beta)(v)) | v \in V\}.$$

If Π_1 and Π_2 are étale finite, we may assume that $(\alpha, \beta) = (1, \beta)$ with $\beta \in \mathfrak{O}_{(p)}^\times$.

Proof. If (DE) fails, $Y = X = V^2$ by Proposition 3.11 (3); so, the assertion follows trivially. We may assume (DE), (A) and (B) as indicated after stating (DE). We follow an argument of Chai which is a version of the argument in [C4] Section 8 adjusted to our self-product of the Hilbert modular variety. Since the two projections $\Pi_j : Y \to V_j$ are dominant, we have $\mathrm{End}(\mathbb{Y}_j) \otimes \mathbb{Q} = F$ for $\mathbb{Y}_j = \Pi_j^* \mathbf{A} = \mathbf{A} \times_{V_j, \Pi_j} Y$. Let $\mathbb{Y}_{/Y} = \mathbb{Y}_1 \times_Y \mathbb{Y}_2$. Thus there are only two possibilities for $\mathrm{End}^{\mathbb{Q}}(\mathbb{Y}) = \mathrm{End}(\mathbb{Y}_{/Y}) \otimes \mathbb{Q}$: Either $\mathrm{End}^{\mathbb{Q}}(\mathbb{Y}) = F \times F$ or $\mathrm{End}^{\mathbb{Q}}(\mathbb{Y}) = M_2(F)$. Suppose that $\mathrm{End}^{\mathbb{Q}}(\mathbb{Y}) = M_2(F)$. By semi-simplicity of the category of abelian schemes, we have two commuting idempotent $e_j \in \mathrm{End}^{\mathbb{Q}}(\mathbb{Y})$ such that $e_j(\mathbb{Y}) = \mathbb{Y}_j$. Since $\mathrm{End}^{\mathbb{Q}}(\mathbb{Y}) = M_2(F)$, we can find an invertible element $\widetilde{\beta}$ in $GL_2(O_{(p)}) \subset M_2(F)$ such that $\widetilde{\beta} \circ e_1 = e_2$; so, $\widetilde{\beta} : \mathbb{Y}_1 \to \mathbb{Y}_2$ is an isogeny, whose specialization to the fiber of $\mathbb{Y}_j$ $(j = 1, 2)$ at y gives rise to an endomorphism $\beta \in \mathrm{End}(A_x) \otimes \mathbb{Q}$. Thus the isogeny $\widetilde{\beta}$ is induced by $\rho(\beta)$ (this point is explained more carefully after proving $\mathrm{End}^{\mathbb{Q}}(\mathbb{Y}) = M_2(F)$).

We suppose $\mathrm{End}^{\mathbb{Q}}(\mathbb{Y}) = F \times F$ and try to get a contradiction (in order to prove that $\mathrm{End}^{\mathbb{Q}}(\mathbb{Y}) = M_2(F)$). We pick a sufficiently small $K_1 = K_2 = K \subset G(\mathbb{A}^{(p\infty)})$ maximal at p so that V_K is smooth. For the moment, we assume that K is open compact. The variety V_K is naturally defined over a finite extension

$\mathbb{F}_q/\mathbb{F}_p$ as the solution of the moduli problem $\mathcal{P}'_{K,\mathfrak{c}}$ in (3.6) for the polarization ideal $\mathfrak{c}$ of A_x (the minimal choice of $\mathbb{F}_q$ is the residue field of $\mathcal{W}\cap k_K$ for Shimura's field k_K of definition of $V_K \subset Sh_{K/\mathbb{Q}}$). The universal abelian scheme $\mathbf{A}_K$ is therefore defined over $V_{K/\mathbb{F}_q}$, and $\mathbf{A}_K$ is a variety of finite type over $\mathbb{F}_q$. Replacing q by its finite power, we may assume that $X_{12} \subset V_K \times V_{K/\mathbb{F}_q}$ is stable under the Galois action $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$, and hence it has a unique geometrically irreducible model $X_{12/\mathbb{F}_q} \subset V_K \times V_{K/\mathbb{F}_q}$ defined over $\mathbb{F}_q$. Let $Y_{12/\mathbb{F}_q}$ be the normalization of $X_{12/\mathbb{F}_q}$. Then $Y_{12} = Y_{12/\mathbb{F}_q} \times_{\mathbb{F}_q} \mathbb{F}$. We write $\mathbb{Y}_{12/Y_{12/\mathbb{F}_q}}$ for the abelian scheme $\mathbf{A}_K^2 \times_{(V_K\times V_K)} Y_{12}$. Then $\mathbb{Y}_{12}$ is an abelian scheme over the variety $Y_{12/\mathbb{F}_q}$ of finite type over $\mathbb{F}_q$. Let η be the generic point of $Y_{12/\mathbb{F}_q}$, and write $\overline{\eta}$ for the geometric point over η and $\mathbb{F}_q(\overline{\eta})^{sep}$ for the separable algebraic closure $\mathbb{F}_q(\overline{\eta})^{sep}$ of $\mathbb{F}_q(\eta)$ in $\mathbb{F}_q(\overline{\eta})$. Take an odd prime $\ell \neq p$, and consider the ℓ–adic Tate module $T_\ell(\mathbb{Y}_{\overline{\eta}})$ for the generic fiber $\mathbb{Y}_{\overline{\eta}}$ of $\mathbb{Y}$. We consider the image of the Galois action $\mathrm{Im}(\mathrm{Gal}(\mathbb{F}_q(\overline{\eta})^{sep}/\mathbb{F}_q(\eta)))$ in $GL_{O_\ell\times O_\ell}(T_\ell(\mathbb{Y}_{\overline{\eta}}))$. Then by a result of Zarhin ([Z] and [DAV] Theorem V.4.7), the Zariski closure over $\mathbb{Q}$ of $\mathrm{Im}(\mathrm{Gal}(\mathbb{F}_q(\overline{\eta})^{sep}/\mathbb{F}_q(\eta)))$ is a reductive subgroup $\mathcal{G}$ of $GL_{F_\ell\times F_\ell}(T_\ell(\mathbb{Y}_{\overline{\eta}})\otimes\mathbb{Q})$, and $\mathrm{Im}(\mathrm{Gal}(\mathbb{F}_q(\overline{\eta})^{sep}/\mathbb{F}_q(\eta)))$ is an open subgroup of $\mathcal{G}(\mathbb{Q}_\ell)$. Moreover, by Zarhin's theorem, the centralizer of $\mathcal{G}$ in $GL_{F_\ell\times F_\ell}(T_\ell(\mathbb{Y}_{\overline{\eta}})\otimes\mathbb{Q})$ is $\mathrm{End}(\mathbb{Y})\otimes\mathbb{Q}_\ell$. Since the reductive subgroups of $GL(2)$ are either tori or contain $SL(2)$, the derived group $\mathcal{G}_1(\mathbb{Q}_\ell)$ of $\mathcal{G}(\mathbb{Q}_\ell)$ has to be $SL_2(F_\ell\times F_\ell)$. By Chebotarev's density, we can find a set of closed points $u \in Y_{12}(\mathbb{F})$ with positive density such that the Zariski closure in $\mathcal{G}$ of the subgroup generated by the Frobenius element $Frob_u \in \mathrm{Im}(\mathrm{Gal}(\mathbb{F}_q(\overline{\eta})^{sep}/\mathbb{F}_q(\eta)))$ at u with $\Pi(u) = (u_1, u_2)$ ($u_j \in V(\mathbb{F})$) is a torus containing a maximal torus $T_u = (T_{u_1} \times T_{u_2}) \cap \mathcal{G}_1$ of the derived group $\mathcal{G}_1$ of $\mathcal{G}$. In particular the centralizer of T_u in $\mathcal{G}_1$ is itself. Thus $\mathbb{Y}_u$ is isogenous to a product of two non-isogenous absolutely simple abelian varieties $Y_1 = A_{u_1}$ and $Y_2 = A_{u_2}$ with multiplication by F defined over a finite field. The endomorphism algebra $M_j = \mathrm{End}^{\mathbb{Q}}(Y_j)$ is a CM quadratic extension of F generated over $\mathbb{Q}$ by the relative Frobenius map ϕ_j induced by $Frob_u$. The relative Frobenius map $Frob_u$ acting on $X_*(\widehat{V}_{u_1}) \cong O_p$ has $[F:\mathbb{Q}]$ distinct eigenvalues $\{\phi_1^{(1-c)\sigma} | \sigma \in \Sigma_1\}$ for the CM type Σ_1 of Y_1, which differ from the eigenvalues of $\phi_2 \in \mathrm{End}(Y_2)$ on $X_*(\widehat{V}_{u_2}) \cong O_p$. Since we have proven that over the open dense subscheme $U = Y^{lin}$ of Y, the formal completion of U at $u \in U$ with $u = (u_1, u_2) \in X \subset V^2$ is canonically isomorphic to a formal subtorus $\widehat{Z} \subset \widehat{V}_{u_1} \times \widehat{V}_{u_2}$ with co-character group $X_*(\widehat{Z}) \cong O_p$, we may assume that our point $u = (u_1, u_2)$ as above is in the (open dense) image U_{12} of U in X_{12} (because the set of such u's has positive density). Projecting $X_*(\widehat{Z})$ down to the left and the right factors V_K, the projection map $X_*(\widehat{Z}) \to X_*(\widehat{V}_{u_j})$ is actually an injection commuting with the action of $Frob_u$. Thus $Frob_u$ has more than $[F:\mathbb{Q}]$ distinct eigenvalues on $X_*(\widehat{Z})$, which is a contradiction. Thus we conclude that $\mathrm{End}^{\mathbb{Q}}(\mathbb{Y}) = M_2(F)$ for any choice of small open compact subgroups K maximal at p. Passing to the limit, we may assume that $K = G(\mathbb{Z}_p)$ (as we do hereafter), and we still have $\mathrm{End}^{\mathbb{Q}}(\mathbb{Y}) = M_2(F)$.

As we have remarked at the beginning, $\mathrm{End}^{\mathbb{Q}}(\mathbb{Y}) = M_2(F)$ implies that we have an isogeny $\widetilde{\beta} : \mathbb{Y}_1 \to \mathbb{Y}_2$ over Y. Writing $\eta_j^{(p)}$ for the prime-to–p

level structure of $\mathbb{Y}_j$ inducing the prime-to–p level structure already chosen for $A_x = \mathbb{Y}_{1,y} = \mathbb{Y}_{2,y}$ at $y \in Y$, we find that $\widetilde{\beta} \circ \eta_1^{(p)} = \eta_2^{(p)} \circ g$ for $g \in G(\mathbb{A}^{(p\infty)})$. Specializing at y, we have $g = \rho(\beta)$ for $\beta \in \mathrm{End}^{\mathbb{Q}}(A_x) = M$. Thus $\widehat{Y}_y \subset \widehat{S} \times \widehat{S}$ is given by the equation $t' = t^{\beta^{1-c}}$ for nonzero $\beta \in \mathfrak{O}_{(p)}$ for the Serre–Tate coordinate t resp. t' with respect to the ordinary level p-structure η_p^{ord} of $A_x = \mathbb{Y}_{1,y}$ resp. $\mathbb{Y}_{2,y}$. By (A) (and Remark 3.12), $\beta^{1-c} \in O_p^\times$; so, we may assume that $\beta \in \mathfrak{O}_{(p)}^\times$ (and hence $\widetilde{\beta}$ is a prime-to-p isogeny). As in the proof of Proposition 3.11 (1), we have $\widehat{X}_{(x,x)} = \bigcup_{L\in I} \widehat{\mathbb{G}}_m \otimes L$ for finitely many O_p-direct summands L of $X_*(\widehat{S}^2)$. As we have shown in the proof of Proposition 3.11 (1), points of Y above (x,x) is indexed by $L \in I$. Suppose that y corresponds to L. Then $\widehat{Y}_y \subset \widehat{S} \times \widehat{S}$ coincides with $\widehat{\mathbb{G}}_m \otimes L$. On the other hand, we have the skew-diagonal $\Delta_\beta = \Delta_{1,\beta} = \{(z, \rho(\beta)(z)) | z \in V\} \subset V \times V$. The formal completion $\widehat{\Delta}_\beta$ along (x,x) therefore coincides with $\widehat{Y}_y$ and $\widehat{\mathbb{G}}_m \otimes L \subset \widehat{X}_{(x,x)}$ inside $\widehat{S}^2$. Thus $\Delta_\beta \subset X$. By the irreducibility of X, we conclude $X = \Delta_\beta$. Since Δ_β is smooth, $\Delta_\beta = Y$, and hence X is smooth everywhere.

If Condition (A) fails, as explained after the statement (DE), the morphisms $\rho(\alpha)^{-1} \circ \Pi_1$ and $\rho(\beta')^{-1} \circ \Pi_2$ for suitable nonzero $\alpha, \beta' \in \mathfrak{O}_{(p)}$ are étale; so, $(\rho(\alpha) \times \rho(\beta'))^{-1}(X) = \Delta_{1,\beta}$ by the above argument; so, $X = \Delta_{\alpha,\beta'\beta}$. This finishes the proof. □

Here are two technical lemmas, before going into the case where $m > 2$.

Lemma 3.17. *Let $N_i = A$ for a commutative ring A ($i = 1, 2, \ldots, m$). Let $N \subset N_1 \times N_2 \times \cdots \times N_m = A^m$ be an A-free submodule of A^m with $m \geq 2$. If A is a product of finitely many local rings and the projection of N to $N_i \times N_m$ is surjective for all $i = 1, 2, \ldots, m-1$ and the projection π' of N to $N' := N_1 \times N_2 \times \cdots \times N_{m-1}$ is surjective, we have $N = A^m$.*

Proof. We may assume that A is a local ring. Tensoring its residue field, by Nakayama's lemma, we may assume that A is a field k. Suppose that $\dim N \leq m-1$. Since π' is surjective, π' is an isomorphism, and $N \cap (N' \times z_m)$ is either empty or $z \in N \cap (N' \times z_m)$ is a unique point with $\pi'(z) = z_m$. Since $N \to N_1 \times N_m$ is surjective, there exists at least two points in $N \cap (N' \times z_m)$, a contradiction. Thus $\dim N = m$ and $N = k^m$. □

For a scheme morphism $f : Z \to Z'$, write $f(Z)$ for the Zariski-closure in Z' of the image of the topological space of Z by f with reduced scheme structure.

Lemma 3.18. *Let the assumption be as in* (N0–3). *Let $S^2 \subset S^m$ be a factor and $\pi : S^m \twoheadrightarrow S^2$ be the projection. Write $\pi(\mathcal{X}) = \mathrm{Spec}(B_0)$ for a local ring $B_0 \subset \mathcal{O}_{\mathcal{X}}$ (with maximal ideal $\mathfrak{m}_0$). Write $\widehat{\mathcal{X}} = \bigcup_{L\in I} \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} L$. Then*

(1) *For $\widehat{B}_0 = \varprojlim_n B_0/\mathfrak{m}_0^n$, there exists a finite set I_1 of O_p-direct summands $\ell \subset X_*(\widehat{S}^2)$ such that $\mathrm{Spf}(\widehat{B}_0) = \bigcup_{\ell\in I_1} \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \ell$, and $\mathrm{rank}_{\mathbb{Z}_p} \ell = \dim B_0$.*

(2) *$\mathrm{rank}_{\mathbb{Z}_p} \pi_*(L)$ is independent of $L \in I$ and is equal to $\dim B_0 = \dim \pi(\mathcal{X})$.*

Proof. From the definition of the image $\pi(\mathcal{X})$, B_0 is given by the image of the composite $\mathcal{O}_{S^2} \xrightarrow{\pi^*} \mathcal{O}_{S^m} \twoheadrightarrow \mathcal{O}_{\mathcal{X}}$. Let $X_K \subset V_K^m$ be as in (N3) such that $\widehat{X}_K = \widehat{X}$. Then $\pi(X_K)$ is an excellent scheme since it is of finite type over a field $\mathbb{F}$ ([EGA] IV.7.8.3 (ii)). For the projection $\mathcal{X}_K$ of $\mathcal{X}$ to S_K^m, write $\pi(\mathcal{X}_K) = \mathrm{Spec}(B_{0,K})$. Then $B_{0,K}$ is a localization of $\pi(X_K)$ at $(x,x) \in S_K^2$ ($S_K = \mathrm{Spec}(\mathcal{O}_{V_K,x})$) and is an excellent integral local ring ([EGA] IV.7.8.3 (ii)). By [EGA] IV.7.8.3 (vii) and (x), $\widehat{B}_0 = \widehat{B}_{0,K}$ is reduced equidimensional. Since $\pi(\mathcal{X}) \subset S^2$ is irreducible and stable under $\mathbb{T}$, $\mathrm{Spf}(\widehat{B}_0) = \bigcup_{\ell \in I_1} \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \ell$ with $\mathrm{rank}_{\mathbb{Z}_p} \ell = \dim B_0$, by Proposition 3.11 (1), for finitely many O_p-direct summands $\ell \subset X_*(\widehat{S}^2)$. This proves (1).

Write $\mathcal{X} = \mathrm{Spec}(A_0)$, $\mathcal{Y} = \mathrm{Spec}(A)$ and $\mathrm{Spec}(B)$ (resp. $\mathrm{Spec}(B_K)$) for the normalization of $\pi(\mathcal{X})$ (resp. $\pi(\mathcal{X}_K)$). Then $\mathrm{Spec}(B_K)/\mathrm{Spec}(B_{0,K})$ is finite ([EGA] IV.7.8.3 (vi)) and hence $\mathrm{Spec}(B)/\mathrm{Spec}(B_0)$ is finite. We have the commutative diagram:

$$\begin{array}{ccccc} B_0 & \xrightarrow{\hookrightarrow} & A_0 & \xrightarrow{\subset} & \widehat{A}_0 \\ \cap\downarrow & & \cap\downarrow & & \downarrow\cap \\ B & \xrightarrow[\hookrightarrow]{} & A & \xrightarrow[\subset]{} & \widehat{A}. \end{array}$$

Here $\widehat{A} = \varprojlim_n A/\mathfrak{m}_{B_0}^n A$ (so, $\widehat{\mathcal{Y}} = \mathrm{Spf}(\widehat{A})$). Write $\pi_0(Z)$ for the set of connected components of a scheme Z. Since $I \cong \pi_0(\mathrm{Spf}(\widehat{A}))$ and $I_1 \cong \pi_0(\mathrm{Spf}(\widehat{B}))$, we have a natural surjection $I = \pi_0(\mathrm{Spf}(\widehat{A})) \xrightarrow{\pi_*} \pi_0(\mathrm{Spf}(\widehat{B})) = I_1$, and $L, L' \in I$ corresponds to a single ℓ if $\pi_*(L) \otimes_O F \subset \ell \otimes_O F$ and $\pi_*(L') \otimes_O F \subset \ell \otimes_O F$. We have $\widehat{A} = \prod_{L \in I} A_L$ with $\widehat{\mathbb{G}}_m \otimes L = \mathrm{Spf}(A_L)$ and $\widehat{B} = \prod_{\ell \in I_1} B_\ell$ with $\widehat{\mathbb{G}}_m \otimes \ell = \mathrm{Spf}(B_\ell)$. Fix ℓ and let $J \subset I$ be the collection of all $L \in I$ such that $\pi_*(L) \otimes_O F \subset \ell \otimes_O F$. Then we have a morphism $B_\ell \to A_\ell := \prod_{L \in J} A_L$, and the projection $B_\ell \to A_L$ to the L-component is an injection. Indeed, the image $\overline{B}_L$ of B_ℓ in A_L is given by $\varprojlim_n B/(\mathfrak{m}_L^n \cap B)$ for the maximal ideal $\mathfrak{m}_L$ of A_L. Since $B/(\mathfrak{m}_L^n \cap B)$ is a finite dimensional $\mathbb{F}$-vector space, writing $\mathfrak{m} = \mathfrak{m}_\ell \cap B = \mathfrak{m}_L \cap B$ for the maximal ideal $\mathfrak{m}_\ell$ of B_ℓ, it is killed by $\mathfrak{m}^N$ ($0 \ll N \in \mathbb{Z}$). Thus $\mathfrak{m}^N \subset (\mathfrak{m}_L^n \cap B)$. The filtrations $\{\mathfrak{m}_L^n \cap B\}_n$ and $\{\mathfrak{m}^n\}_n$ give the same topology on B, since $\mathfrak{m}^n \subset \mathfrak{m}_L^n \cap B$. Thus $B_\ell = \varprojlim_n B/\mathfrak{m}^n \cong \varprojlim_n B/(\mathfrak{m}_L^n \cap B) = \overline{B}_L$. The ring $\overline{B}_L$ is the power series ring over $\mathbb{F}$ with d variables for $d = \mathrm{rank}_{\mathbb{Z}_p} \pi_*(L)$, and B_ℓ is the power series ring with $\mathrm{rank}_{\mathbb{Z}_p} \ell$ variables. This shows $\dim B_0 = \mathrm{rank}_{\mathbb{Z}_p} \ell = \dim B_\ell = \mathrm{rank}_{\mathbb{Z}_p}(\pi_*(L))$ as desired.

We can give a more elementary proof of (2). Let $\pi_V : V_K^m \to V_K^2$ be the projection which induces π. Let $U = \mathrm{Spec}(\mathcal{B}_0) \subset \pi(X_K)$ be a sufficiently small affine open neighborhood of (x,x) such that its normalization $\mathcal{B}/\mathcal{B}_0$ is finite. We can find such $\mathcal{B}_0$ because B/B_0 is finite by (1). We take an affine open neighborhood $U' = \mathrm{Spec}(\mathcal{A}_0) \subset \pi_V^{-1}(U)$ of x^m such that the normalization $\mathcal{A}/\mathcal{A}_0$ is finite and $\mathcal{A}_0$ is of finite type over $\mathcal{B}_0$; so, $\mathcal{A}$ is a noetherian domain of finite type over $\mathcal{B}$. As already explained, $\mathcal{A}$ and $\mathcal{B}$ are excellent. Let $\widetilde{U} = \mathrm{Spec}(\mathcal{B})$ and $\widetilde{U}' = \mathrm{Spec}(\mathcal{A})$. Since $\mathcal{B} \subset \mathcal{A}$ and $\mathcal{A}$ and $\mathcal{B}$ are integral domains, the mor-

phism: $\mathrm{Spec}(\mathcal{A}) \to \mathrm{Spec}(\mathcal{B})$ is generically flat. The non-flat locus $\widetilde{U}'_{nf}$ (which is the Zariski closure of $\{P \in \widetilde{U}' | \mathcal{A}_P \text{ is not flat over } \mathcal{B}_{\pi'(P)}\}$) is a proper closed subscheme of $\widetilde{U}'$. If $\mathrm{rank}_{\mathbb{Z}_p} \pi_*(L) < \mathrm{rank}_{\mathbb{Z}_p} \ell$, the formal completion $\widehat{\mathbb{G}}_m \otimes L$ of $\mathcal{A}$ along the point $y_L \in \widetilde{U}'$ corresponding to L is not flat over the formal completion $\widehat{\mathbb{G}}_m \otimes \ell$ of $\mathcal{B}$ along the image $\pi(y_L)$. Thus $\widetilde{U}'_{nf}$ contains a closed subscheme of maximal dimension, a contradiction against the irreducibility of $\widetilde{U}'$. Thus $\mathrm{rank}_{\mathbb{Z}_p} \pi_*(L) = \mathrm{rank}_{\mathbb{Z}_p} \ell$ as desired. □

Here is the corollary showing $Y = X = V^{m-2} \times \Delta_{(\alpha,\beta)}$ for $m > 2$:

Corollary 3.19. *Let the notation be as in* (N0–3) *and the assumption be as in* Proposition 3.11. *Then X is smooth everywhere, and X^{ord} is Tate O-linear. If $\mathcal{X}$ is finite over S', X is given either by $V^{m-1} \times \{x\}$ or identical to $V^{m-2} \times \Delta_{(\alpha,\beta)}$ for some non-zero $\alpha, \beta \in \mathfrak{O}_{(p)}$ (after permuting first $m-1$ factors).*

Proof. We use the symbols introduced in Proposition 3.11 and its proof; in particular, $S = \mathrm{Spec}(\mathcal{O}_{V,x})$ and $S^m = S' \times S'' \subset V^m$ with $S'' = S$. By Proposition 3.8 ($m = 1$) and by Corollary 3.16 ($m = 2$), we may assume that $m > 2$. Assume that $\mathcal{X} \neq S^m$. If the projection of $\mathcal{X}$ to S'' is a proper closed subscheme in S'', by applying Proposition 3.8 to the image of $\mathcal{X}$ in S'', we find that $\mathcal{X} = S' \times \{x\}$; so, we are done. Thus we may assume that the projection of $\mathcal{X}$ to S'' is dominant and that $\mathcal{X}$ is finite over S' (Proposition 3.11 (3)).

There are two ways to prove the assertion now. We first describe a way of reducing the assertion to Corollary 3.16 which is closer to the treatment in the earlier version of this paper (putting off a brief description of the second method due to Chai after the first). Let $\Pi : Y \to X$ and $\Pi_{\mathcal{Y}} : \mathcal{Y} \to \mathcal{X}$ be the normalization. Then by Proposition 3.11 (3), $\mathcal{Y}$ is finite flat over S', and Y is Tate O-linear at every point $y \in Y$ above $x^m \in X$ (abusing the terminology).

Pick a point $y \in Y$ above x^m, and write $\widehat{Y}_y = \widehat{\mathbb{G}}_m \otimes L$ and $\widehat{X} = \bigcup_{L \in I} \widehat{\mathbb{G}}_m \otimes L$. We write $S_i = S$ be the i-th component of S^m. Let $\pi_{i,m} : S^m \to S_i \times S''$ with $i < m$ be the projection. We regard $\widehat{Y}_y = \widehat{\mathbb{G}}_m \otimes L \subset \widehat{S}^m$. If $\pi_{i,m,*} : L \otimes_O F \to X_*(\widehat{S}_i \times \widehat{S}'') \otimes_O F$ is surjective for all $i < m$, by Lemma 3.17 applied to $A = O_p \otimes_O F = F_p$ and $N_i = X_*(\widehat{S}_i) \otimes_O F$, we find that $L \otimes_O F = X_*(\widehat{S}^m) \otimes_O F$; so, $\widehat{Y}_y = \widehat{S}^m$, and hence $Y = X = V^m$, and we are done. Thus we assume that that the projection $\pi_{i,m,*} : L \otimes_O F \to X_*(\widehat{S}_i \times \widehat{S}'') \otimes_O F$ is not surjective for an $i < m$, and

$$\mathrm{rank}_{\mathbb{Z}_p} \pi_{i,m,*}(L) < \mathrm{rank}_{\mathbb{Z}_p} X_*(\widehat{S}_i \times \widehat{S}'').$$

Recall that $f(Z)$ denotes the Zariski-closure in Z' of the image of the underlying topological space of Z for a morphism $f : Z \to Z'$ of schemes. If $Z = \mathrm{Spec}(A)$ and $Z' = \mathrm{Spec}(B)$, then the topological space of $f(Z)$ is that of $\mathrm{Spec}(f^*(B))$. Write $\Pi_{i,m} := \pi_{i,m} \circ \Pi : \mathcal{Y} \to S_i \times S''$. By Lemma 3.18 (2),

$$\dim \pi_{i,m}(\mathcal{X}) = \mathrm{rank}_{\mathbb{Z}_p} \pi_{i,m,*}(L) < \mathrm{rank}_{\mathbb{Z}_p}(X_*(\widehat{S}_i \times \widehat{S}'')) = \dim(S_i \times S''). \quad (3.27)$$

By (3.27), the reduced image $\pi_{i,m}(\mathcal{X}) \subset S_i \times S''$ is an irreducible proper closed subscheme invariant under $\mathbb{T}$. Applying Corollary 3.16 to $\pi_{i,m}(\mathcal{X})$, we

find that $\pi_{i,m}(\mathcal{X}) = \mathrm{Spec}(\mathcal{O}_{\Delta_{\alpha,\beta},x^2})$ and $\pi_{i,m}(X) \subset \Delta_{\alpha,\beta}$ for some non-zero $\alpha, \beta \in \mathfrak{O}_{(p)}$. Permuting indices to bring i to $m-1$, we conclude

$$X \subset \pi_{m-1}^{-1}(\Delta_{\alpha,\beta}) = V^{m-2} \times \Delta_{\alpha,\beta}.$$

Since $\dim X = (m-1)\dim V$ (by Proposition 3.11 (3)), we conclude by irreducibility $X = V^{m-2} \times \Delta_{\alpha,\beta}$ as desired.

Here is a brief sketch of the second proof, which is based on the argument in [C4] Section 8. By using Chai's globalization of the Serre–Tate coordinate in [C4] Section 2, we find a dense open subscheme $Y^{lin} \subset Y$ ($y \in Y^{lin}$) such that Y is Tate O-linear at every closed point of Y^{lin}. The existence of $Y^{lin} \subset Y$ follows basically from Proposition 5.3 in [C4] and its proof (applied to Y not X). Consider the abelian scheme $\mathbb{Y} = \mathbf{A}^m \times_{V^m} Y$. Then $\mathrm{End}^{\mathbb{Q}}(\mathbb{Y}_{/Y})$ is either isomorphic to F^m or $F^{m-2} \times M_2(F)$, because Y is dominant over V^{m-1} and V by the two projections. Then in a similar manner to the proof of Corollary 3.16, we can prove the impossibility of $\mathrm{End}^{\mathbb{Q}}(\mathbb{Y}_{/Y}) \cong F^m$. Thus we have $\mathrm{End}^{\mathbb{Q}}(\mathbb{Y}_{/Y}) \cong F^{m-2} \times M_2(F)$; in other words, for an index $i < m$, the i-th factor $\mathbb{Y}_i$ obtained by pulling back the i-th factor $\mathbf{A}$ of $\mathbf{A}^m$ to Y is isogenous to the last factor $\mathbb{Y}_m$. This isogeny is induced by a nonzero $\alpha/\beta \in M \cong \mathrm{End}_O^{\mathbb{Q}}(A_x)$ with $\alpha, \beta \in \mathfrak{O}$. Then we conclude that $\widehat{\mathcal{X}} \supset \widehat{S}^{m-1} \times \widehat{\Delta}_{(\alpha,\beta)}$ for $\Delta_{(\alpha,\beta)}$ plugged in the product of the i-th and the m-th copy of V in V^m, and hence X has the desired form. □

The subgroup $\mathfrak{O}_{(p)}^{\times} \cong \rho(\mathcal{T}(\mathbb{Z}_{(p)}))$ in $\overline{\mathcal{E}}(G, \mathfrak{X})$ fixes $x \in Ig(\mathbb{F})$ (Lemma 3.3) and hence acts on the stalk $\mathcal{O}_{V,x}$ and the stalk $\mathcal{O}_{Ig,x}$ of x on the Igusa tower $Ig_{/V}$. The group $\mathcal{T}(\mathbb{Z}_{(p)})$ is embedded into $T(\mathbb{Z}_p) = O_p^{\times}$ as in (3.18). Then the action of $\mathcal{T}(\mathbb{Z}_{(p)})$ extends to its p–adic completion $O_p^{\times} = T(\mathbb{Z}_p) = \mathrm{Aut}(\widehat{S})$ for $\widehat{S} = \mathrm{Spf}(\widehat{\mathcal{O}}_{V,x}) = \mathrm{Spf}(\widehat{\mathcal{O}}_{Ig,x})$. Each $a \in O_p^{\times}$ acts on the formal completion $\widehat{\mathcal{O}}_{Ig,x}$ as automorphisms sending the canonical coordinate t to t^a for $a \in \mathcal{T}(\mathbb{Z}_p) = O_p^{\times}$. Each diagonal element $g = \mathrm{diag}[a,d] \in T^{\delta}(\mathbb{Z}_p)$ for the diagonal torus $T^{\delta} \subset G$ also acts on Ig by the change of level structure $\eta_p^{ord} \mapsto \eta_p^{ord} \circ g$. The image of $T^{\delta}(\mathbb{Z}_p) \cap \mathcal{G}$ in $\overline{\mathcal{E}}(G, \mathfrak{X})$ has trivial intersection with $\mathcal{T}(\mathbb{Z}_{(p)})$ inside $\overline{\mathcal{E}}(G, \mathfrak{X})$, because $T_x(\mathbb{Z}_{(p)})$ is embedded diagonally in $G(\mathbb{A}^{(\infty)})$ by $\rho = \widehat{\rho}_x$, while each element of $T^{\delta}(\mathbb{Z}_p)$ has only nontrivial component at p. Thus the two actions of $a \in \mathcal{T}(\mathbb{Z}_p) = O_p^{\times}$ fixing x and that of $g \in T^{\delta}(\mathbb{Z}_p) \cap \mathcal{G}$ moving x are compatible. In the following theorem, $a \in \mathcal{T}(\mathbb{Z}_p) = O_p^{\times}$ acts on $\widehat{\mathcal{O}}_{Ig,x}$ via $t \mapsto t^a$ for the canonical coordinate t. We are now ready to prove:

Theorem 3.20. *Let $\mathbb{F}$ be the residue field of $\mathcal{W}$ (so it is an algebraic closure of $\mathbb{F}_p$). Let $x \in Ig(\mathbb{F})$ be a closed point (which is fixed by the action of $\mathcal{T}(\mathbb{Z}_{(p)})$ embedded in $\overline{\mathcal{E}}(G, \mathfrak{X})$ by ρ). Let $a_1, \ldots, a_m \in \mathcal{T}(\mathbb{Z}_p)$, and assume that $a_i a_j^{-1} \notin \mathcal{T}(\mathbb{Z}_{(p)})$ for all $i \neq j$. Then $a_j(\mathcal{O}_{Ig,x/\mathbb{F}})$ $(j = 1, 2, \ldots, m)$ are linearly disjoint over $\mathbb{F}$ in $\widehat{\mathcal{O}}_{Ig,x/\mathbb{F}}$, where $\mathcal{O}_{Ig,x}$ is the stalk at x of the Igusa tower over V.*

Let $\mathfrak{b} = \mathrm{Ker}(\phi)$ be the kernel of the homomorphism ϕ in (3.19). Let $\widetilde{\mathfrak{b}}$ be the unique prime ideal of $\mathcal{O}_{Ig,x} \otimes \cdots \otimes \mathcal{O}_{Ig,x}$ over $\mathfrak{b}$. Then $\widetilde{\mathfrak{b}}$ is the kernel of the map $\widetilde{\phi}$ defined in the exactly the same way as ϕ replacing $\mathcal{O}_{V,x}$ by $\mathcal{O}_{Ig,x}$. The

assertion of the theorem is equivalent to $\widetilde{\mathfrak{b}} = 0$. Since $\mathfrak{b} = 0 \Leftrightarrow \widetilde{\mathfrak{b}} = 0$ by the same argument as in the proof of Corollary 3.9, it is enough to prove $\mathfrak{b} = 0$.

Proof. We first suppose that $m = 2$. We use the symbols we introduced in the proof of the above two propositions. In particular, $V = \varprojlim_K V_K$ for K maximal at p. For simplicity, we write $\widehat{\mathcal{O}} = \mathcal{O}_{\widehat{S}/\mathbb{F}}$, $S = \mathrm{Spec}(\mathcal{O})$ and $\widehat{S} = \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}} O$. Suppose $\mathfrak{b} \neq 0$. Applying Proposition 3.11 (3) to the two projections $S \times S \to S$ and $\mathcal{X} = \mathrm{Spec}(\mathcal{R})$ which has two dominant projections onto $S = \mathrm{Spec}(\mathcal{O})$. Then the formal completion $\widehat{\mathcal{X}}$ of $\mathcal{X}$ along $x^2 = (x, x)$ is a formal torus defined by $t'^{u_2} = t^{u_1}$ for $u_1, u_2 \in O_p \cap F_p^\times$ (Corollary 3.16). By our definition of $\mathfrak{b}$, we have $u_2/u_1 = a_2/a_1 \in O_p^\times$; so, we may choose u_1 and u_2 so that $u_1 = 1$ and $u_2 \in O_p^\times$. Let X be the schematic closure of $\mathcal{X}$ in $V \times V$. Then by Corollary 3.16, we find that $X = \Delta_{1,\alpha}$ and hence $u_2 = \alpha^{1-c}$ for $\alpha \in \mathfrak{O}_{(p)}^\times$. Since $\alpha^{1-c} \in \mathcal{T}(\mathbb{Z}_{(p)})$, this contradicts to $a_1/a_2 \notin \mathcal{T}(\mathbb{Z}_{(p)})$. Thus $X = V^2$, and $a_1(\mathcal{O})$ and $a_2(\mathcal{O})$ are linearly disjoint over $\mathbb{F}$.

We now deal with the case where $m > 2$. Recall

$$\phi : \overbrace{\mathcal{O} \otimes_{\mathbb{F}} \mathcal{O} \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} \mathcal{O}}^{m} \to \widehat{\mathcal{O}}$$

in (3.19) which is the $\mathbb{F}$–algebra homomorphism given by $f_1 \otimes f_2 \otimes \cdots \otimes f_m \mapsto \prod_{j=1}^m a_j(f_j) \in \widehat{\mathcal{O}} \otimes_W \mathbb{F}$. Let $\mathcal{R} = \mathrm{Im}(\phi) \subset \widehat{\mathcal{O}} \otimes_W \mathbb{F}$. We regard $\mathcal{X} = \mathrm{Spec}(\mathcal{R})$ as a closed subscheme of S^m for $S = \mathrm{Spec}(\mathcal{O})$. Take the schematic closure X of $\mathcal{X}$ in V^m. By the induction hypothesis on m, $\mathcal{X}$ surjects onto S' and S''. By Proposition 3.11 (3), $\mathcal{X}$ is either finite over S' or $\mathcal{X} = S^m$. In the latter case, we are done; so, we assume that $\mathcal{X}$ is finite over S'. Then by Corollary 3.19, there exists an index $0 < i < m$ so that $\pi_i(X) = \Delta_{\alpha,\beta}$ inside $V^2 = V_i \times V''$ for the i-th factor $V_i = V$ in V^m. We have $\widehat{\mathcal{X}} \supset \widehat{\Delta}$ in Lemma 3.13, and we conclude that $O_p^\times = \mathcal{T}(\mathbb{Z}_p) \ni a_i/a_m = (\alpha/\beta)^{1-c} \in \mathcal{T}(\mathbb{Q})$, which is a contradiction (because $\mathcal{T}(\mathbb{Z}_{(p)}) = \mathcal{T}(\mathbb{Z}_p) \cap \mathcal{T}(\mathbb{Q})$). Thus $\mathcal{X} = S^m$ (and hence $\mathfrak{b} = 0$) and $X = V^m$. □

We can add the datum of a nowhere vanishing differential to our classification problem, looking into the following functor $\mathcal{Q}_K$:

$$U \mapsto \left[(A, \overline{\lambda}, i, \overline{\eta}^{(p)}, \omega)_{/U} \middle| \underline{A} \in \mathcal{P}'_K(U) \cong \mathcal{P}_K^{(p)}(U), \quad \pi_* \Omega_{A/U} = (\mathcal{O}_U \otimes_{\mathbb{Z}} O)\,\omega\right], \tag{3.28}$$

where K is an open compact subgroup of $G(\mathbb{A}^{(\infty)})$ maximal at p and $\underline{A} = (A, \overline{\lambda}, i, \overline{\eta}^{(p)})$ is chosen in $\mathcal{P}'_K(U)$ in (3.6). Then $\mathcal{Q}_K$ is represented by a T–torsor $\mathcal{M}_K(G, X)$ over $Sh^{(p)}(G, X)/K$. The torus $T = \mathrm{Res}_{O/\mathbb{Z}}\mathbb{G}_m$ acts on $\mathcal{Q}_K$ by $\omega \mapsto t\omega$ for $t \in T(\mathcal{O}_U) = (\mathcal{O}_U \otimes_{\mathbb{Z}} O)^\times$. Over W, assuming A_x to be ordinary and choosing a level p^∞–structure η_p^{ord}, it naturally induces an isomorphism of formal group $\eta : A_x^\circ \cong \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}} \mathfrak{d}^{-1} = \mathrm{Spf}(\widehat{W[q^\xi]}_{\xi \in O})$. In other words, $\eta^* \frac{dq}{q}$ gives rise to a canonical differential. Choose a nowhere vanishing differential ω_0 on $A_0 = A_x \otimes_W \mathbb{F}$, and consider the formal completion $\widehat{\mathcal{M}}_K$ of $\mathcal{M}_K$ along its closed point corresponding to $(A_0, \overline{\lambda}_0, i_0, \eta_0^{ord}, \omega_0)$, which is a formal $\widehat{T}$-torsor over $\widehat{S} = \widehat{V}_x$. Here $\widehat{T}$ is the formal completion of T along the origin. Then the

formal $\widehat{T}$-torsor $\widehat{\mathcal{M}}_K$ splits into a product $\widehat{T} \times_W \widehat{S}$ over $\widehat{S} \cong \widehat{\mathbb{G}}_m \otimes \mathfrak{d}^{-1}$. In other words, if we consider the deformation functor:

$$\widehat{\mathcal{Q}}(C) = \left[(A, \overline{\lambda}, i, \eta^{ord}, \omega)_{/C} \middle| (A, \overline{\lambda}, i, \eta^{ord}, \omega) \times_C \mathbb{F} = (A_0, \overline{\lambda}_0, i_0, \eta_0^{ord}, \omega_0)\right]$$

for artinian local W–algebras C with residue field $\mathbb{F}$, $\widehat{\mathcal{Q}}$ is prorepresented by $\widehat{S} \times \widehat{T}$. In the above discussion, we may actually allow K of p–power level in (3.28) as long as K contains the monodromy group U_∞ of the infinity cusp in $G(\mathbb{Z}_p) \cap \mathcal{G}(G, \mathfrak{X})$ by replacing $Sh^{(p)}(G, \mathfrak{X})$ by the Igusa tower $Ig(G, \mathfrak{X})$ and the level structure $\overline{\eta}^{(p)} = \eta^{(p)} K^{(p)}$ by $\eta^{ord} K^{(p)}$. In this slightly more general case, the functor is represented by a formal scheme $\widehat{\mathcal{M}}_K$ which is a $\widehat{T}$–torsor over $\widehat{S} \subset Ig(G, \mathfrak{X})/K$. Therefore in the sequel, we allow modular forms of finite p–power level of type $\Gamma_1(p^r)$.

We identify the character group $X^*(T)$ of T with the module of formal linear combinations $\kappa = \sum_\sigma \kappa_\sigma \sigma$ ($\kappa_\sigma \in \mathbb{Z}$) for field embeddings $\sigma : F \hookrightarrow \overline{\mathbb{Q}}$ so that $x^\kappa = \prod_\sigma \sigma(x)^{\kappa_\sigma}$ ($x \in T(\mathbb{Q})$). For each character κ of T and a p–adic W–algebra C, we write $G_\kappa(C)$ for the κ^{-1}–eigenspace of $\mathcal{O}_{\mathcal{M}/C}$. Thus $G_\kappa(C)$ is the union of C–integral modular forms of weight κ and of finite level (of $\Gamma_1(N)$–type for all positive integers N). Since p is unramified in O, T is smooth over $\mathbb{Z}_p$ and is diagonalizable over $\mathbb{Z}_p$. Therefore we have $\mathcal{O}_{\mathcal{M}/W} = \bigoplus_\kappa G_\kappa(W)$. By the above splitting, we may regard $G_\kappa(C) \subset \mathcal{O}_{\widehat{S}/C}$. In particular, $a \in T_x(\mathbb{Z}_p)$ acts on $f \in G_\kappa(\mathbb{F})$ through the identification $\mathcal{T}(\mathbb{Z}_p) = \mathrm{Aut}_O(\widehat{S}_{/\mathbb{F}})$, and we have $a(f) \in \mathcal{O}_{\widehat{S}/\mathbb{F}}$. We write $t - 1 = (t_j - 1)_j$ for the parameter at 1 of $\widehat{S}$. Each $\phi \in G_\kappa(C)$ has t–expansion given by

$$\phi(t) = \phi(\underline{\mathcal{A}}^{ord}) \in C[[t-1]].$$

The Hasse invariant H satisfies $H(t) \in \mathbb{F}^\times$ (because $\widehat{\mathcal{A}}_{\widehat{S}} \cong \left(\widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}} \mathfrak{d}^{-1}\right) \times_W \widehat{S}$ for the universal deformation $\mathcal{A}_{/\widehat{S}}$). Since H is invertible on Sh^{ord}, for any given *parallel* weight $\kappa = \sum_\sigma k\sigma$ ($k \in \mathbb{Z}$), we have $H_\kappa \in G_\kappa(\mathbb{F})$ such that $H_\kappa(t) = 1$. Indeed, for $k \gg 0$, we can lift H to $E \in G_\kappa(W)$ of level prime to p with $E \equiv H$ mod $\mathfrak{m}_W$ by the ampleness of the weight κ automorphic line bundle. Then allowing p-power level, we can find $H_{\kappa/\mathbb{F}}$ of any parallel positive weight κ by the p-adic density of modular forms of level prime to p in the space of p-adic modular forms (see [PAF] Theorem 4.10). The form $H_{\kappa/\mathbb{F}} \in G_\kappa(\mathbb{F})$ may not have a characteristic 0 lift if $\kappa = 1$ even if we allow the level N divisible by p.

Corollary 3.21. *Fix a parallel weight κ, and let $H_\kappa \in G_\kappa(\mathbb{F})$ be the mod p modular form with $H_\kappa(t) = 1$. Let $a_0, \ldots, a_n \in T_x(\mathbb{Z}_p)$ and suppose that $a_i a_j^{-1} \notin T_x(\mathbb{Q})$ for all $i \neq j$. Let $I \subset \{0, 1, 2, \ldots, n\}$ be a subset of indices. Then if $\{H_\kappa, f_{ij} \in G_\kappa(\mathbb{F})\}_j$ for a fixed $\kappa \neq 0$ are linearly independent over $\mathbb{F}$ for each $i \in I$, then $\{a_i(f_{ij})\}_{i \in I, j}$ in $\mathcal{O}_{\widehat{S}/\mathbb{F}}$ are linearly independent over $\mathbb{F}$.*

Proof. Note that $a(H_\kappa)(t) = H_\kappa(t^a) = 1$. The division by $a(H_\kappa)$ brings the module $a(G_\kappa(\mathbb{F}))$ isomorphically into the ring $a(\mathcal{O}_{Ig,x/\mathbb{F}})$, we may assume that $\kappa = 0$. Then the above theorem (Theorem 3.20) implies the desired result. □

In the introduction, we mentioned linear independence of $a_j(E_{a_j}) = E_{a_j} \circ a_j$ for Eisenstein series E_{a_i} of a weight κ. Strictly speaking, in our application, we prove linear independence of E_{a_j} and H_κ by q-expansion principle, and then apply the above corollary to $\{H_\kappa, E_{a_j} \in G_\kappa(\mathbb{F})\}_j$ to show that $\{a_j(E_{a_j})\}_j$ is linearly independent over $\mathbb{F}$.

4 Eisenstein and Katz Measure

We recall the Fourier expansion of classical Eisenstein series and Eisenstein measure from [HT] Sections 2 and 3. This is based on Katz's theory in [K3], but our exposition slightly differs from it in a fashion adapted to our application. In this section, we do not assume that p is unramified in $F/\mathbb{Q}$.

4.1 Geometric Modular Forms

Let $F/\mathbb{Q}$ be a totally real finite extension with integer ring O. Recall the different $\mathfrak{d}$ of $F/\mathbb{Q}$, and for each ideal $\mathfrak{a}$ we have written $\mathfrak{a}^* = \mathfrak{a}^{-1}\mathfrak{d}^{-1}$. Thus $O^* = \mathfrak{d}^{-1}$. For a nonzero ideal $\mathfrak{N}$ of O, we define a group scheme $\mu_\mathfrak{N}$ over $\mathbb{Z}$ as the Cartier dual of the constant group $O/\mathfrak{N}$. If $\mathfrak{N}$ is generated by an integer $n > 0$), $\mu_\mathfrak{N} \cong O^* \otimes_\mathbb{Z} \mu_n$ canonically by the trace pairing on $O^* \times O$ and the duality between μ_n and $\mathbb{Z}/n\mathbb{Z}$. In general, we can identify $\mu_\mathfrak{N}$ with $\{x \in O^* \otimes \mu_n | \alpha x = 0 \text{ for all } \alpha \in \mathfrak{N}\}$ choosing a positive integer $n \in \mathfrak{N}$. For a fixed fractional ideal $\mathfrak{c}$ of F and an ideal $\mathfrak{N}$ prime to $\mathfrak{c}$, the Hilbert modular variety $\mathfrak{M}(\mathfrak{c}, \mathfrak{N})$ classifies the following triples $(A, \lambda, i)_{/S}$ formed by

- An abelian scheme $\pi : A \to S$ with an algebra homomorphism: $O \hookrightarrow \mathrm{End}(A_{/S})$ making $\pi_*(\Omega_{A/S})$ a locally free $O \otimes_\mathbb{Z} \mathcal{O}_S$–module of rank 1;
- An O–linear polarization $\lambda : A^t \cong A \otimes \mathfrak{c}$. By λ, we identify the O–module of symmetric O–linear homomorphisms $\mathrm{Hom}_{sym}(A_{/S}, A^t_{/S})$ with $\mathfrak{c} = \mathrm{Hom}_{sym}(A_{/S}, A_{/S}) \otimes_O \mathfrak{c}$. Then we require that the (multiplicative) monoid $P(A)$ of symmetric isogenies induced locally by an ample invertible sheaf be identified with the set of totally positive elements $\mathfrak{c}_+ \subset \mathfrak{c}$;
- We have an O-linear closed immersion $i = i_\mathfrak{N} : \mu_\mathfrak{N} \hookrightarrow A[\mathfrak{N}]$ of group schemes.

Thus $\mathfrak{M}(\mathfrak{c}, \mathfrak{N})$ is the coarse moduli scheme of the functor $\mathcal{P}(S) = \left[(A, \lambda, i)_{/S}\right]$ from the category of schemes S into the category $SETS$, where $[\ ] = \{\ \}/\cong$ is the set of isomorphism classes of the objects inside the brackets, and we call $(A, \lambda, i) \cong (A', \lambda', i')$ if we have an O–linear isomorphism $\phi : A_{/S} \to A'_{/S}$ such that $\lambda' = (\phi \otimes 1) \circ \lambda \circ \phi^t$ and $\phi \circ i = i'$. The scheme $\mathfrak{M}(\mathfrak{c}, \mathfrak{N})$ is a fine moduli if $\mathfrak{N}$ is sufficiently deep. In [K3] and [HT], the moduli $\mathfrak{M}(\mathfrak{c}, \mathfrak{N})$ is described as an algebraic space, but it is actually a quasi-projective scheme (e.g. [C1] and [PAF] Chapter 4).

We could insist that $\pi_*(\Omega_{A/S})$ is free over $\mathcal{O}_S \otimes_\mathbb{Z} O$, and taking a generator ω with $\pi_*(\Omega_{A/S}) = (\mathcal{O}_S \otimes_\mathbb{Z} O)\omega$, we may consider the functor classifying

quadruples (A,λ,i,ω):

$$\mathcal{Q}(S)=\left[(A,\lambda,i,\omega)_{/S}\right]. \tag{4.1}$$

Let $T=\mathrm{Res}_{O/\mathbb{Z}}\mathbb{G}_m$. We let $a\in T(S)=H^0(S,(\mathcal{O}_S\otimes_{\mathbb{Z}}O)^\times)$ act on $\mathcal{Q}(S)$ by $(A,\lambda,i,\omega)\mapsto(A,\lambda,i,a\omega)$. By this action, $\mathcal{Q}$ is a T–torsor over $\mathcal{P}$; so, $\mathcal{Q}$ is representable by a scheme $\mathcal{M}=\mathcal{M}(\mathfrak{c},\mathfrak{N})$ affine over $\mathfrak{M}=\mathfrak{M}(\mathfrak{c},\mathfrak{N})$. By definition, $\mathcal{M}$ is a T–torsor over $\mathfrak{M}$. For each character $\kappa\in X^*(T)=\mathrm{Hom}_{gp-sch}(T,\mathbb{G}_m)$ and a given ring R, if $F\neq\mathbb{Q}$, the κ^{-1}–eigenspace of $H^0(\mathcal{M}_{/R},\mathcal{O}_{\mathcal{M}/R})$ is the space of modular forms of weight κ integral over R, where $\mathcal{M}_{/R}=\mathcal{M}\times_{\mathbb{Z}}\mathrm{Spec}(R)$. We write $G_\kappa(\mathfrak{c},\mathfrak{N};R)$ for this space of R–integral modular forms, which is an R–module of finite type. When $F=\mathbb{Q}$, as is well known, we need to take the subsheaf of sections with logarithmic growth towards cusps. To simplify our argument, hereafter in this section, we often assume that $F\neq\mathbb{Q}$, since we do not need to insist on logarithmic growth by the Koecher principle, assuming this condition (in any case we just need to add this growth condition in the elliptic modular case; see [GME] Chapters 2 and 3). Thus $f\in G_\kappa(\mathfrak{c},\mathfrak{N};R)$ is a rule assigning an element in an R–algebra C to each quadruple $(A,\lambda,i,\omega)_{/C}$ (defined over the R–algebra C) satisfying the following three conditions:

(G1) $f(A,\lambda,i,\omega)=f(A',\lambda',i',\omega')\in C$ if $(A,\lambda,i,\omega)\cong(A',\lambda',i',\omega')$ over C;

(G2) $f((A,\lambda,i,\omega)\otimes_{C,\rho}C')=\rho(f(A,\lambda,i,\omega))$ for each $\rho\in\mathrm{Hom}_{R\text{-alg}}(C,C')$;

(G3) $f(A,\lambda,i,a\omega)=\kappa(a)^{-1}f(A,\lambda,i,\omega)$ for $a\in T(C)$.

The sheaf of κ^{-1}–eigenspace $\mathcal{O}_{\mathcal{M}}[\kappa^{-1}]$ under the action of T is an invertible sheaf of weight κ on $\mathfrak{M}$. We write this sheaf as $\underline{\omega}^\kappa$. Then we have

$$G_\kappa(\mathfrak{c},\mathfrak{N};R)=H^0(\mathfrak{M}(\mathfrak{c},\mathfrak{N}),\underline{\omega}^\kappa)$$

as long as $\mathfrak{M}(\mathfrak{c},\mathfrak{N})$ is a fine moduli space. Writing $\underline{\mathbf{A}}=(\mathbf{A},\boldsymbol{\lambda},\mathbf{i},\boldsymbol{\omega})$ for the universal abelian scheme over $\mathcal{M}$, $s=f(\underline{\mathbf{A}})\boldsymbol{\omega}^\kappa$ gives rise to the section of $\underline{\omega}^\kappa$. Conversely, for any section $s\in H^0(\mathfrak{M}(\mathfrak{c},\mathfrak{N}),\underline{\omega}^\kappa)$, taking a unique morphism $\phi:\mathrm{Spec}(C)\to\mathcal{M}$ such that $\phi^*\underline{\mathbf{A}}=\underline{A}$ for $\underline{A}=(A,\lambda,i,\omega)_{/C}$, we can define $f\in G_\kappa$ by $\phi^*s=f(\underline{A})\omega^\kappa$.

Fix a prime p. We fix a fractional ideal $\mathfrak{c}$ prime to $\mathfrak{N}p$ and take two ideals $\mathfrak{a}$ and $\mathfrak{b}$ prime to $\mathfrak{N}p$ such that $\mathfrak{a}\mathfrak{b}^{-1}=\mathfrak{c}$. To this pair $(\mathfrak{a},\mathfrak{b})$, we can attach the Tate AVRM $Tate_{\mathfrak{a},\mathfrak{b}}(q)$ defined over the completed group ring $\mathbb{Z}((\mathfrak{a}\mathfrak{b}))$ made of formal series $f(q)=\sum_{\xi\gg-\infty}a(\xi)q^\xi$ $(a(\xi)\in\mathbb{Z})$. Here ξ runs over all elements in $\mathfrak{a}\mathfrak{b}$, and there exists a positive integer C (dependent on f) such that $a(\xi)=0$ if $\sigma(\xi)+C<0$ for some $\sigma\in I$. We write $R[[(\mathfrak{a}\mathfrak{b})_{\geq0}]]$ for the subring of $R[[\mathfrak{a}\mathfrak{b}]]$ made of formal series f (having coefficients in R) with $a(\xi)=0$ for all ξ with $\sigma(\xi)<0$ for at least one embedding $\sigma:F\hookrightarrow\mathbb{R}$. Actually, we skipped a step of introducing the toroidal compactification of $\mathfrak{M}$ whose (completed) stalk at the cusp corresponding to $(\mathfrak{a},\mathfrak{b})$ actually carries $Tate_{\mathfrak{a},\mathfrak{b}}(q)$. However to make exposition short, we ignore this technically important point, referring the (attentive) reader to the treatment in [K3] Chapter I, [C1], [HT] Section 1 and [H02] Section 4. The scheme $Tate(q)$ can be extended to a semi-abelian scheme over $\mathbb{Z}[[(\mathfrak{a}\mathfrak{b})_{\geq0}]]$ with special fiber $\mathbb{G}_m\otimes\mathfrak{a}^*$ at the augmentation ideal $\mathfrak{A}$.

Since $\mathfrak{a}$ is prime to p, $\mathfrak{a}_p = O_p$. Thus if R is a $\mathbb{Z}_p$–algebra, we have a canonical isomorphism:

$$Lie(Tate_{\mathfrak{a},\mathfrak{b}}(q) \mod \mathfrak{A}) = Lie(\mathbb{G}_m \otimes \mathfrak{a}^*) \cong R \otimes_{\mathbb{Z}} \mathfrak{a}^* \cong R \otimes_{\mathbb{Z}} O^*.$$

By Grothendieck-Serre duality, we have $\Omega_{Tate_{\mathfrak{a},\mathfrak{b}}(q)/R[[(\mathfrak{ab})_{\geq 0}]]} \cong R[[(\mathfrak{ab})_{\geq 0}]]$. Indeed we have a canonical generator ω_{can} of $\Omega_{Tate(q)}$ which induces $\frac{dt}{t} \otimes 1$ on $\mathbb{G}_m \otimes \mathfrak{a}^*$ (writing $\mathbb{G}_m = \mathrm{Spec}(\mathbb{Z}[t, t^{-1}])$; see [K3] (1.1.17) and (1.2.11)). Since $\mathfrak{a}$ is prime to $\mathfrak{N}p$, we have a canonical inclusion $\mu_{\mathfrak{N}} \subset \mu_n \otimes_{\mathbb{Z}} O^* \cong \mu_n \otimes_{\mathbb{Z}} \mathfrak{a}^*$ (for an integer $0 < n \in \mathfrak{N}$ prime to $\mathfrak{a}$) into $\mathbb{G}_m \otimes \mathfrak{a}^*$, which induces a canonical closed immersion $i_{can} : \mu_{\mathfrak{N}} \hookrightarrow Tate(q)$. As described in [K3] (1.1.14) and [HT] page 204, $Tate_{\mathfrak{a},\mathfrak{b}}(q)$ has a canonical $\mathfrak{c}$–polarization λ_{can}. Thus we can evaluate $f \in G_\kappa(\mathfrak{c}, \mathfrak{N}; R)$ at $(Tate_{\mathfrak{a},\mathfrak{b}}(q), \lambda_{can}, i_{can}, \omega_{can})$. The value $f(q) = f_{\mathfrak{a},\mathfrak{b}}(q)$ actually falls in $R[[(\mathfrak{ab})_{\geq 0}]]$ (if $F \neq \mathbb{Q}$: Koecher principle) and is called the q–expansion at the cusp $(\mathfrak{a}, \mathfrak{b})$. When $F = \mathbb{Q}$, we impose f to have values in the ring $R[[(\mathfrak{ab})_{\geq 0}]]$ when we define modular forms (this is the logarithmic growth condition):

(G4) $f_{\mathfrak{a},\mathfrak{b}}(q) \in R[[(\mathfrak{ab})_{\geq 0}]]$ for all $(\mathfrak{a}, \mathfrak{b})$.

Suppose that $\mathfrak{N}$ is prime to p. We can think of a functor

$$\widehat{\mathcal{P}}(R) = \left[(A, \lambda, i_p, i_{\mathfrak{N}})_{/R}\right]$$

similar to $\mathcal{P}$ defined over the category of p–adic rings $R = \varprojlim_n R/p^n R$. The only difference here is that we consider an isomorphism of ind-group schemes $i_p : \mu_{p^\infty} \otimes_{\mathbb{Z}} O^* \cong A[p^\infty]^\circ$ (in place of a differential ω), which induces $\widehat{\mathbb{G}}_m \otimes O^* \cong \widehat{A}$ for the formal completion $\widehat{V}$ at the characteristic p–fiber of a scheme V over $\mathbb{Z}_p$. It is a theorem (due to Deligne–Ribet and Katz) that this functor is representable by the formal completion $\widehat{\mathfrak{M}}(\mathfrak{c}, \mathfrak{N}p^\infty)$ of $\mathfrak{M}(\mathfrak{c}, \mathfrak{N}p^\infty) = \varprojlim_n \mathfrak{M}(\mathfrak{c}, \mathfrak{N}p^n)$ along its mod p fiber. Thus we can think of p–adic modular forms $f_{/R}$ for a p–adic ring R which are functions of $(A, \lambda, i_p, i_{\mathfrak{N}})_{/C}$ (for any p–adic R–algebra C) satisfying the following conditions:

(G_p1) $f(A, \lambda, i_p, i_{\mathfrak{N}}) = f(A', \lambda', i'_p, i'_{\mathfrak{N}}) \in C$ if $(A, \lambda, i_p, i_{\mathfrak{N}})_{/C} \cong (A', \lambda', i'_p, i'_{\mathfrak{N}})_{/C}$;

(G_p2) $f((A, \lambda, i_p, i_{\mathfrak{N}}) \otimes_{C,\rho} C') = \rho(f(A, \lambda, i_p, i_{\mathfrak{N}}))$ for each p-adically continuous R-algebra homomorphism $\rho : C \to C'$;

(G_p3) $f_{\mathfrak{a},\mathfrak{b}}(q) \in R[[(\mathfrak{ab})_{\geq 0}]]$ for all $(\mathfrak{a}, \mathfrak{b})$ prime to $\mathfrak{N}p$.

We write $V(\mathfrak{c}, \mathfrak{N}; R)$ for the space of p–adic modular forms satisfying (G_p1-3). This $V(\mathfrak{c}, \mathfrak{N}; R)$ is a p–adically complete R–algebra.

We have the q–expansion principle valid both for classical modular forms and p–adic modular forms f,

(q-exp) *The q–expansion: $f \mapsto f_{\mathfrak{a},\mathfrak{b}}(q) \in R[[(\mathfrak{ab})_{\geq 0}]]$ determines f uniquely.*

This follows from the irreducibility of the Hilbert modular Igusa tower proven in [DR] (see also [PAF] Theorem 4.21 and [H08] for other proofs).

Since $\mathbb{G}_m \otimes \mathfrak{d}^{-1} = \mathrm{Spec}(\mathbb{Z}[t^\xi]_{\xi\in O})$ has a canonical invariant differential $\frac{dt}{t}$, we have $\omega_p = i_{p,*}\frac{dt}{t}$ on A. This allows us to regard each $f \in G_\kappa(\mathfrak{c}, \mathfrak{N}; R)$ a p–adic modular form by putting

$$f(A, \lambda, i_p, i_{\mathfrak{N}}) = f(A, \lambda, i_{\mathfrak{N}}, \omega_p).$$

By (q-exp), this gives an injection of $G_\kappa(\mathfrak{c}, \mathfrak{N}; R)$ into the space of p–adic modular forms $V(\mathfrak{c}, \mathfrak{N}; R)$ (for a p–adic ring R) preserving q–expansions.

Over $\mathbb{C}$, the category of quadruples (A, λ, i, ω) is equivalent to the category of triples $(\mathcal{L}, \lambda, i)$ made of the following data (cf. [ABV] I): $\mathcal{L}$ is an O–lattice in $O \otimes_{\mathbb{Z}} \mathbb{C} = \mathbb{C}^I$, an alternating form $\lambda : \mathcal{L} \wedge_O \mathcal{L} \cong \mathfrak{c}^*$ and $i : \mathfrak{N}^*/O^* \hookrightarrow F\mathcal{L}/\mathcal{L}$. The form λ is supposed to be positive in the sense that $\lambda(u, v)/\mathrm{Im}(uv^c)$ is totally positive in $O \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{R}^I$. Via polarization λ, we can define theta functions as described in [ABV] Chapter I by which we can embed the complex torus $\mathbb{C}^I/\mathcal{L}$ into a projective space $\mathbf{P}^N(\mathbb{C})$ for sufficiently large dimension N. Then by Chow's theorem, the image A is a projective algebraic variety defined over $\mathbb{C}$ with group structure, in short, an abelian variety over $\mathbb{C}$. The differential ω can be recovered by $\iota : A(\mathbb{C}) = \mathbb{C}^I/\mathcal{L}$ so that $\omega = \iota^* du$ where $u = (u_\sigma)_{\sigma\in I}$ is the variable on $\mathbb{C}^I$.

Conversely, if we start with a triple $(A, \lambda, \omega)_{/\mathbb{C}}$,

$$\mathcal{L}_A = \left\{ \int_\gamma \omega \in O \otimes_{\mathbb{Z}} \mathbb{C} \middle| \gamma \in H_1(A(\mathbb{C}), \mathbb{Z}) \right\}$$

is a lattice in $\mathbb{C}^I$, and the polarization $\lambda : A^t \cong A \otimes \mathfrak{c}$ induces $\mathcal{L} \wedge \mathcal{L} \cong \mathfrak{c}^*$.

Using this equivalence, we can relate our geometric definition of Hilbert modular forms with the classical analytic definition. Recall $\mathfrak{Z}$ which is the product of I copies of the upper half complex plane: $\mathfrak{Z} = \mathfrak{H}^I$ for

$$\mathfrak{H} = \{z = x + \sqrt{-1}y \in \mathbb{C} | y = \mathrm{Im}(z) > 0\}.$$

We regard $\mathfrak{Z} \subset O \otimes_{\mathbb{Z}} \mathbb{C} = \mathbb{C}^I$ made up of $z = (z_\sigma)_{\sigma\in I}$ with totally positive imaginary part. Thus we can think of the submodule $\mathfrak{b}z \subset F \otimes_{\mathbb{Q}} \mathbb{C} = \mathbb{C}^I$ for a cusp $(\mathfrak{a}, \mathfrak{b})$. For each $z \in \mathfrak{Z}$, we define $\mathcal{L}_z = 2\pi\sqrt{-1}(\mathfrak{b}z + \mathfrak{a}^*) \subset \mathbb{C}^I$,

$$\lambda_z(2\pi\sqrt{-1}(az + b), 2\pi\sqrt{-1}(cz + d)) = -(ad - bc) \in \mathfrak{c}^*$$

and $i_z : \mathfrak{N}^*/O^* = (\mathfrak{N}\mathfrak{a})^*/\mathfrak{a}^* \to F\mathcal{L}_z/\mathcal{L}_z$ by $i_z(a \mod O^*) = 2\pi\sqrt{-1}a \mod \mathcal{L}_z$.

Consider the following congruence subgroup $\Gamma_{11}(\mathfrak{N}; \mathfrak{a}, \mathfrak{b})$ given by

$$\left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(F) \middle| a, d \in O,\ b \in (\mathfrak{a}\mathfrak{b})^*,\ c \in \mathfrak{N}\mathfrak{a}\mathfrak{b}\mathfrak{d} \text{ and } d - 1 \in \mathfrak{N} \right\}.$$

Write $\Gamma_{11}(\mathfrak{c}; \mathfrak{N}) = \Gamma_{11}(1; O, \mathfrak{c}^{-1})$. We let $g = (g_\sigma) \in SL_2(F \otimes_{\mathbb{Q}} \mathbb{R}) = SL_2(\mathbb{R})^I$ act on $\mathfrak{Z}$ by linear fractional transformation of g_σ on each component z_σ. Then

$$(\mathcal{L}_z, \lambda_z, i_z) \cong (\mathcal{L}_w, \lambda_w, i_w) \iff w = \gamma(z) \text{ for } \gamma \in \Gamma_{11}(\mathfrak{N}; \mathfrak{a}, \mathfrak{b}).$$

Here an isomorphism between $(\mathcal{L}_z, \lambda_z, i_z)$ and $(\mathcal{L}_w, \lambda_w, i_w)$ is supposed to preserve the decomposition $\mathcal{L}_z \cong \mathcal{L}_w \cong \mathfrak{b} \oplus \mathfrak{a}^*$. The set of pairs $(\mathfrak{a}, \mathfrak{b})$ with $\mathfrak{a}\mathfrak{b}^{-1} = \mathfrak{c}$

is in bijection with the set of cusps of $\Gamma_{11}(\mathfrak{c};1)$. Two cusps are equivalent if they transform each other by an element in $\Gamma_{11}(\mathfrak{c};\mathfrak{N})$. A standard choice is $(O,\mathfrak{c}^{-1})$, which we call the infinity cusp of $\mathfrak{M}(\mathfrak{c},\mathfrak{N})$. For each ideal $\mathfrak{t}$, $(\mathfrak{t},\mathfrak{t}^{-1}\mathfrak{c}^{-1})$ gives another cusp. The two cusps $(\mathfrak{t},\mathfrak{t}^{-1}\mathfrak{c}^{-1})$ and $(\mathfrak{s},\mathfrak{s}^{-1}\mathfrak{c}^{-1})$ are equivalent under $\Gamma_{11}(\mathfrak{c};\mathfrak{N})$ if $\mathfrak{t}=\alpha\mathfrak{s}$ for an element $\alpha\in F^\times$ with $\alpha\equiv 1 \mod \mathfrak{N}$ in $F^\times_{\mathfrak{N}}$. We have

$$\mathfrak{M}(\mathfrak{c},\mathfrak{N})(\mathbb{C})\cong \Gamma_{11}(\mathfrak{c};\mathfrak{N})\backslash\mathfrak{Z}, \text{ canonically.}$$

Let $G=\mathrm{Res}_{O/\mathbb{Z}}GL(2)$. Take an open compact subgroup $K\subset G(\mathbb{A}^{(\infty)})$ such that $u\in K$ if and only if the following two conditions are satisfied:

1. $u\in \left(\begin{smallmatrix} d_F & 0\\ 0 & 1\end{smallmatrix}\right)^{-1} G(\widehat{\mathbb{Z}}) \left(\begin{smallmatrix} d_F & 0\\ 0 & 1\end{smallmatrix}\right)$ for an idele $d_F\in\widehat{O}$ with $d_F\widehat{O}=\widehat{\mathfrak{d}}$;
2. $\left(\begin{smallmatrix} d_F & 0\\ 0 & 1\end{smallmatrix}\right) u \left(\begin{smallmatrix} d_F & 0\\ 0 & 1\end{smallmatrix}\right)^{-1} \mod \mathfrak{N}$ is congruent to an upper unipotent matrix in $GL_2(O/\mathfrak{N})$ modulo $\mathfrak{N}$.

Then taking an idele c with $c\widehat{O}=\widehat{\mathfrak{c}}$, we see that

$$\Gamma_{11}(\mathfrak{c};\mathfrak{N})\subset \left(\left(\begin{smallmatrix} c & 0\\ 0 & 1\end{smallmatrix}\right) K \left(\begin{smallmatrix} c & 0\\ 0 & 1\end{smallmatrix}\right)^{-1}\cap G(\mathbb{Q})_+\right)\subset O^\times\Gamma_{11}(\mathfrak{c};\mathfrak{N})$$

for $G(\mathbb{Q})_+$ made up of all elements in $G(\mathbb{Q})$ with totally positive determinant. Choosing a representative set of the strict ray class group $Cl_F(\mathfrak{N})$ by finite ideles in $F^\times_{\mathbb{A}}$, we find by the approximation theorem that

$$G(\mathbb{A})=\bigsqcup_{c\in Cl_F(\mathfrak{N})} G(\mathbb{Q})\left(\begin{smallmatrix} c & 0\\ 0 & 1\end{smallmatrix}\right) K\cdot G(\mathbb{R})_+$$

for the identity connected component $G(\mathbb{R})_+$ of the Lie group $G(\mathbb{R})$. This shows

$$G(\mathbb{Q})\backslash(\mathfrak{X}\times G(\mathbb{A}^{(\infty)}))/K\cong G(\mathbb{Q})_+\backslash(\mathfrak{Z}\times G(\mathbb{A}^{(\infty)}))/K\cong \bigsqcup_{c\in Cl_F(\mathfrak{N})}\mathfrak{M}(\mathfrak{c},\mathfrak{N})(\mathbb{C}), \tag{4.2}$$

where $G(\mathbb{A})_+=G(\mathbb{A}^{(\infty)})G(\mathbb{R})_+$ and $\mathfrak{X}$ and $\mathfrak{Z}$ is as in (3.1). The $Cl_F(\mathfrak{N})$–tuple $(f_{\mathfrak{c}})_{\mathfrak{c}}$ with $f_{\mathfrak{c}}\in G_\kappa(\mathfrak{c},\mathfrak{N};\mathbb{C})$ can be viewed as a single automorphic form giving a section of a line bundle over $Sh_K(\mathbb{C})=G(\mathbb{Q})_+\backslash(\mathfrak{Z}\times G(\mathbb{A}^{(\infty)}))/K$.

Recall the identification $X^*(T)$ with $\mathbb{Z}[I]$ so that $\kappa(x)=\prod_\sigma \sigma(x)^{\kappa_\sigma}$. Regarding $f\in G_\kappa(\mathfrak{c},\mathfrak{N};\mathbb{C})$ as a holomorphic function of $z\in\mathfrak{Z}$ by $f(z)=f(\mathcal{L}_z,\lambda_z,i_z)$, it satisfies the following automorphic property:

$$f(\gamma(z))=f(z)\prod_\sigma (c^\sigma z_\sigma+d^\sigma)^{\kappa_\sigma} \quad \text{for all } \gamma=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\Gamma_{11}(\mathfrak{c};\mathfrak{N}). \tag{4.3}$$

The holomorphy of f is a consequence of the functoriality (G2). Each $f\in G_\kappa(\mathfrak{c},\mathfrak{N};\mathbb{C})$ has the Fourier expansion

$$f(z)=\sum_{\xi\in(\mathfrak{a}\mathfrak{b})_{\geq 0}} a(\xi)\mathbf{e}_F(\xi z)$$

at the cusp corresponding to $(\mathfrak{a},\mathfrak{b})$. Here $\mathbf{e}_F(\xi z)=\exp(2\pi\sqrt{-1}\sum_\sigma \xi^\sigma z_\sigma)$. This Fourier expansion equals the q–expansion $f_{\mathfrak{a},\mathfrak{b}}(q)$ replacing $\mathbf{e}_F(\xi z)$ by q^ξ.

Shimura studied in his theory of arithmetic of Hecke L–values the effect on modular forms of the following differential operators on $\mathfrak{Z}$ indexed by $\kappa \in \mathbb{Z}[I]$:

$$\delta_\kappa^\sigma = \frac{1}{2\pi\sqrt{-1}}\left(\frac{\partial}{\partial z_\sigma} + \frac{\kappa_\sigma}{2y_\sigma\sqrt{-1}}\right) \quad \text{and} \quad \delta_\kappa^k = \prod_\sigma \left(\delta^\sigma_{\kappa_\sigma+2k_\sigma-2}\cdots\delta^\sigma_{\kappa_\sigma}\right), \tag{4.4}$$

where $k \in \mathbb{Z}[I]$ with $k_\sigma \geq 0$. An important point is that the differential operator preserves rationality property at CM points of (arithmetic) modular forms, although it does not preserve holomorphy (see [AAF] III and [Sh3]). To describe the rationality, we recall the two embeddings $i_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $i_p : \overline{\mathbb{Q}} \hookrightarrow \widehat{\mathbb{Q}}_p$ fixed in the introduction. Recall $\mathcal{W} = i_p^{-1}(W)$, which is a *discrete* valuation ring. Let $(A, \lambda, \omega, i)_{/\mathcal{W}}$ be an ordinary quadruple of CM type (M, Σ) (having complex multiplication by the integer ring $\mathfrak{O} \subset M$). The complex uniformization: $\iota : A(\mathbb{C}) \cong \mathbb{C}^\Sigma/\Sigma(\mathfrak{A})$ induces a canonical base $\omega_\infty = \iota^* du$ of $\Omega_{A/\mathbb{C}}$ over $\mathfrak{O} \otimes_{\mathbb{Z}} \mathbb{R}$, where $u = (u_\sigma)_{\sigma\in\Sigma}$ is the standard variable on $\mathbb{C}^\Sigma$ and $\Sigma(\mathfrak{A}) = \{(\sigma(a))_{\sigma\in\Sigma} \in \mathbb{C}^\Sigma | a \in \mathfrak{A}\}$. We define the periods $\Omega_\infty \in \mathbb{C}^\Sigma = O \otimes_{\mathbb{Z}} \mathbb{C}$ by $\omega = \Omega_\infty\omega_\infty$. The level p–structure $i_p : \mu_{p^\infty} \otimes \mathfrak{d}^{-1} \hookrightarrow A[p^\infty]$ induces an isomorphism $\iota_p : \mathrm{Spf}(\widehat{W[q^\xi]}_{\xi\in O}) = \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}} \mathfrak{d}^{-1} \cong \widehat{A}$ for the p–adic formal group $\widehat{A}_{/W}$ at the origin. Then $\omega = \Omega_p\omega_p$ ($\Omega_p \in O \otimes_{\mathbb{Z}} W = W^\Sigma$) for $\omega_p = \iota_{p,*}\frac{dq}{q}$. Here is the rationality result of Shimura for $f \in G_\kappa(\mathfrak{c}, \mathfrak{N}; \mathcal{W})$:

$$\frac{(\delta_\kappa^k f)(A, \lambda, \omega_\infty, i)}{\Omega_\infty^{\kappa+2k}} = (\delta_\kappa^k f)(A, \lambda, \omega, i) \in \overline{\mathbb{Q}}. \tag{S}$$

Katz interpreted the differential operator in terms of the Gauss-Manin connection of the universal AVRM over $\mathfrak{M}$ and gave a purely algebro-geometric definition of the operator (see [K3] Chapter II and [HT] Section 1 for a summary of the result of Katz). Using this algebraization of δ_κ^k, he extended the operator to geometric modular forms and p–adic modular forms. We write his operator corresponding to Shimura's operator δ_*^k as $d^k : V(\mathfrak{c}, \mathfrak{N}; R) \to V(\mathfrak{c}, \mathfrak{N}; R)$. An important formula given in [K3] (2.6.7) is: for $f \in G_\kappa(\mathfrak{c}, \mathfrak{N}; \mathcal{W})$,

$$\frac{(d^k f)(A, \lambda, \omega_p, i)}{\Omega_p^{\kappa+2k}} = (d^k f)(A, \lambda, \omega, i) = (\delta_\kappa^k f)(A, \lambda, \omega, i) \in \mathcal{W}. \tag{K}$$

Let t be the canonical variable of the Serre–Tate deformation space $\widehat{S}$. Identifying $\widehat{S}$ with $\widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}} \mathfrak{d}^{-1}$, t is the character $1 \in O = X^*(\mathbb{G}_m \otimes_{\mathbb{Z}} \mathfrak{d}^{-1}) = \mathrm{Hom}(\mathbb{G}_m \otimes_{\mathbb{Z}} \mathfrak{d}^{-1}, \mathbb{G}_m)$. Write $S = \mathbb{G}_m \otimes_{\mathbb{Z}} \mathfrak{d}^{-1}$. We have $\widehat{S} = \mathrm{Spf}(\widehat{W[X(S)]})$ for the completion $\widehat{W[X(S)]}$ at the augmentation ideal of the monoid algebra $W[X(S)] = W[O]$ ($X(S) = X^*(S) = \mathrm{Hom}_{alg-gp}(S, \mathbb{G}_m)$), where $W[O]$ is the ring made up of formal finite sums $\sum_{\xi\in O} a(\xi)t^\xi$ ($a(\xi) \in W$). We have the following interpretation of d^κ:

$$d^\kappa \sum_\xi a(\xi)t^\xi = \sum_\xi a(\xi)\xi^\kappa t^\xi. \tag{4.5}$$

To see this formula, let us recall the construction of d^κ. Let $\underline{\mathcal{A}} = (\mathcal{A}, \widehat{\lambda}, \widehat{i})$ be the universal deformation of $\underline{A} = (A, \lambda, i)$ on $\widehat{S}$. Since A is ordinary, the level

p–structure $i_p : \mu_{p^\infty} \otimes_{\mathbb{Z}} \mathfrak{d}^{-1} \hookrightarrow \mathcal{A}$ gives the identification of formal groups $\widehat{i}_p : \widehat{\mathbb{G}}_m \otimes \mathfrak{d}^{-1}_{/\widehat{S}} \cong \widehat{\mathcal{A}}$. Note that $\widehat{\mathbb{G}}_m \otimes \mathfrak{d}^{-1}_{/\widehat{S}}$ is isomorphic to $\widehat{S} \times_W \widehat{S}$ over W; so, we write the standard variable on the base $\widehat{S}$ as t and on the fiber $\widehat{S}$ as s. Then for each $a \in O_p$, we have a unique section $\omega(a) = (a\widehat{i}_p)_* \frac{ds}{s}$ of $\underline{\omega}_{/\widehat{S}}$. The action of a is just $s \mapsto s^a$; so, $\omega(a) = \frac{ds^a}{s^a}$. By [K2] 4.3.1, the differential operator is $\widehat{S}$–invariant, and the canonical variable t is normalized so that $dt^a = at^a \iff d = t\frac{d}{dt}$. In other words, by the construction of d, choosing a parameter t of $\widehat{S}$ so that $\widehat{S} = \mathrm{Spf}(\widehat{W[t^\xi]}_{\xi \in O})$, we have $d = a^{-1}t\frac{d}{dt}$ on $\widehat{S}$ for a unit $a \in W^\times$. Thus changing t by t^a, we have an exact identity as above. This change of variable does not cause much trouble in the computation we execute later (because everything involving t is brought to that of t^a by the variable change). Thus we may assume $d^\kappa t^\xi = \xi^\kappa t^\xi$.

There is another short cut showing (4.5): It is known that d induces a base of invariant differentials on the base $\mathrm{Spf}(\widehat{W[\mathfrak{a}\mathfrak{b}]})$ of the Tate AVRM, regarding it as $\widehat{\mathbb{G}}_m \otimes (\mathfrak{a}\mathfrak{b})$; so, d^σ coincides with δ_0^σ. From this, we can also conclude that d^κ induces an invariant differential.

For each $f \in V(\mathfrak{c}, \mathfrak{N}; R)$ (for a p–adic algebra R), we call the expansion

$$f(t) := f(\mathcal{A}, \widehat{\lambda}, \widehat{i}) = \sum_{\xi \in O} a(\xi, f) t^\xi$$

as an element of $\widehat{R[O]}$ a *t–expansion* of f. Hereafter, we write this ring symbolically as $R[[t^\xi]]_{\xi \in O}$. Choosing a $\mathbb{Z}$–base $\{a_j\}$ of O, $T_j = t^{a_j} - 1$ gives a complete set of local parameters at the point $x \in \widehat{\mathfrak{M}}(\mathfrak{c}, \mathfrak{N})_{/R}$ given by $\underline{A}$ and $\widehat{R[O]} \cong R[[T_1, \ldots, T_d]]$. We have the following t–expansion principle:

(t-exp) *The t–expansion: $f \mapsto f(t) \in R[[t^\xi]]_{\xi \in O}$ determines f uniquely.*

The Taylor expansion of f with respect to the variables $T = (T_j)$ can be computed by applying differential operators $\partial_j = \frac{\partial}{\partial T_j}$ and evaluating the result at $x = \underline{A}$. Since ∂_j is a linear combination of the d^σ's in the field of fractions of R as long as R is of characteristic 0, we have, for $f, g \in V(\mathfrak{c}, \mathfrak{N}; W)$,

$$d^\kappa f(\underline{A}) = d^\kappa g(\underline{A}) \text{ for all } \kappa \geq 0 \iff f(t) = g(t). \tag{4.6}$$

What we have described is actually an oversimplified description of Katz's theory, and the reader is referred to [K3] and [HT] Section 1 for a more rigorous explanation on the subject.

4.2 q–Expansion of Eisenstein Series

Let $\phi : \{O_p \times (O/\mathfrak{f}')\} \times \{O_p \times (O/\mathfrak{f}'')\} \to \mathbb{C}$ be a locally constant function such that $\phi(\varepsilon^{-1}x, \varepsilon y) = N(\varepsilon)^k \phi(x, y)$ for all $\varepsilon \in O^\times$, where k is a positive integer and $\mathfrak{f}'$ and $\mathfrak{f}''$ are integral ideals prime to p. We put $\mathfrak{f} = \mathfrak{f}' \cap \mathfrak{f}''$ and suppose that all $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$ are prime to $\mathfrak{f}p$. We regard ϕ as a function on $X \times Y$ with

$X = Y = O_p \times (O/\mathfrak{f})$ via the natural projection of $\{O_p \times (O/\mathfrak{f})\} \times \{O_p \times (O/\mathfrak{f})\}$ to $\{O_p \times (O/\mathfrak{f}')\} \times \{O_p \times (O/\mathfrak{f}'')\}$. We put $X_\alpha = (O/p^\alpha O) \times (O/\mathfrak{f})$ and define the partial Fourier transform

$$P\phi : \left\{(F_p/O_p^*) \times (\mathfrak{f}^*/O^*)\right\} \times Y = \left\{\bigcup_\alpha p^{-\alpha}\mathfrak{f}^*/O^*\right\} \times Y \to \mathbb{C}$$

of ϕ, taking α so that ϕ factors through $X_\alpha \times Y$, by

$$P\phi(x,y) = \begin{cases} p^{-\alpha[F:\mathbb{Q}]} N(\mathfrak{f})^{-1} \sum_{a \in X_\alpha} \phi(a,y)\mathbf{e}_F(ax) & \text{if } x \in p^{-\alpha}\mathfrak{f}^*/O^*, \\ 0 & \text{if } x \notin p^{-\alpha}\mathfrak{f}^*/O^*, \end{cases} \tag{4.7}$$

where $\mathbf{e}_F$ is the standard additive character of $F_{\mathbb{A}}$ restricted to the local component $F_{p\mathfrak{f}}$ at $p\mathfrak{f}$. This definition does not depend on the choice of α.

We construct an Eisenstein series $E_k(z;\phi)$ for a positive integer k and ϕ as above as a function of triples $(\mathcal{L}, \lambda, i)$ we have studied in the previous subsection. Actually k indicates the parallel weight $\sum_\sigma k\sigma$. Here $i : F_p/O_p^* \times (\mathfrak{f})^*/O^* \hookrightarrow p^{-\infty}\mathcal{L}/\mathcal{L} \times \mathfrak{f}^{-1}\mathcal{L}/\mathcal{L}$ is the level $p^\infty\mathfrak{f}^2$–structure. The $\mathfrak{f}$–part $i_\mathfrak{f}$ of i induces, via polarization, the dual map $i'_\mathfrak{f} : \mathcal{L}/\mathfrak{f}\mathcal{L} \twoheadrightarrow O/\mathfrak{f}$, and hence having $i_\mathfrak{f}$ is equivalent to having a pair $(i_\mathfrak{f}, i'_\mathfrak{f})$, which is literally of level $\mathfrak{f}^2$ (not just of level $\mathfrak{f}$). We define an $O_{\mathfrak{f}p}$–submodule $PV(\mathcal{L}) \subset \mathcal{L} \otimes_O F_{\mathfrak{f}p}$ specified by the following conditions:

(pv1) $PV(\mathcal{L}) \supset \mathcal{L} \otimes_O O_{\mathfrak{f}p}$;

(pv2) $PV(\mathcal{L})/\left(\mathcal{L} \otimes_O O_{\mathfrak{f}p}\right) = \mathrm{Im}(i)$.

By definition, we may regard

$$i^{-1} : PV(\mathcal{L}) \twoheadrightarrow PV(\mathcal{L})/\left(\mathcal{L} \otimes_O O_{\mathfrak{f}p}\right) \cong F_p/O_p^* \times \mathfrak{f}^*/O^*.$$

By Pontryagin duality under $\mathrm{Tr} \circ \lambda$, the dual map i' of i gives rise to

$$i' : PV(\mathcal{L}) \twoheadrightarrow O_p \times (O/\mathfrak{f}).$$

See [HT] page 206 for details for i' which is written as π' there. Then we may regard $P\phi$ as a function on $p^{-\infty}\mathfrak{f}^{-1}\mathcal{L} \cap PV(\mathcal{L}) = \left(\bigcup_\alpha p^{-\alpha}\mathfrak{f}^{-1}\mathcal{L}\right) \cap PV(\mathcal{L})$ by

$$P\phi(w) = \begin{cases} P\phi(i^{-1}(w), i'(w)) & \text{if } (w \mod \mathcal{L}) \in \mathrm{Im}(i), \\ 0 & \text{otherwise.} \end{cases} \tag{4.8}$$

For each $w = (w_\sigma) \in F \otimes_\mathbb{Q} \mathbb{C} = \mathbb{C}^I$, the norm map $N(w) = \prod_{\sigma \in I} w_\sigma$ is well defined. Writing $\underline{\mathcal{L}} = (\mathcal{L}, \lambda, i)$ for simplicity, we define the value $E_k(\underline{\mathcal{L}}; \phi, \mathfrak{c})$ by

$$E_k(\underline{\mathcal{L}}; \phi, \mathfrak{c}) = \frac{\{(-1)^k\Gamma(k+s)\}^{[F:\mathbb{Q}]}}{\sqrt{|D_F|}} {\sum_{w \in p^{-\infty}\mathfrak{f}^{-2}\mathcal{L}/O^\times}}' \frac{P\phi(w)}{N(w)^k|N(w)|^{2s}}\Big|_{s=0}. \tag{4.9}$$

Here "$\sum'$" indicates that we are excluding $w = 0$ from the summation. As shown by Hecke, this type of series is convergent when the real part of s is

sufficiently large and is continued to a meromorphic function well defined at $s = 0$ (as long as either $k \geq 2$ or $\phi(a,0) = 0$ for all a). If either $k \geq 2$ or $\phi(a,0) = 0$ for all a, the function $E_k(\mathfrak{c}, \phi)$ gives an element in $G_\kappa(\mathfrak{c}, \mathfrak{f}^2 p^\infty; \mathbb{C})$ ($\kappa = k \sum_{\sigma: F \hookrightarrow \overline{\mathbb{Q}}} \sigma$), whose q–expansion computed in [HT] Section 2 is given by

$$N(\mathfrak{a})^{-1} E_k(\phi, \mathfrak{c})_{\mathfrak{a},\mathfrak{b}}(q) = 2^{-[F:\mathbb{Q}]} L(1-k; \phi, \mathfrak{a}) + \sum_{0 \ll \xi \in \mathfrak{a}\mathfrak{b}} \sum_{\substack{(a,b) \in (\mathfrak{a} \times \mathfrak{b})/O^\times \\ ab = \xi}} \phi(a,b) \frac{N(a)^k}{|N(a)|} q^\xi, \quad (4.10)$$

where $L(s; \phi, \mathfrak{a})$ is the partial L–function given by the Dirichlet series:

$$\sum_{\xi \in (\mathfrak{a} - \{0\})/O^\times} \phi(\xi, 0) \left(\frac{N(\xi)}{|N(\xi)|} \right)^k |N(\xi)|^{-s}.$$

4.3 Eisenstein Measure

We recall the definition of the Eisenstein measure with values in $V(\mathfrak{c}, \mathfrak{f}\mathfrak{s}; W)$ for a p–adic algebra W given below. Recall the fixed algebraic closure $\mathbb{F}$ of $\mathbb{F}_p$ and the ring $W(\mathbb{F})$ of Witt vectors with coefficients in $\mathbb{F}$. We consider $W(\mathbb{F})$ as a subring of the p–adic completion $\widehat{\mathbb{Q}}_p$ of $\overline{\mathbb{Q}}_p$. Let W be a discrete valuation ring finite flat over $W(\mathbb{F})$ inside $\widehat{\mathbb{Q}}_p$. For any fractional ideal $\mathfrak{a}$, write its prime decomposition as $\prod_\mathfrak{q} \mathfrak{q}^{\varepsilon_\mathfrak{q}}$; so, $\varepsilon_\mathfrak{q}$ is an integer with $\varepsilon_\mathfrak{q} = 0$ for primes $\mathfrak{q}$ which do not show up in the prime decomposition. We denote $\varepsilon(\mathfrak{a}) = \{\varepsilon_\mathfrak{q}\}_\mathfrak{q}$ for this set of exponents. We abbreviate the product $\prod_\mathfrak{q} \mathfrak{q}^{\varepsilon_\mathfrak{q}}$ as $\mathfrak{q}^{\varepsilon(\mathfrak{a})}$, which is equal to $\mathfrak{a}$.

Let $\mathfrak{s}|\mathfrak{f}$ be two integral ideals of F prime to p. We consider the space $\widetilde{\mathfrak{O}} = (O_p \times (O/\mathfrak{f})) \times (O_p \times (O/\mathfrak{s}))$ and write the variable on $\widetilde{\mathfrak{O}}$ as $(x, a; y, b)$ for $x, y \in O_p$ and $a \in O/\mathfrak{f}$ and $b \in O/\mathfrak{s}$. We regard $\widetilde{\mathfrak{O}}$ as a ring; then $\widetilde{\mathfrak{O}}^\times$ is the group of invertible elements in $\widetilde{\mathfrak{O}}$. Embedding $O^\times$ into $\widetilde{\mathfrak{O}}$ diagonally, we can take the closure $\overline{O^\times}$ under the profinite topology of $\widetilde{\mathfrak{O}}$. We also write $\widetilde{T} = (O_p^\times \times (O/\mathfrak{f})) \times (O_p^\times \times (O/\mathfrak{s}))$. We let $\varepsilon \in \overline{O^\times}$ act on $\widetilde{\mathfrak{O}}$ by $\varepsilon(x, a; y, b) = (\varepsilon x, \varepsilon a; \varepsilon y, \varepsilon b)$. Then we define $T = \widetilde{T}/\overline{O^\times}$ and $T^\times = \widetilde{\mathfrak{O}}^\times/\overline{O^\times}$. These are the profinite compact spaces carrying the Eisenstein measure.

For each continuous function $\phi(x, a; y, b)$ on $\widetilde{T}$, we consider the following partial Fourier transform:

$$\phi^\circ(x, a; y, b) = \sum_{u \in O/\mathfrak{f}} \phi(x^{-1}, u; y, b) \mathbf{e}_F(-ua\varpi^{-\varepsilon(\mathfrak{d}\mathfrak{f})}), \quad (4.11)$$

where we have chosen for each prime $\mathfrak{q}$ of F, a prime element $\varpi_\mathfrak{q}$ in $F_\mathfrak{q}^\times \subset F_\mathbb{A}^\times$ and put $\varpi^e = \prod_\mathfrak{q} \varpi_\mathfrak{q}^{e_\mathfrak{q}}$ in $F_\mathbb{A}^\times$ for each exponent $e = \{e_\mathfrak{q} \in \mathbb{Z}\}_\mathfrak{q}$ with $e_\mathfrak{q} = 0$ almost everywhere (that is, except for finitely many primes). The map $\phi \mapsto \phi^\circ$ is a linear operator acting on the space $\mathcal{C}(\widetilde{T}; W)$ of all continuous functions on

$\widetilde{T}$ with values in W (and is invertible by the Fourier inversion formula). If ϕ factors through T, then ϕ° satisfies the following property

$$\phi^\circ(\varepsilon x, \varepsilon a; \varepsilon^{-1}y, \varepsilon^{-1}b) = \phi^\circ(x, a; y, b) \text{ for all } \varepsilon \in \overline{O^\times}.$$

This is the property required to define Eisenstein series (for even weight k) in the previous subsection. Then there exists a unique measure $\mathbf{E}_{\mathfrak{c}} : \mathcal{C}(T; W) \to V(\mathfrak{c}, \mathfrak{fs}; W)$ with the following two properties:

(E1) If ϕ has values in $\overline{\mathbb{Q}}$ equipped with the discrete topology, then for each positive integer $k > 0$,

$$\mathbf{E}_{\mathfrak{c}}(N^{-k}\phi) = E_k(\phi^\circ; \mathfrak{c}),$$

where $N : \widetilde{T} \to \mathbb{Z}_p^\times$ is given by $N(x, a; y, b) = N_{F/\mathbb{Q}}(x)$ for the norm map $N_{F/\mathbb{Q}} : O_p \to \mathbb{Z}_p$. Note here that $N^{-k}\phi^\circ$ (for any positive integer k) factors through $T \iff \phi^\circ$ satisfies invariance under $O^\times$ required for the definition of the Eisenstein series;

(E2) The q–expansion of $\mathbf{E}_{\mathfrak{c}}(\phi)$ at the cusp $(\mathfrak{a}, \mathfrak{b})$ is given by

$$N(\mathfrak{a}) \sum_{0 \ll \xi \in \mathfrak{ab}} q^\xi \sum_{(a,b) \in (\mathfrak{a}\times\mathfrak{b})/O^\times, ab=\xi} \phi^\circ(a; b)|N(a)|^{-1},$$

where $|N(a)|$ is the (complex) absolute value of the norm $N(a)$ of $a \in \mathfrak{a}$, $\mathfrak{a} \times \mathfrak{b}$ is embedded in $\widetilde{T}$ by $(a, a \mod \mathfrak{f}; b, b \mod \mathfrak{s})$, and $\varepsilon \in O^\times$ acts on (a, b) by $(a, b) \mapsto (\varepsilon a, \varepsilon^{-1}b)$.

The existence and the uniqueness of the measure satisfying (E1-2) is a consequence of the q–expansion principle and the q–expansion of the classical Eisenstein series given in the previous section (see [K3] Chapter III and [HT] Section 3). Although it is assumed that $\mathfrak{f} = \mathfrak{s}$ in [HT], there is no difficulty extending the construction to the general case, since for any function factoring through T as in (E1), the corresponding Eisenstein series can be checked to be of level $\mathfrak{fs}$.

When confusion is unlikely, we write $E(\phi)$ for $E_0(\phi; \mathfrak{c})$ to simplify our notation (though $E(\phi)$ fully depends on $\mathfrak{c}$).

4.4 Katz Measure

We can evaluate p–adic modular forms f at any test object $(A, \lambda, i)_{/W}$ defined over W. This gives rise to a linear form $Ev : V(\mathfrak{c}, \mathfrak{fs}; W) \to W$ given by $Ev(f) = f(A, \lambda, i)$. Thus we can think of the evaluation $Ev \circ \mathbf{E}_{\mathfrak{c}}$, which is a bounded measure on $\mathcal{C}(T; W)$ with values in W.

Now we choose a specific test object. Let $x = [z, g]$ be an ordinary CM point of the Shimura variety. We take the abelian scheme (A, λ, i) sitting over $x \in \mathfrak{M}(\mathfrak{c}, \mathfrak{fs})$. Thus A has complex multiplication by a CM field $M = M_x$ with a CM type Σ. We write M' for the reflex field of (M, Σ) (see [ACM] Section 8). We suppose that p is unramified in M (and hence in M'). The complex manifold $A(\mathbb{C})$ is given by $\mathbb{C}^\Sigma/\Sigma(\mathfrak{A})$ for a lattice $\mathfrak{A} \subset M$, and we can find a model A

defined over an abelian extension k of M' such that all torsion points of A are rational over an abelian extension of M' ([ACM] 18.6 and 21.1). The model is unique if the field contains the field of moduli of the sufficiently deep level structure i. By a theorem of Serre–Tate, making i deep (for example, making it of level $\mathfrak{N}$ for a deep $\mathfrak{N}$ prime to p), A has good reduction over $W \cap k$. Here we can insist that k is unramified at p if M is unramified at p. Thus we may assume that (A, λ, i) is defined over W, and if p is unramified in $F/\mathbb{Q}$, we may assume that $W = W(\mathbb{F})$. We further assume that the special fiber $\widetilde{A}$ at p of A is an ordinary abelian variety. Since the residue field $\mathbb{F}$ of W is algebraically closed, $\widetilde{A}[p^\infty] \cong \left(\mu_{p^\infty} \otimes \mathfrak{d}^{-1}\right) \times (F_p/O_p)$. Thus $\widetilde{A}$ has level p^∞ structure i_0 defined over $\mathbb{F}$. By Serre–Tate deformation theory, A sits at the origin $1 \in \widehat{S}$. Thus we can uniquely lift i_0 to a level p^∞–structure i_p; so, we may assume that i contains a level p^∞ structure defined over W.

We would like to recall briefly the construction of the Katz measure interpolating the L–values of arithmetic Hecke characters of conductor dividing $\mathfrak{C}p^\infty$, where $\mathfrak{C}$ is an integral ideal of M prime to p. We write $\mathfrak{O}$ for the integer ring of M. We decompose $\mathfrak{C} = \mathfrak{F}\mathfrak{F}_c\mathfrak{I}$. Here $\mathfrak{I}$ consists of inert or ramified primes over F, $\mathfrak{F}\mathfrak{F}_c$ consists split primes over F and

$$\mathfrak{F} + \mathfrak{F}_c = \mathfrak{F} + \mathfrak{F}^c = \mathfrak{F}_c + \mathfrak{F}_c^c = \mathfrak{O} \quad \text{and} \quad \mathfrak{F}_c \supset \mathfrak{F}^c.$$

We put $\mathfrak{f} = \mathfrak{F}\mathfrak{I} \cap F$, $\mathfrak{s} = \mathfrak{F}_c \cap F$ and $\mathfrak{i} = \mathfrak{I} \cap F$. We have $\mathfrak{O}_p = \mathfrak{O} \otimes_{\mathbb{Z}} \mathbb{Z}_p = \mathfrak{O}_{\mathbf{p}} \times \mathfrak{O}_{\mathbf{p}^c}$, where $\mathfrak{O}_{\mathbf{p}} = \prod_{\mathfrak{P} \in \Sigma_p} \mathfrak{O}_{\mathfrak{P}}$. We suppose the following four conditions:

1. The lattice $\mathfrak{A}$ is a fractional ideal of M prime to $\mathfrak{C}p$; so, we write $A = A(\mathfrak{A})$ (so, $A(\mathfrak{A})(\mathbb{C}) = \mathbb{C}^\Sigma/\Sigma(\mathfrak{A})$).

2. Choose $\delta \in M$ so that $\delta^c = -\delta$ and $\mathrm{Im}(\sigma(\delta)) > 0$ for all $\sigma \in \Sigma$, and have the alternating form $\langle u, v \rangle = (u^c v - u v^c)/2\delta$ induce $\mathfrak{O} \wedge \mathfrak{O} \cong \mathfrak{d}^{-1}\mathfrak{c}^{-1}$. Then this pairing induces $\mathfrak{c}(\mathfrak{A}\mathfrak{A}^c)^{-1}$–polarization $\lambda = \lambda(\mathfrak{A})$.

3. The inclusion $F \hookrightarrow M$ induces a canonical isomorphism $O_p \cong \mathfrak{O}_{\mathbf{p}}$, which in turn induces $i'_p : F_p/O_p^* = F_p/O_p \cong M_\Sigma/\mathfrak{A}\mathfrak{O}_{\mathbf{p}} \subset \mathbb{C}^\Sigma/\Sigma(\mathfrak{A})$. We put $i_p(\mathfrak{A})(x) = i'_p(2\delta x)$. This is the p–part of the level structure $i(\mathfrak{A})$.

4. The prime-to–p part $i^{(p)}$ of $i(\mathfrak{A})$ is defined as follows. Choose an idele d_F of F such that $d_F\widehat{O} = \widehat{\mathfrak{d}}$, the prime-to–$\mathfrak{f}p\mathfrak{d}$ component $d_F^{(\mathfrak{f}p\mathfrak{d})}$ is trivial and $d_{F,\mathfrak{q}} = (2\delta)_{\mathfrak{Q}}$ for prime ideal $\mathfrak{Q}$ with $\mathfrak{Q}|\mathfrak{F}$, where $\mathfrak{q} = \mathfrak{Q} \cap F$. Then $x \mapsto d_F x$ induces $(\mathfrak{f}^2)^*/O^* \hookrightarrow (\mathfrak{F}\mathfrak{i})^{-2}\mathfrak{A}/\mathfrak{A}$, which is the $i_{\mathfrak{f}}$.

In addition to the data $(A(\mathfrak{A}), \lambda(\mathfrak{A}), i(\mathfrak{A}))$, assuming that p is unramified in $F/\mathbb{Q}$, for our later use, we choose the differential $\omega(\mathfrak{A})$ on $A(\mathfrak{A})$ as follows:

5. We choose and fix a differential $\omega = \omega(\mathfrak{O})$ on $A(\mathfrak{O})_{/\mathcal{W}}$ so that

$$H^0(A(\mathfrak{O}), \Omega_{A(\mathfrak{O})/\mathcal{W}}) = (\mathcal{W} \otimes_{\mathbb{Z}} O)\omega.$$

   Since $\mathfrak{A}_p = \mathfrak{O}_p$, $A(\mathfrak{O} \cap \mathfrak{A})$ is an étale covering of both $A(\mathfrak{A})$ and $A(\mathfrak{O})$; so, $\omega(\mathfrak{O})$ induces a differential $\omega(\mathfrak{A})$ first by pull-back to $A(\mathfrak{O} \cap \mathfrak{A})$ and then by pull-back inverse from $A(\mathfrak{O} \cap \mathfrak{A})$ to $A(\mathfrak{A})$.

As long as the projection $\pi : A(\mathfrak{O} \cap \mathfrak{A}) \twoheadrightarrow A(\mathfrak{A})$ is étale, the pull-back inverse $(\pi^*)^{-1} : \Omega_{A(\mathfrak{O}\cap\mathfrak{A})/\mathcal{W}} \to \Omega_{A(\mathfrak{A})/\mathcal{W}}$ is a surjective isomorphism. We thus have

$$H^0(A(\mathfrak{A}), \Omega_{A(\mathfrak{O})/\mathcal{W}}) = (\mathcal{W} \otimes_{\mathbb{Z}} O)\omega(\mathfrak{A}).$$

Let $Cl_M(\mathfrak{i})$ be the ideal class group of M modulo $\mathfrak{i}$ and $Cl^-(\mathfrak{i})$ be the quotient of $Cl_M(\mathfrak{i})$ by the image of $(O/\mathfrak{i})^\times$. Identifying $\mathfrak{O}_{\mathfrak{P}}$ (resp. $\mathfrak{O}_{\mathfrak{P}^c}$) with the first (resp. last) component of O_p of $\widetilde{\mathfrak{O}}$ and embedding $O/\mathfrak{f}$ into $\mathfrak{O}/\mathfrak{F}\mathfrak{I}$ (resp. identifying $O/\mathfrak{s}$ with $\mathfrak{O}/\mathfrak{F}_c$)) through the inclusion $O \hookrightarrow \mathfrak{O}$, we embed $T^\times$ into $Z = Cl_M(\mathfrak{C}p^\infty) = \varprojlim_n Cl_M(\mathfrak{C}p^n)$. Then we have the exact sequence:

$$T^\times \xrightarrow{\iota} Z \to Cl^-(\mathfrak{i}) \to 1,$$

and the kernel of ι is a finite group. We write $[\mathfrak{A}]$ for the image of the class of an ideal $\mathfrak{A}$ prime to $\mathfrak{C}p$ in Z. For $\alpha \in \mathfrak{O}$, we have $[(\alpha)] = \alpha^{-1}$, where the right-hand side is the image of the inclusion $\mathfrak{O}^\times_{\mathfrak{C}p} \to Z$. Choosing a complete representative set $\{\mathfrak{A}\}$ for $Cl^-(\mathfrak{i})$, we have a decomposition

$$Z = \bigsqcup_{\mathfrak{A}} \mathrm{Im}(\iota)[\mathfrak{A}]^{-1}.$$

For each function $\phi \in \mathcal{C}(Z; W)$, we define $\phi_{\mathfrak{A}} \in \mathcal{C}(T; W)$ in the following way: $\phi_{\mathfrak{A}}(t) = \phi(t[\mathfrak{A}]^{-1})$ for $t \in T^\times$ and extend it by 0 outside $T^\times$. Then define

$$\int_Z \phi d\varphi = \sum_{\mathfrak{A}} \int_T \phi_{\mathfrak{A}} d\mathbf{E}_{\mathfrak{c}_{\mathfrak{A}}}(A(\mathfrak{A}), \lambda(\mathfrak{A}), i(\mathfrak{A})), \tag{4.12}$$

where $\mathfrak{c}_{\mathfrak{A}} = \mathfrak{c}(\mathfrak{A}\mathfrak{A}^c)^{-1}$. We write $E_{\mathfrak{A}}(\phi)$ for $\mathbf{E}_{\mathfrak{c}_{\mathfrak{A}}}(\phi)$ for functions $\phi \in \mathcal{C}(T^\times; W)$.

In [K3] Chapter V and [HT] Section 4, computation of $\int_Z \widehat{\lambda} d\varphi$ is made for the p–adic avatar $\widehat{\lambda}$ of an arithmetic Hecke character λ of conductor a factor of $\mathfrak{C}p$. The result is as described in the introduction. Since there are many misprints in [HT] (though all minor), we have added at the end of this paper a correction table of misprints in [HT].

5 Proof of Theorem I

Recall the quadratic CM extension M/F introduced in Section 1, and write Σ (resp. Σ_p) for the CM type (resp. the p–adic CM type) we fixed there. We now prove Theorem I, and the proof concludes in Subsection 5.4. We assume that p is unramified in $F/\mathbb{Q}$ and write $W = W(\mathbb{F})$.

5.1 Splitting the Katz Measure

We start with a general argument. We assume that $p > 2$. Let the triple $(A(\mathfrak{A}), \lambda(\mathfrak{A}), i(\mathfrak{A}))$ be the abelian variety of CM type (M, Σ) as in Section 4.4. We consider the measure $E_{\mathfrak{A}} : \phi \mapsto \int_T \phi d\mathbf{E}_{\mathfrak{c}_{\mathfrak{A}}}(A(\mathfrak{A}), \lambda(\mathfrak{A}), i(\mathfrak{A}))$ (on the image of $T^\times$ in $Cl_M(\mathfrak{C}p^\infty)$) for the polarization ideal $\mathfrak{c}_{\mathfrak{A}} = \mathfrak{c}(\mathfrak{A}\mathfrak{A}^c)^{-1}$. For $\alpha \in M$

prime to $\mathfrak{C}p$, $u \mapsto \alpha u$ induces an isomorphism: $A(\mathfrak{A}) \cong A(\alpha\mathfrak{A})$. This multiplication by α sends $i(\mathfrak{A})$ (resp. $\lambda(\mathfrak{A})$ and $\omega(\mathfrak{A})$) to $\alpha \circ i(\alpha\mathfrak{A})$ which sends an element $x \in (F_p/O^*) \times (\mathfrak{f}^2)^*/O^*$ to $\alpha d_F x \mod \alpha\mathfrak{A}$ (resp. $\alpha\alpha^c\lambda(\mathfrak{A}) = \lambda(\alpha\mathfrak{A})$ and $\alpha\omega(\alpha\mathfrak{A})$). This shows that

$$\int_{T^\times} \phi(\alpha t) dE_{\mathfrak{A}}(t) = \int_{T^\times} \phi(t) dE_{\alpha\mathfrak{A}}(t), \tag{5.1}$$

where $\alpha(x, a; y, b) = (\alpha x, \alpha a; \alpha^c y, \alpha^c b)$ for $t = (x, a; y, b)$. This tells us how the piece of the integral corresponding to $\mathfrak{A}$ in the definition of the Katz measure $d\varphi$ transforms if we change $\mathfrak{A}$ in its ideal class.

This formula (5.1) can be verified functorially using the fact:

$$(A(\alpha\mathfrak{A}), \lambda(\alpha\mathfrak{A}), \alpha i(\alpha\mathfrak{A})) \cong (A(\mathfrak{A}), (\alpha\alpha^c)\lambda(\mathfrak{A}), i(\mathfrak{A})) \text{ by } \alpha u \mapsto u,$$

but there is an easy short-cut: for $k \gg 0$,

$$\begin{aligned}\int_T \phi N^{-k}(t) dE_{\mathfrak{A}}(t) &= \sum_{w \in \mathfrak{A}} P(N^{-k}\phi)^\circ(i^{-1}(w), i'(w)) \\ &= \sum_{w \in \alpha\mathfrak{A}} P(N^{-k}\phi)^\circ(\alpha^{-1}i^{-1}(w), \alpha^{-c}i'(w)) \\ = \int_T \phi N^{-k}(\alpha x, \alpha^{-c}y, \alpha^{-1}a, \alpha^{-c}b) dE_{\alpha\mathfrak{A}} &= \int_T \phi N^{-k}(\alpha^{-1}t) dE_{\alpha\mathfrak{A}}(t),\end{aligned}$$

where $N(x, a; y, b) = N(x) = \prod_\sigma \sigma(x)$. For each function ϕ on $\mathrm{Im}(\iota)[\mathfrak{A}]^{-1}$, we define $\phi_{\mathfrak{A}}(x) = \phi(x[\mathfrak{A}]^{-1})$. Now we decompose, for an open subgroup H of $T^\times$ containing $\mathrm{Ker}(\iota)$,

$$\iota(T^\times)[\mathfrak{A}]^{-1} = \bigsqcup_{\mathfrak{B}} \iota(H)[\mathfrak{B}]^{-1} \iff T^\times = \bigsqcup_{\mathfrak{B}} H[\mathfrak{B}^{-1}\mathfrak{A}].$$

Thus, we have

$$\begin{aligned}\int_{T^\times} \phi_{\mathfrak{A}}(t) dE_{\mathfrak{A}}(t) &= \sum_{\mathfrak{B}} \int_{T^\times} \chi_{H[\mathfrak{B}^{-1}\mathfrak{A}]} \phi_{\mathfrak{A}}(t) dE_{\mathfrak{A}}(t) \\ = \sum_{\mathfrak{B}} \int_{T^\times} \chi_H(t[\mathfrak{B}\mathfrak{A}^{-1}]) \phi_{\mathfrak{B}}(t[\mathfrak{B}\mathfrak{A}^{-1}]) dE_{\mathfrak{A}}(t) &= \sum_{\mathfrak{B}} \int_T \chi_H(t) \phi_{\mathfrak{B}}(t) dE_{\mathfrak{B}}(t),\end{aligned}$$

where χ_H is the characteristic function of H. Note here that we have $\mathfrak{B} = \alpha\mathfrak{A}$ for $\alpha \in M^\times$.

For the moment, we assume that $\mathfrak{C}$ is stable under complex conjugation c. For simplicity, we write Z for $Cl_M(\mathfrak{C}p^\infty)$. We take a subgroup $\Gamma = \Gamma_{\mathfrak{C}} \subset Z$ of finite index satisfying the following two conditions:

1. $Z = \Gamma \times \Delta$ with torsion-free Γ and a finite group $\Delta = \Delta_{\mathfrak{C}}$;
2. Γ and Δ is stable under c.

Under the assumption: $p > 2$, we can choose the splitting $Z = \Gamma \times \Delta$ stable under c. This fact can be shown as follows: We can first split $Z = Z_p \times \Delta'$ so that Z_p is the maximal p–profinite subgroup. This splitting is canonical; so, it is stable under c. Since p is odd, we can split $Z_p = Z_p^+ \times Z_p^-$ so that c acts through multiplication by ± 1 on $Z_p^{\pm}$. Then we just split $Z_p^{\pm} = \Gamma^{\pm} \times \Delta_{\pm}$ for torsion-free $\Gamma^{\pm}$ and finite groups $\Delta_{\pm}$. Thus we can achieve c–stability of $\Gamma = (\Gamma^+ \times \Gamma^-) \subset Z$.

For each $z \in Z$, we define $\pi_-(z) = [z]^- := z^{1-c}$. Recall the torus $T_x = \mathrm{Res}_{M/\mathbb{Q}}\mathbb{G}_m \subset G$ fixing the closed point $x \in Sh^{ord}$ over $(A(\mathfrak{O}), \lambda(\mathfrak{O}), i(\mathfrak{O})) \in \mathfrak{M}(\mathfrak{c}, \mathfrak{f}^2 p^\infty)$ and its quotient $\mathcal{T}$ as in (3.18) (with the injection: $\mathcal{T}(\mathbb{Z}_{(p)}) \hookrightarrow T(\mathbb{Z}_p) = O_p^\times$ sending $\alpha \in T_x(\mathbb{Z}_{(p)})$ to $\alpha^{1-c} \in O_p^\times$). We have a natural exact sequence:

$$1 \to \left(\mathfrak{O}_p^\times \times (\mathfrak{O}/\mathfrak{C})^\times\right)/\overline{\mathfrak{O}^\times} \to Z \to Cl_M \to 1,$$

where Cl_M is the class group of M. Since $O^\times$ is a subgroup of $\mathfrak{O}^\times$ of finite index and p is unramified in $M/\mathbb{Q}$, $\pi_-(\mathfrak{O}^\times)$ is a finite group of order prime to p. By this fact, we see that

$$\Gamma^- \cap \pi_- \left(\left(\mathfrak{O}_p^\times \times (\mathfrak{O}/\mathfrak{C})^\times\right)/\overline{\mathfrak{O}^\times}\right) \hookrightarrow \mathfrak{O}_p^\times[-1] \cong O_p^\times,$$

where $\mathfrak{O}_p^\times[-1] = \{a \in \mathfrak{O}_p^\times | c(a) = a^{-1}\}$. In particular, identifying $\mathfrak{O}_{\mathfrak{p}}$ with O_p, for a principal ideal (α) prime to $\mathfrak{C}p$, $[(\alpha)]^- = \alpha_{\mathfrak{p}}^{c-1} \in T(\mathbb{Z}_p) = \mathfrak{O}_{\mathfrak{p}}^\times = O_p^\times$, where $\mathfrak{p} = \prod_{\mathfrak{p} \in \Sigma_p} \mathfrak{p}$. Therefore, writing $[\mathfrak{A}]$ for the image in Z of an ideal $\mathfrak{A}$ prime to $\mathfrak{C}p$, we have, regarding $\mathcal{T}(\mathbb{Z}_{(p)}) \subset T(\mathbb{Z}_p) = O_p^\times$ by (3.18),

$$[\mathfrak{A}]^- \in \mathcal{T}(\mathbb{Z}_{(p)}) \iff [\mathfrak{A}] \in [\mathfrak{O}_{(p\mathfrak{C})}^\times]\left(\Gamma^+ \times \Delta^+\right), \tag{5.2}$$

where $\Delta^+ = H^0(\mathrm{Gal}(M/F), \Delta)$, and $\mathfrak{O}_{(p\mathfrak{C})} \subset M$ is the localization (not the completion) of $\mathfrak{O}$ at $p\mathfrak{C}$.

We now allow the case $\mathfrak{C} \neq \mathfrak{C}^c$. In any case, we have a canonical splitting of Z into the prime-to–p subgroup $\Delta^{(p)}$ and the p–profinite subgroup Z_p. We fix a splitting $Z_p = \Delta_p \times \Gamma_{\mathfrak{C}}$ so that the natural projection $\pi : Z \to Cl_M(p^\infty)$ induces an isomorphism of $\Gamma_{\mathfrak{C}}$ onto the torsion free part $\Gamma = \Gamma_{\mathfrak{O}}$ of $Cl_M(p^\infty)$ we have already chosen in the above discussion. We then define $\Delta_{\mathfrak{C}} = \Delta^{(p)} \times \Delta_p$.

The translation $\phi(z) \mapsto \phi(z\zeta)$ by $\zeta \in \Delta_{\mathfrak{C}}$ gives an action of $\Delta_{\mathfrak{C}}$ on the space of continuous functions $\mathcal{C}(Z; W)$ on Z with values in W. For each character ψ of $\Delta_{\mathfrak{C}}$, we write $\mathcal{C}(Z; \mathcal{O})[\psi]$ for the ψ–eigenspace for the action of $\Delta_{\mathfrak{C}}$. Then the restriction of continuous functions on Z to $\Gamma_{\mathfrak{C}}$ gives rise to an isomorphism $\mathrm{Res}_\psi : \mathcal{C}(Z; W)[\psi] \cong \mathcal{C}(\Gamma_{\mathfrak{C}}; W)$. We write Inf_ψ for Res_ψ^{-1}.

For a given measure φ on Z, the ψ–component $\varphi_\psi \in W[[\Gamma]]$ is defined by

$$\int_\Gamma \phi d\varphi_\psi = \int_Z \mathrm{Inf}_\psi\, \phi d\varphi.$$

In terms of group algebras, $\widetilde{\psi} : Z \to W[[\Gamma]]$ given by $\widetilde{\psi}(\zeta\gamma) = \psi(\zeta)\pi(\gamma)$ for $\gamma \in \Gamma_{\mathfrak{C}}$ and $\zeta \in \Delta_{\mathfrak{C}}$ induces a continuous W–algebra homomorphism $W[[Z]] \twoheadrightarrow W[[\Gamma]]$ (still written as $\widetilde{\psi}$), and we can verify that $\varphi_\psi = \widetilde{\psi}(\varphi)$. If one chooses

another splitting of Z into a product of a torsion-free group and a finite group, they differ by a character of Γ into $\Delta_{\mathfrak{C}}$. In other words, $i'i^{-1} = \varepsilon$ is a character of Γ for two sections $i, i' : \Gamma \to Z$ of the projection: $Z \twoheadrightarrow \Gamma$. Then $\mathrm{Inf}_{\psi}\, \phi$ for two different splittings differ by multiplication by $\psi \circ \varepsilon$; hence, the invariant $\mu(\varphi_{\psi})$ is independent of the choice of splitting. Hereafter we stop worrying about the choice of splitting, fixing it once and for all. We write Δ for $\Delta_{\mathfrak{C}}$ hereafter.

5.2 Good Representatives

We would like to choose a representative set D for Δ so that the projection $\pi_{\Delta} : Z \to \Delta$ induces an isomorphism $D \cong \Delta$ if $p \nmid |Cl_M|$. In general, $D \cong Z/\Gamma'$ for the intersection Γ' of Γ and the image of $\mathfrak{O}^{\times}_{\mathfrak{C}p}$. We would like to choose D so that our computation of q–expansion (of Eisenstein series) becomes easier.

Let $\mathcal{I}(\mathfrak{C}p)$ be the group of fractional ideals of M prime to $\mathfrak{C}p$, and define

$$\mathcal{I}(\mathfrak{C}p)^{+} = \left\{ \mathfrak{A} \in \mathcal{I}(\mathfrak{C}p) \middle| \mathfrak{A}^{1-c} = \alpha^{1-c}\mathfrak{O} \text{ for } \alpha \in M^{\times} \right\}.$$

Suppose for the moment that $\mathfrak{C}$ does not contain primes ramifying in M/F. Since $\mathfrak{A}$ is prime to $\mathfrak{C}p$, α^{1-c} is prime to $\mathfrak{C}p$. Thus if a prime factor $\mathfrak{Q}$ of $\mathfrak{C}p$ divides the principal ideal (α), its conjugate $\mathfrak{Q}^c$ divides (α) with the equal multiplicity. Thus $\alpha = \beta\gamma$ for $\gamma \in F^{\times}$ with β prime to $\mathfrak{C}p$. In other words, $(\beta^{1-c}) = (\alpha^{1-c}) = \mathfrak{A}^{1-c}$, and hence we can write

$$\mathcal{I}(\mathfrak{C}p)^{+} = \left\{ \mathfrak{A} \in \mathcal{I}(\mathfrak{C}p) \middle| \mathfrak{A}^{1-c} = \alpha_{\mathfrak{A}}^{1-c}\mathfrak{O} \text{ for } \alpha_{\mathfrak{A}} \in M^{\times} \text{ prime to } \mathfrak{C}p \right\} \tag{5.3}$$

if $\mathfrak{C}$ does not contain primes ramifying in M/F. Without assuming the above condition, we can always write

$$\mathcal{I}(\mathfrak{C}p)^{+} = \left\{ \mathfrak{A} \in \mathcal{I}(\mathfrak{C}p) \middle| \mathfrak{A}^{1-c} = \alpha_{\mathfrak{A}}^{1-c}\mathfrak{O} \text{ for } \alpha_{\mathfrak{A}} \in M^{\times} \text{ prime to } \mathfrak{C}'p \right\},$$

where $\mathfrak{C}'$ is the maximal factor of $\mathfrak{C}$ prime to the relative discriminant of M/F.

The quotient of $\mathcal{I}(\mathfrak{C}p)^{+}$ by principal ideals prime to $\mathfrak{C}p$ is a subgroup of the class group Cl_M of M, which we write Cl_M^{+}. We see easily that

$$\overline{Cl}_F = \text{the image of } Cl_F \subset Cl_M^{+} \subset H^0(\mathrm{Gal}(M/F), Cl_M).$$

If the group $O_{+}^{\times}$ of totally positive units of O coincides with the group of square units, the equality $Cl_M^{+} = H^0(\mathrm{Gal}(M/F), Cl_M)$ holds. If further the class number of M is odd, the three groups are all equal. We take a complete representative set D^{-} (resp. D^{+}) for Cl_M/Cl_M^{+} (resp. Cl_M^{+} in $\mathcal{I}(\mathfrak{C}p)^{+}$).

When the class number of M is odd, we choose D^{+} among fractional ideals of F and D^{-} among primes of M split over F. If the class number is even, supposing that $\mathfrak{C}$ is prime to the discriminant of M/F, we choose $D^{+} \cup D^{-}$ among primes of M split over F.

We write Γ' for the intersection of Γ with the image of $\mathfrak{O}^{\times}_{\mathfrak{C}p}$ in the group $Z = Cl_M(\mathfrak{C}p^{\infty})$. Then we put $\mathcal{D}$ for a complete representative set in the localization (not the completion) $\mathfrak{O}^{\times}_{(\mathfrak{C}p)}$ for $\pi(\mathfrak{O}^{\times}_{\mathfrak{C}p})/\Gamma'$ with the projection $\pi : \mathfrak{O}^{\times}_{\mathfrak{C}p} \to Z$ if $\mathfrak{C}$ does not contain primes ramifying in M/F. When $\mathfrak{C}$ is divisible by a prime

ramified in M/F, things get more complicated, because we need to include in $\mathcal{D}$ elements $\alpha \in \mathfrak{O}$ divisible by some ramified primes in $\mathfrak{C}$. So until Subsection 5.5, we assume that $\mathfrak{C}$ is prime to the relative discriminant of M/F. Then we have

$$\int_Z \phi d\varphi = \sum_{\mathfrak{A}\in D^+} \sum_{\alpha\in\mathcal{D}} \sum_{\mathfrak{B}\in D^-} \int_{\Gamma'} \phi_{\mathfrak{A}\alpha\mathfrak{B}}(z) dE_{\mathfrak{A}\alpha\mathfrak{B}}$$

We compute $(A(\mathfrak{BA}), \lambda(\mathfrak{AB}), i(\mathfrak{AB}))$ for $\mathfrak{A} \subset F$. Since we have

$$A(\mathfrak{BA})(\mathbb{C}) = \mathbb{C}^\Sigma/\Sigma(\mathfrak{AB}) = \mathbb{C}^\Sigma/\Sigma(\mathfrak{B}) \otimes_O \mathfrak{A} = A(\mathfrak{B})(\mathbb{C}) \otimes_O \mathfrak{A},$$

we conclude $A(\mathfrak{AB}) = A(\mathfrak{B}) \otimes_O \mathfrak{A}$. There is another construction if we choose $\mathfrak{A} \subset O$: Tensoring $A(\mathfrak{B})$ to the exact sequence: $0 \to \mathfrak{A} \to O \to O/\mathfrak{A} \to 0$, we get another exact sequence:

$$0 \to \mathrm{Tor}_1(O/\mathfrak{A}, A(\mathfrak{B})) \xrightarrow{i} A(\mathfrak{B}) \otimes_O \mathfrak{A} \to A(\mathfrak{B}) \to 0.$$

Since O is a Dedekind domain, we have $\mathrm{Tor}_1(O/\mathfrak{A}, A(\mathfrak{B})) \cong A(\mathfrak{B})[\mathfrak{A}]$ canonically. Thus i brings $A(\mathfrak{B})[\mathfrak{A}]$ onto $(A(\mathfrak{B}) \otimes_O \mathfrak{A})[\mathfrak{A}]$. Since $\lambda(\mathfrak{B})$ is a $\mathfrak{c}_\mathfrak{B}$–polarization for $\mathfrak{c}_\mathfrak{B} = \mathfrak{c}(\mathfrak{BB}^c)^{-1}$, we have $A(\mathfrak{B})^t \xrightarrow{\lambda(\mathfrak{B})} A(\mathfrak{B}) \otimes \mathfrak{c}_\mathfrak{B}$. This induces

$$\begin{aligned}\lambda(\mathfrak{B}) \otimes \mathfrak{A} : (A(\mathfrak{B}) \otimes \mathfrak{A})^t &\cong A(\mathfrak{B})^t/A(\mathfrak{B})^t[\mathfrak{A}] \\ &\cong (A(\mathfrak{B}) \otimes_O \mathfrak{c}_\mathfrak{B}) \otimes_O \mathfrak{A}^{-1} = (A(\mathfrak{B}) \otimes_O \mathfrak{A}) \otimes \mathfrak{c}_{\mathfrak{AB}}.\end{aligned}$$

We can check that $\lambda(\mathfrak{B}) \otimes \mathfrak{A} = \lambda(\mathfrak{AB})$. Since $\mathfrak{A}$ is prime to $\mathfrak{C}p$, the quotient process by the $\mathfrak{A}$–torsion subgroup does not alter the level structure; so, $i(\mathfrak{B})$ induces $i(\mathfrak{AB}) = i(\mathfrak{B}) \otimes \mathfrak{A}$.

The above process of making $(A(\mathfrak{AB}), \lambda(\mathfrak{AB}), i(\mathfrak{AB}))$ can be performed (without any modification) for general triples (A, λ, i) (even without complex multiplication) and yields a functorial map from test objects (A, λ, i) with polarization ideal $\mathfrak{c}$ to test objects $(A \otimes_O \mathfrak{A}, \lambda \otimes \mathfrak{A}, i \otimes \mathfrak{A})$ with polarization ideal $\mathfrak{c}\mathfrak{A}^{-2}$. For a p–adic modular form $f \in V(\mathfrak{c}\mathfrak{A}^{-2}, \mathfrak{N}; R)$, we define $f|\langle\mathfrak{A}\rangle \in V(\mathfrak{c}, \mathfrak{N}; R)$ by

$$f|\langle\mathfrak{A}\rangle(A, \lambda, i) = f(A \otimes_O \mathfrak{A}, \lambda \otimes \mathfrak{A}, i \otimes \mathfrak{A}) \tag{5.4}$$

for a fractional ideal $\mathfrak{A}$ of F prime to $\mathfrak{N}$ (see [PAF] 4.1.9). This shows

$$\int_{\Gamma'} \phi_{\mathfrak{AB}} dE_{\mathfrak{AB}} = (\mathbf{E}(\chi_{\Gamma'}\phi_{\mathfrak{AB}})|\langle\mathfrak{A}\rangle)\,(A(\mathfrak{B}), \lambda(\mathfrak{B}), i(\mathfrak{B})) \ \text{ if } \mathfrak{A} \subset F, \tag{5.5}$$

where $\mathbf{E} = \mathbf{E}_{\mathfrak{c}_{\mathfrak{A}\,\mathfrak{B}}}$.

By adding level, we can construct another operator $[\mathfrak{q}] : V(\mathfrak{cq}, \mathfrak{N}; R) \to V(\mathfrak{c}, \mathfrak{Nq}; R)$ in the following way. Here we assume that $\mathfrak{q}$ is an integral ideal prime to $\mathfrak{c}p$. This goes as follows: For each test object: $(A, \lambda, \omega, i)_{/C}$ (over a p-adic R-algebra C) of level $\mathfrak{Nq}p^\infty$ with polarization ideal $\mathfrak{c}$, we define a new test object $(A', \lambda', \omega', i')$. First define $A' = A/i(\mathfrak{q}^*/O^*)$. The quotient exists

over C, since $i(\mathfrak{q}^*/O^*)$ is an étale subgroup of A (because C is a p–adic ring). The level structure

$$i : \left(F_p/O_p^*\right) \times ((\mathfrak{N}\mathfrak{q})^*/O^*) \to A$$

composed with the quotient map $\pi : A \to A'$ induces, modulo $\mathfrak{q}^*/O^*$, the level structure $i' : F_p/O_p^* \times \mathfrak{N}^*/O^* \to A'$ defined over C. The $\mathfrak{c}\mathfrak{q}$–polarization $\lambda' : A'^t \cong A' \otimes \mathfrak{c}\mathfrak{q}$ is defined as follows: Tensoring the exact sequence $0 \to \mathfrak{q} \to O \to O/\mathfrak{q} \to 0$ with $A^t = A \otimes \mathfrak{c}$, we have another exact sequence:

$$0 \to A \otimes \mathfrak{c}\mathfrak{q}[\mathfrak{q}] \to A \otimes \mathfrak{c}\mathfrak{q} \to A \otimes \mathfrak{c} \to 0.$$

Taking dual of the quotient map $\pi : A \to A'$, we have one more exact sequence:

$$0 \to \mathrm{Hom}(i(\mathfrak{q}^*/O^*), \mathbb{G}_m) \to A'^t \xrightarrow{\pi^t} A^t \to 0,$$

which gives rise to the following exact sequence

$$0 \to \mathrm{Hom}(i(\mathfrak{q}^*/O^*), \mathbb{G}_m) \to A'^t[\mathfrak{q}] \xrightarrow{\lambda\circ\pi^t} i(\mathfrak{q}^*/O^*) \otimes \mathfrak{c} \to 0.$$

Since $\mathfrak{q}$ is prime to $\mathfrak{c}$, the kernel of the composite: $(\pi \otimes \mathrm{id}) \circ \lambda \circ \pi^t : A'^t \to A \otimes \mathfrak{c}$ is the entire $\mathfrak{q}$–torsion subgroup $A'^t[\mathfrak{q}]$. Since $A'^t/A'^t[\mathfrak{q}] = A'^t \otimes \mathfrak{q}^{-1}$, we have constructed an isomorphism:

$$(\pi \otimes \mathrm{id}) \circ \lambda \circ \pi^t : A'^t \otimes \mathfrak{q}^{-1} \cong A \otimes \mathfrak{c}.$$

Tensoring $\mathfrak{q}$ with this isomorphism, we get the desired $\lambda' : A'^t \cong A' \otimes \mathfrak{c}\mathfrak{q}$. Since $\mathfrak{q}$ is prime to p, on a p–adic algebra C, $Lie(A) \cong Lie(A')$, which implies that $\omega' = \pi^*\omega$ is well defined generator of $\Omega_{A'/C}$. The association $(A, \lambda, \omega, i)_{/C} \mapsto (A', \lambda', \omega', i')_{/C}$ is functorial (i.e., a morphism between the functors $\mathcal{Q}$ in (4.1) with respect to $(\mathfrak{c}, \mathfrak{N}\mathfrak{q}p^\infty)$ and $(\mathfrak{c}\mathfrak{q}, \mathfrak{N}p^\infty)$). We have

$$[\mathfrak{q}] : V(\mathfrak{c}\mathfrak{q}, \mathfrak{N}; R) \to V(\mathfrak{c}, \mathfrak{N}\mathfrak{q}; R) \ \text{ and } \ [\mathfrak{q}] : G_\kappa(\mathfrak{c}\mathfrak{q}, \mathfrak{N}; R) \to G_\kappa(\mathfrak{c}, \mathfrak{N}\mathfrak{q}; R)$$

by $f|[\mathfrak{q}](A, \lambda, \omega, i) = f(A', \lambda', \omega', i')$.

We compute $[\mathfrak{q}](A(\mathfrak{A}), \lambda(\mathfrak{A}), i(\mathfrak{A}))_{/\mathcal{W}}$ for a fractional ideal $\mathfrak{A} \subset M$, supposing that all prime factors of $\mathfrak{q}$ are split in M/F. Choose an integral ideal $\mathfrak{Q}$ in M such that the inclusion $O \hookrightarrow \mathfrak{O}$ induces $O/\mathfrak{q} \cong \mathfrak{O}/\mathfrak{Q}$. Then $\mathfrak{Q} + \mathfrak{Q}^c = \mathfrak{O}$. Consider $(A(\mathfrak{A}), \lambda(\mathfrak{A}), i(\mathfrak{A}))$ with the level $\mathfrak{f}\mathfrak{s}\mathfrak{q}p^\infty$–structure $i(\mathfrak{A})$ sending $x \in \mathfrak{q}^*/O^*$ to $2\delta x \in \mathfrak{Q}^{-1}\mathfrak{A}/\mathfrak{A}$. Then $A(\mathfrak{A})[\mathfrak{Q}] = i(\mathfrak{A})(\mathfrak{q}^*/O^*)$ and hence $A(\mathfrak{A})/i(\mathfrak{A})(\mathfrak{q}^*/O^*) = A(\mathfrak{A}\mathfrak{Q}^{-1})$ and $i(\mathfrak{A})' = i(\mathfrak{A}\mathfrak{Q}^{-1})$, which are the level $\mathfrak{f}\mathfrak{s}p^\infty$–structure. Since $\mathfrak{q}$ is prime to $p\mathfrak{c}$, using the fact that $\mathfrak{Q}\mathfrak{Q}^c = \mathfrak{q}$, we can verify that

$$[\mathfrak{q}](A(\mathfrak{A}), \lambda(\mathfrak{A}), \omega(\mathfrak{A}), i(\mathfrak{A}))_{/\mathcal{W}} \cong (A(\mathfrak{A}\mathfrak{Q}^{-1}), \lambda(\mathfrak{A}\mathfrak{Q}^{-1}), \omega(\mathfrak{A}\mathfrak{Q}^{-1}), i(\mathfrak{A}\mathfrak{Q}^{-1}))_{/\mathcal{W}}, \tag{5.6}$$

where $i(\mathfrak{A})$ is the level $\mathfrak{f}\mathfrak{q}p^\infty$–structure as above and $i(\mathfrak{A}\mathfrak{Q}^{-1})$ is the induced level $\mathfrak{f}p^\infty$–structure. We can always choose $\mathfrak{Q} \in D^+$ so that $\mathfrak{Q}^c + \mathfrak{Q} = \mathfrak{O}$ and $\mathfrak{O}/\mathfrak{Q} \cong O/\mathfrak{q}$ for $\mathfrak{q} = \mathfrak{Q} \cap F$. This shows

$$\int_{\Gamma'} \phi_{\mathfrak{Q}^{-1}\mathfrak{B}} dE_{\mathfrak{Q}^{-1}\mathfrak{B}} = \left(\mathbf{E}(\chi_{\Gamma'}\phi_{\mathfrak{Q}^{-1}\mathfrak{B}})|[\mathfrak{q}]\right)(A(\mathfrak{B}), \lambda(\mathfrak{B}), i(\mathfrak{B}))$$

for $\mathfrak{Q}$ and $\mathfrak{q}$ as above, (5.7)

where $\mathbf{E} = \mathbf{E}_{\mathfrak{c}_{\mathfrak{A}\mathfrak{B}}}$. As for the effect of $\alpha \in M^\times$, we may assume either $\alpha \in O$ or $\alpha \in \mathfrak{O} - O$. Then we have, for the characteristic function $\chi_{\Gamma'}$ of Γ',

$$\int_{\Gamma'} \phi_{\alpha\mathfrak{B}} dE_{\alpha\mathfrak{B}} = (\mathbf{E}(\chi_{\Gamma'}\phi_{\alpha\mathfrak{B}})|\langle\alpha\rangle)\,(A(\mathfrak{B}), \lambda(\mathfrak{B}), i(\mathfrak{B})) \quad \text{if } \alpha \in O \cap F^\times, \quad (5.8)$$

$$\int_{\Gamma'} \phi_{\alpha^{-1}\mathfrak{B}} dE_{\alpha^{-1}\mathfrak{B}} = (\mathbf{E}(\chi_{\Gamma'}\phi_{\alpha^{-1}\mathfrak{B}})|[\alpha\alpha^c])\,(A(\mathfrak{B}), \lambda(\mathfrak{B}), i(\mathfrak{B})) \quad \text{if } \alpha \notin O, \quad (5.9)$$

where $\mathbf{E} = \mathbf{E}_{\mathfrak{c}}$ for $\mathfrak{c} = \mathfrak{c}_{\alpha\mathfrak{B}}$ for (5.8) and $\mathfrak{c} = \mathfrak{c}_{\alpha^{-1}\mathfrak{B}}$ for (5.9).

5.3 Computation of q–Expansions

Pick an element $g \in G(\mathbb{A}^{(\infty)})$ with totally positive $\det(g) \in F$. Then g induces an automorphism of the Shimura variety (see (3.9)), and hence the functorial action of g on test objects. We write

$$g(A, \lambda, i) = (A, \lambda_g, i_g)$$

for the image of a test object (A, λ, i) under the action of g. Here, writing $T(A) = \varprojlim_N A[N]$ for the Tate module, the level structure is an isomorphism $i : F^2_{\mathbb{A}^{(\infty)}} \cong T(A) \otimes_{\mathbb{Q}} \mathbb{A}^{(\infty)}$, where $F^2_{\mathbb{A}^{(\infty)}}$ is made up of row vectors on which $G(\mathbb{A}^{(\infty)})$ acts from the right. Then we have $i_g = i \circ g$ and $\lambda_g = \det(g)\lambda$. When $g = \gamma \in G(\mathbb{Q})_+$, we have an isogeny $\widetilde{\gamma} : (A', \lambda', \widetilde{\gamma} \circ i') \to (A, \lambda_\gamma, i_\gamma = i \circ \gamma)$ for a suitable A' (see below). Thus we can interpret the action as an action of an isogeny in this case. This follows from the following three facts for $\gamma \in G(\mathbb{Q})_+$ and test objects over $\mathbb{C}$:

(L1) Writing $\mathcal{L}_z^{\mathfrak{b},\mathfrak{a}^*} = (\mathfrak{b}, \mathfrak{a}^*)^t(z, 1) = \mathfrak{b}z + \mathfrak{a}^*$ and

$$i_z((b, a) \mod \mathfrak{b} \oplus \mathfrak{a}^*) = bz + a \mod \mathcal{L}_z^{\mathfrak{b},\mathfrak{a}^*},$$

we have $\mathcal{L}_{\gamma(z)}^{(\mathfrak{b},\mathfrak{a}^*)\gamma^{-1}} \cong \mathcal{L}_z^{\mathfrak{b},\mathfrak{a}^*}$ by $w \mapsto w(cz + d)^{-1}$, where $\gamma = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)$.

(L2) $i_{\gamma(z)} = (cz + d)i_z \circ \gamma$; so, $A' = \mathbb{C}^I/\mathcal{L}_{\gamma(z)}^{(\mathfrak{b},\mathfrak{a}^*)\gamma^{-1}}$ and

$$\widetilde{\gamma}(w \mod \mathcal{L}_{\gamma(z)}^{(\mathfrak{b},\mathfrak{a}^*)\gamma^{-1}}) = (cz + d)^{-1}w \mod \mathcal{L}_z^{\mathfrak{b},\mathfrak{a}^*}.$$

(L3) We have the identity of the Tate module via i_z:

$$T(\mathbb{C}^I/\mathcal{L}_z^{\mathfrak{b},\mathfrak{a}^*}) \cong \widehat{\mathfrak{b}} \oplus \widehat{\mathfrak{a}}^* \text{ and } T(\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)) = \widehat{\mathfrak{b}} \oplus \widehat{\mathfrak{a}}^* \quad (\widehat{\mathfrak{x}} = \mathfrak{x} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}).$$

If we have an isogeny $\alpha : A \to A$, we have $\alpha(A, \lambda, i) = (A, \lambda', i')$ given by $\lambda' = \alpha\alpha^*\lambda$ and $i'(x) = \alpha i(x)$. Here $\alpha^* = \lambda \circ \alpha^t \circ \lambda^{-1}$, which is $\alpha\alpha^*\mathfrak{c}$–polarization. In other words, defining $\rho(\alpha) \in G(\mathbb{Q})_+$ by $\alpha i = i \circ \rho(\alpha)$, we find that $\rho(\alpha)^{-1}(\alpha(A, \lambda, i)) = (A, \lambda, i)$. Since the Shimura variety classifies the triples up to isogeny, $\alpha(A, \lambda, i)$ and (A, λ, i) are equal as a point of $Sh(G, \mathfrak{X})$,

and $\mathrm{Im}(\rho)$ gives rise to the stabilizer of the point of $Sh(G,\mathfrak{X})$ represented by (A,λ,i) (see Corollary 3.5).

When we consider the level structure i modulo a compact subgroup $K \subset G(\mathbb{A}^{(\infty)})$, we write (A,λ,i_K). Then for $g \in G(\mathbb{A}^{(\infty)})$ with $\det(g) \in F_+^\times$, $g(A,\lambda,i_K) = (A,\lambda_g,(i\circ g)_{g^{-1}Kg})$ is well defined (solely depending on g).

We now consider the Tate AVRM: $Tate_{\mathfrak{a},\mathfrak{b}}(q)$. For each positive integer N, we have a canonical exact sequence:

$$1 \to \mu_N \otimes \mathfrak{a}^* \to Tate_{\mathfrak{a},\mathfrak{b}}(q)[N] \to \mathfrak{b}/N\mathfrak{b} \to 0.$$

We therefore have a canonical level structure i_{can} modulo an (integral) upper unipotent subgroup $U = U(\widehat{\mathbb{Z}}) \subset G(\mathbb{A}^{(\infty)})$, which is represented by the following exact sequence tensored by $\mathbb{A}^{(\infty)}$ (over $\widehat{\mathbb{Z}}$):

$$0 \to \widehat{\mathfrak{a}^*}(1) \to T(Tate_{\mathfrak{a},\mathfrak{b}}(q)) \to \widehat{\mathfrak{b}} \to 0,$$

where $\widehat{\mathfrak{b}} = \widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathfrak{b}$ and $\widehat{\mathfrak{a}^*}(1) = \widehat{\mathbb{Z}}(1) \otimes_{\mathbb{Z}} \mathfrak{a}^*$. Let $K \subset G(\mathbb{A}^{(\infty)})$ be the stabilizer of the row vector space $\widehat{\mathfrak{b}} \oplus \widehat{\mathfrak{a}^*}$, that is,

$$K = K_{\mathfrak{a},\mathfrak{b}} = \left\{ g \in G(\mathbb{A}^{(\infty)}) \middle| (\widehat{\mathfrak{b}} \oplus \widehat{\mathfrak{a}^*})g = \widehat{\mathfrak{b}} \oplus \widehat{\mathfrak{a}^*} \right\}.$$

Thus $\Gamma_{11}(O;\mathfrak{a},\mathfrak{b}) = SL_2(F) \cap K_{\mathfrak{a},\mathfrak{b}}$. Define

$$K(\mathfrak{N}) = K_{\mathfrak{a},\mathfrak{b}}(\mathfrak{N}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in K_{\mathfrak{a},\mathfrak{b}} \middle| c \in \mathfrak{N}\widehat{\mathfrak{a}\mathfrak{b}\mathfrak{d}},\ a \equiv d \equiv 1 \mod \mathfrak{N}\widehat{O} \right\}.$$

Then we have $\Gamma_{11}(\mathfrak{N};\mathfrak{a},\mathfrak{b}) = SL_2(F) \cap K(\mathfrak{N})$. For each given $g \in G(\mathbb{A}^{(\infty)})$ with totally positive $\det(g) \in F$ (so, $g \in \overline{\mathcal{E}}(G,\mathfrak{X})$), we can find finite ideles $a(g), b(g) \in \mathbb{A}^{(\infty)}$ such that $g = u(g)\left(\begin{smallmatrix} b(g) & * \\ 0 & a(g) \end{smallmatrix}\right)$ with $u(g) \in K \cap SL_2(F_{\mathbb{A}})$ and $\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) \in K$. Let (A,λ,i) be as in (L1–3), and put $i_{K(\mathfrak{N})} = (i \mod K(\mathfrak{N}))$. Having $(A,\lambda,i_{K(\mathfrak{N})})$ is equivalent to having $T(A) = i(\widehat{\mathfrak{b}} \oplus \widehat{\mathfrak{a}}^*)$ and $i_{K(\mathfrak{N})} : (\mathfrak{N}\mathfrak{a})^*/\mathfrak{a}^* \hookrightarrow A[\mathfrak{N}]$. The ideles $a(g)$ and $b(g)$ are determined uniquely modulo multiple of units in $\widehat{O}$. We assume here that $a(g)_{\mathfrak{N}} = b(g)_{\mathfrak{N}} = 1$.

Write simply $\mathfrak{a}' = a(g)^{-1}\mathfrak{a}$ and $\mathfrak{b}' = b(g)\mathfrak{b}$ and $K^g = g^{-1}Kg$. We have a canonical identification $\widehat{\mathfrak{b}'} \oplus \widehat{\mathfrak{a}'}^*(1) = T(Tate_{\mathfrak{a}',\mathfrak{b}'}(q))$ and

$$i_{can,K^g}^{\mathfrak{a}',\mathfrak{b}'} : \mu_{\mathfrak{N}} \otimes (\mathfrak{N}\mathfrak{a}')^*/\mathfrak{a}'* \hookrightarrow \mathbb{G}_m \otimes (\mathfrak{a}')^* \twoheadrightarrow Tate_{\mathfrak{a}',\mathfrak{b}'}(q).$$

Since $\mathfrak{b}' \oplus \mathfrak{a}'^*$ and $\mathfrak{b} \oplus \mathfrak{a}^*$ are commensurable, the two Tate AVRM's $Tate_{\mathfrak{a},\mathfrak{b}}(q)$ and $Tate_{\mathfrak{a}',\mathfrak{b}'}(q)$ are in the same isogeny class (over $\mathbb{Z}[[(\mathfrak{a}\mathfrak{b} + \mathfrak{a}'\mathfrak{b}')_{\geq 0}]]$). Since $\widehat{(a(g)\mathfrak{a}^*)} = a(g)^{-1}\widehat{\mathfrak{a}}^*$ and $b(g)\widehat{\mathfrak{b}} = \widehat{b(g)\mathfrak{b}}$, up to isogenies, we have from (L3)

$$\begin{aligned} &g(Tate_{\mathfrak{a},\mathfrak{b}}(q), \lambda_{can}^{\mathfrak{a},\mathfrak{b}}, i_{can,K(\mathfrak{N})}^{\mathfrak{a},\mathfrak{b}}) = \\ &(Tate_{a(g)^{-1}\mathfrak{a},b(g)\mathfrak{b}}(q), \det(g)\lambda_{can}^{\mathfrak{a},\mathfrak{b}} = \lambda_{can}^{a(g)^{-1}\mathfrak{a},b(g)\mathfrak{b}}, i_{can,K(\mathfrak{N})^g}^{a(g)^{-1}\mathfrak{a},b(g)\mathfrak{b}} \circ u(g)). \quad (5.10) \end{aligned}$$

If $g \in F^\times$, then $a(g) = b(g) = g$, and we have an isogeny

$$g : (Tate_{\mathfrak{a},\mathfrak{b}}(q), \lambda_{can}^{\mathfrak{a},\mathfrak{b}}, i_{can}^{\mathfrak{a},\mathfrak{b}}) \to (Tate_{g^{-1}\mathfrak{a},g\mathfrak{b}}(q), \lambda_{can}^{g^{-1}\mathfrak{a},g\mathfrak{b}}, g \circ i_{can}^{g^{-1}\mathfrak{a},g\mathfrak{b}})$$

induced by $q \mapsto q^g$ (or equivalently, by $\mathbb{G}_m \otimes \mathfrak{a}^* \to \mathbb{G}_m \otimes (g^{-1}\mathfrak{a}^*)$ given by $x \otimes a \mapsto x \otimes ga$). Therefore the central rational element acts on the Tate AVRM trivially.

For the Eisenstein series $E(\phi) = E_0(\phi; \mathfrak{c})$ (weight 0) of a function $\phi(x, a; y, b)$ ($(x, a; y, b) \in (\mathfrak{a}_p \times (\mathfrak{a}/\mathfrak{f}\mathfrak{a})) \times (\mathfrak{b}_p \times (\mathfrak{b}/\mathfrak{f}\mathfrak{b}))$), we find from the above computation (assuming $a(g)_{\mathfrak{N}} = b(g)_{\mathfrak{N}} = 1$ for $\mathfrak{N} = p\mathfrak{f}$):

$$\begin{aligned} &E(\phi)(g(Tate_{\mathfrak{a},\mathfrak{b}}(q), \lambda_{can}^{\mathfrak{a},\mathfrak{b}}, i_{can}^{\mathfrak{a},\mathfrak{b}})) \\ &\qquad = E(\phi|u(g))(Tate_{a(g)^{-1}\mathfrak{a},b(g)\mathfrak{b}}(q), \lambda_{can}^{a(g)^{-1}\mathfrak{a},b(g)\mathfrak{b}}, i_{can}^{a(g)^{-1}\mathfrak{a},b(g)\mathfrak{b}}), \quad (5.11) \end{aligned}$$

where $\phi|u(g)(x, a; y, b) = P^{-1}(P\phi((x, a; y, b)u(g))$ (letting the 2×2–matrix $u(g)$ act from the right on the row vector $(x, a; y, b)$) for the partial Fourier transform $\phi \mapsto P\phi$ as in Section 4.2.

We compute the q–expansion of $E(\phi)|\langle\mathfrak{A}\rangle$ for a fractional ideal $\mathfrak{A}$ of F. This is the special case of (5.10) when g is a scalar matrix $\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$ with $a_{\mathfrak{N}} = 1$ (and $a\widehat{O} = \widehat{\mathfrak{A}}$). By construction, we have a homomorphism $\underline{q} : \mathfrak{b} \hookrightarrow \mathbb{G}_m \otimes_{\mathbb{Z}} \mathfrak{a}^*$. Since the $\mathfrak{A}$–torsion points of $Tate_{\mathfrak{a},\mathfrak{b}}(q)$ is given by $\underline{q}(\mathfrak{b}\mathfrak{A}^{-1}/\mathfrak{b}) \oplus (\mu_{\mathfrak{N}} \otimes \mathfrak{a}^*)[\mathfrak{A}]$. Thus

$$Tate_{\mathfrak{a},\mathfrak{b}}(q) \otimes \mathfrak{A}^{-1} = Tate_{\mathfrak{a},\mathfrak{b}}(q)/Tate_{\mathfrak{a},\mathfrak{b}}(q)[\mathfrak{A}] = Tate_{\mathfrak{a}\mathfrak{A},\mathfrak{b}\mathfrak{A}^{-1}}(q).$$

From this, it is easy to see (cf. [PAF] (4.53))

$$(Tate_{\mathfrak{a},\mathfrak{b}}(q) \otimes \mathfrak{A}, \lambda_{can}^{\mathfrak{a},\mathfrak{b}} \otimes \mathfrak{A}, i_{can}^{\mathfrak{a},\mathfrak{b}} \otimes \mathfrak{A}) = (Tate_{\mathfrak{a}\mathfrak{A}^{-1},\mathfrak{b}\mathfrak{A}}(q), \lambda_{can}^{\mathfrak{a}\mathfrak{A}^{-1},\mathfrak{b}\mathfrak{A}}, i_{can}^{\mathfrak{a}\mathfrak{A}^{-1},\mathfrak{b}\mathfrak{A}}), \quad (5.12)$$

where the superscript: "$\mathfrak{a}, \mathfrak{b}$" is to indicate that the attached object is relative to the Tate curve $Tate_{\mathfrak{a},\mathfrak{b}}(q)$.

We compute $[\mathfrak{q}](Tate_{\mathfrak{a},\mathfrak{b}}(q), \lambda_{can}, \omega_{can}, i_{can})$ for an ideal $\mathfrak{q} \subset O$. Recall that

$$Tate_{\mathfrak{a},\mathfrak{b}}(q) = \mathbb{G}_m \otimes \mathfrak{a}^*/\underline{q}(\mathfrak{b}).$$

Tensoring $\mathbb{G}_m \otimes \mathfrak{a}^*$ with the exact sequence: $0 \to O \to \mathfrak{q}^{-1} \to \mathfrak{q}^{-1}/O \to 0$, we have another exact sequence:

$$0 \to (\mathbb{G}_m \otimes \mathfrak{a}^*)[\mathfrak{q}] \to \mathbb{G}_m \otimes \mathfrak{a}^* \to \mathbb{G}_m \otimes (\mathfrak{a}\mathfrak{q})^* \to 0.$$

Taking the quotient by $\underline{q}(\mathfrak{b})$, we get the following exact sequence:

$$0 \to (\mathbb{G}_m \otimes \mathfrak{a}^*)[\mathfrak{q}] \xrightarrow{i_{can}} Tate_{\mathfrak{a},\mathfrak{b}}(q) \to Tate_{\mathfrak{a}\mathfrak{q},\mathfrak{b}}(q) \to 0.$$

Then going back to the construction of the Tate quadruples in [K3] 1.1 (and [HT] 1.7), we can verify

$$\begin{aligned} &[\mathfrak{q}](Tate_{\mathfrak{a},\mathfrak{b}}(q), \lambda_{can}^{\mathfrak{a},\mathfrak{b}}, \omega_{can}^{\mathfrak{a},\mathfrak{b}}, i_{can,K_{\mathfrak{a},\mathfrak{b}}(\mathfrak{N}\mathfrak{q})}) \\ &\qquad = (Tate_{\mathfrak{a}\mathfrak{q},\mathfrak{b}}(q), \lambda_{can}^{\mathfrak{a}\mathfrak{q},\mathfrak{b}}, \omega_{can}^{\mathfrak{a}\mathfrak{q},\mathfrak{b}}, i_{can,K_{\mathfrak{a}\mathfrak{q},\mathfrak{b}}(\mathfrak{N})}). \quad (5.13) \end{aligned}$$

The above action $[\mathfrak{q}]$ corresponds to the action of $g = \left(\begin{smallmatrix} 1 & 0 \\ 0 & q^{-1} \end{smallmatrix}\right)$ for a finite idele q with $q\widehat{O} = \widehat{\mathfrak{q}}$ and $q_{\mathfrak{N}} = 1$. This follows from (5.10) combined with the fact that $K_{\mathfrak{a},\mathfrak{b}}(\mathfrak{N}\mathfrak{q})^g = K_{\mathfrak{a}\mathfrak{q},\mathfrak{b}}(\mathfrak{N})$.

Now we further suppose that $K_{\mathfrak{a},\mathfrak{b}}g = K_{\mathfrak{a},\mathfrak{b}}\gamma$ for $\gamma \in G_+(\mathbb{Q})$ and $g_{\mathfrak{N}} = 1$. Then $u = g\gamma^{-1} \in K_{\mathfrak{a},\mathfrak{b}}$, and hence $u_{\mathfrak{N}} = \gamma_{\mathfrak{N}}^{-1}$. This shows

$$g\gamma^{-1}(Tate_{\mathfrak{a},\mathfrak{b}}, \lambda_{can}^{\mathfrak{a},\mathfrak{b}}, \omega_{can}^{\mathfrak{a},\mathfrak{b}}, i_{can,K_{\mathfrak{a},\mathfrak{b}}(\mathfrak{N})}) = (Tate_{\mathfrak{a},\mathfrak{b}}, \lambda_{can}^{\mathfrak{a},\mathfrak{b}}, \omega_{can}^{\mathfrak{a},\mathfrak{b}}, i_{can,K_{\mathfrak{a},\mathfrak{b}}(\mathfrak{N})} \circ \gamma_{\mathfrak{N}}^{-1}). \quad (5.14)$$

5.4 Linear Independence of Eisenstein Series

Recall that $D \cong Z/\Gamma'$ for the intersection Γ' (in Z) of Γ with the image of $\mathfrak{O}^\times_{\mathfrak{C}_p}$. Let $\chi_{\Gamma'}$ be the characteristic function of $\Gamma' \subset Z$. We put $\phi = \mathrm{Inf}_\psi \chi_{\Gamma'}$ for a character $\psi : \Delta \to W^\times$. We regard ψ as a character of Z composing with the projection: $Z \twoheadrightarrow \Delta$. Although the Eisenstein series $\mathbf{E}_{\mathfrak{c}}(\phi)$ is of weight 0 and is not classical, we take actually $\mathbf{E}_{\mathfrak{c}}(N^{-k}\phi)$ for a positive k so that $N^{-k} \equiv 1 \mod p$ on Z. Then $\mathbf{E}_{\mathfrak{c}}(N^{-k}\phi) \equiv \mathbf{E}_{\mathfrak{c}}(\phi) \mod p$, and hence, just to compute the q–expansion mod p, we can treat $\mathbf{E}_{\mathfrak{c}}(\phi)$ as if it is classical. Thus we can apply Corollary 3.21 to $\mathbf{E}_{\mathfrak{c}}(\phi)$. Recall that we have written $E(\Phi^\circ)$ for $E_0(\Phi^\circ; \mathfrak{c}) = \mathbf{E}_{\mathfrak{c}}(\Phi)$ for a suitable choice of $\mathfrak{c}$ in the context (making $\mathfrak{c}$ explicit is left to the reader since it complicates the symbols attached to the Eisenstein series).

Recall the decomposition $\mathfrak{C} = \mathfrak{F}\mathfrak{F}_c\mathfrak{I}$ in the introduction satisfying

$$\mathfrak{F} + \mathfrak{F}^c = \mathfrak{O},\ \mathfrak{F} + \mathfrak{F}_c = \mathfrak{O},\ \mathfrak{F}_c + \mathfrak{F}_c^c = \mathfrak{O} \text{ and } \mathfrak{F}_c \supset \mathfrak{F}^c, \quad (5.15)$$

$$\text{every prime factor of } \mathfrak{I} \text{ is inert or ramified over } F. \quad (5.16)$$

Recall $\mathfrak{f} = \mathfrak{F}\mathfrak{I} \cap F$, $\mathfrak{s} = \mathfrak{F}_c \cap F$ and $T = (O_p^\times \times (O/\mathfrak{f})) \times (O_p^\times \times (O/\mathfrak{s}))/\overline{O^\times}$. Then the variable on T is written as $(x, a; y, b)$ with $x, y \in O_p$. Write prime decomposition of $\mathfrak{f}$ as $\prod_{\mathfrak{q}} \mathfrak{q}^{\varepsilon(\mathfrak{q})}$. We choose a prime element $\varpi_{\mathfrak{q}}$ in $O_{\mathfrak{q}}$, and we define $\varpi^{\varepsilon(\mathfrak{f})}$ as an idele whose $\mathfrak{q}$–component is given by $\varpi_{\mathfrak{q}}^{\varepsilon(\mathfrak{q})}$. Let

$$\chi^\circ_{\Gamma'}(x, a; y, b) = \sum_{u \in O/\mathfrak{f}} \chi_{\Gamma'}(x^{-1}, u; y, b)\mathbf{e}_F(-\frac{ua}{\varpi^{\varepsilon(\mathfrak{d}\mathfrak{f})}}) = \chi(x, 1; y, b)\mathbf{e}_F(-\frac{a}{\varpi^{\varepsilon(\mathfrak{d}\mathfrak{f})}}),$$

where $\mathbf{e}_F : F_{\mathbb{A}}/F \to \overline{\mathbb{Q}}^\times$ is the standard additive character having the value $\mathbf{e}_F(x_\infty) = \exp(2\pi\sqrt{-1}\mathrm{Tr}(x_\infty))$ at ∞, and χ is the characteristic function of

$$\{(x, a; y, b) | \pi(x, 1; y, b) \in \Gamma'\} \text{ for } \pi : \mathfrak{O}^\times_{\mathfrak{C}_p} \to Z.$$

We split further $\mathcal{D} = \bigsqcup_{\alpha \in \mathcal{D}^-} \alpha\mathcal{D}^+$ where $\mathcal{D}^+$ is the subset of $\mathcal{D}$ represented by elements of $F^\times$:

$$\mathcal{D}^+ = \left\{\alpha \in \mathcal{D} \middle| \alpha\Gamma' = \beta\Gamma' \text{ with } \beta \in F^\times \cap \mathfrak{O}^\times_{\mathfrak{C}_p}\right\}.$$

Recall that we assumed that $\mathfrak{C}$ is prime to the relative discriminant $\mathfrak{D}$ of M/F. We choose $\alpha \in \mathcal{D}^-$ so that $(\alpha) = \mathfrak{Q}$ is a prime ideal split in M over F. Then

$K_{\mathfrak{a},\mathfrak{b}}\rho(\alpha) = K_{\mathfrak{a},\mathfrak{b}}g_{\mathfrak{D}}$ for $g_{\mathfrak{D}} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & q \end{smallmatrix}\right)$ for a finite idele $q \in \widehat{O}$ with $q\widehat{\mathfrak{O}} = \widehat{\mathfrak{Q}}\widehat{\mathfrak{Q}}^c$ and $q\mathfrak{C}_p = 1$. Define $\mathcal{R}$ (resp. $\mathcal{S}$) by a subset $\{\alpha_{\mathfrak{C}p} | \alpha \in \mathcal{D}^+\}$ (resp. $\{\beta_{\mathfrak{f}p} | \beta \in \mathcal{D}^-\}$) in the completion $\mathfrak{O}^\times_{\mathfrak{C}p}$ (resp. $\mathfrak{O}^\times_{\mathfrak{f}P}$).

Hereafter until 5.5, we assume $\mathfrak{I} = 1$. By (5.5), (5.8) and (5.14), we see that

$$\sum_{\beta\in\mathcal{D}^+} \psi(\beta)^{-1} E(\chi^\circ_{\Gamma'})|\langle(\beta)\rangle|\beta = E(\Phi^\circ_+),$$

where "$|\beta$" is the action of the scalar element $\beta \in Z(\mathbb{Q})$, and

$$\Phi^\circ_+(x,a;y,b) = \sum_{s\in\mathcal{S}} \psi(s)\chi^\circ(s^{-1}(x^{-1},a); s(y,b)), \tag{5.17}$$

since $s = \beta_{\mathfrak{f}p} \in \mathcal{S} \subset \mathfrak{O}^\times_{\mathfrak{f}p}$ (for $\beta \in \mathcal{D}^+$) by (5.14). Here note that $\psi(s) = \psi(\beta)^{-1}$ because $s = \beta_{\mathfrak{f}p}$. We further sum up over $\mathcal{D}^-$:

$$\sum_{\alpha\in\mathcal{D}^-} \psi(\alpha)E(\Phi^\circ_+)|[\alpha\alpha^c]|\rho(\alpha)^{-1} = E(\Phi^\circ), \tag{5.18}$$

where we have chosen (α) to be an integral ideal with $\mathfrak{O}/(\alpha) \cong O/((\alpha)\cap F)$ and $(\alpha)\cap F = (\alpha\alpha^c)$, and Φ° is given by

$$\Phi^\circ(x,a;y,b) = \sum_{r\in\mathcal{R}} \psi(r)^{-1}\Phi^\circ_+(r(x,a;y,b)), \tag{5.19}$$

because $\psi(r) = \psi(\alpha)^{-1}$ ($r = \alpha_{\mathfrak{C}p} \in \mathfrak{O}_{\mathfrak{C}p}$). Since we have $E(\Phi^\circ_+(r(x,a;y,b))) = E(\Phi^\circ_+)|\rho(r)$ for $\rho(r) \in K_{\mathfrak{a},\mathfrak{b}}$, we have by (5.14) that

$$\sum_{\alpha\in\mathcal{D}^-} \psi(\alpha)\left(\sum_{\beta\in\mathcal{D}^+} \psi(\beta)^{-1}E(\chi^\circ_{\Gamma'})|\langle\beta\rangle|\beta\right)|[\alpha\alpha^c]|\rho(\alpha)^{-1}$$
$$= \sum_{r\in\mathcal{R}} \psi(r)^{-1}E(\Phi^\circ_+)|\rho(r)^{-1}. \tag{5.20}$$

We have computed $E(\Phi^\circ)$ as a linear combination of transforms of the Eisenstein series $E(\chi^\circ_{\Gamma'})$. On the other hand, by definition of φ_ψ, Φ as above is the restriction of $\mathrm{Inf}_\psi\,\chi_{\Gamma'}$ to $Z_0 = \left(\mathfrak{O}_p^\times \times (\mathfrak{O}/\mathfrak{C})^\times\right)/\overline{\mathfrak{O}^\times} \subset Z$.

Recall that $\mathfrak{A} \in D^+$ is chosen out of fractional ideals of F if $|Cl_M|$ is odd; so, in such a case, the operator $\langle\mathfrak{A}\rangle$ makes sense. Similarly, if $|Cl_M|$ is even, we have chosen $\mathfrak{A} \in D^+ \cup D^-$ among prime ideals of M split over F; so, the operator $[\mathfrak{A}\mathfrak{A}^c]$ regarding $\mathfrak{A}\mathfrak{A}^c$ as a prime ideal of F also makes sense.

Theorem 5.1. *Suppose $p > 2$. Let t be the canonical variable of the Serre–Tate deformation space $\widehat{S} = \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}} \mathfrak{d}^{-1}$ of $(A(\mathfrak{O}), \lambda(\mathfrak{O}), i(\mathfrak{O}))_{/W}$ so that the parameters $(t^{a_1}-1, \dots, t^{a_d}-1)$ (for a base $\{a_i\}_i$ of O over $\mathbb{Z}$) give the coordinate around the origin $1 \in \widehat{S}$. Suppose that $\mathfrak{I} = 1$, and write Φ for the restriction of $\mathrm{Inf}_\psi\,\chi_{\Gamma'}$ to $Z_0 \subset Z$. Put for each $\mathfrak{B} \in D^-$*

$$E_{\mathfrak{B}}(t) = \sum_{\mathfrak{A}\in D^+} \psi(\mathfrak{A})^{-1}E(\Phi^\circ)|\langle\mathfrak{A}\rangle(t) \in \mathcal{O}_{\widehat{S}}$$

if the relative class number of M/F is odd, where we have chosen $\mathfrak{A} \subset F$ prime to $\mathfrak{C}\mathfrak{C}^c p$. Otherwise, we put for each $\mathfrak{B} \in D^-$

$$E_{\mathfrak{B}}(t) = \sum_{\mathfrak{A} \in D^+} \psi(\mathfrak{A}) E(\Phi^\circ)|[\mathfrak{A}\mathfrak{A}^c]|\rho(\alpha_{\mathfrak{A}})^{-1}(t) \in \mathcal{O}_{\widehat{S}},$$

where $[\mathfrak{A}]^- = \alpha_{\mathfrak{A}}^{1-c}$ with $\alpha_{\mathfrak{A}} \in M^\times$ prime to $\mathfrak{C}\mathfrak{C}^c p$. Then the t–expansion of

$$\mathcal{E} = \sum_{\mathfrak{B} \in D^-} \psi(\mathfrak{B}) E_{\mathfrak{B}}|[\mathfrak{B}\mathfrak{B}^c](t^{[\mathfrak{B}]^-})$$

at $(A(\mathfrak{O}), \lambda(\mathfrak{O}), i(\mathfrak{O}))$ gives (up to an automorphism of $W[[Z]]$) the t–expansion of the anticyclotomic measure $\varphi^-_{\psi,\mathfrak{C}}$. In particular, supposing that $p > 2$ is unramified in $F/\mathbb{Q}$, we have the vanishing of the μ–invariant: $\mu(\varphi^-_{\psi,\mathfrak{C}}) = 0$, unless the following three conditions are satisfied:

(M1) *M/F is unramified everywhere (so the strict class number of F is even);*

(M2) *The strict ideal class of the polarization ideal $\mathfrak{c}$ in F is not a norm class of an ideal class of M ($\Leftrightarrow \left(\frac{M/F}{\mathfrak{c}}\right) = -1$);*

(M3) *$\mathfrak{a} \mapsto (\psi(\mathfrak{a}) N_{F/\mathbb{Q}}(\mathfrak{a}) \mod \mathfrak{m}_W)$ is the quadratic character of M/F, which is equivalent to $\psi^* \equiv \psi \mod \mathfrak{m}_W$.*

Under (M1-3)*, the invariant $\mu(\varphi^-_{\psi,\mathfrak{C}})$ is positive and is given by $\mu(\psi)$ in* (5.27).

The Eisenstein series $E_{\mathfrak{B}}$ defined in the theorem really depends on $\mathfrak{B} \in D^-$ since the polarization ideal of $E(\Phi^\circ)$ in the sum depends on $\mathfrak{B}$.

Proof. We first show that the t–Expansion of $\mathcal{E}$ gives (up to an automorphism of $W[[Z]]$) the t–expansion of the Katz measure. We said "up to an automorphism of $W[[Z]]$", because of the following reason: In the definition of the level structure $i(\mathfrak{A})$, $x \in (\mathfrak{f}^2)^*/O^*$ is sent to $2\delta x \in \mathfrak{F}^{-2}/\mathfrak{A}$. This has the following harmless effect: The t–expansion of $E(\Phi^\circ_+((x, a; y, b)\,\mathrm{diag}[2\delta, (2\delta)^{-1}]))$ actually coincides with the t–expansion of the measure. The variable change $(x, a; y, b) \mapsto (x, a; y, b)\,\mathrm{diag}[2\delta, 2\delta^{-1}]$ corresponds to the automorphism: $z \mapsto 2\delta z$ of the topological space Z (since 2δ is chosen to be prime to $p\mathfrak{C}$), which gives rise to an automorphism of $W[[Z]]$. So we forget about the effect of this unit 2δ.

Since the argument is simpler in the case where the relative class number is odd, we assume that the relative class number is even. We are going to compute the κ–derivatives of the Eisenstein series at $\underline{A}(\mathfrak{A}^{-1})$ for applying the t–expansion principle. Let ψ_κ be a unique Hecke character of Z such that $\psi_\kappa(\beta) = \beta^{(1-c)\kappa}\psi(\beta)$ for all $\beta \equiv 1 \mod \mathfrak{C}p$, $\psi_\kappa|_\Delta = \psi$ and $\psi_\kappa(\mathfrak{A}) = \alpha_{\mathfrak{A}}^{(1-c)\kappa}$ with $\alpha_{\mathfrak{A}}$ as in (5.3) for all $\mathfrak{A}$ as above, choosing $\mathfrak{A} \in D^+$ so that $[\mathfrak{A}]^- \in \Gamma'$. We write $\langle z \rangle$ for the projection of $z \in W(\mathbb{F})^\times$ to the p–profinite part of $W(\mathbb{F})^\times$. Then we have

$$\left\langle (x, y)^{\kappa(c-1)}\phi \right\rangle^\circ = \left\langle (x^{-1}y)^\kappa \phi \right\rangle^\circ = \langle (xy)^\kappa \rangle \phi^\circ.$$

We now replace each term $\psi(\alpha\beta^{-1})E(\chi^{\circ}_{\Gamma'})|\langle\beta\rangle|\beta|[\alpha\alpha^c]|\rho(\alpha)^{-1}$ of (5.20) by

$$\begin{aligned}
&\psi(\alpha\beta^{-1})d^{\kappa}(E(\chi^{\circ}_{\Gamma'})|\langle\beta\rangle|\beta|[\alpha\alpha^c]|\rho(\alpha)^{-1})(\underline{A}(\mathfrak{A}^{-1}))\\
&\quad\overset{(*)}{=}\psi(\alpha\beta^{-1})(\alpha\beta^{-1})^{\kappa(c-1)}E((xy)^{\kappa}(\chi^{\circ}_{\Gamma'}|[\alpha\beta^{-1}]))(\underline{A}((\alpha^{-1}\beta)\mathfrak{A}^{-1}))\\
&\quad=\psi(\alpha\beta^{-1})(\alpha\beta^{-1})^{\kappa(c-1)}E(((x^{-1}y)^{\kappa}(\chi_{\Gamma'}\circ(\alpha\beta^{-1}))^{\circ})(\underline{A}((\alpha^{-1}\beta)\mathfrak{A}^{-1}))\\
&\quad\overset{(**)}{=}\psi_{\kappa}(\alpha\beta^{-1})E((\langle(x^{-1}y)^{\kappa}\rangle\chi_{\Gamma'})^{\circ}|[\alpha\beta^{-1}]))(\underline{A}((\alpha^{-1}\beta)\mathfrak{A}^{-1}))\\
&\quad=\psi_{\kappa}(\alpha\beta^{-1})E((\langle(x^{-1}y)^{\kappa}\rangle\chi_{\Gamma'})^{\circ}))(\underline{A}(\mathfrak{A}^{-1})),\quad(5.21)
\end{aligned}$$

where $\phi|[\alpha](x,a;y,b)=\phi(\alpha^{-1}x,\alpha^{-1}a;\alpha y,\alpha b)$, $\phi\circ\alpha(t)=\phi(\alpha t)$ for $t\in T$ and $\underline{A}(\mathfrak{A}^{-1})=(A(\mathfrak{A}^{-1}),\lambda(\mathfrak{A}^{-1}),i(\mathfrak{A}^{-1}))$. The above equality indicated by $(*)$ (resp. $(**)$) follows from (5.1) and the formulas: $d^{\kappa}(q^{xy})=(xy)^{\kappa}q^{xy}$ and $d^{\kappa}(t^a)=a^{\kappa}t^a$ (resp. the fact that $\chi_{\Gamma'}\circ(\alpha\beta^{-1})$ is the characteristic function of $(\alpha^{-1}\beta)\Gamma'$). To avoid this type of complicated computation for $\alpha_{\mathfrak{A}}$, we choose $\mathfrak{A}$ so that $\langle\alpha_{\mathfrak{A}}^{1-c}\rangle=\alpha_{\mathfrak{A}}^{1-c}$ and $[\mathfrak{A}]^-\in\Gamma'$ (this is always possible). Let $\mathcal{F}=\left(\mathrm{Inf}_{\psi}\left((x^{-1}y)^{\kappa}\chi_{\Gamma'}\right)|_{Z_0}\right)^{\circ}=\langle(xy)^{\kappa}\rangle\Phi^{\circ}$. By the computation given in [HT] (4.9) (where the ideal denoted by $\mathfrak{A}$ is actually $\mathfrak{A}^{-1}$ in this paper), the partial L–value for the character ψ_{κ} and for the ideal class of $\mathfrak{A}^{-1}$ is given by, for $E(\mathcal{F})=E_{\mathfrak{c}_{\mathfrak{A}^{-1}}}(\mathcal{F})$,

$$\psi_{\kappa}(\mathfrak{A})E(\mathcal{F})(\underline{A}(\mathfrak{A}^{-1}))=\psi_{\kappa}(\mathfrak{A})d^{\kappa}(E(\Phi^{\circ})|[\mathfrak{A}\mathfrak{A}^c]|\rho(\alpha_{\mathfrak{A}})^{-1})(\underline{A}(\mathfrak{O}))$$

and

$$d^{\kappa}E(\underline{A}(\mathfrak{O}))=\sum_{\mathfrak{A}\in D^+}\psi_{\kappa}(\mathfrak{A})E(\mathcal{F})(\underline{A}(\mathfrak{A}^{-1}))\qquad(5.22)$$

for all $\kappa\geq 0$. This is because $\rho(\alpha_{\mathfrak{A}})$ fixes the test object $(A(\mathfrak{O}),\lambda(\mathfrak{O}),i(\mathfrak{O}))$ and convert the variable t into $t^{\alpha^{1-c}}$ ($\alpha=\alpha_{\mathfrak{A}}$; Corollary 3.5), and hence we have

$$E(\phi)|[\mathfrak{A}\mathfrak{A}^c]|\rho(\alpha_{\mathfrak{A}})^{-1}(\underline{A}(\mathfrak{O}))=E(\phi)(\underline{A}(\mathfrak{A}^{-1})).$$

In [HT], bottom of page 215, $a^{(c-1)2\kappa}$ appears instead of the single power $a^{(c-1)\kappa}$ ($a=\alpha_{\mathfrak{A}}^{-1}$ in our computation here), but as can be easily checked (and as is obvious from the evaluation formula of the Katz measure in the introduction), this is a misprint, and the above single power $a^{(c-1)\kappa}$ is the correct one.

Now we apply the operator $[\mathfrak{B}\mathfrak{B}^c]$ and make variable change: $t\mapsto t^{[\mathfrak{B}]^-}$ in (5.22). We may again assume that $[\mathfrak{B}]^-\in\Gamma'$. The operator $[\mathfrak{B}\mathfrak{B}^c]$ (resp. the variable change: $t\mapsto t^{[\mathfrak{B}]^-}$) plays the role of $[\mathfrak{A}\mathfrak{A}^c]$ (resp. $\rho(\alpha_{\mathfrak{A}})^{-1}$) in the above computation, and we obtain by the effect of the differential operator d^{κ} again

$$d^{\kappa}(E_{\mathfrak{B}}|[\mathfrak{B}\mathfrak{B}^c](t^{[\mathfrak{B}]^-}))(\underline{A}(\mathfrak{O}))=\sum_{\mathfrak{A}\in D^+}\psi_{\kappa}(\mathfrak{A}\mathfrak{B})E(\mathcal{F})(\underline{A}((\mathfrak{A}\mathfrak{B})^{-1})).\qquad(5.23)$$

This combined with the evaluation formula (1.3) (and [HT] (4.9)) shows that the function in the theorem, after applying d^{κ} and evaluating at $\underline{A}(\mathfrak{O})$, has the property satisfied by the measure $\varphi^{-}_{\psi,\mathfrak{C}}$; so, the first assertion follows from (4.6).

As explained above Corollary 3.21, we have a unique element $H_{\kappa}\in G_{\kappa}(\mathbb{F})$ whose t-expansion is the constant 1 (identical to the t-expansion of the Hasse

invariant). Abusing terminology, we call H_κ the Hasse invariant. We want to apply Corollary 3.21 taking $\{a_i\}_i = \{[\mathfrak{B}]^-\}_{\mathfrak{B}\in D^-}$ and $\{f_{ij}\} = \{E_{\mathfrak{B}}|[\mathfrak{B}\mathfrak{B}^c] \bmod \mathfrak{m}_W\}_{\mathfrak{B}\in D^-}$. Here for each index i with $a_i = [\mathfrak{B}]^-$, $\{f_{ij}\}$ is given by the single element $\overline{E}_{\mathfrak{B}} := (E_{\mathfrak{B}}|[\mathfrak{B}\mathfrak{B}^c] \mod \mathfrak{m}_W)$.

To verify the assumption (of Corollary 3.21) of linear independence (over $\mathbb{F}$) of $\{H_\kappa, f_{ij}\}_j$ for each i, we need to show that for each $\mathfrak{B} \in D^-$, $\overline{E}_{\mathfrak{B}}$ is linearly independent from the Hasse invariant $H_\kappa(t) = 1$, unless (M1-3) are satisfied. Here we use the t–expansion principle and the q–expansion principle. Once this is done, by Corollary 3.21, $\{\overline{E}_{\mathfrak{B}}(t^{[\mathfrak{B}]^-})\}_{\mathfrak{B}\in D^-}$ is linearly independent over $\mathbb{F}$, and hence we conclude the nonvanishing of $\mathcal{E}$ (i.e., the vanishing of the μ-invariant) by Corollary 3.21 (which requires the unramifiedness of p in $F/\mathbb{Q}$), since elements in $\{[\mathfrak{B}]^-\}_{\mathfrak{B}\in D^-}$ are distinct modulo $T_x(\mathbb{Q})$. We show the linear independence of $E_{\mathfrak{B}}$ from H_κ by finding a totally positive $\xi \in F$ such that the q–expansion coefficient $a(\xi, E_{\mathfrak{B}}) \not\equiv 0 \mod \mathfrak{m}_W$.

We may assume that ψ has conductor divisible by $\mathfrak{C}$. Write $\pi : T^\times \to \Delta$ for the projection $Z \to \Delta$ composed with $\iota : T^\times \to Z$. Let Ψ be the function on $T^\times$ given by $\Psi(x, a; y, b) = \psi \circ \pi(x^{-1}, a^{-1}; y, a)$. By our assumption, $\Phi^\circ(x, a; y, b) = G(\psi_{\mathfrak{f}})\Psi$, where the Gauss sum $G(\psi_{\mathfrak{f}})$ is given by

$$\sum_{u \mod \mathfrak{f}} \psi(u)^{-1}\mathbf{e}_F(-u\varpi^{-\varepsilon(\mathfrak{f}\mathfrak{d})}).$$

This number is a p–adic unit; so, for our purpose, we can forget about it. The q–expansion coefficient of $\xi \in \mathfrak{a}\mathfrak{b}$ of $E(\Psi)$ at the cusp $(\mathfrak{a}, \mathfrak{b})$ is given by

$$\sum_{(a,b)\in(\mathfrak{a}\times\mathfrak{b})/O^\times, ab=\xi} \Psi(a, b)|N(a)|^{-1}.$$

We fix $\mathfrak{B} \in D^-$. This determines $\mathfrak{c}_{\mathfrak{B}^{-1}}$ which is the polarization ideal of $\lambda(\mathfrak{B}^{-1})$ on $A(\mathfrak{B}^{-1})$. If we write $\mathfrak{c}$ for the polarization ideal of $\lambda(\mathfrak{O})$, we know $\mathfrak{c}_{\mathfrak{B}^{-1}} = \mathfrak{c}(\mathfrak{B}\mathfrak{B}^c)$. We choose $\mathfrak{c}_{\mathfrak{B}^{-1}}^{-1}$ to be a prime $\mathfrak{l}$ prime to $p\mathfrak{f}$ (this is possible by changing it in its strict ideal class and choosing $\delta \in M$ suitably).

We first assume that the class number is even. We have chosen $\mathfrak{A}$ to be a prime $\mathfrak{O}$ of M split over F. Then $\mathfrak{O}^{1-c} = \alpha_{\mathfrak{O}}^{1-c}$, and $\mathfrak{O}\alpha_{\mathfrak{O}}^{-1}$ is a product of primes in F and ramified primes in M/F. If $\mathfrak{O}\alpha_{\mathfrak{O}}^{-1} = \mathfrak{u}$ does not contain ramified primes, then the operator $[\mathfrak{O}\mathfrak{O}^c] = [\mathfrak{q}]$ ($\mathfrak{q} = \mathfrak{O}\cap F$) is given by $g \in G(\mathbb{A}^{(\infty)})$ with $g_{p\mathfrak{f}} = 1$, and $\rho(\alpha_{\mathfrak{O}})g^{-1} \in K_{\mathfrak{a},\mathfrak{b}}(\mathfrak{f}\mathfrak{s})Z(\mathbb{A}^{(\infty)})$. Thus $f|[\mathfrak{O}\mathfrak{O}^c]|\rho(\alpha_{\mathfrak{O}})^{-1} = f|\langle\mathfrak{u}\rangle$ for an integral ideal $\mathfrak{u}$ of F and modular form f on $K_{\mathfrak{a},\mathfrak{b}}(\mathfrak{f}\mathfrak{s})$.

We assume that $\mathfrak{O}\alpha_{\mathfrak{O}}^{-1}$ contains ramified primes. Then we may assume that $\mathfrak{O}\alpha_{\mathfrak{O}}^{-1} = \mathfrak{u}\mathfrak{R}$ for a square-free product $\mathfrak{R}$ of ramified primes $\mathfrak{L}$ and an ideal $\mathfrak{u} \subset F$. For each ramified prime $\mathfrak{L}$, we may assume that

$$\rho(\mathfrak{O}_{\mathfrak{L}}) = \left\{ \left(\begin{smallmatrix} a & b \\ b\varpi_{\mathfrak{l}} & a \end{smallmatrix}\right) | a, b \in O_{\mathfrak{l}} \right\},$$

where $\mathfrak{l} = \mathfrak{L} \cap F$. Let $g \in G(\mathbb{A}^{(\infty)})$ be the element whose action on the Hilbert modular variety coincides with $[\mathfrak{q}]$. As already seen, $g^{(\mathfrak{q})} = 1$ outside $\mathfrak{q}$ and $g_{\mathfrak{q}} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi_{\mathfrak{q}}^{-1} \end{smallmatrix}\right)$. Recalling the relative discriminant $\mathfrak{D}$ of M/F, by

the definition of the level-$\mathfrak{q}$-structure of $\underline{A}(\mathfrak{B})$, we find that $g^{-1}\rho(\alpha_{\mathfrak{D}})^{(\mathfrak{D})} \in K_{\mathfrak{a},\mathfrak{b}}(\mathfrak{fs})Z(\mathbb{A}^{(\infty)})$, and writing $\mathfrak{R} = \prod_{\mathfrak{L}} \mathfrak{L}$ for ramified primes $\mathfrak{L}$ in M/F, we may assume that $\rho(\alpha_{\mathfrak{D}})_{\mathfrak{l}}^{-1} = \left(\begin{smallmatrix} 0 & 1 \\ \varpi_{\mathfrak{l}} & 0 \end{smallmatrix}\right)$ for each prime factor $\mathfrak{l}$ of $\mathfrak{r} = \mathfrak{R} \cap F$. Note that $K_{\mathfrak{a},\mathfrak{b}}(\mathfrak{fs})_{\mathfrak{l}}\rho(\alpha_{\mathfrak{D}})_{\mathfrak{l}}^{-1} = \left(\begin{smallmatrix} \varpi_{\mathfrak{l}} & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Therefore, $f|[\mathfrak{q}]|\rho(\alpha_{\mathfrak{D}})^{-1} = f|\langle \mathfrak{u} \rangle| \left(\begin{smallmatrix} \varpi_{\mathfrak{r}}^{e(\mathfrak{r})} & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and the operator $\rho(\alpha_{\mathfrak{D}})^{-1} \circ [\mathfrak{q}]$ brings $Tate_{\mathfrak{a},\mathfrak{b}}(q)$ to $Tate_{\mathfrak{u}^{-1}\mathfrak{a},\mathfrak{u}\mathfrak{b}\mathfrak{r}}(q)$ (see (5.10)). We can thus rewrite the sum defining $E_{\mathfrak{B}}$ in the theorem as:

$$E_{\mathfrak{B}} = \sum_{\mathfrak{u}} \psi(\mathfrak{u})^{-1} \sum_{\mathfrak{r}} \psi(\mathfrak{R}) E(\Phi^{\circ})|\langle \mathfrak{u} \rangle|[\mathfrak{r}], \tag{5.24}$$

where $\mathfrak{u}$ runs over a complete representative set (of F–ideals) for the image $\iota(Cl_F)$ of Cl_F in Cl_M, $\mathfrak{r} = \mathfrak{R} \cap F$, and $\mathfrak{r}$ runs over a complete representative set for $Cl_M^+/\iota(Cl_F)$ made of square-free factors of $\mathfrak{D}$. To make our treatment uniform, even if the class number is odd, we change notation in the sum defining $E_{\mathfrak{B}}$ and rewrite it as

$$E_{\mathfrak{B}} = \sum_{\mathfrak{u}} \psi(\mathfrak{u})^{-1} E(\Phi^{\circ})|\langle \mathfrak{u} \rangle, \tag{5.25}$$

where $\mathfrak{u}$ runs over a complete representative set (of F–ideals) for the image $\iota(Cl_F)$ of Cl_F in Cl_M. Hereafter we use the notation $\mathfrak{u}$ to indicate an ideal representing a class in $\iota(Cl_F)$ and stop assuming that the class number is even.

We now take a totally positive $0 \ll \xi \in O$ so that $(\xi) = \mathfrak{l}\mathfrak{n}$ ($\mathfrak{l} = \mathfrak{c}_{\mathfrak{B}^{-1}}^{-1}$: a prime by our choice) for an integral ideal $\mathfrak{n}$ prime to $\mathfrak{C}p\mathfrak{D}$. We pick a pair $(a, b) \in F^2$ with $ab = \xi$ for $a \in \mathfrak{u}^{-1}$ and $b \in \mathfrak{l}\mathfrak{u}\mathfrak{r}$. Then $(a) = \mathfrak{u}^{-1}\mathfrak{x}$ for an integral ideal $\mathfrak{x}$ and $(b) = \mathfrak{u}\mathfrak{l}\mathfrak{r}\mathfrak{y}$. Since $(ab) = \mathfrak{l}\mathfrak{n}$, we find that $\mathfrak{x}\mathfrak{y} = \mathfrak{n}$ and hence $\mathfrak{r} = O$ because $\mathfrak{n}$ is prime to $\mathfrak{D}$. Thus for each factor $\mathfrak{x}$ of $\mathfrak{n}$, we could have a pair $(a_{\mathfrak{x}}, b_{\mathfrak{x}})$ with $a_{\mathfrak{x}} b_{\mathfrak{x}} = \xi$ such that

$$((a_{\mathfrak{x}}) = \mathfrak{u}_{\mathfrak{x}}^{-1}\mathfrak{x},\ (b_{\mathfrak{x}}) = (\xi a_{\mathfrak{x}}^{-1}) = \mathfrak{u}_{\mathfrak{x}}\mathfrak{l}\mathfrak{n}\mathfrak{x}^{-1})$$

for $\mathfrak{u}_{\mathfrak{x}} \in D^+$ representing the ideal class of the ideal $\mathfrak{x}$. We then write down the q–expansion coefficient of q^{ξ} at the cusp $(O, \mathfrak{l})$ of $E_{\mathfrak{B}}$ as in the theorem:

$$\begin{aligned}
G(\psi_{\mathfrak{f}})^{-1} a(\xi, E_{\mathfrak{B}}) &= G(\psi_{\mathfrak{f}})^{-1} \sum_{\mathfrak{x}|\mathfrak{n}} \psi^{-1}(\mathfrak{u}_{\mathfrak{x}}) a(\xi, E(\Phi^{\circ})|\langle \mathfrak{u}_{\mathfrak{x}} \rangle) \\
&\overset{(E2)}{=} \sum_{\mathfrak{x}|\mathfrak{n}} N(\mathfrak{u}_{\mathfrak{x}})^{-1} \psi^{-1}(\mathfrak{u}_{\mathfrak{x}}) \psi_{\mathfrak{F}}(a_{\mathfrak{x}}) \psi_{\mathfrak{F}_c}^{-1}(\xi {a_{\mathfrak{x}}}^{-1}) \frac{1}{|N(a_{\mathfrak{x}})|} \\
&= \psi_{\mathfrak{F}_c}^{-1}(\xi) \sum_{\mathfrak{x}|\mathfrak{n}} N(\mathfrak{u}_{\mathfrak{x}})^{-1} \psi^{-1}(\mathfrak{u}_{\mathfrak{x}}) \psi_{\mathfrak{C}}(a_{\mathfrak{x}}) \frac{1}{|N(a_{\mathfrak{x}})|} \\
&= \psi_{\mathfrak{F}_c}^{-1}(\xi) \sum_{\mathfrak{x}|\mathfrak{n}} \frac{1}{\psi(\mathfrak{x}) N(\mathfrak{x})} \\
&= \psi_{\mathfrak{F}_c}^{-1}(\xi) \prod_{\mathfrak{q}|\mathfrak{n}} \left(\sum_{j=0}^{e(\mathfrak{q})} (\psi(\mathfrak{q}) N(\mathfrak{q}))^{-j} \right) \\
&= \psi_{\mathfrak{F}_c}^{-1}(\xi) \psi(\mathfrak{n})^{-1} N(\mathfrak{n})^{-1} \prod_{\mathfrak{q}|\mathfrak{n}} \frac{1 - (\psi(\mathfrak{q}) N(\mathfrak{q}))^{e(\mathfrak{q})+1}}{1 - \psi(\mathfrak{q}) N(\mathfrak{q})},
\end{aligned} \tag{5.26}$$

where $\mathfrak{n} = \prod_{\mathfrak{q}|\mathfrak{n}} \mathfrak{q}^{e(\mathfrak{q})}$ is the prime factorization of $\mathfrak{n}$.

Suppose that $\mathfrak{n}$ is a prime $\mathfrak{q}$. Then by (5.26), we have $G(\psi_{\mathfrak{f}})^{-1}a(\xi, E_{\mathfrak{B}}) = \psi_{\mathfrak{F}_c}^{-1}(\xi)(1+(\psi(\mathfrak{q})N(\mathfrak{q}))^{-1})$. If $\psi(\mathfrak{q})N(\mathfrak{q}) \equiv -1 \mod \mathfrak{m}_W$ for all prime ideals $\mathfrak{q}$ in the strict class of $\mathfrak{c}_{\mathfrak{B}^{-1}}$, the character $\mathfrak{a} \mapsto \psi(\mathfrak{a})N(\mathfrak{a}) \mod \mathfrak{m}_W$ is of conductor 1, and the strict class number has to be even.

We define, for the valuation v of W (normalized so that $v(p) = 1$)

$$\mu(\psi) = \mathrm{Inf}_{\mathfrak{n}}\, v\left(\prod_{\mathfrak{q}|\mathfrak{n}} \frac{1-(\psi(\mathfrak{q})N(\mathfrak{q}))^{e(\mathfrak{q})+1}}{1-\psi(\mathfrak{q})N(\mathfrak{q})}\right), \tag{5.27}$$

where $\mathfrak{n}$ runs over all integral ideals prime to $\mathfrak{D}$ of the form $\mathfrak{c}(\mathfrak{A}\mathfrak{A}^c)$ for ideals $\mathfrak{A}$ of M. Here $\mathfrak{c}$ is the polarization ideal of $A(\mathfrak{O})$. Then, by moving around $\mathfrak{B}$ in D^-, the μ–invariant $\mu(\varphi^-_\psi)$ of the ψ–branch of the anticyclotomic Katz measure φ^- is less than or equal to $\mu(\psi)$, and $\mu(\varphi^-_\psi) = \mu(\psi)$ if M/F is everywhere unramified. In particular, if $\widetilde{\psi} = \psi N \mod \mathfrak{m}_W$ as a character of $F^\times_{\mathbb{A}}$ has non-trivial conductor, $0 \le \mu(\varphi^-_\psi) \le \mu(\psi) = 0$. We may therefore assume that $\widetilde{\psi}$ has conductor 1 and that $\widetilde{\psi}(\mathfrak{c}) = -1$.

We now recall the conditions (V) and (M1–3) stated in the introduction:

(V) $\quad \psi^* \equiv \psi \mod \mathfrak{m}_W$ and $N(\mathfrak{c})^{-1}\psi(\mathfrak{c}^{-1})W'(\psi) \equiv -1 \mod \mathfrak{m}_W$.

and

(M1) M/F is unramified at every finite place;

(M2) The strict ideal class of the polarization ideal $\mathfrak{c}$ in F is not a norm class of an ideal class of M $\left(\Leftrightarrow \left(\frac{M/F}{\mathfrak{c}}\right) = -1\right)$;

(M3) $\mathfrak{a} \mapsto (\psi(\mathfrak{a})N_{F/\mathbb{Q}}(\mathfrak{a}) \mod \mathfrak{m}_W)$ is the character $\left(\frac{M/F}{\cdot}\right)$ of M/F.

We first give a direct proof of the equivalence of (V) and (M1–3) as a lemma (following the suggestion of one of the referees of this paper), and after that, we shall give an indirect proof of the same fact using $\mu(\psi)$ defined above.

Lemma 5.2. *Let the assumption be as in* Theorem I. *Then we have an equivalence:* (V) $\iff$ (M1–3).

Proof. Suppose (M1–3). Write $\overline{\psi} = (\psi \mod \mathfrak{m}_W)$ and $\overline{\omega} = (N_{F/\mathbb{Q}} \mod \mathfrak{m})$. Then $\overline{\omega}(xx^c) = \overline{N}(x) := (N(x) \mod \mathfrak{m}_W)$ for the p-adic cyclotomic character N of $M^\times_{\mathbb{A}}$. By (M3) and class field theory, we have $\overline{\psi}\overline{\omega}(xx^c) = 1$ for $x \in M^\times_{\mathbb{A}}$. This implies $\overline{\psi}^*(x) = \overline{\psi}(x^{-c})\overline{N}(x)^{-1} = \overline{\psi}(x)\overline{\omega}(xx^c)\overline{N}(x)^{-1} = \overline{\psi}(x)$, which proves the first part of (V). By (M2–3), $\widetilde{\psi}(\mathfrak{c}) = \overline{\psi N}(\mathfrak{c}) = -1$; so, we need to prove $W'(\psi) \equiv 1 \mod \mathfrak{m}_W$. Since $\overline{\psi}^* = \overline{\psi}$, we have $\overline{\psi}(x) = \overline{\psi}(x^{-c})$ for $x \in \widehat{\mathfrak{O}}^\times$ with $x_p = 1$. Thus for a prime ideal $\mathfrak{Q}|\mathfrak{C}$ of M outside p, $\mathfrak{q} = \mathfrak{Q} \cap F$ splits as $\mathfrak{q} = \mathfrak{Q}\overline{\mathfrak{Q}}$ in M. Identifying $\mathfrak{O}_{\mathfrak{Q}} \cong O_{\mathfrak{q}} \cong \mathfrak{O}_{\overline{\mathfrak{Q}}}$ and writing $g(\overline{\psi}) = \sum_{u \in (\mathfrak{O}_{\mathfrak{Q}}/\mathfrak{Q}^e)^\times} \lambda_{\mathfrak{Q}}(u)\mathbf{e}_M(\varpi_{\mathfrak{Q}}^{-e}d_{\mathfrak{Q}}^{-1}u)$, we have $g(\overline{\psi}_{\mathfrak{Q}})g(\overline{\psi}_{\overline{\mathfrak{Q}}}) = g(\overline{\psi}_{\mathfrak{Q}})g(\overline{\psi}_{\mathfrak{Q}}^{-1})$ by

$\overline{\psi}(u) = \overline{\psi}(u^{-c})$, and hence, we get $g(\overline{\psi}_{\mathfrak{Q}})g(\overline{\psi}_{\overline{\mathfrak{Q}}}) = N(\mathfrak{Q}^e)\overline{\psi}_{\mathfrak{Q}}(-1)$. Since $\widetilde{\psi} = \overline{\psi N}$ is everywhere unramified by (M1), we have $\overline{\psi}_{\mathfrak{Q}}(-1) = 1$; so, finally we get $g(\overline{\psi}_{\mathfrak{Q}})g(\overline{\psi}_{\overline{\mathfrak{Q}}}) = N(\mathfrak{Q}^e)$. We may take $\varpi_{\overline{\mathfrak{Q}}} = \varpi_{\mathfrak{Q}}^c$ in the definition of $G(d_{\overline{\mathfrak{Q}}}, \psi_{\overline{\mathfrak{Q}}})$. Then we have

$$\overline{\psi}(\varpi_{\mathfrak{Q}})\overline{\psi}(\varpi_{\overline{\mathfrak{Q}}})\overline{N}(\mathfrak{Q}) = \overline{\psi}(N_{M/F}(\varpi_{\mathfrak{Q}}))\overline{\omega}(N_{M/F}(\varpi_{\mathfrak{Q}})) = \left(\frac{M/F}{N_{M/F}(\varpi_{\mathfrak{Q}})}\right) = 1.$$

Since $G(d_{\mathfrak{Q}}, \psi_{\mathfrak{Q}})G(d_{\overline{\mathfrak{Q}}}, \psi_{\overline{\mathfrak{q}}}) = \psi(\varpi_{\mathfrak{Q}}^e)\psi(\varpi_{\overline{\mathfrak{Q}}}^e)g(\psi_{\mathfrak{Q}})g(\psi_{\overline{\mathfrak{Q}}})$ (if $\mathfrak{Q}^e \parallel \mathfrak{C}$), we find that $G(d_{\mathfrak{Q}}, \psi_{\mathfrak{Q}})G(d_{\overline{\mathfrak{Q}}}, \psi_{\overline{\mathfrak{q}}}) \equiv 1 \mod \mathfrak{m}_W$ for all $\mathfrak{Q}|\mathfrak{C}$ prime to p. Thus we get $W'(\psi) = \prod_{\mathfrak{Q}} G(d_{\mathfrak{Q}}, \psi_{\mathfrak{Q}})G(d_{\overline{\mathfrak{Q}}}, \psi_{\overline{\mathfrak{q}}}) \equiv 1 \mod \mathfrak{m}_W$.

Now we assume (V). Since $\overline{\psi}^*(x) = \overline{\psi}(x^{-c})\overline{N}(x)^{-1} = \overline{\psi}(x)$, we find that $\overline{\psi}(xx^c) = \overline{N}(x)^{-1} = \overline{\omega}(xx^c)^{-1}$, which implies $\overline{\psi}\overline{\omega}$ is a global Hecke character trivial on $N_{M/F}(M_{\mathbb{A}}^{\times})$. Then by class field theory, $\overline{\psi}\overline{\omega}$ is either trivial or equal to $\left(\frac{M/F}{}\right)$. Since ψ is a character of totally imaginary M with connected $M_{\infty}^{\times}$, $\psi_{\infty} = 1$. Since $\overline{\omega}$ has nontrivial at ∞ (because F is real and $p > 2$), we find that $\overline{\psi}\overline{\omega} = \left(\frac{M/F}{}\right)$. Since the conductor of $\overline{\psi}\overline{\omega}$ is concentrated on $\mathfrak{C}p$ which is prime to the conductor of $\left(\frac{M/F}{}\right)$ (because $p\mathfrak{C}$ is made up of split primes in M/F), we find that $\left(\frac{M/F}{}\right)$ is everywhere unramified. Thus we get (M1) and (M3). By $N(\mathfrak{c})^{-1}\psi(\mathfrak{c}^{-1})W'(\psi) \equiv -1 \mod \mathfrak{m}_W$, the condition (M2) follows from the fact $W'(\psi) \equiv 1 \mod \mathfrak{m}_W$ by the computation of the first part which uses only (M1) and (M3) already proven. □

Here is the indirect argument: We are going to show that if $\mu(\psi) > 0$, M/F is unramified everywhere and $\widetilde{\psi} \equiv \left(\frac{M/F}{}\right) \mod \mathfrak{m}_W$. We have already proven that if $\mu(\psi) > 0$, $\widetilde{\psi}$ is unramified and $\widetilde{\psi}(\mathfrak{c}) = -1$ before the lemma. We now choose two prime ideals $\mathfrak{q}$ and $\mathfrak{q}'$ so that $\mathfrak{l}\mathfrak{q}\mathfrak{q}' = (\xi)$. Then by (5.26), we have

$$G(\psi_{\mathfrak{f}})^{-1}a(\xi, E_{\mathfrak{B}}) = \psi_{\mathfrak{F}_c}^{-1}(\xi)\left(1 + \frac{1}{\psi(\mathfrak{q})N(\mathfrak{q})}\right)\left(1 + \frac{1}{\psi(\mathfrak{q}')N(\mathfrak{q}')}\right). \tag{5.28}$$

Since $\widetilde{\psi}(\mathfrak{q}\mathfrak{q}') = \widetilde{\psi}(\mathfrak{l}^{-1}) = -1$, we find that if $a(\xi, E_{\mathfrak{B}}) \equiv 0 \mod \mathfrak{m}_W$,

$$-1 = \widetilde{\psi}(\mathfrak{q}/\mathfrak{q}') = \widetilde{\psi}(\mathfrak{l}^{-1})\widetilde{\psi}(\mathfrak{q}^2) = -\widetilde{\psi}(\mathfrak{q}^2).$$

Since we can choose $\mathfrak{q}$ arbitrary, we find that $\widetilde{\psi}$ is quadratic.

The polarization ideal of $\lambda(\mathfrak{B}^{-1})$ is $\mathfrak{c}(\mathfrak{B}\mathfrak{B}^c)$ as already remarked. Since the strict ideal classes $\{[\mathfrak{B}\mathfrak{B}^c]\}_{\mathfrak{B}\in D^-}$ together with Cl_F^2 covers all the classes in $N_{M/F}(Cl_M)$ in the strict ideal class group Cl_F, we find that $-1 = \widetilde{\psi}(\mathfrak{c}) = \widetilde{\psi}(\mathfrak{c}(\mathfrak{B}\mathfrak{B}^c))$ implies that $\widetilde{\psi}$ is trivial on $N_{M/F}(Cl_M)$ but non-trivial on Cl_F. This implies, by class field theory, $\widetilde{\psi}$ is the quadratic character $\left(\frac{M/F}{}\right)$ of the quadratic extension M/F. In particular, M/F is unramified everywhere.

Since (M1) and (M3) are established under the condition $\mu(\psi) > 0$, by the above lemma (or rather by its proof), we have $W'(\psi) \equiv 1 \mod \mathfrak{m}_W$. We thus

find $\mu(\varphi_\psi^-) > 0 \Leftrightarrow$ the three conditions (M1–3) are satisfied. Under these three conditions, by the q–expansion principle, we find $\mu(\psi) = \mu(\varphi_\psi^-)$, which is finite.

We show $\mu(\psi) > 0$ under (M1-3) (without using the identity: $\mu(\psi) = \mu(\varphi_\psi^-)$). Since $\widetilde{\psi}(\mathfrak{n}) = \left(\frac{M/F}{\mathfrak{c}}\right) = -1$ for $\mathfrak{n}$ appearing in the definition of $\mu(\psi)$, for odd number of prime factors $\mathfrak{q}|\mathfrak{n}$ has odd exponent $e(\mathfrak{q})$. Thus $\widetilde{\psi}(\mathfrak{q})^{e(\mathfrak{q})+1} = 1$, and hence the factor $1 - (\psi(\mathfrak{q})N(\mathfrak{q}))^{e(\mathfrak{q})+1}$ in the definition vanishes; so, $\mu(\psi) \geq 1$. This finishes the proof. □

To show that the condition (M2) really depends on the CM-type Σ, we give an example. We take $F = \mathbb{Q}[\sqrt{21}]$. This real quadratic field has strict class number 2 (so has class number 1). We thus have a unique everywhere unramified CM quadratic extension $M = \mathbb{Q}[\sqrt{-3}, \sqrt{-7}]$. Define two CM types of M: Σ_3 (resp. Σ_7) to be the inflation to M of the identity inclusion of $\mathbb{Q}[\sqrt{-3}]$ (resp. $\mathbb{Q}[\sqrt{-7}]$) into $\mathbb{C}$. Then we can chose $\delta = \delta_\ell$ for Σ_ℓ to be $\sqrt{-\ell}$. Since $(2\delta_\ell)^{-1} = \mathfrak{d}_{M/F}(2\delta_\ell)^{-1} = \mathfrak{c}^{-1}\mathfrak{d}^{-1}$, we find $\mathfrak{c} = ((7+\sqrt{21})\sqrt{21}^{-1})$ for Σ_7 and hence $\left(\frac{M/F}{\mathfrak{c}}\right) = -1$ in this case because $(7+\sqrt{21})$ is totally positive. Contrary to this, we find $\mathfrak{c} = ((3+\sqrt{21})\sqrt{21}^{-1})$ and $\left(\frac{M/F}{\mathfrak{c}}\right) = 1$ for Σ_3.

5.5 Non-Vanishing of the μ–Invariant

Define

$$\Gamma_0(\mathfrak{N};\mathfrak{a},\mathfrak{b}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(F) \middle| ad - bc \gg 0,\ a,d \in O,\ c \in \mathfrak{N}\mathfrak{a}\mathfrak{b}\mathfrak{d},\ b \in (\mathfrak{a}\mathfrak{b})^* \right\}, \tag{5.29}$$

where $a \gg 0$ indicates that a is totally positive. We let the congruence subgroup $\Gamma_0(\mathfrak{f}\mathfrak{s};\mathfrak{a},\mathfrak{b})$ act on

$$PV(T) = \left\{ (y,x) \middle| x \in F_p \times ((\mathfrak{f}\mathfrak{a})^*/\mathfrak{a}^*),\ y \in \mathfrak{b}_p \times (\mathfrak{b}/\mathfrak{s}\mathfrak{b}) \right\}$$

by $(y,x) \mapsto (y,x)\gamma$. It is easy to check that this action is well defined. Note that for a function ϕ on T, we have $E(\phi)|\gamma = E(P^{-1}(P\phi \circ \gamma))$ if $\gamma \in GL_2(F)$ has $\det(\gamma) \gg 0$ and preserves the lattice ${}^t(\mathfrak{a}^* \oplus \mathfrak{b})$ made of column vectors.

We give an example of a branch with positive μ–invariant if $\mathfrak{C}$ contains a prime inert in M/F. We assume that $\mathfrak{Q}$ is a prime factor of $\mathfrak{I}$ prime to $\mathfrak{I}/\mathfrak{Q}$ such that $\psi_{\mathfrak{Q}} = 1$; so, ψ is imprimitive at $\mathfrak{Q}$. For the moment, we further assume that $\mathfrak{Q}$ is an inert prime of M over F generated by a totally positive element $\varpi \in O$. This assumption of principality is for simplicity in order to have well defined Hecke operator $T(\mathfrak{q})$ ($\mathfrak{q} = \mathfrak{Q} \cap F$) acting on $G_k(\Gamma_0(\mathfrak{N};\mathfrak{a},\mathfrak{b}))$, because otherwise $T(\mathfrak{q})$ brings $G_k(\Gamma_0(\mathfrak{N};\mathfrak{a},\mathfrak{b}))$ into $G_k(\Gamma_0(\mathfrak{N};\mathfrak{q}^{-1}\mathfrak{a},\mathfrak{b}))$.

We put $f = E(\Phi_+^\circ)$ as in (5.17). Since $P\Phi_+^\circ \circ \gamma = \psi(d)P\Phi_+^\circ$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(\mathfrak{f}p^r;\mathfrak{a},\mathfrak{b})$ (for a suitable $r > 0$), we know $f|\gamma = \psi(\gamma)f$, where $\psi(\gamma) = \psi(d)$. The number r is the exponent such that the character: $Z \twoheadrightarrow \Delta \xrightarrow{\psi} W^\times$ factors through $Cl_M(\mathfrak{C}p^r)$.

We are going to see that $f \mapsto \sum_{\alpha\in\mathcal{R}} f|[\alpha\alpha^c]|\rho(\alpha)^{-1}$, with the notation (in particular $\mathcal{R}$) in (5.19), factors through the level-lowering trace operator Tr from $\Gamma_0(\mathfrak{f}p^r;\mathfrak{a},\mathfrak{b})$ to its subgroup $\Gamma_0(\mathfrak{f}p^r/\mathfrak{q};\mathfrak{a},\mathfrak{b})$. The regular representation $\rho:\mathfrak{O}_{\mathfrak{Q}}\to M_2(O_{\mathfrak{q}})$ induces $\overline{\rho}:\mathfrak{O}/\mathfrak{Q}\to M_2(O/\mathfrak{q})$. Let $C=\overline{\rho}((\mathfrak{O}/\mathfrak{Q})^\times)$. We have the following decomposition:

$$GL_2(O/\mathfrak{q}) = BC = CB$$

for the upper triangular Borel subgroup B. Note that the image of C in $PGL(2)$ is the maximal quotient (we call the "$-$" quotient) on which $c\in\mathrm{Gal}(M/F)$ acts by -1. The "$-$" quotient has order $N(\mathfrak{q})+1$. By this Iwasawa decomposition, it is easy to see that

$$(g|[\mathfrak{q}])|\mathrm{Tr} = N(\mathfrak{q})g|T(\mathfrak{q})$$

for the Hecke operator $T(\mathfrak{q})$ (of level prime to $\mathfrak{q}$).

Since $\mathcal{D}^-\subset M^\times/F^\times$, complex conjugation acts on $\mathcal{D}^-$ by "-1" (writing additively). Thus we can identify $\mathcal{D}^-$ with the "$-$" quotient of $\pi(\mathfrak{O}_{\mathfrak{f}p})/\Gamma'$ for the natural map $\pi:\mathfrak{O}^\times_{\mathfrak{f}p}\to Z$. So ignoring the effect of the action of the finite group $\mathfrak{O}^\times/O^\times$, we can decompose $\mathcal{R}$ into a product of subsets $\mathcal{R}_{\mathfrak{l}}$ for prime factors $\mathfrak{l}|\mathfrak{f}p$:

$$\bigsqcup_{\varepsilon\in\mathfrak{O}^\times/O^\times}\varepsilon\mathcal{R}\cong\prod_{\mathfrak{l}|\mathfrak{f}p}\mathcal{R}_{\mathfrak{l}}=\mathcal{R}'$$

with $\mathcal{R}_{\mathfrak{l}}\subset\mathfrak{O}^\times_{\mathfrak{l}}$.

The sum over $\mathcal{R}$ in (5.20) is still valid even if $\mathfrak{I}\neq 1$, and we have

$$\sum_{r\in\mathcal{R}}\psi(r)^{-1}E|\rho(r)^{-1} = (\mathfrak{O}^\times:O^\times)^{-1}\left(\sum_{r\in\mathcal{R}'}\psi(r)^{-1}E|\rho(r)^{-1}\right).$$

Defining the trace map by the summation of translation by $\rho(r)$ over $r\in\mathcal{R}_{\mathfrak{Q}}\cong C$ modulo center, we then have

$$\sum_{r\in\mathcal{R}}\psi(r)^{-1}E|\rho(r)^{-1} = (\mathfrak{O}^\times:O^\times)^{-1}\left(\sum_{r\in\mathcal{R}''}\psi(r)^{-1}E|\rho(r)^{-1}\right)|\mathrm{Tr},$$

where $\mathcal{R}''=\prod_{\mathfrak{l}\neq\mathfrak{q}}\mathcal{R}_{\mathfrak{l}}$. The prime factor of $(\mathfrak{O}^\times:O^\times)$ is either even or ramified in M, which is excluded by our assumption; so, division by the index is harmless for us (in the computation of the μ–invariant).

For simplicity, we write Φ for Φ°_+ and write $\Phi_{\mathfrak{q}}$ for the restriction of Φ to the factor $(O/\mathfrak{q})$. Then we see

$$\Phi_{\mathfrak{q}}(a)=\begin{cases}-1 & \text{if } a\in(O/\mathfrak{q})^\times,\\ N(\mathfrak{q})-1 & \text{if } a=0 \text{ in } O/\mathfrak{q}.\end{cases}$$

This shows that, $f=g|[\mathfrak{q}]-g$ for $g=E(\phi)$, where ϕ is the function Φ°_+ of outside–p–level $\mathfrak{f}/\mathfrak{q}$ defined for the character ψ_0 modulo $(\mathfrak{C}/\mathfrak{Q})p^r$ inducing the character: $Z\twoheadrightarrow\Delta\xrightarrow{\psi}W^\times$. We can check

$$g|T(\mathfrak{q}) = \left(1+\frac{\psi_0(\mathfrak{q})}{N(\mathfrak{q})}\right)g, \tag{5.30}$$

since the partial Fourier transform $P\phi$ at $\mathfrak{s}\mathfrak{f}$ is basically a constant multiple of $\psi_{0,\mathfrak{f}\mathfrak{s}} : (O/\mathfrak{f})^\times \times (O/\mathfrak{s})^\times \to W^\times$ (up to translation by an $\mathfrak{s}\mathfrak{f}$–adic unit) on $PV(\mathcal{L})_\mathfrak{f}/\operatorname{Ker}((i_\mathfrak{f}^{-1}, i'_\mathfrak{s})) \cong (O/\mathfrak{f}) \times (O/\mathfrak{s})$ for test objects $(\mathcal{L}, \lambda, i)$. Thus we see

$$\begin{aligned}\operatorname{Tr}(f) &= N(\mathfrak{q})g|T(\mathfrak{q}) - (N(\mathfrak{q})+1)g \\ &= N(\mathfrak{q})(1+\psi_0(\mathfrak{q})N(\mathfrak{q})^{-1})g - (N(\mathfrak{q})+1)g = (\psi_0(\mathfrak{Q})-1)g.\end{aligned}$$

This shows that the function in (5.20) just vanishes if $\psi_0(\mathfrak{Q}) = 1$.

The above argument works without assuming the principality of the prime $\mathfrak{Q}$, after summing up over Cl_M^+. We explain briefly the reason. For each split prime $\mathfrak{A} \in \mathcal{I}(\mathfrak{I}p)^+$, put $f_\mathfrak{A} = E(\Phi_+^\circ)|[\mathfrak{A}\mathfrak{A}^c]|\rho(\alpha_\mathfrak{A})^{-1}$ where $\alpha_\mathfrak{A}^{1-c} = [\mathfrak{A}]^-$. Then $f_\mathfrak{A}$ only depends on the class of $\mathfrak{A}$ in $CL_M = Cl_M(1)$. Then we find

$$g_\mathfrak{A} = E(\phi)|[\mathfrak{A}\mathfrak{A}^c]|\rho(\alpha_\mathfrak{A})^{-1} \in G_k(\Gamma_0(\mathfrak{f}/\mathfrak{q}; \mathfrak{a}\mathfrak{A}, \mathfrak{b}))$$

such that $f_\mathfrak{A} = g_{\mathfrak{A}_{\mathfrak{q}^{-1}}}|[\mathfrak{q}] - g_\mathfrak{A}$ and

$$g_\mathfrak{A}|T(\mathfrak{q}) = (1+\psi_0(\mathfrak{q})N(\mathfrak{q})^{-1})g_{\mathfrak{A}_\mathfrak{q}},$$

where $\mathfrak{A} \mapsto \mathfrak{A}_\mathfrak{q}$ is the permutation on $Cl_M^+(1)$ induced by $\mathfrak{A} \mapsto \mathfrak{Q}\mathfrak{A}$. We make a sum over the ideal classes in Cl_M^+:

$$\begin{aligned}&\sum_\mathfrak{A} \psi(\mathfrak{A})(g_{\mathfrak{A}_{\mathfrak{q}^{-1}}}|[\mathfrak{q}] - g_\mathfrak{A})|\operatorname{Tr} \\ &\qquad = \sum_\mathfrak{A} \psi(\mathfrak{A})N(\mathfrak{q})g_{\mathfrak{A}_{\mathfrak{q}^{-1}}}|T(\mathfrak{q}) - (N(\mathfrak{q})+1)\sum_\mathfrak{A}\psi(\mathfrak{A})g_\mathfrak{A} \\ &\quad = \sum_\mathfrak{A} \psi(\mathfrak{A})N(\mathfrak{q})(1+\psi_0(\mathfrak{q})N(\mathfrak{q})^{-1})g_\mathfrak{A} - (N(\mathfrak{q})+1)\sum_\mathfrak{A}\psi(\mathfrak{A})g_\mathfrak{A} \\ &\qquad = (\psi_0(\mathfrak{Q})-1)\sum_\mathfrak{A}\psi(\mathfrak{A})g_\mathfrak{A}.\end{aligned}$$

We get the following fact for inert primes $\mathfrak{I}$ observed first by Gillard ([G2] Proposition 2):

Proposition 5.3. *Let $\mathfrak{Q}$ be a prime of M inert over F and assume that $\mathfrak{C} = \mathfrak{C}'\mathfrak{Q}$ with $\mathfrak{C}' + \mathfrak{Q} = \mathfrak{O}$. Suppose the following two conditions are satisfied:*

(I1) *$\psi \mod \mathfrak{m}_W$ is imprimitive (induced by a character modulo $\mathfrak{C}'p^\infty$). Thus $\psi \equiv \psi_0 \mod \mathfrak{m}_W$ for a character ψ_0 of $Cl_M(\mathfrak{C}'p^\infty)$;*

(I2) *$\psi_0(\mathfrak{Q}) \equiv 1 \mod \mathfrak{m}_W$.*

Then the anti-cyclotomic branch $\varphi^-_{\psi,\mathfrak{f}}$ has positive μ–invariant. If further ψ itself is imprimitive induced by a character ψ_0 of $Cl_M(\mathfrak{C}'p^\infty)$ and $\mathfrak{I} = \mathfrak{Q}$ (so, $\mathfrak{C}'$ is made up of split primes), the invariant $\mu(\varphi^-_\psi)$ is given by the sum of the additive p–adic valuation of $(\psi_0(\mathfrak{Q})-1)$ and $\mu(\psi_0)$ as in (5.27).

The condition (I1) implies that the order of ψ is divisible by p if ψ is primitive at i; so, either $N(\mathfrak{q})-1$ or $N(\mathfrak{q})+1$ is divisible by p.

Proof. When ψ itself is imprimitive, by the above calculation, the positivity: $\mu(\varphi^-_\psi) > 0$ is clear. The last assertion is a consequence of the linear independence of $g_{\mathfrak{A}}$ (for $\mathfrak{A}$ running through Cl^+_M) over $\mathbb{F}$, which follow from an argument similar to the proof of Theorem 5.1. When ψ is primitive at $\mathfrak{Q}$, we choose a character ψ_0 of $Cl_M(\mathfrak{C}'p^\infty)$ with $\psi \equiv \psi_0 \mod \mathfrak{m}_W$. Then $\varphi^-_{\psi,\mathfrak{C}} \equiv \varphi^-_{\psi_0,\mathfrak{C}}$; so, the positivity of the μ–invariant of $\varphi^-_{\psi,\mathfrak{C}}$ follows from that of $\varphi^-_{\psi_0,\mathfrak{C}}$. □

Ramified primes $\mathfrak{I}$ can be treated modifying the above argument. Here we shall give a sketch of the argument; so, suppose that $\mathfrak{I}$ is a ramified prime. For simplicity, we assume that $\mathfrak{i} = \mathfrak{I} \cap F$ is unramified in $F/\mathbb{Q}$. By the definition of the level structure $i = i(\mathfrak{O})$, we have $i(\mathfrak{i}^{-1}/O) \oplus \mathfrak{I}^{-1}/\mathfrak{O} = \mathfrak{i}^{-1}/\mathfrak{O}$. Thus writing $\mathfrak{O} = \mathfrak{a}^* + \mathfrak{b}z = \mathcal{L}_z$ for fractional ideals $\mathfrak{a}$ and $\mathfrak{b}$ of F prime to $\mathfrak{c}ip$, we may assume that $(z) = \mathfrak{I}\mathfrak{x}$ for $\mathfrak{x}$ prime to $\mathfrak{I}$. The stabilizer in $SL_2(F)$ of the lattice $\mathfrak{O} = \mathcal{L}_z$ is given by $\Gamma_0(\mathfrak{i};\mathfrak{a},\mathfrak{b})$. Since $z\mathfrak{b} \subset \mathfrak{I}$ and $\mathfrak{b}$ is prime to $\mathfrak{i}$, we find that $\rho(\mathfrak{O}_\mathfrak{I})$ is made of matrices $\left(\begin{smallmatrix} a & \varpi_\mathfrak{i} b \\ b & a \end{smallmatrix}\right)$ for $a, b \in O_\mathfrak{i}$. We also suppose that $\mathfrak{C} = \mathfrak{I}$ for simplicity. Since complex conjugation acts on $(\mathfrak{O}/\mathfrak{I})^\times$ trivially, $\mathcal{D}^-$ is made of two elements 1 and z as above. In other words, $\mathfrak{x}^{1-c} = z^{1-c}$, and we may assume that the operator $\rho(z)^{-1} \circ [\mathfrak{x}\mathfrak{x}^c]$ is the action of the normalizer $\tau = \left(\begin{smallmatrix} 0 & \varpi_\mathfrak{i} \\ 1 & 0 \end{smallmatrix}\right)$ of $\Gamma_0(\mathfrak{i};\mathfrak{a},\mathfrak{b})$. Note that we have a natural map $Cl_F(\mathfrak{i}) \to Cl_M(\mathfrak{I})$. Let ψ be an imprimitive character modulo a p–power. Writing

$$f_\mathfrak{A} = E(\Phi^\circ_+)|[\mathfrak{A}\mathfrak{A}^c]|\rho(\alpha_\mathfrak{A})^{-1} = g_{\mathfrak{A}_\mathfrak{i}^{-1}}|[\mathfrak{i}] - g_\mathfrak{A}$$

for $g_\mathfrak{A}$ with $g_\mathfrak{A}|[\mathfrak{i}]|\tau = \psi_0(\mathfrak{I})g_{\mathfrak{A}_\mathfrak{i}}$, we do the same computation as in the inert case:

$$\sum_\mathfrak{A} \psi(\mathfrak{A})(g_{\mathfrak{A}_\mathfrak{i}^{-1}}|[\mathfrak{i}] - g_\mathfrak{A})|\tau$$
$$= \sum_\mathfrak{A} \psi(\mathfrak{A})\psi_0(\mathfrak{I})g_\mathfrak{A} - \sum_\mathfrak{A} \psi(\mathfrak{A})g_\mathfrak{A} = (\psi_0(\mathfrak{I}) - 1)\sum_\mathfrak{A} \psi(\mathfrak{A})g_\mathfrak{A}.$$

This formula (and its generalization for $\mathfrak{C} \neq \mathfrak{I}$) for imprimitive character proves the following fact whose proof is left to the reader:

Proposition 5.4. *Let $\mathfrak{Q}$ be a prime of M ramified over F and assume that $\mathfrak{C} = \mathfrak{C}'\mathfrak{Q}$ with $\mathfrak{C}' + \mathfrak{Q} = \mathfrak{O}$. Suppose the following two conditions are satisfied:*

(R1) *$\psi \mod \mathfrak{m}_W$ is imprimitive (induced by a character modulo $\mathfrak{C}'p^\infty$. Thus $\psi \equiv \psi_0 \mod \mathfrak{m}_W$ for a character ψ_0 of $Cl_M(\mathfrak{C}'p^\infty)$;*

(R2) *$\psi_0(\mathfrak{Q}) \equiv 1 \mod \mathfrak{m}_W$.*

Then the anti-cyclotomic branch $\varphi^-_{\psi,\mathfrak{C}}$ has positive μ–invariant. If further ψ itself is not primitive induced by a character ψ_0 of $Cl_M(\mathfrak{C}'p^\infty)$ and $\mathfrak{I} = \mathfrak{Q}$ (so $\mathfrak{C}'$ is made up of split primes), we have $\mu(\varphi^-_\psi)$ is given by the sum of the additive p–adic valuation of $(\psi_0(\mathfrak{Q}) - 1)$ and $\mu(\psi_0)$ as in (5.27).

Presumably, if one is able to carry out the computation of the q–expansion of the sum $E = E(\Phi^\circ)$, one should be able to get an exact formula of the μ–invariant of φ^-_ψ without restriction to its conductor. However, the q–expansion is

rather complicated, or at least, the process of computation looks rather involved (when ψ is primitive at some inert or ramified primes). This is natural since we have the cases of positive μ–invariant as described above. We hope to come back this question in future, hopefully proving the conjecture by Gillard asserting the vanishing of $\mu(\varphi_{\psi}^{-})$ (except in the case specified by (M1-3)) when ψ is primitive of order prime to p (see [G2] Conjecture).

6 Appendix: Correction to [HT]

Here is a table of misprints in [HT], and "P.3 L.5b" indicates fifth line from the bottom of the page three.

page and line	Read	Should Read
P.192 L.1b	$\prod_{\mathfrak{l}\|\mathfrak{I}} G((2\delta)^c, \lambda_{\mathfrak{l}}^{-1})$	$\prod_{\mathfrak{Q}\|\mathfrak{I}} G((2\delta)^c, \lambda_{\mathfrak{Q}}^{-1})$
P.194 L.13b	$x_{\mathfrak{q}}$	x_σ
P.199 L.5b	$\underline{\omega}(\Sigma) \otimes \underline{\omega}(\Sigma^c)$	$\underline{\omega}(\Sigma) \oplus \underline{\omega}(\Sigma^c)$
P.199 L.1b	$\delta(k, \rho_x)(f(X, \lambda, \omega, i))$	$\delta(k, \rho_x)^d(f(X, \lambda, \omega, i))$
P.200 (1.12)	$\underline{\omega}_{/\mathfrak{A}_p} \otimes U_{/\mathfrak{A}_p}$	$\underline{\omega}_{/\mathfrak{A}_p} \oplus U_{/\mathfrak{A}_p}$
P.201 L.3b	$\mathfrak{c}^{-1}$	$\mathfrak{c}^*$
P.202 L.1	$\mathfrak{ab})^*$	$(\mathfrak{ab})^*$
P.202 L.10	$\mathfrak{O}_{\mathrm{Z}}$	$\mathfrak{O}_{\mathfrak{Z}}$
P.204 L.7	$\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)$	$\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)[M]$
P.206 L.3b	$2\pi i(\mathfrak{a}^* z + \mathfrak{b})$	$2\pi i(\mathfrak{b}z + \mathfrak{a}^*)$
P.206 L.1b	$\sqrt{F_F}$	$\sqrt{\lvert D_F\rvert}$
P.207 L.10b	$\sum_{a\in\mathfrak{a}} C_k(1, z, s)$	$\sum_{a\in\mathfrak{a}} C_k(a, z, s)$
P.207 L.4b	$\mathbf{e}_F(x, a_0)$	$\mathbf{e}_F(xa_0)$
P.208 L.5	$\mathbf{e}_F(x, bz)$	$\mathbf{e}_F(xbz)$
P.211 L.1	$\mathfrak{Q}\|\mathfrak{F}$	$\mathfrak{Q}\|\mathfrak{F}p$
P.211 L.2	$(\mathfrak{FI})^{-2}\mathfrak{A}/\mathfrak{A}$	$(\mathfrak{Fi})^{-2}\mathfrak{A}/\mathfrak{A}$ $(\mathfrak{i} = \mathfrak{I} \cap F)$
P.213 L.3b	$\prod_{\mathfrak{L}\|\mathfrak{F}\mathfrak{l}}(1 - \lambda^*(\mathfrak{L} - \lambda^*(\mathfrak{L}))^{-1}$	$\prod_{\mathfrak{L}\|\mathfrak{F}\mathfrak{I}}(1 - \lambda^*(\mathfrak{L}))^{-1}$
P.214 L.12	$\lambda_\Sigma(x_p)$	$\lambda_\Sigma^{-1}(x_p)$
P.215 L.2	$\lambda_{\mathfrak{P}^c}^{-1}(a_{\mathfrak{P}^c)\lambda\mathfrak{P}}^{-1} a\mathfrak{P})$	$\lambda_{\mathfrak{P}^c}^{-1}(a\mathfrak{P}^c)\lambda_{\mathfrak{P}}^{-1}(a\mathfrak{P})$
P.215 (4.9)	$\frac{\int_{G_\infty} \widehat{\lambda} d\varphi}{\Omega_p^{m_0\Sigma+2}}$	$\frac{\int_{G_\infty} \widehat{\lambda} d\varphi}{\Omega_p^{m_0\Sigma+2d}}$
P.215 L.7b, 3b P.217 L.11b, 9b	$a^{-m_0\Sigma-2d(1-c)}$	$a^{-m_0\Sigma-d(1-c)}$
P.216 L.8	$\chi_{\mathfrak{P}}(a_{\mathfrak{P}}).N(\mathfrak{P})^{-1}$	$\chi_{\mathfrak{P}}(a_{\mathfrak{P}}) - N(\mathfrak{P})^{-1}$
P.235 L.1b	$\lambda_p(x)$	$\widehat{\lambda}(x)$
P.241 L.9	$\eta(\varpi_{\mathfrak{q}}) + \eta(\varpi_{\mathfrak{q}})$	$\eta(\varpi_{\mathfrak{q}}) + \eta'(\varpi_{\mathfrak{q}})$ or $\eta(\varpi_{\mathfrak{q}})$
P.241 L.14	$(1 - \overline{\alpha}_{\mathfrak{q}}\beta_{\mathfrak{q}}X)(1 - \alpha_{\mathfrak{q}}\overline{\beta}_{\mathfrak{q}}X)$	$(1 - \overline{\alpha}_{\mathfrak{q}}\beta_{\mathfrak{q}}X)(1 - X)$ $\times(1 - \alpha_{\mathfrak{q}}\overline{\beta}_{\mathfrak{q}}X)$
P.241 L.14	spherical	minimal principal
P.241 L.15	special	minimal special
P.241 L.15b	$L(s, \mathbf{f})$	$L(s, Ad(\mathbf{f}))$
P.241 L.2b	$\mathfrak{q} \in \Xi$	$\mathfrak{q} \in \Xi = \Xi_p \cup \Xi_s$
P.245 L.4b	$\sum_{\mathfrak{q}\|C}$	$\prod_{\mathfrak{q}\|C}$
P.249 L.1b	$(\nu_Q^0 c)^{-1}$	$(\nu_Q \circ c)^{-1}$
P.250 L.8b	$\mathbf{2}_F$	$\mathbf{e}_F$
P.250 L.4b	$(\theta(\lambda_P))$	$W'(\theta(\lambda_P))$
P.251 L.10	$(\lambda_P^0 c)$ at three places	$(\lambda_P \circ c)$
P.256 L.14b	$\psi^*(C_f)$	$\psi_*(C_f)$
P.257 L.12b	$W\mathbf{P}^m$	$X\mathbf{P}^m$

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