Companion forms and classicality in the $\mathrm{GL}_2(\mathbb{Q})$-case

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To Prof. T.C. Vasudevan for his sixtieth birthday

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1 Introduction

In this paper we give proofs of the existence of companion forms in the elliptic modular case. We study two situations. The first concerns modulo p classical cusp eigenforms in weight $k \in [2, p-1]$; there, our proof is based on the modulo p étale-crystalline comparison theorem, in the spirit of the approaches of [14], [11] and [4] (instead of the original one [17]). The second concerns p-adic overconvergent forms in weight $k = 1$. In this case the proof is based on a deformation theory argument, as in the paper of Buzzard-Taylor [7] and its generalization [8], both devoted to the study of Artin's conjecture in degree 2. Our approach of the first part differs from the previous ones by its use of integral and rigid structures of the dual BGG complex which clarify in our opinion some calculations of [11] and [4]. Let us recall that we already used this tool in an essential way in another situation, in [27]. This work is a translation in the elliptic modular case of a work in progress in the genus two Siegel modular case, where we establish similar results in the course of the study of Yoshida's conjecture ([34] and [32]).

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2 Modular forms with p-small weight $k \geq 2$ and cohomology

Let $N \geq 1$ and $\Gamma_1(N) = \{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}); d \equiv 1, c \equiv 0 \pmod{N}\}$.

Let f be a fixed cusp eigenform of level group $\Gamma_1(N)$ with $N \geq 1$ prime to p, of weight $k \geq 2$ and character $\epsilon : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$. We shall write in the sequel $k = n + 2$ with $n \geq 0$. Let p be a prime which does not divide N and such that $k < p$ that is, $k - 1 = n + 1 < p - 1$. We fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$; we endow it with the p-adic valuation ord_p such that $\mathrm{ord}_p(p) = 1$. We fix an embedding ι_p of the field $\overline{\mathbb{Q}}$ of algebraic complex numbers into $\overline{\mathbb{Q}}_p$. We assume that f is ordinary with respect to this embedding; this means that $f|T_p = a_p \cdot f$ and $\mathrm{ord}_p(\iota_p(a_p)) = 0$. Let us put $a_p = \alpha + \beta$ with $\alpha\beta = p^{k-1}\epsilon(p)$ and $\mathrm{ord}_p(\iota_p(\alpha)) = 0$.

Let $\rho_f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to {}_2(\overline{\mathbb{Q}}_p)$ be the p-adic Galois representation associated to (f, p). Let us fix a p-adic discrete valuation subring $\mathcal{O} \subset \overline{\mathbb{Q}}_p$ containing the Hecke eigenvalues of f and over which the representation ρ_f is defined. Let $\chi : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Z}_p^\times$ resp. $\omega : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{F}_p^\times$, be the p-adic, resp. modulo p cyclotomic character.

Recall that P. Deligne in a letter to Serre (May 28th, 1974) proved that the assumption of ordinarity of f at ι_p implies that the restriction of ρ_f to a decomposition group D_p at p has a special form. For any $\gamma \in \mathcal{O}^\times$, we define $\xi(\gamma) : D_p \to \mathcal{O}^\times$ to be the unramified character sending a geometric Frobenius at p to γ. Then,

$$\rho_f|_{D_p} \sim \begin{pmatrix} \xi(\alpha) & * \\ 0 & \xi(\frac{\beta}{p^{k-1}})\chi^{1-k} \end{pmatrix}$$

Let ϖ be a uniformizing parameter of $\mathcal{O}$ and $\mathbb{F} = \mathcal{O}/(\varpi)$ its residue field. We assume throughout this paper that the residual representation $\overline{\rho}_f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{F})$ is irreducible. The first part of the paper (Sections 2,3,4) is devoted to the proof of the following Theorem, due to Gross [17].

Theorem 1 *If $\overline{\rho}_f$ is tamely ramified, and if $k < p-1$, then there exists an ordinary cusp eigenform g of level $\Gamma_1(N)$ and weight $k' = p+1-k$ with Hecke eigenvalues b_n, such that for any $n \geq 1$, prime to pN, we have $a_n \equiv n^{k-1} b_n \pmod{\mathfrak{p}}$, where $\mathfrak{p}$ is the prime of $\overline{\mathbb{Z}}$ determined by ι_p.*

If $k = p-1$, the same conclusion holds except that the ordinary cusp form g of weight $k' = 2$ may have level N or Np, in which case we can replace it by a form of weight $p+1$ and level N.

This can be reformulated as a confirmation of the (now proved) Serre's Residual Modularity Conjecture as follows. Indeed, by Deligne's formula above, we have

$$\overline{\rho}_f|_{D_p} \sim \begin{pmatrix} \xi(\overline{\alpha}) & * \\ 0 & \xi(\overline{\frac{\beta}{p^{k-1}}})\omega^{1-k} \end{pmatrix}$$

with the upper right shoulder being zero; consider the Galois representation $\overline{\rho} = \overline{\rho}_f \otimes \omega^{k-1}$; it is odd, continuous, irreducible and its restriction to the inertia subgroup I_p is conjugated to

$$\overline{\rho}|_{I_p} \sim \begin{pmatrix} 1 & * \\ 0 & \omega^{1-k'} \end{pmatrix}$$

(with $* = 0$ and $k' = p+1-k \in [2, p-1]$), so its Fontaine-Serre's weights are 0 and $k'-1$; it must therefore come from a ι_p-ordinary cusp eigenform g of level N prime to p, and weight $k' < p$:

$$\overline{\rho}_f \otimes \omega^{k-1} = \overline{\rho}_g.$$

Comparing the traces on Frobenius elements at $\ell \not| Np$, and using Brauer-Nesbitt theorem, we see that this statement is equivalent to Gross theorem as stated above: $a_n \equiv b_n n^{k-1} \pmod{\mathfrak{p}}$ for all n's prime to Np.

2.1 Integral modular forms

We briefly recall some classical definitions. More details can be found in [24], [18], [19] and [20]. Let X be the modular curve defined over $\mathbb{Z}[\frac{1}{N}]$ which classifies elliptic curves endowed with a point of order exactly N. If $N > 4$, it is a fine moduli scheme. We assume this for the moment. Then, it is a geometrically connected smooth quasiprojective scheme endowed with a universal elliptic curve $f : \mathbf{E} \to X$. Let $\overline{X}$ be the arithmetic smooth compactification over $\mathbb{Z}[\frac{1}{N}]$ of X obtained by adding the relative divisor $C = \mathbf{Cusps}$ of cusps (see [24]); the divisor C is finite flat over $\mathbb{Z}[\frac{1}{N}]$. This compactification is endowed with a universal one dimensional semi-abelian group scheme (the neutral connected component of the universal generalized elliptic curve) $f : \mathcal{G} \to \overline{X}$ with a section of order exactly N. Let e be the zero section of f. The sheaf of relative differentials $\omega = e^*\Omega_{\mathcal{G}/\overline{X}}$ is locally free of rank one. Let $\pi : \mathcal{T}_\omega \to \overline{X}$ be the $\mathbb{G}_m$-torsor given by $\mathcal{T}_\omega = \mathrm{Isom}_{\overline{X}}(\mathcal{O}_{\overline{X}}, \omega)$. The $\mathcal{O}_{\overline{X}}$-Module $\pi_*\mathcal{O}_{\mathcal{T}_\omega}$ is a representation of the group scheme $\mathbb{G}_m$; any such representation is totally decomposed by the characters of the torus $X^*(\mathbb{G}_m) = \mathbb{Z}$; for any $k \in \mathbb{Z}$, let $\chi_k : t \mapsto t^k$ be the corresponding character of $\mathbb{G}_m$. Then we have $\pi_*\mathcal{O}_{\mathcal{T}_\omega} = \bigoplus_{k\in\mathbb{Z}} \pi_*\mathcal{O}_{\mathcal{T}_\omega}[\chi_{-k}]$. Let us put $\omega^k = \pi_*\mathcal{O}_{\mathcal{T}_\omega}[\chi_{-k}]$; it is an invertible sheaf; we define the subsheaf $\omega_k = \omega^k(-\mathbf{Cusps})$. For any $\mathbb{Z}[\frac{1}{N}]$-algebra R, we define the modules of arithmetic modular resp. cuspidal forms by

$$M_k(\Gamma, R) = H^0(\overline{X} \times R, \omega^k) \quad \text{and} \quad S_k(\Gamma, R) = H^0(\overline{X} \times R, \omega_k).$$

2.2 Hodge filtration and Gauss-Manin connection

We recall definitions by stressing the fact that they are valid over the ring $\mathbb{Z}[\frac{1}{N}]$, hence over any $\mathbb{Z}[\frac{1}{N}]$-algebra.

Let $\overline{f} : \overline{\mathbf{E}} \to \overline{X}$ be the Kuga-Sato compactification of $\mathbf{E} \to X$; the relative surface $\overline{\mathbf{E}}$ is regular and the smooth locus of $\overline{f}$ is $\mathcal{G} \subset \overline{\mathbf{E}}$; moreover, the inverse image $D = \overline{f}^{-1}(C) = \overline{\mathbf{E}} \backslash \mathbf{E}$ is a divisor with normal crossings in $\overline{\mathbf{E}}$. Let $\mathcal{H} = R^1\overline{f}_*\Omega^\bullet_{\overline{\mathbf{E}}/\overline{X}}(d\log D/C)$. It is a rank two locally free $\mathcal{O}_{\overline{X}}$-sheaf together with a two-step filtration -the Hodge filtration, and a log-connection -the Gauss-Manin connection, $\nabla : \mathcal{H} \to \mathcal{H} \otimes \Omega_{\overline{X}}(d\log C)$. Let us recall the definition of this filtration and this log-connection.

First, the Hodge filtration is simply given by the short exact sequence

$$0 \to \overline{f}_*\Omega^1_{\overline{\mathbf{E}}/\overline{X}}(d\log D/C) \to R^1\overline{f}_*\Omega^\bullet_{\overline{\mathbf{E}}/\overline{X}}(d\log D/C) \to R^1\overline{f}_*\mathcal{O}_{\overline{\mathbf{E}}} \to 0$$

that is, $\mathrm{Fil}^i = \mathcal{H}$ if $i \leq 0$, $\mathrm{Fil}^1 = \overline{f}_*\Omega^1_{\overline{\mathbf{E}}/\overline{X}}(d\log D/C)$, and $\mathrm{Fil}^i = 0$ if $i > 1$. Note that a trivial calculation in local charts at cusps shows that

$\overline{f}_*\Omega^1_{\overline{\mathbf{E}}/\overline{X}}(d\log D/C) = e^*\Omega_{\mathcal{G}/\overline{X}} = \omega$. Then by Serre duality, we have $R^1\overline{f}_*\mathcal{O}_{\overline{\mathbf{E}}} \cong \omega^{-1}$, hence, $gr^0\mathcal{H} = \omega^{-1}$ and $gr^1\mathcal{H} = \omega$.

The Gauss-Manin connection is defined as follows. Consider the short exact sequence

$$0 \to \overline{f}^*\Omega^1_{\overline{X}}(d\log C) \to \Omega^1_{\overline{\mathbf{E}}}(d\log D) \to \Omega^1_{\overline{\mathbf{E}}/\overline{X}}(d\log D/C) \to 0$$

It gives rise to a filtration of the log-de Rham complex $\Omega^\bullet_{\overline{\mathbf{E}}}(d\log D)$ by $\mathrm{Fil}^i = \mathrm{Im}\,(\overline{f}^*\Omega^i_{\overline{X}}(d\log C) \otimes \Omega^{\bullet-i}_{\overline{\mathbf{E}}}(d\log D) \to \Omega^\bullet_{\overline{\mathbf{E}}}(d\log D))$ for $i = 0, 1, 2$.

The associated graded pieces are

$$gr^i = \overline{f}^*\Omega^i_{\overline{X}}(d\log C) \otimes \Omega^{\bullet-i}_{\overline{\mathbf{E}}/\overline{X}}(d\log D/C)$$

$(i = 0, 1)$.

In particular we have a short exact sequence of complexes

$$0 \to \overline{f}^*\Omega^1_{\overline{X}}(d\log C)\otimes\Omega^{\bullet-1}_{\overline{\mathbf{E}}/\overline{X}}(d\log D/C) \to \Omega^\bullet_{\overline{\mathbf{E}}}(d\log D) \to \Omega^\bullet_{\overline{\mathbf{E}}/\overline{X}}(d\log D/C) \to 0$$

The log-connection ∇ is defined as the connecting morphism associated to $R\overline{f}_*$ applied to this short exact sequence; it reads:

$$R^1\overline{f}_*\Omega^\bullet_{\overline{\mathbf{E}}/\overline{X}}(d\log D/C) \to R^2\overline{f}_*(\Omega^{\bullet-1}_{\overline{\mathbf{E}}/\overline{X}}(d\log D/C) \otimes \overline{f}^*\Omega^1_{\overline{X}}(d\log C))$$

that is,

$$\nabla : \mathcal{H} \to \mathcal{H} \otimes \Omega^1_{\overline{X}}(d\log C)$$

The locally free sheaf $\mathcal{H}_n = \mathrm{Sym}^n\mathcal{H}$ of rank $n+1$ inherits a "tensor product" Hodge filtration $(\mathrm{Fil}^i_n)_i$ and a "tensor product" log-connection ∇_n; the filtration (Fil^i_n) $(= i = 0, \ldots n+1)$ is given by $\mathrm{Fil}^i_n\mathcal{H}_n = \sum_{k_1,\ldots,k_n} \mathrm{Fil}^{k_1}\mathcal{H} \otimes \ldots \otimes \mathrm{Fil}^{k_n}\mathcal{H}$ the sum being taken over all n-uples of integers k_j's, equal to 0, 1 or 2, such that $k_1 + \ldots + k_n = i$. It follows easily from the definition that $gr^i\mathcal{H}_n = \omega^{-n+2i}$ for $i = 0, \ldots, n$, and $gr^i\mathcal{H}_n = 0$ otherwise. Similarly, the log-connection $\nabla_n : \mathcal{H}_n \to \mathcal{H}_n \otimes \Omega^1_{\overline{X}}(d\log C)$ is given by $\nabla_n(v_1 \otimes \ldots \otimes v_n) = \sum_{i=1}^n v_1 \otimes \ldots \otimes \nabla(v_i) \otimes \ldots \otimes v_n$. It obviously satisfies Griffiths transversality: $\nabla_n(\mathrm{Fil}^i_n) \subset \mathrm{Fil}^{i-1}_n$. We denote by $\mathcal{H}(n)$ the de Rham complex $\nabla_n : \mathcal{H}_n \to \mathcal{H}_n \otimes \Omega^1_{\overline{X}}(d\log C)$ concentrated in degrees n and $n+1$. It is filtered by the absolute Hodge filtration ${\mathrm{Fil}_{\mathrm{abs}}}^i$ $(i = 0, \ldots, n+2)$ defined as the convolution of the "filtration bête" on $\mathcal{O}_{\overline{X}} \to \Omega_{\overline{X}}(d\log C)$ with $\mathrm{Fil}^\bullet_n$, namely:

$${\mathrm{Fil}_{\mathrm{abs}}}^i\mathcal{H}(n) : \mathrm{Fil}^i_n \to \mathrm{Fil}^{i-1}_n \otimes \Omega_{\overline{X}}(d\log C)$$

for $i = 0, \ldots, n+1$ and ${\mathrm{Fil}_{\mathrm{abs}}}^{n+2}\mathcal{H}(n) = 0$. Hence, it gives rise to a sequence of complexes ${\mathrm{gr}_{\mathrm{abs}}}^i\mathcal{H}(n)$ (with differential ${\mathrm{gr}_{\mathrm{abs}}}^i\nabla_n$):

$$\begin{array}{cccc}
{\mathrm{gr}_{\mathrm{abs}}}^0: & \omega^{-n} & \to & 0 \\
{\mathrm{gr}_{\mathrm{abs}}}^1: & \omega^{-n+2} & \to & \omega^{-n}\otimes\Omega_{\overline{X}}(d\log C) \\
\vdots & & & \\
{\mathrm{gr}_{\mathrm{abs}}}^i: & \omega^{-n+2i} & \to & \omega^{-n+2i-2}\otimes\Omega_{\overline{X}}(d\log C) \\
\vdots & & & \\
{\mathrm{gr}_{\mathrm{abs}}}^n: & \omega^{n} & \to & \omega^{n-2}\otimes\Omega_{\overline{X}}(d\log C) \\
{\mathrm{gr}_{\mathrm{abs}}}^{n+1}: & 0 & \to & \omega^{n}\otimes\Omega_{\overline{X}}(d\log C)
\end{array}$$

We then remark that for $i = 1, \ldots, n$, the differential $gr^i\nabla_n$ of the i-th complex ${\mathrm{gr}_{\mathrm{abs}}}^i\mathcal{H}(n)$ is an isomorphism. More precisely, the construction of the Kodaira-Spencer homomorphism $KS : \omega \to \omega^{-1}\otimes\Omega^1_{\overline{X}}(d\log C)$ from the Gauss-Manin connection shows that ${\mathrm{gr}_{\mathrm{abs}}}^i\nabla_n = \mathrm{Id}_{\omega^{-n+2i-1}}\otimes KS$. Since KS is an isomorphism, the same holds for $gr^i\nabla_n$.

Let us make these data explicit.

Recall the situation for $n = 0$. For each $\mathbb{Z}[\frac{1}{N}]$-affine neighborhood V of the divisor C of cusps in $\overline{X}$, we consider the affine covering (X, V) of $\overline{X}$; we form the Cech bicomplex :

$$\begin{array}{ccc}
\mathcal{O}_{\overline{X}}(X\cap V) & \stackrel{d}{\to} & \Omega^1_{\overline{X}}(X\cap V) \\
\uparrow & & \uparrow \\
\mathcal{O}_{\overline{X}}(X)\oplus\mathcal{O}_{\overline{X}}(V) & \stackrel{d}{\to} & \Omega^1_{\overline{X}}(X)\oplus\Omega^1_{\overline{X}}(d\log C)(V)
\end{array}$$

The associated total complex (in degrees $0, 1, 2$) calculates $H^1_{log-dR}(\overline{X})$.

Similarly, for $n > 0$, the total complex (in degrees $n, n+1, n+2$) associated to the Cech bicomplex

$$\begin{array}{ccc}
\mathcal{H}_n(X\cap V) & \stackrel{\nabla_n}{\to} & \mathcal{H}_n\otimes\Omega^1_{\overline{X}}(X\cap V) \\
\uparrow & & \uparrow \\
\mathcal{H}_n(X)\oplus\mathcal{H}_n(V) & \stackrel{\nabla_n}{\to} & \mathcal{H}_n\otimes\Omega^1_{\overline{X}}(X)\oplus\mathcal{H}_n\otimes\Omega^1_{\overline{X}}(d\log C)(V)
\end{array}$$

calculates the $\mathbb{Z}[\frac{1}{N}]$-module $H^\bullet_{log-dR}(\overline{X}, \mathcal{H}_n)$. This cohomology module is filtered by the total Hodge filtration on the complex $\mathcal{H}(n)$ defined by the subcomplexes (concentrated in degrees n and $n+1$) $\mathrm{Fil}^i\mathcal{H}(n) = \mathrm{Fil}^i\mathcal{H}_n \oplus \mathrm{Fil}^{i-1}\mathcal{H}_n\otimes\Omega^1_{\overline{X}}(d\log C)$ for $i = 0, \ldots, n+1$; moreover, for $i = 1, \ldots, n$, $gr^i\mathcal{H}(n) = \omega^{-n+2i}\oplus\omega^{-n+2i-2}\otimes\Omega^1_{\overline{X}}(d\log C)$, while $gr^0\mathcal{H}(n) = \omega^{-n}\oplus 0$ and

$gr^{n+1}\mathcal{H}(n) = 0 \oplus \omega^n \otimes \Omega^1_{\overline{X}}(d\log C)$ We can now determine the spectral sequence associated to this filtration

$$E_1^{i,j} = H^{i+j}(\overline{X}, gr^i\mathcal{H}(n)) \Rightarrow H^{i+j-n}_{log-dR}(\overline{X}, \mathcal{H}_n)$$

The determination above of $gr^i\nabla_n$ leads to the

Corollary 2.1 *We have a natural short exact sequence of* $\mathbb{Z}[\frac{1}{N}]$*-modules*

$$0 \to E_1^{n+1,0} \to H^1_{log-dR}(\overline{X}, \mathcal{H}_n) \to E_1^{0,n+1} \to 0$$

where $E_1^{n+1,0} = H^0(\overline{X}, \omega^{n+2})$ *and* $E_1^{0,n+1} = H^1(\overline{X}, \omega^{-n})$*. Here natural means compatible to algebraic correspondences.*

Proof: Indeed, since $gr^i\nabla_n$ are isomorphisms for $i = 0, \ldots, n$, we notice that $E_1^{i,j} = 0$ unless $i = 0$ and $j = n+1$ or $i = n+1$ and $j = 0$, in which case $E_1^{n+1,0} = H^1(\overline{X}, gr^{n+1}\mathcal{H}(n)) = H^0(\overline{X}, \omega^n \otimes \Omega^1_{\overline{X}}(d\log C)) = H^0(\overline{X}, \omega^{n+2})$ by Kodaira-Spencer isomorphism. Similarly, $E_1^{0,n+1} = H^1(\overline{X}, gr^0\mathcal{H}(n))$ which is equal to $H^1(\overline{X}, \omega^{-n})$.

3 de Rham cohomology and dual BGG complex

Let p be a prime number not dividing $2N$ and such that $n+1 < p-1$. We fix as base ring the $\mathbb{Z}[\frac{1}{N}]$-algebra $\mathbb{Z}_p$. In this section all modules and sheaves of modules are $\mathbb{Z}_p$-modules. We recall how to compute the Hodge filtration on $\mathcal{H}_n$ and on $H^1_{log-dR}(\overline{X}, \mathcal{H}_n)$ in group-theoretic terms, via the dual BGG complex.

3.1 Relative Koszul and BGG complexes

For any affine $\mathbb{Z}_p$-group scheme $H = \operatorname{Spec} A$, let $\mathfrak{h}$ be its Lie algebra, $U\mathfrak{h}$ its universal enveloping algebra, and $\mathcal{U}(H)$ its distribution algebra ([21] Chapter 7, or [12] II.1.12). Let J be the kernel of the counit character $A \to \mathbb{Z}_p$; recall that $\mathcal{U}(H) = \bigcup_k \mathcal{U}_k(H)$ where $\mathcal{U}_k(H) = \operatorname{Hom}_{\mathbb{Z}_p}(A/J^{k+1}, \mathbb{Z}_p)$. The product (of convolution) of two distributions $\mu, \nu : A \to \mathbb{Z}_p$ is the composition of their tensor product $\mu \otimes \nu : A \otimes A \to \mathbb{Z}_p$ with the comultiplication $c : A \to A \otimes A$. Since $\mathcal{U}_1(H) = \operatorname{Hom}_{\mathbb{Z}_p}(A/J^2, \mathbb{Z}_p) = \mathbb{Z}_p \oplus \mathfrak{h}$, there is a canonical filtered algebras homomorphism $U(\mathfrak{h}) \to \mathcal{U}(H)$; it is an isomorphism over a field of characteristic zero, hence induces an injective $\mathbb{Z}_p$-algebra homomorphism $U\mathfrak{h} \hookrightarrow \mathcal{U}(H)$ Its reduction modulo p, however, is not injective: over a field of characteristic p, the kernel is the ideal $U\mathfrak{h}^{[p]}$ generated by p-th powers (see [21] II.1.12). This implies therefore that for any $k < p$, the injection $U_k\mathfrak{h} \hookrightarrow \mathcal{U}_k(H)$ is actually an isomorphism.

For any representation V of H defined over $\mathbb{Z}_p$, the natural action of $U\mathfrak{h}$ on V extends naturally to $\mathcal{U}(H)$. Namely, consider the comorphism $c_V : A(V) \to A \otimes_{\mathbb{Z}_p} A(V)$ of the action $H \times V \to V$ and for any $\mu \in \mathcal{U}(H)$, let us form $(\mu \otimes Id) \circ c_V$; this is the comorphism of the action of μ on V.

Recall also the Invariance Theorem ([12] II.4.6) that the ring of left-invariant differential operators on H is isomorphic to $\mathcal{U}(H)$.

If H is a split Chevalley group over $\mathbb{Z}_p$, one can give a rather explicit description of the inclusions $U\mathfrak{h} \subset \mathcal{U}(H) \subset U\mathfrak{h} \otimes \mathbb{Q}_p$. Namely, let $B = (Z_i, H_j, X_k)$ be a Chevalley basis of $\mathfrak{h}$ (with Z_i central, H_j semisimple and X_k nilpotent) then, $\mathcal{U}(H)$ is generated by the divided powers $\frac{X_k^n}{n!}$, and the divided difference products $\begin{pmatrix} Z_i \\ n \end{pmatrix}$ and $\begin{pmatrix} H_j \\ n \end{pmatrix}$ for all $n \geq 0$ (see [21], II.1.12 and [9] or [3] Chapt.VIII,Sect.12).

Consider the $\mathbb{Z}_p$-group scheme $G = \mathrm{GL}_2$; let T be its standard torus, consisting of diagonal matrices, and B its standard Borel, consisting of upper triangular matrices in G. Let V be the standard two dimensional representation of G and $V_n = \mathrm{Sym}^n V$; under the assumption $n < p$, this module can be called unambiguously the irreducible $\mathbb{Z}_p$-representation of highest weight n (see [29] Sect. 1.9, Lemma. Let $\mathfrak{g}$, $\mathfrak{b}$ resp. $\mathfrak{t}$ be the $\mathbb{Z}_p$-Lie algebra of G, B resp. of T. For any $\mathbb{Z}_p$-representations Y of G and Y' of B, recall the Garland-Lepowsky "tensor identity" (see [15] Prop.1.7 and [29] Sect.2.2). It provides a canonical isomorphism of $U\mathfrak{g}$-modules (and similarly of $\mathcal{U}(G)$-modules):

$$(TI1) \quad (U\mathfrak{g} \otimes_{U\mathfrak{b}} Y') \otimes_{\mathbb{Z}_p} Y \cong U\mathfrak{g} \otimes_{U\mathfrak{b}} (Y' \otimes_{\mathbb{Z}_p} Y|_{\mathfrak{b}})$$

resp.

$$(TI2) \quad (\mathcal{U}(G) \otimes_{\mathcal{U}(B)} Y') \otimes_{\mathbb{Z}_p} Y \cong \mathcal{U}(G) \otimes_{\mathcal{U}(B)} (Y' \otimes_{\mathbb{Z}_p} Y|_{\mathfrak{b}})$$

where $Y|_{\mathfrak{b}}$ denotes the $\mathfrak{b}$-module restriction of the action of $\mathfrak{g}$ on Y to $\mathfrak{b}$.

Consider the natural morphism $\mathfrak{g}/\mathfrak{b} \to U\mathfrak{g} \otimes_{U\mathfrak{b}} \mathbb{Z}_p$ which sends $\overline{X}$ to $X \otimes 1$, where $X \in \mathfrak{g}$ is any lifting of $\overline{X} \in \mathfrak{g}/\mathfrak{b}$. It gives rise to a resolution by (relative) Verma modules of the trivial representation $\mathbb{Z}_p$, called the relative Koszul resolution (see [1] or [29] Sect.2.2).

$$0 \to U\mathfrak{g} \otimes_{U\mathfrak{b}} \mathfrak{g}/\mathfrak{b} \overset{d_0}{\to} U\mathfrak{g} \otimes_{U\mathfrak{b}} \mathbb{Z}_p \overset{d_{-1}}{\to} \mathbb{Z}_p \to 0$$

where d_{-1} is induced by the augmentation character $U\mathfrak{g} \to \mathbb{Z}_p$.

By tensoring by a representation Y and applying the isomorphisms (TI1), one obtains a resolution of the $\mathfrak{g}$-representation Y (relative Koszul complex):

$$U\mathfrak{g} \otimes_{U\mathfrak{b}} (\mathfrak{g}/\mathfrak{b} \otimes_{\mathbb{Z}_p} Y|_{\mathfrak{b}}) \xrightarrow{d_0} U\mathfrak{g} \otimes_{U\mathfrak{b}} Y|_{\mathfrak{b}}$$

with the maps $d_0(\xi \otimes \overline{X} \otimes v) = \xi X \otimes v - \xi \otimes Xv$ and $d_{-1}(\xi \otimes v) = \xi v$.

For purpose of coherence with the Hodge filtration later, we consider this complex as placed in degree $-(n+1)$ and $-n$.

Replacing $U\mathfrak{h}$ by $\mathcal{U}(H)$ for $H = G, B$, and using (TI2) instead of (TI1), we get another relative Koszul complex. We shift it by n, so that it is now concentrated in degrees $-1-n$ and $-n$. The resulting complex is denoted $\mathcal{S}(G, B, Y)$:

$$\mathcal{S}(G,B,Y) \quad \mathcal{U}(G) \otimes_{\mathcal{U}(B)} (\mathfrak{g}/\mathfrak{b} \otimes Y|_{\mathfrak{b}}) \to \mathcal{U}(G) \otimes_{\mathcal{U}(B)} Y$$

of generalized relative Verma $\mathbb{Z}_p$-modules (in the terminology of [29] Sect.4.1). Since we deal with $G = \mathrm{GL}_2$, so that G/B is one-dimensional, it turns out that this complex is actually a resolution of the G-representation Y (by generalized relative Verma modules), although it is not so for higher dimensional Chevalley groups.

Let (X_-, X_+, H, Z) be the standard basis of the Lie algebra $\mathfrak{g}$ of G, with $X_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $X_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $Z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; let $S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $W = \{1, S\}$; note that $ad(S)(H) = -H$; finally, let $\mathcal{Z}$ be the center of $\mathcal{U}(G)$; as mentioned previously, it is generated by $\begin{pmatrix} Z \\ n \end{pmatrix}$ $(n \geq 0)$ and the center $\mathcal{Z}'$ of $\mathcal{U}(G')$ (distribution algebra of the derived group G' of G);

Consider the (arithmetic) Harish-Chandra isomorphism

$$\gamma_o : \mathcal{Z}' \otimes \mathbb{Q}_p \cong \mathbb{Q}_p[H]^W = \mathbb{Q}_p[\frac{1}{2}H^2 + H], \quad z = P(H) + y.X_+ \mapsto \gamma_o(z) = P(H),$$

where the action of the symmetry $S \in W$ is by $H \mapsto -2 - H$. The image by γ_o of the Casimir element $C = X_+X_- + X_-X_+ + \frac{1}{2}H^2$ is $\frac{1}{2}H^2 + H$.

Recall that the infinitesimal character of a Verma module $\mathcal{M}$ over $\mathbb{Z}_p$ or of a $\mathfrak{g}$-irreducible $\mathbb{Z}_p$-representation Y is the character of $\mathcal{Z}$ giving its action on $\mathcal{M}$ resp. Y.

By the very definition of the Harish-Chandra isomorphism, the infinitesimal character $\chi_{(n;c)}$ of $\mathcal{M}_{(n;c)} = \mathcal{U}(G) \otimes_{\mathcal{U}(B)} \mathbb{Z}_p w_{n,c}$ for $w_{n,c}$ a vector such that $Hw_{n,c} = nw_{n,c}$, $Zw_{n,c} = cw_{n,c}$ and $X_+ w_{n,c} = 0$, $(n, c \in \mathbb{Z})$, is given by $\chi_{(n;c)}(C) = \frac{1}{2}n^2 + n$, $\chi_{(n;c)}(Z) = c$.

Moreover, the Verma modules $\mathcal{M}_{(n;c)}$ and $\mathcal{M}_{(n';c)}$ have the same infinitesimal character if and only if $c' = c$ and $n' = n$ or $n' = -2-n$; in particular, the infinitesimal character $\chi_Y : \mathcal{Z} \to \mathbb{Z}_p$ of an irreducible representation (Y, ρ_Y) of G is uniquely determined by the pair of eigenvalues $(\rho_Y(H); \rho_Y(Z))$ of (H, Z) on a highest weight vector $w \in Y$: if $\rho_Y(H) = n$ and $\rho_Y(Z) = c$, then $\chi_Y(C) = \frac{1}{2}n^2 + n$ (and $\chi_Y(Z) = c$).

Let us describe explicitely the subcomplex of $\mathcal{S}(G, B, V_n^\vee)$ on which $\mathcal{Z}$ acts by the infinitesimal character of $V_n^\vee$ (up to semisimplification). Via the description above, the infinitesimal character of V_n is given by the pair $(n; n)$ hence that of $V_n^\vee$ is described by $(n; -n)$. On the other hand, for any pair of integers $(m, c) \in \mathbb{Z}^2$ such that $c \equiv m \pmod 2$, let $W_{(m;c)}$ be the one-dimensional representation of T given by the character $\mathrm{diag}(t_1, t_2) \mapsto t_1^m (t_1 t_2)^{\frac{c-m}{2}}$. Recall that $\mathrm{Hom}_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{\mathcal{U}(B)} W_{(m;c)}, \mathcal{U}(G) \otimes_{\mathcal{U}(B)} W_{(m';c')})$ does not vanish if and only if $c = c'$ and $m = m'$ or $m = -m' - 2$, in which cases it is free of rank one (see for instance [16]). For $m' = n$ and $m = -n-2$, a basis of this module is provided by $\xi \otimes 1 \mapsto \xi X_-^{n+1} \otimes 1$. Indeed, for any choice of a basis w_k of $W_{(k;c)}$, the map $w_{-n-2} \mapsto X_-^{n+1} \otimes w_n$ is B-linear To see this, one applies the formulae in $U\mathfrak{g}$: $HX_-^k = X_-^k H - 2kX_-^k$ and $X_+ X_-^k = X_-^k X_+ + kX_-^{k-1}H - k(k-1)X_-^{k-1}$ in order to get $H \cdot (X_-^{n+1} \otimes w_n) = -(n+2)X_-^{n+1} \otimes w_n$ and $X_+ \cdot (X_-^{n+1} \otimes w_n) = 0$.

The complex $\mathcal{S}(G, B, V_n^\vee)$ is filtered by complexes of Verma modules; let $\mathcal{S}(G, B, V_n^\vee)_{\chi_{(n;-n)}}$ be the subcomplex cut by the infinitesimal character $\chi_{(n;-n)}$ of $V_n^\vee$. It is given by the last line of the diagram below. Note that it consists in $\mathcal{U}(G)$-simple generalized Verma modules. Since $n < p$, we get a direct factor subcomplex (see [29] Sect.2.7, Corollary).

$$
\begin{array}{ccc}
((\mathcal{U}(G) \otimes_{\mathcal{U}(B)} \mathfrak{g}/\mathfrak{b}) \otimes_{\mathbb{Z}_p} Y)_{\chi_{(n;-n)}} & \to & ((\mathcal{U}(G) \otimes_{\mathcal{U}(B)} \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} Y)_{\chi_{(n;-n)}} \\
|| & & || \\
((\mathcal{U}(G) \otimes_{\mathcal{U}(B)} W_{(-2;0)}) \otimes_{\mathbb{Z}_p} Y)_{\chi_{(n;-n)}} & & ((\mathcal{U}(G) \otimes_{\mathcal{U}(B)} \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W_{(n;-n)})_{\chi_{(n;-n)}} \\
|| & & || \\
\mathcal{U}(G) \otimes_{\mathcal{U}(B)} W_{(-n-2;-n)} & \to & \mathcal{U}(G) \otimes_{\mathcal{U}(B)} W_{(n;-n)}
\end{array}
$$

where the equality of $\mathcal{U}(G)$-modules

$$((\mathcal{U}(G) \otimes_{\mathcal{U}(B)} W_{(-2;0)}) \otimes_{\mathbb{Z}_p} V_n^\vee)_{\chi_{(n;-n)}} = \mathcal{U}(G) \otimes_{\mathcal{U}(B)} W_{(-n-2;-n)}$$

follows from the formula, valid for any $\mathfrak{b}$-representation T:

$$(\mathcal{U}(G) \otimes_{\mathcal{U}(B)} T)_{\chi_{(n;-n)}} = \mathcal{U}(G) \otimes_{\mathcal{U}(B)} T_{\chi_{(n;-n)}},$$

together with the fact that only the last term of the filtration by $\mathfrak{b}$-submodules of $W_{(-2;0)} \otimes_{\mathbb{Z}_p} V_n^\vee$ corresponds to an action of $\mathcal{Z}$ by the character $\chi_{(n;-n)}$.

Similar remark for the equality

$$(\mathcal{U}(G) \otimes_{\mathcal{U}(B)} V_n^\vee)_{\chi_{(n;-n)}} = \mathcal{U}(G) \otimes_{\mathcal{U}(B)} W_{(n;-n)}$$

Moreover, the bottom horizontal arrow is an injection given by $\xi \otimes w \mapsto \xi X_-^{n+1} \otimes w$.

Definition 3.1 *The BGG complex $BGG(V_n^\vee)$ is the filtered subcomplex of $\mathcal{S}(G, B, V_n^\vee)$ given by:*

$$BGG(V_n^\vee): \quad \mathcal{U}(G) \otimes_{\mathcal{U}(B)} W_{(-n-2;-n)} \to \mathcal{U}(G) \otimes_{\mathcal{U}(B)} W_{(n;-n)}$$

placed in degrees $-n$ and $-(n+1)$.

Comments:

Its differential, which is of order $n+1$ is given by $\xi \otimes w_{-n-2} \mapsto \xi X_-^{n+1} \otimes w_n$. Note that as a $\mathfrak{b}$-representation, $V_n^\vee$ admits $W_{(n;-n)}$ as a subrepresentation by sending w_n to v_n so we get

$$i_0 : U\mathfrak{g} \otimes_{U\mathfrak{b}} W_{(n;-n)} \hookrightarrow U\mathfrak{g} \otimes_{U\mathfrak{b}} V_n^\vee, \quad 1 \otimes w_n \mapsto 1 \otimes v_n.$$

However, $\mathfrak{g}/\mathfrak{b} \otimes V_n^\vee$ does not admit $W_{(-n-2;-n)}$ as $\mathfrak{b}$-subrepresentation. This apparent paradox is solved as follows. In fact, $U\mathfrak{g} \otimes_{U\mathfrak{b}} \mathfrak{g}/\mathfrak{b} \otimes V_n^\vee$ admits several copies of $W_{(-n-2;-n)}$ as $\mathfrak{t}$-subrepresentations, namely the lines generated by the vectors $X_-^i \otimes \overline{X}_- \otimes X_-^{n-i} v_n$ (for $i = 0, \ldots, n$) but a simple calculation shows that it admits only one copy of $W_{(-n-2;-n)}$ as $\mathfrak{b}$-submodule, namely the line generated by

$$\sum_{i=0}^{n} X_-^i \otimes \overline{X}_- \otimes X_-^{n-i} v_n$$

The meaning of this statement is that this vector is the unique linear combination of the $X_-^i \otimes \overline{X}_- \otimes X_-^{n-i} v_n$'s annihilated by X_+. In particular, we have a morphism of $U\mathfrak{g}$-modules

$$i_1 : U\mathfrak{g} \otimes_{U\mathfrak{b}} W_{(-n-2;-n)} \hookrightarrow U\mathfrak{g} \otimes_{U\mathfrak{b}} \mathfrak{g}/\mathfrak{b} \otimes V_n^\vee$$

given by $1 \otimes w_{-n-2} \mapsto -\sum_{i=1}^n X_-^i \otimes \overline{X}_- \otimes X_-^{n-i} v_n$.

The inclusion of the BGG complex as a subcomplex of the relative Koszul complex is given by (i_1, i_0). Indeed one checks easily that

$$i_0(X_-^{n+1} \otimes w_n) = d_0(i_1(1 \otimes w_{-n-2})).$$

We define the increasing H-filtration Fil_i^H ($i = 0, 1, \ldots, n{+}2$) on $\mathcal{S}(G, B, V_n^\vee)$ by putting Fil_i^H to be the submodule in

$$\mathcal{U}(G) \otimes_{\mathcal{U}(B)} (\mathfrak{g}/\mathfrak{b} \otimes_{\mathbb{Z}_p} V_n^\vee) \oplus \mathcal{U}(G) \otimes_{\mathcal{U}(B)} V_n^\vee$$

induced from $\mathcal{U}(B)$ to $\mathcal{U}(G)$ of the B-submodules sum of the H-eigenspaces corresponding to eigenvalues $\geq n + 2 - 2i$:

$$\mathrm{Fil}_i^H: \quad \mathcal{U}(G) \otimes_{\mathcal{U}(B)} (\mathfrak{g}/\mathfrak{b} \otimes_{\mathbb{Z}_p} V_n^\vee)_{\geq n+2-2i} \to \mathcal{U}(G) \otimes_{\mathcal{U}(B)} (V_n^\vee)_{\geq n+2-2i}$$

For $i \neq 1, n+2$, its i-th graded complex gr_i^H is given by an isomorphism

$$\mathrm{gr}_i^H : \mathcal{U}(G) \otimes_{\mathcal{U}(B)} W_{(n-2i;-n)} \cong \mathcal{U}(G) \otimes_{\mathcal{U}(B)} W_{(n-2i;-n)}$$

while

$$\mathrm{gr}_1^H : 0 \to \mathcal{U}(G) \otimes_{\mathcal{U}(B)} W_{(n;-n)}$$

and

$$\mathrm{gr}_{n+2}^H : \mathcal{U}(G) \otimes_{\mathcal{U}(B)} W_{(-n-2;-n)} \to 0$$

Moreover, this filtration induces the following very simple filtration on $BGG(V_n^\vee)$:

$\mathrm{Fil}_0^H BGG = 0$, $\mathrm{Fil}_1^H BGG : 0 \to \mathcal{U}(G) \otimes_{\mathcal{U}(B)} W_{(n;-n)}$ and $\mathrm{Fil}_{n+2}^H BGG = BGG(V_n^\vee)$.

For this filtration, we have the same graded pieces as above

$$\mathrm{gr}_1^H BGG : 0 \to \mathcal{U}(G) \otimes_{\mathcal{U}(B)} W_{(n;-n)}$$

and

$$\mathrm{gr}_{n+2}^H BGG : \mathcal{U}(G) \otimes_{\mathcal{U}(B)} W_{(-n-2;-n)} \to 0$$

Note that the differential is zero for each of these complexes.

We shall relate this filtration to the Hodge filtration in the following section.

3.2 (log-)de Rham and dual BGG complexes

On $\mathbb{Z}_p^2$ (column vectors), there is a standard symplectic pairing given by the determinant, preserved by the special linear group G', and a standard filtration, preserved by B; similarly, on the sheaf $\mathcal{H}$ defined in Sect.2.2, there is the Poincaré symplectic pairing and the Hodge filtration. Let us consider the $\mathbb{Z}_p$-scheme $\mathcal{T}_{\mathcal{H}} = \mathrm{Isom}_{\overline{X},\mathrm{sympl},\mathrm{fil}}(\mathcal{O}_{\overline{X}}^2, \mathcal{H})$, where the index "sympl" and "fil" refers to isomorphisms compatible both to the symplectic pairings and the filtrations; note that on the target module, we view these data via the identification $\mathcal{O}^2_{\overline{X}}=\mathbb{Z}_p^2\otimes\mathcal{O}_{\overline{X}}$ (see [27] Sect.5.2 for more details).

Let us put $B'=B\cap G'$; the morphism $\mathcal{T}_{\mathcal{H}}\to\overline{X}$ defines a right B'-torsor. it is related to the (right) $\mathbb{G}_m$-torsor $\mathcal{T}_\omega$ that we defined in Sect.2.1. Namely, there is a forgetful morphism $\mathcal{T}_{\mathcal{H}}\to\mathcal{T}_\omega$ obtained by restricting a filtered isomorphism to the Fil^1 of the standard filtration of $\mathcal{O}^2_{\overline{X}}$ (taking values in $\omega\subset\mathcal{H}$).

The torsor $\mathcal{T}_{\mathcal{H}}$ allows us to define a covariant sheafification functors $F_{\mathbb{Z}_p}$ from the category of finite free $\mathbb{Z}_p$-representations of B' to the category $\mathcal{FF}$ of finite locally free $\mathcal{O}_{\overline{X}}$-modules with filtration.

Note that we should consider representations of B, that is, we should include the action of the center, but we shall omit it for simplicity, so that we write W_m instead of $W_{(m;c)}$, by forgetting about the central character.

For any object W of $Rep_{\mathbb{Z}_p}(B')$, we define $F_{\mathbb{Z}_p}(W)$ as the sheaf of sections of the bundle

$$\mathcal{T}_{\mathcal{H}}\overset{B'}{\times}W\to\overline{X}$$

Actually, in the sequel, we'll also consider the contravariant version of $F_{\mathbb{Z}_p}$:

$$F^\vee_{\mathbb{Z}_p}=\mathrm{Hom}_{\mathcal{O}_{\overline{X}}}(-,\mathcal{O}_{\overline{X}})\circ F_{\mathbb{Z}_p}$$

which is exact on the full subcategory of B'-representations which are finite free over $\mathbb{Z}_p$. For such a representation W, we thus have

$F^\vee_{\mathbb{Z}_p}(W)=Hom_{\mathcal{O}_{\overline{X}}}(F_{\mathbb{Z}_p}(W),\mathcal{O}_{\overline{X}})$.

Remark: Denoting by T the Levi quotient of B, and by $\mathbb{G}_m$ the "semisimple" subtorus of T, so that $B'/N=\mathbb{G}_m$, we see that if W is a $\mathbb{G}_m$-module, then the forgetful morphism above induces a vector bundle isomorphism

$$(CompTors)\quad \mathcal{T}_{\mathcal{H}}\overset{B'}{\times}W\cong\mathcal{T}_\omega\overset{\mathbb{G}_m}{\times}W.$$

We see immediately that $F_{\mathbb{Z}_p}(\mathbb{Z}_p)=\mathcal{O}_{\overline{X}}$ and more generally for any $n\in\mathbb{Z}$,

$F_{\mathbb{Z}_p}(W_n)=\omega^n$, so that $F^\vee_{\mathbb{Z}_p}(W_n)=\omega^{-n}$. This can be viewed from the definition since it is obvious that $F_{\mathbb{Z}_p}(W_1)=\omega$; it can also be checked over $\mathbb{C}$: the pull-back of $\mathcal{T}_\omega$ to the upper half plane P is given by $P\times\mathbb{C}^\times$ with left action of the discrete group Γ given by the automorphic factor $j(\gamma,z)$, hence W_n is sent to ω^n.

By the Kodaira-Spencer isomorphism, this implies that $F_{\mathbb{Z}_p}(\mathfrak{g}/\mathfrak{b})=T_{\overline{X}}(-log\,C)$ namely, the subsheaf of the tangent bundle consisting of sections which, near the cusps, are multiple of $q\frac{d}{dq}$; it is also the dual sheaf of $\Omega_{\overline{X}}(d\log C)$.

For $n < p$, the $\mathbb{Z}_p$-representation V_n of G is defined unambiguously and we have by definition $F^\vee_{\mathbb{Z}_p}(V_n^\vee) = \mathcal{H}_n$.

We want to extend the functor $F_{\mathbb{Z}_p}$ to the category of induced modules $M(V) = U\mathfrak{g} \otimes_{U\mathfrak{b}} V$, resp. $\mathcal{M}(V) = \mathcal{U}(G) \otimes_{\mathcal{U}(B)} V$ (V a $\mathfrak{b}$-module of weights $0 \leq \ell < p$),

by the same formula

$$F_{\mathbb{Z}_p}(M(V)) = \mathcal{T}_\mathcal{H} \overset{B}{\times} U\mathfrak{g} \otimes_{U\mathfrak{b}} V$$

resp.

$$F_{\mathbb{Z}_p}(\mathcal{M}(V)) = \mathcal{T}_\mathcal{H} \overset{B}{\times} \mathcal{U}(G) \otimes_{\mathcal{U}(B)} V$$

it will take values in the category of finite modules over the ring $\overline{\mathcal{D}}_{\overline{X}}$ of log C-differential operators, resp. $\widetilde{\mathcal{D}}_{\overline{X}}$ of log C-divided power differential operators, on $\overline{X}$ (see [13] p.217 and [27] Sect.4.2).

For this, we need to recall some algebra.

Let $G = \operatorname{Spec} A$, hence $A = \mathcal{O}_G$; consider the diagram of short exact sequences

$$\begin{array}{ccccccc} 0 \to & I & \to & A \otimes_{\mathbb{Z}_p} A & \overset{m}{\to} & A & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & J & \to & A & \overset{\epsilon}{\to} & \mathbb{Z}_p & \to 0 \end{array}$$

where m denotes the multiplication, ϵ the counit morphism and the vertical maps are $1 \otimes f - f \otimes 1 - \mapsto f - \epsilon(f)$, resp. $\epsilon \otimes Id$, resp.ϵ.

Recall that $\mathcal{U}(G)$ is the union of the submodules $\mathcal{U}_k(G) = \operatorname{Hom}_{\mathbb{Z}_p}(A/J^{k+1}, \mathbb{Z}_p)$.

The left A-module $\mathcal{D}$ of differential operators on G is the union of $\mathcal{D}_k = \operatorname{Hom}_A((A \otimes A)/I^{k+1}, A)$, where the left-$A$-module structure on $A \otimes A$ is by multiplication on the first factor. We have therefore a natural map $\Phi : \mathcal{U}(G) \to \mathcal{D}$ given by $\Phi(\xi)(f \otimes g) = f\xi(g)$ The image of Φ is precisely the subring of left-invariant differential operators: $\eta \in \mathcal{D}$ such that for the left translation by any $g \in G$, $L_g^* \circ \eta = \eta \circ (L_g^* \otimes Id)$.

Similarly, let us compute $\mathcal{U}(G) \otimes_{\mathcal{U}(B)} \mathbb{Z}_p$ as a left B-module.

Let $\mathcal{J}_B$ be the kernel of the map $\mathcal{O}_B \to \mathbb{Z}_p$ of evaluation at 1. The big cell $N^- \times B$ of G is the open set of matrices $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ such that $a_{11} \neq 0$. Hence, the restriction to this cell induces an inclusion with $\mathbb{Z}_p$-flat cokernel $\mathcal{O}_G \hookrightarrow \mathcal{O}_G[\frac{1}{a_{11}}] = \mathcal{O}_{N^-} \otimes \mathcal{O}_B$. Therefore $\mathcal{O}_G \cap (1 \otimes \mathcal{J}_B)$ is $\mathbb{Z}_p$-direct factor in $\mathcal{O}_G$ and we have a commutative diagram

$$\begin{array}{ccccccc}
0 \to & \mathrm{Hom}((\mathcal{O}_{N^-}\otimes \mathcal{J}_B)_{<0},\mathbb{Z}_p) & \to & \mathrm{Hom}(\mathcal{O}_{N^-}\otimes \mathcal{J}_B,\mathbb{Z}_p) & \to & \mathrm{Hom}(\mathcal{O}_G\cap(\mathcal{O}_{N^-}\otimes \mathcal{J}_B),\mathbb{Z}_p) \\
 & \downarrow & & \downarrow & & \downarrow \\
0 \to & \mathrm{Hom}((\mathcal{O}_{N^-}\otimes \mathcal{O}_B)_{<0},\mathbb{Z}_p) & \to & \mathrm{Hom}(\mathcal{O}_{N^-}\otimes \mathcal{O}_B,\mathbb{Z}_p) & \to & \mathrm{Hom}(\mathcal{O}_G,\mathbb{Z}_p) \\
 & \downarrow & & \downarrow & & \downarrow \\
 & 0 & \to & \mathrm{Hom}(\mathcal{O}_{N^-}\otimes 1,\mathbb{Z}_p) & = & \mathrm{Hom}(\mathcal{O}_{N^-},\mathbb{Z}_p)
\end{array}$$

where the upper line of vertical arrows are injections, and the lower are surjections. We wrote $(\mathcal{O}_{N^-}\otimes \mathcal{O}_B)_{<0}$ for the $\mathbb{Z}_p$-submodule of $\mathcal{O}_{N^-}\otimes \mathcal{O}_B$ where the discrete valuation $\mathrm{ord}_{a_{11}}$ at a_{11} is strictly negative. The exactness of the horizontal lines, resp. vertical lines come from the $\mathbb{Z}_p$-linear decomposition $\mathcal{O}_{N^-}\otimes\mathcal{O}_B=\mathcal{O}_G\oplus(\mathcal{O}_{N^-}\otimes\mathcal{O}_B)_{<0}$ resp. $\mathcal{O}_{N^-}\otimes\mathcal{O}_B=(\mathcal{O}_{N^-}\otimes\mathcal{J}_B)\oplus(\mathcal{O}_{N^-}\otimes 1)$ from which it follows that $(\mathcal{O}_{N^-}\otimes \mathcal{J}_B)_{<0}=(\mathcal{O}_{N^-}\otimes \mathcal{O}_B)_{<0}$.

Thus, $\mathcal{U}(G)\otimes_{\mathcal{U}(B)}\mathbb{Z}_p=\bigcup_k \mathrm{Hom}(\mathcal{O}_{N^-}/\mathcal{J}_{N^-}^{k+1},\mathbb{Z}_p)$. Note that we have the following $\mathbb{Z}_p$-linear decomposition:

$$\mathrm{Hom}(\mathcal{O}_{N^-}/\mathcal{J}_{N^-}^{k+1},\mathbb{Z}_p)=\bigoplus_{\ell\leq k}\mathbb{Z}_p\frac{1}{\ell!}\overline{X}_-^\ell$$

where X_- denotes the linear form sending $f\in\mathcal{O}_{N^-}$ to the value at $0\in\mathbb{G}_a$ of its derivative, once one uses the 1-parameter group $U_{-\alpha}:\mathbb{G}_a\to N^-$ sending x to $\begin{pmatrix}1&0\\x&1\end{pmatrix}$ to identify $\mathbb{G}_a$ to N^-. One can then describe explicitely the left action of B by the formulas $H\overline{X}_-^k=-2k\overline{X}_-^k$, $X_+\overline{X}_-^k=-k(k-1)\overline{X}_-^k$.

By what precedes, the product $\mathcal{T}_{\mathcal{H}}\times\mathrm{Hom}(\mathcal{O}_{N^-}/\mathcal{J}_{N^-}^{k+1},\mathbb{Z}_p)$ identifies to the bundle of left B-invariant log C-PD differential operators of order $\leq k$ over $\mathcal{T}_{\mathcal{H}}$. Hence, the contracted product $\mathcal{T}_{\mathcal{H}}\overset{B'}{\times}\mathrm{Hom}(\mathcal{O}_{N^-}/\mathcal{J}_{N^-}^{k+1},\mathbb{Z}_p)$ identifies to the bundle of log C-PD differential operators of order $\leq k$ on $\overline{X}$.

Once this is fixed, one can determine the image of $\mathcal{M}(\mathbb{Z}_p)$ and $\mathcal{M}(\mathfrak{g}/\mathfrak{q})$ As left $\mathcal{O}_{\overline{X}}$-Module, we have

$$F_{\mathbb{Z}_p}(\mathcal{M}(\mathbb{Z}_p))=\bigoplus_k\frac{1}{k!}T_{\overline{X}}(-d\log C)^{\otimes k}=\widetilde{\mathcal{D}}_{\overline{X}}$$

and similarly

$$F_{\mathbb{Z}_p}(\mathcal{M}(\mathfrak{g}/\mathfrak{b}))=\widetilde{\mathcal{D}}_{\overline{X}}\otimes_{\mathcal{O}_{\overline{X}}}T_{\overline{X}}(-d\log C).$$

More generally, for any $\mathbb{Z}_p$-finite free B-module W (with fixed central action), put $F_{\mathbb{Z}_p}(W)=\mathcal{W}$; then

$$F_{\mathbb{Z}_p}(\mathcal{M}(W)) = \widetilde{\mathcal{D}}_{\overline{X}} \otimes_{\mathcal{O}_{\overline{X}}} \mathcal{W}.$$

Remark: The rings $\mathcal{D}_{\overline{X}}$, $\overline{\mathcal{D}}_{\overline{X}}$ and $\widetilde{\mathcal{D}}_{\overline{X}}$ are $\mathcal{O}_{\overline{X}}$-bimodules, but not $\mathcal{O}_{\overline{X}}$-algebras (non-commutativity of the functions with the differential operators).

Lemma 3.2 *1) $F_{\mathbb{Z}_p}(M(\mathbb{Z}_p)) = \overline{\mathcal{D}}_{\overline{X}}$, $F_{\mathbb{Z}_p}(\mathcal{M}(\mathbb{Z}_p)) = \widetilde{\mathcal{D}}_{\overline{X}}$ and*

$$F_{\mathbb{Z}_p}(M(\mathfrak{g}/\mathfrak{b})) = \overline{\mathcal{D}}_{\overline{X}} \otimes_{\mathcal{O}_{\overline{X}}} T_{\overline{X}}(-d\log C), \quad F_{\mathbb{Z}_p}(\mathcal{M}(\mathfrak{g}/\mathfrak{b})) = \widetilde{\mathcal{D}}_{\overline{X}} \otimes_{\mathcal{O}_{\overline{X}}} T_{\overline{X}}(-d\log C),$$

2) The image by $F^\vee_{\mathbb{Z}_p}$ of the relative Koszul resolution of $\mathbb{Z}_p$: $M(\mathfrak{g}/\mathfrak{b}) \to M(\mathbb{Z}_p)$ yields the log-de Rham complex $\mathcal{O}_{\overline{X}} \xrightarrow{d} \Omega_{\overline{X}}(d\log C)$.

3) For any $n \in [0, p[$, the image by $F^\vee_{\mathbb{Z}_p} = \mathrm{Hom}_{\mathcal{O}_{\overline{X}}}(-, \mathcal{O}_{\overline{X}})$ of the relative Koszul resolution of $V_n^\vee$: $M(\mathfrak{g}/\mathfrak{b} \otimes V_n^\vee) \to M(V_n^\vee)$ yields the log-de Rham complex $\mathcal{H}(n)$ for $\mathcal{H}_n$ with its Gauss-Manin log-connection ∇_n:

$$\mathcal{H}_n \xrightarrow{\nabla_n} \mathcal{H}_n \otimes_{\mathcal{O}_{\overline{X}}} \Omega_{\overline{X}}(d\log C)$$

Proof: 1) is already proved. For 2) and 3), let us briefly recall how from a morphism of degree 1

$$(*) \quad \overline{\mathcal{D}}_{\overline{X}} \otimes_{\mathcal{O}_{\overline{X}}} \mathcal{W}_1 \to \overline{\mathcal{D}}_{\overline{X}} \otimes_{\overline{X}} \mathcal{W}_0$$

gives rise to a (log-)connection $\mathcal{W}_0^\vee \to \mathcal{W}_1^\vee \otimes_{\overline{X}} \Omega_{\overline{X}}(d\log C)$.

Let $\overline{\mathcal{D}}_{\overline{X},i}$, $i = 0$ resp. 1, be the module of log C-differential operators of degree 0, resp. at most 1. Since the morphism $(*)$ has degree 1, it preserves $\overline{\mathcal{D}}_{\overline{X},1}$. We have $\mathcal{O}_{\overline{X}} = \overline{\mathcal{D}}_{\overline{X},0}$ while $\overline{\mathcal{D}}_{\overline{X},1}$ inserts in a short exact sequence of $\mathcal{O}_{\overline{X}}$-modules

$$0 \to T_{\overline{X}}(-d\log C) \xrightarrow{a} \overline{\mathcal{D}}_{\overline{X},1} \xrightarrow{b} \mathcal{O}_{\overline{X}} \to 0$$

By composing with the inclusion $a \otimes Id_{\mathcal{W}}$ and the projection $b \otimes Id_{\mathcal{W}}$, the map $(*)$ gives rise to

$$T_{\overline{X}}(-d\log C) \otimes \mathcal{W}_1 \to \mathcal{W}_0$$

whose dual provides the desired log-connection.

Let us apply this to the image by $F_{\mathbb{Z}_p}$ of $M(\mathfrak{g}/\mathfrak{b} \otimes V_n^\vee) \to M(V_n^\vee)$. The morphism is given, for $\xi \in U\mathfrak{g}$, $\overline{Y} \in \mathfrak{g}/\mathfrak{b}$ and $v^\vee \in V_n^\vee$, by $\xi \otimes \overline{Y} \otimes v^\vee \mapsto \xi Y \otimes v^\vee - \xi \otimes Y v^\vee$ hence it is degree 1, and after dualizing we find a morphism

$$\delta : \mathcal{H}_n \to \mathcal{H}_n \otimes_{\mathcal{O}_{\overline{X}}} \Omega_{\overline{X}}(d\log C)$$

given for $f \in \mathcal{O}_{\overline{X}}$ and $v \in \mathcal{H}_n$ by $fv \mapsto (\overline{Y} \otimes v^\vee \mapsto Yf \cdot \langle v, v^\vee \rangle - f\langle Yv, v^\vee \rangle)$, in this formula we recognize the Gauss-Manin connection ∇_n as desired.QED.

We can also define a similar functor $F_{\mathbb{Z}/p\mathbb{Z}}$ for the categories of $\mathbb{Z}/p\mathbb{Z}$-representations resp. $\mathbb{Z}/p\mathbb{Z}$-generalized Verma modules. By construction, we have

$$F_{\mathbb{Z}/p\mathbb{Z}}(W/pW) = F_{\mathbb{Z}_p}(W) \otimes \mathbb{Z}/p\mathbb{Z}, \quad \text{and}\, F_{\mathbb{Z}/p\mathbb{Z}}(\mathcal{M}/p\mathcal{M}) = F_{\mathbb{Z}_p}(\mathcal{M}) \otimes \mathbb{Z}/p\mathbb{Z}.$$

It sends $S(G, B, V_n^\vee)$ to the logarithmic de Rham complex on $\overline{X}$.

Let $\mathcal{K}^\bullet(n) = (\mathcal{K}_n^0 \to \mathcal{K}_n^1)$ be the subcomplex of $\mathcal{H}(n)$ defined as the image of the complex $BGG(G, B, V_n^\vee)$ by $F_{\mathbb{Z}_p}^\vee$ (resp.$F_{\mathbb{Z}/p\mathbb{Z}}$). The complex $\mathcal{K}^\bullet(n)$ is called the dual BGG complex.

One has $\mathcal{K}_n^0 = \omega^{-n}$ and $\mathcal{K}_n^1 = \omega^{n+2}$ with a differential induced by ∇_n. By Definition 3.1, this differential is dual to $w_{-n-2} \mapsto X_-^{n+1} \otimes w_n$. If we want to take into account the determinant character, we put $\omega^s(t) = \mathcal{W}_{s;s+2t}$; then, the exact formula for the dual BGG complex is

$$(**) \quad \mathcal{K}_n^\bullet : \omega^{-n}(n+1) \to \omega^{n+2}$$

This is important for the future action of the Hecke operators on the complex.

Theorem 2 *The complex $\mathcal{K}^\bullet(n)$ is a direct factor filtered subcomplex of $\mathcal{H}^\bullet(n)$ which is quasi-isomorphic to $\mathcal{H}^\bullet(n)$.*

The Hodge filtration on $\mathcal{H}^\bullet(n)$ induces on $\mathcal{K}^\bullet(n)$ the filtration

$$\begin{array}{rccc} \mathrm{Fil}^0: & \omega^{-n} & \to & \omega^{n+2} \\ \mathrm{Fil}^{n+1}: & 0 & \to & \omega^{n+2} \\ Fil^{n+2}: & 0 & \to & 0 \end{array}$$

The proof is given for more general groups in [27] Theorem 6 Sect.5.4. This is where one needs to introduce the divided power versions of the Koszul resp. BGG complexes (that is to replace $U\mathfrak{g}$ by $\mathcal{U}(G)$), and also to introduce a functor, called the L functor, sending $\overline{\mathcal{D}}_{\overline{X}}$ to $\widetilde{\mathcal{D}}_{\overline{X}}$, and the logarithmic de Rham complex to its divided power analogue in order to transform the log de Rham complex and the BGG dual complex into résolutions.

3.3 The ordinary locus

We use the notations of Section 2.1. Let $\overline{X} \otimes \mathbb{Z}/p\mathbb{Z}$ be the modulo p modular curve and $\mathcal{G} \to \overline{X} \otimes \mathbb{Z}/p\mathbb{Z}$ the reduction modulo p of the semi-abelian curve, neutral component of the universal generalized elliptic curve. Let $\mathcal{G}[p]^0$ be the neutral component of the quasi-finite group scheme $\mathcal{G}[p]$ of p-torsion in $\mathcal{G}$. It is a finite flat group scheme over $\overline{X} \otimes \mathbb{Z}/p\mathbb{Z}$. The ordinary locus S of $\overline{X} \otimes \mathbb{Z}/p\mathbb{Z}$ is the largest subscheme over which $\mathcal{G}[p]^0$ is of multiplicative type.

The multiplication by p on $\mathcal{G}$ factors through the relative Frobenius morphism $F : \mathcal{G} \to \mathcal{G}^{(p)}$, where $\mathcal{G}^{(p)}$ denotes the pull-back of $\mathcal{G}$ by the absolute Frobenius on $\overline{X} \otimes \mathbb{Z}/p\mathbb{Z}$. Let V (the Verschiebung morphism) be the quotient morphism $V : \mathcal{G}^{(p)} \to \mathcal{G}$. Recall that for any line bundle $\mathcal{L}$ on $\overline{X} \otimes \mathbb{Z}/p\mathbb{Z}$, there is a canonical isomorphism $\mathcal{L}^{(p)} \cong \mathcal{L}^{\otimes p}$ The Hasse invariant H is the modulo p modular form of weight $p-1$ defined as the morphism $V^* : \omega \to \omega^{(p)}$ deduced from V. By definition of S, this is the locus where V is étale, that is, where H does not vanish. Hence S is open. It is non-empty hence dense, by existence of the Tate curve. Since ω is ample on $\overline{X} \otimes \mathbb{Z}/p\mathbb{Z}$, the open subscheme S is affine.

3.4 Frobenius and U_p

Let $\overline{X}_0(p)$ be the (compactified) modular curve classifying p-isogenies ($E \to E'$) over $\overline{X}$ over $\mathbb{Z}_p$ (see [24]). Note that by definition of $\overline{X}$, an auxiliary $\Gamma_1(N)$-level structure ($N \geq 4$) with p prime to N, is present but not mentioned in the sequel. We have two degeneracy maps π_1 sending ($E \to E'$) to E and π_2 sending ($E \to E'$) to E'. An algebraic correspondence (π_1, π_2, α) acting on the pair $(\overline{X}, \mathcal{F})$ for a sheaf $\mathcal{F}$ on $\overline{X}$ is determined by (π_1, π_2) and a morphism $\alpha : \pi_2^*\mathcal{F} \to \pi_1^*\mathcal{F}$ For $\mathcal{F} = \mathcal{H}_n$, the correspondence T_p acts on $H^\bullet_{\mathrm{dR}}(\overline{X}, \mathcal{H}_n)$ by $\pi_{1,*}\circ\alpha_*\circ\pi_2^*$ where $\alpha : \pi_2^*\mathcal{H}_n \to \pi_1^*\mathcal{H}_n$ is defined by $Sym^n \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} : V_n^\vee \to V_n^\vee$. Let us consider the "Frobenius" subcorrespondence ϕ obtained by restricting this diagram to the ordinary locus S of $\overline{X} \otimes \mathbb{Z}/p\mathbb{Z}$ and by restricting π_1 and π_2 to the multiplicative type part $\overline{X}^m$ of $\pi_1^{-1}(S)$ (one knows that the ordinary locus of $\overline{X}_0(p)$ is the disjoint union of two components: $\overline{X}^m$ where the kernel of the isogeny is of multiplicative type, and $\overline{X}^e$, where it is étale). This subcorrespondence is of degree p and is actually a purely inseparable morphism because π_2 is purely inseparable of degree p while π_1 is an isomorphism $\overline{X}^m \cong S$.

Let $\overline{X}\widehat{}_{(p)}$ be the formal scheme completion along the special fiber of $\overline{X}_{\mathbb{Z}_p}$ and S_∞ be its open formal subscheme with underlying scheme S. Let H the canonical subgroup $H \subset \mathcal{G}$ lifting the kernel of Frobenius in the universal (semi-)elliptic curve $\mathcal{G} \to S_\infty$ (see Sect.2.1 for the notations); the Frobenius morphism ϕ lifts as a morphism $\phi : S_\infty \to S_\infty$: $\phi(x) = y$ if and only if $E_y = E_x/H_x$; this morphism is finite and flat. It extends to the Igusa tower $T_{\infty,\infty} \to S_\infty$ as a finite flat endomorphism, so we can form ϕ_* as an endomorphism of the

p-adic algebra $V(N)$ of p-adic modular forms; it leaves stable its submodules $H^0(S_\infty, \omega^k)$ for any $k \in \mathbb{Z}$.

We define U_p as $\frac{1}{p}\phi_*$. The q-expansion principle together with the formula $\sum_{n\geq 1} a_n(f|U_p)q^n = \sum_{n\geq 1} a_{np}(f)q^n$ implies that U_p preserves integrality hence preserves $V(N)$ and its submodules $H^0(S_\infty, \omega^k)$ for any $k \in \mathbb{Z}$.

If f is a classical form of level $\Gamma_1(Np)$, it can be viewed as a p-adic modular form of level prime to p as usual (see Hida [18]). Then, the definition above of U_p is compatible with the definition of this operator on classical forms.

Finally, let us recall that we defined in Sect.2 an integral version of the Eichler-Shimura morphism for $k = n + 2 \geq 2$:

$$H^0(\overline{X}, \omega^k) \hookrightarrow H^1_{log-dR}(\overline{X}, \mathcal{H}_n)$$

Here, we view both modules over $\mathbb{Z}_p$. By functoriality, this morphism is Hecke-equivariant for all T_n's (n prime to N) including T_p. Moreover in the restriction morphism

$$Res : H^0(\overline{X}_{\mathbb{Z}_p}, \omega^k) \to H^0(S_\infty, \omega^k),$$

$Res(f|T_p)$ is congruent to $Res(f)|U_p$ mod. p because $k \geq 2$.

4 Comparison Theorem and companion forms in p-small weight $k \geq 2$

Let f be our weight k new form of level N. Recall that we fixed a p-adic embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$; let $K \subset \overline{\mathbb{Q}}_p$ be a finite extension of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}$ containing the eigenvalues of f such that the Galois representation ρ_f is defined over $\mathcal{O}$; let $\mathbb{F} = \mathcal{O}/(\varpi)$ be its residue field. let V_f be the rank 2 free $\mathcal{O}$-module of this representation. Recall its cohomological definition. For $A = \mathbb{Q}_p, \mathbb{Z}_p, K, \mathcal{O}, \mathbb{F}$, let $V_n(A) = \mathrm{Symm}^n A^2$ viewed as a representation of G over A and $H^1(A) = H^1_{\mathrm{et}}(X \otimes \mathrm{Spec}\,\overline{\mathbb{Q}}, V_n(A))$; it is a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representation defined over A. Let $V_f(K)$ be the 2-dimensional vector space cut by the idempotent 1_f associated to f in the Hecke algebra of level N and weight k (which acts faithfully on $H^1(K)$). Then, define V_f as $V_f(K) \cap H^1(\mathcal{O})$. By definition, it is a direct factor in $H^1(\mathcal{O})$. Under the assumptions that $(p, N) = 1$ and $n+1 < p-1$, we know that $H^i(\mathcal{O})$ is torsion-free for $i = 0, 1, 2$. This implies that $H^1(\mathcal{O})/\varpi H^1(\mathcal{O}) \cong H^1(\mathbb{F})$ and that $V_f \hookrightarrow H^1(\mathcal{O})$ induces an injection $\overline{V}_f = V_f/\varpi V_f \hookrightarrow H^1(\mathbb{F})$.

4.1 The integral comparison theorem

Under the assumption $n+1 < p-1$, Falting's log-crys comparison theorem states that, for $A = \mathbb{Q}_p, \mathbb{Z}_p, \mathbb{F}_p$, the contravariant Fontaine-Laffaille functor D sends the restriction to $D_p = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ of $H^1(A)$ to the log-crystalline cohomology module $H^1_{\mathrm{lc}}(A) = H^1_{\mathrm{log-crys}}(\overline{X}/A, \mathcal{H}(n))$, viewed as a ϕ-filtered A-module. Moreover, as a filtered A-module, $H^1_{\mathrm{log-cr}}(\overline{X}/A, \mathcal{H}(n))$ coincides with $H^1_{\mathrm{log-dR}}(\overline{X}/A, \mathcal{H}(n))$. Moreover, by this functor, the D_p-submodule V_f, resp. $\overline{V}_f$, of $H^1(\mathcal{O})$ resp. $H^1(\mathcal{O})$, is sent to the ϕ-filtered submodule M_f resp. $\overline{M}_f$ of $H^1_{\mathrm{lc}}(\mathcal{O}) = H^1_{\mathrm{lc}}(\mathbb{Z}_p) \otimes \mathcal{O}$ resp. $H^1_{\mathrm{lc}}(\mathbb{F}) = H^1_{\mathrm{lc}}(\mathbb{F}_p) \otimes \mathbb{F}$.

The Hodge filtration on $H^1_{\mathrm{lc}}(A)$ has been determined in Sect. 2 using the quasi-isomorphism between the de Rham and the dual BGG complex. We found $H^1_{\mathrm{lc}}(A) = H^1_{\mathrm{log-cr}}(\overline{X}/A, \mathcal{H}(n)) = H^1_{\mathrm{log-cr}}(\overline{X}/A, \mathcal{K}(n))$, so $\mathrm{Fil}^0 = \ldots = \mathrm{Fil}^n = H^1_{\mathrm{lc}}(A)$, $\mathrm{Fil}^{n+1} = \mathrm{Im}\left(H^0(\overline{X}_A, \omega^{n+2}) \to H^1_{\mathrm{lc}}(A)\right)$ and $\mathrm{Fil}^{n+2} = 0$.

Since M_f is a sub-filtered module of $H^1_{lc}(\mathcal{O})$, we see that $\mathrm{Fil}^{n+1} M_f$ is $\mathcal{O}$-free of rank one, generated by the image $[f]$ of the differential form $\omega_f = (2i\pi)^{n+2} f(z) dv^{n+2}$ defined over $\mathcal{O}$ (as usual in this notation, the variable z is the uniformizing variable of the complex modular curve and v is the uniformizing of the elliptic curve above z). By Scholl's theorem [30], the ($\mathcal{O}$-linear) crystalline Frobenius ϕ on M_f is annihilated by $(X-\alpha)(X-\beta) \in \mathcal{O}[X]$. Moreover, it induces two maps $\phi_0 : \mathrm{Fil}^0(\overline{M}_f) \to \overline{M}_f$ and $\phi_{n+1} : \mathrm{Fil}^{n+1}(\overline{M}_f) \to \overline{M}_f$ the latter being induced by $p^{-(n+1)}\phi | \mathrm{Fil}^{n+1}$. Recall that the weak admissibility condition on $\overline{M}_f$ reads

$$\overline{M}_f = \phi_0(\mathrm{Fil}^0(\overline{M}_f)) + \phi_{n+1}(\mathrm{Fil}^{n+1}(\overline{M}_f))$$

Moreover, as explained in Perrin-Riou's paper [28], on the filtered module side, the ordinarity of the representation V_f at p is equivalent to the following compatibility between the Hodge filtration $\mathrm{Fil}^\bullet(M_f)$ and the slope filtration. Recall that the steps of the slope filtration $\mathrm{Fil}^\bullet_\phi$ are defined by $\mathrm{Fil}^i_\phi(M_f)$ is the sum of the ϕ-eigenspaces where the slope is $\geq i$. In this 2-dimensional situation, let (e_1, e_2) an $\mathcal{O}$-basis of M_f made of ϕ-eigenvectors: $\phi(e_1) = \alpha e_1$ and $\phi(e_2) = \beta e_2$; then, the compatibility reads simply as

$$M_f = \mathrm{Fil}^{n+1}(M_f) \oplus \mathcal{O} \cdot e_1$$

Moreover the decomposability assumption of $\overline{V}_f$ translates as the decomposability of $\overline{M}_f$, that is,

$$\mathrm{Fil}^{n+1}(\overline{M}_f) = \mathbb{F} \cdot \bar{e}_2$$

Under this assumption, we see that $[f] \in \mathrm{Fil}^{n+1}(\overline{M}_f)$ happens to be an eigenvector for ϕ_{n+1}, with eigenvalue the reduction modulo ϖ of $\frac{\beta}{p^{n+1}}$. That is,

$\phi_{n+1}([f]) - \frac{\beta}{p^{n+1}}[f] = 0$ in $H^1_{\mathrm{lc}}(\mathbb{F})$. In order to exploit this formula we need to introduce the ordinary locus of $\overline{X}_{\mathbb{F}_p}$.

4.2 Dual BGG and companion forms

Recall that the invertible sheaf ω on $\overline{X}_{\mathbb{Z}_p}$ is relatively ample. Let H be the Hasse invariant on $\overline{X}_{\mathbb{F}_p}$; it is a non zero global section of ω^{p-1}; therefore, the locus S where it does not vanish is a non-empty affine open subset of $\overline{X}_{\mathbb{F}_p}$. It is called the ordinary locus because its geometric points correspond to (generalized) elliptic curves which are ordinary. Since it is affine, we have

- $H^i(S, \mathcal{F}) = 0$ for every $i > 0$ and for any coherent sheaf $\mathcal{F}$ on S, and
- $H^i(S, \mathcal{F}^\bullet) = H^i(\mathcal{F}^\bullet(V))$ for any complex $\mathcal{F}^\bullet$ of coherent sheaves.

Applying this to the dual BGG complex, we find that

$$H^1(S, \mathcal{H}(n)) = H^1(S, \mathcal{K}(n)) = \omega^{n+2}(S)/\nabla_n(\omega^{-n}(S)).$$

Moreover, the Frobenius morphism ϕ_S on S induces an automorphism of $H^0(S, \omega^m)$ for any m, in such a way that the short exact sequence

$$Ex(S) \quad 0 \to H^0(S, \omega^{-n}) \to H^0(S, \omega^{n+2}) \to H^1(S, \mathcal{H}(n)) \to 0$$

is ϕ-equivariant. Let $H^0(]S[, \omega^m)$ be the rigid cohomology of the affinoid tube $]S[$.

Hence, the differential form $\eta = \frac{\phi}{p^{k-1}}(\omega_f) - \frac{\beta}{p^{k-1}}\omega_f \in H^0(S, \omega^k)$ maps to $[\eta] = 0$ in $H^1(S, \mathcal{H}(n))$. By exactness of $Ex(S)$, there exists $\omega' \in H^0(S, \omega^{-n}(n+1))$ such that $\eta = \nabla_n(\omega')$ (here, we mention the twist by $n+1$, because it becomes important in what follows).

A priori, ω' is only defined over S. Let us first assume it is eigen for U_p with a p-adic unit eigenvalue; this assumption is motivated by a formula of Coleman whose proof is recalled in [4] Sect.3, $\phi = p^{k-1} < p > U_p^*$ on $]S[$, so $\frac{\phi}{p^{k-1}}(\omega_f) - \frac{\beta}{p^{k-1}}\omega_f = (unit) \cdot \omega_{f-\alpha f(pz)}$; the form $f - \alpha f(pz)$ is eigen for U_p with eigenvalue β. Since $p^{k-1}d \circ U_p = U_p \circ d$ (for the BGG dual differential d), we would find $\omega'|U_p = \overline{\frac{\beta}{p^{k-1}}} \cdot \omega'$ if we could divide by p^{k-1} (that is, in characteristic zero), and ω' would be ordinary. Thus, by Jochnowitz's lemma (or Hida Theory [20]), the form $H \cdot \omega' \in H^0(S, \omega^{p-1-n})$, being of weight $k' = p + 1 - k \geq 2$ and ordinary, would extend to the whole curve $\overline{X}_{\mathbb{F}}$. Moreover, by Hida Theory [20], it would even be associated to the reduction modulo p, say $\overline{g}$, of a characteristic zero ordinary cusp eigenform g of weight k': $\nabla_n(\omega_{\overline{g}}) = \omega'$.

However, this argument does not go through because we are in characteristic p so that the ordinarity does not seem to follow from the information above. However, let us prove that $\omega' H$ does extend to the whole curve $\overline{X}_{\mathbb{F}}$. Let $t \geq 1$ be the order of pole of the meromorphic fomr ω' along the supersingular divisor $Z \subset \overline{X}$. We need to see that $t \leq 1$. The endomorphism $\frac{\phi}{p^{k-1}}$ of $H^0(V, \omega^k)$ maps $H^0(\overline{X}, \omega^k)$ to $H^0(\overline{X}, \omega^{pk})$; therefore, by the q-expansion principle applied to $H^0(\overline{X}, \omega^{pk})$, we have the following equality of forms in $H^0(\overline{X}, \omega^{pk})$:

$$\overline{\omega}_f^{(p)} - \delta_1 H^k \overline{\omega}_f = H^k d\omega'$$

where $\overline{\omega}_f^{(p)} = \frac{\phi}{p^{k-1}} \overline{\omega}_f$. The left-hand side has no pole along Z. By a calculation of Katz [23], the operator d increases the order of pole at a supersingular point x by $k-1$: if $\mathrm{ord}_x(\omega') = t$, $\mathrm{ord}_x(d\omega') = t + k - 1$. By a theorem of Igusa, $\mathrm{ord}_x(H) = 1$, hence the right hand side has a pole of order at x equal to $t-1$. We must therefore have $t \leq 1$. That is, $\omega' H$ extends to the whole curve modulo p.

On the other hand, ω' belongs to the localization $H^0(V, \omega^{-n}(n+1))_{\mathfrak{m}}$; since $H \equiv 1 \pmod p$, the multiplcation by H commutes to Hecke operators outside Np, so that $\omega' H$ belongs to the localization $H^0(\overline{X}, \omega^{p-1-n}(n+1))_{\mathfrak{m}}$. This artinian module is non zero, hence its socle (that is, the part annihilated by $\mathfrak{m}$) is also non zero. In other words, we can assume that $H\omega'$ is eigen for the twisted action of the Hecke algebra.

Recall that for any prime ℓ different from p, we do have $\ell^{k-1} d \circ T_\ell = T_\ell \circ d$. Therefore, after untwisting, we see that the eigenvalues of $H\omega'$ are the characteristic p numbers $b_\ell = \frac{a_\ell}{\ell^{k-1}}$. By Deligne-Serre's lemma, this eigensystem lifts to characteristic zero so that there exists an eigenform g of weight $p+1-k$ and a prime $\mathfrak{p}$ above p in the field of coefficients of g, so that the eigenvalues B_ℓ of g satisfy

$$B_\ell \equiv \frac{a_\ell}{\ell^{k-1}} \pmod{\mathfrak{p}} \quad \text{for all } \ell \neq p.$$

We conclude by irreducibility of $\overline{\rho}_f$ that the Galois representation $\overline{\rho}_f \otimes \omega^{k-1}$ is modular:

$$\overline{\rho}_f \otimes \omega^{k-1} = \overline{\rho}_g$$

5 Overconvergent companion forms in weight 1

In their work on the degree 2 Artin-Langlands conjecture, Buzzard and Taylor for their construction of a weight one cusp eigenform f realizing a given degree two Artin representation, made the assumption that this representation is residually modular. More precisely, they proved the following

Theorem 3 *Let $p \geq 5$ and let $\mathcal{O}$ be a discrete valuation ring with uniformizing parameter ϖ which is finite and flat over $\mathbb{Z}_p$; let $\rho : \mathrm{Gal}\,(\overline{\mathbb{Q}}, \mathbb{Q}) \to \mathrm{GL}_2(\mathcal{O})$ be a continuous representation such that*

- *ρ is unramified outside a finite set of primes*
- *$\rho \pmod{\varpi}$ is modular and absolutely irreducible*
- *ρ is unramified at p and $\rho(Frob_p)$ has eigenvalues α and β which are distinct modulo ϖ,*

then, there exists a modular form of weight one and an embedding $i : \mathbb{Q}(f) \hookrightarrow \mathcal{O}$ such that $\rho_f = \rho$. In particular, ρ has finite image and $L(\rho, s) = L(f, s)$ is entire.

Their method of proof consists in using these assumptions to construct two distinct Hida families; the authors then specialize these two families in weight one Then, they proceed to glue the overconvergent weight one eigenforms thus obtains, to construct the sought for classical eigenform f.

In the subsequent section, we explain how in the construction of the two families and their specialization in weight one as overconvergent forms, we may avoid the assumption of unramifiedness at p and the use of weight 2 and, sometimes, the use of a level divisible by p, by allowing a higher weight $k \in [2, p[$. We don't discuss the deepest problem, namely, the analytic continuation of these overconvergent forms.

5.1 Galois deformations and modularity

Let $p \geq 5$ be a prime; let f be a cusp eigenform of level N prime to p, character ϵ_f; assume that f is p-ordinary and of weight $k \in [2, p[$. Let$\mathcal{O}$ be the ring of integers of a p-adic field containing the eigenvalues of f such that $a_p(f) = \alpha + \beta$ with $\alpha \in \mathcal{O}^\times$ and $\alpha\beta = p^{k-1}\epsilon_f(p)$; let ϖ a uniformizing parameter of $\mathcal{O}$ and $\kappa = \mathcal{O}/(\varpi)$ its residue field.

Assume that the residual representation $\overline{\rho}_f$ splits at p:

$$\overline{\rho}_f | D_p = \xi(\overline{\alpha}) \oplus \xi(\overline{\frac{\beta}{p^{k-1}}})\omega^{1-k}$$

Let us consider the deformation problem over local artinian $\mathcal{O}$-algebras with residue field κ, which send such an algebra A to the set of strict conjugacy classes of Galois representations $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(A)$ deforming $\overline{\rho}_f$: $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\kappa)$ such that

- ρ is unramified outside Np,
- ρ is ordinary at p; moreover, its restriction to D_p is of the form

$$\rho|D_p \sim \begin{pmatrix} \xi(\alpha_\rho) & * \\ 0 & * \end{pmatrix}$$

  with $\alpha_\rho \equiv \alpha \pmod{\mathfrak{m}_A}$

This deformation problem is prorepresentable over a universal deformation ring R_α^{univ}. By a theorem of Diamond (in weight 2) and R. Ramakrishna's thesis (for generalization to higher weights $k \in [2,p[$), R_α^{univ} is a "natural" quotient of Hida's universal ordinary Hecke algebra $h^0(N)$ with auxiliary level N; by "natural", we mean, as on p.911 of [7], that the quotient map sends T_ℓ resp. S_ℓ to $\rho^{\mathrm{univ}}(Frob_\ell)$ resp. $\det \rho^{\mathrm{univ}}(Frob_\ell)$ for ℓ prime to Np, resp. U_ℓ to 0 for ℓ dividing N, and sends U_p to $\alpha^{\mathrm{univ}} \in (R_\alpha^{\mathrm{univ}})^\times$, where all the Frobeniuses are geometric and where α^{univ} is the scalar by which $Frob_p$ acts on the unramified line $H^0(I_p, \rho^{\mathrm{univ}})$. This theorem is actually a direct application of the Taylor-Wiles method (using Ramakrishna's thesis) if one assumes that the deformations are minimal at primes ℓ dividing N (see Taylor-Wiles, or see [32]). In particular, the representation ρ_f provides the classical eigenform of weight k and level Np: $f`_{\alpha,k} = f - \beta f(pz)$ which satisfies $f_{\alpha,k}|U_p = \alpha f_{\alpha,k}$.

But, more interestingly, given an Artin representation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathcal{O})$ such that

$$\rho|D_p \sim \begin{pmatrix} \xi(\alpha_\rho) & 0 \\ 0 & \xi(\beta_\rho)\omega^{1-k} \end{pmatrix}$$

and $\overline{\rho} = \overline{\rho}_f$, there exists a character $\lambda_\alpha : h^0(N) \to \mathcal{O}$ which factors through R^{univ}, say, as $\lambda_\alpha : R^{\mathrm{univ}} \to \mathcal{O}$ and such that $\rho_\alpha^{\mathrm{univ}} \otimes_{R_\alpha^{\mathrm{univ}},\lambda_\alpha} \mathcal{O} = \rho$. This provides by Hida theory a p-adic cusp eigenform f_α of weight 1, such that $f_\alpha|U_p = \alpha_\rho \cdot f_\alpha$. This form being p-ordinary is overconvergent (as explained in [**?**] Lemma 1, Sect.2), hence defines a section of the sheaf ω on an affinoid neighborhood of the rigid ordinary locus $]S[$ in $\overline{X}^{\mathrm{rig}}$.

On the other hand, by the theorem on companion forms proven above, there exists a p-ordinary cusp eigenform g of level N and weight $k' = p+1-k$ in $[2,p[$, such that $\overline{\rho} \otimes \omega^{1-k} = \overline{\rho}_g$.

Let us consider the deformation problem which sends a local $\mathcal{O}$-algebra A as before to the set of strict conjugacy classes of Galois representations $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(A)$ deforming $\overline{\rho}_g : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\kappa)$ such that

- ρ is unramified outside Np,
- ρ is ordinary at p and its restriction to D_p is of the form

$$\rho|D_p \sim \begin{pmatrix} \xi(\beta_\rho) & * \\ 0 & * \end{pmatrix}$$

with $\beta_\rho \equiv \frac{\beta}{p^{k-1}} \pmod{\mathfrak{m}_A}$

The representation $\rho \otimes \omega^{k-1}$ provides such a deformation of $\overline{\rho}_g$. The universal deformation ring R_β^{univ} is again a "natural" quotient of $h^0(N)$. This is defined by the same condition for T_ℓ, S_ℓ, U_ℓ for $\ell \neq p$, but at p the condition is that U_p maps to $\beta^{\mathrm{univ}} \in (R_\beta^{\mathrm{univ}})^\times$ such that $\beta^{\mathrm{univ}} \equiv \frac{\beta}{p^{k-1}} \pmod{\mathfrak{m}_\beta^{\mathrm{univ}}}$.

Again, this implies there exists a character of $\mathcal{O}$-algebras $\lambda_\beta : h^0(N) \to \mathcal{O}$ which factors through R_β^{univ}, say, as as $\lambda_\beta : R_\beta^{\mathrm{univ}} \to \mathcal{O}$, such that $\rho_\beta^{\mathrm{univ}} \otimes_{R_\beta^{\mathrm{univ}}, \lambda_\beta} \mathcal{O} = \rho$. This character gives rise to a p-adic cusp eigenform f_β with eigenvalues $\lambda_\beta(T_\ell)$ for ℓ prime to Np and such that $f_\beta|U_p = \beta_\rho f_\beta$.

Thus, we have proven:

Proposition 5.1 *Given a degree two odd Artin representation* $\rho : \mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathcal{O})$ *finitely ramified, such that*

$$\rho|D_p \sim \begin{pmatrix} \xi(\alpha_\rho) & 0 \\ 0 & \xi(\beta_\rho)\omega^{1-k} \end{pmatrix}$$

and which is residually irreducible and such that $\overline{\rho} = \overline{\rho}_f$ *for a cusp eigenform of level* N *prime to* p *and weight* $k \in [2, p[$, *there exist two* p*-adic overconvergent cusp eigenforms* f_{α_ρ} *and* f_{β_ρ} *of weight* 1 *and auxiliary level* N *such that* $f_{\alpha_\rho}|U_p = \alpha_\rho \cdot f_{\alpha_\rho}$, *resp.* $f_{\beta_\rho}|U_p = \beta_\rho \cdot f_{\beta_\rho}$.

This result has been recently generalized to the $GSp(4, \mathbb{Q})$-case in [32] and [33] when one replaces degree two odd Artin representations by degree four symplectic p-adic representations $\rho_{A,p}$ associated to irreducible abelian surfaces over $\mathbb{Q}$. The proof follows the same plan as the $\mathrm{GL}_2(\mathbb{Q})$-case: one constructs a companion form g to a p-ordinary cusp eigenform f of level N prime to p and p-small weight (k_1, k_2) (that is, $k_1 \geq k_2 \geq 3$ and $k_1 + k_2 - 3 < p - 1$), under the assumption that its residual representation $\overline{\rho}_f$ "partially splits" at p. Then, one considers two deformation problems consisting in p-ordinary symplectic rank four representations deforming $\overline{\rho}_f$ resp. $\overline{\rho}_g$. Then, we use a result of the type $R = T$ generalizing Diamond's theorem, but only in the minimal deformation case. Its specialization associated to $\rho_{A,p}$ resp. $\rho_{A,p} \otimes \omega^{k_2-2}$ yields the two overconvergent p-adic cusp eigenforms f_{α_A} resp. f_{β_A} of weight $(2,2)$ analogue to those in the proposition above.

However, the next point in the Artin case, treated in [7] and [8], is to make use of these forms to produce a classical cusp eigenform f_1 of level N or Np such that $\rho_{f_1} = \rho$. In the $GSp(4,\mathbb{Q})$-case, this question remains open.

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