PARITY INDUCED GROWTH

OF SELMER GROUPS OF

HILBERT MODULAR FORMS

OVER RING CLASS FIELDS

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1. GENERAL SETUP

- L, F number fields, $G_F = \mathrm{Gal}(\overline{F}/F)$
- M motive **over** F, **coefficients** in L
- M **self-dual** $M \xrightarrow{\sim} M^*(1)$ (skew-symmetric)
- M **pure** (of weight -1)
- $L(M,s) = \sum_{n\geq 1} a_n\, n^{-s} \qquad (a_n \in L)$
- $L(\iota M,s) = \sum_{n\geq 1} \iota(a_n)\, n^{-s} \qquad (\iota : L \hookrightarrow \mathbf{C})$
- $L_\infty(\iota M,s)$ Γ-factor (independent of ι)
- **Conjectural functional equation** (C_{FE}):

$$(L_\infty \cdot L)(\iota M,s) \stackrel{?}{=} \varepsilon(M)c(M)^{-s}(L_\infty \cdot L)(\iota M,-s)$$

$$\Longrightarrow \quad (-1)^{\mathrm{ord}_{s=0} L(\iota M,s)} = \varepsilon(M) = \pm 1$$

- **$\mathfrak{p}$-adic étale realization** of M ($\mathfrak{p} \mid p$ in L):

  geometric $L_{\mathfrak{p}}[G_F]$-module $M_{\mathfrak{p}}$

- skew-symmetric **self-duality** $M_{\mathfrak{p}} \xrightarrow{\sim} M_{\mathfrak{p}}^*(1)$

- **Bloch-Kato Selmer groups:**

$$H^1_f(F, M_{\mathfrak{p}}) \subset H^1(F, M_{\mathfrak{p}})$$

- **Conjecture of Bloch and Kato** (C_{BK}):

$$h^1_f(F, M_{\mathfrak{p}}) := \dim_{L_{\mathfrak{p}}} H^1_f(F, M_{\mathfrak{p}}) \stackrel{?}{=} \mathrm{ord}_{s=0} L(\iota M, s)$$

- $E_{/F}$ **elliptic curve,** $M = h^1(E)(1)$**,** $L = \mathbf{Q}$**,**

$\mathfrak{p} = p$**,** $M_{\mathfrak{p}} = V_p(E)$**,** $L(M, s) = L(E/F, s+1)$**,**

$$h^1_f(F, M_{\mathfrak{p}}) = \mathrm{rk}_{\mathbf{Z}}\, E(F) + \mathrm{cork}_{\mathbf{Z}_p}\, Ш(E/F)[p^\infty]$$

- **Parity conjecture for Selmer groups** (C_{par}):

$(-1)^{h^1_f(F, M_\mathfrak{p})} \stackrel{?}{=} \varepsilon(M) = \varepsilon(M_\mathfrak{p}) = \prod_v \varepsilon_v(M_\mathfrak{p})$

$M_\mathfrak{p}$ self-dual pure geometric $L_\mathfrak{p}[G_F]$-module

- $(C_{par}) \stackrel{(C_{FE})}{\Longleftrightarrow} (C_{BK} \pmod 2)$

- $L(\iota M, s)$ automorphic $\stackrel{\text{often}}{\Longrightarrow}$ automorphic (C_{FE})

- $\varepsilon_v(M_\mathfrak{p}) \stackrel{?}{=}$ automorphic ε_v (often OK if $v \nmid p$)

2. HILBERT MODULAR FORMS

- F totally real

- $g \in S_k(\mathfrak{n}, 1)$ cuspidal Hilbert newform

over F of level $\mathfrak{n}$, **trivial character,**

parallel (even) weight k

- $\pi = \pi(g)$ automorphic representation

of $PGL_2(\mathbf{A}_F)$ attached to g

- K CM field, $[K : F] = 2$

- $\mathrm{rec}_K : \mathbf{A}_K^* \longrightarrow \mathrm{Gal}(K^{\mathrm{ab}}/K)$

- $K[\infty] = (K^{\mathrm{ab}})^{\mathrm{rec}_K(\mathbf{A}_F^*)}$

- $\chi : \mathrm{Gal}(K[\infty]/K) \longrightarrow \mathbf{C}^*$ continuous

- $K_\chi = K[\infty]^{\mathrm{Ker}(\chi)}$, **dihedral over** F

- Theta series $\theta_\chi \in M_1(\mathfrak{n}_\chi, \eta)$,

$\eta = \eta_{K/F} : \mathbf{A}_F^*/K^* N_{K/F}(\mathbf{A}_K^*) \xrightarrow{\sim} \{\pm 1\}$

- $L(\pi \times \chi, s) := L(\pi(g) \times \pi(\theta_\chi), s)$

- $L(\pi \times \chi, s) = \varepsilon(\pi \times \chi, \frac{1}{2})\, c^{-s} L(\pi \times \chi, 1 - s)$

- Fix: $\iota : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$, $\iota_p : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$, $L \subset \overline{\mathbf{Q}}$ s.t.

$\iota(L) \supset \mathrm{Im}(\chi) \cup$ all Hecke eigenvalues of g

- $\iota_p : L \hookrightarrow L_{\mathfrak{p}} \subset \overline{\mathbf{Q}}_p$, $\quad V_{\mathfrak{p}}(g)$ $L_{\mathfrak{p}}[G_F]$-module

- $V = V_{\mathfrak{p}}(g)(k/2) \xrightarrow{\sim} V^*(1) \qquad (\dim_{L_{\mathfrak{p}}}(V) = 2)$

- $L(\pi \times \chi, s + \frac{1}{2}) = L(\iota M, s)$, $\quad V \otimes \mathrm{Ind}_{K/F}(\chi) = M_{\mathfrak{p}}$

- $H^1_f(F, M_{\mathfrak{p}}) = H^1_f(K, V \otimes \chi) = H^1_f(K_\chi, V)^{(\chi^{-1})}$

- $h^1_f(K, V \otimes \chi) = \dim_{L_{\mathfrak{p}}} H^1_f(K, V \otimes \chi)$

$(C_{BK}) \qquad \mathrm{ord}_{s=1/2} L(\pi \times \chi, s) \stackrel{?}{=} h^1_f(K, V \otimes \chi)$

THM 1. If g is **potentially p-ordinary**

$(\exists \varphi \quad g \otimes \varphi$ is p-ordinary) and has no CM,

$2 \nmid \mathrm{ord}_{s=1/2} L(\pi \times \chi, s) \Longrightarrow 2 \nmid h^1_f(K, V \otimes \chi)$.

- OK if g has CM by $K' \not\subset K_\chi$ and $p \neq 2$

Cor. If $K \subset K_0 \subset K[\infty]$, $[K_0 : K] < \infty$ and g is potentially p-ordinary with no CM, $h^1_f(K_0, V) \geq |X^-(g, K_0)|$, where $X^\pm(g, K_0) =$

$= \{\chi : \mathrm{Gal}(K_0/K) \longrightarrow \mathbf{C}^* \mid \varepsilon(\pi \times \chi, \frac{1}{2}) = \pm 1\}$

- OK if g has CM by $K' \not\subset K_0$ and $p \neq 2$
- There is **no general formula** for $\varepsilon(\pi \times \chi, \frac{1}{2})$

Ex. $q \neq 2$ prime, $F \cap \mathbf{Q}(\mu_{q^\infty}) = \mathbf{Q}$, $a \in O_F$, $a \notin F^{*q}$, $\forall v \nmid q \quad \mathrm{ord}_v(a) < q$.

THM 2. If $g \in S_k(\mathfrak{n}, 1)$ is ‘‘nice" at $v \mid qa$, has no CM, is potentially p-ordinary,

$d = (-1)^{[F:\mathbf{Q}]} N(\mathfrak{n}/(\mathfrak{n},(aq)^\infty))$, $\left(\frac{d}{q}\right) = -1$, then

$\forall r \geq 0 \quad h^1_f(F(\mu_{q^r}, \sqrt[q^r]{a}), V) - h^1_f(F,V) \geq q^r - 1$,

$h^1_f(F(\sqrt[q^r]{a}), V) - h^1_f(F,V) \geq r, \ \equiv r \pmod 2$.

THM 1 $\Longrightarrow$ THM 2:

- $F_r = F\mathbf{Q}(\mu_{q^r})^+ \subset K_r = F(\mu_{q^r}) \qquad (r \geq 1)$
- Fix: $\quad \chi_r : \mathrm{Gal}(K_r(\sqrt[q^r]{a})/K_r) \xrightarrow{\sim} \mu_{q^r}$

$H^1_f(F(\sqrt[q^r]{a}), V) = H^1_f(F,V) \oplus \bigoplus_{s=1}^r H^1_f(K_s, V \otimes \chi_s)$

- If $\forall v \mid q \ \ \pi_v$ is in principal series
- and $\forall v \mid a \ \ \pi_v$ is not supercuspidal, then
- $\varepsilon(BC_{F_s/F}(\pi) \times \chi_s, \frac{1}{2}) = \left(\frac{d}{q}\right)$

3. ELLIPTIC CURVES

THM 3. F totally real, $p \neq 2$ if $F \neq \mathbf{Q}$,

F_0/F finite abelian, F'/F_0 Galois of odd

degree, E elliptic curve over F, either

(1) $2 \nmid [F:\mathbf{Q}]$ and E is modular over F, or

(2) $j(E) \notin O_F$. Then:

$$\mathrm{ord}_{s=1} L(E/F', s) \equiv h^1_f(F', V_p(E)) \pmod 2$$

4. TOWARDS $(C_{par}(M_{\mathfrak{p}}))$: $(-1)^{h^1_f(F, M_{\mathfrak{p}})} \stackrel{?}{=} \varepsilon(M_{\mathfrak{p}})$

(I) **Deformation:** $C_{par}(M_{\mathfrak{p}}) \overset{\text{often}}{\Longleftrightarrow} C_{par}(M'_{\mathfrak{p}})$

if $M_{\mathfrak{p}}, M'_{\mathfrak{p}} \in$ self-dual **$\mathfrak{p}$-adic family**

(II) **Euler system** (ES) for $M'_{\mathfrak{p}}$

(III) **Non-triviality** of this (ES)

- (II), (III) $\stackrel{\text{often}}{\Longrightarrow}$ $h^1_f(F, M'_{\mathfrak{p}}) = 1$, $\varepsilon(M'_{\mathfrak{p}}) = -1$

- **Examples:** (Ia) **Hida theory** (twisted)
  - (Ib) **Dihedral Iwasawa theory**
  - (II) **CM points** (Kolyvagin, ...)
  - (III) **CM points** (Cornut-Vatsal, ...)

5. TERMINOLOGY

- V geometric $L_{\mathfrak{p}}[G_F]$-module (‘‘$V = M_{\mathfrak{p}}$")

- $V_v = V|_{G_{F_v}}$, $G_{F_v} \supset W_{F_v}$ Weil group ($v \nmid \infty$)

- $WD(V_v)$ repr. of the Weil-Deligne group

attached to V_v (use $D_{pst}(V_v)$ if $v \mid p$)

- Monodromy filtration M_n on $WD(V_v)$:

$NM_n \subseteq M_{n-2}$, $N^r : gr_r^M \xrightarrow{\sim} gr_{-r}^M$ $(r \geq 0)$

- V_v is **pure of weight** $w \in \mathbf{Z}$ iff $\forall r \in \mathbf{Z}$ gr_r^M

is pure of weight $w + r$ as a W_{F_v}-repr.

- V is **pure of weight** $w \in \mathbf{Z}$ iff each V_v is.

Monodromy weight conjecture: $X_{/F}$ proper smooth $\Longrightarrow$ $H^m_{et}(X_{\overline{F}}, L_{\mathfrak{p}})(n)$ pure of wt $m - 2n$

- Known: at v where X has good reduction
  - $X = \prod X_i$, X_i AV or $\dim(X_i) \leq 2$

- V satisfies **Pančiškin's condition** at $v \mid p$:

$0 \longrightarrow V_v^+ \longrightarrow V_v \longrightarrow V_v^- \longrightarrow 0$ ($L_{\mathfrak{p}}[G_{F_v}]$-modules), $D^0_{dR}(V_v^+) = 0 = D_{dR}(V_v^-)/D^0_{dR}(V_v^-)$

- **Ex.** E elliptic curve over F, $V = V_p(E)$

- **split mult. red. at v:** $V_v^+(-1) = V_v^- = \mathbf{Q}_p$

- **good ord. red. at v:** $V_v^+(-1) = V_v^{-*} = A$

  (A unramified, pure of weight 1)

- **ε-factors (self-dual case):**

- $V \xrightarrow{\sim} V^*(1)$ (skew-symmetric isomorphism)

- $\varepsilon(V) = \prod_w \varepsilon_w(V) = \pm 1$

- $\forall w \nmid \infty \quad \varepsilon_w(V) = \varepsilon(WD(V_w), \psi, d\mu_\psi)$,

- ψ non-trivial additive character of F_w

- $d\mu_\psi$ ψ-self-dual Haar measure on F_w

- $\varepsilon_w(V) = \pm 1$ **does not depend on ψ**

- Fix: $\lambda : \overline{\mathbf{Q}}_p \xrightarrow{\sim} \mathbf{C}$ s.t. $\lambda^{-1}(\iota(L)) \subset L_{\mathfrak{p}}$

- $S_p = \{v \mid p \text{ in } F\}$, $\quad S_\infty = \{w \mid \infty \text{ in } F\}$

- $r_p : \mathrm{Hom}(F, \overline{\mathbf{Q}}_p) \longrightarrow S_p$

- $r_\infty : \mathrm{Hom}(F, \mathbf{C}) \longrightarrow S_\infty$

- $\forall v \in S_p \qquad S_\infty(v) = r_\infty \circ \lambda_*(r_p^{-1}(v)) \subset S_\infty$

- $\forall v \in S_p \qquad \prod_{w \in S_\infty(v)} \varepsilon_w(V) = \pm 1$ is defined

in terms of $\dim_{L_{\mathfrak{p}}} D^i_{dR}(V_v)/D^{i+1}_{dR}(V_v)$

6. DEFORMATION RESULT

THM 4 (Doc. Math. 12 (2007) + Erratum).

If V, V' are (geometric, pure of weight -1)

regular members of a self-dual 1-parameter

p-adic family $\mathcal{T}$ of repr. of $\mathrm{Gal}(F_S/F)$

satisfying Pančiškin's condition $\forall v \mid p$,

$$(-1)^{h^1_f(F,V)}/\varepsilon(V) = (-1)^{h^1_f(F,V')}/\varepsilon(V').$$

- **Ex. (Dihedral Iwasawa theory)**

- $F \subset F_0 \subset F_\infty$, $\quad \Gamma = \mathrm{Gal}(F_\infty/F_0)$ **abelian**,

Γ_{tors} finite, $\quad \Gamma/\Gamma_{\mathrm{tors}} \xrightarrow{\sim} \mathbf{Z}_p^r$

$\mathrm{Gal}(F_\infty/F) = \Gamma \rtimes \{1, c\}$, $\quad c^2 = 1$, $\quad c\gamma c^{-1} = \gamma^{-1}$

- $V \xrightarrow{\sim} V^*(1)$ (skew-symmetric), V geometric

$L_{\mathfrak{p}}[G_F]$-module, pure (of weight -1),

satisfying Pančiškin's condition $\forall v \mid p$.

Cor. If $\chi, \chi' : \Gamma \longrightarrow L_{\mathfrak{p}}^*$ are characters of finite order such that $\chi|_{\Gamma_{\rm tors}} = \chi'|_{\Gamma_{\rm tors}}$, then

$(-1)^{h^1_f(F_0, V\otimes\chi)}/\varepsilon(F_0, V\otimes\chi) =$

$(-1)^{h^1_f(F_0, V\otimes\chi')}/\varepsilon(F_0, V\otimes\chi')$.

PROOF OF THM 4

(1) Extended Selmer groups $\widetilde{H}^1_f(F,V)$

- $H^1_f(F,V)$ related to **complex L-functions**

- $\widetilde{H}^1_f(F,V)$ related to **p-adic L-functions**

$$0 \longrightarrow \bigoplus_{v|p} H^0(F_v, V_v^-) \longrightarrow \widetilde{H}^1_f(F,V) \longrightarrow$$
$$\longrightarrow H^1_f(F,V) \longrightarrow 0$$

- $\widetilde{\varepsilon}_v(V) = \varepsilon_v(V) \times \begin{cases} (-1)^{\dim_{L_\mathfrak{p}} H^0(F_v, V_v^-)}, & v \mid p \\ 1 & v \nmid p \end{cases}$

- $(-1)^{h^1_f(F,V)}/\varepsilon(V) = (-1)^{\widetilde{h}^1_f(F,V)}/\widetilde{\varepsilon}(V)$

(2) $\forall v \nmid p\infty \qquad \varepsilon_{0,v}(V) = \varepsilon_{0,v}(V') \qquad$ (easy)

purity of $V, V' \implies \varepsilon_v(V) = \varepsilon_v(V')$

(3) $\forall v \mid p \qquad \widetilde{\varepsilon}_v(W) \prod_{w \in S_\infty(v)} \varepsilon_w(W)$ **is the**

same for $W = V, V'$ (explicit formulas)

- (2) + (3) $\implies \widetilde{\varepsilon}(V) = \widetilde{\varepsilon}(V')$

- Morally, $\widetilde{\varepsilon}(V)$ is the **"ε-factor of the p-adic L-function attached to the family $\mathcal{T}$"**

(4) $\mathcal{T}$ is an $R[\mathrm{Gal}(F_S/F)]$-module, $R \supset \mathbf{Z}_p$

complete local noeth. domain, $\dim(R) = 2$

- $\mathcal{V} = \mathcal{T} \otimes_R \mathrm{Frac}(R) \xrightarrow{\sim} \mathcal{V}^*(1)$ (skew-symm.)

- $P, P' \in \mathrm{Spec}(R)$, $\qquad R_P, R_{P'}$ **DVR**

- $V = \mathcal{T}_P / P\mathcal{T}_P\,, \qquad V' = \mathcal{T}_{P'} / P'\mathcal{T}_{P'}$

- $\forall v \mid p \qquad 0 \longrightarrow \mathcal{V}_v^+ \longrightarrow \mathcal{V}_v \longrightarrow \mathcal{V}_v^- \longrightarrow 0$

inducing Pančiškin's condition for V_v, V'_v

$$0 \longrightarrow \widetilde{H}^1_f(\mathcal{T})_P/P \longrightarrow \widetilde{H}^1_f(V) \longrightarrow \widetilde{H}^2_f(\mathcal{T})_P[P] \longrightarrow 0$$

- There is a **symplectic pairing**

$$\left(\widetilde{H}^2_f(\mathcal{T})_P\right)_{\mathrm{tors}} \times \left(\widetilde{H}^2_f(\mathcal{T})_P\right)_{\mathrm{tors}} \longrightarrow \mathrm{Frac}(R)/R_P$$

on an R_P-module of finite length

$$\Longrightarrow \quad \widetilde{h}^1_f(V) \equiv \mathrm{rk}_R\, \widetilde{H}^1_f(\mathcal{T}) \pmod 2$$

- The same holds for V'

$$\Longrightarrow \quad \widetilde{h}^1_f(V) \equiv \widetilde{h}^1_f(V') \pmod 2 \quad \Longrightarrow \quad \textbf{THM 4}$$

THM 1: $\varepsilon(\pi(g)\otimes\chi,\frac{1}{2}) = -1 \Longrightarrow 2 \nmid h^1_f(K, V\otimes\chi)$

(1) **If $k > 2$, reduce to $k = 2$:**

- $\exists f = g\otimes\varphi \in S_k(\mathfrak{n}_f, \varphi^2)$ p-ordinary

- p-stabilization of f $\in$ Hida family $\mathcal{F}$

- $\exists_\infty f' \in \mathcal{F}$ of weight 2 s.t. $g' = f'\otimes(\varphi')^{-1}$

$\in S_2(\mathfrak{n}', 1)$, $\varepsilon(\pi(g')\otimes\chi,\frac{1}{2}) = \varepsilon(\pi(g)\otimes\chi,\frac{1}{2})$

- **THM 4** $\Longrightarrow$ (**THM 1** for g $\Longleftrightarrow$ for g')

(2) **Reduction to $|S_p| > 1$:**

- If $S_p = \{v\}$, $\exists$ F'/F cyclic of odd degree

in which v splits (F' totally real)

- $K' = KF'$, $\chi' = \chi\circ N_{K'/K}$

- $\pi(g') = BC_{F'/F}(\pi(g))$, g' pot. p-ordinary

- $\varepsilon(\pi(g') \otimes \chi', \frac{1}{2}) = \varepsilon(\pi(g) \otimes \chi, \frac{1}{2})$

- $h^1_f(K', V \otimes \chi') \equiv h^1_f(K, V \otimes \chi) \pmod 2$

(3) If $k = 2$, $\varepsilon(\pi \otimes \chi, \frac{1}{2}) = -1$, then

- $\exists$ a **Shimura curve** N^*_H over F

- $\exists$ a **simple quotient** $\mathrm{Jac}(N^*_H) \longrightarrow A_0$

$\Longrightarrow$ $\alpha : N^*_H \longrightarrow \mathrm{Jac}(N^*_H) \longrightarrow A_0$ (over F)

- $\exists\, L_0 \subset L$ tot. real, $O_{L_0} = \mathrm{End}_F(A_0)$

$\iota(L_0) = \mathbf{Q}(\text{all Hecke eigenvalues of } g)$

- $V = V_{\mathfrak{p}}(A)$, $\qquad A = O_L \otimes_{O_{L_0}} A_0$

- $S_\infty = \{\tau_1, \ldots, \tau_d\}$, $\qquad d = [F : \mathbf{Q}]$

- B a **quaternion algebra over** F s.t.

$$B_{\tau_j} = B \otimes_{F,\tau_j} \mathbf{R} \xrightarrow{\sim} \begin{cases} M_2(\mathbf{R}), & j = 1 \\ \mathbf{H}, & j > 1 \end{cases}$$

- $H \subset \widehat{B}^* = (B \otimes \widehat{\mathbf{Z}})^*$ **open compact**

- $(N_H \otimes_{F,\tau_1} \mathbf{C})(\mathbf{C}) = B^* \backslash (\mathbf{C} - \mathbf{R}) \times \widehat{B}^* / H \widehat{F}^*$

$[z,b]_H \in (N_H \otimes_{F,\tau_1} \mathbf{C})(\mathbf{C})$, $[\cdot g] : [z,b] \mapsto [z,bg]$

- $N_H^* = N_H$ ($\cup \{\text{cusps}\}$ if $B \xrightarrow{\sim} M_2(\mathbf{Q})$)

CM points on N_H:

- Given $t : K \hookrightarrow B$, $t|_F = \text{id}$; set $\widehat{t} : \widehat{K} \hookrightarrow \widehat{B}$

- $\exists! \ z \in \mathbf{C} \qquad \text{Im}(z) > 0, \quad t(K^*)z = z$

- $CM_K(N_H) = \{[z,b]_H \mid b \in \widehat{B}^*\}$

- **Shimura's reciprocity law:**

$$\forall a \in \widehat{K}^* \qquad \mathrm{rec}_K(a)\,[z,b]_H = [z, \widehat{t}(a)b]_H$$

$$\Longrightarrow \quad CM_K(N_H) \subset N_H(K[\infty])$$

- `Given` $x \in CM_K(N_H)$ `defined over`

$K(x) \supset K_\chi = K[\infty]^{\mathrm{Ker}(\chi)}$`, then`

- $\chi : G_x = \mathrm{Gal}(K(x)/K) \longrightarrow O_L^*$

- $e_{\overline{\chi}} := \sum_{g \in G_x} \chi(g)\, g \in O_L[G_x]$

THM 5 (Proc. Durham 2004). `If` g `has`

`CM by` K'`, assume` $K' \not\subset K_\chi$`.  Then:`

$$e_{\overline{\chi}}(1 \otimes \alpha(x)) \notin A_{\mathrm{tors}} \implies h^1_f(K, V \otimes \chi) = 1$$

WELL KNOWN FACTS:

(F1) If $e_{\overline{\chi}}(1 \otimes \alpha(x)) \notin A_{\rm tors}$, then $\forall v \nmid \infty$

($\star_v$) $\quad \mathrm{inv}_v(B_v) = \eta_v(-1)\,\varepsilon_v(\pi \otimes \chi, \frac{1}{2})$

$\Longrightarrow$ B is **determined** by K, χ and $\pi = \pi(g)$

(above, $\eta = \eta_{K/F} : \mathbf{A}_F^* \longrightarrow \{\pm 1\}$)

(F2) $e_{\overline{\chi}}(1 \otimes \alpha(x)) \notin A_{\rm tors} \Longrightarrow \varepsilon(\pi \otimes \chi, \frac{1}{2}) = -1$

(4) **Back to THM 1:** need to construct

- $K \subset K_\chi \subset K_\infty \subset K[\infty]$ s.t.

$\Gamma = \mathrm{Gal}(K_\infty/K) \xrightarrow{\sim} \Gamma_{\rm tors} \times \mathbf{Z}_p^r$, $\Gamma_{\rm tors}$ finite

- CM point $x' \in CM_K(N_H^*)$ defined over K_∞

- $\chi' : \Gamma \longrightarrow \overline{\mathbf{Q}}^* \subset \overline{\mathbf{Q}}_p^*$ of finite order s.t.

$\chi'|_{\Gamma_{\rm tors}} = \chi|_{\Gamma_{\rm tors}}, \qquad e_{\overline{\chi}'}(1 \otimes \alpha(x')) \notin A_{\rm tors}$

$\overset{THM5}{\Longrightarrow}$ **THM 1** `for` χ' $\overset{THM4}{\Longrightarrow}$ **THM 1** `for` χ

`(5)` **Invariant linear forms** `(`$v \nmid \infty$`)`

- $j_v : K_v = K \otimes_F F_v \hookrightarrow B_v = B \otimes_F F_v$
- π'_v **smooth irred. gen. repr. of** B_v^*/F_v^*
- $\varphi_v : K_v^*/F_v^* \longrightarrow \mathbf{C}^*$ `(continuous)`

THM (Tunnell, Waldspurger, H. Saito)

$\mathrm{Hom}_{j_v(K_v^*)}(\pi'_v \otimes \varphi_v, \mathbf{C}) = \mathbf{C}\ell_v$ `if`

$\mathrm{inv}_v(B_v) = \eta_v(-1)\, \varepsilon_v(\pi'_v \otimes \varphi_v, \frac{1}{2})$ `(`$= 0$ `if not)`

[T-W-S] THM $\Longrightarrow$ FACT (F1)

`(`$\star\star_v$`)` $\qquad \mathrm{inv}_v(B_v) = \eta_v(-1)\, \varepsilon_v(\pi'_v \otimes \varphi_v, \frac{1}{2})$

• v **splits** in $K/F \implies \mathrm{inv}_v(B_v) = \eta_v(-1) =$

$= \varepsilon_v(\pi \otimes \chi, \frac{1}{2}) = 1 \implies$ $(\star\star_v)$

• v **does not split** in $K/F \implies K_v^*/F_v^*$ is

pro-finite, so $\mathrm{Hom}_{j_v(K_v^*)}(\pi_v' \otimes \varphi_v, \mathbf{C}) \neq 0$

$\iff$ $(\star\star_v)$ $\iff$ $(\pi_v' \otimes \varphi_v)^{j_v(K_v^*)} \neq 0$

FROM LEFT TO RIGHT

• $x = [z, b]_H \in CM_K(N_H^*)$, $e_{\overline{\chi}}(1 \otimes \alpha(x)) \notin A_{\mathrm{tors}}$

• Set $j = \mathrm{Ad}(b)^{-1} \circ \widehat{t} : \widehat{K} \hookrightarrow \widehat{B}$

$\forall a \in \widehat{K}^* \quad \mathrm{rec}_K(a)\,[z, b]_H = [z, \widehat{t}(a)b]_H = [z, bj(a)]_H$

• $\exists$ finite $\Delta \subset \widehat{K}^*$, $\exists$ open compact $H_1 \subset \widehat{B}^*$

$f_a : N_{H_1}^* \xrightarrow{[\cdot j(a)]} N_{j(a)^{-1}H_1 j(a)}^* \longrightarrow N_H^* \xrightarrow{\alpha} A_0$ $(a \in \Delta)$

$n \cdot e_{\overline{\chi}}(1 \otimes \alpha(x)) = \sum_{a \in \Delta} \chi(a) f_a(x_1), \quad x_1 = [z, b]_{H_1}$

- $n \cdot e_{\overline{\chi}}(1 \otimes \alpha(x)) = \alpha_1(x_1)$ $\qquad$ ($n \geq 1$)

- $\alpha_1 = \sum_{a \in \Delta} \chi(a) f_a : N^*_{H_1} \longrightarrow A$

- $\alpha_1(x_1) \notin A_{\text{tors}} \implies \alpha_1 \neq$ constant $\implies$

$\exists\, \omega \in \Gamma(A^{\text{an}}, \Omega^{\text{an}}) \quad \alpha_1^*(\omega) \neq 0$

- $\Gamma(A^{\text{an}}, \Omega^{\text{an}})$ can be written in terms of

$(\pi'_f)^H$ (π' automorphic representation of

$B^*_{\mathbf{A}}$, $JL(\pi') = \pi$) and its Galois conjugates

$\implies 0 \neq \alpha_1^*(\omega) \in (\pi'_f \otimes \chi)^{j(\widehat{K}^{*\circ})} \implies$ (F1)

NON-TRIVIALITY OF CM POINTS

- $K[\infty] = \bigcup K[c]$, $c \subset O_F$, $O_c = O_F + cO_K$

$\mathrm{Gal}(K[c]/K) \xrightarrow{\sim} \widehat{K}^*/K^*\widehat{F}^*(\widehat{O}_c)^*$

- $(c, P_1 \cdots P_s) = 1$ ($s \geq 1$), $\forall i \ \mathrm{inv}_{P_i}(B_{P_i}) = 1$
- $K_\infty = K[c(P_1 \cdots P_s)^\infty]$, $\Gamma = \mathrm{Gal}(K_\infty/K)$
- $\mathcal{C} = \{[z, b_1 \cdots b_s b]_H \mid b_i \in B_{P_i}^*\} \subset CM_K(N_H^*)$

Want: $\forall \chi_0 : \Gamma_{\mathrm{tors}} \to O_L^* \; \exists\, x \in \mathcal{C} \; e_{\overline{\chi}_0}(1 \otimes \alpha(x)) \notin A_{\mathrm{tors}}$

METHOD OF CORNUT AND VATSAL

- $\exists\, \Gamma_1 \subset \Gamma_0 = \Gamma_{\mathrm{tors}}$ acting in the same

"geometric" way on all $x \in \mathcal{C}$:

$e_{\overline{\chi}_0}(1 \otimes \alpha(x)) = \sum_{\sigma \in \mathcal{R}} \chi_0(\sigma)\, \alpha_1(\sigma(x_1))$

- $\mathcal{R} \subset \Gamma_0$ representatives of Γ_0/Γ_1
- $x_1 \in \mathcal{C}_1 \subset CM_K(N_{H_1}^*)$

• $\{\sigma \in \mathcal{R}\}$ act **"independently"** on $\mathcal{C}$

(uses a known case of the **André-Oort**

conjecture, proved by Edixhoven-Yafaev)

• **Conclusion:** if $\alpha_1 \neq$ constant, then

$\exists\, x \in \mathcal{C} \qquad e_{\overline{\chi}_0}(1 \otimes \alpha(x)) \notin A_{\mathrm{tors}}$

Proposition (Aflalo - J.N.) Assumptions $(\star_v)$

for $v \in S$ and existence of $\omega \in \Gamma(A^{\mathrm{an}}, \Omega^{\mathrm{an}}) \cap$

$(\pi'_f)^H$ s.t. $\quad \forall v \in S \quad \ell_v(\omega) \neq 0 \implies \alpha_1^*(\omega) \neq 0$

END OF PROOF OF THM 1

• **Choose very carefully:**

• c, $P_1, \ldots, P_s \mid p$ s.t. $\quad K_\chi \subset K[c(P_1 \cdots P_s)^\infty]$

- B, $H \subset \widehat{B}^*$, $\alpha : N_H^* \longrightarrow A_0$

- $\mathcal{C} \subset CM_K(N_H^*)$ ($P_1 \cdots P_s$-isogeny class)

$\Longrightarrow$ get $\chi' : \Gamma = \mathrm{Gal}(K[c(P_1 \cdots P_s)^\infty]/K) \longrightarrow \overline{\mathbf{Q}}^*$

s.t. $\chi_0 := \chi|_{\Gamma_{\mathrm{tors}}} = \chi'|_{\Gamma_{\mathrm{tors}}}$ and $x \in \mathcal{C}$ s.t.

$e_{\overline{\chi}'}(1 \otimes \alpha(x)) \notin A_{\mathrm{tors}}$

PROOF OF THM 3

- Reduce to $F' = K$, $\varepsilon(\pi \otimes 1, \frac{1}{2}) = -1$

- Use Cornut-Vatsal

- Replace THM 4 by a result of Mazur-Rubin

on $h_f^1(-, V_p(E))$ in dihedral extensions

of order $2p^n$