Families of Galois representations and families of automorphic forms - an introduction

Tuesday, March 25, 2008
12:29 PM

Frank Calegari Page 1

Calegari $k/\mathcal{O}$ Galois extn. $G = \text{Gal}(k/\mathcal{O})$.

$\mathfrak{m}$ a Galois extn. $\text{Gal}(k/\mathcal{O}_p) \to \text{Gal}(k/\mathcal{O})$.

$p$ unramified over this is an isomorphism.

In this case, we get a distinguished element in $\text{Gal}(k/\mathcal{O})$, the Frobenius $\sigma_p$, denoted $\sigma_p \in \text{Gal}(k/\mathcal{O}_p) \cong \text{Gal}(k/\mathcal{O})$.

If $\tau, \tau' \in k^\times$, then $\tau \sigma_p = \tau' \sigma_p^{-1}$.

So given $p \not\mid \mathcal{O}$, we get a conjugacy class $\sigma_p < G$.

By Chebotarev Density Theorem, every element in $G$ is a Frobenius for some conjunctive classes.

Furthermore, the collection $\sigma_p$ together with $G$ determine $k$.

The same analysis applies to any Galois extension $k/F$ ($F \neq k(\mathcal{O})$), now the Frobenius conjugacy classes are indexed by primes in $F$.

Fix a set of primes $S$ of $F$, set $C_{F, S} = \{ \sigma \in \text{Gal}(k/F) \mid \text{$\sigma$ unramified outside $S$} \}$.

$C_{F, S}$ is a topological group with neighborhood basis around the identity given by the kernels of maps $C_{F, S} \to \text{Gal}(k/\mathcal{O})$.

Topologically generated by Frobenius classes over primes outside $S$. 

(… 0 1 0 1 …)
\( G_{F,S} \) has a finite class \( \langle p \rangle \) for \( p \in G_F, p \notin S. \)

Assumption: \( |S| < \infty \) (\( S \) topologically generate \( G_{F,S} \))

Example: \( S = \{a\}, F = \emptyset, G_{F,S} = \{1\} \) (Nikolski)
A natural way to study a topological group is to study its representations.

\[ S = \mathbb{Z}_p \times, \quad G_{\mathbb{Z}_p} \cong G(\mathbb{Z}_p^\times/\mathbb{Z}_p) \cong \text{Gal}(\mathbb{Q}(\mu_p^\times)/\mathbb{Q}) \]

\[ \cong \mathbb{Z}_p^\times. \]

If \( \text{Gal}(\mathbb{Q}(\mu_p^\times)/\mathbb{Q}) \), then what is the image of \( \sigma_p \) in \( \mathbb{Z}_p^\times \)? Answer: \( l \).

\[ G_{\mathbb{Z}_p} \cong \mathbb{Z}_p^\times \]

\[ \langle \sigma_p \rangle \mapsto l \quad (\text{for } l \neq p). \]
Kronecker-Hecke:

$$\mathbb{Z}_p^* \cong \mathbb{Q}_p \times \mathbb{Z}_p^*.$$  
(Dirichlet's theorem on a.p. \(l\)'s generate \(\mathbb{Z}_p^*\).

(Chabauty's theorem can be viewed as a generalization of Dirichlet's theorem to the non-abelian case.)

We want to study \(G_{E_{1}}\) by considering its continuous representations.

\[p: G_{E_{1}} \to GL_n(\mathbb{E})\]

together with \(\langle p, \sigma \rangle\).

(i) \(E = \mathbb{C}\)

(ii) \(E = \mathbb{F}_q \cong \overline{\mathbb{F}_q}\)

(iii) \(E = \) finite extension of \(\mathbb{Q}_p\)

In (i), (ii), \(\operatorname{Im} p\) is finite.

In (iii) \(\operatorname{Im} p\) may be infinite. \(E = \mathbb{Q}_p,\)

\[\chi: G_{\mathbb{Q}_p} \rightarrow \mathbb{Z}_p^*\]
\[\chi: GL(\mathbb{Q}_p) = \mathbb{Q}_p^*\]

Lemma. Let \([E: \mathbb{Q}_p] < \infty\). Let \(p: G_{E_{1}} \rightarrow GL_n(E)\) be continuous, then after conjugation, the image lies in \(GL_n(\mathbb{E})\).

Think of \(p_\mathbb{E}\) for \(\mathbb{E} \cong \mathbb{F}_q\), as an endomorphism on a vector space \(V\) of dimension \(n\) over \(E\). Choose a \(\mathbb{E}\)-lattice inside \(V\), say \(L\). Consider the \(\mathbb{E}[G_{\mathbb{F}_q}]\)-module generated by \(L\), call it \(L'\). By continuity, \(L'\) is compact \(\Rightarrow L' = \frac{1}{p^r} L\) for some \(r\).
continuity, \( L' \) is compact \( \Rightarrow \) \( L' \leq \frac{1}{\rho} L \) for some \( \rho \).

5. \( L' \) is a lattice. \( \square \).

The upshot of this is that we can consider the reduction

\[
\rho: G_{F,r} \rightarrow \text{GL}_n(C_\infty)
\]

\[
\text{GL}_n(C_{F,1}/C_{F,2}) \quad \text{or} \quad \text{GL}_n(F_q).
\]

In summary, we can break down our problem of studying \( \rho \) into:

(a) Understand all maps \( \tilde{\rho}: G_{F,r} \rightarrow \text{GL}_n(F_q) \) (Hard)

(b) Given \( \tilde{\rho} \), understand all “lifts” \( \rho \Rightarrow \tilde{\rho}: G_{F,r} \rightarrow \text{GL}_n(C_{F,1}) \) (Easier)

(Why: \( \ker ( \text{GL}_n(C_{F,1}) \rightarrow \text{GL}_n(F_q) ) \) is solvable, since it's pro-\( \rho \).

One therefore can utilize induction and CFT.)
Problem 2: \[
\bar{\rho}: G_{F^s} \to GL_n(k), \quad k = F^s.
\]

As a special case, consider lifting the Galois action on the reduction of a zero dimensional variety over $G_{F^s}$. The answer is provided by Hensel's Lemma.

We will only consider the case that $\bar{\rho}$ is absolutely irreducible.

$(\bar{\rho}: G_{F^s} \to GL_n(k) \to GL_n(F^s))$ is irreducible.

(1) Try to lift to $GL_n(k^{F^s/s})$.

Let $\mathcal{C}$ be the category of complete local quasi-regular rings $(A, m)$ with $A/m = k$.

(With a morphism $A \to k$)

the maps between objects are given by

\[
A \to A', \\
\downarrow 2 \downarrow \\
A/m = k \to A'/m = k
\]

A representation $\rho: G_{F^s} \to GL_n(A)$ is the same as an $A^s$-module $V_k$ with a continuous action by $G_{F^s}$.

(through $\rho$)

$\bar{\rho} \mapsto V_k$ over $k$.

Def: $V_k$ is a deformation of $V_k$ if $V_k \otimes_{A^s} A/m \approx V_k$.

Def: $V_k$ and $V_{k'}$ are strictly equivalent if $\exists$ a commutative diagram $V_k \approx V_{k'}$ such that...
\[
\begin{align*}
V_k & \rightarrow V_k \\
\downarrow & \quad \downarrow \\
V_k & \rightarrow V_k
\end{align*}
\]

We have a function \( \mathcal{D} : \gamma \rightarrow \text{Sets} \)

\[ A \mapsto \mathcal{D}(A) = \{ \text{the strict equivalence class of deformations } [V_a] \} \]

The concrete problem is: (taking \( A = k[5] / \gamma^2 \))

What is \( \mathcal{D}(k[5] / \gamma^2) \)?

Study \( A = k[5] / \gamma^2 \):

\[ 0 \rightarrow k \rightarrow A \rightarrow k \rightarrow 0 \quad \text{a.c.e.s. } \& \ \text{a } k\text{-module} \]

\[ 0 \rightarrow V_k \rightarrow V_k \rightarrow V_k \rightarrow 0 \]

It follows that, given \([c] \in \mathcal{D}(k[5] / \gamma^2)\), \([c]\) can be viewed as an element in \( \text{Ext}^1_{k\left[G_{F_5}\right]}(V_k, V_k) \).

Given \([c]\), choose a basis for \( V_k \). \( \sigma \in G_{F_5} \) then acts as

\[
\begin{pmatrix}
X_\sigma \\
Y_\sigma \\
X_\sigma
\end{pmatrix}
\]

a map from \( V_k \rightarrow V_k \)

Fix a (choice space) splitting \( \psi^{-1} : V_k \rightarrow V_h \), then

\[ \sigma \in G_{F_5} \mapsto \text{Hom}(V_k, V_k) \]

\[
(\sigma; \psi^{-1}(\sigma) \mapsto \psi^{-1}(\sigma) \circ \psi(\sigma))
\]

\[ \text{more explicitly } (Y_{\text{top}}, X_{\text{top}}) \mapsto (0, X_{\text{top}}) \]

from a subgroup \( \psi^{-1} \subset V_k \).

\[ \therefore \text{given a class } \gamma \text{ on a co-cycle } \]

\[ G_{F_5} \rightarrow \text{Hom}(V_k, V_k) \]

\[ \therefore H^1(G_{F_5}, \text{Hom}(V_k, V_k)) \]

Suppose \( \psi \in V_k \otimes V_k \) such that

\[
(\sigma; (\sigma, (0, \psi)) \mapsto \sigma \cdot \psi, \text{top}) \]

\[ \text{for each } \sigma \in G_{F_5}. \]

\[ \therefore \text{given a class } \gamma \text{ on a co-cycle } \]

\[ G_{F_5} \rightarrow \text{Hom}(V_k, V_k) \]

\[ \therefore H^1(G_{F_5}, \text{Hom}(V_k, V_k)) \]
\[ D(\mathbb{K}\ell/\ell^2) = \text{Ext}_K^{\mathfrak{g}}(V_k, V_k) = H^1(\mathfrak{g}_{\mathbb{F}, k}, \text{Ad}(V_k)) \]

- gives \( K \)-vector space structure to \( D(\mathbb{K}\ell/\ell^2) \)

- we see that \( H^1(\mathfrak{g}_{\mathbb{F}, k}, \text{Ad}(V_k)) \) is finite.

Let \( \mathfrak{g}_{\mathbb{F}, k} \) act on \( \text{Ad}(V_k) \) via \( \text{Gal}(\mathbb{K}/\ell) \), then

\[ \text{inf} \quad \text{finite} \]

\[ \text{Hom}_k(\mathfrak{g}_{\mathbb{F}, k}, \text{Ad}(V_k)) \]

Thin (Mumford): The functor \( D \) is representable, i.e., \( \exists A \in \mathfrak{g} \) s.t. \( D(A) = \text{Hom}_k (\mathbb{K}, A) \).

Interpretation: \( \exists \text{Proj} : \mathfrak{g}_{\mathbb{F}, k} \rightarrow \text{GL}(V_k) \) lifting \( \tilde{\Gamma} \)

i.e., given any \( \tilde{\gamma} \) satisfying \( \tilde{\Gamma} \),

\[ \tilde{\gamma} \rightarrow \gamma \rightarrow \text{GL}(V_k) \]

\[ \exists A \in \mathfrak{g} \text{ s.t. } \mathfrak{g}_{\mathbb{F}, k} \rightarrow \text{GL}(\mathbb{R}) \rightarrow \text{GL}(A) \]

\[ \tilde{\Gamma} \circ \text{Proj} = \tilde{\gamma} \quad (\text{up to equivalence}) \]
Given $R$, what is $O(k_{\mathbb{F}_p}/\mathbb{Z}_p^2)$?

\[ \text{Hom}(CR, k_{\mathbb{F}_p}/\mathbb{Z}_p^2) \]
\[ m_k \rightarrow k_{\mathbb{F}_p}/\mathbb{Z}_p^2 \]
\[ R \rightarrow k_{\mathbb{F}_p}/\mathbb{Z}_p^2 \]
\[ k = k \]

So \[ \text{Hom}(C_{\mathbb{F}_p}, k) = \text{Hom}(C_{\mathbb{F}_p}/\mathbb{Z}_p^2, k) = O(k_{\mathbb{F}_p}/\mathbb{Z}_p^2) \]

If \[ R = \mathbb{Z}_p[[T]] \], \[ m_{\mathbb{F}_p}/\mathbb{Z}_p^2 \leq \mathbb{F}_p \leftrightarrow \rightarrow O(k_{\mathbb{F}_p}/\mathbb{Z}_p^2) = \mathbb{F}_p. \]

Also can be realized as:

- \( (\text{Hom}_{\mathbb{F}_p}(\mathbb{Z}_p[[T]], \mathbb{F}_p)) \leq \mathbb{F}_p. \)
- \( \text{Ext}^1_{\mathbb{F}_p(C_{\mathbb{F}_p}, k)}(\mathbb{F}_p, \mathbb{F}_p) \)
- \( \text{H}^1(C_{\mathbb{F}_p}, k_{\mathbb{F}_p}) \)
Let \( d_1 = \dim H^1(G_{F_5}, \text{Ad} V_k) = \dim \text{Hom}_k(R, kG_{F_5}^{1/2}) \)

\[ \mapsto R = \text{NC}(E_{T_1, T_2, \ldots, T_r}) / I. \]

(Became \( \text{Hom}_k(R, kG_{F_5}^{1/2}) \)),

\text{Tangent space of } R \text{ at closed pt.}

What is the obstruction to lifting? 

\( \phi: G_{F_5} \to GL_n(A/L) \)

\( \text{consider} \quad \downarrow \quad \text{a \it set-theoretic lifting} \)

\( \psi: \quad \downarrow \quad \text{GL}_n(A/L) \)

\( c(s, \tau) = \varphi(\tau) \circ \varphi(s^{-1}) \circ \varphi^{-1} \)

\( \in H^1 + M_n(I) \text{ and } \mathfrak{I}^2 \)

\( I \otimes \text{Ad}(\psi(E)) \quad \text{non-nilpotent with entries in } I \)

\( \in H^2(G_{F_5}, \text{Ad}(\text{Ad} V_k / L) \otimes I) \)

If \( H^2(G_{F_5}, \text{Ad} V_k) = 0 \), then \( R = \text{smooth} \Rightarrow I = 0 \).

Let \( d_2 = \dim H^2(G_{F_5}, \text{Ad} V_k) \)

\( I \otimes R^{1/2} \text{ has dimension } = d_2 \).

* of generators for \( I = d_2 \).

We can say much about \( d_1, d_2 \).

But we can say something about \( d_1 - d_2 \). (Using Euler characteristic)

Assume \( S \) prime dividing \( p \). Then \( d_1 - d_2 = \sum_{v \mid p} \dim H^1(G_v, \text{Ad} V_k) \)

\( = n^2 \text{ if } G_v = \mathbb{F}_p \)
(If \( n = 2 \), then \( d_1 - d_2 = 1r_1 + 2r_2 - r_1 - r_2 \))

\[
\dim(\text{Adj} V_3) = \begin{cases} \{e\} \\
\{e\} \cup \{b_2\} \end{cases}
\]

(\( \text{depends on the sign of the rep} \ a \))

\( n^2 \) if even

\( n \) if odd \((?)\)

**Case:** \( n = 1 \), \( d_1 - d_2 = \text{kernel dim of } R/R \)

\( \leq \text{we know} \)

(\( \text{Hand: even for } n \neq 2, \text{ correct kernel dim (i.e. equality above) } \Rightarrow \text{superlative case} \))

\[ n = 2, \quad \overline{\rho}(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{even} \quad \overline{\rho}(e) = \begin{pmatrix} 1 & \quad \text{odd} \\ -1 \end{pmatrix}
\]

\[ d_1 - d_2 = 1t + 4 - 1 = 3 \]

\[ d_1 - d_2 = 1t + 4 - 2 = 3 \]

==

**Roughly how to prove \( R = T \)?**

- \( \dim \text{ of the target space: } \dim_k \text{ Hom}_{\text{rep}}(R, k[S/\mathfrak{s}]) \)
- replace \( R \) by something closer to the smooth one ...