On deformations of Eisenstein cohomology classes and applications

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Let

- $G_f$ be reductive.
- $p$ prime.
- $G_f / \mathbb{Q}_p$ split.

$p$-adic variation of cohomological automorphic rep'n of $G_f(\mathbb{A})$.

$k_f \subseteq G_f(\mathbb{A}_f)$

$G_{\infty} = G_f(\mathbb{R})$

$U_{\mathbb{Z}_p}$

$\tilde{\mathcal{M}}$ = local system on $S_{G_f}(k_f) = G_f(\mathbb{A}_f)^{G_f(\mathbb{Z})} / k_{\mathbb{Z}_p} \mathbb{Z}_p k_f$

$H^\bullet_c \left( S_{G_f}(k_f), \tilde{\mathcal{M}} \right)$

$\mathcal{M} = \text{alg, repn}$. 

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$\chi \in X(T)^+$ are irreducible repn of $G$ of highest wt $\chi$.

$\chi \in X(G_f)$

$A_{\chi}$ = locally analytic induction $Ind_{G_{E_{\chi}}}^{G_f} \chi$

$\lambda$ = analytic

$I = Iwahori subgrp

D_{\chi} = continuous dual of $A_{\chi}$.

\[ H^*(G_f, \psi_{\chi}^\wedge) \leftarrow \widetilde{H^*}(S_{\chi}(G_f), d\mathcal{J}_\chi) \]

See Stevens

\[ H^\vee \text{cusp} \simeq \bigoplus_{\Pi \text{irred}} \pi_{\Pi}^{k_{\Pi}} \otimes H^*(\text{Lie } G_\Pi, \mathbb{R}, \pi_{\Pi} \otimes \psi_{\pi_{\Pi}}^\wedge) \]

\[ H^\vee \text{cusp} \simeq \bigoplus_{\Pi \text{irred}} \pi_{\Pi}^{k_{\Pi}} \otimes H^*(\text{Lie } G_\Pi, \mathbb{R}, \pi_{\Pi} \otimes \psi_{\pi_{\Pi}}^\wedge) \]
Emerton: \[ \lim_{m \to \infty} \lim_{k \to k_f} H^*(S_g(k_f), \mathbb{Z}/m) \]

Ash-Stevens: look at \( H^*(S_g(k_f, I), D) \)

Relate these using derived categories ...
Eigen varieties:

\[ S \text{: finite set of primes containing all the } \pi \text{ at which } k_f \text{ is ramified.} \]

\[ T^+ = \left\{ \pi \in \mathbb{Z}_p \mid \pi \mathbb{Z}_p \cap \mathbb{Z}_p = \mathbb{Z}_p \right\} \]

\[ R_{k_f} = \mathbb{E}_c^\infty \left( \mathbb{G}(\mathbb{A}_f^{1,5}) \big/ k_f^{1,5}, \mathbb{Z} \right) \otimes \mathbb{Z}[T^+] \]

\[ (f^\times \otimes t \rightarrow f^\times \otimes 1_{k_f^{1,5}}) \]

\[ H^*(\mathbb{G}(k_f^{1,5}), \mathbb{D}_x) \]

\[ W_x \]

\[ \mathcal{E}_{k_f^{1,5}, S} = \left\{ x = (\theta, \lambda) \mid \lambda \in \mathbb{C}(\overline{\Omega}_p) \right\} \]

\[ \theta : R_{k_f} \rightarrow \overline{\Omega}_p \]

locally rigid analytic ...
Eisenstein cohomology:

\[ G > P \quad \text{max. parabolic.} \]

\[ MN. \]

\[ H^*(K_f, W_f^\vee) = \bigoplus \left( \text{Ind}_{P(P)}^{G(G)} \sigma \right)_{P} \otimes H^*(\text{Lie } M_w, K_w, \tau_w) \]

\[ \oplus \]

\[ \text{can be composed.} \]

\[ \oplus \]

\[ \text{Lie } W_f^* = W_f/W_f. \]

\[ \text{alg. irreducible repn of } M \]

\[ \text{of highest wt. } w(n+p)-p. \]

\[ \pi \quad \text{cohomological automorphic repn.} \]

\[ \pi = E_f \otimes \pi_w \quad \text{of wt. } x \]

\[ \text{to make the } \mathfrak{p}\text{-adic variation, one needs to make a choice} \]

\[ \pi_f \mid E_f(C_f^*) = \chi \text{ character of } T(C_f^*). \]

\[ \chi \text{ unramified, } \pi_f^I \text{ has dim. } \# W. \]

\[ \oplus \]

\[ \# W \times \text{ a repn of } T^+. \]

This choice determines the type of deformation.

For \( \text{GL}_n(\mathbb{Q}_p) \), if level \( N \) prime to \( p \), \( \pi_f \rightarrow \pi \) where \( a, b \) are roots of the Hecke char. polynomial \( \Delta_f \).

In the case of \( \mathbb{E}_8 \) the choice is very important.

\[ E_8 \quad \text{of level } 2. \]

\[ \frac{u(1, k)}{2} \cdot \sum_{n=0}^{k-1} \frac{q^n}{n!}, \quad \text{for } k \geq 1. \]

\[ E_8(q) \rightarrow \text{eigenvalue of } U_f \text{ is } 1 \rightarrow \text{slope } = 0 \quad \text{family of Eisenstein} \]
$E_{0x}(g) \to$ eigenvalue of $U^x$ is $L \to \text{ slope } = 0 \quad \text{family of Eisenstein series.}$

$E_{0x}(g) \to \text{ slope } L \to \text{ family which is generically cuspidal.}$

**Example:** an arithmetic case: repn of $GL_2(K), \ K$ imag. quad. field.

$GL_2(K) \qquad \text{quasisplit unitary repn of $GL_2(\mathbb{R})$.}$

$P \subset G \text{ parabolic, } \quad P \subset \text{MN}$

$p = 2 \left( \frac{x}{2} \right) \qquad M \in GL_2(K): \begin{pmatrix} g & \ast \\ 0 & g^{-1} \end{pmatrix}$

Consider $\Theta \text{ a } p$-stabilization of $\text{Ind}_{P(K')}^G(1)$.

Assume that the weight of $\sigma$ is regular.

$H^r_{p_f}(\mathbb{R}; \mathcal{O}_K) = \left( \text{Ind}_{P(K')}^G(1) \right)^1 \quad \Theta \left( \text{Ind}_{P(K')}^G(1) \right) \quad \text{for large degree.}$

$\otimes H^*(GL_2(\mathbb{R}), \mathbb{A}_K, H^* (\text{Lie}N, W_{\Theta}))$

$E_{0x}$ Eisenstein class in $H^*(SG_k(\mathbb{R}), W_{\Theta})$ a regular $\Theta \otimes W_{\Theta}$

Can apply the previous result to determine a deformation of $\Theta$.

If Glob. repn of $\Theta$ exists, say $\Theta_0$, then the Glob. repn attached to $\Theta$ is $\Theta \cong \Theta_0 \otimes \tilde{\Theta}_0$, where

$\tilde{\Theta}_0 : G_{\mathbb{F}_p} \to GL_2(F_p).$

If $\Theta$ is "far" from being ordinary, we should expect to have a generically cuspidal family.

But: $H^i(GL_2(\mathbb{R}), \cdots) \neq 0 \text{ for } i=1,2$

$\Rightarrow m(K_0^i, L, \Theta) = 0 \quad \Rightarrow \text{ the deformation is not of full dimension.}$
(So we cannot apply the thin) but we expect it to be $G$-cyclic 1, because $o$ is going to occur in 2 consecutive degrees.

However, there is no reason to think that the pts in this family are classical, therefore, we need to find information regarding Calecic reps. for this pt. to order to apply this strategy.

Galois representation:

If $G$ is cyclic, then $G = \text{unitary}_G$. we expect \( R_{\text{par}} \mid G \rightarrow \text{GL}(n, \mathbb{Q}_p) \).

Question: What about any pt \( x \in \mathbb{Q}_p, s? \)

If \( x \in \mathbb{Q}_p, s \), should we able to attach Galois reps. (by analytic appr.)

0%, we do not know. (No clean expressions on strategy)

Case 1: If \( x \in \mathbb{Q}_p, s \), then there should exist

\( R_{\text{par}} : G \rightarrow \text{GL}(n, \mathbb{Q}_p) \)

attached to $x$.

Refine: \( R_{\text{par}} \) exists and $R_{\text{par}} \mid \text{Gal}(\overline{\mathbb{Q}_p})$ is a

triangular rep (\( (\mathbb{Q}_p, \mathfrak{m}) \)-module attached to this is triangular)
Can define a Galois action on $\hat{H}^0(S_0(K_f), \mathcal{D}_x)[\theta] = R_p(x, \lambda^\theta)$

If $R_p(x)$ exists, $R_p(x, \lambda^\theta)$ should be $= \lambda^\theta R_p(x)$.

Consequence relation on $R_p(x, \lambda^\theta)$

Claim. Let $M(x, \lambda^\theta)$ be the $(G, T)$ module attached to $R_p(x, \lambda^\theta)$.

Then $M(x, \lambda^\theta)$ is triangular, with character $\delta_i^\theta$ among the

set of characters $\delta_1^\theta, \delta_2^\theta, \ldots, \delta_r^\theta$, $i_1 < i_2 < \ldots < i_r$.

Claim $\Rightarrow$ if $\theta$ is “far” from very ordinary, then $R_p(x, \lambda^\theta)$ needs to be big enough...

If we deform the Eisenstein class in a very non-ordinary way...

expect to get a big family of Galois reps $\rightarrow \rho_\lambda$. 