QUADRATIC EXERCISES IN IWASAWA THEORY

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1. INTRODUCTION

The anticyclotomic main conjecture of p-ordinary CM fields M was proven in [11] under some hypothesis. In this paper, we propose the removal of the following hypothesis we made in [11]:

(S) *The prime-to-p part of the conductor of the branch character ψ is a product of primes of M of relative degree one over the maximal totally real subfield F of M.*

We shall remove this assumption by reducing the conjecture to the result in [11] under (S) by a simple argument of quadratic (automorphic) base-change. The condition (S) is equivalent to having the automorphic induction of the character everywhere in the principal series (cf. [26]).

Write R (resp. O) for the integer ring of M (resp. F). The field F is totally real, and M is a totally imaginary quadratic extension of F (inside a fixed algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$). We fix an odd prime $p > 2$ *unramified* in $M/\mathbb{Q}$. The field M is called p-ordinary if there exists an abelian variety with complex multiplication by M having ordinary good reduction at p. This property can be detected by the CM type of the abelian variety. To describe this fact, fix an embedding $i_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. A CM type Σ is associated to an abelian variety with CM by M having ordinary good reduction at i_p if the embeddings $i_p \circ \sigma$ for $\sigma \in \Sigma$ induce exactly a half Σ_p of

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the p–adic places of M. Such a CM type is called a p-ordinary CM type, and we fix a p-ordinary CM type Σ and the set Σ_p of associated p-adic places. We identify Σ_p with a subset of prime factors of p in M. Fix $c \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$ inducing the generator of $\mathrm{Gal}(M/F)$. The disjoint union $\Sigma_p \sqcup \Sigma_p^c$ is the set of all prime factors of p in M.

We study the arithmetic of the unique $\mathbb{Z}_p^{[F:\mathbb{Q}]}$–extension M_∞^- of M (unramified outside p and ∞) on which $c\sigma c^{-1} = \sigma^{-1}$ for all $\sigma \in \Gamma_M^- = \mathrm{Gal}(M_\infty^-/M)$. We choose a complete discrete valuation ring W inside $\overline{\mathbb{Q}}_p$ finite flat and unramified over $\mathbb{Z}_p$. A Hecke character ψ of $M^\times \backslash M_\mathbb{A}^\times$ is called *anticyclotomic* if $\psi(x^c) = \psi(x)^{-1}$. By class field theory, if ψ is of finite order, we often regard ψ as a character of $\mathrm{Gal}(\overline{\mathbb{Q}}/M)$, and then anticyclotomy of ψ can be interpreted as $\psi(c\sigma c^{-1}) = \psi(\sigma)^{-1}$. We write the conductor of ψ as $\mathfrak{c}\mathfrak{P}^e$ for an ideal $\mathfrak{c}$ prime to p. Here for a multi-exponent $e = \sum_{\mathfrak{P}|p} e(\mathfrak{P})\mathfrak{P} \in \mathbb{Z}[\Sigma_p \sqcup \Sigma_p^c]$, we write $\mathfrak{P}^e$ for $\prod_{\mathfrak{P}|p} \mathfrak{P}^{e(\mathfrak{P})}$.

Suppose that ψ has finite order. Let $M(\psi)/M$ be the class field cut out by ψ; in other words, ψ induces the isomorphism $\mathrm{Gal}(M(\psi)/M) \cong \mathrm{Im}(\psi)$. Consider the extension $M_\infty^- M(\psi)/M(\psi)$ which is called the anticyclotomic tower over $M(\psi)$. Let $L_\infty/M_\infty^- M(\psi)$ be the maximal p–abelian extension unramified outside Σ_p. Each $\gamma \in \mathrm{Gal}(L_\infty/M)$ acts on the normal subgroup $X = X_M := \mathrm{Gal}(L_\infty/M_\infty^- M(\psi))$ continuously by conjugation, and by the commutativity of X, this action factors through $\mathrm{Gal}(M(\psi)M_\infty^-/M)$. We fix a splitting $\mathrm{Gal}(M(\psi)M_\infty^-/M) = \Gamma_M^- \times G_{tor}(\psi)$ for the maximal torsion subgroup $G_{tor}(\psi) \cong \mathrm{Im}(\psi)$. Then we look into the Γ_M^-–module: $X[\psi] = X_M[\psi] := X \otimes_{G_{tor}(\psi),\psi} W$.

As is well known, $X[\psi]$ is a $W[[\Gamma_M^-]]$–module of finite type, and if ψ is anticyclotomic nontrivial over $\mathrm{Gal}(\overline{\mathbb{Q}}/F[\sqrt{p^*}])$ with $p^* = (-1)^{(p-1)/2}p$, it is proven to be a torsion $W[[\Gamma_M^-]]$–module by a result of Fujiwara (cf. [10] Corollary 5.4 and [17] Theorem 5.33) generalizing the fundamental work of Wiles [27] and Taylor-Wiles [25]. Thus we can think of the characteristic element $\mathcal{F}_M^-(\psi) \in W[[\Gamma_M^-]]$ of the module $X[\psi]$. As we have seen in [18] and [19], we have the anticyclotomic p–adic Hecke L–function $L_M^-(\psi) \in \overline{W}[[\Gamma_M^-]]$ (constructed by Katz), where $\overline{W}$ is the completed p–adic integer ring of the maximal unramified extension of $\mathbb{Q}_p$ inside $\overline{\mathbb{Q}}_p$. We regard $W \subset \overline{W}$. Then the anticyclotomic main conjecture can be stated:

Anticyclotomic Main Conjecture. *We have the identity:* $\mathcal{F}^-(\psi) = L_M^-(\psi)$ *up to a unit in* $\overline{W}[[\Gamma_M^-]]$.

The main conjecture for imaginary quadratic fields (including the cyclotomic $\mathbb{Z}_p$–extension) and its anticyclotomic version for imaginary quadratic fields have been proved by K. Rubin refining Kolyvagin's method of Euler systems, and after that, the anticyclotomic version was again treated by J. Tilouine (for imaginary quadratic cases) by a method similar to the one exploited in this paper combined with the class number formula of the ring class fields. A partial result towards the general conjecture was studied in [18], [19], [11] and [12].

We fix a continuous *anticyclotomic* character $\psi : \mathrm{Gal}(\overline{F}/M) \to W^\times$ of finite order. We shall prove

Theorem. *Assume that* $p > 3$ *and the following three conditions:*

1. *The anticyclotomic character* ψ *has order prime to* p.
2. *The local character* $\psi_\mathfrak{P}$ *is non-trivial over* $M_\mathfrak{P}^\times$ *for all* $\mathfrak{P} \in \Sigma_p$.
3. *The restriction* ψ^* *of* ψ *to* $\mathrm{Gal}(\overline{F}/M[\sqrt{p^*}])$ *is non-trivial.*

Then the anticyclotomic main conjecture holds.

This theorem was proven in [11] under the aforementioned extra assumption (S) which we remove in this paper. In a forthcoming paper, resorting to a framed version of the "$R = T$" theorem in [3], we plan to remove the assumption (2). The assumption (3) is at this moment difficult to remove (because it is fundamental for the Taylor-Wiles system to work in proving the necessary "$R = T$" theorems). The assumption (1) can be removed, but it is a technical endeavor; so, for simplicity, we assume it in this paper.

To give a short description of the idea of the proof, decompose $\mathfrak{c} = \mathfrak{F}\mathfrak{F}_c\mathfrak{I}\mathfrak{R}$ so that $\mathfrak{I}$ is a product of inert prime factors, $\mathfrak{R}$ is a product of ramified prime factors $\mathfrak{F}+\mathfrak{F}_c = R$ and $\mathfrak{F}_c^c \supset \mathfrak{F}$. Let $\mathfrak{i} = \mathfrak{I} \cap O$. Write $h_{\mathfrak{i}}(M/F) = (h(M)/h(F)) \prod_{\mathfrak{l}|\mathfrak{i}}(N(\mathfrak{l})+1)$, where $h(M)$ and $h(F)$ are class numbers of M and F, respectively. Up to a p-adic unit, $h_{\mathfrak{i}}(M/F)$ is the relative class number of the ray class group modulo $\mathfrak{i}$, that is the ratio of the order of the ray class group $Cl_M(\mathfrak{i})$ of M and that of $Cl_F(\mathfrak{i})$ of F.

The idea of the proof is to reduce the conjecture to the case under (S) treated in [11] by quadratic base change to a well chosen totally real quadratic extension F'/F and a further refinement, eliminating the assumption (S), of the method exploited in [18] and [12] Theorem 5.1, where we have proven $L_M^-(\psi)|\mathcal{F}_M^-(\psi)$ in $\overline{W}[[\Gamma_M^-]]$ under (S). As is well known (e.g., [17] Lemma 5.31), we can always find an algebraic Hecke character φ of M such that $\varphi^- = \psi$, where $\varphi^- = \varphi^c\varphi^{-1}$ with $\varphi^c(x) = \varphi(x^c)$. We choose φ well. Then one of the main ingredients of the proof is the congruence power series $H(\psi) \in W[[\Gamma_M^-]]$ of the CM–component of the universal nearly ordinary Hecke algebra $\mathbf{h}$ for $GL(2)_{/F}$ associated to the theta series of φ. In the joint works with Tilouine, we took $\mathbf{h}$ of (outside p) level $N_{M/F}(\mathfrak{C})d(M/F)$ for the conductor $\mathfrak{C}$ of φ and the relative discriminant $d(M/F)$ of M/F. In this paper, as in [11] and [12] Section 2.10, we take the Hecke algebra of level $\mathfrak{N}(\psi)$ which is a product of $\mathfrak{c} \cap F$ and $d(M/F)$ (introducing a new type of Neben character determined by φ with $\varphi^- = \psi$). Fujiwara formulated his results in [4] using such level groups. Another important ingredient is the following divisibility assertion (without assuming (S)) in Corollary 3.8 which will be proven refining the proof of a similar result in [12] Corollary 5.6 under (S):

$$h_{\mathfrak{i}}(M/F)L_M^-(\psi)\big|H(\psi) \text{ in } \overline{W}[[\Gamma_M^-]]. \tag{A}$$

Since in [18], the assertion (A) is proven in $W[[\Gamma_M^-]] \otimes_{\mathbb{Z}} \mathbb{Q}$ inverting p, applying the method of [12] to the p-primary parts, we prove the μ-invariant inequality

$$\mu(h_{\mathfrak{i}}(M/F)L_M^-(\psi)) \leq \mu(H(\psi)) \tag{1.1}$$

in Corollary 3.8, and the vanishing of the μ-invariant (of p-adic Hecke L-functions under (S)) studied in [13] is essential in the proof. On the other hand, Fujiwara's result already quoted implies (see [4], [19], [10] and [17] Sections 3.2–3 and 5.3):

$$H(\psi) = h_{\mathfrak{i}}(M/F)\mathcal{F}_M^-(\psi) \text{ up to units in } \overline{W}[[\Gamma_M^-]]. \tag{B}$$

Thus we get (see Theorem 3.1):

$$L_M^-(\psi)|\mathcal{F}_M^-(\psi) \text{ in } \overline{W}[[\Gamma_M^-]]. \tag{C}$$

To conclude the theorem, we choose an appropriate totally real quadratic extension F'/F and put $E = F'M$. Since $\mathrm{Gal}(E/F) = \mathrm{Gal}(F'/F) \times \mathrm{Gal}(M/F) \cong (\mathbb{Z}/2\mathbb{Z})^2$, there exists a third quadratic extension M'/F in E/F. Note that M' is a CM field.

In summary, we have

$$\begin{tikzcd}
 & M' \arrow[dr] & \\
F \arrow[ur] \arrow[r, hook] \arrow[dr] & F' \arrow[r, hook] & E. \\
 & M \arrow[ur] &
\end{tikzcd}$$

We can arrange F'/F so that

- (F1) for any prime $\mathfrak{r}$ of F ramifying in M, the inertia group of $\mathfrak{r}$ in $\mathrm{Gal}(E/F)$ is given by $\mathrm{Gal}(E/M')$ (so, any prime factor of $\mathfrak{r}$ in F' splits in E/F'),
- (F2) for any prime $\mathfrak{q}$ inert in M/F in the conductor $\mathfrak{c}$ of ψ, the decomposition group of $\mathfrak{q}$ in $\mathrm{Gal}(E/F)$ is given by $\mathrm{Gal}(E/M')$ (so, any prime factor of $\mathfrak{q}$ in F' splits in E/F'),
- (F3) E/M ramifies at least one finite place outside p.

We consider the base-change $\psi_E = \psi \circ N_{E/M}$. Then by (F1–2), ψ_E has conductor whose prime factors all split in E/F' (so, (S) for ψ_E is satisfied). Thus by the main theorem of [11], we have $L^-_E(\psi_E) = \mathcal{F}^-_E(\psi_E)$ up to units in $\overline{W}[[\Gamma^-_E]]$. We have the restriction map $\mathrm{Res} : \Gamma^-_E \to \Gamma^-_M$. By (F3), Res is a surjection. We can verify (see Section 4) that $\mathrm{Res}(\mathcal{F}^-_E(\psi_E)) = \mathcal{F}^-_M(\psi)\mathcal{F}^-_M(\psi\alpha)$ for $\alpha = \left(\frac{E/M}{}\right)$ and $\mathrm{Res}(L^-_E(\psi_E)) = L^-_M(\psi)L^-_M(\psi\alpha)$ up to units in $\overline{W}[[\Gamma^-_M]]$. Thus

$$L^-_M(\psi)L^-_M(\psi\alpha) = \mathcal{F}^-_M(\psi)\mathcal{F}^-_M(\psi\alpha)$$

up to units in $\overline{W}[[\Gamma^-_M]]$. Note here that $\psi\alpha$ remains anticyclotomic (because α has order 2). By (C): $L^-_M(\psi)|\mathcal{F}^-_M(\psi)$ and $L^-_M(\psi\alpha)|\mathcal{F}^-_M(\psi\alpha)$, we conclude the individual identify:

$$L^-_M(\psi) = \mathcal{F}^-_M(\psi) \quad\text{and}\quad L^-_M(\psi\alpha) = \mathcal{F}^-_M(\psi\alpha)$$

up to units in $\overline{W}[[\Gamma^-_M]]$. This finishes the proof of the theorem.

The proof of (1.1) again involves a base-change technique to quadratic extensions fitting into a diagram similar to the above (since the main result in [13] is again proved under (S)), but we need to choose the real quadratic extension F_1/F more carefully than F'/F. We will postpone the exact specification of F_1/F to Section 3.3, since the choice is subtle and rather technical.

Notation: Here is a basic notation we use without explaining much. We write $\mathbb{A} \subset \mathbb{R} \times \prod_l \mathbb{Q}_l$ for the adele ring, and $\mathbb{A}^{(\infty)}$ is the ring of finite adeles; so, $\mathbb{A} = \mathbb{A}^{(\infty)} \oplus \mathbb{R}$. We regard $GL(2)$ as a linear algebraic group defined over O and write G for $\mathrm{Res}_{O/\mathbb{Z}}GL(2)$. Write I for the set of all embeddings of F into $\overline{\mathbb{Q}}$, and define $\mathfrak{Z} = \mathfrak{H}^I$ for the product of I copies of the upper half complex plane $\mathfrak{H}$. A classical Hilbert modular form is a holomorphic function on $\mathfrak{Z}$ with certain automorphy property (see (2.6)), and an adelic Hilbert modular form (whose precise definition we will give later) is a function on the idele group $G(\mathbb{A})$.

2. Hilbert modular forms and Hecke algebras

Let us give a short description of the adelic/classical Hilbert modular forms and their Hecke algebra of level $\mathfrak{N}$ (cf. [12], [8] Sections 2.2-4, [16] Sections 4.2.8–4.2.12 and [17] Section 2.3), limiting ourselves to the extent logically necessary to understand our proof of the key assertion (C) and the theorem. If the reader is familiar with the theory introduced in the above papers and books, he/she can skip this section.

2.1. **Double coset rings.** We first recall formal Hecke rings of double cosets. For that, we fix a prime element $\varpi_{\mathfrak{q}}$ of $O_{\mathfrak{q}}$ for every prime ideal $\mathfrak{q}$ of O. We consider the following open compact subgroup of $G(\mathbb{A}^{(\infty)})$:

$$\begin{aligned} U_0(\mathfrak{N}) &= \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in GL_2(\widehat{O}) \middle| c \equiv 0 \mod \mathfrak{N}\widehat{O} \right\}, \\ U_1^1(\mathfrak{N}) &= \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in U_0(\mathfrak{N}) \middle| a \equiv d \equiv 1 \mod \mathfrak{N}\widehat{O} \right\}, \end{aligned} \tag{2.1}$$

where $\widehat{O} = O \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ and $\widehat{\mathbb{Z}} = \prod_\ell \mathbb{Z}_\ell$. Then we introduce the following semi-group

$$\Delta_0(\mathfrak{N}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in G(\mathbb{A}^{(\infty)}) \cap M_2(\widehat{O}) \middle| c \equiv 0 \mod \mathfrak{N}\widehat{O}, d_{\mathfrak{N}} \in O_{\mathfrak{N}}^\times \right\}, \tag{2.2}$$

where $d_{\mathfrak{N}}$ is the projection of $d \in \widehat{O}$ to $\prod_{\mathfrak{q}|\mathfrak{N}} O_{\mathfrak{q}}$ for prime ideals $\mathfrak{q}$. Writing T_0 for the maximal diagonal torus of $GL(2)_{/O}$ and putting

$$D_0 = \left\{ \mathrm{diag}[a,d] = \left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix} \right) \in T_0(F_{\mathbb{A}^{(\infty)}}) \cap M_2(\widehat{O}) \middle| d_{\mathfrak{N}} = 1 \right\}, \tag{2.3}$$

we have (e.g. [15] 3.1.6 and [16] Section 5.1)

$$\Delta_0(\mathfrak{N}) = U_0(\mathfrak{N}) D_0 U_0(\mathfrak{N}). \tag{2.4}$$

In this section, writing $\mathfrak{p}^\alpha = \prod_{\mathfrak{p}|p} \mathfrak{p}^{\alpha(\mathfrak{p})}$ with $\alpha = (\alpha(\mathfrak{p}))$, the group U is assumed to be a subgroup of $U_0(\mathfrak{N}\mathfrak{p}^\alpha)$ with $U \supset U_1^1(\mathfrak{N}\mathfrak{p}^\alpha)$ for some multi-exponent α (though we do not assume that $\mathfrak{N}$ is prime to p). Formal finite linear combinations $\sum_\delta c_\delta U\delta U$ of double cosets of U in $\Delta_0(\mathfrak{N}\mathfrak{p}^\alpha)$ form a ring $R(U, \Delta_0(\mathfrak{N}\mathfrak{p}^\alpha))$ under convolution product (see [23] Chapter 3 or [15] 3.1.6). The algebra is commutative and is isomorphic to the polynomial ring over the group algebra $\mathbb{Z}[U_0(\mathfrak{N}\mathfrak{p}^\alpha)/U]$ with variables $\{T(\mathfrak{q}), T(\mathfrak{q},\mathfrak{q})\}_{\mathfrak{q}}$ for primes $\mathfrak{q}$, $T(\mathfrak{q})$ corresponding to the double coset $U \left(\begin{smallmatrix} \varpi_{\mathfrak{q}} & 0 \\ 0 & 1 \end{smallmatrix} \right) U$ and $T(\mathfrak{q},\mathfrak{q})$ (for primes $\mathfrak{q} \nmid \mathfrak{N}\mathfrak{p}^\alpha$) corresponding to $U_0 \varpi_{\mathfrak{q}} U$. Here we have chosen a prime element $\varpi_{\mathfrak{q}}$ in $O_{\mathfrak{q}}$. The group element $u \in U_0(\mathfrak{N}\mathfrak{p}^\alpha)/U$ in $\mathbb{Z}[U_0(\mathfrak{N}\mathfrak{p}^\alpha)/U]$ corresponds to the double coset UuU (cf. [16] Lemma 5.2).

2.2. **Adelic Hilbert modular forms.** The double coset ring $R(U, \Delta_0(\mathfrak{N}\mathfrak{p}^\alpha))$ naturally acts on the space of modular forms on U whose definition we now recall. Recall that T_0 is the diagonal torus of $GL(2)_{/O}$; so, $T_0 = \mathbb{G}_{m/O}^2$. Since $T_0(O/\mathfrak{N}')$ is canonically a quotient of $U_0(\mathfrak{N}')$ for an ideal $\mathfrak{N}'$, a character $\epsilon : T_0(O/\mathfrak{N}') \to \mathbb{C}^\times$ can be considered as a character of $U_0(\mathfrak{N}')$. Writing $\epsilon\left(\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix} \right)\right) = \epsilon_1(a)\epsilon_2(d)$, if $\epsilon^- = \epsilon_1^{-1}\epsilon_2$ factors through $O/\mathfrak{N}$ for $\mathfrak{N}|\mathfrak{N}'$, then we can extend the character ϵ of $U_0(\mathfrak{N}')$ to $U_0(\mathfrak{N})$ by putting $\epsilon(u) = \epsilon_1(\det(u))\epsilon^-(d)$ for $u = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in U_0(\mathfrak{N})$. In this sense, we hereafter assume that ϵ is defined modulo $\mathfrak{N}$ and regard ϵ as a character of $U_0(\mathfrak{N})$. We choose a Hecke character $\epsilon_+ : F_{\mathbb{A}}^\times / F^\times \to \mathbb{C}^\times$ with infinity type $(1 - [\kappa])I$ (for an integer $[\kappa]$) such that $\epsilon_+(z) = \epsilon_1(z)\epsilon_2(z)$ for $z \in \widehat{O}^\times$. We also write ϵ_+^t for the restriction of ϵ_+ to the maximal torsion subgroup $\Delta_F(\mathfrak{N})$ of $Cl_F^+(\mathfrak{N}p^\infty)$ (the strict ray class group modulo $\mathfrak{N}p^\infty : \varprojlim_n Cl_F^+(\mathfrak{N}p^n)$).

Writing T^2 for $\mathrm{Res}_{O/\mathbb{Z}} T_0$ (the diagonal torus of G), the group of geometric characters $X^*(T^2)$ is isomorphic to $\mathbb{Z}[I]^2$ so that the character $(m,n) \in \mathbb{Z}[I]^2$ has value $x^m y^n = \prod_{\sigma \in I} (\sigma(x)^{m_\sigma} \sigma(y)^{n_\sigma})$ at each diagonal matrix $\mathrm{diag}[x,y] := \left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix} \right) \in T^2$.

Taking $\kappa = (\kappa_1, \kappa_2) \in \mathbb{Z}[I]^2$, we assume $[\kappa]I = \kappa_1 + \kappa_2$ (identifying I with $\sum_{\sigma \in I} \sigma$), and we associate with κ a factor of automorphy:

$$J_\kappa(g, \tau) = \det(g_\infty)^{\kappa_2 - I} j(g_\infty, \tau)^{\kappa_1 - \kappa_2 + I} \quad \text{for } g \in G(\mathbb{A}) \text{ and } \tau \in \mathfrak{Z}, \tag{2.5}$$

where $\det(g_\infty)^{\kappa_2 - I} = \prod_{\sigma \in I} \det(g_\sigma)^{\kappa_{2,\sigma} - 1}$, $j(\left(\begin{smallmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{smallmatrix}\right), \tau_\sigma) = c_\sigma \tau_\sigma + d_\sigma$ and

$$j(g_\infty, \tau)^{\kappa_1 - \kappa_2 + I} = \prod_{\sigma \in I} j(g_\sigma, \tau_\sigma)^{\kappa_{1,\sigma} - \kappa_{2,\sigma} + 1}.$$

Let $C_{\mathbf{i}} = \{g \in G(\mathbb{R}) | g(\mathbf{i}) = \mathbf{i}\}$ for $\mathbf{i} = (\sqrt{-1}, \cdots, \sqrt{-1}) \in \mathfrak{Z}$. We define $S_\kappa(U, \epsilon; \mathbb{C})$ by the space of functions $f : G(\mathbb{A}) \to \mathbb{C}$ satisfying the following three conditions (e.g. [8] Section 2.2 and [16] Section 4.3.1):

(S1) $f(\alpha z x u) = \epsilon(u)\epsilon_+^t(\delta) f(x) J_\kappa(u, \mathbf{i})^{-1}$ for all $\alpha \in G(\mathbb{Q})$ and all $u \in U \cdot C_{\mathbf{i}}$ and z is any element in the center $F_{\mathbb{A}}^\times \subset G(\mathbb{A})$ representing an element δ in the maximal torsion subgroup $\Delta_F(\mathfrak{N})$ of $Cl_F^+(\mathfrak{N}p^\infty)$;

(S2) Choose $u \in G(\mathbb{R})$ with $u(\mathbf{i}) = \tau$ for $\tau \in \mathfrak{Z}$, and put $f_x(\tau) = f(xu)J_\kappa(u, \mathbf{i})$ for each $x \in G(\mathbb{A}^{(\infty)})$ (which only depends on τ). Then f_x is a holomorphic function on $\mathfrak{Z}$ for all x;

(S3) $f_x(\tau)$ for each x is rapidly decreasing as $\eta_\sigma \to \infty$ ($\tau = \xi + \mathbf{i}\eta$) for all $\sigma \in I$ uniformly.

It is easy to check (e.g. [15] 3.1.5) that the function f_x in (S2) satisfies the classical automorphy condition:

$$f(\gamma(\tau)) = \epsilon(x^{-1}\gamma x) f(\tau) J_\kappa(\gamma, \tau) \quad \text{for all } \gamma \in \Gamma_x(U), \tag{2.6}$$

where $\Gamma_x(U) = xUx^{-1}G(\mathbb{R})^+ \cap G(\mathbb{Q})$. Also by (S3), f_x is rapidly decreasing towards all cusps of Γ_x (e.g. [15] (3.22)); so, it is a cusp form.

(ϵ_+) Imposing that f have the central character ϵ_+ in addition to the action of $\Delta_F(\mathfrak{N})$ in (S1), we define the subspace $S_\kappa(\mathfrak{N}, \epsilon_+; \mathbb{C})$ of $S_\kappa(U_0(\mathfrak{N}), \epsilon; \mathbb{C})$.

The symbols $\kappa = (\kappa_1, \kappa_2)$ *and* $(\varepsilon_1, \varepsilon_2)$ *here correspond to* (κ_2, κ_1) *and* $(\varepsilon_2, \varepsilon_1)$ *in* [16] Section 4.2.6 (page 171) and [17] Section 2.3 because of a different notational convention in [16] and [17].

We identify I with $\sum_\sigma \sigma$ in $\mathbb{Z}[I]$. We have $S_\kappa = 0$ unless $\kappa_1 + \kappa_2 = [\kappa_1 + \kappa_2]I$ for $[\kappa_1 + \kappa_2] \in \mathbb{Z}$, because $I - (\kappa_1 + \kappa_2)$ is the infinity type of the central character of automorphic representations generated by S_κ. We write simply $[\kappa]$ for $[\kappa_1 + \kappa_2] \in \mathbb{Z}$ assuming $S_\kappa \neq 0$. The $SL(2)$–weight of the central character of an irreducible automorphic representation π generated by $f \in S_\kappa(U, \epsilon; \mathbb{C})$ is given by $k := \kappa_1 - \kappa_2 + I$ (which specifies the infinity type of π_∞ as a discrete series representation of $SL_2(F_\mathbb{R})$). There is a geometric meaning of the weight κ: the Hodge weight of the motive attached to π (cf. [2]) is given by $\{(\kappa_{1,\sigma}, \kappa_{2,\sigma}), (\kappa_{2,\sigma}, \kappa_{1,\sigma})\}_\sigma$, and thus, the requirement $\kappa_1 - \kappa_2 \geq I$ is the regularity assumption for the motive (and is equivalent to the classical weight $k \geq 2I$ condition).

Choose a prime element $\varpi_{\mathfrak{q}}$ of $O_{\mathfrak{q}}$ for each prime $\mathfrak{q}$ of F. We extend $\epsilon^- : \widehat{O}^\times \to \mathbb{C}^\times$ to $F^\times_{\mathbb{A}^{(\infty)}} \to \mathbb{C}^\times$ just by putting $\epsilon^-(\varpi_{\mathfrak{q}}^m) = 1$ for $m \in \mathbb{Z}$. This is possible because $F_{\mathfrak{q}}^\times = O_{\mathfrak{q}}^\times \times \varpi_{\mathfrak{q}}^{\mathbb{Z}}$ for $\varpi_{\mathfrak{q}}^{\mathbb{Z}} = \{\varpi_{\mathfrak{q}}^m | m \in \mathbb{Z}\}$. Similarly, we extend ϵ_1 to $F^\times_{\mathbb{A}^{(\infty)}}$. Then we define $\epsilon(u) = \epsilon_1(\det(u))\epsilon^-(a_{\mathfrak{N}})$ for $u = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Delta_0(\mathfrak{N})$. Let $\mathcal{U}$ be the unipotent algebraic subgroup of $GL(2)_{/O}$ defined by $\mathcal{U}(A) = \left\{\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right) \middle| a \in A\right\}$. For each $UyU \in R(U, \Delta_0(\mathfrak{N}\mathfrak{p}^\alpha))$, we decompose $UyU = \bigsqcup_{t \in D_0, u \in \mathcal{U}(\widehat{O})} utU$ for finitely many u and t (see [23] Chapter 3 or [15] 3.1.6) and define

$$f|[UyU](x) = \sum_{t,u} \epsilon(t)^{-1} f(xut). \tag{2.7}$$

We check that this operator preserves the space $S_\kappa(\mathfrak{N}, \epsilon; \mathbb{C})$ of cusp forms. This action for y with $y_{\mathfrak{N}} = 1$ is independent of the choice of the extension of ϵ to $T_0(F_{\mathbb{A}})$. When $y_{\mathfrak{N}} \neq 1$, we may assume that $y_{\mathfrak{N}} \in D_0 \subset T_0(F_{\mathbb{A}})$, and in this case, t can be chosen so that $t_{\mathfrak{N}} = y_{\mathfrak{N}}$ (so $t_{\mathfrak{N}}$ is independent of single right cosets in the double coset). If we extend ϵ to $T_0(F_{\mathbb{A}}^{(\infty)})$ by choosing another prime element $\varpi'_{\mathfrak{q}}$ and write the extension as ϵ', then we have

$$\epsilon(t_{\mathfrak{N}})[UyU] = \epsilon'(t_{\mathfrak{N}})[UyU]',$$

where the operator on the right-hand-side is defined with respect to ϵ'. Thus the sole difference is the root of unity $\epsilon(t_{\mathfrak{N}})/\epsilon'(t_{\mathfrak{N}})$. Since it depends on the choice of $\varpi_{\mathfrak{q}}$, we make the choice once and for all, and write $T(\mathfrak{q})$ for $\left[U\left(\begin{smallmatrix}\varpi_{\mathfrak{q}} & 0\\ 0 & 1\end{smallmatrix}\right)U\right]$ (if $\mathfrak{q}|\mathfrak{N}$). By linearity, these action of double cosets extends to the ring action of the double coset ring $R(U, \Delta_0(\mathfrak{N}\mathfrak{p}^\alpha))$.

2.3. **Fourier and q-expansion.** To introduce rationality of modular forms, we recall Fourier expansion of adelic modular forms (cf. [8] Sections 2.3–4). Recall the embedding $i_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, and identify $\overline{\mathbb{Q}}$ with the image of i_∞. Recall also the differental idele $d \in F_{\mathbb{A}}^\times$ with $d^{(\mathfrak{d})} = 1$ and $d\widehat{O} = \mathfrak{d}\widehat{O}$. Each member f of $S_\kappa(U, \epsilon; \mathbb{C})$ has its Fourier expansion:

$$f\left(\begin{smallmatrix} y & x\\ 0 & 1\end{smallmatrix}\right) = |y|_{\mathbb{A}} \sum_{0 \ll \xi \in F} a(\xi y d, f)(\xi y_\infty)^{-\kappa_2} \mathbf{e}_F(i\xi y_\infty)\mathbf{e}_F(\xi x), \tag{2.8}$$

where $\mathbf{e}_F : F_{\mathbb{A}}/F \to \mathbb{C}^\times$ is the additive character with $\mathbf{e}_F(x_\infty) = \exp(2\pi i \sum_{\sigma \in I} x_\sigma)$ for $x_\infty = (x_\sigma)_\sigma \in \mathbb{R}^I = F \otimes_{\mathbb{Q}} \mathbb{R}$. Here $y \mapsto a(y, f)$ is a function defined on $y \in F_{\mathbb{A}}^\times$ only depending on its finite part $y^{(\infty)}$. The function $a(y, f)$ is supported by the set $(\widehat{O} \times F_\infty) \cap F_{\mathbb{A}}^\times$ of *integral* ideles.

Let $F[\kappa]$ be the field fixed by $\{\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F) | \kappa\sigma = \kappa\}$, over which the character $\kappa \in X^*(T^2)$ is rational. Write $O[\kappa]$ for the integer ring of $F[\kappa]$. We also define $O[\kappa, \epsilon]$ for the integer ring of the field $F[\kappa, \epsilon]$ generated by the values of ϵ over $F[\kappa]$. For any $F[\kappa, \epsilon]$–algebra A inside $\mathbb{C}$, we define

$$S_\kappa(U, \epsilon; A) = \left\{ f \in S_\kappa(U, \epsilon; \mathbb{C}) \middle| a(y, f) \in A \ \text{ for all integral ideles } y \right\}. \tag{2.9}$$

We can interpret $S_\kappa(U, \epsilon; A)$ as the space of A–rational global sections of a line bundle of a variety defined over A (e.g., [17] Chapter 4); so, we have, by the flat base-change theorem (e.g. [14] Lemma 1.10.2),

$$S_\kappa(\mathfrak{N}, \epsilon; A) \otimes_A \mathbb{C} = S_\kappa(\mathfrak{N}, \epsilon; \mathbb{C}) \tag{2.10}$$

The Hecke operators preserve A–rational modular forms (e.g., [16] 4.2.9). We define the Hecke algebra $h_\kappa(U, \epsilon; A) \subset \mathrm{End}_A(S_\kappa(U, \epsilon; A))$ by the A–subalgebra generated by the Hecke operators of $R(U, \Delta_0(\mathfrak{N}\mathfrak{p}^\alpha))$.

For any $\overline{\mathbb{Q}}_p$–algebras A, we define

$$S_\kappa(U, \epsilon; A) = S_\kappa(U, \epsilon; \overline{\mathbb{Q}}) \otimes_{\overline{\mathbb{Q}}, i_p} A. \tag{2.11}$$

By linearity, $y \mapsto a(y, f)$ extends to a function on $F_{\mathbb{A}}^{\times} \times S_\kappa(U, \epsilon; A)$ with values in A. We define the q–expansion coefficients (at p) of $f \in S_\kappa(U, \epsilon; A)$ by

$$\mathbf{a}_p(y, f) = y_p^{-\kappa_2} a(y, f). \tag{2.12}$$

The formal q–expansion of an A–rational f has values in the space of functions on $F_{\mathbb{A}^{(\infty)}}^{\times}$ with values in the formal monoid algebra $A[[q^\xi]]_{\xi \in F_+}$ of the multiplicative semi-group F_+ made up of totally positive elements, which is given by

$$f(y) = \mathcal{N}(y)^{-1} \sum_{\xi \gg 0} \mathbf{a}_p(\xi y d, f) q^\xi, \tag{2.13}$$

where $\mathcal{N} : F_{\mathbb{A}}^{\times}/F^{\times} \to \overline{\mathbb{Q}}_p^{\times}$ is the character given by $\mathcal{N}(y) = y_p^{-I} |y^{(\infty)}|_{\mathbb{A}}^{-1}$.

We now define for any p–adically complete $O[\kappa, \epsilon]$–algebra A in $\mathbb{C}_p$

$$S_\kappa(U, \epsilon; A) = \left\{ f \in S_\kappa(U, \epsilon; \mathbb{C}_p) \middle| \mathbf{a}_p(y, f) \in A \text{ for integral } y \right\}. \tag{2.14}$$

These spaces have geometric meaning as the space of A–integral global sections of a line bundle defined over A of the Hilbert modular variety of level U (see [16] Section 4.2.6), and the q–expansion above for a fixed $y = y^{(\infty)}$ gives rise to the geometric q–expansion at the infinity cusp of the classical modular form f_x for $x = \left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right)$ (see [7] (1.5) and [16] (4.63)).

We choose a complete representative set $\{c_i\}_{i=1,\dots,h}$ in finite ideles for the strict idele class group $F^\times \backslash F_{\mathbb{A}}^{\times}/\widehat{O}^\times F_{\infty+}^{\times}$, where h is the strict class number of F. Let $\mathfrak{c}_i = c_i O$. Put $t_i = \left(\begin{smallmatrix} c_i d^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and consider $f_i = f_{t_i}$ as defined in (S2). The collection $(f_i)_{i=1,\dots,h}$ determines f, because of the approximation theorem. Then $f(c_i d^{-1})$ gives the q–expansion of f_i at the Tate abelian variety with $\mathfrak{c}_i$–polarization $\mathrm{Tate}_{\mathfrak{c}_i^{-1}, O}(q)$ ($\mathfrak{c}_i = c_i O$). By the q–expansion principle (e.g., [13] Section 4.1 or [17] 4.2.6), the q–expansion $f(y)$ determines f uniquely.

2.4. **Hilbert modular Hecke algebras.** We write $T(y)$ for the Hecke operator acting on $S_\kappa(U, \epsilon; A)$ corresponding to the double coset $U \left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right) U$ for an integral idele y. We renormalize $T(y)$ to have a p–integral operator $\mathbb{T}(y)$: $\mathbb{T}(y) = y_p^{-\kappa_2} T(y)$. Since this only affects $T(y)$ with $y_p \neq 1$, $\mathbb{T}(\mathfrak{q}) = T(\varpi_\mathfrak{q}) = T(\mathfrak{q})$ if $\mathfrak{q} \nmid p$. However $\mathbb{T}(\mathfrak{p}) \neq T(\mathfrak{p})$ for primes $\mathfrak{p}|p$. The renormalization is optimal to have the stability of the A–integral spaces under Hecke operators. We define $\langle \mathfrak{q} \rangle = N(\mathfrak{q}) T(\mathfrak{q}, \mathfrak{q})$ for $\mathfrak{q} \nmid \mathfrak{N}\mathfrak{p}^\alpha$, which is equal to the central action of a prime element $\varpi_\mathfrak{q}$ of $O_\mathfrak{q}$ times $N(\mathfrak{q}) = |\varpi_\mathfrak{q}|_{\mathbb{A}}^{-1}$. We have the following formula of the action of $T(\mathfrak{q})$ and $T(\mathfrak{q}, \mathfrak{q})$ (e.g., [16] Section 4.2.10):

$$\mathbf{a}_p(y, f|\mathbb{T}(\mathfrak{q})) = \begin{cases} \mathbf{a}_p(y\varpi_\mathfrak{q}, f) + \mathbf{a}_p(y\varpi_\mathfrak{q}^{-1}, f|\langle \mathfrak{q} \rangle) & \text{if } \mathfrak{q} \text{ is outside } \mathfrak{n} \\ \mathbf{a}_p(y\varpi_\mathfrak{q}, f) & \text{otherwise,} \end{cases} \tag{2.15}$$

where the level $\mathfrak{n}$ of U is the ideal maximal under the condition: $U_1^1(\mathfrak{n}) \subset U \subset U_0(\mathfrak{N})$. Thus $\mathbb{T}(\varpi_\mathfrak{q}) = U(\mathfrak{q})$ (up to p–adic units) when $\mathfrak{q}$ is a factor of the level of U (even when $\mathfrak{q}|p$; see [16] (4.65–66)). Writing the level of U as $\mathfrak{N}\mathfrak{p}^\alpha$, we assume

$$\text{either } p|\mathfrak{N}\mathfrak{p}^\alpha \text{ or } [\kappa] \geq 0, \tag{2.16}$$

since $\mathbb{T}(\mathfrak{q})$ and $\langle \mathfrak{q} \rangle$ preserve the space $S_\kappa(U, \epsilon; A)$ under this condition (see [16] Theorem 4.28). We define the Hecke algebra $h_\kappa(U, \epsilon; A)$ (resp. $h_\kappa(\mathfrak{N}, \epsilon_+; A)$)

with coefficients in A by the A–subalgebra of the A–linear endomorphism algebra $\mathrm{End}_A(S_\kappa(U,\epsilon;A))$ (resp. $\mathrm{End}_A(S_\kappa(\mathfrak{N},\epsilon_+;A))$) generated by the action of the finite group $U_0(\mathfrak{N}\mathfrak{p}^\alpha)/U$, $\mathbb{T}(\mathfrak{q})$ and $\langle\mathfrak{q}\rangle$ for all $\mathfrak{q}$.

We have canonical projections:

$$R(U_1^1(\mathfrak{N}\mathfrak{p}^\alpha),\Delta_0(\mathfrak{N}\mathfrak{p}^\alpha)) \twoheadrightarrow R(U,\Delta_0(\mathfrak{N}\mathfrak{p}^\alpha)) \twoheadrightarrow R(U_0(\mathfrak{N}p^\beta),\Delta_0(\mathfrak{N}p^\beta))$$

for all $\alpha \geq \beta$ ($\Leftrightarrow$ $\alpha(\mathfrak{p}) \geq \beta(\mathfrak{p})$ for all $\mathfrak{p}|p$) taking canonical generators to the corresponding ones, which are compatible with inclusions

$$S_\kappa(U_0(\mathfrak{N}p^\beta),\epsilon;A) \hookrightarrow S_\kappa(U,\epsilon;A) \hookrightarrow S_\kappa(U_1^1(\mathfrak{N}\mathfrak{p}^\alpha),\epsilon;A).$$

We get a projective system of Hecke algebras $\{h_\kappa(U,\epsilon;A)\}_U$ (U running through open subgroups of $U_0(\mathfrak{N}p)$ containing $U_1^1(\mathfrak{N}p^\infty)$), whose projective limit (when $\kappa_1 - \kappa_2 \geq I$) gives rise to the universal Hecke algebra $\mathbf{h}(\mathfrak{N},\epsilon;A)$ for a complete p–adic algebra A. This algebra is known to be independent of κ (as long as $\kappa_1 - \kappa_2 \geq I$) and has canonical generators $\mathbb{T}(y)$ over $A[[\mathbf{G}]]$ (for $\mathbf{G} = (O_p \times (O/\mathfrak{N}^{(p)}))^\times \times Cl_F^+(\mathfrak{N}p^\infty)$), where $\mathfrak{N}^{(p)}$ is the prime-to-p part of $\mathfrak{N}$. Here note that the operator $\langle\mathfrak{q}\rangle$ is included in the action of $\mathbf{G}$, because $\mathfrak{q} \in Cl_F^+(\mathfrak{N}p^\infty)$. We write $h_\kappa^{n.ord}(U,\epsilon;A)$, $h_\kappa^{n.ord}(\mathfrak{N}p^\alpha,\epsilon_+;A)$ and $\mathbf{h}^{n.ord} = \mathbf{h}^{n.ord}(\mathfrak{N},\epsilon;A)$ for the image of the (nearly) ordinary projector $e = \lim_n \mathbb{T}(p)^{n!}$. The algebra $\mathbf{h}^{n.ord}$ is by definition the universal nearly ordinary Hecke algebra over $A[[\mathbf{G}]]$ of level $\mathfrak{N}$ with "Neben character" ϵ. We also note here that this algebra $\mathbf{h}^{n.ord}(\mathfrak{N},\epsilon;A)$ is exactly the one $\mathbf{h}(\psi^+,\psi')$ employed in [18] page 240 (when specialized to the CM component there) if A is a complete p–adic discrete valuation ring.

Let $\mathbf{\Lambda}_A = A[[\mathbf{\Gamma}]]$ for the maximal torsion-free quotient $\mathbf{\Gamma}$ of $\mathbf{G}$. We fix a splitting $\mathbf{G} = \mathbf{\Gamma} \times \mathbf{G}_{tor}$ for a finite group $\mathbf{G}_{tor}$. If A is a complete p–adic valuation ring, then $\mathbf{h}^{n.ord}(\mathfrak{N},\epsilon;A)$ is a torsion-free $\mathbf{\Lambda}_A$–algebra of finite rank and is $\mathbf{\Lambda}_A$–free under some mild conditions on $\mathfrak{N}$ and ϵ ([16] 4.2.12). Take a point $P \in \mathrm{Spf}(\mathbf{\Lambda})(A) = \mathrm{Hom}_{cont}(\mathbf{\Gamma},A^\times)$. Regarding P as a character of $\mathbf{G}$, we call P *arithmetic* if it is given locally by an algebraic character $\kappa(P) \in X^*(T^2)$ with $\kappa_1(P) - \kappa_2(P) \geq I$. Thus if P is arithmetic, $\epsilon_P = P\kappa(P)^{-1}$ is a character of $T_0(O/\mathfrak{p}^\alpha\mathfrak{N})$ for some multi-exponent $\alpha \geq 0$. Similarly, the restriction of P to $Cl_F^+(\mathfrak{N}p^\infty)$ is a p–adic Hecke character ϵ_{P+} induced by an arithmetic Hecke character of infinity type $(1 - [\kappa(P)])I$. As long as P is arithmetic, we have a canonical specialization morphism:

$$\mathbf{h}^{n.ord}(\mathfrak{N},\epsilon;A) \otimes_{\mathbf{\Lambda}_A,P} A \twoheadrightarrow h_{\kappa(P)}^{n.ord}(\mathfrak{N}\mathfrak{p}^\alpha,\epsilon_{P+};A),$$

which is an isogeny (surjective and of finite kernel) and is an isomorphism if $\mathbf{h}^{n.ord}$ is $\mathbf{\Lambda}_A$–free. The specialization morphism takes the generators $\mathbb{T}(y)$ to $\mathbb{T}(y)$.

3. Anticyclotomic Iwasawa series

We fix a conductor $\mathfrak{C}$ which is a nonzero R-ideal. We decompose $\mathfrak{C} = \mathfrak{F}\mathfrak{F}_c\mathfrak{I}\mathfrak{R}$ so that $\mathfrak{F}\mathfrak{F}_c$ consists of split primes over F, $\mathfrak{I}$ (resp. $\mathfrak{R}$) consists of inert (resp. ramified) primes over F, $\mathfrak{F} + \mathfrak{F}_c = R$ and $\mathfrak{F}_c^c \supset \mathfrak{F}$. In this section, we redo the computation done in [12] Section 5 allowing the case $\mathfrak{I}\mathfrak{R} \neq 1$. In [12], we implicitly used the following inclusion diagram of semisimple extensions:

$$\begin{tikzcd}
 & & M_1 = M \arrow[drr] & & \\
F \arrow[urr] \arrow[drr] \arrow[r, hook] & F \oplus F \arrow[rrr, hook] & & & M \oplus M, \\
 & & M \arrow[urr] & &
\end{tikzcd}$$

where $M_1 (\cong M)$ is embedded into $M \oplus M$ by $\xi \mapsto (\xi, \xi^c)$ and M is embedded into $M \oplus M$ diagonally. We replace this diagram by well chosen field extensions:

$$\begin{tikzcd}
 & M_1 \arrow[dr] & \\
F \arrow[ur] \arrow[dr] \arrow[r, hook] & F_1 \arrow[r, hook] & K. \\
 & M \arrow[ur] &
\end{tikzcd}$$

This new choice allows us to prove the assertion (A): $h_{\mathfrak{i}}(M/F)L^-_M(\psi)|H(\psi)$ without assuming (S).

The cuspidal automorphic induction $\pi(\varphi)$ of φ is supercuspidal at prime factors $\mathfrak{l}$ of $\mathfrak{I}\mathfrak{N}$, but by definition, the local Galois representation at $\mathfrak{l}$ associated to $\pi(\varphi)$ is the induced representation of the character $\varphi_{\mathfrak{l}}$ of the quadratic extension $M_{\mathfrak{l}}/F_{\mathfrak{l}}$. This fact (the condition (ind) below) still allows us to easily determine the exact Euler factor at $\mathfrak{l}$ of the p-adic Rankin product studied in [7]. The local computation at primes where $\pi(\varphi)$ is non-super-cuspidal (so, principal) has been done in [12] Section 5, but for the sake of completeness, we will repeat some details.

We consider $Z = Z(\mathfrak{C}) = \varprojlim_n Cl_M(\mathfrak{C}p^n)$ for the ray class group $Cl_M(\mathfrak{x})$ of M modulo $\mathfrak{x}$. We split $Z(\mathfrak{C}) = \Delta_{\mathfrak{C}} \times \Gamma_{\mathfrak{C}}$ for a finite group $\Delta = \Delta_{\mathfrak{C}}$ and a torsion-free subgroup $\Gamma_{\mathfrak{C}}$. Since the projection: $Z(\mathfrak{C}) \twoheadrightarrow Z(1)$ induces an isomorphism $\Gamma_{\mathfrak{C}} = Z(\mathfrak{C})/\Delta_{\mathfrak{C}} \cong Z(1)/\Delta_1 = \Gamma_1$, we identify $\Gamma_{\mathfrak{C}}$ with Γ_1 and write it as Γ_M, which has a natural action of $\mathrm{Gal}(M/F)$. We define $\Gamma^+_M = H^0(\mathrm{Gal}(M/F), \Gamma)$ and $\Gamma^-_M = \Gamma_M/\Gamma^+_M$. Write $\pi_- : Z \to \Gamma^-_M$ and $\pi_\Delta : Z \to \Delta$ for the two projections. Take a character $\varphi : \Delta \to \overline{\mathbb{Q}}^\times$, and regard it as a character of Z through the projection: $Z \twoheadrightarrow \Delta$. The Katz measure $\mu_{\mathfrak{C}}$ on $Z(\mathfrak{C})$ associated to the p–adic CM type Σ_p as in [18] Theorem II induces the anticyclotomic φ–branch μ^-_φ by

$$\int_{\Gamma^-_M} \phi d\mu^-_\varphi = \int_{Z(\mathfrak{C})} \phi(\pi_-(z))\varphi(\pi_\Delta(z))d\mu_{\mathfrak{C}}(z).$$

We write $L^-_M(\varphi)$ for this measure $d\mu^-_\varphi$ regarding it as an element of the algebra $\Lambda^- = W[[\Gamma^-_M]]$ made up of measures with values in W.

We look into the arithmetic of the unique $\mathbb{Z}_p^{[F:\mathbb{Q}]}$–extension M^-_∞ of M on which we have $c\sigma c^{-1} = \sigma^{-1}$ for all $\sigma \in \mathrm{Gal}(M^-_\infty/M)$ for complex conjugation c. The extension M^-_∞/M is called the anticyclotomic tower over M. Writing $M(\mathfrak{C}p^\infty)$ for the ray class field over M modulo $\mathfrak{C}p^\infty$, we identify $Z(\mathfrak{C})$ with $\mathrm{Gal}(M(\mathfrak{C}p^\infty)/M)$ via the Artin reciprocity law. Then $\mathrm{Gal}(M(\mathfrak{C}p^\infty)/M^-_\infty) = \Gamma^+_M \times \Delta_{\mathfrak{C}}$ and $\mathrm{Gal}(M^-_\infty/M) = \Gamma^-_M$. We then define M_Δ by the fixed field of $\Gamma_{\mathfrak{C}}$ in $M(\mathfrak{C}p^\infty)$; so, $\mathrm{Gal}(M_\Delta/M) = \Delta$. Since φ is a character of Δ, φ factors through $\mathrm{Gal}(M^-_\infty M_\Delta/M)$. Let $L_\infty/M^-_\infty M_\Delta$ be the maximal p–abelian extension unramified outside Σ_p. Each $\gamma \in \mathrm{Gal}(L_\infty/M)$ acts on the normal subgroup $X = \mathrm{Gal}(L_\infty/M^-_\infty M_\Delta)$ continuously by conjugation, and by the commutativity of X, this action factors through $\mathrm{Gal}(M_\Delta M_\infty^-/M)$. Then we look into the Γ_M^-–module: $X[\varphi] = X \otimes_{\Delta_{\mathfrak{C}},\varphi} W$.

As is well known, $X[\varphi]$ is a Λ^-–module of finite type, and in many cases, it is torsion by a result of Fujiwara (cf. [4], [10] Corollary 5.4 and [17] Section 5.3) generalizing the fundamental work of Wiles [27] and Taylor-Wiles [25]. If one assumes the Σ–Leopoldt conjecture for abelian extensions of M, we know that $X[\varphi]$ is a torsion module over Λ^- unconditionally (see [19] Theorem 1.2.2). If $X[\varphi]$ is a torsion Λ^-–module, we can think of the characteristic element $\mathcal{F}^-(\varphi) \in \Lambda^-$ of the module $X[\varphi]$. If $X[\varphi]$ is not of torsion over Λ^-, we simply put $\mathcal{F}^-(\varphi) = 0$. A character φ of Δ is called *anticyclotomic* if $\varphi(c\sigma c^{-1}) = \varphi^{-1}(\sigma)$.

We are going to prove in this section the following theorem:

Theorem 3.1. *Let ψ be an anticyclotomic character of Δ. If $p > 3$ is unramified in $M/\mathbb{Q}$, then the anticyclotomic p–adic Hecke L–function $L_M^-(\psi)$ is a factor of $\mathcal{F}^-(\psi)$ in Λ^-.*

Regarding φ as a Galois character, we define $\varphi^-(\sigma) = \varphi(c\sigma c^{-1}\sigma^{-1})$ for $\sigma \in \mathrm{Gal}(\overline{M}/M)$. Then φ^- is anticyclotomic. By enlarging $\mathfrak{C}$ if necessary, we can find a character φ such that $\psi = \varphi^-$ for any given anticyclotomic ψ (e.g. [14] page 339 or [17] Lemma 5.31). Thus we may always assume that $\psi = \varphi^-$.

It is proven in [18] and [19] that $L_M(\varphi^-)$ is a factor of $\mathcal{F}^-(\varphi^-)$ in $\Lambda^- \otimes_{\mathbb{Z}} \mathbb{Q}$. Thus the improvement concerns the p–factor of $L_M^-(\varphi^-)$, which has been shown to be trivial in [13] under the assumption (S) except for the rare cases of positive μ by a trivial reason, but it can be often nontrivial without the assumption (S) as such examples are given in [13] at the end. The main point of this section is to give a new proof of the assertion (A) reducing it to the vanishing of the μ-invariant of the p-adic Hecke L-functions in [13] Theorem I (which still assumes (S) but as we have already explained, by a quadratic base-change we can reduce things to the nonvanishing result over a quadratic CM extension K of M where (S) is satisfied). We will restate the assertion (A) as Corollary 3.8. The proof is similar to the argument in [18], [19] and [12], but the use of $\mu(L_K^-) = 0$ is a new point. We first deduce a refinement of the result in [18] Section 7 using a unique Hecke eigenform (in a given automorphic representation) of minimal level at nonsupercuspidal places and new at supercuspidal places. The minimal level is possibly a proper factor of the conductor of the representation.

Here we describe how to reduce Theorem 3.1 to Corollary 3.8. Since the result is known for $F = \mathbb{Q}$ by the works of Rubin and Tilouine, we may assume that $F \neq \mathbb{Q}$. Put $\Lambda = W[[\Gamma_M]]$. By definition, for the universal Galois character $\widetilde{\varphi} : \mathrm{Gal}(M(\mathfrak{C}p^\infty)/M) \to \Lambda^\times$ sending $\delta \in \Delta_{\mathfrak{C}}$ to $\varphi(\delta)$ and $\gamma \in \Gamma_M$ to the group element $\gamma \in \Gamma_M \subset \Lambda$, the Pontryagin dual of the adjoint Selmer group $\mathrm{Sel}(Ad(\mathrm{Ind}_M^F \widetilde{\varphi}))$ defined in [15] 5.2 is isomorphic to the direct sum of $X[\psi] \otimes_{\Lambda^-} \Lambda$ and $\frac{Cl_M(\mathfrak{i})\otimes_{\mathbb{Z}}\Lambda}{Cl_F(\mathfrak{i})\otimes_{\mathbb{Z}}\Lambda}$ for the ray class groups $Cl_M(\mathfrak{i})$ and $Cl_F(\mathfrak{i})$ modulo $\mathfrak{i}$, respectively, for M and F (see [19] Proposition 3.32 and also [17] Theorem 5.33). Thus the characteristic power series of the Selmer group is given by $h_{\mathfrak{i}}(M/F)\mathcal{F}^-(\psi)$.

To relate this power series $h_{\mathfrak{i}}(M/F)\mathcal{F}^-(\psi)$ ($\psi = \varphi^-$) to congruence among automorphic forms, put $\mathfrak{f} = \mathfrak{F} \cap O$ and $\mathfrak{f}' = \mathfrak{F}_c \cap O$, and we identify $O_{\mathfrak{f}} \cong R_{\mathfrak{F}}$ and $O_{\mathfrak{f}'} \cong R_{\mathfrak{F}_c}$. Note that $\mathfrak{f} \subset \mathfrak{f}'$. Recall the maximal diagonal torus $T_0 \subset GL(2)_{/O}$. Thus φ restricted to $(R_{\mathfrak{F}} \times R_{\mathfrak{F}_c})^\times$ gives rise to the character φ of $T_0(O_{\mathfrak{f}})$. We then extend φ to a character φ_F of $T_0(O_{\mathfrak{f}} \times O_{d(M/F)})$ by $\varphi_F(\mathrm{diag}[x_{\mathfrak{f}}, y_{\mathfrak{f}}], \mathrm{diag}[x', y']) =$

$\varphi(\mathrm{diag}[x_{\mathfrak{f}}, y_{\mathfrak{f}}])\left(\frac{M/F}{y'}\right)$. Then we define the level ideal $\mathfrak{N}$ by $(\mathfrak{C}(\varphi^-)\cap F)d(M/F)$ and consider the Hecke algebra $\mathbf{h}^{n.ord} = \mathbf{h}^{n.ord}(\mathfrak{N}, \varphi_F; W)$. It is easy to see that there is a unique $W[[\mathbf{\Gamma}]]$–algebra homomorphism $\lambda : \mathbf{h}^{n.ord} \to \Lambda$ such that the associated Galois representation ρ_λ ([8] 2.8) is $\mathrm{Ind}_M^F \widetilde{\varphi}$. Here $\mathbf{\Gamma}$ is the maximal torsion-free quotient of $\mathbf{G}$ introduced in Section 2. Note that the restriction of ρ_λ to the inertia group $I_{\mathfrak{q}}$ at each prime $\mathfrak{q}|\mathfrak{f}$ is the diagonal representation $\left(\begin{smallmatrix} \varphi_{F,1} & 0 \\ 0 & \varphi_{F,2} \end{smallmatrix}\right)$ with values in $GL_2(W)$. For supercuspidal primes $\mathfrak{q}$, $\rho_{\mathfrak{q}} = \rho_\lambda|_{D_{\mathfrak{q}}} = \mathrm{Ind}_{M_{\mathfrak{Q}}}^{F_{\mathfrak{q}}} \varphi_{\mathfrak{Q}}$ for the unique prime $\mathfrak{Q}$ of M above $\mathfrak{q}$ (in this case, $\rho_{\mathfrak{q}}$ is absolutely irreducible). We write $H(\psi)$ for the congruence power series $H(\lambda)$ of λ (see [8] Section 2.9, where $H(\lambda)$ is written as $\eta(\lambda)$). Writing $\mathbb{T}$ for the local ring of $\mathbf{h}^{n.ord}$ through which λ factors, the divisibility: $H(\psi)|h_{\mathfrak{i}}(M/F)\mathcal{F}^-(\psi)$ follows from the surjectivity onto $\mathbb{T}$ of the natural morphism from the universal nearly ordinary deformation ring $R^{n.ord}$ of $\mathrm{Ind}_F^M \varphi \mod \mathfrak{m}_W$ (without deforming $\rho_{\mathfrak{q}}$ for each $\mathfrak{q}|\mathfrak{N}$ prime to p and the restriction of the determinant character to the torsion-part of $Cl_F^+(\mathfrak{N}p^\infty)$). See [19] Sections 3.3 and 6.2 for details of this implication. The surjectivity is obvious from our construction of $\mathbf{h}^{n.ord}(\mathfrak{N}, \psi_F; W)$ because it is generated by $\mathrm{Tr}(\rho_\lambda(Frob_{\mathfrak{q}}))$ for primes $\mathfrak{q}$ outside $p\mathfrak{N}$ and by the diagonal entries of ρ_λ restricted to $D_{\mathfrak{q}}$ for $\mathfrak{q}|p\mathfrak{N}$. Thus we prove the assertion (A): $h_{\mathfrak{i}}(M/F)L_M^-(\psi)|H(\psi)$ as Corollary 3.8, which will be proven in the rest of this section. As a final remark, if we write $\mathbb{T}^\chi$ for the quotient of $\mathbb{T}$ which parameterizes all p–adic modular Galois representations congruent to $\mathrm{Ind}_M^F \varphi$ with a given (compatible) determinant character χ, we have $\mathbb{T} \cong \mathbb{T}^\chi \widehat{\otimes}_W W[[\mathbf{\Gamma}^+]] = \mathbb{T}^\chi[[\mathbf{\Gamma}^+]]$ for the maximal torsion-free quotient $\mathbf{\Gamma}^+$ of $Cl_F^+(\mathfrak{N}p^\infty)$ (cf. [15] Theorem 5.44). This implies $H(\psi) \in W[[\Gamma_M^-]]$.

3.1. **Adjoint square L–values as Petersson metric.** Let $G = \mathrm{Res}_{O/\mathbb{Z}} GL(2)$. Let S be a finite set of finite places of F. Let π be a cuspidal automorphic representation of $G(\mathbb{A})$ which are everywhere principal at finite places outside S, supercuspidal at all places of S and in holomorphic discrete series at archimedean places. Since π is associated to holomorphic automorphic forms on $G(\mathbb{A})$, π is rational over the Hecke field generated by eigenvalues of the primitive Hecke eigenform in π. We have $\pi = \pi^{(\infty)} \otimes \pi_\infty$ for representations $\pi^{(\infty)}$ of $G(\mathbb{A}^{(\infty)})$ and π_∞ of $G(\mathbb{R})$. We further decompose

$$\pi^{(\infty)} = \bigotimes_{\mathfrak{q}\notin S} \pi(\epsilon_{1,\mathfrak{q}}, \epsilon_{2,\mathfrak{q}}) \otimes \bigotimes_{\mathfrak{l}\in S} \pi_{\mathfrak{l}}$$

for the principal series representation $\pi(\epsilon_{1,\mathfrak{q}}, \epsilon_{2,\mathfrak{q}})$ of $GL_2(F_{\mathfrak{q}})$ with two characters $\epsilon_{1,\mathfrak{q}}, \epsilon_{2,\mathfrak{q}} : F_{\mathfrak{q}}^\times \to \overline{\mathbb{Q}}^\times$. By the rationality of π, these characters have values in $\overline{\mathbb{Q}}$. Write ϵ_+ for the central character of $\pi^{(\infty)}$, and its local component $\epsilon_{+,\mathfrak{q}}$ for $\mathfrak{q} \notin S$ is given by $\epsilon_{1,\mathfrak{q}}\epsilon_{2,\mathfrak{q}}$. For $\mathfrak{l} \in S$, we put $\epsilon_{1,\mathfrak{l}} = 1$ and $\epsilon_{2,\mathfrak{l}} = \epsilon_{+,\mathfrak{l}}$. Thus $\epsilon_+ = \epsilon_1\epsilon_2$ for the product $\epsilon_j = \prod_{\mathfrak{q}} \epsilon_{j,\mathfrak{q}}$ over all finite places $\mathfrak{q}$. Write $\epsilon_{+,S} = \prod_{\mathfrak{l}\in S} \epsilon_{+,\mathfrak{l}}$ as a character of $O_S^\times = \prod_{\mathfrak{l}\in S} O_{\mathfrak{l}}^\times$. For any $d \in \widehat{O}^\times$, we write $d_S \in O_S^\times$ for the projection, and we put $d^{(S)} = d/d_S$.

In the space of automorphic forms in π, there is a unique normalized Hecke eigenform $f = f_\pi$ of *minimal* level satisfying the following conditions (see [6] Corollary 2.2):

(L1) The level $\mathfrak{N}$ is given by $\mathfrak{c}(\epsilon^-)\prod_{\mathfrak{l}\in S}\mathfrak{c}(\pi_{\mathfrak{l}})$ for the conductor $\mathfrak{c}(\epsilon^-)$ of the character ϵ^- of $(\widehat{O}^{(S)})^\times = \widehat{O}^\times/O_S^\times$ and the conductor $\mathfrak{c}(\pi_{\mathfrak{l}})$ of $\pi_{\mathfrak{l}}$ (for $\mathfrak{l} \in S$).

(L2) Note that $\epsilon_\pi : \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \epsilon_1((ad-bc)^{(S)})\epsilon^-(d)$ is a character of $U_0(\mathfrak{N})$ whose restriction to $U_0(C(\pi))$ for the conductor $C(\pi)$ of π induces the "Neben" character $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \epsilon_1(a^{(S)})\epsilon_2(d^{(S)})\epsilon_{+,S}(d_S)$. Then $f : G(\mathbb{Q})\backslash G(\mathbb{A}) \to \mathbb{C}$ satisfies $f(xu) = \epsilon_\pi(u)f(x)$.

(L3) The cusp form f gives rise (in the manner described in (S3)) to holomorphic cusp forms of weight $\kappa = (\kappa_1, \kappa_2) \in \mathbb{Z}[I]^2$.

In short, f_π is a cusp form in $S_\kappa(\mathfrak{N}, \epsilon; \mathbb{C})$. It is easy to see that $\Pi = \pi \otimes \epsilon_2^{-1}$ has conductor $\mathfrak{N}$ and that $v \otimes \epsilon_2$ is a constant multiple of f for the new vector v of Π (note here that Π may not be automorphic, but Π is an admissible irreducible representation of $G(\mathbb{A})$; so, the theory of new vectors still applies). Since the conductor $C(\pi)$ of π is given by the product of the conductors of ϵ_1 and ϵ_2, the minimal level $\mathfrak{N}$ is a factor of the conductor $C(\pi)$ and is often a proper divisor of $C(\pi)$.

By (L2), the Fourier coefficient $a(y, f)$ satisfies $a(uy, f) = \epsilon_1(u^{(S)})a(y, f)$ for $u \in \widehat{O}^\times$ ($\widehat{O} = O \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$). In particular, the function: $y \mapsto a(y, f)\overline{a(y, f)}$ only depends on the fractional ideal yO. Thus writing $a(\mathfrak{a}, f)\overline{a(\mathfrak{a}, f)}$ for the ideal $\mathfrak{a} = yO$, we defined in [7] the self Rankin product by

$$D(s - [\kappa] - 1, f, f) = \sum_{\mathfrak{a} \subset O} a(\mathfrak{a}, f)\overline{a(\mathfrak{a}, f)} N(\mathfrak{a})^{-s},$$

where $N(\mathfrak{a}) = [O : \mathfrak{a}] = |O/\mathfrak{a}|$. We have a shift: $s \mapsto s - [\kappa] - 1$, because in order to normalize the L–function, we used in [7] (4.6) the unitarization $\pi^u = \pi \otimes |\cdot|_{\mathbb{A}}^{([\kappa]-1)/2}$ in place of π to define the Rankin product. The weight κ^u of the unitarization satisfies $[\kappa^u] = 1$ and $\kappa^u \equiv \kappa \mod \mathbb{Q}I$. Note that (cf. [7] (4.2a))

$$f_\pi^u(x) := f_{\pi^u}(x) = D^{-([\kappa]+1)/2} f_\pi(x) |\det(x)|_{\mathbb{A}}^{([\kappa]-1)/2}. \tag{3.1}$$

We are going to define Petersson metric on the space of cusp forms satisfying (L1-3). For that, we write

$$X_0 = X_0(\mathfrak{N}) = G(\mathbb{Q})_+ \backslash G(\mathbb{A})_+ / U_0(\mathfrak{N}) F_{\mathbb{A}}^\times SO_2(F_{\mathbb{R}}).$$

We define the inner product (f, g) by

$$(f, g)_{\mathfrak{N}} = \int_{X_0(\mathfrak{N})} \overline{f(x)} g(x) |\det(x)|_{\mathbb{A}}^{[\kappa]-1} dx \tag{3.2}$$

with respect to the invariant measure dx on X_0 as in [7] page 342. In exactly the same manner as in [7] (4.9) (under the notational convention there), we obtain

$$\begin{aligned} D^s(4\pi)^{-I(s+1)-(\kappa_1-\kappa_2)} \Gamma_F((s+1)I + (\kappa_1 - \kappa_2)) \zeta_F^{(\mathfrak{N})}(2s+2) D(s, f, f) \\ = N(\mathfrak{N})^{-1} D^{-[\kappa]-2} (f, f\mathbb{E}_{0,0}(x, \mathbf{1}, \mathbf{1}; s+1))_{\mathfrak{N}}, \end{aligned}$$

where D is the discriminant $N(\mathfrak{d})$ of F, $\zeta_F^{(\mathfrak{N})}(s) = \zeta_F(s) \prod_{\mathfrak{q}|\mathfrak{N}} (1 - N(\mathfrak{q})^{-s})$ for the Dedekind zeta function $\zeta_F(s)$ of F and $\mathbb{E}_{k,w}(x, \mathbf{1}, \mathbf{1}; s)$ ($k = \kappa_1 - \kappa_2 + I$ and $w = I - \kappa_2$) is the Eisenstein series of level $\mathfrak{N}$ defined above (4.8e) of [7] for the identity characters $(\mathbf{1}, \mathbf{1})$ in place of $(\chi^{-1}\psi^{-1}, \theta)$ there.

By the residue formula at $s = 1$ of $\zeta_F^{(\mathfrak{N})}(2s)\mathbb{E}_{0,0}(x, \mathbf{1}, \mathbf{1}; s)$ (e.g. (RES2) in [9] page 173), we find

$$\begin{aligned}(3.3)\quad &(4\pi)^{-I-(\kappa_1-\kappa_2)}\Gamma_F(I + (\kappa_1 - \kappa_2))\mathrm{Res}_{s=0}\zeta_F^{(\mathfrak{N})}(2s + 2)D(s, f, f)\\ &\qquad = D^{-[\kappa]-2}N(\mathfrak{N})^{-1}\prod_{\mathfrak{q}|\mathfrak{N}}(1 - N(\mathfrak{q})^{-1})\frac{2^{[F:\mathbb{Q}]-1}\pi^{[F:\mathbb{Q}]}R_\infty h(F)}{w\sqrt{D}}(f, f)_{\mathfrak{N}},\end{aligned}$$

where $w = 2$ is the number of roots of unity in F, $h(F)$ is the class number of F and R_∞ is the regulator of F.

Since f corresponds to $v \otimes \epsilon_2$ for the new vector $v \in \Pi = \pi \otimes \epsilon_2^{-1}$ of the principal series representation $\Pi^{(S\infty)}$ of minimal level in its twist class $\{\Pi \otimes \eta\}$ (η running over all finite order characters of $F^\times_{\mathbb{A}^{(S\infty)}}$), by making product $\overline{f} \cdot f$, the effect of tensoring ϵ_2 disappears. Thus we may compute the Euler factor of $D(s, f, f)$ as if f were a new vector of the minimal level representation. Then for each prime factor $\mathfrak{q}|\mathfrak{c}(\epsilon^-)$, the Euler $\mathfrak{q}$–factor of $\zeta_F^{(\mathfrak{N})}(2s + 2)D(s, f, f)$ is given by

$$\sum_{\nu=0}^{\infty} a(\mathfrak{q}^\nu, f)\overline{a(\mathfrak{q}^\nu, f)}N(\mathfrak{q})^{-\nu s} = \left(1 - N(\mathfrak{q})^{[\kappa]-s}\right)^{-1},$$

because $a(\mathfrak{q}, f)\overline{a(\mathfrak{q}, f)} = N(\mathfrak{q})^{[\kappa]}$ by [5] Lemma 12.2. We now look into the local factor at $\mathfrak{l} \in S$. We suppose that

(ind) the local representation $\sigma(\pi_\mathfrak{l})$ of $\mathrm{Gal}(\overline{F}_\mathfrak{l}/F_\mathfrak{l})$ (for each place $\mathfrak{l} \in S$), associated to $\pi_\mathfrak{l}$ by the Langlands functoriality, is of the form $\mathrm{Ind}_{M_\mathfrak{l}}^{F_\mathfrak{l}} \xi_\mathfrak{l}$ for a quadratic extension $M_\mathfrak{l}/F_\mathfrak{l}$.

The above condition is satisfied always for any odd prime in S (see [26]). Since $\pi_\mathfrak{l}$ ($\mathfrak{l} \in S$) is supercuspidal, $\mathrm{Ind}_{M_\mathfrak{l}}^{F_\mathfrak{l}} \xi_\mathfrak{l}$ is irreducible, for $\tau \in \mathrm{Gal}(\overline{F}_\mathfrak{l}/F_\mathfrak{l})$ inducing the generator of $\mathrm{Gal}(M_\mathfrak{l}/F_\mathfrak{l})$ on $M_\mathfrak{l}$, $\xi_\mathfrak{l}^\tau(\sigma) = \xi_\mathfrak{l}(\tau\sigma\tau^{-1})$ is not equal to $\xi_\mathfrak{l}$. Take

$$V = \mathrm{Hom}(\mathrm{Ind}_{M_\mathfrak{l}}^{F_\mathfrak{l}} \xi_\mathfrak{l}, \mathrm{Ind}_{M_\mathfrak{l}}^{F_\mathfrak{l}} \xi_\mathfrak{l})$$

and consider it as $\mathrm{Gal}(\overline{F}_\mathfrak{l}/F_\mathfrak{l})$-module by $\sigma f = \sigma \circ f \circ \sigma^{-1}$.

If $M_\mathfrak{l}/F_\mathfrak{l}$ is unramified, for the inertia subgroup $I_\mathfrak{l}$ of $D_\mathfrak{l} := \mathrm{Gal}(\overline{F}_\mathfrak{l}/F_\mathfrak{l})$, we find

$$\begin{aligned}V^{I_\mathfrak{l}} &= \mathrm{Hom}_{I_\mathfrak{l}}(\mathrm{Ind}_{M_\mathfrak{l}}^{F_\mathfrak{l}} \xi_\mathfrak{l}, \mathrm{Ind}_{M_\mathfrak{l}}^{F_\mathfrak{l}} \xi_\mathfrak{l})\\ &\qquad = \mathrm{Hom}_{I_\mathfrak{l}}(\xi_\mathfrak{l} \oplus \xi_\mathfrak{l}^\phi, \xi_\mathfrak{l} \oplus \xi_\mathfrak{l}^\phi) = \mathrm{End}_{I_\mathfrak{l}}(\xi_\mathfrak{l}) \oplus \mathrm{End}_{I_\mathfrak{l}}(\xi_\mathfrak{l}^\phi),\end{aligned}$$

where we may take τ to be the Frobenius map ϕ of $D_\mathfrak{l}/I_\mathfrak{l}$ and $\xi_\mathfrak{l}^\phi(x) = \xi_\mathfrak{l}(\phi x\phi^{-1})$. Thus $\dim V^{I_\mathfrak{l}} = 2$. Since

$$V = \mathrm{End}(\mathrm{Ind}_{M_\mathfrak{l}}^{F_\mathfrak{l}} \xi_\mathfrak{l}) \text{ and } Ad(\mathrm{Ind}_{M_\mathfrak{l}}^{F_\mathfrak{l}} \xi_\mathfrak{l}) = \{x \in \mathrm{End}(\mathrm{Ind}_{M_\mathfrak{l}}^{F_\mathfrak{l}} \xi_\mathfrak{l})|\mathrm{Tr}(x) = 0\},$$

we have $\dim Ad(\mathrm{Ind}_{M_\mathfrak{l}}^{F_\mathfrak{l}} \xi_\mathfrak{l})^{I_\mathfrak{l}} = 1$. Writing 1 (resp. 1_ϕ for the identity map of $\mathrm{End}_{I_\mathfrak{l}}(\xi_\mathfrak{l})$ (resp. $\mathrm{End}_{I_\mathfrak{l}}(\xi_\mathfrak{l}^\phi)$), $Ad(\mathrm{Ind}_{M_\mathfrak{l}}^{F_\mathfrak{l}} \xi_\mathfrak{l})^{I_\mathfrak{l}}$ is generated by $v = 1 \oplus -1_\phi$ and ϕ interchanges the two components; so, we have $\phi v = -v$. Thus the corresponding Euler factor of $L(s, Ad(\mathrm{Ind}_{M_\mathfrak{l}}^{F_\mathfrak{l}} \xi_\mathfrak{l}))$ at $\mathfrak{l}$ is given by $(1 + N(\mathfrak{l})^{-s})$. If $M_\mathfrak{l}/F_\mathfrak{l}$ is a ramified quadratic extension, we see easily that $\sigma(\pi_\mathfrak{l})|_{I_\mathfrak{l}}$ is still irreducible, and hence, $\dim Ad(\mathrm{Ind}_{M_\mathfrak{l}}^{F_\mathfrak{l}} \xi_\mathfrak{l})^{I_\mathfrak{l}} = 0$, and the Euler factor is trivial.

Split $S = S^{ur} \bigsqcup S^r$ for the collection S^{ur} of $\mathfrak{l} \in S$ such that $M_\mathfrak{l}/F_\mathfrak{l}$ is unramified. Then

(1) at $\mathfrak{q}|\mathfrak{c}(\epsilon^-)$, the zeta function $\zeta_F^{(\mathfrak{N})}(2s+2)D(s,f,f)$ has the single Euler factor $(1-N(\mathfrak{q})^{-s-1})^{-1}$, and the zeta function $\zeta_F(s+1)L(s+1,Ad(f))$ has its square $(1-N(\mathfrak{q})^{-s-1})^{-2}$ at $\mathfrak{q}|\mathfrak{c}(\epsilon^-)$, because $L(s+1,Ad(f))$ contributes one more factor $(1-N(\mathfrak{q})^{-s-1})^{-1}$;
(2) at $\mathfrak{l} \in S^{ur}$, $\zeta_F^{(\mathfrak{N})}(2s+2)D(s,f,f)$ has the trivial Euler factor 1, and the zeta function $\zeta_F(s+1)L(s+1,Ad(f))$ has $(1-N(\mathfrak{l})^{-s-1})^{-1}(1+N(\mathfrak{l})^{-s-1})^{-1}$;
(3) at $\mathfrak{l} \in S^r$, $\zeta_F^{(\mathfrak{N})}(2s+2)D(s,f,f)$ has the trivial Euler factor 1, and the zeta function $\zeta_F(s+1)L(s+1,Ad(f))$ has the factor $(1-N(\mathfrak{l})^{-s-1})^{-1}$.

The Euler factors outside $\mathfrak{N}$ are the same by the standard computation. Therefore, under (ind), the left-hand-side of (3.3) is given by

$$\begin{aligned}(3.4)\quad &\zeta_F^{(\mathfrak{N})}(2s+2)D(s,f,f)\\ &= \left(\prod_{\mathfrak{q}|\mathfrak{N}}(1-N(\mathfrak{q})^{-s-1})\cdot\prod_{\mathfrak{l}\in S^{ur}}(1+N(\mathfrak{q})^{-s-1})\right)\zeta_F(s+1)L(s+1,Ad(f)).\end{aligned}$$

By comparing the residue at $s=0$ of (3.4) with (3.3) (in view of (3.1)), we get

$$\begin{aligned}(3.5)\quad (f_\pi^u,f_\pi^u)_{\mathfrak{N}} &= D^{-[\kappa]-1}(f_\pi,f_\pi)_{\mathfrak{N}}\\ &= \frac{D\cdot\Gamma_F((\kappa_1-\kappa_2)+I)N(\mathfrak{N})}{2^{2((\kappa_1-\kappa_2)+I)+1}\pi^{((\kappa_1-\kappa_2)+2I)}}\prod_{\mathfrak{l}\in S^{ur}}(1+N(\mathfrak{q})^{-1})L(1,Ad(f))\end{aligned}$$

for the primitive adjoint square L–function $L(s,Ad(f))$ (e.g. [9] Section 2.3). Here we have written $x^s=\prod_\sigma x^{s_\sigma}$ for $s=\sum_\sigma s_\sigma\sigma\in\mathbb{C}[I]$, and $\Gamma_F(s)=\prod_\sigma\Gamma(s_\sigma)$ for the Γ–function $\Gamma(s)=\int_0^\infty e^{-t}t^{s-1}dt$. This formula is consistent with the one given in [18] Theorem 7.1 (but is much simpler).

3.2. **Primitive p–Adic Rankin product.** Let $\mathfrak{N}$ and $\mathfrak{J}$ be integral ideals of F prime to p. We shall use the notation introduced in Section 2. Thus, for a p–adically complete valuation ring $W\subset\mathbb{C}_p$, $\mathbf{h}^{n.ord}(\mathfrak{N},\epsilon;W)$ and $\mathbf{h}^{n.ord}(\mathfrak{J},\varepsilon;W)$ are the universal nearly ordinary Hecke algebra with level $(\mathfrak{N},\epsilon)$ and $(\mathfrak{J},\varepsilon)$, respectively. The character $\epsilon=(\epsilon_1,\epsilon_2,\epsilon_+^t)$ is made of the characters of ϵ_j of $T_0(O_p\times(O/\mathfrak{N}'^{(p)}))$ (for an ideal $\mathfrak{N}'\subset\mathfrak{N}$) of finite order and for the restriction ϵ_+^t to $\Delta_F(\mathfrak{N})$ (the torsion part of $Cl_F^+(\mathfrak{N}'p^\infty)$) of a Hecke character ϵ_+ extending $\epsilon_1\epsilon_2$. Similarly we regard ε as a character of $\mathbf{G}(\mathfrak{J}')$ for an ideal $\mathfrak{J}'\subset\mathfrak{J}$); so, $\epsilon^-=\epsilon_1^{-1}\epsilon_2$ and ε^- are well defined (finite order) character of $T_0(O_p\times(O/\mathfrak{N}))$ and $T_0(O_p\times(O/\mathfrak{J}))$, respectively. In particular we have $\mathfrak{C}^{(p)}(\epsilon^-)|\mathfrak{N}$ and $\mathfrak{C}^{(p)}(\varepsilon^-)|\mathfrak{J}$, where $\mathfrak{C}^{(p)}(\epsilon^-)$ is the prime-to–p part of the conductor $\mathfrak{C}(\epsilon^-)$ of ϵ^-. We assume that

$$(3.6)\qquad \mathfrak{C}^{(p)}(\epsilon^-)\supset\mathfrak{N},\ \mathfrak{C}^{(p)}(\varepsilon^-)=\mathfrak{J}\ \text{ and }\ \epsilon_1=\varepsilon_1\ \text{ on }\widehat{O}^\times.$$

For the moment, we also assume for simplicity that

$$(3.7)\qquad \epsilon_{\mathfrak{q}}^-\neq\varepsilon_{\mathfrak{q}}^-\ \text{ on }O_{\mathfrak{q}}^\times\text{ for }\mathfrak{q}|\mathfrak{J}\mathfrak{N}.$$

Let $\lambda:\mathbf{h}^{n.ord}(\mathfrak{N},\epsilon;W)\to\Lambda$ and $\lambda':\mathbf{h}^{n.ord}(\mathfrak{J},\varepsilon;W)\to\Lambda'$ be $\mathbf{\Lambda}$–algebra homomorphisms for integral domains Λ and Λ' finite torsion-free over $\mathbf{\Lambda}$. Write $\mathfrak{N}=\mathfrak{C}^{(p)}(\epsilon^-)\mathfrak{s}$ (thus, $\mathfrak{s}$ is the level at supercuspidal places for λ). Let $\rho_\lambda:\mathrm{Gal}(\overline{\mathbb{Q}}/F)\to GL_2(Q(\Lambda))$ be the Galois representation associated to λ (so, $\rho_\lambda(Frob_{\mathfrak{q}})=T(\mathfrak{q})$ for almost all primes $\mathfrak{q}$), where $Q(\Lambda)$ is the quotient field of Λ. Consider its restriction $\rho_{\lambda,\mathfrak{l}}$ to $\mathrm{Gal}(\overline{\mathbb{Q}}_{\mathfrak{l}}/F_{\mathfrak{l}})$ for a prime factor $\mathfrak{l}$ of $\mathfrak{s}$. We suppose to have a quadratic extension $M_{\mathfrak{l}}/F_{\mathfrak{l}}$ such that

(SC) $\rho_{\lambda,\mathfrak{l}}$ is isomorphic to an irreducible induced representation $\mathrm{Ind}_{M_{\mathfrak{l}}}^{F_{\mathfrak{l}}} \xi_{\mathfrak{l}}$ for a Galois character $\xi_{\mathfrak{l}} : \mathrm{Gal}(\overline{\mathbb{Q}}_{\mathfrak{l}}/M_{\mathfrak{l}}) \to \Lambda^{\times}$ at each prime factor $\mathfrak{l}|\mathfrak{s}$.

Since we only deal with automorphic induction from a quadratic extension of F in our application, this condition is always satisfied (and as we mentioned already, it hold for any odd prime factor of $\mathfrak{s}$ by [26]).

For each arithmetic point $P \in \mathrm{Spf}(\Lambda)(\overline{\mathbb{Q}}_p)$, let $f_P \in S_{\kappa(P)}(U_0(\mathfrak{N}p^{\alpha}), \epsilon_P; \overline{\mathbb{Q}}_p)$ be the normalized Hecke eigenform of minimal level belonging to λ at P. In other words, for $\lambda_P = P \circ \lambda : \mathbf{h}^{n.ord} \to \overline{\mathbb{Q}}_p$, we have $a(y, f_P) = \lambda_P(T(y))$ for all integral idele y with $y_p = 1$. In the automorphic representation generated by f_P, we can find a unique automorphic form f_P^{ord} with $a(y, f_P^{ord}) = \lambda(T(y))$ for all y, which we call the (nearly) *ordinary projection* of f_P. Similarly, using λ', we define $g_Q \in S_{\kappa(Q)}(U_0(\mathfrak{J}p^{\beta}), \varepsilon_Q; \overline{\mathbb{Q}}_p)$ for each arithmetic point $Q \in \mathrm{Spf}(\Lambda')(\overline{\mathbb{Q}}_p)$. Recall that we have two characters $(\epsilon_{P,1}, \epsilon_{P,2})$ of $T_0(\widehat{O})$ associated to ϵ_P. Recall $\epsilon_P = (\epsilon_{P,1}, \epsilon_{P,2}, \epsilon_{P+}) : T_0(\widehat{O})^2 \times (F_{\mathbb{A}}^{\times}/F^{\times}) \to \mathbb{C}^{\times}$. The $\mathfrak{p}$-component (for a prime $\mathfrak{p}|p$) of the automorphic representation $\pi(\lambda_P)$ generated by the nearly ordinary form f_P is necessarily principal or special, because $\lambda(T(p))$ is a p-adic unit. For simplicity, we assume

(PR) for each prime $\mathfrak{p}|p$, the $\mathfrak{p}$-components of $\pi(\lambda_P)$ and $\pi(\lambda'_Q)$ are principal.

Since we only deal with automorphic induction from a p-ordinary CM quadratic extension of F in our application, this condition is always satisfied. This condition combined with $\mathfrak{C}^{(p)}(\varepsilon^-) = \mathfrak{J}$ in (3.6) implies that all local factors of $\pi(\lambda'_Q)$ at finite places are in principal series.

The central character ϵ_{P+} of f_P coincides with $\epsilon_{P,1}\epsilon_{P,2}$ on $\widehat{O}^{\times}$ and has infinity type $(1 - [\kappa(P)])I$. We suppose

(3.8) The character $\epsilon_{P,1}\varepsilon_{Q,1}^{-1}$ is induced by a global finite order character θ.

The assumption (3.6) implies that θ is unramified outside p. As seen in [7] 7.F, we can find an automorphic form $g_Q|\theta^{-1}$ on $G(\mathbb{A})$ whose Fourier coefficients are given by $a(y, g_Q|\theta^{-1}) = a(y, g_Q)\theta^{-1}(yO)$, where $\theta(\mathfrak{a}) = 0$ if $\mathfrak{a}$ is not prime to $\mathfrak{C}(\theta)$. The above condition implies, as explained in the previous subsection,

$$y \mapsto a(y, f_P)\overline{a(y, g_Q|\theta^{-1})\theta(y)}$$

factors through the ideal group of F. Note that

$$a(y, f_P)\overline{a(y, g_Q|\theta^{-1})\theta(y)} = a(y, f_P)\overline{a(y, g_Q)}$$

as long as y_p is a unit. We thus write $a(\mathfrak{a}, f_P)\overline{a(\mathfrak{a}, g_Q|\theta^{-1})\theta(\mathfrak{a})}$ for the above product when $yO = \mathfrak{a}$ and define

$$\begin{aligned}(3.9)\quad D(s - \frac{[\kappa(P)] + [\kappa(Q)]}{2} &- 1, f_P, g_Q|\theta^{-1}, \theta^{-1}) \\ &= \sum_{\mathfrak{a}} a(\mathfrak{a}, f_P)\overline{a(\mathfrak{a}, g_Q|\theta^{-1})\theta(\mathfrak{a})}N(\mathfrak{a})^{-s}.\end{aligned}$$

Hereafter we write $\kappa = \kappa(P)$ and $\kappa' = \kappa(Q)$ if confusion is unlikely.

Note that for $g'_Q(x) = g_Q|\theta^{-1}(x)\theta(\det(x))$,

$$D(s, f_P, g'_Q) := D(s, f_P, g'_Q, \mathbf{1}) = D(s, f_P, g_Q|\theta^{-1}, \theta^{-1}).$$

Though the introduction of the character θ further complicates our notation, we can do away with it just replacing g_Q by g'_Q, since the local component $\pi(\varepsilon'_{Q,1,\mathfrak{q}}, \varepsilon'_{Q,2,\mathfrak{q}})$ of the automorphic representation generated by g'_Q satisfies $\varepsilon'_{1,Q} = \epsilon_{1,P}$, and hence without losing much generality, we may assume a slightly stronger condition:

$$\epsilon_{P,1} = \varepsilon_{Q,1} \text{ on } \widehat{O}^{\times}. \tag{3.10}$$

in our computation.

For each holomorphic Hecke eigenform f, we write $M(f)$ for the rank 2 motive attached to f (see [2]), $\widetilde{M}(f)$ for its dual, ρ_f for the $\mathfrak{p}$–adic Galois representation of $M(f)$ and $\widetilde{\rho}_f$ for the contragredient of ρ_f. Here $\mathfrak{p}$ is the p–adic place of the Hecke field of f induced by $i_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. Thus $L(s, M(f))$ coincides with the standard L–function of the automorphic representation generated by f, and the Hodge weight of $M(f_P)$ is given by $\{(\kappa_{1,\sigma}, \kappa_{2,\sigma}), (\kappa_{2,\sigma}, \kappa_{1,\sigma})\}_\sigma$ for each embedding $\sigma : F \hookrightarrow \mathbb{C}$. We have $\det(\rho_{f_P}(Frob_{\mathfrak{q}})) = \epsilon^u_P(\mathfrak{q})N(\mathfrak{q})^{[\kappa]}$ ($\mathfrak{p} \neq \mathfrak{q}$; see [17] 2.3.8).

Lemma 3.2. *Suppose* (3.6) *and* (3.8). *Write* $\mathfrak{N} = \mathfrak{C}^{(p)}(\epsilon^-)\mathfrak{s}$*, and assume that at primes* $\mathfrak{l}|\mathfrak{s}$*, the* $\mathfrak{l}$*-factor* $\pi_{\mathfrak{l}}$ *of the automorphic representation generated by* f_P *is supercuspidal and is an automorphic induction of a character of a quadratic extension of* $F_{\mathfrak{l}}$*. Then, for primes* $\mathfrak{q} \nmid p$*, the Euler* $\mathfrak{q}$*–factor of*

$$L^{(\mathfrak{N}\mathfrak{J})}(2s - [\kappa] - [\kappa'], \epsilon^u_P\varepsilon^{-u}_Q)D(s - \frac{[\kappa] + [\kappa']}{2} - 1, f_P, g'_Q)$$

is equal to the Euler $\mathfrak{q}$*–factor of* $L_{\mathfrak{q}}(s, M(f_P) \otimes \widetilde{M}(g_Q))$ *given by*

$$\det\left(1 - (\rho_{f_P} \otimes \widetilde{\rho}_{g_Q})(Frob_{\mathfrak{q}})\big|_{V^I} N(\mathfrak{q})^{-s})\right)^{-1},$$

where V *is the space of the* $\mathfrak{p}$*–adic Galois representation of the tensor product:* $\rho_{f_P} \otimes \widetilde{\rho}_{g_Q}$ *and* $V^I = H^0(I, V)$ *for the inertia group* $I \subset \mathrm{Gal}(\overline{\mathbb{Q}}/F)$ *at* $\mathfrak{q}$*.*

Proof. As already explained, we may assume (3.10) instead of (3.8). Let $\mathfrak{q} \nmid \mathfrak{s}$ be a prime. By abusing the notation, we write $\pi(\epsilon_{P,1,\mathfrak{q}}, \epsilon_{P,2,\mathfrak{q}})$ (resp. $\pi(\varepsilon_{Q,1,\mathfrak{q}}, \varepsilon_{Q,2,\mathfrak{q}})$) for the $\mathfrak{q}$–factor of the representation generated by f_P (resp. g_Q). By the work of Carayol, R. Taylor and Blasius–Rogawski combined with a recent work of Blasius [1], the restriction of ρ_{f_P} to the decomposition group at $\mathfrak{q}$ is isomorphic to $\mathrm{diag}[\epsilon_{P,1,\mathfrak{q}}, \epsilon_{P,2,\mathfrak{q}}]$ (regarding $\epsilon_{P,i,\mathfrak{q}}$ as Galois characters by local class field theory). The same fact is true for g_Q. If $\mathfrak{q}|\mathfrak{J}\mathfrak{N}$ but $\mathfrak{q} \nmid \mathfrak{s}$, then V^I is one dimensional on which $Frob_{\mathfrak{q}}$ acts by $\epsilon_{P,1,\mathfrak{q}}(\varpi_{\mathfrak{q}})\overline{\varepsilon}_{Q,1,\mathfrak{q}}(\varpi_{\mathfrak{q}}) = a(\mathfrak{q}, f_P)\overline{a(\mathfrak{q}, g_Q)}$ because $\epsilon_{i,P,\mathfrak{q}}\overline{\varepsilon}_{j,Q,\mathfrak{q}}$ is ramified unless $i = j = 1$ ($\Leftrightarrow \epsilon_1 = \varepsilon_1$ on $\widehat{O}^{\times}$ and $\epsilon^-_{\mathfrak{q}} \neq \epsilon^-_{\mathfrak{q}}$). If $\mathfrak{q} \nmid \mathfrak{J}\mathfrak{N}$, both $\pi(\epsilon_{P,1,\mathfrak{q}}, \epsilon_{P,2,\mathfrak{q}}) \otimes \epsilon^{-1}_{P,1,\mathfrak{q}} = \pi(1, \epsilon^-_{P,\mathfrak{q}})$ and $\pi(\varepsilon_{Q,1,\mathfrak{q}}, \varepsilon_{Q,2,\mathfrak{q}}) \otimes \varepsilon^{-1}_{Q,1,\mathfrak{q}} = \pi(1, \varepsilon^-_{Q,\mathfrak{q}})$ are unramified principal series. By $\epsilon_{P,1,\mathfrak{q}} = \varepsilon_{Q,1,\mathfrak{q}}$:(3.10), we have an identity:

$$\rho_{f_P} \otimes \rho_{g^c_Q} \cong (\rho_{f_P} \otimes \epsilon^{-1}_{P,1,\mathfrak{q}}) \otimes (\rho_{g^c_Q} \otimes \varepsilon_{Q,1,\mathfrak{q}})$$

on the inertia group, which is unramified. Therefore V is unramified at $\mathfrak{q}$. At the same time, the L–function has full Euler factor at $\mathfrak{q} \nmid \mathfrak{J}\mathfrak{N}$.

Now assume that $\mathfrak{l}$ is a prime factor of $\mathfrak{s}$. Then $\rho_{f_P}|_{D_{\mathfrak{l}}} \cong \mathrm{Ind}^D_{D'}\,\xi$ for a character ξ of D' for a subgroup D' of $D = D_{\mathfrak{l}}$ of index 2. Write $D' = \mathrm{Gal}(\overline{\mathbb{Q}}_{\mathfrak{l}}/M_{\mathfrak{l}})$ for a quadratic extension $M_{\mathfrak{l}}/F_{\mathfrak{f}}$. By the super-cuspidality assumption (SC), $\mathrm{Ind}^D_{D'}\,\xi$ is absolutely irreducible, and hence, the character ξ does not have an extension to D; so, $\rho_{f_P}|_{D'} = \xi \oplus \xi'$ with $\xi \neq \xi'$ for $\xi'(x) = \xi(\sigma x \sigma^{-1})$ for $\sigma \in D$ nontrivial on $M_{\mathfrak{l}}$. If $M_{\mathfrak{l}}/F_{\mathfrak{l}}$ is ramified, $\rho_{f_P}|_I$ for the inertia group $I \subset D$ is irreducible; so, $\rho_{f_P} \otimes \rho_{g_Q^c}|_I \cong (\varepsilon_{Q,1,\mathfrak{l}}^{-1} \otimes \rho_{f_P}|_I) \oplus (\varepsilon_{Q,2,\mathfrak{l}}^{-1} \otimes \rho_{f_P}|_I)$ does not have any I-invariant. In particular, the two Euler factors we are comparing are both trivial. Suppose that $M_{\mathfrak{l}}/F_{\mathfrak{l}}$ is unramified; so, $I \subset D'$. Since $\xi \neq \xi'$, the two sets of characters $A := \{\xi, \xi'\}$ and $B := \{\varepsilon_{Q,1,\mathfrak{l}}, \varepsilon_{Q,2,\mathfrak{l}}\}$ of I has empty intersection, because if they have nontrivial intersection, ξ has an extension (given by one of the elements in B) to $D = \langle I, \phi \rangle$, where ϕ is the Frobenius element. Since $\rho_{f_P} \otimes \rho_{g_Q^c}|_I \cong \bigoplus_{\xi \in A, \eta \in B} \xi \eta^{-1}$ does not have any nontrivial I-invariant subspace. Thus the two Euler factors we are comparing are both trivial and again identical. □

We continue to assume (SC) that the $\mathfrak{l}$-component $\pi_{P,\mathfrak{l}}$ of the automorphic representation generated by f_P is supercuspidal for all primes $\mathfrak{l}|\mathfrak{s}$. We would like to compute $f_P|\tau(x) := \epsilon_P^u(\det(x))^{-1} f_P(x\tau)$ for $\tau(N) = \left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right) \in G(\mathbb{A}^{(\infty)})$ for an idele $N = N(P)$ with $N^{(p\mathfrak{N})} = 1$ and $NO = \mathfrak{C}(\epsilon_P^-)\mathfrak{s}$ (whose prime-to–p factor is $\mathfrak{N}$). We continue to abuse notation and write, at a prime $\mathfrak{q} \nmid \mathfrak{s}$, $\pi(\epsilon_{P,1,\mathfrak{q}}, \epsilon_{P,2,\mathfrak{q}}) \otimes \epsilon_{P,1,\mathfrak{q}}^{-1}$ as $\pi(1, \epsilon_{P,\mathfrak{q}}^-)$ (thus $\epsilon_{P,\mathfrak{q}}^-$ is the character of $F_{\mathfrak{q}}^\times$ inducing the original $\epsilon_{P,\mathfrak{q}}^-$ on $O_{\mathfrak{q}}^\times$). We write $(\epsilon_{P,\mathfrak{q}}^-)^u = \epsilon_{P,\mathfrak{q}}^- / |\epsilon_{P,\mathfrak{q}}^-|$ (which is a unitary character). In the Whittaker model $V(1, \epsilon_{P,\mathfrak{q}}^-)$ of $\pi(1, \epsilon_{P,\mathfrak{q}}^-)$ (realized in the space of functions on $GL_2(F_{\mathfrak{q}})$), we have a unique function $\phi_{\mathfrak{q}}$ on $GL_2(F_{\mathfrak{q}})$ whose Mellin transform gives rise to the local L–function of $\pi(1, \epsilon_{\mathfrak{q}}^-)$. In particular, we have (cf. [7] (4.10b))

$$\phi_{\mathfrak{q}}|\tau_{\mathfrak{q}}(x) := (\epsilon_P^-)^u(\det(x))^{-1}\phi_{\mathfrak{q}}(x\tau_{\mathfrak{q}}) = W(\phi_{\mathfrak{q}})|\epsilon_P^-(N_{\mathfrak{q}})|^{1/2}\overline{\phi}_{\mathfrak{q}}(x),$$

where $\overline{\phi}_{\mathfrak{q}}$ is the complex conjugate of $\phi_{\mathfrak{q}}$ belonging to the representation space $V(1, \overline{\epsilon}_{P,\mathfrak{q}}^-)$, and $W(\phi_{\mathfrak{q}})$ is the epsilon factor of the representation $\pi(1, \epsilon_{P,\mathfrak{q}}^-)$ as in [7] (4.10c) (so, $|W(\phi_{\mathfrak{q}})| = 1$). Then $\phi_{\mathfrak{q}} \otimes \epsilon_{P,1,\mathfrak{q}}(x) := \epsilon_{P,1,\mathfrak{q}}(\det(x))\phi_{\mathfrak{q}}(x)$ is in $V(\epsilon_{P,1,\mathfrak{q}}, \epsilon_{P,2,\mathfrak{q}})$ and gives rise to the $\mathfrak{q}$–component of the global Whittaker model of the representation π generated by f_P. Similarly, for a prime factor $\mathfrak{l}|\mathfrak{s}$, in the Whittaker model of $\pi_{P,\mathfrak{l}}$ we have a unique function $\phi_{\mathfrak{l}}$ on $GL_2(F_{\mathfrak{l}})$ whose Mellin transform gives rise to the local L–function of $\pi_{P,\mathfrak{l}}$, and we have

$$\phi_{\mathfrak{l}}|\tau_{\mathfrak{l}}(x) := (\epsilon_P^-)^u(\det(x))^{-1}\phi_{\mathfrak{l}}(x\tau_{\mathfrak{l}}) = W(\phi_{\mathfrak{l}})|\epsilon_P^-(N_{\mathfrak{l}})|^{1/2}\overline{\phi}_{\mathfrak{l}}(x),$$

where $\overline{\phi}_{\mathfrak{l}}$ is the complex conjugate of $\phi_{\mathfrak{l}}$, and $W(\phi_{\mathfrak{l}})$ is the epsilon factor of the representation $\pi_{P,\mathfrak{l}}$ as in [7] (4.10c) (so, $|W(\phi_{\mathfrak{l}})| = 1$).

The above formula then implies

$$\begin{aligned}(\phi_{\mathfrak{q}} \otimes \epsilon_{P,1,\mathfrak{q}})|\tau_{\mathfrak{q}} &:= \epsilon_P^u(\det(x))^{-1}(\phi_{\mathfrak{q}} \otimes \epsilon_{P,1,\mathfrak{q}})(x\tau_{\mathfrak{q}}) \\ &= \epsilon_{P,1,\mathfrak{q}}^u(N_{\mathfrak{q}})|N_{\mathfrak{q}}|_{\mathfrak{q}}^{(1-[\kappa])/2} W(\phi_{\mathfrak{q}})(\overline{\phi}_{\mathfrak{q}} \otimes \overline{\epsilon}_{P,1,\mathfrak{q}})(x).\end{aligned}$$

Define the root number $W_{\mathfrak{q}}(f_P) = W(\phi_{\mathfrak{q}})$ and $W(f_P) = \prod_{\mathfrak{q}} W_{\mathfrak{q}}(f_P)$. Here note that $W_{\mathfrak{q}}(f_P) = 1$ if the prime $\mathfrak{q}$ is outside $\mathfrak{C}(\epsilon_{P,1})\mathfrak{C}(\epsilon_{P,2})\mathfrak{s}D$. We conclude from the above computation the following formula:

$$f_P|\tau(x) := \epsilon_P^u(\det(x))^{-1} f_P(x\tau) = W(f_P)\epsilon_{P,1}^u(N)|N|_{\mathbb{A}}^{(1-[\kappa])/2} f_P^c(x), \tag{3.11}$$

where f_P^c is determined by $a(y, f_P^c) = \overline{a(y, f_P)}$ for all $y \in F_{\mathbb{A}}^\times$. This shows

$$W(f_P)W(f_P^c) = \epsilon_{P,\infty}^u(-1) = \epsilon_{P,\infty}^+(-1). \tag{3.12}$$

Using the formula (3.11) instead of [7] (4.10b), we prove in the same manner as in [7] Theorem 5.2 the following result (which is identical in appearance to Theorem 5.3 of [12] even if we allow mild super-cuspidal places satisfying (SC)):

Theorem 3.3. *Suppose* (SC), (PR), (3.6) *and* (3.7). *There exists a unique element* $\mathcal{D}$ *in the field of fractions of* $\Lambda\widehat{\otimes}_W\Lambda'$ *satisfying the following interpolation property: Let* $(P,Q)\in\mathrm{Spf}(\Lambda)\times\mathrm{Spf}(\Lambda')$ *be an arithmetic point such that*

(W) $\kappa_1(P)-\kappa_1(Q)>0\geq\kappa_2(P)-\kappa_2(Q)$ *and* $\epsilon_{P,1}=\varepsilon_{Q,1}$ *on* $\widehat{O}^\times$.

Then $\mathcal{D}$ *is finite at* (P,Q) *and we have*

$$\mathcal{D}(P,Q)=W(P,Q)C(P,Q)S(P)^{-1}E(P,Q)\frac{L^{(p)}(1,M(f_P)\otimes\widetilde{M}(g_Q))}{(f_P,f_P)},$$

where, writing $k(P)=\kappa_1(P)-\kappa_2(P)+I$,

$$W(P,Q)=\frac{(-1)^{k(Q)}}{(-1)^{k(P)}}\frac{N(\mathfrak{J})^{([\kappa(Q)]+1)/2}}{N(\mathfrak{N})^{([\kappa(P)]-1)/2}}\cdot\prod_{\mathfrak{p}|p}\frac{\varepsilon_Q^u(d_\mathfrak{p})G(\varepsilon_{Q,1,\mathfrak{p}}^{-1}\epsilon_{P,1,\mathfrak{p}})G(\varepsilon_{Q,2,\mathfrak{p}}^{-1}\epsilon_{P,1,\mathfrak{p}})}{\epsilon_P^u(d_\mathfrak{p})G((\epsilon_{P,\mathfrak{p}}^-)^{-1})}$$

$$C(P,Q)=2^{([\kappa(P)]-[\kappa(Q)])I-2k(P)}\pi^{2\kappa_2(P)-([\kappa(Q)]+1)I}$$
$$\times\Gamma_F(\kappa_1(Q)-\kappa_2(P)+I)\Gamma_F(\kappa_2(Q)-\kappa_2(P)+I),$$

$$S(P)=\prod_{\mathfrak{p}\nmid\mathfrak{C}_p(\epsilon_P^-)}\left(\epsilon_P^-(\varpi_\mathfrak{p})-1\right)\left(1-\epsilon_P^-(\varpi_\mathfrak{p})|\varpi_\mathfrak{p}|_\mathfrak{p}\right)\prod_{\mathfrak{p}|\mathfrak{C}_p(\epsilon_P^-)}(\epsilon_P^-(\varpi_\mathfrak{p})|\varpi_\mathfrak{p}|_\mathfrak{p})^{\delta(\mathfrak{p})},$$

$$E(P,Q)=\prod_{\mathfrak{p}\nmid\mathfrak{C}_p(\varepsilon_Q^-)}\frac{(1-\varepsilon_{Q,1}\epsilon_{P,1}^{-1}(\varpi_\mathfrak{p}))(1-\varepsilon_{Q,2}\epsilon_{P,1}^{-1}(\varpi_\mathfrak{p}))}{(1-\varepsilon_{Q,1}^{-1}\epsilon_{P,1}(\varpi_\mathfrak{p})|\varpi_\mathfrak{p}|_\mathfrak{p})(1-\varepsilon_{Q,2}^{-1}\epsilon_{P,1}(\varpi_\mathfrak{p})|\varpi_\mathfrak{p}|_\mathfrak{p})}$$
$$\times\prod_{\mathfrak{p}|\mathfrak{C}_p(\varepsilon_Q^-)}\frac{\varepsilon_{Q,2}\epsilon_{P,1}^{-1}(\varpi_\mathfrak{p}^{\gamma(\mathfrak{p})})(1-\varepsilon_{Q,1}\epsilon_{P,1}^{-1}(\varpi_\mathfrak{p}))}{(1-\varepsilon_{Q,1}^{-1}\epsilon_{P,1}(\varpi_\mathfrak{p})|\varpi_\mathfrak{p}|_\mathfrak{p})}.$$

Here $\mathfrak{C}_p(\epsilon_P^-):=\mathfrak{C}(\epsilon_P^-)+(p)=\prod_{\mathfrak{p}|p}\mathfrak{p}^{\delta(\mathfrak{p})}$ *and* $\mathfrak{C}_p(\varepsilon_P^-):=\mathfrak{C}(\varepsilon_Q^-)+(p)=\prod_{\mathfrak{p}|p}\mathfrak{p}^{\gamma(\mathfrak{p})}$. *Moreover for the congruence power series* $H(\lambda)$ *of* λ, $H(\lambda)\mathcal{D}\in\Lambda\widehat{\otimes}_W\Lambda'$.

The expression of p–Euler factors and root numbers is simpler than the one given in [7] Theorem 5.1, because automorphic representation of g_Q is everywhere principal at finite places (by (3.6)). The shape of the constant $W(P,Q)$ appears to be slightly different from [7] Theorem 5.2. Firstly the present factor $(-1)^{k(P)+k(Q)}$ is written as $(\varepsilon_{Q+}\epsilon_{P+})_\infty(-1)$ in [7]. Secondly, in [7], it is assumed that $\varepsilon_{Q,1}^{-1}$ and $\epsilon_{P,1}^{-1}$ are both induced by a global character ϵ'_P and ϵ'_P unramified outside p. Thus the factor $(\varepsilon'_{Q,\infty}\epsilon'_{P,\infty})(-1)$ appears there. This factor is equal to $(\varepsilon_{Q,1,p}\epsilon_{P,1,p})(-1)=\theta_p(-1)$, which is trivial because of the condition (W). We do not need to assume the individual extensibility of $\varepsilon_{Q,1}$ and $\epsilon_{P,1}$. This extensibility is assumed in order to have a global Hecke eigenform $f_P^\circ=f_P^u\otimes\epsilon'_P$. However this assumption is redundant, because all computation we have done in [7] can be done locally using the local Whittaker model. Also $C(P,Q)$ in the above theorem is slightly different from the one in [7] Theorem 5.2, because $(f_P,f_P)=D^{[\kappa(P)]+1}(f_P^\circ,f_P^\circ)$ for f_P° appearing in the formula of [7] Theorem 5.2.

Proof. We start with a slightly more general circumstance. We shall use the symbol introduced in [7]. Suppose $\mathfrak{C}(\epsilon^-)|\mathfrak{N}$ and $\mathfrak{C}(\varepsilon^-)|\mathfrak{J}$, and take normalized Hecke eigenforms $f\in S_\kappa(\mathfrak{N},\epsilon_+;\mathbb{C})$ and $g\in S_{\kappa'}(\mathfrak{J},\varepsilon_+;\mathbb{C})$. Suppose $\epsilon_1=\varepsilon_1$.

We define $f^c \in S_\kappa(\mathfrak{N}, \overline{\epsilon}_+; \mathbb{C})$ by $a(y, f^c) = \overline{a(y, f)}$. Then $f^c(w) = \overline{f(j^{-1}wj)}$ for $j = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. We put $\Phi(w) = (\overline{f^c}g^c)(w)$. Then we see $\Phi(wu) = \epsilon(u)\varepsilon^{-1}(u)\Phi(w)$ for $u \in U = U_0(\mathfrak{N}') \cap U_0(\mathfrak{J}')$. Since $\epsilon_1 = \varepsilon_1$, we find that $\epsilon(u)\varepsilon(u)^{-1} = \epsilon^-(\varepsilon^-)^{-1}(d) = \epsilon^u(\varepsilon^u)^{-1}(d)$ if $u = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. We write simply ω for the central character of $\overline{f}^c g^c$, which is the Hecke character $\epsilon_+^u(\varepsilon_+^u)^{-1}|\cdot|_{\mathbb{A}}^{-[\kappa']-[\kappa]}$. Then we have $\Phi(zw) = \omega(z)\Phi(w)$, and $\Phi(wu_\infty) = \overline{J_\kappa(u_\infty, \mathbf{i})}^{-1} J_{\kappa'}(u_\infty, \mathbf{i})^{-1}\Phi(w)$. We then define $\omega^*(w) = \omega(d_{\mathfrak{N}'\mathfrak{J}'})$ for $w = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in B(\mathbb{A})U \cdot G(\mathbb{R})^+$. Here B is the algebraic subgroup of G made of matrices of the form $\left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right)$. We extend $\omega^* : G(\mathbb{A}) \to \mathbb{C}$ outside $B(\mathbb{A})U \cdot G(\mathbb{R})^+$ just by 0. Similarly we define $\eta : G(\mathbb{A}) \to \mathbb{C}$ by

$$\eta(w) = \begin{cases} |y|_{\mathbb{A}} & \text{if } g = \left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right) zu \text{ with } z \in F_{\mathbb{A}}^\times \text{ and } u \in U \cdot SO_2(F_{\mathbb{R}}), \\ 0 & \text{otherwise.} \end{cases}$$

For each $\mathbb{Q}$–subalgebra $A \subset \mathbb{A}$, we write $B(A)_+ = B(A) \cap G(\mathbb{A}^{(\infty)}) \times G(\mathbb{R})^+$. Note that $\Phi(w)\overline{\omega}^*(w)\eta(w)^{s-1}$ for $s \in \mathbb{C}$ is left invariant under $B(\mathbb{Q})_+$. Then we compute

$$\mathcal{Z}(s, f, g) = \int_{B(\mathbb{Q})_+ \backslash B(\mathbb{A})_+} \Phi(w)\overline{\omega}^*(w)\eta(w)^{s-1} d\varphi_B(w)$$

for the measure $\varphi_B \left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right) = |y|_{\mathbb{A}}^{-1} dx \otimes d^\times y$ defined in [7] page 340. We have

$$\begin{aligned} \mathcal{Z}(s, f, g) &= \int_{F_{\mathbb{A}+}^\times} \int_{F_{\mathbb{A}}/F} \Phi \left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right) dx |y|_{\mathbb{A}}^{s-1} d^\times y \\ &= D^{\frac{1}{2}} \int_{F_{\mathbb{A}}^\times} a(dy, f)\overline{a(dy, g)} \mathbf{e}_F(2\sqrt{-1}y_\infty) y_\infty^{-(\kappa_2+\kappa_2')} |y|_{\mathbb{A}}^s d^\times y \\ &\overset{dy \mapsto y}{=} D^{s+\frac{1}{2}} \int_{F_{\mathbb{A}}^\times} a(y, f)\overline{a(y, g)} \mathbf{e}_F(2\sqrt{-1}y_\infty) y_\infty^{-(\kappa_2+\kappa_2')} |y|_{\mathbb{A}}^s d^\times y \\ &= D^{s+\frac{1}{2}} (4\pi)^{-sI+\kappa_2+\kappa_2'} \Gamma_F(sI - \kappa_2 - \kappa_2') D(s - \frac{[\kappa] + [\kappa']}{2} - 1, f, g), \end{aligned}$$

where $D(s - \frac{[\kappa]+[\kappa']}{2} - 1, f, g)$ is defined in (3.9). Define $C_{\infty+} \subset G(\mathbb{R})^+$ by the stabilizer in $G(\mathbb{R})^+$ of $\mathbf{i} \in \mathfrak{Z}$. We now choose an invariant measure φ_U on $X(U) = G(\mathbb{Q})_+ \backslash G(\mathbb{A})_+ / UC_{\infty+}$ so that

$$\int_{X(U)} \sum_{\gamma \in O^\times B(\mathbb{Q})_+ \backslash G(\mathbb{Q})_+} \phi(\gamma w) d\varphi_U(w) = \int_{B(\mathbb{Q})_+ \backslash B(\mathbb{A})_+} \phi(b) d\varphi_B(b)$$

whenever ϕ is supported on $B(\mathbb{A}^{(\infty)})u \cdot G(\mathbb{R})^+$ and the two integrals are absolutely convergent. There exists a unique invariant measure φ_U as above (see [7] page 342 where the measure is written as μ_U). On $B(\mathbb{A})_+$,

$$\Phi(w) = \overline{f}^c g^c(w) \overline{J_\kappa(w, \mathbf{i})} J_{\kappa'}(w, \mathbf{i})$$

and the right-hand-side is left $C_{\infty+}$ invariant (cf. (S2) in Section 2.2). Then by the definition of φ_U, we have

$$\int_{B(\mathbb{Q})_+ \backslash B(\mathbb{A})_+} \Phi \overline{\omega}^* \eta^{s-1} d\varphi_B = \int_{X(U)} \overline{f}^c(w) g^c(w) E(w, s-1) d\varphi_U(w), \tag{3.13}$$

where

$$E(w, s) = \sum_{\gamma \in O^\times B(\mathbb{Q})_+ \backslash G(\mathbb{Q})_+} \overline{\omega}^*(\gamma w)\eta(\gamma w)^s \overline{J_\kappa(\gamma w, \mathbf{i})} J_{\kappa'}(\gamma w, \mathbf{i}).$$

Note that $E(zw,s) = ({\epsilon_+}^{-u}\varepsilon_+^u)(z)E(w,s)$ for $z \in \widehat{O}^\times$. By definition, $E(\alpha x) = E(x)$ for $\alpha \in G(\mathbb{Q})_+$; in particular, it is invariant under $\alpha \in F^\times$. For $z \in F_\mathbb{R}^\times$, $E(zw,s) = N(z)^{[\kappa]+[\kappa']-2}E(w,s)$. Thus $|\det(w)|_\mathbb{A}^{1-([\kappa]+[\kappa])/2}E(w,s)$ has eigenvalue $\epsilon_+^{-u}\varepsilon_+^u(z)$ under the central action of $z \in F^\times\widehat{O}^\times F_\mathbb{R}^\times$. The averaged Eisenstein series:

$$\begin{aligned}\mathcal{E}(w,s) &= \sum_{a \in Cl_F} \epsilon_+^u\varepsilon_+^{-u}(a)|\det(aw)|_\mathbb{A}^{1-([\kappa]+[\kappa'])/2}E(aw,s)\\ &= |\det(w)|_\mathbb{A}^{1-([\kappa]+[\kappa'])/2}\sum_{a \in Cl_F}\epsilon_+\overline{\varepsilon}_+(a)E(aw,s)\end{aligned}$$

satisfies $\mathcal{E}(zw,s) = \epsilon_+^{-u}\varepsilon_+^u(z)\mathcal{E}(w,s)$, where a runs over complete representative set for $F_\mathbb{A}^\times/F^\times\widehat{O}^\times F_\mathbb{R}^\times$ and ϵ_+ is the central character of f and $\overline{\varepsilon}_+$ is the central character of g^c. Defining the PGL_2 modular variety $\overline{X}(U) = X(U)/F_\mathbb{A}^\times$, by averaging (3.13), we find

$$\begin{aligned}&D^{s+\frac{1}{2}}(4\pi)^{-sI+\kappa_2+\kappa'_2}\Gamma_F(sI-\kappa_2-\kappa'_2)D(s-\frac{[\kappa]+[\kappa']}{2}-1,f,g)\\ &\qquad = \int_{\overline{X}(U)}\overline{f}^c(w)g^c(w)\mathcal{E}(w,s-1)|\det(w)|_\mathbb{A}^{([\kappa]+[\kappa'])/2-1}d\varphi_U(w).\end{aligned}\tag{3.14}$$

Writing $U = U_0(\mathfrak{L})$ and writing $r = \kappa'_2 - \kappa_2$, we define an Eisenstein series $\mathbb{E}_{k-k',r}(\overline{\omega}^u,\mathbf{1};s)$ by

$$N(\mathfrak{L})^{-1}\sqrt{D}|\det(w)|_\mathbb{A}^{\frac{[\kappa']-[\kappa]}{2}}L^{(\mathfrak{L})}(2s,\omega^u)\mathcal{E}(w,s+\frac{[\kappa]+[\kappa']}{2}-1),$$

where $k = \kappa_1 - \kappa_2$ and $k' = \kappa'_1 - \kappa'_2$. The ideal $\mathfrak{L}$ is given by $\mathfrak{N} \cap \mathfrak{J}$. Then, changing variable $s - \frac{[\kappa]+[\kappa']}{2} - 1 \mapsto s$, we can rewrite (3.14) as

$$\begin{aligned}&D^{s+\frac{1}{2}}(4\pi)^{-sI-\frac{k+k'}{2}}\Gamma_F(sI+\frac{k+k'}{2})L^{(\mathfrak{L})}(2s+2,\epsilon^u\varepsilon^{-u})D(s,f,g)\\ &= N(\mathfrak{L})^{-1}D^{-(3+[\kappa]+[\kappa'])/2}(f^c, g^c\mathbb{E}_{k-k',r}(\overline{\omega}^u,\mathbf{1};s+1))_\mathfrak{L}\\ &= N(\mathfrak{L})^{-s-\frac{[\kappa]+[\kappa']}{2}}D^{-(3+[\kappa]+[\kappa'])/2}(f^c|\tau,(g^c|\tau)\mathbf{G}_{k-k',r}(\omega^u,\mathbf{1};s+1))_\mathfrak{L},\end{aligned}\tag{3.15}$$

where $\mathbf{G}_{k-k',r}(\omega^u,\mathbf{1};s) = N(\mathfrak{L})^{s-1+\frac{[\kappa']-[\kappa]}{2}}\mathbb{E}_{k-k',r}(\overline{\omega}^u,\mathbf{1};s)|\tau$ for τ of level $\mathfrak{L}$, and

$$(\phi,\phi')_\mathfrak{L} = \int_{\overline{X}(U)}\overline{\phi}(w)\phi'(w)|\det(w)|_\mathbb{A}^{[\kappa]-1}d\varphi_U(w)$$

is the normalized Petersson inner product on $S_\kappa(U,\epsilon;\mathbb{C})$. This formula is (essentially) equivalent to the formula in [7] (4.9) (although we have more general forms f and g with character ϵ and ε not considered in [7]). In [7] (4.9), k' is written as κ and r is written as $w - \omega$.

Let E be the Eisenstein measure of level $\mathfrak{L} = \mathfrak{N} \cap \mathfrak{J}$ defined in [7] Section 8, where $\mathfrak{L}$ is written as L. We take an idele L with $LO = \mathfrak{L}$ and $L^{(\mathfrak{L})} = 1$. Similarly we take ideles J and N replacing in the above formula $\mathfrak{L}$ by $\mathfrak{J}$ and $\mathfrak{N}$, respectively, and L by the corresponding J and N, respectively.

The algebra homomorphism $\lambda' : \mathbf{h}^{n.ord}(\mathfrak{J},\varepsilon;W) \to \Lambda'$ induces, by the W–duality, $\lambda'^* : \Lambda'^* \hookrightarrow \mathbf{S}^{n.ord}(\mathfrak{J},\varepsilon;W)$, where $\mathbf{S}^{n.ord}(\mathfrak{J},\varepsilon;W)$ is a subspace of p–adic modular forms of level $(\mathfrak{J},\varepsilon)$ (see [8] 2.6). We then consider the p-adic convolution as in [7]

Section 9 (page 382):

$$\mathcal{D} = \lambda * \lambda' = \frac{E *_\lambda ([L/J] \circ \lambda'^*)}{H(\lambda) \otimes 1} \quad \text{for } E *_\lambda ([L/J] \circ \lambda'^*) \in \Lambda \widehat{\otimes} \Lambda',$$

where $[L/J]$ is the operator defined in [7] Section 7.B and all the ingredient of the above formula is as in [7] page 383. An important point here is that we use the congruence power series $H(\lambda) \in \Lambda$ (so $H(\lambda) \otimes 1 \in \Lambda \widehat{\otimes} \Lambda'$) defined with respect to $\mathbf{h}^{n.ord}(\mathfrak{N}, \epsilon; W)$ instead of $\mathbf{h}(\epsilon^u, \epsilon_1)$ considered in [7] page 379 (so, $H(\lambda)$ is actually a factor of H in [7] page 379, which is an improvement).

We write the minimal level of f_P^{ord} as $\mathfrak{N}\mathfrak{p}^\alpha$ for $\mathfrak{p}^\alpha = \prod_{\mathfrak{p}|p} \mathfrak{p}^{\alpha(\mathfrak{p})}$. Then we define $\varpi^\alpha = \prod_{\mathfrak{p}|p} \varpi_{\mathfrak{p}}^{\alpha(\mathfrak{p})}$. The integer $\alpha(\mathfrak{p})$ is given by the exponent of $\mathfrak{p}$ in $\mathfrak{C}(\epsilon_P^-)$ or 1 whichever larger. We now compute $\mathcal{D}(P, Q)$. We shall give the argument only when $j = [\kappa(P)] - [\kappa(Q)] \geq 1$, since the other case can be treated in the same manner as in [7] Case II (page 387). Put $\mathbf{G} = \mathbf{G}_{jI,0}\left(\varepsilon_Q^{-u} \epsilon_P^u, \mathbf{1}; 1 - \frac{j}{2}\right)$. We write $(\cdot, \cdot)_{\mathfrak{N}\mathfrak{p}^\alpha} = (\cdot, \cdot)_\alpha$ and put $\mathfrak{m} = L\mathfrak{p}^\alpha$. As before, $m = L\varpi^\alpha$ satisfies $mO = \mathfrak{m}$ and $m^{(\mathfrak{m})} = 1$. Put $r(P, Q) = \kappa_2(Q) - \kappa_2(P)$, which is non-negative by the weight condition (W) in the theorem. Then in exactly the same manner as in [7] Section 10 (page 386), we find, for $c = (2\sqrt{-1})^{j[F:\mathbb{Q}]} \pi^{[F:\mathbb{Q}]}$,

$$\begin{aligned} (3.16) \quad (-1)^{k(Q)} N(\mathfrak{J}/\mathfrak{L})^{-1} N(\mathfrak{L}\mathfrak{p}^\alpha)^{[\kappa(Q)]-1} c((f_P^{ord})^c | \tau(N\varpi^\alpha), f_P^{ord})_\alpha \mathcal{D}(P, Q) \\ = ((f_P^{ord})^c | \tau(m), (g_Q^{ord} | \tau(J\varpi^\alpha) | \tau(m)) \cdot (\delta_{jI}^{r(P,Q)} \mathbf{G}))_{\mathfrak{m}}. \end{aligned}$$

By [7] Corollary 6.3, we have, for $r = r(P, Q)$,

$$\delta_{jI}^r \mathbf{G} = \Gamma_F(r + I)(-4\pi)^{-r} \mathbf{G}_{jI+2r,r}\left(\varepsilon_Q^{-u} \epsilon_P^u, \mathbf{1}; 1 - \frac{j}{2}\right).$$

Then by (3.15), we get

$$\begin{aligned} &(3.17) \\ &c(-1)^{k(Q)} N(\mathfrak{J}\mathfrak{p}^\alpha)^{-1} C(P, Q)^{-1} (f_P^{ord} | \tau(N\varpi^\alpha), f_P^{ord})_\alpha \mathcal{D}(P, Q) \\ &= L^{(\mathfrak{m})}(2 - [\kappa(P)] + [\kappa(Q)], \omega_{P,Q}) D(\frac{[\kappa(Q)] - [\kappa(P)]}{2}, f_P^{ord}, (g_Q^{ord} | \tau(J\varpi^\alpha))^c), \end{aligned}$$

where $\omega_{P,Q} = \varepsilon_{Q+}^{-1} \epsilon_{P+}$ for the central characters ε_{Q+} of g_Q and ϵ_{P+} of f_P.

Now we compute the Petersson inner product $(f_P^{ord} | \tau(N\varpi^\alpha), f_P^{ord})_\alpha$ in terms of (f_P, f_P). Note that for $f, g \in S_\kappa(U_0(N), \epsilon; \mathbb{C})$

$$(3.18) \qquad (f^u)|\tau(N) = |N|_{\mathbb{A}}^{([\kappa]-1)/2} (f | \tau(N))^u \quad \text{and} \quad (f, g)_{\mathfrak{N}} = D^{[\kappa]+1}(f^u, g^u).$$

The computation we have done in [7] page 357 in the proof of Lemma 5.3 (vi) is valid without any change for each $\mathfrak{p}|p$, since at p–adic places, f_P in [7] has the Neben type we introduced in this paper also for places outside p. The difference is that we compute the inner product in terms of (f_P, f_P) not (f_P°, f_P°) as in [7] Lemma 5.3 (vi), where f_P° is the primitive form associated to $f_P^u \otimes \epsilon_{P,1}^u$ assuming that $\epsilon_{P,1}^u$ lifts to a global finite order character (the character $\epsilon_{P,1}^{-u}$ is written as ϵ' in the proof of Lemma 5.3 (vi) of [7]). Note here $f_P^\circ = f_P \otimes \epsilon_{P,1}^{-1}$ by definition and hence $(f_P^\circ, f_P^\circ) = (f_P^u, f_P^u)$, because tensoring a unitary character to a function does not alter the hermitian inner product. Thus we find

$$\frac{(f_P^{ord,u}|\tau(N\varpi^\alpha), f_P^{ord,u})}{(f_P^\circ, f_P^\circ)} = |N\varpi^\alpha|_{\mathbb{A}}^{([\kappa(P)]-1)/2}\frac{(f_P^{ord}|\tau(N\varpi^\alpha), f_P^{ord})}{(f_P, f_P)}. \tag{3.19}$$

A key point of the proof of Lemma 5.3 (vi) is the formula writing down $f_P^{ord,u}\otimes\epsilon_{P,1}^{-u}$ in terms of f_P°. Even without assuming the liftability of $\epsilon_{P,1}^u$ to a global character, the same formula is valid for f_P^{ord} and f_P before tensoring $\epsilon_{P,1}^{-1}$ (by computation using the local Whittaker model). We thus have $f_P^{ord} = f_P|R$ for a product $R = \prod_{\mathfrak{p}|p} R_{\mathfrak{p}}$ of local operators $R_{\mathfrak{p}}$ given as follows: If the prime $\mathfrak{p}$ is a factor of $\mathfrak{C}(\epsilon_P^-)\mathfrak{s}$, then $R_{\mathfrak{p}}$ is the identity operator. If $\mathfrak{p}$ is prime to $\mathfrak{C}(\epsilon_P^-)\mathfrak{s}$ ($\Leftrightarrow \pi(1,\epsilon_{P,\mathfrak{p}}^-)$ is spherical), then $f|R_{\mathfrak{p}} = f - \epsilon_{P,2}(\varpi_{\mathfrak{p}})f|[\varpi_{\mathfrak{p}}]$, where $f|[\varpi_{\mathfrak{p}}](x) = |\varpi_{\mathfrak{p}}|_{\mathfrak{p}}f|g$ with $f|g(x) = f(xg)$ for $g = \left(\begin{smallmatrix}\varpi_{\mathfrak{p}}^{-1} & 0\\ 0 & 1\end{smallmatrix}\right)$. Writing U for the level group of f_P and $U' = U\cap U_0(\mathfrak{p})$, we note $f|T(\mathfrak{p}) = \mathrm{Tr}_{U/U'}(f|g^{-1})$. This shows

$$\begin{aligned}(f_P|[\varpi_{\mathfrak{p}}], f_P)_{U'} &= |\varpi_{\mathfrak{p}}|_{\mathfrak{p}}^{[\kappa(P)]}(f_P, f_P|g^{-1})_{U'}\\ &= |\varpi_{\mathfrak{p}}|_{\mathfrak{p}}^{[\kappa(P)]}(f_P, \mathrm{Tr}_{U/U'}(f_P|g^{-1}))_U = (a+b)(f_P, f_P)_U,\end{aligned}$$

where $a = |\varpi_{\mathfrak{p}}|_{\mathfrak{p}}^{[\kappa(P)]}\epsilon_{P,1}(\varpi_{\mathfrak{p}})$, $b = |\varpi_{\mathfrak{p}}|_{\mathfrak{p}}^{[\kappa(P)]}\epsilon_{P,2}(\varpi_{\mathfrak{p}})$, and $(\cdot,\cdot)_U$ is the Petersson metric on $\overline{X}(U)$. Similarly we have,

$$\begin{aligned}(f_P, f_P|[\varpi_{\mathfrak{p}}])_{U'} &= \overline{(a+b)}(f_P, f_P)_U\\ \text{and}\ \ (f_P|[\varpi_{\mathfrak{p}}], f_P|[\varpi_{\mathfrak{p}}])_{U'} &= |\varpi_{\mathfrak{p}}|_{\mathfrak{p}}^{[\kappa(P)]}(|\varpi_{\mathfrak{p}}|_{\mathfrak{p}}+1)(f_P, f_P)_U\end{aligned}$$

By (3.11) and by (3.19), we conclude from [7] Lemma 5.3 (vi)

$$\begin{aligned}\frac{((f_P^{ord})^c|\tau(N\varpi^\alpha), f_P^{ord})_\alpha}{(f_P, f_P)} &= |N|_{\mathbb{A}}^{(1-[\kappa(P)])/2}(-1)^{k(P)}\epsilon_P^u(d_p)W'(f_P)S(P)\\ &\quad\times\epsilon_{P,2}(\varpi^\alpha)\prod_{\mathfrak{p}|((p)+\mathfrak{C}(\epsilon_P^-))}G(\epsilon_{P,2,\mathfrak{p}}^{-1}\epsilon_{P,1,\mathfrak{p}})\end{aligned} \tag{3.20}$$

for $\mathfrak{p}$ running over the prime factors of p.

We now give a brief description of the computation of the extra Euler factors: $E(P,Q)$ and $W(P,Q)$. Again the computation is the same as in [7] Lemma 5.3 (iii)-(v), because the level structure and the Neben character at p–adic places are the same as in [7] for f_P and g_Q and these factors only depend on p–adic places. Then we get the Euler p–factor $E(P,Q)$ and $W(P,Q)$ as in the theorem from [7] lemma 5.3. □

Remark 3.4. We assumed the condition (3.7) to make the proof of the theorem simpler. We now remove this condition. Let $\mathcal{E}$ be the set of all prime factors $\mathfrak{q}$ of $\mathfrak{I}\mathfrak{N}$ outside $\mathfrak{s}$ such that $\varepsilon_{\mathfrak{q}}^- = \epsilon_{\mathfrak{q}}^-$ on $O_{\mathfrak{q}}^\times$. Thus we assume that $\mathcal{E}\neq\emptyset$. Then in the proof of Lemma 3.2, the inertia group at $\mathfrak{q}\in\mathcal{E}$ fixes a two-dimensional subspace of $\rho_{f_P}\otimes\rho_{g_Q^c}$, one corresponding to $\epsilon_{1,\mathfrak{q}}\otimes\varepsilon_{1,Q}^{-1}$ and the other coming from $\epsilon_{2,\mathfrak{q}}\otimes\varepsilon_{2,Q}^{-1}$. The Euler factor corresponding to the latter does not appear in the Rankin product process; so, we get an imprimitive L–function, whose missing Euler factors are

$$E'(P,Q)^{-1} = \prod_{\mathfrak{q}\in\mathcal{E}}(1-\epsilon_{2,\mathfrak{q}}\varepsilon_{2,Q}^{-1}(\varpi_{\mathfrak{q}}))^{-1}.$$

Thus the final result is identical to Theorem 3.3 if we multiply $E(P,Q)$ by $E'(P,Q)$ in the statement of the theorem. In our application, λ and λ' will be automorphic inductions of Λ–adic characters $\widetilde{\varphi} : \mathrm{Gal}(\overline{M}/M) \to \Lambda^\times$ and $\widetilde{\xi} : \mathrm{Gal}(\overline{M}_1/M) \to \Lambda^\times$ for two ordinary CM fields M/F and M_1/F. If the prime-to–p conductor of $\widetilde{\xi}^-$ are made of primes split in M/F, $\mathcal{E}$ is the set of primes ramifying commonly in M/F and M_1/F outside $\mathfrak{s}$. Then $E'(P,Q)$ is the specialization of

$$E' = \prod_{\mathfrak{q}\in\mathcal{E}} (1 - \widetilde{\varphi} \otimes \widetilde{\xi}^{-1}(Frob_{\mathfrak{q}}))$$

at (P,Q), and $E' \in \Lambda\widehat{\otimes}\Lambda'$ is not divisible by the prime element of W (that is, the μ–invariant of E' vanishes). Actually, we can choose $\widetilde{\varphi}$ and $\widetilde{\xi}$ so that $\widetilde{\varphi}\widetilde{\xi}^{-1}(Frob_{\mathfrak{q}}) \not\equiv 1 \mod \mathfrak{m}_\Lambda$, and under this choice, we may assume that $E' \in (\Lambda\widehat{\otimes}\Lambda')^\times$.

3.3. **Comparison of p–adic L–functions.** For each character $\varphi : \Delta_{\mathfrak{C}} \to W^\times$, we have the extension $\widetilde{\varphi} : \mathrm{Gal}(\overline{\mathbb{Q}}/M) \to \Lambda$ sending $(\gamma,\delta) \in \Gamma_M \times \Delta_{\mathfrak{C}}$ to $\varphi(\delta)\gamma$ for the group element $\gamma \in \Gamma_M$ inside the group algebra Λ. Regarding $\widetilde{\varphi}$ as a character of $\mathrm{Gal}(\overline{\mathbb{Q}}/M)$, the induced representation $\mathrm{Ind}_M^F \widetilde{\varphi}$ is modular nearly ordinary at p, and hence, for a suitably chosen ϵ dependent on φ, by the universality of the nearly p–ordinary Hecke algebra $\mathbf{h} := \mathbf{h}^{n.ord}(\mathfrak{N}, \epsilon; W)$ defined in Section 2.4, we have a unique algebra homomorphism $\lambda = \lambda_\varphi : \mathbf{h} \to \Lambda$ such that $\mathrm{Ind}_M^F \widetilde{\varphi} \cong \lambda \circ \rho^{Hecke}$ for the universal nearly ordinary modular Galois representation ρ^{Hecke} with coefficients in $\mathbf{h}$, where $\mathfrak{N}$ is the prime-to–p part of $d(M/F)N_{M/F}(\mathfrak{C}(\varphi^-))$ for the relative discriminant $d(M/F)$ of M/F and for the conductor $\mathfrak{C}(\varphi^-)$ of the anticyclotomic projection φ^-. Thus for each arithmetic point $P \in \mathrm{Spf}(\Lambda)(\overline{\mathbb{Q}}_p)$ (in the sense of [8] 2.7), we have a classical Hecke eigenform $f_P = f(\widetilde{\varphi}_P)$ of weight $\kappa(P)$, which is a (nonstandard) theta series of the Galois character $\widetilde{\varphi}_P = (\widetilde{\varphi} \mod P)$ introduced in [12] 5.3. We write the automorphic induction of the complex Hecke character associated (via global class field theory) to the Galois character $\widetilde{\varphi}_P$ as $\pi(\widetilde{\varphi}_P)$ (which was written as $\pi(\lambda_P)$ before). Then $f(\widetilde{\varphi}_P)$ is the normalized Hecke eigenform in $\pi(\widetilde{\varphi}_P)$ minimal at nonsupercuspidal places and new at supercuspidal places. Hereafter, we use the same symbol $\widetilde{\varphi}_P$ for the complex Hecke character associated to the Galois character $\widetilde{\varphi}_P$.

To give an explicit description of $\pi(\widetilde{\varphi}_P)$ and $f(\widetilde{\varphi}_P)$, decompose the conductor $\mathfrak{C}(\psi)$ of $\psi = \varphi^-$ into the product $\mathfrak{F}\mathfrak{F}_c\mathfrak{I}\mathfrak{R}$ as in the introduction so that $\mathfrak{F}_c^c \supset \mathfrak{F}$, $\mathfrak{F}_c + \mathfrak{F} = R$, $\mathfrak{I}$ is a product of inert primes in M/F and $\mathfrak{R}$ is the product of primes ramified in M/F. Since the case of $\mathfrak{I}\mathfrak{R} = R$ has been dealt with in [12], we assume that $\mathfrak{I}\mathfrak{R} \subsetneq R$. Then the automorphic representation $\pi(\widetilde{\varphi}_P)$ of weight $\kappa(P)$ has minimal prime-to-p level $\mathfrak{N} = \mathfrak{N}(\varphi)$ given by $N_{M/F}(\mathfrak{F}^{(p)})N_{M/F}(\mathfrak{I}\mathfrak{R})d(M/F)$, where $\mathfrak{F}^{(p)}$ is the prime-to–p part of $\mathfrak{F}$. The set S of super cuspidal places for $\pi(\widetilde{\varphi}_P)$ is made up of primes $\mathfrak{l}$ of O appearing in $N_{M/F}(\mathfrak{I}\mathfrak{R})$. By [8] 7.1, the Hecke character $\widetilde{\varphi}_P$ has infinity type

$$\infty(\widetilde{\varphi}_P) = -\sum_{\sigma\in\Sigma} (\kappa_1(P)_{\sigma|_F}\sigma + \kappa_2(P)_{\sigma|_F}\sigma c).$$

In the automorphic induction $\pi(\widetilde{\varphi}_P)$, we have a unique normalized Hecke eigenform $f(\widetilde{\varphi}_P)$ of minimal level.

The prime-to–p level of the cusp form $f(\widetilde{\varphi}_P)$ is $\mathfrak{N}$ as above, and it satisfies (L1-3) in 3.1 for $\epsilon = \epsilon_P$ given as follows. To describe the local component of ϵ, we use local class field theory and identify local characters of $R_{\mathfrak{L}}^\times$ with the corresponding characters of the inertia group $I_{\mathfrak{L}} \subset \mathrm{Gal}(\overline{M}_{\mathfrak{L}}/M_{\mathfrak{L}})$ (resp. $I_{\mathfrak{q}} \subset \mathrm{Gal}(\overline{M}_{\mathfrak{L}}/F_{\mathfrak{q}})$) for $\mathfrak{q} = \mathfrak{L} \cap O$) for each prime $\mathfrak{L}$ of R. Here is the description:

$$\begin{aligned}
\epsilon_{1,\mathfrak{q}} &= \begin{cases} \widetilde{\varphi}_P|_{I_{\mathfrak{L}}} & \text{if } \mathfrak{q} = \mathfrak{L}\overline{\mathfrak{L}}, \\ \text{an extension of } \widetilde{\varphi}_P|_{I_{\mathfrak{L}}} \text{ to } I_{\mathfrak{q}} & \text{if } \mathfrak{q} \notin S,\ \mathfrak{q} = \mathfrak{L}^2 \text{ or } \mathfrak{L}, \\ \text{the trivial character} & \text{if } \mathfrak{q} \in S \end{cases} \\
\epsilon_{2,\mathfrak{q}} &= \begin{cases} \widetilde{\varphi}_P|_{I_{\overline{\mathfrak{L}}}} & \text{if } \mathfrak{q} = \mathfrak{L}\overline{\mathfrak{L}}, \\ \epsilon_{1,\mathfrak{q}} \left(\frac{M_{\mathfrak{L}}/F_{\mathfrak{q}}}{}\right) & \text{if } \mathfrak{q} \notin S,\ \mathfrak{q} = \mathfrak{L}^2 \text{ or } \mathfrak{L}, \\ \text{the determinant character of } \mathrm{Ind}_{M_{\mathfrak{L}}}^{F_{\mathfrak{q}}} \widetilde{\varphi}_P & \text{if } \mathfrak{q} \in S, \end{cases}
\end{aligned} \tag{3.21}$$

where $\mathfrak{L}$ and $\overline{\mathfrak{L}}$ are distinct primes in M.

Write $\mathfrak{N}' = d(M/F)N_{M/F}(\mathfrak{C}(\varphi))$, and write ϵ_+^t for the restriction of $\epsilon_+ = \epsilon_1\epsilon_2$ to $\Delta_F(\mathfrak{N}')$, which is independent of P (because it factors through the torsion part of $Cl_F^+(\mathfrak{N}'p^\infty)$). Since λ_φ is of minimal level, the congruence module $C_0(\lambda_\varphi; \Lambda)$ is a well defined Λ–module of the form $\Lambda/H(\psi)\Lambda$ (see [8] 2.9, and recall here $\psi = \varphi^-$). As we already remarked, we can choose $H(\psi)$ in $\Lambda^- = W[[\Gamma_M^-]]$ (see [15] Theorem 5.44). The element $H(\psi)$ is called the *congruence power series* of λ_φ (identifying Λ^- with a power series ring over W of $[F:\mathbb{Q}]$ variables).

We now choose well a totally real quadratic extension F_1/F, and put $K = F_1M$. Then in the composite $K = F_1M$, there are three quadratic extensions M, F_1 and M_1 of F inside K:

$$\begin{tikzcd}
 & M_1 \arrow[dr] & \\
F \arrow[ur] \arrow[r, hook] \arrow[dr] & F_1 \arrow[r, hook] & K. \\
 & M \arrow[ur] &
\end{tikzcd}$$

We impose $F_{1,\mathfrak{L}} = M_{\mathfrak{L}}$ and $M_{1,\mathfrak{L}} = F_{\mathfrak{L}}$ for all primes in K over $\mathfrak{l} \in S$ and $F_{1,\mathfrak{P}} = F_{\mathfrak{P}}$ for all primes $\mathfrak{P}|p$ in K. In other words,

(1) M_1/F is a CM field;
(2) For primes $\mathfrak{q} \in S \cup \{\mathfrak{p}|p\}$ in F, the decomposition group of $\mathfrak{q}$ in $\mathrm{Gal}(K/F)$ is given by $\mathrm{Gal}(K/M_1)$.

Since we impose how places decompose in F_1/F only at the finite set $S \cup \{\mathfrak{p}|p\} \cup \{v|\infty\}$ of places of F, there will be infinitely many choices of F_1. The field M_1/F and hence K/F_1 are a p-ordinary CM field in which all primes in S and over p split. Our choice of F_1 and M_1 could be different from the choice (F', M') we made in the introduction; so, we use different symbols.

Write R_1 for the integer ring of M_1. We choose a conductor $\mathfrak{C}'$ (an R_1-ideal prime to p) made of primes *split* in M_1/F. Then in the same manner as above, we define the groups $(\Gamma_{M_1}, {\Gamma_{M_1}}^-, {\Gamma_{M_1}}^+, \Delta'_{\mathfrak{C}'})$ for M_1 in place of $(\Gamma_M, \Gamma_M^-, \Gamma_M^+, \Delta_{\mathfrak{C}})$ for M. Choose a character $\xi : \Delta_{\mathfrak{C}'} \to W^\times$ of conductor $\mathfrak{C}'$. Let $\mathfrak{J} = N_{M_1/F}(\mathfrak{C}(\xi^-))d(M_1/F)$, and choose a p-ordinary CM type Σ' of M_1. We then put $\Lambda' = W[[\Gamma_{M_1}]]$ and define the character $\widetilde{\xi} : \Gamma_{M_1} \times \Delta'_{\mathfrak{C}'} \to \Lambda'^\times$ for ξ in the same way of the construction of $\widetilde{\varphi}$. The Hecke character $\widetilde{\xi}_Q$ has infinity type

$$\infty(\widetilde{\xi}_Q) = -\sum_{\sigma \in \Sigma'} (\kappa_1(Q)_{\sigma|_F}\sigma + \kappa_2(Q)_{\sigma|_F}\sigma c).$$

In the automorphic induction $\pi(\widetilde{\xi}_Q)$ relative to M_1/F, we have a unique normalized Hecke eigenform $g(\widetilde{\xi}_Q)$ of minimal level. Thus we get an algebra homomorphism $\lambda_\xi : \mathbf{h} = \mathbf{h}^{n.ord}(\mathfrak{J}, \varepsilon; W) \to \Lambda'$ which gives rise to the family of minimal modular forms $g(\widetilde{\xi}_Q)$ for each arithmetic points $Q \in \mathrm{Spec}(\Lambda')$.

The prime-to–p level of the cusp form $g(\widetilde{\xi}_Q)$ is $\mathfrak{J}$ as above, and it satisfies the corresponding conditions (L1-3) in 3.1 for $S = \emptyset$ and $(\kappa_1, \kappa_2) = (\kappa_1(Q), \kappa_2(Q))$ (after replacing M/F by M_1/F). Since we need the explicit form of the Neben character $\varepsilon = \varepsilon_Q$ of $g(\widetilde{\xi}_Q)$ later, we repeat its description, although it is the same as the one given for $f(\widetilde{\varphi}_P)$ (replacing the data concerning φ_P by those of $\widetilde{\xi}_Q$), and the description is indeed simpler, since $\pi(\widetilde{\xi}_Q)$ does not have supercuspidal places. To write down the Neben character $\varepsilon = \varepsilon_Q$ of $g(\widetilde{\xi}_Q)$, as before for φ, we use local class field theory and identify local characters of $R_{1,\mathfrak{L}}^\times$ with the corresponding characters of the inertia group $I_{1,\mathfrak{L}} \subset \mathrm{Gal}(\overline{M}_{1,\mathfrak{L}}/M_{1,\mathfrak{L}})$ (resp. $I_\mathfrak{q} \subset \mathrm{Gal}(\overline{M}_{1,\mathfrak{L}}/F_\mathfrak{q})$) for $\mathfrak{q} = \mathfrak{L} \cap O$) for each prime $\mathfrak{L}$ of R_1. Here is the description:

$$\begin{aligned} \varepsilon_{1,\mathfrak{q}} &= \begin{cases} \widetilde{\xi}_Q|_{I_{1,\mathfrak{L}}} & \text{if } \mathfrak{q} = \mathfrak{L}\overline{\mathfrak{L}}, \\ \text{an extension of } \widetilde{\xi}_Q|_{I_{1,\mathfrak{L}}} \text{ to } I_\mathfrak{q} & \text{if } \mathfrak{q} = \mathfrak{L}^2 \text{ or } \mathfrak{L}, \end{cases} \\ \varepsilon_{2,\mathfrak{q}} &= \begin{cases} \widetilde{\xi}_Q|_{I_{1,\overline{\mathfrak{L}}}} & \text{if } \mathfrak{q} = \mathfrak{L}\overline{\mathfrak{L}}, \\ \varepsilon_{1,\mathfrak{q}} \left(\frac{M_{\overline{\mathfrak{L}}}/F_\mathfrak{q}}{}\right) & \text{if } \mathfrak{q} = \mathfrak{L}^2 \text{ or } \mathfrak{L}, \end{cases} \end{aligned} \tag{3.22}$$

where $\mathfrak{L}$ and $\overline{\mathfrak{L}}$ are distinct primes in M_1.

By Theorem 3.3 and Remark 3.4, we have the (imprimitive) p–adic Rankin product $\mathcal{D} = \lambda_\varphi * \lambda_\xi$ with missing Euler factor $E' \in (\Lambda\widehat{\otimes}\Lambda')^\times$ as in Remark 3.4 for two characters $\varphi : \Delta_\mathfrak{C} \to W^\times$ and $\xi : \Delta_{\mathfrak{C}'} \to W^\times$. Writing $\mathcal{R} = \mathcal{D} \cdot H(\psi) \in \Lambda\widehat{\otimes}_W\Lambda'$, we have $\mathcal{D} = \frac{\mathcal{R}}{H(\psi)}$.

Let $\widehat{\xi}_Q = \widetilde{\xi}_Q|_{\mathrm{Gal}(\overline{\mathbb{Q}}/K)}$ and $\widehat{\varphi}_P = \widetilde{\varphi}_Q|_{\mathrm{Gal}(\overline{\mathbb{Q}}/K)}$. We define a p–adic L–functions $\mathcal{L}_K(\widehat{\varphi}^{-1}\widehat{\xi})$ for $\widehat{\xi} = \xi|_{\mathrm{Gal}(\overline{\mathbb{Q}}/K)}$ and $\widehat{\varphi} = \varphi|_{\mathrm{Gal}(\overline{\mathbb{Q}}/K)}$ by

$$\mathcal{L}_p(\widehat{\varphi}^{-1}\widehat{\xi})(P, Q) = E'(P, Q) L_K(\widehat{\varphi}_P^{-1}\widehat{\xi}_Q)$$

for the Katz p–adic L–function L_K for the CM field K.

We follow the argument in [7], [18] and [8] to show the following identity of p–adic L–functions:

Theorem 3.5. *Let φ (resp. ξ) be a character of $\Delta_\mathfrak{C}$ (resp. $\Delta'_{\mathfrak{C}'}$) with values in $W^\times$. Suppose the following two conditions:*

(i) *$\mathfrak{C}(\xi^-)$ is prime to any inert or ramified prime of M_1/F;*

(ii) *$\{\epsilon_{1,\mathfrak{q}}|_{O_\mathfrak{q}^\times}, \epsilon_{2,\mathfrak{q}}|_{O_\mathfrak{q}^\times}\} \cap \{\varepsilon_{1,\mathfrak{q}}|_{O_\mathfrak{q}^\times}, \varepsilon_{2,\mathfrak{q}}|_{O_\mathfrak{q}^\times}\}$ is nonempty at each prime factor $\mathfrak{q}$ of $N_{M/F}(\mathfrak{C}(\varphi))N_{M_1/F}(\mathfrak{C}(\xi))d(M/F)d(M_1/F)$.*

Then we have, for a power series $\mathcal{L} \in \Lambda\widehat{\otimes}_W\Lambda'$,

$$\frac{\mathcal{L}}{H(\psi)} = \frac{\mathcal{L}_p(\widehat{\varphi}^{-1}\widehat{\xi})}{h_\mathfrak{i}(M/F)L_M^-(\psi)},$$

where $\psi = \varphi^-$ and $h_\mathfrak{i}(M/F) = h(M)/h(F)\prod_{\mathfrak{l} \in S^{ur}}(N(\mathfrak{l}) + 1)$. The power series $\mathcal{L}$ is equal to $(\lambda_\varphi * \lambda_\xi) \cdot H(\psi)$ up to units in $\Lambda\widehat{\otimes}_W\Lambda'$.

In our earlier papers [18] Theorem 8.1 and [12] Theorem 5.5, we have taken $M = M_1$; so, this theorem is a version of the result there taking different CM fields M and M_1 (well chosen). We have an extra error factor $\prod_{\mathfrak{l}\in S^{ur}}(N(\mathfrak{l})+1)$ in $h_{\mathfrak{i}}(M/F)$ which does not appear in [12] Theorem 5.5). As noticed by R. Gillard and described in [13] at the end, if we take p to be a prime appearing in this factor, the anticyclotomic p-adic Hecke L-function $L^-_M(\varphi^-)$ often has a positive μ-invariant. The above theorem is not empty because of

Lemma 3.6. *The assumption* (ii) *of the above theorem can be achieved by choosing* (F_1,ξ) *well. In particular, for any finite set* S_1 *of primes outside* S *and* $pN_{M/F}(\mathfrak{C}(\varphi))$ *and given semi-simple quadratic extensions* $L_{\mathfrak{l}}/F_{\mathfrak{l}}$ *for* $\mathfrak{l}\in S_1$*, we may impose to have* $M_{1,\mathfrak{l}}=L_{\mathfrak{l}}$ *for all* $\mathfrak{l}\in S_1$ *and* $\xi|_{\widehat{O}^\times}=\varphi|_{\widehat{O}^\times}$ *to find such a pair* (F_1,ξ)*.*

Proof. For prime factors $\mathfrak{q}$ of $d(M/F)d(M_1/F)$, we may choose $\varepsilon_{1,\mathfrak{q}}$ and $\epsilon_{1,\mathfrak{q}}$ to be both the trivial character; so, we need to choose ξ so that $\varepsilon_{1,\mathfrak{q}}=\epsilon_{1,\mathfrak{q}}$ for a factor $\mathfrak{q}$ of $N_{M/F}(\mathfrak{C})N_{M_1/F}(\mathfrak{C}')$ unramified in K/F. We can choose F_1 so that $M_{1,\mathfrak{l}}$ is as specified for $\mathfrak{l}\in S_1$ and that all prime factors in $\mathfrak{C}(\varphi)$ split over F split also in M_1. For such split prime factors $\mathfrak{q}$ of F, we can identify $R_{1,\mathfrak{q}}$ and $R_{\mathfrak{q}}$ and we impose $\varphi_{\mathfrak{q}}=\xi_{\mathfrak{q}}$ on $R^\times_{\mathfrak{q}}=R^\times_{1,\mathfrak{q}}$. For $\mathfrak{l}\in S_1$, we choose $\xi_{\mathfrak{l}}$ to be unramified. For prime factors $\mathfrak{l}$ of $N_{M/F}(\mathfrak{C}(\varphi))$ non split in M, $\mathfrak{l}$ splits into $\mathfrak{L}\overline{\mathfrak{L}}$ in M_1/F_1 by our choice of F_1 already made. Then we choose $\xi_{\mathfrak{L}}=1$ on $R^\times_{1,\mathfrak{L}}$, and we impose $\xi_{\overline{\mathfrak{L}}}|_{O^\times_{\mathfrak{l}}}=\varphi_{\mathfrak{l}}|_{O^\times_{\mathfrak{l}}}$. Then by our definition, $\epsilon_{1,\mathfrak{q}}=\varepsilon_{1,\mathfrak{q}}$ on $O^\times_{\mathfrak{q}}$ for all $\mathfrak{q}|N_{M/F}(\mathfrak{C}(\varphi))N_{M_1/F}(\mathfrak{C}(\xi))d(M/F)d(M_1/F)$, and $\varphi|_{\widehat{O}^\times}=\xi|_{\widehat{O}^\times}$. By adding ramification to F_1 outside S_1 and $d(M/F)\mathfrak{C}(\varphi)$ if necessary, we may assume that ${R_1}^\times=O^\times$. Since $\varphi|_{\widehat{O}^\times}$ is trivial over $R^\times\supset O^\times={R_1}^\times$, we can extend $\varphi|_{\widehat{O}^\times}$ to a finite order character of $\widehat{R_1}^\times/{R_1}^\times$, which can be extended to a finite order character of $M^\times_{1,\mathbb{A}}/{M_1}^\times$ without adding any more ramification, as desired. $\square$

Proof of Theorem 3.5: The proof of the theorem is identical to the proof given in [8] Section 7.4 (and of [12] Theorem 5.5), since the factors $C(P,Q)$, $W(P,Q)$, $S(P)$ and $E(P,Q)$ appearing in Theorem 3.3 are identical to the one appearing in [7] Theorem 5.2 except for the power of the discriminant D, which is compensated by the difference of (f°_P,f°_P) appearing [7] Theorem 5.2 from (f_P,f_P) in Theorem 3.3. Since we do not lose any Euler factors in (3.5) (thanks to our minimal level structure and the assumption (3.6) assuring principality everywhere for g_Q) and the epsilon factor of $\pi(\widetilde{\varphi}_P)\otimes\pi(\widetilde{\xi}_Q)$ is the product of the individual one associated to $\pi(\widetilde{\varphi}_P)$ and $\pi(\widetilde{\xi}_Q)$, we can write the numerator of the Rankin product as $\mathcal{L}_p(\widehat{\varphi}^{-1}\widehat{\xi})$. This finishes the proof of Theorem 3.5. $\square$

Lemma 3.7. *Choosing* (F_1,ξ) *well, if* $p>3$*, we may assume that the* μ*-invariant of* $\mathcal{L}_p(\widehat{\varphi}^{-1}\widehat{\xi})$ *projected to* $W[[\Gamma^-_L]]$ *vanishes.*

Proof. By our choice of F_1 and M_1, the conductor of $\widehat{\varphi}^{-1}\widehat{\xi}$ is made of primes splits over F_1; so, the condition (S) with respect to K/F_1 is satisfied by $\widehat{\varphi}^{-1}\widehat{\xi}$. In particular, all primes ramifying in M/F splits in M_1/F. Thus we do not need to multiply the Katz p-adic L-function $L_K(\widehat{\varphi}^{-1}\widehat{\xi})$ by the extra Euler factor $E'(P,Q)$ in Remark 3.4 to get $\mathcal{L}_p(\widehat{\varphi}^{-1}\widehat{\xi})$, and hence, we need to prove the vanishing of μ of the anticyclotomic projection $L^-_K(\widehat{\varphi}^{-1}\widehat{\xi})$, which is studied in [13].

In [13] Theorem I, under the split prime-to-p conductor assumption (S) (with respect to K/F_1) of $\widehat{\varphi}\widehat{\xi}^{-1}$ (which holds by our choice of F_1/F as we have seen), a set of three conditions (M1–3) equivalent to the non-vanishing of $\mu(L_K^-(\widehat{\varphi}^{-1}\widehat{\xi}))$ is stated. One of them is the modulo p identity of Hecke characters of F_1:

(M3) $(\varphi^{-1} \circ N_{K/M})_1(\xi \circ N_{K/M_1})_1\omega \equiv \left(\frac{K/F_1}{}\right)$ mod $\mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal of W, $?_1$ is the restriction to the Hecke character to $F_{1,\mathbb{A}}^\times$ and ω is the Teichmüller character of F_1 of conductor p.

We further restricts these characters to $\widehat{O}^\times$. Since we may assume $\xi|_{\widehat{O}^\times} = \varphi|_{\widehat{O}^\times}$ by the above lemma, we get $\omega \equiv \left(\frac{K/F_1}{}\right)$ mod $\mathfrak{m}$ over $N_{K/M_1}(\widehat{O}^\times) = N_{K/M}(\widehat{O}^\times) = (\widehat{O}^\times)^2$, which is not the case if $p > 3$. Indeed, $\omega|_{(\widehat{O}^\times)^2} = \omega^2|_{\widehat{O}^\times}$ is ramified at p and $\left(\frac{K/F_1}{}\right)$ is unramified at p, since p is unramified in $K/\mathbb{Q}$ by our choice of M and F_1. Thus $\mu(L_K^-(\widehat{\varphi}^{-1}\widehat{\xi})) = 0$ holds under $\xi|_{\widehat{O}^\times} = \varphi|_{\widehat{O}^\times}$ (and such a choice is possible by the above lemma). □

This implies the following corollary:

Corollary 3.8. *We have*

$$h_{\mathfrak{i}}(M/F)L_M^-(\psi)\big|H(\psi)$$

in Λ^- *if* $p > 3$ *is unramified in* M.

Proof. Let μ be the μ-invariant of $h_{\mathfrak{i}}(M/F)L_M^-(\psi)$. Then $h_{\mathfrak{i}}(M/F)L_M^-(\psi) = p^\mu \cdot \Phi$ with $\Phi \in W[[\Gamma_M^-]]$ prime to p, and by Theorem 3.5 combined with the above lemma, we get $p^\mu|H(\psi)$ ($\Leftrightarrow \mu \le \mu(H(\psi))$). On the other hand, $\Phi|H(\psi)$ is the main theorem of [18]: Theorem I (strictly speaking, we should have said that $H(\psi)$ and the congruence power series H used in [18] have the common prime to p-part or that the same proof as in [18] page 257 just above Theorem 8.2 works in the present case taking $K = M \oplus M$ and $M_1 = M$ as described in the beginning of Section 3). This finishes the proof. □

Since $H(\psi)|h_{\mathfrak{i}}(M/F)\mathcal{F}_M^-(\psi)$ as explained in the introduction, this implies Theorem 3.1.

4. Behaviour of p-adic L-functions under base-change

The following theorem is the only ingredient left to be proven in the proof of the anticyclotomic main conjecture given in the introduction:

Theorem 4.1. *Assume* (F1–3) *for* E, M', F *in the introduction. Then we have* $\mathrm{Res}(\mathcal{F}_E^-(\psi_E)) = \mathcal{F}_M^-(\psi)\mathcal{F}_M^-(\psi\alpha)$ *and* $\mathrm{Res}(L_E^-(\psi_E)) = L_M^-(\psi)L_M^-(\psi\alpha)$ *up to units in* $\overline{W}[[\Gamma_M^-]]$*, where* $\alpha = \left(\frac{E/M}{}\right)$.

Proof. We work in the setting and with the symbols we defined in the introduction. Let $X(R)$ be the CM abelian variety defined over $\mathcal{W} = \overline{W} \cap \overline{\mathbb{Q}}$ with $X(R)(\mathbb{C}) = \mathbb{C}^\Sigma/R^\Sigma$. Let R_E be the integer ring of E. Then $X(R_E) = X(R) \otimes_R R_E$. This shows that the CM (Néron) eigen period $(\Omega_\sigma)_{\sigma\in\Sigma}$ of $X(R)$ defined in [20] (2.6.31) (where Ω is written as c) to construct the p-adic L-function $L_M^-(\psi)$ can be used to construct $L_E(\psi_E)$. In the construction of the p-adic L-function, the polarization ideal of $X(R)$ contributes (see the interpolation formula (1.3) of [13]). Indeed, the polarization is induced by a Riemann form $\langle x, y\rangle = \mathrm{Tr}_{M/\mathbb{Q}}(xy^c/2\delta)$ for a well chosen purely imaginary element $0 \ne \delta \in M$ (see (d1–2) in [13]). We can use the same δ to give a polarization of $X(R_E)$ because R_E/R is unramified at p. For any Hecke character ψ'_E whose p-adic avatar factoring through Res is of the form $\psi'_E = \psi' \circ N_{E/M}$ for a Hecke character ψ' of M. Since the archimedean L-function $L(s, \psi'_E)$ is factored into the product $L(s, \psi')L(s, \psi'\alpha)$, it is easy to verify by the interpolation formulas of [20], [18] and [13] (1.3) of $L_E(\psi_E)$ that the same factorization $\mathrm{Res}(L^-_E(\psi_E)) = L^-_M(\psi)L^-_M(\psi\alpha)$ holds up to units in $W[[\Gamma^-_M]]$.

To show the identity on the Galois side, we write $L^E_\infty/E^-_\infty E(\psi_E)$ for the maximal p-abelian extension unramified outside Σ_p. Note that $E(\psi_E)$ is the composite of $M(\psi)$ and $M(\psi\alpha)$. Indeed, by (F3), α ramifies at a prime where ψ is unramified; so, $E = M(\alpha)$ and $M(\psi)$ are linearly disjoint, and hence, $M(\alpha\psi)M(\psi) = E(\psi_E)$.

Let L^ψ_∞ (resp. $L^{\alpha\psi}_\infty$) be the maximal p-abelian extension of $M(\psi)M^-_\infty$ (resp. $M(\psi\alpha)M^-_\infty$) unramified outside Σ_p. We can also think the maximal p-abelian extension $L'_\infty/M(\psi)M(\psi\alpha)M^-_\infty$ inside L^E_∞. Then L'_∞ is the composite of L^ψ_∞ and $L^{\alpha\psi}_\infty$. The Galois group $\mathrm{Gal}(L'_\infty E^-_\infty/E^-_\infty)$ is isomorphic to

$$X^E_M := X_E \otimes_{\mathbb{Z}_p[[\Gamma^-_E]],\mathrm{Res}} \mathbb{Z}_p[[\Gamma^-_M]],$$

since E/M ramifies at some places outside p by (F3). Put

$$X^E_M[\psi_E] := X_E[\psi_E] \otimes_{W[[\Gamma^-_E]],\mathrm{Res}} W[[\Gamma^-_M]].$$

Let σ be the generator of $\mathrm{Gal}(E/M)$. Taking the extension $\widetilde{\sigma}$ of σ to $\overline{\mathbb{Q}}$, we can let $\mathrm{Gal}(E/M)$ acts on X^E_M and $X^E_M[\psi_E]$ by $\sigma \cdot x = \widetilde{\sigma} x \widetilde{\sigma}^{-1}$. Then $X_M[\psi] \cong X^E_M[\psi_E]/(\sigma - 1)X^E_M[\psi_E]$ and $X_M[\psi\alpha] \cong X^E_M[\psi_E]/(\sigma + 1)X^E_M[\psi_E]$ as $W[[\Gamma^-_M]]$-modules. This shows

$$X_E[\psi_E] \otimes_{W[[\Gamma^-_E]],\mathrm{Res}} W[[\Gamma^-_M]] \cong X_M[\psi] \oplus X_M[\psi\alpha]$$

as $W[[\Gamma^-_M]]$-modules. Thus we have the identity of the Fitting ideals (see [21] Appendix 4):

$$\mathrm{Res}(\mathrm{Fitt}_{W[[\Gamma^-_E]]}(X_E[\psi_E])) = \mathrm{Fitt}_{W[[\Gamma^-_M]]}(X_M[\psi]) \, \mathrm{Fitt}_{W[[\Gamma^-_M]]}(X_M[\psi\alpha]).$$

By [17] Theorem 5.33 and Lemma 5.23 (2), $X_E[\psi_E]$ is a torsion module of finite type over $W[[\Gamma^-_E]]$ with homological dimension 1 (so, it does not have any pseudo-null submodules non-null). Then $\mathrm{Fitt}_{W[[\Gamma^-_E]]}(X_E[\psi_E])$ is a principal ideal generated by the characteristic power series $\mathcal{F}^-_E(\psi_E)$. Thus we find

$$\mathrm{Fitt}_{W[[\Gamma^-_M]]}(X_M[\psi]) \, \mathrm{Fitt}_{W[[\Gamma^-_M]]}(X_M[\psi\alpha]) = \mathrm{Res}(\mathcal{F}^-_E(\psi_E)).$$

Since the reflexive closure of

$$\mathrm{Fitt}_{W[[\Gamma^-_M]]}(X_M[\psi]) \, \mathrm{Fitt}_{W[[\Gamma^-_M]]}(X_M[\psi\alpha]) = \mathrm{Fitt}_{W[[\Gamma^-_M]]}(X_M[\psi] \oplus X_M[\psi\alpha])$$

is generated by $\mathcal{F}^-_M(\psi)\mathcal{F}^-_M(\psi\alpha)$, we conclude the identity of the principal ideals:

$$(\mathcal{F}^-_M(\psi)\mathcal{F}^-_M(\psi\alpha)) = (\mathrm{Res}(\mathcal{F}^-_E(\psi_E)))$$

as desired, and moreover $\mathrm{Fitt}_{W[[\Gamma^-_M]]}(X_M[\psi] \oplus X_M[\psi\alpha])$ is principal and is generated by $\mathcal{F}^-_M(\psi)\mathcal{F}^-_M(\psi\alpha)$ (though this can be also proven by the result of Fujiwara [4] following the argument proving Theorem 5.33 in [17] Chapter 5, removing some of the simplifying assumptions (h1–4) made there). $\square$

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