FACTORIZATION OF THE $\mathcal{L}$-INVARIANT OF TATE CURVES

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1. Lecture 4

We continue to assume that p totally splits in $F/\mathbb{Q}$ in the proofs.

1.1. **Greenberg's $\mathcal{L}$–invariant in a different point of view.** We start with a slightly more general setting: We let $V = \rho_{2n,n} = Sym^{\otimes 2n}\rho_E \otimes \det \rho_E^{-n}$ for odd n (so, V is critical at $s = 1$). Let $\mathcal{F}^+V = \mathcal{F}^1V$ and $\mathcal{F}^-V = \mathcal{F}^0V$. Them $\dim \mathcal{F}^-_{\mathfrak{p}}V/\mathcal{F}^+_{\mathfrak{p}}V = 1$. Recall

$$\overline{L}_{\mathfrak{p}_j}(V) = \mathrm{Ker}(H^1(F_{\mathfrak{p}_j}, V) \to H^1(F_{\mathfrak{p}_j}, \frac{V}{\mathcal{F}^+_{\mathfrak{p}_j}(V)})) \ \text{ if } j \le b;$$

otherwise,

$$\overline{L}_{\mathfrak{p}_j}(V) = L_{\mathfrak{p}_j}(V) = \mathrm{Ker}(H^1(F_{\mathfrak{p}_j}, V) \to H^1(I_{\mathfrak{p}_j}, \frac{V}{\mathcal{F}^+_{\mathfrak{p}_j}(V)})).$$

The fact $\overline{L}_{\mathfrak{p}_j}(V) = L_{\mathfrak{p}_j}(V)$ follows from the following F-version of the argument in [G] page 160:

Lemma 1.1. *Let $V = \rho_{2n,n}$ for odd n. Then we have*

$$\overline{L}_{\mathfrak{p}}(V) = L_{\mathfrak{p}}(V) \quad \text{and} \quad \overline{\mathrm{Sel}}_F(V) = \mathrm{Sel}_F(V).$$

Proof. Since we have $\overline{L}_{\mathfrak{p}}(V) = L_{\mathfrak{p}}(V)$ by definition if $\mathfrak{p} = \mathfrak{p}_j$ with $j > b$; so, we may assume that $j \le b$. Write $H^\bullet(M)$ for $H^\bullet(F_{\mathfrak{p}}, M)$ for $\mathrm{Gal}(\overline{F}_{\mathfrak{p}}/F_{\mathfrak{p}})$-modules M. Write $r : H^1(V) \to H^1(I_{\mathfrak{p}}, \overline{V})$ for $\overline{V} = V/\mathcal{F}^+V$. Thus $L_{\mathfrak{p}}(V) = \mathrm{Ker}(r)$. We can factor the map r as $r = \mathrm{Res} \circ \gamma$ for $\gamma : H^1(V) \to H^1(\overline{V})$ and $\mathrm{Res} : H^1(\overline{V}) \to H^1(I_{\mathfrak{p}}, \overline{V})$. Since $\mathrm{Ker}(\gamma) = \overline{L}_{\mathfrak{p}}(V)$, we need to show that $\mathrm{Im}(\gamma) \cap \mathrm{Ker}(\mathrm{Res}) = 0$.

Writing $Y = \mathcal{F}^-_{\mathfrak{p}}V/\mathcal{F}^2_{\mathfrak{p}}V$ and $\overline{Y} = \mathcal{F}^-_{\mathfrak{p}}V/\mathcal{F}^+_{\mathfrak{p}}V$, we have exact sequences of $D_{\mathfrak{p}}$-modules: $Y \hookrightarrow V/\mathcal{F}^2_{\mathfrak{p}}V \twoheadrightarrow V/\mathcal{F}^-_{\mathfrak{p}}V$ and $\overline{Y} \hookrightarrow \overline{V} \twoheadrightarrow V/\mathcal{F}^-_{\mathfrak{p}}V$. Note that $Y \cong \left(\begin{smallmatrix} \mathcal{N} & \xi_q \\ 0 & 1 \end{smallmatrix}\right) \cong \rho_E$ as $D_{\mathfrak{p}}$-modules by a simple computation (if $j \le b$). Since $H^0(V/\mathcal{F}^-_{\mathfrak{p}}V) = 0$, by the long exact sequences of the above two short exact sequences, we find that the natural maps $H^1(Y) \to H^1(V/\mathcal{F}^2_{\mathfrak{p}}V)$ and $H^1(\overline{Y}) \to H^1(\overline{V})$ are injective. Identify $H^1(\overline{Y})$ with its image in $H^1(\overline{V})$. We have

$$\mathrm{Im}(\overline{\gamma}) = \mathrm{Im}(\overline{\gamma} : H^1(Y) \to H^1(\overline{Y})) \subset H^1(\overline{V}).$$

Since $H^2(V/\mathcal{F}^-_{\mathfrak{p}}V) \cong H^0((V/\mathcal{F}^-_{\mathfrak{p}}V)^*(1)) = H^0(\mathcal{F}^-_{\mathfrak{p}}V) = 0$, the $\gamma' : H^1(V) \to H^1(V/\mathcal{F}^-_{\mathfrak{p}}V)$ is surjective with kernel $H^1(\mathcal{F}^-_{\mathfrak{p}}V)$; so, $\mathrm{Im}(\gamma) \cap H^1(\overline{Y}) = \mathrm{Im}(\overline{\gamma})$.

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By the inflation-restriction sequence,

$$\mathrm{Ker}(\mathrm{Res}) = H^1(D_{\mathfrak{p}}/I_{\mathfrak{p}}, \overline{V}^{I_{\mathfrak{p}}}) = \overline{V}^{I_{\mathfrak{p}}}/(Frob_{\mathfrak{p}} - 1)\overline{V}^{I_{\mathfrak{p}}} = \mathcal{F}^-_{\mathfrak{p}}V/\mathcal{F}^+_{\mathfrak{p}}V.$$

Similarly

$$\begin{aligned}\mathrm{Ker}(\mathrm{Res}_Y : H^1(\overline{Y}) \to H^1(I_{\mathfrak{p}}, \overline{Y})) &= H^1(D_{\mathfrak{p}}/I_{\mathfrak{p}}, \overline{Y}^{I_{\mathfrak{p}}}) \\ &= \overline{Y}^{I_{\mathfrak{p}}}/(Frob_{\mathfrak{p}} - 1)\overline{Y}^{I_{\mathfrak{p}}} = \mathcal{F}^-_{\mathfrak{p}}Y/\mathcal{F}^+_{\mathfrak{p}}Y = \mathcal{F}^-_{\mathfrak{p}}V/\mathcal{F}^+_{\mathfrak{p}}V.\end{aligned}$$

Thus inside $H^1(\overline{V})$, $\mathrm{Ker}(\mathrm{Res}) = \mathrm{Ker}(\mathrm{Res}_Y) \subset H^1(\overline{Y})$. This shows

$$\mathrm{Im}(\gamma) \cap \mathrm{Ker}(\mathrm{Res}) = 0 \Leftrightarrow \mathrm{Im}(\gamma) \cap \mathrm{Ker}(\mathrm{Res}) \cap H^1(\overline{Y}) = 0 \Leftrightarrow \mathrm{Im}(\overline{\gamma}) \cap \mathrm{Ker}(\mathrm{Res}) = 0,$$

because $\mathrm{Im}(\gamma) \cap H^1(\overline{Y}) = \mathrm{Im}(\overline{\gamma})$. We therefore need to show that

$$\mathrm{Im}(\overline{\gamma} : H^1(Y) \to H^1(\overline{Y})) \cap \mathrm{Ker}(\mathrm{Res} : H^1(\overline{Y}) \to H^1(I_{\mathfrak{p}}, \overline{Y})) = 0.$$

We have the long exact sequence attached to $\mathcal{F}^+_{\mathfrak{p}}Y (= \mathbb{Q}_p(1)) \hookrightarrow Y \twoheadrightarrow \overline{Y} (= \mathbb{Q}_p)$:

$$0 \to \overline{Y} = H^0(\overline{Y}) \to H^1(\mathcal{F}^+_{\mathfrak{p}}Y) \to H^1(Y) \xrightarrow{\overline{\gamma}} H^1(\overline{Y}) \to H^2(\mathcal{F}^+_{\mathfrak{p}}Y) \to H^2(Y) = 0.$$

By the non-splitting of the short sequence, $H^0(\overline{Y})$ injects into $H^1(\mathcal{F}^+_{\mathfrak{p}}Y)$. By the local Tate duality,

$$\dim H^2(Y) = \dim H^0(\mathrm{Hom}_{\mathbb{Q}_p}(Y, \mathbb{Q}_p(1))) = 0,\ \dim H^2(\mathcal{F}^+_{\mathfrak{p}}Y) = \dim \mathcal{F}^-_{\mathfrak{p}}V/\mathcal{F}^+_{\mathfrak{p}}V = 1.$$

This shows that $\dim H^1(Y) = 2$ and $\dim \mathrm{Im}(\overline{\gamma}) = 1$, because by Kummer's theory, noting $F_{\mathfrak{p}} = \mathbb{Q}_p$,

$$H^1(\mathcal{F}^+_{\mathfrak{p}}Y) = H^1(\mathbb{Q}_p(1)) = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_n F^\times_{\mathfrak{p}}/(F^\times_{\mathfrak{p}})^{p^n} \cong \mathbb{Q}_p^2$$

and $H^1(\overline{Y}) = H^1(\mathbb{Q}_p) \cong \mathrm{Hom}((F_{\mathfrak{p}})^\times, \mathbb{Q}_p) \cong \mathbb{Q}_p^2$. By the inflation-restriction sequence, we have

$$\mathrm{Ker}(\mathrm{Res}_Y) = \mathrm{Ker}(H^1(\overline{Y}) \to H^1(I_{\mathfrak{p}}, \overline{Y})) \cong H^1(D_{\mathfrak{p}}/I_{\mathfrak{p}}, \overline{Y}^{I_{\mathfrak{p}}}) \cong \overline{Y} \cong \mathbb{Q}_p.$$

Thus $\dim \mathrm{Ker}(\mathrm{Res}_Y) + \dim \mathrm{Im}(\overline{\gamma}) = \dim H^1(\overline{Y})$. Thus we need to show $\mathrm{Ker}(\mathrm{Res}_Y) + \mathrm{Im}(\overline{\gamma}) = H^1(\overline{Y})$. By the local Tate duality, noting $Y^* \cong Y = \mathbb{Q}_p$, this statement is equivalent to

$$\mathrm{Ker}(\delta : H^1(\mathcal{F}^+_{\mathfrak{p}}Y) \to H^1(Y)) \cap \mathrm{Ker}(\mathrm{Res}_Y)^\perp = 0.$$

Here $\mathrm{Ker}(\mathrm{Res}_Y)^\perp = H^1_{fl}(\mathcal{F}^+_{\mathfrak{p}}Y) = \overline{Y} \otimes_{\mathbb{Z}_p} \varprojlim_n O^\times_{\mathfrak{p}}/(O^\times_{\mathfrak{p}})^{p^n} \subset H^1(\overline{Y}(1))$, because of the explicit form of Tate duality: $\langle \phi, [\xi_u] \rangle = \phi(\sigma_u)$ for $\phi \in \mathrm{Hom}(D_p, \mathbb{Q}_p) = H^1(\mathbb{Q}_p)$. Since $\mathrm{Ker}(\delta)$ gives rise to the subspace spanned by extension class of $\mathbb{Q}_p(1) = \mathcal{F}^+_{\mathfrak{p}}Y \hookrightarrow Y \twoheadrightarrow \overline{Y} \cong \mathbb{Q}_p$, it is given by the cocycles in $\xi_q \otimes \overline{Y}$ for the Tate period q of E at $\mathfrak{p} = \mathfrak{p}_j$. In particular, $(\overline{Y} \otimes \xi_q) \cap H^1_{fl}(\mathcal{F}^+_{\mathfrak{p}}Y)$ is given by

$$(q \otimes \overline{Y}) \cap (\overline{Y} \otimes_{\mathbb{Z}_p} \varprojlim_n O^\times_{\mathfrak{p}}/(O^\times_{\mathfrak{p}})^{p^n})$$

inside $\overline{Y} \otimes_{\mathbb{Z}_p} \varprojlim_n F^\times_{\mathfrak{p}}/(F^\times_{\mathfrak{p}})^{p^n}$, which is trivial (because q is a nonunit). $\square$

Return to $\rho_{2,1} = Ad(\rho_E)$. Suppose $R_n \cong K[[X_{\mathfrak{p}}]]_{\mathfrak{p}|p}$. Let us recall the definition of $\mathcal{L}(Ad(\rho_E))$. Then, as already seen, we have a unique subspace $\mathbb{H}$ of $H^1(\mathfrak{G}, V)$ projecting down onto

$$\prod_{\mathfrak{p}} \mathrm{Im}(\iota_{\mathfrak{p}}) \hookrightarrow \prod_{\mathfrak{p}} \frac{H^1(F_{\mathfrak{p}}, V)}{\overline{L}_{\mathfrak{p}}(V)}.$$

Then by the restriction, $\mathbb{H}$ gives rise to a subspace $L = L_V$ of

$$\prod_{\mathfrak{p}} \mathrm{Hom}(D_p^{ab}, \mathcal{F}_{\mathfrak{p}}^- V/\mathcal{F}_{\mathfrak{p}}^+ V) \cong \prod_{\mathfrak{p}} (\mathcal{F}_{\mathfrak{p}}^- V/\mathcal{F}_{\mathfrak{p}}^+ V)^2$$

isomorphic to $\prod_{\mathfrak{p}} (\mathcal{F}_{\mathfrak{p}}^- V/\mathcal{F}_{\mathfrak{p}}^+ V)$. If a cocycle c representing an element in $\mathbb{H}$ is unramified, it gives rise to an element in $\mathrm{Sel}_F(V)$. By the vanishing of $\mathrm{Sel}_F(V)$ (Vanishing lemma), this implies $c = 0$; so, the projection of L to the first factor $\prod_{\mathfrak{p}} \frac{\mathcal{F}_{\mathfrak{p}}^- V}{\mathcal{F}_{\mathfrak{p}}^+ V}$ (via $\phi \mapsto (\phi([\gamma, F_{\mathfrak{p}}])/\log_p(\gamma))_{\mathfrak{p}}$) is surjective. Thus this subspace L is a graph of a K–linear map

$$\mathcal{L} : \prod_{\mathfrak{p}} \mathcal{F}_{\mathfrak{p}}^- V/\mathcal{F}_{\mathfrak{p}}^+ V \to \prod_{\mathfrak{p}} \mathcal{F}_{\mathfrak{p}}^- V/\mathcal{F}_{\mathfrak{p}}^+ V. \tag{1.1}$$

We then define $\mathcal{L}(V) = \det(\mathcal{L}) \in K$. This is a description of the direct construction of $\mathbb{H}$. We recall the following lemma we proved (under an extra assumption that ρ_E is unramified outside p):

Lemma 1.2. *Let $V = Ad(\rho_E)$, and assume that $\mathrm{Sel}_F(V) = 0$. The space $\mathbb{H}$ defined above consists of cohomology classes of 1-cocycles $c : \mathrm{Gal}(\overline{F}/F) \to V$ such that*

1. *c is unramified outside p;*
2. *c restricted to the decomposition subgroup $\mathrm{Gal}(\overline{F}_{\mathfrak{p}}/F_{\mathfrak{p}}) \cong D_{\mathfrak{p}} \subset \mathrm{Gal}(\overline{F}/F)$ at each $\mathfrak{p}|p$ has values in $\mathcal{F}_{\mathfrak{p}}^- V$ and $c|_{D_{\mathfrak{p}}}$ modulo $\mathcal{F}_{\mathfrak{p}}^+ V$ becomes unramified over $F_{\mathfrak{p}}[\mu_{p^\infty}]$ for all $\mathfrak{p}|p$.*

1.2. **Factorization of $\mathcal{L}$-invariants.** In this section, we factorize $\mathcal{L}(Ad(\rho_E))$ into the product over multiplicative places and the contribution of the good reduction part. We keep notation introduced in the previous section; so, $V = Ad(\rho_E)$. For simplicity, we assume that $F_{\mathfrak{p}} = \mathbb{Q}_p$ for all $\mathfrak{p}|p$.

Proposition 1.3. *Let V be $Ad(\rho_E)$. Suppose $b > 0$, and fix an index k with $1 \le k \le b$. Let $a \in \prod_{i=1}^e \mathrm{Hom}(D_{\mathfrak{p}_i}, \mathcal{F}_{\mathfrak{p}_i}^- V/\mathcal{F}_{\mathfrak{p}_i}^+ V)$ be induced by $c \in \mathbb{H}$ such that $c \in \mathbb{H}$ restricts down trivially to $\frac{H^1(F_{\mathfrak{p}_i}, V)}{\overline{L}_{\mathfrak{p}_i}(V)}$ for all $i \neq k$. Then we have $a([\gamma_{\mathfrak{p}_i}, F_{\mathfrak{p}_i}]) = 0$ for all $i \neq k$ and $a([p, F_{k'}]) = 0$ for all $k' \neq k$ with $k' \le b$.*

Proof. Take a cocycle $c \in \mathbb{H}$ restricting down to $\frac{H^1(F_k, V)}{\overline{L}_{\mathfrak{p}_k}(V)}$ trivially to $\frac{H^1(F_{\mathfrak{p}_i}, V)}{\overline{L}_{\mathfrak{p}_i}(V)}$ for all $i \neq k$. Since $\mathbb{H} \cong \prod_{i=1}^e \mathrm{Im}(\iota_{\mathfrak{p}_i})$ by the restriction map, such cocycles c form a direct summand of $\mathbb{H}$ isomorphic to $\mathrm{Im}(\iota_{\mathfrak{p}_k})$.

If $i > b$, $L_{\mathfrak{p}_i}(V)$ is made of classes of cocycles becoming unramified modulo those with values in $\mathcal{F}_{\mathfrak{p}_i}^+ V$; so, even if $c|_{D_{\mathfrak{p}_i}}$ vanishes in $\frac{H^1(F_{\mathfrak{p}_i}, V)}{L_{\mathfrak{p}_i}(V)}$ (that is, $c|_{D_{\mathfrak{p}_i}} \in L_{\mathfrak{p}_i}(V)$), we cannot pull out much information on the value $a([p, F_{\mathfrak{p}_i}])$ because of the ambiguity modulo unramified cocycles with values in $\mathcal{F}_{\mathfrak{p}_i}^- V/\mathcal{F}_{\mathfrak{p}_i}^+ V$. Anyway, $a([\gamma_{\mathfrak{p}_i}, F_{\mathfrak{p}_i}]) = 0$ because $[\gamma_{\mathfrak{p}_i}, F_{\mathfrak{p}_i}] \in I_{\mathfrak{p}_i}$.

Write $\mathcal{F}^+_{\mathfrak{p}}H^1(F_{\mathfrak{p}}.V)$ for the image of $H^1(F_{\mathfrak{p}},\mathcal{F}^+_{\mathfrak{p}}V)$ in $H^1(F_{\mathfrak{p}},V)$. For the index $k \leq b$, $\overline{L}_{\mathfrak{p}}(V)$ is exactly $\mathcal{F}^+_{\mathfrak{p}_k}H^1(F_k,V)$. Thus for $i \leq b$ with $i \neq k$, $\overline{L}_{\mathfrak{p}_i}(V)$ is made of cocycles of $D_{\mathfrak{p}_i}$ with values in $\mathcal{F}^+_{\mathfrak{p}_i}V$, and the condition that $c|_{D_{\mathfrak{p}_i}} \in \overline{L}_{\mathfrak{p}_i}(V)$ implies the vanishing of $a(\sigma) = c(\sigma) \mod \mathcal{F}^+_{\mathfrak{p}_i}V$ for all $\sigma \in D_{\mathfrak{p}_i}$. This shows the last assertion: $a([p,F_{k'}]) = 0$. □

By the above lemma, we get immediately the following fact.

Corollary 1.4. *Let the notation be as in Proposition 1.3. Then the linear operator $\mathcal{L}$ acting on $\prod_{\mathfrak{p}}\mathcal{F}^-_{\mathfrak{p}}V/\mathcal{F}^+_{\mathfrak{p}}V$ preserves the following exact sequence:*

$$0 \to \prod_{i>b}\mathcal{F}^-_{\mathfrak{p}_i}V/\mathcal{F}^+_{\mathfrak{p}_i}V \to \prod_{\mathfrak{p}}\mathcal{F}^-_{\mathfrak{p}}V/\mathcal{F}^+_{\mathfrak{p}}V \to \prod_{k\leq b}\mathcal{F}^-_{\mathfrak{p}_k}V/\mathcal{F}^+_{\mathfrak{p}_k}V \to 0,$$

and $\mathcal{L}$ acting on the quotient $\prod_{k\leq b}\mathcal{F}^-_{\mathfrak{p}_k}V/\mathcal{F}^+_{\mathfrak{p}_k}V$ sends $\mathcal{F}^-_{\mathfrak{p}_k}V/\mathcal{F}^+_{\mathfrak{p}_k}V$ into itself for each $k \leq b$.

Definition 1.1. Define $\mathcal{L}(1)$ (resp. $\mathcal{L}_k(V)$) by

$$\det\left(\mathcal{L}|_{\prod_{i>b}\mathcal{F}^-_{\mathfrak{p}_i}V/\mathcal{F}^+_{\mathfrak{p}_i}V}\right) \in K$$

for $V = Ad(\rho_E)$ (resp. the determinant of the linear operator induced by $\mathcal{L}$ on

$$\prod_{\mathfrak{p}}\mathcal{F}^-_{\mathfrak{p}}V/\mathcal{F}^+_{\mathfrak{p}}V/\prod_{i\neq k}\mathcal{F}^-_{\mathfrak{p}_i}V/\mathcal{F}^+_{\mathfrak{p}_i}V$$

for $V = Ad(\rho_E)$).

Corollary 1.5. *Let the notation be as above. Then we have*

$$\mathcal{L}(Ad(\rho_E)) = \mathcal{L}(1)\prod_{k=1}^{b}\mathcal{L}_k(Ad(\rho_E))$$

for odd $n \geq 1$.

Proposition 1.6. *Suppose $n = 1$. Then for $k \leq b$, we have $\mathcal{L}_k(\rho_{2,1}) = \frac{\log_p(Q_k)}{\mathrm{ord}_p(Q_k)}$, where $Q_k = N_{F_k/\mathbb{Q}_p}(q_k)$ for the Tate period q_k of $E_{/F_k}$.*

This follows from the argument in the third lecture. In fact, this type of result is valid for $\rho_{2n,n}$ for all odd n under a suitable higher dimensional generalization of the deformation conjecture.

References

[G] R. Greenberg, Trivial zeros of p–adic L–functions, Contemporary Math. **165** (1994), 149–174

[H07b] H. Hida, $\mathcal{L}$-invariants of Tate curves, 2006 (posted at `www.math.ucla.edu/~hida`)

[HMI] H. Hida, *Hilbert modular forms and Iwasawa theory*, Oxford University Press, 2006

[MFG] H. Hida, *Modular Forms and Galois Cohomology*, Cambridge Studies in Advanced Mathematics **69**, 2000, Cambridge University Press

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