$\mathcal{L}$-INVARIANT OF p-ADIC L-FUNCTIONS

HARUZO HIDA

1. Lecture 1

Let $\overline{\mathbb{Q}} \subset \mathbb{C}$ be the field of all algebraic numbers. We fix a prime $p > 2$ and a p-adic absolute value $|\cdot|_p$ on $\overline{\mathbb{Q}}$. Then $\mathbb{C}_p$ is the completion of $\overline{\mathbb{Q}}$ under $|\cdot|_p$. We write $W = \{x \in K \big| |x|_p < 1\}$ for the p-adic integer ring of sufficiently large extension $K/\mathbb{Q}_p$ inside $\mathbb{C}_p$. We write $\overline{\mathbb{Q}}_p$ for the field of all numbers in $\mathbb{C}_p$ algebraic over $\mathbb{Q}_p$. Start with a strictly compatible system $\{\rho_{\mathfrak{l}}\}$ of semi-simple Galois representations $\rho_{\mathfrak{l}} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_d(E_{\mathfrak{l}})$ for primes $\mathfrak{l}$ of the coefficient field $E \subset \overline{\mathbb{Q}}$. We assume that ρ does not contain the trivial representation as a subquotient. We write S for the finite set of ramification of ρ and $\rho_{\mathfrak{l}}$ is unramified outside $S \cup \{\infty, \ell\}$, where ℓ is the residual characteristic of $\mathfrak{l}$. We write $\mathfrak{p} = \{\xi \in O_E \big| |\xi|_p < 1\}$ and often write $W := O_{E,\mathfrak{p}}$, where O_E is the integer ring of E. Often we just write ρ for $\rho_{\mathfrak{p}}$ which acts on $V = E_{\mathfrak{p}}^d$.

For simplicity, we assume that $p \notin S$. Let $E_\ell(X) = \det(1 - \rho_{\mathfrak{q}}(Frob_\ell)|_{V_{I_\ell}} X) \in E[X]$ (assuming $\mathfrak{q} \nmid \ell$). We always assume that $\rho_{\mathfrak{p}}$ is ordinary in the following sense: ρ restricted to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is upper triangular with diagonal characters $\mathcal{N}^{a_j}$ on the inertia I_p for the p-adic cyclotomic character $\mathcal{N}$ ordered from top to bottom as $a_1 \geq a_2 \geq \cdots \geq 0 \geq \cdots \geq a_d$. Thus

$$\rho|_{I_p} = \begin{pmatrix} \mathcal{N}^{a_1} & * & \cdots & * \\ 0 & \mathcal{N}^{a_2} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{N}^{a_d} \end{pmatrix}.$$

In other words, we have a decreasing filtration $\mathcal{F}^{i+1}V \subset \mathcal{F}^i V$ stable under $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ such that the Tate twists $gr^i(V)(-i) := (F^iV/F^{i+1}V)(-i)$ is unramified. Define

$$H_p(X) = \prod_i \det(1 - Frob_p|_{gr^i(V)(-i)} p^i X) = \prod_{j=1}^d (1 - \alpha_j X).$$

Then it is believed to be $E_p(X) = H_p(X)$ if $p \notin S$ and $E_p(X)|H_p(X)$ otherwise. In any case, $\mathrm{ord}_p(\alpha_j) \in \mathbb{Z}$. Let us define

$$\beta_j = \begin{cases} \alpha_j & \text{if } \mathrm{ord}_p(\alpha_j) \geq 1, \\ p\alpha_j^{-1} & \text{if } \mathrm{ord}_p(\alpha_j) \leq 0 \end{cases}$$

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and put $e = |\{j|\beta_j = p\}|$.

$$\mathcal{E}(\rho) = \prod_{j=1}^{d}(1 - \beta_j p^{-1}) \text{ and } \mathcal{E}^+(\rho) = \prod_{j=1,\beta_j \neq p}^{d}(1 - \beta_j p^{-1}).$$

Then the complex L-function is defined by $L(s,\rho) = \prod_\ell E_\ell(\ell^{-s})^{-1}$ whose value at 1 is critical. We suppose to have an algebraicity result (basically conjectured by Deligne) that for a well defined period $c^+(\rho)(1)) \in \mathbb{C}^\times$ such that $\frac{L(s,\rho\otimes\varepsilon)}{c^+(\rho(1))} \in \overline{\mathbb{Q}}$ for all finite order characters $\varepsilon : \mathbb{Z}_p^\times \to \mu_{p^\infty}(\overline{\mathbb{Q}})$. Then we should have

Conjecture 1.1 ($\mathcal{L}$-invariant)**.** *There exist a power series $\Phi^{an}(X) \in W[[X]]$ and a p-adic L-function $L_\rho^{an}(s,\rho) = \Phi_\rho^{an}(\gamma^{1-s} - 1)$ interpolating $L(1,\rho\otimes\varepsilon)$ for p-power order character ε such that $\Phi_\rho^{an}(\varepsilon(\gamma) - 1) \sim \mathcal{E}(\rho\otimes\varepsilon)\frac{L(1,\rho\otimes\varepsilon)}{c^+(\rho(1))}$ with the modifying p-factor $\mathcal{E}(\rho)$ as above (putting $\mathcal{E}(\rho\otimes\varepsilon) = 1$ if $\varepsilon \neq 1$). The L-function $L_p^{an}(s,\rho)$ has zero of order $e + \mathrm{ord}_{s=1} L(s,\rho)$ for a nonzero constant $\mathcal{L}^{an}(\rho) \in \mathbb{C}_p^\times$ (called the analytic $\mathcal{L}$-invariant), we have*

$$\lim_{s\to 1}\frac{L_p^{an}(s,\rho)}{(s-1)^e} = \mathcal{L}^{an}(\rho)\mathcal{E}^+(\rho)\frac{L(1,\rho)}{c^+(\rho(1))},$$

where "$\lim_{s\to1}$" is a p-adic limit, $c^+(\rho(1))$ is the transcendental factor of the critical complex L-value $L(1,\rho)$ and $\mathcal{E}^+(\rho)$ is the product of nonvanishing modifying p-factors.

When $e > 0$, we call that $L_p^{an}(s,\rho)$ has an exceptional zero at $s = 1$. Here is an example. Start with a Dirichlet character $\chi : (\mathbb{Z}/N\mathbb{Z}) \to \overline{\mathbb{Q}}^\times$ with $\chi(-1) = -1$. Then $c(\rho^+(1)) = (2\pi i)$. If we suppose $\chi = \left(\frac{-D}{\cdot}\right)$ for a square free positive integer D, the modifying Euler factor vanishes at $s = 1$ if the Legendre symbol $\left(\frac{-D}{p}\right) = 1 \Leftrightarrow (p) = \mathfrak{p}\overline{\mathfrak{p}}$ in $O_{\mathbb{Q}[\sqrt{-D}]}$ with $\mathfrak{p} = \left\{x \in O_{\mathbb{Q}[\sqrt{-D}]} \middle| |x|_p < 1\right\}$. By a work of Kubota–Leopoldt and Iwasawa, we have a p-adic analytic L-function $L_p^{an}(s,\chi) = \Phi^{an}(\gamma^{1-s} - 1)$ for a power series $\Phi^{an}(X) \in \Lambda = W[[X]]$ and $\gamma = 1 + p$ such that for $\mathcal{E}(\chi\mathcal{N}^m) = (1 - \chi(p)p^{m-1})$

$$L_p^{an}(m,\chi) = \Phi^{an}(\gamma^{1-m} - 1) = \mathcal{E}(\chi\mathcal{N}^m)L(1-m,\chi) \sim \mathcal{E}(\chi\mathcal{N}^m)\frac{L(m,\chi)}{(2\pi i)^m}$$

for all positive integer m as long as $|n^m - n|_p < 1$ for all n prime to p. If we have an exceptional zero at 1, it appears that we lose the exact connection of the p-adic L-value and the corresponding complex L-value. However, the conjecture says we can recover the complex L-value via an appropriate derivative of the p-adic L-function as long as we can compute $\mathcal{L}^{an}(\rho)$. We may regard χ as a Galois character $\mathrm{Gal}(\mathbb{Q}[\mu_N]/\mathbb{Q}) = (\mathbb{Z}/N\mathbb{Z})^\times \xrightarrow{\chi} \{\pm1\}$, and we remark that $\chi(Frob_p) = 1$ to have the exceptional zero. For our later use, for the class number h of $\mathbb{Q}[\sqrt{-D}]$, we write the generator of $\overline{\mathfrak{p}}^h$ as ϖ; so, $\overline{\mathfrak{p}}^h = (\varpi)$ for $\varpi \in \mathbb{Q}[\sqrt{-D}]$.

Though we formulated conjecture for $p \notin S$, if $\rho_\mathfrak{p}$ is ordinary semi-stable, we have the same phenomena and can formulate the conjecture. Here is such an example. Start with an elliptic curve $E_{/\mathbb{Q}}$, which yields a compatible system $\rho_E := \{T_\ell E\}$ given by the ℓ-adic Tate module $T_\ell E$. Suppose that E has split multiplicative reduction at p. In this case, $H_p(X) = (1-X)(1-pX)$ and $E_p(X) = (1-X)$, $\mathcal{E}(\rho_E) = 0$ and $\mathcal{E}^+(\rho) = 1$. Then by the solution of the Shimura-Taniyama conjecture by Wiles et al, this L-function has a p-adic analogue constructed by Mazur such that we have $\Phi_E^{an}(X) \in \Lambda$ with $\Phi_E^{an}(\varepsilon(\gamma)-1) = \mathcal{E}(\rho_E \otimes \varepsilon)\frac{G(\varepsilon^{-1})L(1,E,\varepsilon)}{\Omega_E}$ for all p-power order character $\varepsilon : \mathbb{Z}_p^\times \to W^\times$; in other words, $L_p^{an}(s,E) = \Phi_E^{an}(\gamma^{1-s}-1)$. Here Ω_E is the period of the Néron differential of E. Thus if $Frob_p$ has eigenvalue 1 on $T_\ell E$, the exceptional zero appears at $s=1$ as in the case of Dirichlet character. The $Frob_p$ has eigenvalue 1 if and only if E has multiplicative reduction mod p.

The problem of $\mathcal{L}$-invariant is to compute explicitly the $\mathcal{L}$-invariant $\mathcal{L}^{an}(\rho)$. The $\mathcal{L}$-invariant in the cases where $\rho = \chi = \left(\frac{-D}{\cdot}\right)$ as above and $\rho = \rho_E$ for E with split multiplicative reduction is computed in the 1970s to 90s, and the results are

Theorem 1.2. *Let the notation and the assumption be as above.*

(1) $\mathcal{L}^{an}(\chi) = \frac{\log_p(q)}{\mathrm{ord}_p(q)} = \frac{\log_p(q)}{h}$ *for* $q \in \mathbb{C}_p$ *given by* $q = \varpi/\overline{\varpi}$ ($\overline{\mathfrak{p}}^h = (\varpi)$) *and the class number* h *of* $\mathbb{Q}[\sqrt{-D}]$ *(Gross–Koblitz and Ferrero–Greenberg);*

(2) *For* E *split multiplicative at* p, *writing* $E(\mathbb{C}_p) = \mathbb{C}_p^\times/q^{\mathbb{Z}}$ *for the Tate period* $q \in \mathbb{Q}_p^\times$, *we have* $\mathcal{L}^{an}(\rho_E) = \frac{\log_p(q)}{\mathrm{ord}_p(q)}$. *This was conjectured by Mazur–Tate–Teitelbaum and later proven by Greenberg–Stevens.*

Here $\log_p$ *is the Iwasawa logarithm and* $|x|_p = p^{-\mathrm{ord}_p(x)}$.

1.1. **Arithmetic $\mathcal{L}$-invariant.** Starting with an ordinary p-adic Galois representation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to G(W)$ for a split reductive group $G_{/\mathbb{Z}_p}$, there is a systematic way to create many Galois representations whose eigenvalues of $Frob_p$ contain 1. Indeed, let ρ acts on the Lie algebra $\mathfrak{s}$ of the derived group of G by the adjoint action. If $G = GL(n)$, $\mathfrak{s} = \{x \in M_n(W) | \mathrm{Tr}(x) = 0\}$ and the adjoint action is by conjugation. Write this Galois representation as $Ad_G(\rho)$ (or just $Ad(\rho)$). If $G = GL(n)$, since

$$M_n(W) = Ad(\rho) \oplus \{\text{scalar matrices}\},$$

the action of $Ad(\rho)(Frob_p)$ on $Ad(W)$ has eigenvalue 1 with multiplicity $\geq n-1$. If G is symplectic or orthogonal, $Ad(\rho)$ is often critical at $s=1$.

Now require that $\rho_F : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to GL_2(W)$ be a totally odd p-ordinary Galois representation of dimension 2 over a totally real field F. We make $Ad(\rho_F)$ and consider the induced representation $\rho := \mathrm{Ind}_F^{\mathbb{Q}} Ad(\rho_F)$ whose eigenvalues of $Frob_p$ has 1 with multiplicity e for the number e of prime factors of p in F.

Returning to a general ordinary representation $\rho = \rho_{\mathfrak{p}}$, we describe an arithmetic way of constructing p-adic L-function due to Iwasawa and others. We can define Galois cohomologically the Selmer group

$$\mathrm{Sel}(\rho \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p) \subset H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}_\infty), \rho \otimes \mathbb{Q}_p/\mathbb{Z}_p)$$

for the $\mathbb{Z}_p$-extension $\mathbb{Q}_\infty/\mathbb{Q}$ inside $\mathbb{Q}(\mu_{p^\infty})$. The Galois group $\Gamma = \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$ acts on $H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}_\infty), \rho \otimes \mathbb{Q}_p/\mathbb{Z}_p)$ and hence on $\mathrm{Sel}(\rho)$, making it as a discrete module over the group algebra $W[[\Gamma]] = \varprojlim_n W[\Gamma/\Gamma^{p^n}]$. Identifying Γ with $1 + p\mathbb{Z}_p$ by the cyclotomic character, we may regard $\gamma \in \Gamma$. Then $W[[\Gamma]] \cong \Lambda$ by $\gamma \mapsto 1 + X$. By the classification theory of compact Λ-modules, the Pontryagin dual $\mathrm{Sel}^*(\rho)$ has a Λ-linear map into $\prod_{f \in \Omega} \Lambda/f\Lambda$ with finite kernel and cokernel for a finite set $\Omega \subset \Lambda$. The power series $\Phi_\rho = \prod_{f \in \Omega} f(X)$ is uniquely determined up to unit multiple. We then define $L_p(s,\rho) = \Phi_\rho(\gamma^{1-s}-1)$. Greenberg gave a recipe of defining $\mathcal{L}(\rho)$ for this $L_p(s,\rho)$ and verified in 1994 the conjecture for this $L_p(s,\rho)$ except for the nonvanishing of $\mathcal{L}(\rho)$

(under some restrictive conditions). For the adjoint square $Ad(\rho_F)$ for ρ_F associated to a Hilbert modular form, the conjecture (except for the nonvanishing of $\mathcal{L}(\rho)$) was again proven in my paper in Israel journal (in 2000) under the milder conditions, for example, $\overline{\rho}_F = (\rho_F \mod \mathfrak{m}_W)$ is absolutely irreducible over $\mathrm{Gal}(\overline{\mathbb{Q}}/F[\mu_p])$ and the p-distinguishedness condition for $\overline{\rho}_F|_{\mathrm{Gal}(\overline{F}_\mathfrak{p}/F_\mathfrak{p})}$ for all $\mathfrak{p}|p$ (which we recall later). If there exists an analytic p-adic L-function $L_p^{an}(s,\rho) = \Phi_\rho^{an}(\gamma^{1-s}-1)$ interpolating complex L-values, the main conjecture of Iwasawa's theory confirms $\Phi_\rho = \Phi_\rho^{an}$ up to unit multiple.

Suppose now that ρ_F is associated to a Hilbert modular Hecke eigenform of weight $k \geq 2$ over a totally real field F. Following Greenberg's recipe, we try to compute $\mathcal{L}(Ad(\rho_F)) = \mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}} Ad(\rho_F))$ explicitly. By ordinarity, we have $\rho_F|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/F_\mathfrak{p})} \cong \left(\begin{smallmatrix} * & * \\ 0 & \alpha_\mathfrak{p} \end{smallmatrix}\right)$ with distinct diagonal characters factoring through $I_\mathfrak{p} \to \mathrm{Gal}(F_\mathfrak{p}[\mu_{p^\infty}]/F_\mathfrak{p})$ for the inertia group $I_\mathfrak{p}$ for all $\mathfrak{p}|p$. We consider the universal nearly ordinary deformation $\boldsymbol{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(R)$ over K with the pro-Artinian local universal K-algebra R. This means that for any Artinian local K-algebra A with maximal ideal $\mathfrak{m}_A$ and any Galois representation $\rho_A : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to GL_2(A)$ such that

(1) unramified outside ramified primes for ρ_F;
(2) $\rho_A|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/F_\mathfrak{p})} \cong \left(\begin{smallmatrix} * & * \\ 0 & \alpha_{A,\mathfrak{p}} \end{smallmatrix}\right)$ with $\alpha_{A,\mathfrak{p}} \equiv \alpha_\mathfrak{p} \mod \mathfrak{m}_A$ such that the diagonal characters factor through $I_\mathfrak{p} \to \mathrm{Gal}(F_\mathfrak{p}[\mu_{p^\infty}]/F_\mathfrak{p})$ for all $\mathfrak{p}|p$;
(3) $\det(\rho_A) = \det \rho_F$;
(4) $\rho_A \equiv \rho_F \mod \mathfrak{m}_A$,

there exists a unique K-algebra homomorphism $\varphi : R \to A$ such that $\varphi \circ \boldsymbol{\rho} \cong \rho_A$. Write $\Gamma_\mathfrak{p} \cong \mathbb{Z}_p$ for the p-profinite part of the inertia subgroup $\overline{I}_\mathfrak{p}$ of $\mathrm{Gal}(F_\mathfrak{p}[\mu_{p^\infty}]/F_\mathfrak{p})$. Choose a generator $\gamma_\mathfrak{p}$ of $\Gamma_\mathfrak{p}$ and identify $W[[\Gamma_\mathfrak{p}]]$ with $W[[X_\mathfrak{p}]]$ by $\gamma_\mathfrak{p} \leftrightarrow 1 + X_\mathfrak{p}$. Since $\boldsymbol{\rho}|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/F_\mathfrak{p})} \cong \left(\begin{smallmatrix} * & * \\ 0 & \boldsymbol{\delta}_\mathfrak{p} \end{smallmatrix}\right)$, $\boldsymbol{\delta}_\mathfrak{p}\alpha_\mathfrak{p}^{-1} : \overline{I}_\mathfrak{p} \to R$ factors through $\Gamma_\mathfrak{p}$ and induces an algebra structure on R over $W[[X_\mathfrak{p}]]$. Thus R is an algebra over $K[[X_\mathfrak{p}]]_{\mathfrak{p}|p}$. If we write $\varphi_\rho : R \to K$ for the morphism with $\varphi_\rho \circ \boldsymbol{\rho} \cong \rho_F$, by our construction, $\mathrm{Ker}(\varphi_\rho) \supset (X_\mathfrak{p})_{\mathfrak{p}|p} = (X)$. We state a conjecture:

Conjecture 1.3 (Deformation)**.** *We have $R \cong K[[X_\mathfrak{p}]]_{\mathfrak{p}|p}$.*

The conjecture is verified by Kisin when $F = \mathbb{Q}$ via Serre's modularity conjecture proven by Khare/Wintenberger/Kisin. For general $F \neq \mathbb{Q}$, this follows from the work of Skinner-Wiles, Kisin, Lin Chen and Fujiwara generalizing the fundamental work by Wiles and Taylor-Wiles if $\mathrm{Im}(\rho_E \mod p)$ is nonsoluble. Here is my theorem:

Theorem 1.4. *Assume Conjecture 1.3. Then for the local Artin symbol $[p, F_\mathfrak{p}] = Frob_\mathfrak{p}$, we have*

$$\mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}} Ad(\rho_F)) = \det\left(\frac{\partial \boldsymbol{\delta}_\mathfrak{p}([p,F_\mathfrak{p}])}{\partial X_{\mathfrak{p}'}}\right)_{\mathfrak{p},\mathfrak{p}'}\Big|_{X=0} \prod_\mathfrak{p} \log_p(\gamma_\mathfrak{p})\alpha_\mathfrak{p}([p,F_\mathfrak{p}])^{-1},$$

where $\gamma_\mathfrak{p}$ is the generator of $\Gamma_\mathfrak{p}$ by which we identify the group algebra $W[[\Gamma_\mathfrak{p}]]$ with $W[[X_\mathfrak{p}]]$.

Here are some examples showing usefulness of this theorem: Take a totally imaginary quadratic extension M/F in which all prime factors $\mathfrak{p}|p$ in F splits as $\mathfrak{P}\overline{\mathfrak{P}}$. Take a set $\Sigma = \{\mathfrak{P}|p\}$ so that $\Sigma \sqcup \overline{\Sigma}$ is the set of all prime factors of p in M. Write h for the class number of M and choose $\varpi(\mathfrak{P}) \in M$ so that $\mathfrak{P}^h = (\varpi(\mathfrak{P}))$ for $\mathfrak{P} \in \overline{\Sigma}$. For any Galois character $\psi : \mathrm{Gal}(\overline{\mathbb{Q}}/M) \to W^\times$ of M with $\psi(\sigma) \neq \psi^c(\sigma)$ for $\psi^c(\sigma) = \psi(c\sigma c^{-1})$ and a complex conjugation $c \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$, we have $Ad(\mathrm{Ind}_M^F \psi) = \chi \oplus \mathrm{Ind}_M^F \psi^{1-c}$ for $\chi = \left(\frac{M/F}{\cdot}\right)$, and we can easily show $\mathcal{L}(\chi) = \mathcal{L}(Ad(\mathrm{Ind}_M^F \psi))$. The arithmetic p-adic L-function $L_p(s,\chi)$ for $\chi = \left(\frac{M/F}{\cdot}\right)$ constructed à la Iwasawa has an exceptional zero of order $\geq e$ for $e = |\Sigma|$. Since we can compute explicitely the universal deformation $\boldsymbol{\rho}$ of $\rho = \mathrm{Ind}_M^F \psi$, we get from the theorem

Corollary 1.5. *We have* $\mathcal{L}(\chi) = \frac{\det\left(\log_p(N_{\mathfrak{P}'}(\varpi(\mathfrak{P})^{(1-c)}))\right)_{\mathfrak{P},\mathfrak{P}' \in \overline{\Sigma}}}{\prod_{\mathfrak{P}\in\overline{\Sigma}} \mathrm{ord}_p(N_{\mathfrak{P}}(\varpi(\mathfrak{P})^{(1-c)})}$ *up to a simple constant, where* $N_{\mathfrak{P}}$ *is the local norm* $N_{M_{\mathfrak{P}}/\mathbb{Q}_p}$.

If $E_{/F}$ is an elliptic curve with split multiplicative reduction at all $\mathfrak{p}|p$, we write $E_{/F_{\mathfrak{p}}}(\overline{F}_{\mathfrak{p}}) \cong \overline{F}_{\mathfrak{p}}^\times / q_{\mathfrak{p}}^{\mathbb{Z}}$ for the Tate period $q_{\mathfrak{p}} \in F_{\mathfrak{p}}^\times$. Then we have directly from the theorem the following

Corollary 1.6 ($\mathcal{L}$-invariant)**.** *The* $\mathcal{L}$*-invariant* $Ad(\rho_E)$ *is given by*

$$\mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}} Ad(\rho_E)) = \prod_{\mathfrak{p}} \frac{\log_p(N_{\mathfrak{p}}(q_{\mathfrak{p}}))}{\mathrm{ord}_p(N_{\mathfrak{p}}(q_{\mathfrak{p}}))},$$

where $N_{\mathfrak{p}}$ *is the local norm* $N_{F_{\mathfrak{p}}/\mathbb{Q}_p}$.

A more general case is treated in my paper appeared in

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(posted in my web page `www.math.ucla.edu/~hida`), and the proof of the case treated in this note is given in my book "Hilbert modular forms and Iwasawa theory" from Oxford University Press. The proof of the factorization of the $\mathcal{L}$-invariant of the above corollary is automorphic in the book (a Hecke algebra argument) and is Galois cohomological in the IMRN paper (and in my papers quoted in the IMRN paper, all posted in my web page).