

L-value formula

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Lecture 4: We sketch the proof of the L-value formula for a division quaternion algebra D/\mathbb{Q} . The algebra D can be definite or indefinite, though we describe mainly the details only for the indefinite case. Let $B = D \otimes_{\mathbb{Q}} E$ for a real semi-simple quadratic extension E . The non-trivial automorphism $\sigma \in \text{Gal}(E/\mathbb{Q})$ acts on B through the factor E . Since the case $E = \mathbb{Q} \times \mathbb{Q}$ is easier, we mainly assume that E is a field. A key point is the use of [the see-saw principle](#) for the decomposition $D_{\sigma} = Z \oplus D_0$, where $D_{\sigma} := \{v \in B \mid v^{\iota} = v^{\sigma}\}$ with the reduced norm $N : D_{\sigma} \rightarrow \mathbb{Q}$ and $Z = D_{\sigma} \cap E$ and $D_0 = \{v \in D_{\sigma} \mid \text{Tr}(v) = 0\}$. We need to use the Siegel–Weil formula for D_0 . For simplicity, we assume $M = \partial$. The details are in Chapter 5, and the case $M = M_2(\mathbb{Q})$ is dealt with in Section 5.5 of the notes.

§0. **An idea of Waldspurger.** For an elliptic cusp form f , an idea of Waldspurger of computing the period of a theta lift of f for a quadratic space $V = W \oplus W^\perp$ over an orthogonal Shimura subvariety $S_W \times S_{W^\perp} \subset S_V$ is two-folds:

(S) Split $\theta(\phi')(\tau, h, h^\perp) = \theta(\phi)(\tau, h) \cdot \theta(\tau, \phi^\perp)(h^\perp)$ ($h^\perp \in \mathcal{O}_{W^\perp}(\mathbb{A})$) for a decomposition $\phi' = \phi \otimes \phi^\perp$ (ϕ and ϕ^\perp Schwartz–Bruhat functions on $W_\mathbb{A}$ and $W_\mathbb{A}^\perp$);

(R) For the theta lift $\theta^*(\phi)(f)(h) = \int_X f(\tau) \theta(\phi)(\tau, h) d\mu$ with an $\mathrm{SL}(2)$ -Shimura curve X , the period P over the Shimura subvariety $S \times S^\perp$ (S for $\mathrm{O}(W)$ and S^\perp for $\mathrm{O}(W^\perp)$) is given by:

$$\begin{aligned} & \int_{S \times S^\perp} \int_X f(\tau) \theta(\phi)(\tau; h) d\mu dh \quad (d\mu = \eta^{-2} d\xi d\eta) \\ &= \int_X f(\tau) \left(\int_{S^\perp} \theta(\phi^\perp)(\tau; h^\perp) dh^\perp \right) \cdot \left(\int_S \theta(\phi_0)(\tau; h_0) dh \right) d\mu. \end{aligned}$$

Then invoke the Siegel–Weil formula to convert inner integrals into the Siegel–Weil Eisenstein series $E(\phi)$ and $E(\phi^\perp)$, reaching Rankin–Selberg integral

$$P = \int_X f(\tau) E(\phi^\perp) E(\phi_0) d\mu = L\text{-value}.$$

§1. **Choice of V :** For a \mathbb{Q} -vector space V and a \mathbb{Q} -algebra A , write $V_A := V \otimes_{\mathbb{Q}} A$. Let $E := \mathbb{Q}[\sqrt{\Delta}]$ be a quadratic extension of \mathbb{Q} with discriminant Δ . Pick a quaternion algebra D over \mathbb{Q} and put $B := D \otimes_{\mathbb{Q}} E$. We let $1 \neq \sigma \in \text{Gal}(E/\mathbb{Q})$ act on D through the factor E . Recall

$$V = D_{\sigma} := \{v \in B | v^{\sigma} = v^{\iota}\} \quad \text{for } v^{\iota} = \text{Tr}_{B/E}(v) - v.$$

The quadratic form is given by $Q(v) = vv^{\sigma} = N(v) \in \mathbb{Q}$. We have two cases of isomorphism classes of $(D_{\mathbb{R}}, E_{\mathbb{R}})$. Note $E_{\mathbb{R}} = \mathbb{R} \times \mathbb{R}$; so, we have two cases Case I and Case H. The symbol “I” (resp. “H”) indicate D is indefinite (resp. definite). The decomposition we take is

$$V = Z \oplus D_0 \quad Z = \mathbb{Q} \text{ with quadratic form } Q_Z(z) = z^2, \text{ and}$$

$$D_0 := \{v_0 \in \sqrt{\Delta}D | \text{Tr}_{D/\mathbb{Q}}(v) = 0\} \text{ with } Q_0(v) = vv^{\sigma} = N(v)$$

Signature of D_0 is (1,2) in Case I and (3,0) in Case H, \mathcal{O}_{D_0} is almost D^{\times} and the same for $\mathcal{O}_{D_{\sigma}}$ and B^{\times} .

§2. Bruhat functions and majorant. On $Z = \mathbb{Q}$, for a Dirichlet character ψ modulo N , we regard ψ as a function supported on $\hat{\mathbb{Z}} \subset Z_{\mathbb{A}(\infty)} = \mathbb{A}^{(\infty)}$. This ψ produces theta series $\sum_{n \in \mathbb{Z}} \psi(n) n^j \mathbf{e}(n^2 \tau)$ on $\Gamma_0(4N^2)$ of character $\psi\left(\frac{-1}{\cdot}\right)$ and of weight $j + \frac{1}{2}$.

Take a maximal order R of D and take the characteristic function ϕ_0 of $D_{0,\mathbb{A}} \cap \sqrt{\Delta} \hat{R}$. Here for any lattice L , $\hat{L} = L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$. This ϕ_0 produces theta series on $\Gamma_0(4\partial\Delta)$ of character $\left(\frac{-\Delta}{\cdot}\right)$.

The theta series for D_σ of $\psi \otimes \phi_0$ has level $M = [4N^2, 4\partial\Delta]$. We choose M so that $C|M$ for the conductor C of F .

A positive definite symmetric matrix $P \in M_n(\mathbb{R})$ (or the symmetric bilinear form on $V_{\mathbb{R}}$ associated to P) is a **positive majorant** of a symmetric matrix S if $PS^{-1} = SP^{-1}$ ($\Leftrightarrow S^{-1}P = P^{-1}S$).

§3. **Schwartz function Ψ on $D_{\sigma, \mathbb{R}}$ in Case I.** The recipe of Hecke–Siegel is to put $\Psi(v) = H(v)e(\xi N(v) + P(v)\eta\sqrt{-1})$ for $e(x) = \exp(2\pi\sqrt{-1}x)$ and a harmonic polynomial H , where $P(v) = \frac{1}{2}p(v, v)$ with a positive majorant p of $s(v, v') = \text{Tr}_{B/E}(v^\iota v')$. All positive majorants form the symmetric space \mathfrak{S} of O_{D_σ} .

We identify $(D_{\sigma, \mathbb{R}}, N) = (M_2(\mathbb{R}), \det)$ by $M_2(\mathbb{R}) \ni v \mapsto (v, v^\iota) \in D_{\sigma, \mathbb{R}} \subset D_{\mathbb{R}} \times D_{\mathbb{R}}$ and put for $(z, w) \in \mathfrak{H} \times \mathfrak{H}$ on which $B^\times \sim \text{GO}_{D_\sigma}$ acts by $\alpha(z, w) = (\alpha(z), \alpha^\sigma(w))$. For $(z, w) \in \mathfrak{H} \times \mathfrak{H}$, a standard harmonic polynomial of $v \in D_\sigma$ of degree k is given by $[v; z, w]^k = s(v, p(z, w))^k$ for $p(z, w) = \begin{pmatrix} z & 1 \\ 1 & w \end{pmatrix} J$. For $0 < k \in \mathbb{Z}$, $\Psi(v; \tau, z, w) = \text{Im}(\tau)[v; z, \bar{w}]^k e(N(v)\bar{\tau} + i \frac{\text{Im}(\tau)}{2|\text{Im}(z)\text{Im}(w)|} |[v; z, \bar{w}]|^2)$, for $(\alpha, \beta) \in GL_2(\mathbb{R}) \times GL_2(\mathbb{R})$ (see §3.2.3),

$$\alpha p(z, w) \beta^\iota = p(\alpha(z), \beta(w)) j(\alpha, z) j(\beta, w).$$

This formula is due to Shimura. This function is not a tensor product of functions on $Z_{\mathbb{R}}$ and $D_{0, \mathbb{R}}$ which causes some difficulty later. For simplicity, we assume $k = 2$. See Section 3.2 for Ψ .

§4. **Theta kernel.** Let ϕ be a Schwartz-Bruhat function on $D_{\sigma, \mathbb{A}}$. Let $\text{Mp}(\mathbb{A}) \twoheadrightarrow \text{SL}_2(\mathbb{A})$ be the metaplectic cover constructed by Weil, and $\phi \mapsto \mathbf{w}(g)\phi$ the Weil representation. Noting $B^\times \twoheadrightarrow \text{GO}_{D_\sigma}$ by $v \mapsto h^\iota v h^\sigma$, Siegel–Weil theta series $\theta(g; h)$ is

$$\sum_{\alpha \in D_\sigma} (\mathbf{w}(g)\phi)(h^\iota \alpha h^\sigma) : \text{SL}_2(\mathbb{Q}) \backslash \text{Mp}(\mathbb{A}) \times B^\times \backslash B_{\mathbb{A}}^\times \rightarrow \mathbb{C}.$$

Write $\hat{\Gamma} = \hat{\Gamma}_\phi = \{u \in B_{\mathbb{A}(\infty)}^\times \mid \theta(g, u^\iota h u^\sigma) = \theta(g, h)\}$.

In Case I, choose $\phi = (\psi \otimes \phi_0)\Psi(v; \sqrt{-1}, \sqrt{-1}, \sqrt{-1})$ and for $g_\tau = \eta^{-1/2} \begin{pmatrix} \eta & \xi \\ 0 & 1 \end{pmatrix}$ ($\tau = \xi + \eta\sqrt{-1} \in \mathfrak{H}$), we specialize g to g_τ and h to (g_z, g_w) for $(\tau, z, w) \in \mathfrak{H} \times \mathfrak{H} \times \mathfrak{H}$. Then

$$\theta(\tau; z, w) := \theta(g_\tau; g_z, g_w) = \sum_{\alpha \in D_\sigma} (\psi \otimes \phi_0)(\alpha) \Psi(\alpha; \tau, z, w).$$

Set $\theta^*(\phi)(f) := \int_{X_0(M)} f(\tau) \theta(\phi)(\tau; z, w) \eta^{k-2} d\xi d\eta$ ($k = 2$). Then $\theta^*(\phi)(f)$ is a weight 2 quaternionic modular form on B^\times holomorphic in z and anti-holomorphic in w for $f \in S_2^-(\Gamma_0(M), \psi^{-1}(\underline{\Delta}))$.

§5. **Theta differential form.** To compute the period on $Sh_D = D_+^\times \setminus (D_{\mathbb{A}(\infty)}^\times \times \mathfrak{H}) \subset Sh_B = B^\times \setminus (B_{\mathbb{A}(\infty)}^\times \times \mathfrak{H}_B)$, we convert $\theta(\tau; z, w)$ into a sheaf valued differential 2-form. If $n = k - 2 > 0$, the sheaf comes from the B^\times -module

$$L_E(n; A) = \sum_{0 \leq i, j \leq n} A X^{n-j} Y^j X^{m-i} Y^i$$

with B^\times -action $\gamma P(X, Y; X', Y') = P((X, Y)^t \gamma^\iota; (X', Y')^t \gamma^{\sigma \iota})$. As we assumed $k = 2$ (i.e., $n = 0$), we have $L(n; A) = A$.

By putting $\Theta = \theta(\phi)(\tau; z, w) dz \wedge d\bar{w}$ for $n = k - 2$, we get \mathbb{C} -valued Γ_ϕ -invariant differential form. The period we like to compute is

$$P = P_1(\theta^*(\phi)(f)) = \int_{Sh_D} \int_{X_0(M)} f(-\bar{\tau}) \Theta(\tau; z, z) d\xi d\eta.$$

We integrate over Sh_D by a measure $d\mu$ given by $y^{-2} dx dy$ over \mathfrak{H} and $\int_{\widehat{\Gamma}} d\mu = 1$.

§6. **Siegel–Weil Eisenstein series**; §4.4.2. Recall the explicit section $\mathbf{r} : B \hookrightarrow \text{Mp}$ of the representation \mathbf{w} as follows:

$$\mathbf{r}(\text{diag}[a, a^{-1}])\phi(v) = |a|_{\mathbb{A}}^{3/2} \phi(av), \quad \mathbf{r}\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right)\phi(v) = \mathbf{e}(uN(v))\phi(v).$$

For the standard Borel subgroup $B \subset \text{SL}_2$, the function $g \mapsto (\mathbf{r}(g)\phi)(0)$ is left $B(\mathbb{Q})$ invariant. Siegel–Weil Eisenstein series is

$$E(\phi)(g; s) = \sum_{\gamma \in B(\mathbb{Q}) \backslash \text{SL}_2(\mathbb{Q})} (\mathbf{w}(\gamma g)\phi)(0) |a(\gamma g)|_{\mathbb{A}}^s,$$

where $g = \text{diag}[a(g), a(g)^{-1}] \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} c$ for an element c in the maximal compact subgroup by Iwasawa decomposition.

The Siegel–Weil formula by Kudla–Rallis and Sweet is

$$E(\phi)(g; 0) = \int_S \theta(\phi)(g, h) d\omega \quad \text{for the Tamagawa measure } d\omega.$$

The ratio $\mathfrak{m} = \mathfrak{m}(\hat{\Gamma}) = d\mu/d\omega$ is the mass of Siegel–Shimura, which is an arithmetic rational number times $\zeta(2)/\pi$ in Case I and $\zeta(2)/\pi^2$ in Case H. Later we describe the Tamagawa measure, the metaplectic group and Weil representation in more details.

§7. **Conclusion in Case I; §5.3.** Decomposing $v = a \oplus v_0$ in $D_\sigma = Z \oplus D_0$, we have for $k = 2$

$$[a + v_0; z, \bar{z}]^k = ([a; z, \bar{z}] + [v_0; z, \bar{z}])^k = \sum_{j=0}^k \binom{k}{j} (a(z - \bar{z}))^{k-j} [v_0; z, \bar{z}]^j.$$

Thus we have $\phi = \sum_{j=0}^k \binom{k}{j} \phi_{k-j}^Z \phi_j^{D_0}$ with infinity part $\psi_j^?$ of $\phi_j^?$ given by

$$\psi_j^{D_0} := (z - \bar{z})^{-j} [v_0; z, \bar{z}]^j \mathbf{e}(N(\mathfrak{x})\bar{\tau} + \frac{i \operatorname{Im}(\tau) |v_0; z, \bar{z}|^2}{2 \operatorname{Im}(z)^2}), \psi_j^Z := a^j \mathbf{e}(a^2 \tau)$$

with $(\mathbf{w}(b) \phi_j^{D_0})(0) = 0$ and $E(\phi_j^{D_0})|_{B(\mathbb{A})} = 0$ unless $j = 0$, and we reach Rankin convolution of $\theta(\phi_k^Z) = \sum_{n \in \mathbb{Z}} \psi(n) n^k \mathbf{e}(n^2 z)$ and f over $B(\mathbb{Q}) \backslash B(\mathbb{A}) / B(\hat{\mathbb{Z}}) \cong [0, 1) \times \mathbb{R}_+^\times$, which produces (see [Sh75])

$$\zeta(2)P = \mathfrak{m} 2^{-2k} * (2\pi)^{-k} \Gamma(k) L^{(s)}(1, \operatorname{Ad}(\rho_f) \otimes \left(\frac{\Delta}{-}\right))$$

with a simple constant $*$. Here $L^{(s)}$ means we remove Euler factors at $p|C$ with either $f|U(p) = 0$ or $\psi(p) = 0$.

§8. **Conclusion in Case H; §5.2.** The choice of the Bruhat function ϕ is the same as in Case I. As a \mathbb{C} -valued function, set

$$\Psi(\tau; v; \mathbf{x}) = e(N(v)\tau).$$

Again in exactly the same way, for

$$\theta^*(\phi)(f) := \int_{X_0(M)} \theta(\phi)(\tau; g) f(\tau) \eta^{k-2} d\xi d\eta \quad (k = 2)$$

and $P = \int_S \theta^*(\phi)(f) d\mu$, we conclude for a simple constant c'

$$\zeta(2)P = 2\mathfrak{m} *' (2\pi)^{-k+1} \Gamma(k) L^{(s)}(1, \text{Ad}(\rho_f) \otimes \left(\frac{\Delta}{-}\right)).$$

Writing the point set $S = \{x\}_{x \in Sh_R}$, $\mathfrak{m}(\hat{\Gamma}) = \sum_{x \in Sh_R} e_x^{-1} \div \zeta(2)$ for $e_x = |\hat{\Gamma} \cap \mathcal{O}_{D_0}(\mathbb{Q})|$ and $P \div \sum_{x \in Sh_R} e_x^{-1} \theta^*(\phi)(f)(x)$.

Thus the period formula is an adjoint analogue of the mass formula of Siegel–Shimura. The determination of $\mathfrak{m}(\hat{\Gamma})$ was finished by Shimura in 1999 for an arbitrary quadratic space over a totally real field (see §5.2.8 for the explicit formula for the mass).

§9. Schwartz–Bruhat functions; §3.1.3. For a \mathbb{Q} -vector space V , write $V_p := V \otimes_{\mathbb{Z}} \mathbb{Z}_p$ (which is a vector space over a local field \mathbb{Q}_p). A Bruhat function on V_p is a locally constant compactly supported function with values in \mathbb{C} . Write $\mathcal{S}(V_p)$ for the space of Bruhat functions on V_p . For a real vector space V_{∞} , we define $\mathcal{S}(V_{\infty})$ to be the Schwartz space of functions on V_{∞} . Thus $\mathcal{S}(V_{\infty})$ is made of C^{∞} -class functions with all derivatives rapidly decreasing as Euclidean norm of $v \in V_{\infty}$ grows. In other words, $\phi \in \mathcal{S}(V_{\infty})$ if and only if ϕ is of C^{∞} -class and for any polynomial $P(v)$ and any m -th derivative Φ of ϕ , $|P(v)\Phi(v)|$ goes to 0 as $|x| \rightarrow \infty$. Writing $V_{\mathbb{A}}$ for the adelization. We pick a lattice L of V and put $\hat{L} = \prod_p L_p \subset V_{\mathbb{A}(\infty)}$ with $L_p = L \otimes_{\mathbb{Z}} \mathbb{Z}_p$. A Schwartz-Bruhat function on $V_{\mathbb{A}}$ is a finite linear combination of the function of the form $\phi(x) = \prod_v \phi_v(x_v)$ with $\phi_v \in \mathcal{S}(V_v)$ and ϕ_p is the characteristic function of L_p for almost all p .

§10. Weil representation. Let $v(u) = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$, $\text{diag}[a, b] = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ and $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Let $e : \mathbb{A}/\mathbb{Q} \rightarrow S^1$ be the additive character with $e(x_\infty) := \exp(2\pi\sqrt{-1}x_\infty)$. We put $e_v := e|_{\mathbb{Q}_v}$ and for a number field F , we write $e_F = e \circ \text{Tr}_{F/\mathbb{Q}}$ and $e_{F_v} = e_v \circ \text{Tr}_{F_v/\mathbb{Q}_v}$. Let $Q : V \rightarrow F$ be a non-degenerate quadratic form with symmetric bilinear form $s(v, w) = Q(v + w) - Q(v) - Q(w)$ over a number field F . We have the following operator $r(g) \in \text{Aut}(\mathcal{S}(V_?))$ for $m = \dim_F V$, $? = v, \mathbb{A}$ with $u \in F_v$ or $F_{\mathbb{A}}$ and $a \in F_v^\times$:

$r(v(u)) = e_F(uQ(v))\phi(v)$, $r(\text{diag}[a, a^{-1}])\phi(v) = |a|_{F_{\mathbb{A}}}^{m/2}\phi(av)$ and $r(J)\phi(v) = \hat{\phi}(-v) := \int_{V_?} e_F(s(w, -v))\phi(w)dw$ (Fourier transform), where dw is normalized so that $\hat{\hat{\phi}}(v) = \phi(-v)$. If $b, b' \in B(F_{\mathbb{A}})$ (upper triangular Borel subgroup), we extend r to $\Omega = B(F_{\mathbb{A}})JB(F_{\mathbb{A}})$ by $r(bJb') := r(b)r(J)r(b')$. Then if $g, h \in \text{SL}_2(F_?)$ either unipotent, diagonal or J , $r(gh) = \kappa(g, h)r(g)r(h)$ for a 2-cocycle κ on SL_2 with values in S^1 . Write $\text{Mp}(F_?) \subset \text{Aut}(\mathcal{S}(V_?))$ for the group generated by these operators. We have an extension $(*)$ $1 \rightarrow S^1 \rightarrow \text{Mp}(F_?) \xrightarrow{\pi_?} \text{SL}_2(F_?) \rightarrow 1$ with $\text{Mp} \ni w(g) \mapsto g \in \text{SL}_2$. Therefore, the group Mp acts on $\mathcal{S}(V_?)$ by a representation w .

§11. **Weil's theta series.** The extension $(*)$ is split in the following cases:

1. $\dim_F V$ is even (the section is unique and if $V = D_{\sigma, F}$, $b \mapsto \chi_V(a)r(b)$ over $B(F_{\mathbb{A}})$ if $b = v(u) \text{diag}[a, a^{-1}]$);
2. $b \mapsto r(b)$ and also $b \mapsto \chi_V(a)r(b)$ as above over $B(F_{\mathbb{A}})$;
3. Over $\hat{\Gamma}_0(4)$ (canonical);
4. Over $SL_2(F)$ (canonical and coincides with $b \mapsto r(b)$ over $B(F)$).

For the orthogonal group O_V for V and $\phi \in \mathcal{S}(V_{\mathbb{A}})$, we define a function on $Mp(F_{\mathbb{A}}) \times O_V(F_{\mathbb{A}})$ by $\theta(\phi)(g, h) = \sum_{\alpha \in V} w(g) L(h) \phi(\alpha)$, where $(L(h)\phi)(v) = \phi(vh)$ (as usual, $O(F_{\mathbb{A}})$ acts on $V_{F_{\mathbb{A}}}$ from the right). Weil showed that $\theta(\phi)(g, h)$ is real analytic on $Mp(F_{\infty}) \times O_V(F_{\infty})$, left invariant under $SL_2(F) \times O_V(F)$ and right invariant under an open subgroup of $Mp(F_{\mathbb{A}(\infty)}) \times O_V(F_{\mathbb{A}(\infty)})$; in short, an automorphic form on $Mp \times O_V$.

All the details are in Chapter 4.