L-value formula

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Lecture 4: We sketch the proof of the L-value formula for a division quaternion algebra $D_{/\mathbb{Q}}$. The algebra D can be definite or indefinite, though we describe mainly the details only for the indefinite case. Let $B = D \otimes_{\mathbb{Q}} E$ for a real semi-simple quadratic extension E. The non-trivial automorphism $\sigma \in \mathrm{Gal}(E/\mathbb{Q})$ acts on B through the factor E. Since the case $E = \mathbb{Q} \times \mathbb{Q}$ is easier, we mainly assume that E is a field. A key point is the use of the see-saw principle for the decomposition $D_\sigma = Z \oplus D_0$, where $D_\sigma := \{v \in B | v^\iota = v^\sigma\}$ with the reduced norm $N : D_\sigma \to \mathbb{Q}$ and $Z = D_\sigma \cap E$ and $D_0 = \{v \in D_\sigma | \mathrm{Tr}(v) = 0\}$. We need to use the Siegel–Weil formula for D_0. For simplicity, we assume $M = \partial$. The details are in Chapter 5, and the case $M = M_2(\mathbb{Q})$ is dealt with in Section 5.5 of the notes.

§0. **An idea of Waldspurger.** For an elliptic cusp form f, an idea of Waldspurger of computing the period of a theta lift of f for a quadratic space $V = W \oplus W^\perp$ over an orthogonal Shimura subvariety $S_W \times S_{W^\perp} \subset S_V$ is two-folds:
(S) Split $\theta(\phi')(\tau, h, h^\perp) = \theta(\phi)(\tau, h) \cdot \theta(\tau, \phi^\perp)(h^\perp)$ ($h^? \in \mathrm{O}_{W^?}(\mathbb{A})$) for a decomposition $\phi' = \phi \otimes \phi^\perp$ (ϕ and $\phi^\perp$ Schwartz–Bruhat functions on $W_{\mathbb{A}}$ and $W_{\mathbb{A}}^\perp$);
(R) For the theta lift $\theta^*(\phi)(f)(h) = \int_X f(\tau)\theta(\phi)(\tau, h)d\mu$ with an SL(2)-Shimura curve X, the period P over the Shimura subvariety $S \times S^\perp$ (S for $\mathrm{O}(W)$ and $S^\perp$ for $\mathrm{O}(W^\perp)$) is given by:

$$\int_{S\times S^\perp}\int_X f(\tau)\theta(\phi)(\tau;h)d\mu dh \quad (d\mu = \eta^{-2}d\xi d\eta)$$
$$= \int_X f(\tau)\left(\int_{S^\perp}\theta(\phi^\perp)(\tau;h^\perp)dh^\perp\right)\cdot\left(\int_S \theta(\phi_0)(\tau;h_0)dh\right)d\mu.$$

Then invoke the Siegel–Weil formula to convert inner integrals into the Siegel-Weil Eisenstein series $E(\phi)$ and $E(\phi^\perp)$, reaching Rankin-Selberg integral

$$P = \int_X f(\tau)E(\phi^\perp)E(\phi_0)d\mu = L\text{-value.}$$

§**1. Choice of V:** For a $\mathbb{Q}$-vector space V and a $\mathbb{Q}$-algebra A, write $V_A := V \otimes_{\mathbb{Q}} A$. Let $E := \mathbb{Q}[\sqrt{\Delta}]$ be a quadratic extension of $\mathbb{Q}$ with discriminant Δ. Pick a quaternion algebra D over $\mathbb{Q}$ and put $B := D \otimes_{\mathbb{Q}} E$. We let $1 \neq \sigma \in \mathrm{Gal}(E/\mathbb{Q})$ act on D through the factor E. Recall

$$V = D_\sigma := \{v \in B | v^\sigma = v^\iota\} \quad \text{for } v^\iota = \mathrm{Tr}_{B/E}(v) - v.$$

The quadratic form is given by $Q(v) = vv^\sigma = N(v) \in \mathbb{Q}$. We have two cases of isomorphism classes of $(D_{\mathbb{R}}, E_{\mathbb{R}})$. Note $E_{\mathbb{R}} = \mathbb{R} \times \mathbb{R}$; so, we have two cases Case I and Case H. The symbol "I" (resp. "H") indicate D is indefinite (resp. definite). The decomposition we take is

$$V = Z \oplus D_0 \quad Z = \mathbb{Q} \text{ with quadratic form } Q_Z(z) = z^2, \text{ and}$$

$$D_0 := \{v_0 \in \sqrt{\Delta} D | \mathrm{Tr}_{D/\mathbb{Q}}(v) = 0\} \text{ with } Q_0(v) = vv^\sigma = N(v)$$

Signature of D_0 is (1,2) in Case I and (3,0) in Case H, O_{D_0} is almost $D^\times$ and the same for O_{D_σ} and $B^\times$.

§2. **Bruhat functions and majorant.** On $Z = \mathbb{Q}$, for a Dirichlet character ψ modulo N, we regard ψ as a function supported on $\widehat{\mathbb{Z}} \subset Z_{\mathbb{A}^{(\infty)}} = \mathbb{A}^{(\infty)}$. This ψ produces theta series $\sum_{n\in\mathbb{Z}} \psi(n) n^j \mathbf{e}(n^2\tau)$ on $\Gamma_0(4N^2)$ of character $\psi\left(\frac{-1}{}\right)$ and of weight $j + \frac{1}{2}$.

Take a maximal order R of D and take the characteristic function ϕ_0 of $D_{0,\mathbb{A}} \cap \sqrt{\Delta}\widehat{R}$. Here for any lattice L, $\widehat{L} = L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$. This ϕ_0 produces theta series on $\Gamma_0(4\partial\Delta)$ of character $\left(\frac{-\Delta}{}\right)$.

The theta series for D_σ of $\psi \otimes \phi_0$ has level $M = [4N^2, 4\partial\Delta]$. We choose M so that $C|M$ for the conductor C of F.

A positive definite symmetric matrix $P \in M_n(\mathbb{R})$ (or the symmetric bilinear form on $V_{\mathbb{R}}$ associated to P) is a positive majorant of a symmetric matrix S if $PS^{-1} = SP^{-1}$ ($\Leftrightarrow S^{-1}P = P^{-1}S$).

§3. **Schwartz function Ψ on $D_{\sigma,\mathbb{R}}$ in Case I.** The recipe of Hecke–Siegel is to put $\Psi(v) = H(v)\mathbf{e}(\xi N(v) + P(v)\eta\sqrt{-1})$ for $\mathbf{e}(x) = \exp(2\pi\sqrt{-1}x)$ and a harmonic polynomial H, where $P(v) = \frac{1}{2}p(v,v)$ with a positive majorant p of $s(v,v') = \mathrm{Tr}_{B/E}(v^\iota v')$. All positive majorants form the symmetric space $\mathfrak{S}$ of O_{D_σ}.

We identify $(D_{\sigma,\mathbb{R}}, N) = (M_2(\mathbb{R}), \det)$ by $M_2(\mathbb{R}) \ni v \mapsto (v, v^\iota) \in D_{\sigma,\mathbb{R}} \subset D_\mathbb{R} \times D_\mathbb{R}$ and put for $(z,w) \in \mathfrak{H} \times \mathfrak{H}$ on which $B^\times \sim \mathrm{GO}_{D_\sigma}$ acts by $\alpha(z,w) = (\alpha(z), \alpha^\sigma(w))$. For $(z,w) \in \mathfrak{H} \times \mathfrak{H}$, a standard harmonic polynomial of $v \in D_\sigma$ of degree k is given by $[v;z,w]^k = s(v, p(z,w))^k$ for $p(z,w) = \left(\begin{smallmatrix} z \\ 1 \end{smallmatrix}\right)(w,1)J$. For $0 < k \in \mathbb{Z}$, $\Psi(v;\tau,z,w) = \mathrm{Im}(\tau)[v;z,\overline{w}]^k \mathbf{e}(N(v)\overline{\tau} + i\frac{\mathrm{Im}(\tau)}{2|\mathrm{Im}(z)\,\mathrm{Im}(w)|}|[v;z,\overline{w}]|^2)$, for $(\alpha,\beta) \in GL_2(\mathbb{R}) \times GL_2(\mathbb{R})$ (see §3.2.3),

$$\alpha p(z,w)\beta^\iota = p(\alpha(z), \beta(w)) j(\alpha,z) j(\beta,w).$$

This formula is due to Shimura. This function is not a tensor product of functions on $Z_\mathbb{R}$ and $D_{0,\mathbb{R}}$ which causes some difficulty later. For simplicity, we assume $k=2$. See Section 3.2 for Ψ.

§**4. Theta kernel.** Let ϕ be a Schwartz-Bruhat function on $D_{\sigma,\mathbb{A}}$. Let $\mathrm{Mp}(\mathbb{A}) \twoheadrightarrow \mathrm{SL}_2(\mathbb{A})$ be the metaplectic cover constructed by Weil, and $\phi \mapsto \mathbf{w}(g)\phi$ the Weil representation. Noting $B^\times \twoheadrightarrow \mathrm{GO}_{D_\sigma}$ by $v \mapsto h^\iota v h^\sigma$, Siegel–Weil theta series $\theta(g;h)$ is

$$\sum_{\alpha \in D_\sigma} (\mathbf{w}(g)\phi)(h^\iota \alpha h^\sigma) : \mathrm{SL}_2(\mathbb{Q})\backslash \mathrm{Mp}(\mathbb{A}) \times B^\times \backslash B^\times_{\mathbb{A}} \to \mathbb{C}.$$

Write $\widehat{\Gamma} = \widehat{\Gamma}_\phi = \{u \in B^\times_{\mathbb{A}(\infty)} | \theta(g, u^\iota h u^\sigma) = \theta(g,h)\}$.

In Case I, choose $\phi = (\psi \otimes \phi_0)\Psi(v; \sqrt{-1}, \sqrt{-1}, \sqrt{-1})$ and for $g_\tau = \eta^{-1/2}\left(\begin{smallmatrix}\eta & \xi \\ 0 & 1\end{smallmatrix}\right)$ $(\tau = \xi + \eta\sqrt{-1} \in \mathfrak{H})$, we specialize g to g_τ and h to (g_z, g_w) for $(\tau, z, w) \in \mathfrak{H} \times \mathfrak{H} \times \mathfrak{H}$. Then

$$\theta(\tau; z, w) := \theta(g_\tau; g_z, g_w) = \sum_{\alpha \in D_\sigma} (\psi \otimes \phi_0)(\alpha)\Psi(\alpha; \tau, z, w).$$

Set $\theta^*(\phi)(f) := \int_{X_0(M)} f(\tau)\theta(\phi)(\tau; z, w)\eta^{k-2} d\xi d\eta$ $(k = 2)$. Then $\theta^*(\phi)(f)$ is a weight 2 quaternionic modular form on $B^\times$ holomorphic in z and anti-holomorphic in w for $f \in S_2^-(\Gamma_0(M), \psi^{-1}\left(\frac{\Delta}{}\right))$.

§**5. Theta differential form.** To compute the period on $Sh_D = D_+^\times\backslash(D^\times_{\mathbb{A}(\infty)} \times \mathfrak{H}) \subset Sh_B = B^\times\backslash(B^\times_{\mathbb{A}(\infty)} \times \mathfrak{Z}_B)$, we convert $\theta(\tau; z, w)$ into a sheaf valued differential 2–form. If $n = k-2 > 0$, the sheaf comes from the $B^\times$-module

$$L_E(n; A) = \sum_{0\le i,j\le n} AX^{n-j}Y^jX'^{n-i}Y'^i$$

with $B^\times$-action $\gamma P(X,Y;X',Y') = P((X,Y)^t\gamma^\iota;(X',Y')^t\gamma^{\sigma\iota})$. As we assumed $k = 2$ (i.e., $n = 0$), we have $L(n; A) = A$.

By putting $\Theta = \theta(\phi)(\tau; z, w)dz \wedge d\overline{w}$ for $n = k-2$, we get $\mathbb{C}$-valued Γ_ϕ-invariant differential form. The period we like to compute is

$$P = P_1(\theta^*(\phi)(f)) = \int_{Sh_D}\int_{X_0(M)} f(-\overline{\tau})\Theta(\tau; z, z)d\xi d\eta.$$

We integrate over Sh_D by a measure $d\mu$ given by $y^{-2}dxdy$ over $\mathfrak{H}$ and $\int_{\widehat{\Gamma}} d\mu = 1$.

§6. Siegel–Weil Eisenstein series; §4.4.2. Recall the explicit section $\mathbf{r} : B \hookrightarrow \mathrm{Mp}$ of the representation $\mathbf{w}$ as follows:

$$\mathbf{r}(\mathrm{diag}[a, a^{-1}])\phi(v) = |a|_{\mathbb{A}}^{3/2}\phi(av), \quad \mathbf{r}\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\phi(v) = \mathbf{e}(uN(v))\phi(v).$$

For the standard Borel subgroup $B \subset \mathrm{SL}_2$, the function $g \mapsto (\mathbf{r}(g)\phi)(0)$ is left $B(\mathbb{Q})$ invariant. Siegel–Weil Eisenstein series is

$$E(\phi)(g; s) = \sum_{\gamma \in B(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{Q})} (\mathbf{w}(\gamma g)\phi)(0)|a(\gamma g)|_{\mathbb{A}}^{s},$$

where $g = \mathrm{diag}[a(g), a(g)^{-1}]\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} c$ for an element c in the maximal compact subgroup by Iwasawa decomposition.

The Siegel–Weil formula by Kudla-Rallis and Sweet is

$$E(\phi)(g; 0) = \int_S \theta(\phi)(g, h)d\omega \quad \text{for the Tamagawa measure } d\omega.$$

The ratio $\mathfrak{m} = \mathfrak{m}(\widehat{\Gamma}) = d\mu/d\omega$ is the mass of Siegel–Shimura, which is an arithmetic rational number times $\zeta(2)/\pi$ in Case I and $\zeta(2)/\pi^2$ in Case H. Later we describe the Tamagawa measure, the metaplectic group and Weil representation in more details.

§7. Conclusion in Case I; §5.3. Decomposing $v = a \oplus v_0$ in $D_\sigma = Z \oplus D_0$, we have for $k = 2$

$$[a + v_0; z, \overline{z}]^k = ([a; z, \overline{z}] + [v_0; z, \overline{z}])^k = \sum_{j=0}^{k} \binom{k}{j} (a(z - \overline{z}))^{k-j} [v_0; z, \overline{z}]^j.$$

Thus we have $\phi = \sum_{j=0}^{k} \binom{k}{j} \phi^Z_{k-j} \phi^{D_0}_j$ with infinity part $\Psi^?_j$ of $\phi^?_j$ given by
$\Psi^{D_0}_j := (z - \overline{z})^{-j} [v_0; z, \overline{z}]^j \mathbf{e}(N(\mathfrak{x})\overline{\tau} + \frac{i\,\mathrm{Im}(\tau)\|[v_0;z,\overline{z}]\|^2}{2\,\mathrm{Im}(z)^2})$, $\Psi^Z_j := a^j \mathbf{e}(a^2\tau)$
with $(\mathbf{w}(b)\phi^{D_0}_j)(0) = 0$ and $E(\phi^{D_0}_j)|_{B(\mathbb{A})} = 0$ unless $j = 0$, and we reach Rankin convolution of $\theta(\phi^Z_k) = \sum_{n \in \mathbb{Z}} \psi(n) n^k \mathbf{e}(n^2 z)$ and f over $B(\mathbb{Q})\backslash B(\mathbb{A})/B(\widehat{\mathbb{Z}}) \cong [0,1) \times \mathbb{R}^\times_+$, which produces (see [Sh75])

$$\zeta(2)P = \mathfrak{m}2^{-2k} * (2\pi)^{-k}\Gamma(k) L^{(s)}(1, Ad(\rho_f) \otimes \left(\frac{\Delta}{}\right))$$

with a simple constant $*$. Here $L^{(s)}$ means we remove Euler factors at $p|C$ with either $f|U(p) = 0$ or $\psi(p) = 0$.

§8. Conclusion in Case H; §5.2. The choice of the Bruhat function ϕ is the same as in Case I. As a $\mathbb{C}$-valued function, set

$$\mathbf{\Psi}(\tau; v; \mathbf{x}) = \mathbf{e}(N(v)\tau).$$

Again in exactly the same way, for

$$\theta^*(\phi)(f) := \int_{X_0(M)} \theta(\phi)(\tau; g) f(\tau)\eta^{k-2} d\xi d\eta \quad (k = 2)$$

and $P = \int_S \theta^*(\phi)(f) d\mu$, we conclude for a simple constant c'

$$\zeta(2)P = 2\mathfrak{m} *' (2\pi)^{-k+1}\Gamma(k)L^{(s)}(1, Ad(\rho_f) \otimes \left(\frac{\Delta}{}\right)).$$

Writing the point set $S = \{x\}_{x \in Sh_R}$, $\mathfrak{m}(\widehat{\Gamma}) = \sum_{x \in Sh_R} e_x^{-1} \doteqdot \zeta(2)$ for $e_x = |\widehat{\Gamma} \cap O_{D_0}(\mathbb{Q})|$ and $P \doteqdot \sum_{x \in Sh_R} e_x^{-1} \theta^*(\phi)(f)(x)$.

Thus the period formula is an adjoint analogue of the mass formula of Siegel–Shimura. The determination of $\mathfrak{m}(\widehat{\Gamma})$ was finished by Shimura in 1999 for an arbitrary quadratic space over a totally real field (see §5.2.8 for the explicit formula for the mass).

§**9. Schwartz–Bruhat functions;** §**3.1.3.** For a $\mathbb{Q}$-vector space V, write $V_p := V \otimes_{\mathbb{Z}} \mathbb{Z}_p$ (which is a vector space over a local field $\mathbb{Q}_p$). A Bruhat function on V_p is a locally constant compactly supported function with values in $\mathbb{C}$. Write $\mathcal{S}(V_p)$ for the space of Bruhat functions on V_p. For a real vector space V_∞, we define $\mathcal{S}(V_\infty)$ to be the Schwartz space of functions on V_∞. Thus $\mathcal{S}(V_\infty)$ is made of C^∞–class functions with all derivatives rapidly decreasing as Euclidean norm of $v \in V_\infty$ grows. In other words, $\phi \in \mathcal{S}(V_\infty)$ if and only if ϕ is of C^∞–class and for any polynomial $P(v)$ and any m–th derivative Φ of ϕ, $|P(v)\Phi(v)|$ goes to 0 as $|x| \to \infty$. Writing $V_{\mathbb{A}}$ for the adelization. We pick a lattice L of V and put $\widehat{L} = \prod_p L_p \subset V_{\mathbb{A}^{(\infty)}}$ with $L_p = L \otimes_{\mathbb{Z}} \mathbb{Z}_p$. A Schwartz-Bruhat function on $V_{\mathbb{A}}$ is a finite linear combination of the function of the form $\phi(x) = \prod_v \phi_v(x_v)$ with $\phi_v \in \mathcal{S}(V_v)$ and ϕ_p is the characteristic function of L_p for almost all p.

§**10. Weil representation.** Let $v(u) = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$, $\mathrm{diag}[a,b] = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ and $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Let $\mathbf{e} : \mathbb{A}/\mathbb{Q} \to S^1$ be the additive character with $\mathbf{e}(x_\infty) := \exp(2\pi\sqrt{-1}x_\infty)$. We put $\mathbf{e}_v := \mathbf{e}|_{\mathbb{Q}_v}$ and for a number field F, we write $\mathbf{e}_F = \mathbf{e} \circ \mathrm{Tr}_{F/\mathbb{Q}}$ and $\mathbf{e}_{F_v} = \mathbf{e}_v \circ \mathrm{Tr}_{F_v/\mathbb{Q}_v}$. Let $Q : V \to F$ be a non-degenerate quadratic form with symmetric bilinear form $s(v,w) = Q(v+w) - Q(v) - Q(w)$ over a number field F. We have the following operator $\mathbf{r}(g) \in \mathrm{Aut}(\mathcal{S}(V_?))$ for $m = \dim_F V$, $? = v, \mathbb{A}$ with $u \in F_v$ or $F_\mathbb{A}$ and $a \in F_v^\times$:
$\mathbf{r}(v(u)) = \mathbf{e}_F(uQ(v))\phi(v)$, $\mathbf{r}(\mathrm{diag}[a,a^{-1}]\phi(v) = |a|_{F_\mathbb{A}}^{m/2}\phi(av)$ and
$\mathbf{r}(J)\phi(v) = \widehat{\phi}(-v) := \int_{V_?} \mathbf{e}_F(s(w,-v))\phi(w)dw$ (Fourier transform), where dw is normalized so that $\widehat{\widehat{\phi}}(v) = \phi(-v)$. If $b, b' \in B(F_\mathbb{A})$ (upper triangular Borel subgroup), we extend $\mathbf{r}$ to $\Omega = B(F_\mathbb{A})JB(F_\mathbb{A})$ by $\mathbf{r}(bJb') := \mathbf{r}(b)\mathbf{r}(J)\mathbf{r}(b')$. Then if $g, h \in \mathrm{SL}_2(F_?)$ either unipotent, diagonal or J, $\mathbf{r}(gh) = \kappa(g,h)\mathbf{r}(g)\mathbf{r}(h)$ for a 2-cocycle κ on SL_2 with values in S^1. Write $\mathrm{Mp}(F_?) \subset \mathrm{Aut}(\mathcal{S}(V_?))$ for the group generated by these operators. We have an extension $(*)$ $1 \to S^1 \to \mathrm{Mp}(F_?) \xrightarrow{\pi_?} \mathrm{SL}_2(F_?) \to 1$ with $\mathrm{Mp} \ni \mathbf{w}(g) \mapsto g \in \mathrm{SL}_2$. Therefore, the group Mp acts on $\mathcal{S}(V_?)$ by a representation $\mathbf{w}$.

§**11. Weil's theta series.** The extension $(*)$ is split in the following cases:

1. $\dim_F V$ is even (non-canonically but the section can be made to coincide with $b \mapsto \mathbf{w}(b)$ over $B(F_{\mathbb{A}})$);
2. $b \mapsto \mathbf{r}(b)$ over $B(F_{\mathbb{A}})$;
3. Over $\widehat{\Gamma}_0(4)$ (canonical);
4. Over $SL_2(F)$ (canonical and coincides with $b \mapsto \mathbf{r}(b)$ over $B(F)$.

For the orthogonal group O_V for V and $\phi \in \mathcal{S}(V_{\mathbb{A}})$, we define a function on $\mathrm{Mp}(F_{\mathbb{A}}) \times O_V(F_{\mathbb{A}})$ by $\theta(\phi)(g,h) = \sum_{\alpha \in V} \mathbf{w}(g) L(h) \phi(\alpha)$, where $(L(h)\phi)(v) = \phi(vh)$ (as usual, $O(F_{\mathbb{A}})$ acts on $V_{F_{\mathbb{A}}}$ from the right). Weil showed that $\theta(\phi)(g,h)$ is real analytic on $\mathrm{Mp}(F_\infty) \times O_V(F_\infty)$, left invariant under $SL_2(F) \times O_V(F)$ and right invariant under an open subgroup of $\mathrm{Mp}(F_{\mathbb{A}(\infty)}) \times O_V(F_{\mathbb{A}(\infty)})$; in short, an automorphic form on $\mathrm{Mp} \times O_V$.

All the details are in Chapter 4.