Hecke equivariance in the simplest case

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Lecture 3: We first reduce the proof of $\sum_{n=1}^{\infty}(\mathcal{F}, \mathcal{G}|T(n))q^n \in S_2(\Gamma_0(M))$ for a definite quaternion algebra $D_{/\mathbb{Q}}$ with Φ in Choice A to a duality theorem between Hecke algebra and the space of cusp forms. In later lectures, we compute more generally the q-expansion of the theta descent of a quaternionic automorphic form via $\theta(\phi)$ which coincides with $\sum_{n=1}^{\infty}(\mathcal{F}, \mathcal{G}|T(n))q^n$ if $\phi = \Phi$ and $E = \mathbb{Q} \times \mathbb{Q}$.

§**0. Reduction to Duality Theorem.** We recall $M = \partial N_0$ with $(N_0, \partial) = 1$. Let $h_k(M; A)$ be the subalgebra of $\mathrm{End}_{\mathbb{C}}(S_k(\Gamma_0(M)))$ generated over A by Hecke operators $T(n)$ and

$$S_k(\Gamma_0(M); A) = S_k(\Gamma_0(M)) \cap A[[q]].$$

Recall

Duality theorem *The space $S := S_k(\Gamma_0(M); A)$ is A-dual of $H := h_k(M; A)$ such that for a linear form $\varphi : h_k(M; A) \to A$,* $\boxed{\sum_{n=1}^{\infty} \varphi(T(n))q^n}$ $\in S_k(\Gamma_0(M); A)$. *Writing $f = \sum_{n=1}^{\infty} a(n, f)q^n \in S$, the pairing $\langle \cdot, \cdot \rangle : H \times S \to A$ is given by $\langle h, f \rangle = a(1, f|h)$.*

By Jacquet-Langlands correspondence, $S(A) = H^0(Sh_R, A)$ is a module over $h_2(M; A)$. Then applying the above theorem to the linear form $h_2(M; A) \ni h \mapsto (\mathcal{F}, \mathcal{G}|h)$, we get the assertion $\theta_*(\Phi)(\mathcal{F} \otimes \mathcal{G}) = \sum_{n=1}^{\infty} (\mathcal{F}, \mathcal{G}|T(n))q^n = \sum_{n=1}^{\infty} (\mathcal{F}|T(n), \mathcal{G})q^n \in S_2^{\partial\text{-new}}(\Gamma_0(M); A)$ for $\mathcal{F} \otimes \mathcal{G} \in S(A) \otimes_A S(A)$ in Theorem A.

§**1. Hecke operators.** Define a semi-group of the Eichler order

$\Delta(M) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in M_2(\mathbb{Z}) \middle| c \equiv 0, (a, M) = 1 \bmod M, ad - bc > 0 \right\}.$

Then $\Gamma_0(M)\mathcal{T}(n)\Gamma_0(M) = \mathcal{T}(n)$ for

$$\mathcal{T}(n) := \{\alpha \in \Delta(M) | \det(\alpha) = n\},$$

and we have a disjoint decomposition [IAT, 3.36]

(C1) $$\mathcal{T}(n) = \bigsqcup_{0<a|n, ad=n, (a,M)=1} \bigsqcup_{b=0}^{d-1} \Gamma_0(M)\alpha_{a,d,b},$$

where $\alpha_{a,d,b} := \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix} \right)$. For $f : \mathfrak{H} \to \mathbb{C}$ and $\alpha \in \Delta(M)$, we define $f|_k\alpha := \det(\alpha)^{k-1} f(\alpha(\tau)) j(\alpha, \tau)^{-k}$ for $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)$, $\alpha(\tau) = \frac{a\tau + b}{c\tau + d}$ and $j(\alpha, \tau) = c\tau + d$. Since $j(\alpha\beta, \tau) = j(\alpha, \beta(\tau)) j(\beta, \tau)$, we have $f|_k(\alpha\beta) = (f|_k\alpha)|_k\beta$. Then $S_k(\Gamma_0(M))$ is made of holomorphic functions with $f|_k\gamma = f$ for all $\gamma \in \Gamma_0(N)$ and converging expansion $f|_k\alpha = \sum_{0<n\in\mathbb{Q}} a(n, f|_k\alpha) q^n$ for $q^n = \exp(2\pi\sqrt{-1}\tau)$ for all $\alpha \in SL_2(\mathbb{Z})$. We define $f|T(n)(\tau) := \sum_{a,d,b} f|_k\alpha_{a,d,b}$ as in the above decomposition of $\mathcal{T}(n)$. Then $T(n) \in \mathrm{End}_{\mathbb{C}}(S_k(\Gamma_0(M)))$.

§2. **Hecke relation and integrality.** If $f(\tau)$ is given by Fourier expansion $f = \sum_{n=1}^{\infty} a(n,f)q^n$, by computation, the decompositions of $\mathcal{T}(n)$ in §1 tells us the following relation

(R1) $$a(m, f|T(n)) = \sum_{0<d|(m,n),(d,M)=1} d^{k-1} a(\frac{mn}{d^2}, f).$$

Form this, it is clear $T(m)T(n) = T(n)T(m)$ and for $l|M$, writing $U(l)$ for $T(l)$, $a(m, f|U(l)) = a(ml, f)$ and hence $U(l^n) = U(l)^n$.

Define $S_k(\Gamma_0(M); A) := S_k(\Gamma_0(M)) \cap A[[q]]$ for a subalgebra $A \subset \mathbb{C}$. By Shimura [EMI,§4.1.5], $S_k(\Gamma_0(M); A) = S(\Gamma_0(M); \mathbb{Z}) \otimes_{\mathbb{Z}} A$, and hence it is legitimate to define

(I) $$S_k(\Gamma_0(M); A) = S(\Gamma_0(M); \mathbb{Z}) \otimes_{\mathbb{Z}} A \subset A[[q]]$$

for any algebra A not necessarily in $\mathbb{C}$. Then we define $h_k(M; A) := A[T(n)|n = 1, 2, \ldots] \subset \mathrm{End}_A(S_k(\Gamma_0(M); A))$, which is a commutative A-algebra.

§3. **Duality Theorem.** *Define the pairing between* $h_k = h_k(M; A)$ *and* $S_k = S_k(\Gamma_0(M); A)$ *by* $\langle h, f\rangle = a(1, f|h)$. *Then the pairing is perfect; i.e.,*

$$\mathrm{Hom}_A(h_k, A) \cong S_k \quad \textit{and} \quad \mathrm{Hom}_A(S_k, A) \cong h_k.$$

In particular, $\varphi \in \mathrm{Hom}_A(h_k, A)$ *is sent to* $\sum_{n=1}^{\infty} \varphi(T(n))q^n$.

We prove this by steps.

Step 1: $A = \mathbb{Q}$**.** Then by (I), it is valid for all $\mathbb{Q}$-algebras A. By (I), $\mathrm{rank}_{\mathbb{Z}} S_k(\Gamma_0(M); \mathbb{Z}) < \infty$; so, $\mathrm{rank}_{\mathbb{Z}} h_k(M; \mathbb{Z}) < \infty$. Tensoring $\mathbb{Q}$, we need to show the pairing is non-degenerate. By (R1), $\langle T(n), f\rangle = a(n, f)$. Thus if $\langle T(n), f\rangle = 0$ for all n, the coefficients $a(n, f) = 0$ for all n, which implies $f = 0$.

Pick $h \in h_k(M; \mathbb{Q})$. Suppose $\langle h, f\rangle = 0$ for all $f \in S_k(\Gamma_0(\mathbb{Q}); \mathbb{Q})$. By $\langle hT(n), f\rangle = a(1, f|hT(n)) = a(1, f|T(n)h) = \langle h, f|T(n)\rangle = 0$ and $\langle hT(n), f\rangle = \langle T(n), f|h\rangle = a(n, f|h)$, we find $f|h = 0$ for all f, and $h = 0$. □

§4. Conclusion of the proof.

Step 2: $A = \mathbb{F}_p$**.** By Step 1, $\dim_{\mathbb{F}_p} S_k(\Gamma_0(M); \mathbb{F}_p) < \infty$; so, $\dim_{\mathbb{F}_p} h_k(M; \mathbb{F}_p) < \infty$, and the same argument proves the non-degeneracy.

Step 3: $A = \mathbb{Z}$**.** Taking a $\mathbb{Z}$-basis $\{h_i\}_i$ of h_k and $\{f_j\}_j$ of S_k. Then the matrix of the pairing $S = (\langle h_i, f_j \rangle)_{i,j}$ satisfies $\det(S) \not\equiv 0 \mod p$ for all prime p by Step 2. Therefore, $\det(S) = \pm 1$, finishing the proof for $A = \mathbb{Z}$. The by (I), we get the result for general A.

Final step. Pick $\varphi \in \mathrm{Hom}_A(h_k, A)$. Then by the perfectness of the paring, we find $f \in S_k(\Gamma_0(M); A)$ such that $\langle h, f \rangle = \varphi(h)$. Since $a(n, f) = \langle T(n), f \rangle = \varphi(T(n))$, we find $f = \sum_{n=1}^{\infty} \varphi(T(n)) q^n$ as desired. $\square$

§**5. Hecke operators on $Sh = D^\times\backslash D^\times_{\mathbb{A}}$.** Recall the Eichler order R of level $M = \partial N_0$. If $p \nmid M$, $R_p = R \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \Delta(M)_p$ for the p-adic closure $\Delta(M)_p$ of $\Delta(M)$ in $M_2(\mathbb{Z}_p)$. If $p|N_0$, by the condition $(a, M) = 1$, $\Delta(M)_p \subsetneq R_p$. Define

$$\widehat{\Delta}^D(M) = \prod_{p\nmid\partial} \Delta(M)_p^\iota \times \prod_{p|\partial} R_p,$$

where $\alpha + \alpha^\iota = \mathrm{Tr}(\alpha)$ and $\alpha\alpha^\iota = N(\alpha)$. Put $\widehat{\mathcal{T}}(n) = \{\alpha \in \widehat{\Delta}^D(M) | \det(\alpha)\widehat{\mathbb{Z}} = n\widehat{\mathbb{Z}}\}$. Then $\widehat{R}^\times\widehat{\mathcal{T}}(n)\widehat{R}^\times = \widehat{\mathcal{T}}(n)$ and similarly to (C1), we have

(C2) $$\widehat{\mathcal{T}}(n) = \bigsqcup_{0<a|n,ad=n} \bigsqcup_{b=0}^{d-1} \alpha_{d,a,b}\widehat{R}^\times.$$

We need to use right coset decomposition for adelic automorphic forms, since $\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})/\widehat{\Gamma}_0(M)\mathrm{SO}_2(\mathbb{R}) \cong \Gamma_0(M)\backslash\mathfrak{H}$ by $x \mapsto x_\infty(\sqrt{-1})$ (assuming $\det(x_\infty) > 0$); in particular, the right multiplication $xa = \alpha xu$ by a for $a \in \widehat{\Delta}^{M_2(\mathbb{Q})} \cap \mathrm{GL}_2(\mathbb{A}^{(\infty)})$ and $\alpha \in \mathrm{GL}_2(\mathbb{Q})$ is converted into the left multiplication $\alpha^{-1}x_\infty(\sqrt{-1})$ on $\mathfrak{H}$.

§6. **Hecke operators on** $S(A)$. For a function $\mathcal{F} : Sh_R = D^\times\backslash D^\times_{\mathbb{A}}/\widehat{R}^\times \to A$, we define $T(n)\mathcal{F}(s) = \sum_{a,d,b}\mathcal{F}(s\alpha_{d,a,b})$. By the above right coset description, we have $T(n)\mathcal{F} \in S(A)$. We may regard $\mathcal{T}(n)$ as the characteristic function T_n of the open set $\mathcal{T}(n)$ of $D^\times_{\mathbb{A}(\infty)}$. For the Haar measure $d\mu$ with $\int_{\widehat{R}^\times} d\mu = 1$, $T(n)\mathcal{F}(x) = \int_{\mathcal{T}(n)} \mathcal{F}(xy)d\mu(y)$ coincide with

$$T_n * \mathcal{F}(x) = \int_{D^\times_{\mathbb{A}(\infty)}} \mathcal{F}(xy)T_n(y)d\mu(y) = \int_{D^\times_{\mathbb{A}(\infty)}} \mathcal{F}(y)T_n(yx^{-1})d\mu(y).$$

Then the convolution product

$$T_n * T_m(x) = \int_{D^\times_{\mathbb{A}(\infty)}} T_n(y)T_m(yx^{-1})d\mu(y) = \int_{D^\times_{\mathbb{A}(\infty)}} T_n(yx)T_m(y)d\mu(y)$$

actually gives the Hecke operator product $T(n)T(m)$. We can verify by computation

(R2) $$T(m)T(n) = \sum_{0<d|(m,n),(d,M)=1} \langle d\rangle T\left(\frac{mn}{d^2}\right),$$

where $\langle d\rangle\mathcal{F}(x) = \mathcal{F}(xd)$.

§7. **Jacquet–Langlands correspondence: Section 3.4.** Here is a version of the Jacquet–Langlands correspondence discussed in §3.4.5:

We have an $\mathbb{C}$-linear isomorphism $JL : S(\mathbb{C}) \to S_2^{\partial\text{-new}}(\Gamma_0(M))$ such that $JL \circ T(n) = T(n) \circ JL$.

This map is not canonical. By the theta correspondence, define $\Theta : S(A) \otimes S(A) \to S_2(\Gamma_0(M); A)$ by

$$\Theta(\mathcal{F} \otimes \mathcal{G}) = \theta_*(\Phi)(\mathcal{F} \otimes \mathcal{G}) = \sum_{n=1}^{\infty} (\mathcal{F}, T(n)\mathcal{G})q^n.$$

Let $S_2^{\partial\text{-new}}(\Gamma_0(N); A) = S_2^{\partial\text{-new}}(\Gamma_0(M)) \cap S_2(\Gamma_0(M); A)$ for $A \subset \mathbb{C}$ and $S_2^{\partial\text{-new}}(\Gamma_0(M); A) := S_2^{\partial\text{-new}}(\Gamma_0(M); \mathbb{Z}) \otimes_{\mathbb{Z}} A$ for general A. The morphism Θ factors through $S_2^{\partial\text{-new}}(\Gamma_0(M); A)$. Define $h(A) := A[T(n)|n = 1, 2, \ldots] \subset \mathrm{End}_A(S(A))$. Then we have

Corollary. *We have a canonical isomorphism $h_2(M; A) \twoheadrightarrow h(A)$ sending $T(n)$ to $T(n)$ for all n given by $h_2(M; A) \ni T(n) \mapsto T(n)|_{S^{\partial\text{-new}}(\Gamma_0(M);A)} \mapsto T(n) \in h(A)$.*

§8. $(\mathcal{F},\mathcal{G}|T(n))$ as a period. Write $\mathcal{T}(n)=\bigsqcup_{a\in S(n)}\widehat{R}^\times a\widehat{R}^\times$. For each coset $[a]:=\widehat{R}^\times a\widehat{R}^\times$, choose a representative set $U(n)$ for $\widehat{R}^\times/\widehat{R}^\times_a$ for $\widehat{R}^\times_a:=(a\widehat{R}^\times a^{-1}\cap\widehat{R}^\times)$; so,

$$\widehat{R}^\times=\bigsqcup_{u\in U(n)}u(a\widehat{R}^\times a^{-1}\cap\widehat{R}^\times).$$

Multiplying $a\widehat{R}^\times a^{-1}$ from the right, we get

$$\widehat{R}^\times a\widehat{R}^\times a^{-1}=\bigsqcup_{u\in U(n)}ua\widehat{R}^\times a^{-1}\Leftrightarrow\widehat{R}^\times a\widehat{R}^\times=\bigsqcup_{u\in U(n)}ua\widehat{R}^\times.$$

We have two morphisms $Sh_a:=Sh_{\widehat{R}^\times_a}=D^\times\backslash D^\times_{\mathbb{A}}/\widehat{R}^\times_a D^\times_\infty\rightrightarrows Sh_R$ given by $x\widehat{R}^\times_a\mapsto x\widehat{R}^\times$ and $x\widehat{R}^\times_a\mapsto xa\widehat{R}^\times$; so, $Sh_a\hookrightarrow Sh_{\widehat{R}^\times}\times Sh_{\widehat{R}^\times}$. Let $Sh_n:=\bigcup_{a\in S(n)}Sh_a\subset Sh_{\widehat{R}^\times}\times Sh_{\widehat{R}^\times}$. One can verify

$$(\mathcal{F},\mathcal{G}|T(n))=\int_{Sh_n}\mathcal{F}(x)\mathcal{G}(x)d\mu'=:(\mathcal{F}\otimes\mathcal{G},Sh_n)\ \text{(homology pairing)}$$

where $d\mu'$ is the Dirac measure on Sh_n. Therefore

$$\theta_*(\Phi)(\mathcal{F}\otimes\mathcal{G})=\sum_{n=1}^{\infty}(\mathcal{F}\otimes\mathcal{G},Sh_n)q^n.$$

§**9. Indefinite case.** Now we assume that $D_\infty = D \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{R})$ (an indefinite division quaternion algebra). We take the Eichler order R of level $M = \partial N_0$ as before, and identifying $D_\infty = M_2(\mathbb{R})$, we define $Sh_R = D_+^\times \backslash (D^\times_{\mathbb{A}(\infty)} \times \mathfrak{H})$, where $D_+^\times = \{\alpha \in D^\times | N(\alpha) > 0\}$ which acts on the upper half complex plane by linear fractional transformation.

Since D is division, Sh_R has the universal abelian surface $\mathbf{A}$ with R-multiplication and a level structure. Let $\omega_{/Sh_R}$ be the Hodge line bundle relative to $\pi : \mathbf{A} \to Sh_R$; so, $\omega^{\otimes 2} = \det(\pi_*\Omega_{\mathbf{A}/Sh_R})$, and $\omega^{\otimes 2} \cong \Omega^1_{X_0(M)/A}$ is $(\pi_*\Omega_{\mathbb{E}/Y_0(M)})^{\otimes 2}$ extended to $X_0(M)$ by 0 at cusps. We consider

$$S_k(Sh_R; A) := H^0(Sh_R, \omega^{\otimes 2}_{/A}),\ H^1(Sh_R, A)$$

on which Hecke operator $T(n)$ acts as correspondences. This definition match with the one already given $S_2(\Gamma_0(M); A)$ by the q-expansion principle.

§**10. Hecke algebras.** We have $S_2(Sh_R;\mathbb{C}) \oplus S_2^-(Sh_R;\mathbb{C}) \cong H^1(Sh_R,\mathbb{C})$ for the complex conjugation $S_2^-(Sh_R;\mathbb{C})$ of $S_2(Sh_R;\mathbb{C})$ by associating the cohomology class $[\mathcal{F}]$ of $\omega(\mathcal{F}) = 2\pi i\mathcal{F}(\tau)d\tau$ (or its complex conjugate). By the Hecke equivariance, we have $H(A) := A[T(n)|n = 1, 2, \ldots] \subset \mathrm{End}_A(S_2(Sh_R; A))$. We have the Poincaré duality

$$(\cdot,\cdot) : H^1(Sh_R,\mathbb{C}) \times H^1(Sh_R,\mathbb{C}) \to H^2(Sh_R,\mathbb{C}(1)) = \mathbb{C}.$$

In the same manner as in §8, we define the correspondence $Sh_n \subset Sh_R \times Sh_R$ as a Shimura subcurve. Choosing a good Schwartz function Φ_∞ we will specify in a later lecture and the Bruhat function $\Phi^{(\infty)}$ in case A, in the same manner as in the definite case, we have $\theta_*(\Phi) : H^1(Sh_R \times Sh_R, A) = H^1(Sh_R, A) \otimes H^1(Sh_R, A) \to S_2^{\partial\text{-new}}(\Gamma_0(M); A)$ with

$$\theta_*(\Phi)(\mathcal{F} \otimes \mathcal{G}) = \sum_{n=1}^{\infty} ([\mathcal{F}] \otimes [\mathcal{G}], Sh_n)q^n = \sum_{n=1}^{\infty} ([\mathcal{F}], [\mathcal{G}]|T(n))q^n.$$

We will prove this for Sh_α for general D_σ in later lectures.

§**11. Period relation.** Complex conjugation c as an element of Aut$(Sh_R(\mathbb{C}))$ acts on $H^1(Sh_R, A)$ whose $\pm$-eigenspace is denoted by $H^1(Sh_R, A)[\pm]$. For a generator $\mathcal{F}$ (resp. $\mathcal{F}_\pm$) of $S_2(Sh_R; \mathcal{W})[\lambda]$ and $H^1(Sh_R, A)[\pm, \lambda]$, define $\omega_\pm(\mathcal{F}) = \omega(\mathcal{F}) \pm \overline{\omega(\mathcal{F})} = \Omega^D_\pm \mathcal{F}_\pm$. If $\theta^*(\Phi)(f) = \Omega^D(\mathcal{F}_+ \otimes \mathcal{F}_-)$ for $\Omega^D \in \mathbb{C}$, again we find $\Omega^D \doteq \Omega_+\Omega_-$ up to units in $\mathcal{W}$ by the $\mathcal{R} = \mathbb{T}$ theorem.

Period Theorem: If $S_2^{new}(\Gamma_0(M); \mathcal{W})[\lambda] = \mathcal{W}f$,

$$\Omega^D = \Omega_+\Omega_- \overset{\text{Faltings}}{=} \Omega^D_+\Omega^D_-$$

up to $\mathcal{W}$-units. This follows from the fact that Shimura's abelian variety A_f in $J_0(M)$ associated to f and $A_\mathcal{F}$ in the jacobian of Sh_R associated to $\mathcal{F}$ have the same Hasse–Weil L-function for H^1, and hence by Faltings, they are isogenous over $\mathbb{Q}$.