EXTENSIONS AND THE EXCEPTIONAL ZERO 
OF THE ADJOINT SQUARE $L$-FUNCTIONS

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Take a totally real field $F$ with integer ring $O$ as a base field. We fix an identification $\iota : \Q_p \cong C \supset \Q$. Fix a prime $p > 2$, and write $\Sigma$ for the set of prime factors of $p$ in $F$. Start with a holomorphic automorphic representation $\pi$ of $GL_2(F_A)$ (a Hilbert modular Hecke eigenform) which is spherical and nearly $p$-ordinary at $\Sigma$. Then we have the compatible system of $\lambda$-adic representations $\rho = \{\rho_\lambda\}_\lambda$ of $\pi$, and if $\lambda \nmid p$, $\rho_\lambda(Frob_p)$ is unramified and has two eigenvalues a $p$-adic unit eigenvalue $\alpha$ (with respect to $\iota$) and a $p$-adic nonunit $\beta$. When we consider a $p$-adic member of $\rho$, it is supposed to be associated to $i_p = \iota^{-1} : \Q \hookrightarrow \Q_p$. We consider the Galois stable subspace $Ad(\rho) \subset \rho \otimes ^t \rho^{-1} = \text{End}(\rho)$ with zero trace (whose Galois action is given by conjugation). The Euler factor at $p$ of $L(s, Ad(\rho))$ is then given by

$$[(1 - \alpha \beta^{-1}p^{-s})(1 - p^{-s})(1 - \alpha^{-1} \beta p^{-s})]^{-1}.$$ 

The $p$-adic $L$-function $L_p(s, Ad(\rho))$, whose value at 1 is a constant multiple of $(1 - pp^{-1})(1 - \alpha^{-1} \beta p^{-1})(1 - p\alpha \beta^{-1}p^{-1})L(1, Ad(\rho))$, has an exceptional 0 at $s = 1$ (corresponding to the Frobenius eigenvalue = 1 at $p|\rho$) whose order is the number of such Euler factors $r = |\Sigma|$ if the $L$-invariant $L(Ad(\rho))$ of $Ad(p_\rho)$ does not vanish. The $L$-invariant $L(Ad(\rho))$ is defined by the following (hypothetical) formula:

$$\frac{d^r L_p(s, Ad(\rho))}{ds^r} \bigg|_{s=1} = L(Ad(\rho)) \frac{L(1, Ad(\rho))}{\text{a period}}.$$ 

The appearance of the trivial zero is always true without assuming unramifiedness of $\pi$ or $\rho$ at $p$ for the adjoint square $L$-functions, and this is a peculiar point when we study the $L$-invariant of the adjoint square $L$. Indeed, by the near ordinarity, $\rho_{p_\rho}|_{D_{p_\rho}} \cong \left( \begin{array}{cc} \delta_\rho & * \\ 0 & \delta_\rho \end{array} \right)$, and hence, the semisimplified $Ad(\rho)|_{D_{p_\rho}}$ has eigenvalue 1 for $Frob_{p_\rho}$. Since Greenberg has given a Galois cohomological definition of the $L$-invariant without recourse to the analytic $p$-adic $L$-function, we can discuss the adjoint square $L$-invariant using his definition, and we would like to relate it to differential calculus of $p$-adic analytic families lifting $\pi$.
For simplicity, we assume that \( p \) totally splits in \( F/\mathbb{Q} \) and \( \pi \) has level 1, which allows us to avoid some technicality. Let \( (\kappa_{1,p} \leq \kappa_{2,p}) \) be the \( p \)-adic Hodge-Tate type of \( \rho_p \) at the place \( \mathfrak{p}|p \). Defining \( \kappa = (\kappa_1, \kappa_2) \) with \( \kappa_j = \sum \kappa_{j,p}\mathfrak{p} \in \mathbb{Z}[\Sigma] \), we call \( \kappa \) the weight of \( \pi \). We suppose \( k_p = \kappa_{2,p} - \kappa_{1,p} + 1 \geq 2 \) (the weight “\( \geq 2 \)” condition). Write the central character of \( \pi \) as \( \varepsilon \); so, \( \det(\rho) = \varepsilon N \) for the cyclotomic character \( N \). The representation \( \pi \) has a \( p \)-normalized vector \( f \in \pi \). The form \( f \) is normalized so that the archimedean Fourier coefficients \( a_{\infty}(y,f) \) gives the Hecke eigenvalue of the Hecke operator \( T(y) \) and \( U(y) (y \in \widehat{O} \cap F_\kappa^\times) \) corresponding to \( \Gamma_0(p) (\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}) \widehat{\Gamma}(p) \) for the adelic open compact subgroup \( \widehat{\Gamma}(p) \) of \( \Gamma_0 \)-type. It is better to introduce \( a_p(y,f) = y^{-\kappa_2}a_{\infty}(y,f) \) which we call the \( q \)-expansion coefficients of \( f \), is the eigenvalue of \( T_p(y) = y^{-\kappa_1}T(y) \) and is \( p \)-integral with \( f|U_p(y) = \delta_p([y,F_p])f \) if \( 0 \neq y \in O_\kappa^\times \). Here \( y^{-\kappa_1} = \prod_{p \mid \mathfrak{p}} N_{F_p/q_p}(y_p)^{-\kappa_1,p} \).

1. Analytic families of automorphic representations

A philosophical interpretation of the zero of \( L_p(s, Ad(\rho)) \) at \( s = 0,1 \) as a factor of \( L_p(s, \text{End}(\rho)) = L_p(s, \text{End}(\pi)) \) is

\[
\zeta \rightarrow \text{rank Ext}^1_{\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}(\rho, \rho) \rightarrow \text{rank Ext}^1_{\text{automorphic rep}}(\pi, \pi) = \text{r}.
\]

Here the extension group \( \text{Ext}^1_{\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} \) is computed in the category of nearly ordinary \( p \)-adic representations unramified outside \( p \) and \( \infty \). To explore this question, it is essential to lift \( \pi \) (or \( f \)) to \( \Lambda \)-adic automorphic representations. Let us describe this point first. Fix a discrete valuation ring \( W \subset \overline{\mathbb{Q}}_p = \mathbb{C} \) (sufficiently large) finite flat over \( \mathbb{Z}_p \) as a base ring. Take an open subgroup \( S \) of \( G^{(\rho)} := \text{GL}_2(F^{(p\infty)}_\Lambda) \). \( p \)-Adic modular forms on \( S \) over a \( p \)-adic \( W \)-algebra \( R = \text{lim} R/p^nR \) classify triples \((X, \overline{\lambda}, \phi)/A\) for \( p \)-adic \( R \)-algebras \( A \). Here \( X \) is an AVRM by \( O \) (so, \( O \leftarrow \text{End}(X/A) \) with \( \Omega_{X/A} \cong O \otimes_\mathbb{Z} A \) locally), \( \overline{\lambda} \) is a polarization class up to prime-to-\( p \) \( O \)-linear isogenies, and \( \phi = (\phi_p, \phi^{(p)}) \) is a pair of level structures \( \phi_p : \mu_{p\infty} \otimes O^* \rightarrow X[p^{\infty}] \) (\( O^* \) is the \( \mathbb{Z} \)-dual of \( O \)) and \( \phi^{(p)} : (F^{(p\infty)}_\Lambda)^2 \cong \text{V}(\rho)(X) = \lim_{\rightarrow} X[N] \otimes A^{(p\infty)} \) modulo \( S \). A \( p \)-adic modular form \( h \) is a functorial rule satisfying

1. \( h((X, \overline{\lambda}, \phi)/A) \in A \) depends only on the prime-to-\( p \) isogeny class of \((X, \overline{\lambda}, \phi)/A\).
2. If \( \varphi : A \rightarrow B \) is a \( p \)-adically continuous \( R \)-algebra homomorphism, then \( h((X, \overline{\lambda}, \phi) \otimes_{A, \varphi} B) = \varphi(h((X, \overline{\lambda}, \phi)/A)) \),
(3) If $z$ is a central element in $G^{(p)}$, $h((X, \overline{\lambda}, \phi_p, \phi^{(p)} \circ z)_{/A}) = \varepsilon(z)h((X, \overline{\lambda}, \phi)_{/A})$.

Writing $\mathcal{V}(S; \varepsilon; R)$ for the space of $p$-adic modular forms on $S$ over $R$, and taking the limit $\mathcal{V}(\varepsilon; R) = \varprojlim S \mathcal{V}(S; \varepsilon; R)$, $g \in G^{(p)}$ acts on $\mathcal{V}(\varepsilon; R)$ by $g \cdot h((X, \overline{\lambda}, \phi_p, \phi^{(p)}) = h((X, \overline{\lambda}, \phi_p, \phi^{(p)} \circ g)_{/A})$ (the $p$-adic automorphic representation). For $u \in O_p^\times$ and $h \in \mathcal{V}(\varepsilon; R)$, define $h|u(X, \overline{\lambda}, \phi_p, \phi^{(p)}) = h(X, \overline{\lambda}, \phi_p \circ u, \phi^{(p)})$, which is an element of $\mathcal{V}(\varepsilon; R)$. Let

$$\mathcal{V}_\kappa(\varepsilon; R) = \{ h \in \mathcal{V}(\varepsilon; R) \mid h|u = u^{-\kappa_1}h \text{ for all } u \in O_p^\times \}.$$ 

Also $U_p(y)$ for $0 \neq y \in O_p$ acts on $\mathcal{V}(\varepsilon; W)$ by

$$a_p(y, h|U_p(\varpi)) = a_p(\varpi y, h).$$

Similarly, $T_p(q)$ acts on $\mathcal{V}(GL_2(\mathcal{O}_q), \varepsilon; W)$ by

$$a_p(y, h|T_p(q)) = a_p(\varpi y, h) + N(q)\varepsilon(q)a_p(y_{\overline{\varpi}_q}, h)$$

for the uniformizer $\varpi_q$ at a prime $q \mid p$. Define the ordinary projector $e = \lim_{n \to \infty} U_p(p)^{ni}$ on $\mathcal{V}(\varepsilon; R)$, and write the image as $\mathcal{V}^n.ord(\varepsilon; R)$. The prime-to-$p$ part $\pi^{(p)}$ of $\pi$ appears as a subquotient of $\mathcal{V}^n.ord(\varepsilon; W)$ generated by translations $g \cdot f$ of $f \in \pi$ regarded as a $p$-adic modular form.

**Theorem 1.1** (multiplicity 1). The automorphic representation of $G^{(p)} = GL_2(F^{(p\infty)}_A)$ on $\mathcal{V}^n.ord(\varepsilon; \overline{\mathbb{Q}_p}) = \mathcal{V}^n.ord(\varepsilon; W) \otimes_W \overline{\mathbb{Q}_p}$ is admissible and is a direct sum of admissible irreducible representations of $G^{(p)}$ with multiplicity at most 1.

Let $\Gamma_F$ be the $p$-profinite part of $O_p^\times$; so, $\Gamma_F = (1 + p\mathbb{Z}_p)^\Sigma$. Let $\Lambda = \Lambda_F$ be the Iwasawa algebra $W[[\Gamma_F]] = \varprojlim_n W[\Gamma_F/\Gamma_F^n]$. Fix a generator $\gamma_p \in 1 + p\mathbb{Z}_p$ of the $p$-component of $\Gamma_F$, and identify $\Lambda = W[[x_p]]_{p \in \Sigma}$ by $\gamma_p \mapsto 1 + x_p$. We have the universal cyclotomic character $\kappa : O_p^\times \to \Lambda^\times$ sending $u \in O_p^\times$ to the projection $\langle u \rangle \in \Gamma_F \subset \Lambda^\times$. Define $V(S, \varepsilon; \Lambda) = V(S, \varepsilon; W) \otimes_W \Lambda = \varprojlim_n V(S, \varepsilon; \Lambda/m^n_\Lambda)$ and $V(\varepsilon; \Lambda) = \varprojlim_S V(S, \varepsilon; \Lambda)$. Again $G^{(p)}$, $U_p(y)$, $T_p(y)$, $u \in O_p^\times$ and the projector $e$ act on $V(\varepsilon; \Lambda)$. Define $V^n.ord$ by the image of $e$ and

$$V^n.ord(\varepsilon; \Lambda) = \{ h \in V^n.ord(\varepsilon; \Lambda) \mid h|u = u^{-\kappa_1}\kappa(u)h \text{ for all } u \in O_p^\times \}$$

on which $G^{(p)}$ and $U_p(y)$ acts. For each $v = \sum_p v_p\mathbb{Z}$, consider $\kappa_v = (\kappa_1 - v, \kappa_2 + v)$ and the algebra homomorphism $v : \Lambda \to W$ given by $v(u) = \prod_p u_{v_p}^p$ for $u = (u_p)_{p \mid p} \in \Gamma_F$. 

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Theorem 1.2. For an algebraic closure \( K \) of \( \text{Frac}(\Lambda) \), the automorphic representation of \( G^{(\nu)} \) on \( V_{\kappa,\text{ord}}(\varepsilon;K) = V_{\kappa,\text{ord}}(\varepsilon;\Lambda) \otimes_{\Lambda} K \) is admissible and is a direct sum of admissible irreducible representations with multiplicity at most 1. For a given \( \pi \) as above, there exists a unique irreducible admissible factor \( \Pi \) of \( V_{\kappa,\text{ord}}(\varepsilon;K) \) defined over a finite flat extension \( \mathcal{I} \) of \( \Lambda \) such that \( \Pi \otimes_{\mathcal{I},\mathbb{R}} W \cong \pi \) for an algebra homomorphism \( P : \mathcal{I} \to W \) extending \( 0 \in \mathbb{Z}[\Sigma] \) on \( \Lambda \). Moreover for each \( v \in \mathbb{Z}[\Sigma] \) with \( k_p + 2v_p \geq 2 \) for all \( p \mid p \) and each \( W \)-algebra homomorphism \( Q : \mathcal{I} \to W \) extending \( v \), \( \pi_Q = \Pi \otimes_{\mathcal{I},Q} W \) is an automorphic representation of \( G^{(\nu)} \) coming from classical Hilbert modular form of weight \( \kappa \).

Thus we get a \( p \)-adic analytic family of automorphic representation \( \{\pi_Q\}_{Q \in \text{Spf}(\mathcal{I})/W} \). A naive question is

Question 1.3. When the minimal ring of definition of \( \Pi \) is not equal to \( \Lambda \)?

We have \( \mathcal{I} = \Lambda \) for almost all the time; however, there are limited examples of nonscalar extension \( \mathcal{I} \neq \Lambda \). Let \( a(q) \in \mathcal{I} \) be the Hecke eigenvalue of \( T_p(q) \) or \( U_p(q) \) (if \( q = p \)) of \( \Pi \). For simplicity, we assume \( \mathcal{I} = \Lambda \) and write \( \Sigma = \{p_1, \ldots, p_d\} \), \( \gamma_j = \gamma_{p_j} \) and \( x = x_{p_j} \in \Lambda \). Here is a naive transcendency questions

Question 1.4. Fix \( v(1 + x_j) = v(\gamma_j) \) for \( j \geq 2 \) and for \( v^{(1)} = \sum_{j \geq 2} v_j p_j \in \mathbb{Z}[\Sigma - \{p_1\}] \) with \( k_j + 2v_j \geq 2 \) \((j \geq 2)\). Regard \( a(q) \) as a function of \( x_1 \).

1. Fix a prime \( q \). Moving \( v = v_1 p_1 + v^{(1)} \) for integers \( v_1 \) with \( k + 2v_1 \geq 2 \), is the set \( \{v(a(q))|v_1 \geq 1 - \frac{k}{2}\} \) an infinite set?

2. Further suppose that \( \Pi \) does not have complex multiplication. Is the field \( \mathbb{Q}[v(a(q))]|v_1 \geq 1 - \frac{k}{2}| \subset \overline{\mathbb{Q}} \) an infinite extension?

As is well known, by Galois deformation theory, if \( \rho \mod \mathfrak{m}_W \) is absolutely irreducible, we have a modular Galois deformation \( \rho_\Pi : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathbb{I}) \) unramified outside \( p \).

2. Extensions of \( \Pi \) and \( \rho_\Pi \)

Recall \( V(S, \varepsilon; \Lambda) = V(S, \varepsilon; W) \hat{\otimes}_W \Lambda \). Thus the \( W \)-derivations \( \partial \in \text{Der}_W(\Lambda, \Lambda) \) acts on \( V(\varepsilon; \Lambda) \). Let \( \partial_j = (1 + x_j) \frac{\partial}{\partial x_j} \in \text{Der}_W(\Lambda, \Lambda) \).

By the formula defining \( T_p(y) \) and \( U_p(y) \), we see easily \( \partial_j(h|T_p(y)) = (\partial_j h)|T_p(y) \) and \( \partial_j(h|U_p(y)) = (\partial_j h)|U_p(y) \). However, this does not mean that \( \partial_j(\pi) \subset \pi \). Indeed, setting \( \partial h = \{\partial_1 h, \ldots, \partial_r h, h\} \in V(\varepsilon; \Lambda)^{r+1} \) and \( \partial a = \{\partial_1 a, \ldots, \partial_r a\} \in \Lambda^r \), if \( \mathbf{f} \) is the \( p \)-normalized
Hecke eigenform in $\Pi$, $f|T_p(q) = a(q)f$; so, applying $\partial_0$, we find
\[(\partial_0 f)|T_p(q) = \begin{pmatrix} a(q)^{-1} & \partial a(q) \\ 0 & a(q) \end{pmatrix} \partial_0 f.\]
This tells us that the translations of components of $\partial f$ under $G^p$ span a constituent $\tilde{\Pi}$ of $V_{n.ord}(\epsilon; K)$ fitting into the following exact sequence of $G^p$-representations
\[0 \rightarrow \Pi' \rightarrow \tilde{\Pi} \rightarrow \Pi \rightarrow 0.\]
This extension is nontrivial because we can find a set of $r$ primes $Q = \{q_1, \ldots, q_r\}$ with $v(\det(\partial a(q_j))) \neq 0$ for any given $v \in \mathbb{Z}[\Sigma]$. Thus specializing the above exact sequence tensoring $\otimes_{\Lambda,v} W$, we find

**Theorem 2.1.** We have $\text{rank} \, \text{Ext}^1_{\text{automorphic rep}}(\pi^{(p)}, \pi^{(p)}) \geq |\Sigma|.$

Here $\pi^{(p)}$ is the prime to $p$-part of $\pi$. Since the existence of the exceptional zero of the adjoint square $L$-function is independent of $\pi$, to have $r$ independent extension as in the theorem, we are forced to have an infinitesimal deformation of $\pi$ with at least $r$ independent variables. This explains the existence of a $r$-variable $p$-adic analytic family containing $\pi$ as a member. Obviously, we may ask

**Question 2.2.** $\text{rank} \, \text{Ext}^1_{\text{automorphic rep}}(\pi^{(p)}, \pi^{(p)}) = |\Sigma|?$

This question has an affirmative answer under the condition that the local ring of the universal Hecke algebra acting nontrivially on the Hecke eigenforms in $\pi^{(p)}$ is isomorphic to an appropriate universal deformation ring (see Section 4.4 of a forthcoming book [HMI] from Oxford University press with title: “Hilbert Modular Forms and Iwasawa Theory”).

We can apply the same trick to the Galois representation $\rho_{\Pi}$. Let $\epsilon_j$ be the class of $y_j$ in $\mathbb{I}[y_j]/(y_j)^2$. Then $\tilde{\rho}_{\Pi} = \rho_{\Pi} + \sum_j \partial_j \rho_{\Pi} \epsilon_j : \text{Gal}(\mathbb{I}[\epsilon_j]_{y_j \in \Sigma})$ gives rise to a nontrivial extension
\[0 \rightarrow \rho^r_{\Pi} \rightarrow \tilde{\rho}_{\Pi} \rightarrow \rho_{\Pi} \rightarrow 0.\]

We write $\tilde{\rho}_{\pi} = \tilde{\rho}_{\Pi} \otimes_{\Lambda,v} W$. The standard Selmer group $\text{Sel}_F(\text{Ad}(\rho_{\pi}))$ is a submodule of the Galois cohomology group $H^1(F, \text{Ad}(\rho_{\pi})) \subset \text{Ext}^1_{\text{Gal}(\mathbb{I}/F)}(\rho_{\pi}, \rho_{\pi})$ spanned by cocycles unramified outside $p$ and unramified modulo upper-nilpotent matrices at $p|p$ identifying $\rho$ with a matrix representation so that $\rho|_{D_p} = \begin{pmatrix} \delta_p & * \\ 0 & \delta_p \end{pmatrix}$. The little bigger “$-$” Selmer group is generated by cocycles unramified outside $p$ and unramified modulo upper-triangular matrices at $p|p$. Then for each submodule $X$ of $\tilde{\rho}$ isomorphic to $\rho^{r-1}$, the extension class $[\tilde{\rho} \mod X] \in \text{Sel}_F(\text{Ad}(\rho_{\pi})).$
Theorem 2.3 (Greenberg). We have \( \text{rank}_W \text{Sel}_F(\text{Ad}(\rho_{i_p})) \geq |\Sigma| \), and the equality holds if \( \text{Sel}_F(\text{Ad}(\rho_{i_p})) \) is finite.

By a work of Fujiwara, \( \text{Sel}_F(\text{Ad}(\rho_{i_p})) \) is finite if \( \overline{\rho} := (\rho_{i_p} \mod m_W) \) is absolutely irreducible over \( F[\mu_p] \). Thus the extension \( \widetilde{\rho}_\Pi \) of \( \rho_\Pi \) is highly nontrivial, because \( \det(\partial_i a(q_j)) \in \mathbb{I}^\times \) for many sets of primes \( Q \) with positive density in \( \{\text{primes}\}^\Sigma \).

The \( p \)-adic \( \mathcal{L} \)-function \( L_p(s, \text{Ad}(\rho)) \) is actually related to the Selmer group \( \text{Sel}_{F_\infty}(\text{Ad}(\rho_{i_p})) \) over the cyclotomic \( \mathbb{Z}_p \)-extension \( F_\infty/F \). Then the eigenvalue \( \alpha_N \) and \( \beta_N \) of the Frobenius \( \text{Frob}_{q_j} \) over a high layer \( F_N/F \) is a high \( p \)-power of \( \alpha_0 \) and \( \beta_0 \); so, \( \det(\partial_i a(q_j)) \) over \( F_N \) is no longer a unit even if it is over \( F \). To guarantee the nontriviality of the extension \( \widetilde{\rho}_\Pi \) over \( F_\infty \), we need to have \( \det(\partial_i a(p_j)) \neq 0 \), which is difficult to prove (but follows if Question 1.4 is affirmative for \( q \in \Sigma \)).

**Question 2.4.** \( \det(\partial_i a(p_j)) \neq 0? \)

### 3. \( \mathcal{L} \)-invariant

Greenberg has given (in his paper in Contemporary Math. 165 149–174) a Galois cohomological definition of the \( \mathcal{L} \)-invariant of \( p \)-adic ordinary representations of \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \). Applying his definition to \( \text{Ind}_F^Q \text{Ad}(\rho_{i_p}) \), we can compute \( \mathcal{L}(\text{Ad}(\rho)) \).

**Theorem 3.1.** If \( \overline{\rho} \) is absolutely irreducible, we have

\[
\mathcal{L}(\text{Ad}(\rho_v)) = v(\det(a(p_i)^{-1}\partial_i a(p_j))) \prod_{p|p} \gamma_p^{-v_p} \log_p(\gamma_p).
\]

As conjectured by Greenberg, we should have \( \mathcal{L}(\text{Ad}(\rho_v)) \neq 0 \), and if it is the case, Question 2.4 will have an affirmative answer. Combining these results with the computation by Greenberg–Stevens and Greenberg of the \( \mathcal{L} \)-invariant of elliptic curves with multiplicative reduction at \( p \), we get

**Corollary 3.2.** Suppose that \( \pi_v \) is associated to an elliptic curve \( E/F \) with split multiplicative reduction at all \( p|p \). Then we have

\[
\mathcal{L}(\text{Ad}(\rho_v)) = \mathcal{L}(E) = \prod_{p|p} \log_p(N_{F_p/Q_p}(q_p)) / \text{ord}_p(N_{F_p/Q_p}(q_p)),
\]

where \( E(F_p) = F_p^\times / q_p^\mathbb{Z} \).

In this case, \( \mathcal{L}(E) \neq 0 \) by the theorem of St. Etienne. The proof of these results will appear in my forthcoming book [HMI] Section 3.4).