EXTENSIONS AND THE EXCEPTIONAL ZERO OF THE ADJOINT SQUARE L-FUNCTIONS

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Take a totally real field F with integer ring O as a base field. We fix an identification $\iota : \overline{\mathbb{Q}}_p \cong \mathbb{C} \supset \overline{\mathbb{Q}}$. Fix a prime $p > 2$, and write Σ for the set of prime factors of p in F. Start with a holomorphic automorphic representation π of $GL_2(F_{\mathbb{A}})$ (a Hilbert modular Hecke eigenform) which is spherical and nearly p-ordinary at Σ. Then we have the compatible system of λ-adic representations $\rho = \{\rho_\lambda\}_\lambda$ of π, and if $\lambda \nmid p$, $\rho_\lambda(Frob_{\mathfrak{p}})$ is unramified and has two eigenvalues a p-adic unit eigenvalue α (with respect to ι) and a p-adic nonunit β. When we consider a p-adic member of ρ, it is supposed to be associated to $i_p = \iota^{-1} : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. We consider the Galois stable subspace $Ad(\rho) \subset \rho \otimes {}^t\rho^{-1} = \mathrm{End}(\rho)$ with zero trace (whose Galois action is given by conjugation). The Euler factor at $\mathfrak{p}$ of $L(s, Ad(\rho))$ is then given by

$$\left[(1-\alpha\beta^{-1}p^{-s})(1-p^{-s})(1-\alpha^{-1}\beta p^{-s})\right]^{-1}.$$

The p-adic L-function $L_p(s, Ad(\rho))$, whose value at 1 is a constant multiple of $(1-pp^{-1})(1-\alpha^{-1}\beta p^{-1})(1-p\alpha\beta^{-1}p^{-1})L(1, Ad(\rho))$, has an exceptional 0 at $s = 1$ (corresponding to the Frobenius eigenvalue $= 1$ at $\mathfrak{p}|p$) whose order is the number of such Euler factors $r = |\Sigma|$ if the $\mathcal{L}$-invariant $\mathcal{L}(Ad(\rho))$ of $Ad(\rho_{i_p})$ does not vanish. The $\mathcal{L}$-invariant $\mathcal{L}(Ad(\rho))$ is defined by the following (hypothetical) formula:

$$\frac{d^r L_p(s, Ad(\rho))}{ds^r}\Big|_{s=1} \stackrel{?}{=} \mathcal{L}(Ad(\rho))\frac{L(1, Ad(\rho))}{\text{a period}}.$$

The appearance of the trivial zero is always true without assuming unramifiedness of π or ρ at p for the adjoint square L-functions, and this is a peculiar point when we study the $\mathcal{L}$-invariant of the adjoint square L. Indeed, by the near ordinarity, $\rho_{i_p}|_{D_{\mathfrak{p}}} \cong \begin{pmatrix} \delta'_{\mathfrak{p}} & * \\ 0 & \delta_{\mathfrak{p}} \end{pmatrix}$, and hence, the semisimplified $Ad(\rho)|_{D_{\mathfrak{p}}}$ has eigenvalue 1 for $Frob_{\mathfrak{p}}$. Since Greenberg has given a Galois cohomological definition of the $\mathcal{L}$-invariant without recourse to the analytic p-adic L-function, we can discuss the adjoint square $\mathcal{L}$-invariant using his definition, and we would like to relate it to differential calculus of p-adic analytic families lifting π.

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For simplicity, we assume that p totally splits in $F/\mathbb{Q}$ and π has level 1, which allows us to avoid some technicality. Let $(\kappa_{1,\mathfrak{p}} \le \kappa_{2,\mathfrak{p}})$ be the p-adic Hodge-Tate type of ρ_{i_p} at the place $\mathfrak{p}|p$. Defining $\kappa = (\kappa_1, \kappa_2)$ with $\kappa_j = \sum_{\mathfrak{p}} \kappa_{j,\mathfrak{p}}\mathfrak{p} \in \mathbb{Z}[\Sigma]$, we call κ the weight of π. We suppose $k_{\mathfrak{p}} = \kappa_{2,\mathfrak{p}} - \kappa_{1,\mathfrak{p}} + 1 \ge 2$ (the weight "≥ 2" condition). Write the central character of π as ε; so, $\det(\rho) = \varepsilon\mathcal{N}$ for the cyclotomic character $\mathcal{N}$. The representation π has a p-normalized vector $f \in \pi$. The form f is normalized so that the archimedean Fourier coefficients $\mathbf{a}_\infty(y, f)$ gives the Hecke eigenvalue of the Hecke operator $T(y)$ and $U(y)$ $(y \in \widehat{O} \cap F_{\mathbb{A}}^\times)$ corresponding to $\widehat{\Gamma}_0(p) \left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right) \widehat{\Gamma}_0(p)$ for the adelic open compact subgroup $\widehat{\Gamma}_0(p)$ of Γ_0-type. It is better to introduce $\mathbf{a}_p(y, f) = y_p^{-\kappa_1}\mathbf{a}_\infty(y, f)$ which we call the q-expansion coefficients of f, is the eigenvalue of $T_p(y) = y_p^{-\kappa_1}T(y)$ and is p-integral with $f|U_p(y) = \delta_{\mathfrak{p}}([y, F_{\mathfrak{p}}])f$ if $0 \ne y \in O_{\mathfrak{p}}^\times$. Here $y_p^{-\kappa_1} = \prod_{\mathfrak{p}|p} N_{F_{\mathfrak{p}}/\mathbb{Q}_p}(y_{\mathfrak{p}})^{-\kappa_{1,\mathfrak{p}}}$.

1. Analytic families of automorphic representations

A philosophical interpretation of the zero of $L_p(s, Ad(\rho))$ at $s = 0, 1$ as a factor of $L_p(s, \mathrm{End}(\rho)) = L_p(s, \mathrm{End}(\pi))$ is

$$\begin{aligned}&\text{an order } r \text{ zero of } L_p(s, Ad(\rho)) = L_p(s, Ad(\pi)) \text{ at } s = 1\\ &\qquad \overset{?}{\leftrightarrow} \operatorname{rank} \mathrm{Ext}^1_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}(\rho, \rho) \overset{?}{=} \operatorname{rank} \mathrm{Ext}^1_{\text{automorphic rep}}(\pi, \pi) = r.\end{aligned}$$

Here the extension group $\mathrm{Ext}^1_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}$ is computed in the category of nearly ordinary p-adic representations unramified outside p and ∞. To explore this question, it is essential to lift π (or f) to Λ-adic automorphic representations. Let us describe this point first. Fix a discrete valuation ring $W \subset \overline{\mathbb{Q}}_p = \mathbb{C}$ (sufficiently large) finite flat over $\mathbb{Z}_p$ as a base ring. Take an open subgroup S of $G^{(p)} := GL_2(F_{\mathbb{A}}^{(p\infty)})$. p-Adic modular forms on S over a p-adic W-algebra $R = \varprojlim R/p^nR$ classify triples $(X, \overline{\lambda}, \phi)_{/A}$ for p-adic R-algebras A. Here X is an AVRM by O (so, $O \hookrightarrow \mathrm{End}(X_{/A})$ with $\Omega_{X/A} \cong O \otimes_{\mathbb{Z}} A$ locally), $\overline{\lambda}$ is a polarization class up to prime-to-p O-linear isogenies, and $\phi = (\phi_p, \phi^{(p)})$ is a pair of level structures $\phi_p : \mu_{p^\infty} \otimes O^* \hookrightarrow X[p^\infty]$ (O^* is the $\mathbb{Z}$-dual of O) and $\phi^{(p)} : (F_{\mathbb{A}}^{(p\infty)})^2 \cong V^{(p)}(X) = (\varprojlim_N X[N]) \otimes \mathbb{A}^{(p\infty)}$ modulo S. A p-adic modular form h is a functorial rule satisfying

1. $h((X, \overline{\lambda}, \phi)_{/A}) \in A$ depends only on the prime-to-p isogeny class of $(X, \overline{\lambda}, \phi)_{/A}$,
2. If $\varphi : A \to B$ is a p-adically continuous R-algebra homomorphism, then $h((X, \overline{\lambda}, \phi)_{/A} \otimes_{A,\varphi} B) = \varphi(h((X, \overline{\lambda}, \phi)_{/A}))$,

(3) If z is a central element in $G^{(p)}$, $h((X,\overline{\lambda},\phi_p,\phi^{(p)}\circ z)_{/A}) = \varepsilon(z)h((X,\overline{\lambda},\phi)_{/A})$.

Writing $\mathcal{V}(S,\varepsilon;R)$ for the space of p-adic modular forms on S over R, and taking the limit $\mathcal{V}(\varepsilon;R)=\varinjlim_S \mathcal{V}(S,\varepsilon;R)$, $g\in G^{(p)}$ acts on $\mathcal{V}(\varepsilon;R)$ by $g\cdot h((X,\overline{\lambda},\phi_p,\phi^{(p)}) = h((X,\overline{\lambda},\phi_p,\phi^{(p)}\circ g)_{/A})$ (the p-adic automorphic representation). For $u\in O_p^\times$ and $h\in\mathcal{V}(\varepsilon;R)$, define $h|u(X,\overline{\lambda},\phi_p,\phi^{(p)}) = h(X,\overline{\lambda},\phi_p\circ u,\phi^{(p)})$, which is an element of $\mathcal{V}(\varepsilon;R)$. Let

$$\mathcal{V}_\kappa(\varepsilon;R)=\{h\in\mathcal{V}(\varepsilon;R)|h|u=u^{-\kappa_1}h \text{ for all } u\in O_p^\times\}.$$

Also $U_p(y)$ for $0\neq y\in O_p$ acts on $\mathcal{V}(\varepsilon;W)$ by

$$\mathbf{a}_p(y,h|U_p(\varpi))=\mathbf{a}_p(\varpi y,h).$$

Similarly, $T_p(\mathfrak{q})$ acts on $\mathcal{V}(GL_2(O_\mathfrak{q}),\varepsilon;W)$ by

$$\mathbf{a}_p(y,h|T_p(\mathfrak{q}))=\mathbf{a}_p(\varpi y,h)+N(\mathfrak{q})\varepsilon(\mathfrak{q})\mathbf{a}_p(\frac{y}{\varpi_\mathfrak{q}},h)$$

for the uniformizer $\varpi_\mathfrak{q}$ at a prime $\mathfrak{q}\nmid p$. Define the ordinary projector $e=\lim_{n\to\infty}U_p(p)^{n!}$ on $\mathcal{V}(\varepsilon;R)$, and write the image as $\mathcal{V}^{n.ord}(\varepsilon;R)$. The prime-to-$p$ part $\pi^{(p)}$ of π appears as a subquotient of $\mathcal{V}_\kappa^{n.ord}(\varepsilon;W)$ generated by translations $g\cdot f$ of $f\in\pi$ regarded as a p-adic modular form.

Theorem 1.1 (multiplicity 1)**.** *The automorphic representation of $G^{(p)}=GL_2(F_\mathbb{A}^{(p\infty)})$ on $\mathcal{V}_\kappa^{n.ord}(\varepsilon;\overline{\mathbb{Q}}_p)=\mathcal{V}_\kappa^{n.ord}(\varepsilon;W)\otimes_W\overline{\mathbb{Q}}_p$ is admissible and is a direct sum of admissible irreducible representations of $G^{(p)}$ with multiplicity at most* 1.

Let Γ_F be the p-profinite part of $O_p^\times$; so, $\Gamma_F=(1+p\mathbb{Z}_p)^\Sigma$. Let $\Lambda=\Lambda_F$ be the Iwasawa algebra $W[[\Gamma_F]]=\varprojlim_n W[\Gamma_F/\Gamma_F^{p^n}]$. Fix a generator $\gamma_\mathfrak{p}\in 1+p\mathbb{Z}_p$ of the $\mathfrak{p}$-component of Γ_F, and identify $\Lambda=W[[x_\mathfrak{p}]]_{\mathfrak{p}\in\Sigma}$ by $\gamma_\mathfrak{p}\leftrightarrow 1+x_\mathfrak{p}$. We have the universal cyclotomic character $\boldsymbol{\kappa}:O_p^\times\to\Lambda^\times$ sending $u\in O_p^\times$ to the projection $\langle u\rangle\in\Gamma_F\subset\Lambda^\times$. Define $V(S,\varepsilon;\Lambda)=\mathcal{V}(S,\varepsilon;W)\widehat{\otimes}_W\Lambda=\varprojlim_n\mathcal{V}(S,\varepsilon;\Lambda/\mathfrak{m}_\Lambda^n)$ and $V(\varepsilon;\Lambda)=\varinjlim_S V(S,\varepsilon;\Lambda)$. Again $G^{(p)}$, $U_p(y)$, $T_p(y)$, $u\in O_p^\times$ and the projector e act on $V(\varepsilon;\Lambda)$. Define $V^{n.ord}$ by the image of e and

$$V_{\boldsymbol{\kappa}}^{n.ord}(\varepsilon;\Lambda)=\left\{h\in V^{n.ord}(\varepsilon;\Lambda)\middle|h|u=u^{-\kappa_1}\boldsymbol{\kappa}(u)h \text{ for all } u\in O_p^\times\right\}$$

on which $G^{(p)}$ and $U_p(y)$ acts. For each $v=\sum_\mathfrak{p}v_\mathfrak{p}\mathfrak{p}\in\mathbb{Z}[\Sigma]$, consider $\kappa_v=(\kappa_1-v,\kappa_2+v)$ and the algebra homomorphism $v:\Lambda\to W$ given by $v(u)=\prod_\mathfrak{p}u_\mathfrak{p}^{v_\mathfrak{p}}$ for $u=(u_\mathfrak{p})_{\mathfrak{p}|p}\in\Gamma_F$.

Theorem 1.2. *For an algebraic closure $\mathcal{K}$ of $Frac(\Lambda)$, the automorphic representation of $G^{(p)}$ on $V^{n.ord}_{\boldsymbol{\kappa}}(\varepsilon;\mathcal{K}) = V^{n.ord}_{\boldsymbol{\kappa}}(\varepsilon;\Lambda)\otimes_\Lambda\mathcal{K}$ is admissible and is a direct sum of admissible irreducible representations with multiplicity at most 1. For a given π as above, there exists a unique irreducible admissible factor Π of $V^{n.ord}_{\boldsymbol{\kappa}}(\varepsilon;\mathcal{K})$ defined over a finite flat extension $\mathbb{I}$ of Λ such that $\Pi\otimes_{\mathbb{I},P}W \cong \pi$ for an algebra homomorphism $P:\mathbb{I}\to W$ extending $0\in\mathbb{Z}[\Sigma]$ on Λ. Moreover for each $v\in\mathbb{Z}[\Sigma]$ with $k_{\mathfrak{p}}+2v_{\mathfrak{p}}\geq 2$ for all $\mathfrak{p}|p$ and each W-algebra homomorphism $Q:\mathbb{I}\to W$ extending v, $\pi_Q = \Pi\otimes_{\mathbb{I},Q}W$ is an automorphic representation of $G^{(p)}$ coming from classical Hilbert modular form of weight κ_v.*

Thus we get a p-adic analytic family of automorphic representation $\{\pi_Q\}_{Q\in\mathrm{Spf}(\mathbb{I})(W)}$. A naive question is

Question 1.3. *When the minimal ring of definition of Π is not equal to Λ?*

We have $\mathbb{I}=\Lambda$ for almost all the time; however, there are limited examples of nonscalar extension $\mathbb{I}\neq\Lambda$. Let $\mathbf{a}(\mathfrak{q})\in\mathbb{I}$ be the Hecke eigenvalue of $T_p(\mathfrak{q})$ or $U_p(\mathfrak{q})$ (if $\mathfrak{q}=\mathfrak{p}$) of Π. For simplicity, we assume $\mathbb{I}=\Lambda$ and write $\Sigma=\{\mathfrak{p}_1,\dots,\mathfrak{p}_d\}$, $\gamma_j=\gamma_{\mathfrak{p}_j}$ and $x=x_{\mathfrak{p}_j}\in\Lambda$. Here is a naive transcendency questions

Question 1.4. *Fix $v(1+x_j)=v(\gamma_j)$ for $j\geq 2$ and for $v^{(1)}=\sum_{j\geq 2}v_j\mathfrak{p}_j\in\mathbb{Z}[\Sigma-\{\mathfrak{p}_1\}]$ with $k_j+2v_j\geq 2$ ($j\geq 2$). Regard $\mathbf{a}(\mathfrak{q})$ as a function of x_1.*

1. *Fix a prime $\mathfrak{q}$. Moving $v=v_1\mathfrak{p}_1+v^{(1)}$ for integers v_1 with $k+2v_1\geq 2$, is the set $\{v(\mathbf{a}(\mathfrak{q}))|v_1\geq 1-\frac{k}{2}\}$ an infinite set?*
2. *Further suppose that Π does not have complex multiplication. Is the field $\mathbb{Q}[v(\mathbf{a}(\mathfrak{q}))|v_1\geq 1-\frac{k}{2}]\subset\overline{\mathbb{Q}}$ an infinite extension?*

As is well known, by Galois deformation theory, if $\rho \mod \mathfrak{m}_W$ is absolutely irreducible, we have a modular Galois deformation $\rho_\Pi:\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\to GL_2(\mathbb{I})$ unramified outside p.

2. Extensions of Π and ρ_Π

Recall $V(S,\varepsilon;\Lambda)=\mathcal{V}(S,\varepsilon;W)\widehat{\otimes}_W\Lambda$. Thus the W-derivations $\partial\in Der_W(\Lambda,\Lambda)$ acts on $V(\varepsilon;\Lambda)$. Let $\partial_j=(1+x_j)\frac{\partial}{\partial x_j}\in Der_W(\Lambda,\Lambda)$. By the formula defining $T_p(y)$ and $U_p(y)$, we see easily $\partial_j(h|T_p(y))=(\partial_j h)|T_p(y)$ and $\partial_j(h|U_p(y))=(\partial_j h)|U_p(y)$. However, this does not mean that $\partial_j(\pi)\subset\pi$. Indeed, setting $\partial_0 h={}^t(\partial_1 h,\dots,\partial_r h,h)\in V(\varepsilon;\Lambda)^{r+1}$ and $\partial\mathbf{a}={}^t(\partial_1\mathbf{a},\dots,\partial_r\mathbf{a})\in\Lambda^r$, if $\mathbf{f}$ is the p-normalized

Hecke eigenform in Π, $\mathbf{f}|T_p(\mathfrak{q}) = \mathbf{a}(\mathfrak{q})\mathbf{f}$; so, applying ∂_0, we find

$$(\partial_0\mathbf{f})|T_p(\mathfrak{q}) = \begin{pmatrix} \mathbf{a}(\mathfrak{q})\cdot 1_r & \partial\mathbf{a}(\mathfrak{q}) \\ 0 & \mathbf{a}(\mathfrak{q}) \end{pmatrix} \partial_0\mathbf{f}.$$

This tells us that the translations of components of $\partial\mathbf{f}$ under $G^{(p)}$ span a constituent $\widetilde{\Pi}$ of $V^{n.ord}(\varepsilon;\mathcal{K})$ fitting into the following exact sequence of $G^{(p)}$-representations

$$0 \to \Pi^r \to \widetilde{\Pi} \to \Pi \to 0.$$

This extension is nontrivial because we can find a set of r primes $Q = \{\mathfrak{q}_1, \ldots, \mathfrak{q}_r\}$ with $v(\det(\partial_i\mathbf{a}(\mathfrak{q}_j))) \neq 0$ for any given $v \in \mathbb{Z}[\Sigma]$. Thus specializing the above exact sequence tensoring $\otimes_{\Lambda,v}W$. we find

Theorem 2.1. *We have* $\operatorname{rank}\operatorname{Ext}^1_{\text{automorphic rep}}(\pi^{(p)}, \pi^{(p)}) \geq |\Sigma|$.

Here $\pi^{(p)}$ is the prime to p-part of π. Since the existence of the exceptional zero of the adjoint square L-function is independent of π, to have r independent extension as in the theorem, we are forced to have an infinitesimal deformation of π with at least r independent variables. This explains the existence of a r-variable p-adic analytic family containing π as a member. Obviously, we may ask

Question 2.2. $\operatorname{rank}\operatorname{Ext}^1_{\text{automorphic rep}}(\pi^{(p)}, \pi^{(p)}) = |\Sigma|$*?*

This question has an affirmative answer under the condition that the local ring of the universal Hecke algebra acting nontrivially on the Hecke eigenforms in $\pi^{(p)}$ is isomorphic to an appropriate universal deformation ring (see Section 4.4 of a forthcoming book [HMI] from Oxford University press with title: "Hilbert Modular Forms and Iwasawa Theory").

We can apply the same trick to the Galois representation ρ_Π. Let ε_j be the class of y_j in $\mathbb{I}[y_j]/(y_j)^2$. Then $\widetilde{\rho}_\Pi = \rho_\Pi + \sum_j \partial_j\rho_\Pi\epsilon_j$: $\mathrm{Gal}(\mathbb{I}[\epsilon_j]_{\mathfrak{p}_j\in\Sigma})$ gives rise to a nontrivial extension

$$0 \to \rho_\Pi^r \to \widetilde{\rho}_\Pi \to \rho_\Pi \to 0.$$

We write $\widetilde{\rho}_v = \widetilde{\rho}_\Pi \otimes_{\Lambda,v} W$. The standard Selmer group $\mathrm{Sel}_F(Ad(\rho_{i_p}))$ is a submodule of the Galois cohomology group $H^1(F, Ad(\rho_{i_p})) \subset \operatorname{Ext}^1_{\mathrm{Gal}(\overline{\mathbb{Q}}/F)}(\rho_{i_p}, \rho_{i_p})$ spanned by cocycles unramified outside p and unramified modulo upper-nilpotent matrices at $\mathfrak{p}|p$ identifying ρ with a matrix representation so that $\rho|_{D_\mathfrak{p}} = \begin{pmatrix} \delta'_\mathfrak{p} & * \\ 0 & \delta_\mathfrak{p} \end{pmatrix}$. The little bigger "$-$" Selmer group is generated by cocycles unramified outside p and unramified modulo upper-triangular matrices at $\mathfrak{p}|p$. Then for each submodule X of $\widetilde{\rho}$ isomorphic to ρ^{r-1}, the extension class $[\widetilde{\rho} \mod X] \in \mathrm{Sel}^-_F(Ad(\rho_{i_p}))$.

Theorem 2.3 (Greenberg)**.** *We have* $\mathrm{rank}_W \mathrm{Sel}^-_F(Ad(\rho_{i_p})) \geq |\Sigma|$, *and the equality holds if* $\mathrm{Sel}_F(Ad(\rho_{i_p}))$ *is finite.*

By a work of Fujiwara, $\mathrm{Sel}_F(Ad(\rho_{i_p}))$ is finite if $\overline{\rho} := (\rho_{i_p} \mod \mathfrak{m}_W)$ is absolutely irreducible over $F[\mu_p]$. Thus the extension $\widetilde{\rho}_\Pi$ of ρ_Π is highly nontrivial, because $\det(\partial_i \mathbf{a}(\mathfrak{q}_j)) \in \mathbb{I}^\times$ for many sets of primes Q with positive density in $\{\text{primes}\}^\Sigma$.

The p-adic L-function $L_p(s, Ad(\rho))$ is actually related to the Selmer group $\mathrm{Sel}_{F_\infty}(Ad(\rho_{i_p}))$ over the cyclotomic $\mathbb{Z}_p$-extension F_∞/F. Then the eigenvalue α_N and β_N of the Frobenius $Frob_{\mathfrak{q}_j}$ over a high layer F_N/F is a high p-power of α_0 and β_0; so, $\det(\partial_i \mathbf{a}(\mathfrak{q}_j))$ over F_N is no longer a unit even if it is over F. To guarantee the nontriviality of the extension $\widetilde{\rho}_\Pi$ over F_∞, we need to have $\det(\partial_i \mathbf{a}(\mathfrak{p}_j)) \neq 0$, which is difficult to prove (but follows if Question 1.4 is affirmative for $\mathfrak{q} \in \Sigma$).

Question 2.4. $\det(\partial_i \mathbf{a}(\mathfrak{p}_j)) \neq 0$*?*

3. $\mathcal{L}$-invariant

Greenberg has given (in his paper in Contemporary Math. **165** 149–174) a Galois cohomological definition of the $\mathcal{L}$-invariant of p-adic ordinary representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Applying his definition to $\mathrm{Ind}_F^{\mathbb{Q}} Ad(\rho_{i_p})$, we can compute $\mathcal{L}(Ad(\rho))$.

Theorem 3.1. *If* $\overline{\rho}$ *is absolutely irreducible, we have*

$$\mathcal{L}(Ad(\rho_v)) = v(\det(\mathbf{a}(\mathfrak{p}_i)^{-1}\partial_i \mathbf{a}(\mathfrak{p}_j))) \prod_{\mathfrak{p}|p} \gamma_{\mathfrak{p}}^{-v_{\mathfrak{p}}} \log_p(\gamma_{\mathfrak{p}}).$$

As conjectured by Greenberg, we should have $\mathcal{L}(Ad(\rho_v)) \neq 0$, and if it is the case, Question 2.4 will have an affirmative answer. Combining these results with the computation by Greenberg–Stevens and Greenberg of the $\mathcal{L}$-invariant of elliptic curves with multiplicative reduction at p, we get

Corollary 3.2. *Suppose that* π_v *is associated to an elliptic curve* $E_{/F}$ *with split multiplicative reduction at all* $\mathfrak{p}|p$. *Then we have*

$$\mathcal{L}(Ad(\rho_v)) = \mathcal{L}(E) = \prod_{\mathfrak{p}|p} \frac{\log_p(N_{F_{\mathfrak{p}}/\mathbb{Q}_p}(q_{\mathfrak{p}}))}{\mathrm{ord}_p(N_{F_{\mathfrak{p}}/\mathbb{Q}_p}(q_{\mathfrak{p}}))},$$

where $E(F_{\mathfrak{p}}) = F_{\mathfrak{p}}^\times / q_{\mathfrak{p}}^{\mathbb{Z}}$.

In this case, $\mathcal{L}(E) \neq 0$ by the theorem of St. Etienne. The proof of these results will appear in my forthcoming book [HMI] Section 3.4).