HECKE ALGEBRAS FOR GL_1 AND GL_2

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§ 0. Introduction.

In our previous papers [6] and [7], we have defined the module of congruences of the Hecke algebra of the space of modular forms of $GL_2(\mathbb{Q})$ and studied its arithmetic relation with the special values of a certain L-function of modular forms. In the present paper, we shall propose the study of the module of relative 1-differentials of the Hecke algebra (over a base ring) as well as its module of congruences. These modules have the same Fitting ideal under certain assumptions (Lemmas 1.8, 1.9 and 1.11). The precise definition of these modules will be given in § 1. Then, we shall define in § 2 the Hecke algebra for GL_1 over arbitrary algebraic number fields F and show that if the level of this algebra is equal to 1, the module of 1-differentials over $\mathbb{Z}_p$ is isomorphic to the p-part of the ideal class group of F and the module of congruences is isomorphic to $\mathbb{Z}_p/h\mathbb{Z}_p$ for the class number h of F (Theorem 2.6). In § 3, we shall define the Hecke algebras for $GL_2(F)$ for an arbitrary totally real field F and list some of their properties without proof. When M is a CM-field quadratic over F, there is a natural algebra homomorphism of the Hecke algebra for $GL_2(F)$ to that of $GL_1(M)$ (4.5). Under a certain assumption (4.2), we shall show that these modules for $GL_1(M)$ are obtained as a quotient of the corresponding one for $GL_2(F)$ (Theorem 4.3). In § 5, we shall investigate the relation between our theory and the Iwasawa theory of the natural $\mathbb{Z}_p \times \mathbb{Z}_p$-extension of imaginary quadratic fields. To explain this, we shall introduce some symbols for $\mathbb{Z}_p$-extensions. Let M be an imaginary quadratic field with class number one. We fix an odd prime p which decomposes in M into the product of two different primes $\mathfrak{p}$ and $\bar{\mathfrak{p}}$. Let F_∞ (resp. K_∞) be the unique $\mathbb{Z}_p$-extension of $\mathbb{Q}$ (resp. M) unramified outside p (resp. $\mathfrak{p}$), and put $M_\infty = M \cdot F_\infty$. Let L_∞ be the maximal abelian extension of M unramified outside p and Y_∞ be the maximal p-abelian extension of L_∞ unramified outside $\mathfrak{p}$. Let C be the centralizer of complex conjugation in $Gal(L_\infty/M)$, and let H_∞ denote the fixed field of C in L_∞ (i.e. the anti-cyclotomic part of L_∞). Finally, let Z_n (resp. X_n) be the maximal p-abelian extension of $H_\infty \cdot M_n$ (resp. M_n) for the n-th layer M_n of M_∞ unramified outside $\mathfrak{p}$. There are restriction morphisms : $Gal(Y_\infty/L_\infty) \longrightarrow Gal(X_n/M_\infty \cdot K_\infty)$ and $Gal(Y_\infty/L_\infty) \longrightarrow Gal(Z_n/M_\infty \cdot H_\infty)$ for $n = 1,2,\ldots,\infty$. Our Theorem 5.5 asserts that the finite module $Gal(X_n/M_n \cdot K_\infty)$ can be obtained as a quotient of the module of differentials of an appropriate Hecke algebra of $GL_2(F_n)$. After taking the projective limit relative to n, the (one variable) Iwasawa module $Gal(X_\infty/M_\infty \cdot K_\infty)$ is again a quotient of the limit of these modules for $GL_2(F_n)$ with respect to the base change lifting maps for GL_2 ([9]). On the other hand, the (two variable) Iwasawa module $Gal(Z_\infty/M_\infty \cdot H_\infty)$ is expected to be related to the kernel of this quotient map. In fact, in the case of $n = 0$ (i.e. the case of the module $Gal(Z_0/M_\infty \cdot H_\infty)$), the result in [5] supports this expectation (for details, see Remark 5.6). Finally in § 6, we shall prove that the local ring of the Hecke algebra of $GL_2(\mathbb{Q})$ obtained from M is a Gorenstein ring under a certain irreducibility assumption of the Galois action. Thus the module of defect N_S defined in [6] vanishes as conjectured there when it corresponds to M.

Originally, the author had a formulation of the results presented here by using certain modules expressed as the first "torsion" (or extension) modules over the Hecke algebra. The use of the module of 1-differentials or the (co)-tangent space was suggested to the author by B. Mazur. Anyway, the two definitions coincide in almost all cases (Lemma 1.1).

In the seminar talk, the author gave an account of the construction of Galois representations into GL_2 over the algebra $\mathbb{Z}_p[[X]]$. A detailed proof of this has already appeared in [7]. We thus take this opportunity to show that the ideas in [7] developed for GL_2 do reveal some new aspect even for the theory of GL_1.

§ 1. Some lemmas and definitions.

Let A be a noetherian integral domain of characteristic 0 and R and B be A-algebras finite over A. We suppose that there is a surjective A-algebra homomorphism $\lambda : R \longrightarrow B$. Let $\Omega^1_{R/A}$ be the module of relative 1-differentials of R over A (cf. [1, § 20]). We recall its definition : Let I be the kernel of the natural A-algebra homomorphism : $R \otimes_A R \longrightarrow R$ given by $a \otimes b \longmapsto a \cdot b \in R$. Then, the R-module I/I^2 is by definition isomorphic to $\Omega^1_{R/A}$. The map $d : R \longrightarrow \Omega^1_{R/A}$ given by $d(a) = a \otimes 1 - 1 \otimes a \bmod I^2$ gives a derivation trivial over A. For each R-module M, the module of A-trivial derivations $Der_A(R,M)$ can be identified with $Hom_R(\Omega^1_{R/A},M)$ by $Hom_R(\Omega^1_{R/A},M) \ni f \longmapsto f \circ d \in Der_A(R,M)$. Thus, the module $\Omega^1_{R/A}$ is the cotangent space of R over A. We shall define a B-module by

$$C_1(\lambda;B) = Tor^R_1(B,B),$$

where we regard B as an R-module through λ.

Lemma 1.1. Let a be the kernel of λ. Then we have that

$$C_1(\lambda;B) \simeq a/a^2 \simeq a \otimes_R B.$$

If B is etale over A, then $C_1(\lambda;B) \simeq \Omega^1_{R/A} \otimes_R B$.

Proof. If B/A is smooth, there is a natural exact sequence (cf. [1, 20, 5.14].

$$0 \longrightarrow a/a^2 \longrightarrow \Omega^1_{R/A} \otimes_R B \longrightarrow \Omega^1_{B/A} \longrightarrow 0.$$

If B is etale over A, then $\Omega^1_{B/A} = 0$ and thus $a/a^2 \simeq \Omega^1_{R/A} \otimes_R B$. We have by definition the exact sequence of R-modules :

$$0 \longrightarrow a \longrightarrow R \xrightarrow{\lambda} B \longrightarrow 0,$$

which yields a long exact sequence of the torsion modules :

$$\begin{array}{ccccccccc} 0 = Tor^R_1(R,B) & \longrightarrow & Tor^R_1(B,B) & \longrightarrow & a \otimes_R B & \longrightarrow & R \otimes_R B & \longrightarrow & B \otimes_R B \longrightarrow 0. \\ & & & & & & \wr\Vert & & \wr\Vert \\ & & & & & & B & = & B \end{array}$$

This shows that $Tor^R_1(B,B) \simeq a \otimes_R B \simeq a \otimes_R R/a \simeq a/a^2$.

The following fact is easy :

Lemma 1.2. Let $A \longrightarrow A'$ be a flat A-algebra morphism and write $\lambda_{A'} : R \otimes_A A' \longrightarrow B \otimes_A A'$ for the base change morphism of λ. Then, we have that $C_1(\lambda_{A'};B\otimes_A A') \simeq C_1(\lambda;B)\otimes_A A'$.

<u>Lemma 1.3</u>. Let $\lambda : R \longrightarrow B$ and $\lambda' : R' \longrightarrow B'$ be surjective A-algebra morphisms in the category of finite flat A-algebras. Then, we have that $C_1(\lambda\otimes_A\lambda';B\otimes_A B') \simeq (C_1(\lambda;B)\otimes_A B')\oplus(C_1(\lambda';B')\otimes_A B)$.

<u>Proof</u>. Put $\mathfrak{a} = \mathrm{Ker}(\lambda)$ and $\mathfrak{a}' = \mathrm{Ker}(\lambda')$. By Lemma 1.2, we may assume that A is a local ring. Then we have by the flatness of B over A decompositions of A-modules : $R \simeq \mathfrak{a}\oplus B$ and $R' \simeq \mathfrak{a}'\oplus B'$. The kernel of $\lambda\otimes\lambda'$ is given by $\mathfrak{A} = (\mathfrak{a}\otimes_A\mathfrak{a}')\oplus(\mathfrak{a}\otimes_A B')\oplus(B\otimes_A\mathfrak{a}') = \mathfrak{a}\otimes_A R' + R\otimes_A\mathfrak{a}'$ in $R\otimes_A R'$. We see easily that

$$\mathfrak{A}^2 = \mathfrak{a}^2\otimes R' + R\otimes\mathfrak{a}^2 + \mathfrak{a}\otimes\mathfrak{a}' = (\mathfrak{a}\otimes\mathfrak{a}')\oplus(\mathfrak{a}^2\otimes_A B')\oplus(B\otimes_A(\mathfrak{a}')^2).$$

Thus we know that $\mathfrak{A}/\mathfrak{A}^2 = ((\mathfrak{a}/\mathfrak{a}^2)\otimes_A B')\oplus((\mathfrak{a}'/(\mathfrak{a}')^2)\otimes_A B)$, which finishes the proof.

Hereafter we suppose that

(1.1a) R <u>and</u> B <u>are flat over</u> A.

Let K be the quotient field of A. We now suppose that we can decompose $R\otimes_A K$ into the K-algebra direct sum :

(1.1b) $R\otimes_A K = B\otimes_A K\oplus D'$ with a K-algebra D'

so that λ induces the projection of R into $B\otimes_A K$. Let ρ be the projection of R into D' and put $D = \rho(R)$. We shall define another B-module :

$$C_0(\lambda;B) = (B\oplus D)/R.$$

This is the module of congruences between B and D. The following result is easy :

<u>Lemma 1.4</u>. Let $\mathfrak{b}$ be the kernel of ρ. Then we have that

$$C_0(\lambda;B) \simeq B/\mathfrak{b} \simeq D/\mathfrak{a} \simeq R/\mathfrak{a}\oplus\mathfrak{b} \quad \text{as R-modules.}$$

For each finite torsion B-module M, let $\phi_B(M)$ be the Fitting ideal of M over B (Northcott [10, Chap. 3]). We recall the definition : we take a finite presentation : $B^a \xrightarrow{\gamma} B^b \longrightarrow M \longrightarrow 0$ (exact). Note that $a \geq b$ since M is B-torsion. We write matricially γ as $C\in M_{a,b}(B)$ and let $\phi_B(M)$ be the ideal of B generated by all $b\times b$-

minors of C. Then $\phi_B(M)$ depends only on M. The following facts are known :

(1.2a) $\phi_B(M_1\oplus M_2) = \phi_B(M_1)\cdot\phi_B(M_2)$

(1.2b) for any multiplicative subset S of B,

$$\phi_{B_S}(M\otimes_B B_S) = \phi_B(M)B_S,$$

where B_S is the localization of B

(1.2c) If $M = B/\mathfrak{b}$ with an ideal $\mathfrak{b}$ of B, we have $\phi_B(M) = \mathfrak{b}$.

<u>Corollary 1.5</u>. $C_1(\lambda;B)$ is annihilated by $\phi_B(C_0(\lambda;B))$.

This follows from (1.2c) and Lemma 1.4.

<u>Lemma 1.6</u>. Suppose that $\mathrm{Hom}_A(R,A)\simeq R$ and $\mathrm{Hom}_A(B,A)\simeq B$ as R-modules. By identifying $\mathrm{Hom}_A(R,A)$ (resp. $\mathrm{Hom}_A(B,A)$) with R (resp. B), let λ^* be the adjoint map of $\lambda : R \longrightarrow B$. Then $\lambda^* : B \longrightarrow R$ is a B-linear map and $\phi_B(C_0(\lambda;B))$ is generated over B by $\lambda\circ\lambda^*\in\mathrm{Hom}_B(B,B) = B$.

<u>Proof</u>. We may assume by (1.2b) that A is a local ring. Let $\langle\,,\,\rangle_R$ (resp. $\langle\,,\,\rangle_B$) be the pairing : $R\times R \longrightarrow A$ (resp. $B\times B \longrightarrow A$) which induces the isomorphism $\mathrm{Hom}_A(R,A)\simeq R$ (resp. $\mathrm{Hom}_A(B,A)\simeq B$). Since A is local, R and B are A-free by (1.1a). Then we see easily that

$$\mathfrak{a} = \mathrm{Ker}(\lambda) = \mathrm{Ann}(\lambda^*(B)),$$

and thus, for $r\in R$, $b, c\in B$, and $b'\in R$ with $\lambda(b') = b$,

$$\langle r,\lambda^*(bc)\rangle_R = \langle b\lambda(r),c\rangle_B = \langle\lambda(b'r),c\rangle_B = \langle r,b'\lambda^*(c)\rangle_R = \langle r,b\lambda^*(c)\rangle_R.$$

This shows the B-linearity of λ^*. For any R-module M, we write $M^* = \mathrm{Hom}_A(M,A)$. Since $\mathfrak{a}$ is A-free, we have an exact sequence :

$$0 \longrightarrow B^* \xrightarrow{\lambda^*} R^* \longrightarrow \mathfrak{a}^* \longrightarrow 0.$$

Thus $\mathfrak{a}^*\simeq D$ as R-module. This shows that $\lambda^*(B) = \mathrm{Ker}(\rho)$, and $\phi_B(C_0(\lambda;B)) = \lambda\circ\lambda^*(B)$ by (1.2c).

<u>Lemma 1.7</u>. Let $\lambda_i : R_i \longrightarrow A$ be surjective homomorphisms satisfying (1.1a,b) for $i = 1,2,\ldots,h$. Put $\lambda = \lambda_1\otimes\ldots\otimes\lambda_h : R = R_1\otimes_A\ldots\otimes_A R_h \longrightarrow A$.

Then λ satisfies (1.1a,b) and if $\mathrm{Hom}_A(R_i,A)\simeq R_i$ as R_i-modules for all i, then $\phi_A(C_0(\lambda;A)) = \phi_A(C_0(\lambda_1;A))\dots\phi_A(C_0(\lambda_h;A))$ and it is a principal ideal.

Proof. The first assertion is obvious. We write $\langle\,,\rangle_i$ for the pairing of R_i over A by which we identify R_i with $\mathrm{Hom}_A(R_i,A)$. Then the pairing $\langle\,,\rangle : R\times R \longrightarrow A$ defined by $\langle a_1\otimes\dots\otimes a_h, b_1\otimes\dots\otimes b_h\rangle = \prod_{i=1}^{h}\langle a_i,b_i\rangle_i$ gives an isomorphism : $R\simeq \mathrm{Hom}_A(R,A)$ as R-modules. Then we see that $\lambda^* = \lambda_1^*\otimes\dots\otimes\lambda_h^*$ and $\lambda\circ\lambda^* = \prod_{i=1}^{h}\lambda_i\circ\lambda_i^*$. This combined with Lemma 1.6 shows the result.

Lemma 1.8. Under the same assumption as in Lemma 1.7, decompose $R_i\otimes_A K = K\oplus D_i'$ as in (1.1b) and let D_i be the image of R_i in D_i'. Suppose that $\mathrm{Hom}_A(D_i,A)\simeq D_i$ as R_i-modules for $i = 1,2,\dots,h$. Then, we have that for $\lambda = \lambda_1\otimes\dots\otimes\lambda_h$,

$$C_1(\lambda;A) = C_0(\lambda_1;A)\oplus\dots\oplus C_0(\lambda_h;A)$$

$$\phi_A(C_1(\lambda;A)) = \phi_A(C_0(\lambda;A)) = \phi_A(C_0(\lambda_1;A))\dots\phi_A(C_0(\lambda_h;A)).$$

Proof. By Lemmas 1.7 and 1.3, we may suppose that $h=1$. Write $R=R_1$ and take $\rho : R\longrightarrow D$ as in (1.1b). Since $D^* = \mathrm{Hom}_A(D,A)\simeq D$ as R-module, we know, in the same manner as in the proof of Lemma 1.6 that for $a=\mathrm{Ker}(\lambda)$,

$$D\simeq a^* \text{ as } R\text{-module}$$

and thus $a\simeq D^*\simeq D$ as R-module. Thus a is generated by $a = \rho\circ\rho^*\in \mathrm{Hom}_R(D,D)=D$, which is not a zero divisor. Thus, we know that

$$C_1(\lambda;A) = a/a^2 = aD/a^2D \simeq D/aD \simeq D/a \simeq C_0(\lambda;A).$$

This shows the result.

Lemma 1.9. Let G be a commutative finite group, and put $R = A[G]$ (the group algebra). For each character $\chi : G\longrightarrow A^\times$, let $\chi_a : R\longrightarrow A$ be the A-algebra homomorphism such that $\chi_a(\sigma)=\chi(\sigma)$ for all $\sigma\in G$. Then, we have that

$$C_0(\chi_a;A)\simeq A/|G|A \quad\text{and}\quad C_1(\chi_a;A)\simeq A\otimes_{\mathbb{Z}} G,$$

where $|G|$ denotes the cardinality of G and we regard G naturally as a $\mathbb{Z}$-module. In particular, we have that

$$\phi_A(C_0(\chi_a;A)) = \phi_A(C_1(\chi_a;A)) = |G|A.$$

Proof. Evidently, $\chi_a(1) = |G|^{-1}\sum_{\sigma\in G}\chi^{-1}(\sigma)\sigma$ in $\chi_a(A)=A$. This shows that $b = R\cap A$ is generated by $\sum_{\sigma\in G}\chi^{-1}(\sigma)\sigma$, and thus, $C_0(\lambda;A)\simeq A/b\simeq A/|G|A$, and $\phi_A(C_0(\lambda;A)) = |G|A$. Write $G = G_1\times\dots\times G_h$, where each G_i is a cyclic group. Then we see that

$$R = A[G_1]\otimes_A\dots\otimes_A A[G_h] \quad\text{and}\quad \chi_a = \chi_{1,a}\otimes\dots\otimes\chi_{h,a},$$

where χ_i is the restriction of χ to G_i. Thus, by Lemma 1.3 and Lemma 1.8, we may assume that G is cyclic of order N. Then $A[G]\simeq A[X]/(X^N-1)$. Let $F(X)$ be a monic polynomial of degree d in $A[X]$. Then $R' = A[X]/(F(X))$ has a basis $1,x,\dots,x^{d-1}$ for the image x of X in R'. For any $\xi\in R'$, write

$$\xi = a_1(\xi)+a_2(\xi)x+\dots+a_d(\xi)x^{d-1} \quad\text{with}\quad a_i(\xi)\in A.$$

Define a pairing $\langle\,,\rangle : R'\times R'\longrightarrow A$ by

$$\langle\xi,\eta\rangle = a_d(\xi\cdot\eta).$$

This gives an isomorphism : $\mathrm{Hom}_A(R',A)\simeq R'$ as R'-modules (in fact, $\langle x^i,x^{d-1-j}\rangle = \delta_{ij}$ if $i\le j$). By this fact, we know that $\mathrm{Hom}_A(R,A)\simeq R$ and $\mathrm{Hom}_A(D,A)\simeq D$ (because D is also generated by one element over A). Then the result follows from Lemma 1.7.

Remark 1.10. We use the same notation as in Lemma 1.9. Then the automorphism group $\mathrm{Aut}(G)$ naturally acts on A-algegra $R = A[G]$. Let $G_\chi = \{\sigma\in\mathrm{Aut}(G)\mid\chi\circ\sigma=\chi\}$. Then $C_0(\chi_a;A)$ and $C_1(\chi_a;A)$ can be naturally regarded as G_χ-module. By definition, G_χ acts trivially on $C_0(\lambda;A)$, and the isomorphism

$$C_1(\chi_a;A)\simeq A\otimes_{\mathbb{Z}} G$$

gives an isomorphism of G_χ-modules. In fact, we see easily that a is generated by $x_\sigma = \sigma-\chi(\sigma)$ for $\sigma\in G$ and a^2 is generated by

$x_{\sigma\tau} - \chi(\sigma)x_\tau - \chi(\tau)x_\sigma$ $(=x_\sigma \cdot x_\tau)$ for $\sigma, \tau \in G$. Thus $C_1(\chi_a;A) \simeq A \otimes_{\mathbb{Z}} G$ as G_χ-modules. More generally, if $\lambda : R \longrightarrow A$ be an A-algebra homomorphism satisfying (1.1a,b) and if $G_\lambda = \{\sigma \in Aut(R/A) | \lambda \circ \sigma = \lambda\}$, then $C_1(\lambda,R)$ and $C_0(\lambda;R)$ are G_λ-modules and G_λ acts trivially on $C_0(\lambda;A)$.

Let W be an abelian p-profinite group, and write W_t for the torsion part of W. Put $W_0 = W/W_t$. Suppose that

(1.3) $\quad W_0 \simeq \mathbb{Z}_p$ and W_t is a finite group.

Let Γ be a subgroup of W_0 of finite index. We may decompose (non-canonically) as $W = W_0 \times W_t$. Let A be a valuation ring finite flat over $\mathbb{Z}_p$, and write $A[[\Gamma]]$, $A[[W_0]]$ and $A[[W]]$ for the p-adically continuous group algebra of Γ, W_0 and W. We write simply Λ for $A[[\Gamma]]$.

<u>Lemma 1.11</u>. Let χ be a character of W/Γ with values in A. We denote by χ_a^Γ for the $A[[W_0]]$-algebra homomorphism

$$\chi_a^\Gamma : A[[W]] \otimes_\Lambda A[[W_0]] \longrightarrow A[[W_0]]$$

defined as follows : for $w = (w_0,w_t) \in W_0 \times W_t$ and $a \in A[[W_0]]$, $\chi_a^\Gamma(w \otimes a) = \chi(w)w_0 \cdot a \in A[[W_0]]$. Then we have that

$$C_1(\chi_a^\Gamma;A[[W_0]]) \simeq A[[W_0]] \otimes_{\mathbb{Z}} (W/\Gamma)$$

$$C_0(\chi_a^\Gamma;A[[W_0]]) \simeq A[[W_0]] \otimes_{A[[W_0]]} \mathbb{Z}/[W:\Gamma]\mathbb{Z}$$

<u>Proof</u>. Let u be a topological generator of W_0. Then we see that $u^{p^r} \in \Gamma$ for $p^r = [W_0:\Gamma]$, and thus $A[[W_0]] = \Lambda[X]/(X^{p^r} - u^{p^r})$. Write $W_t = G_1 \times \dots \times G_r$ for cyclic groups G_i and A for $A[[W_0]]$. Then we see that

$$A[[W]] \otimes_\Lambda A \simeq (A[X]/(X^{p^r} - u^{p^r})) \otimes_A A[G_1] \otimes \dots \otimes A[G_r].$$

Since we know that

$$\frac{d}{dX}(X^{p^r} - u^{p^r}) = p^r X^{p^r - 1},$$

we know that C_1 for $A[X]/(X^{p^r} - u^{p^r})$ is isomorphic to $A/p^rA \simeq A \otimes_{\mathbb{Z}} W_0/\Gamma$

by Lemma 1.1. Then the result follows from Lemma 1.8.

§ 2. <u>The case of</u> GL_1.

Let F be an arbitrary number field of finite degree over $\mathbb{Q}$ and $\mathcal{O}_F$ be the ring of integers of F. We take the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ inside of $\mathbb{C}$ and always consider algebraic number fields in $\overline{\mathbb{Q}}$. We fix a prime number p in $\mathbb{Z}$, and let Ω be the p-adic completion of an algebraic closure of $\mathbb{Q}_p$. The normalized norm of Ω is written as $| \, |_p$. We fix once and for all an embedding of $\overline{\mathbb{Q}}$ into Ω and consider $\overline{\mathbb{Q}}$ also a subfield of Ω. Put $F_\infty = F \otimes_{\mathbb{Q}} \mathbb{R}$ and $F_p = F \otimes_{\mathbb{Q}} \mathbb{Q}_p$, and let $F_A^\times$ be the idele group of F. Let $F_0^\times$ be the finite part of $F_A^\times$, i.e. $F_A^\times = F_0^\times \times F_\infty^\times$. For each prime $\mathfrak{p}$ of F, let $F_{\mathfrak{p}}$ be the $\mathfrak{p}$-adic completion of F and $\mathcal{O}_{F_{\mathfrak{p}}}$ be its $\mathfrak{p}$-adic integer ring. Put $\hat{\mathcal{O}}_F = \prod_{\mathfrak{p}} \mathcal{O}_{F_{\mathfrak{p}}}$. For $x \in F_A^\times$, let x_p, $x_{\mathfrak{p}}$ and x_∞ be its component in $F_p^\times$, $F_{\mathfrak{p}}^\times$ and $F_\infty^\times$, respectively. Let $F_{\infty+}^\times$ be the identity component of $F_\infty^\times$. Let $I = I_F$ be the set of all embeddings of F into $\overline{\mathbb{Q}}$ and $\mathbb{Z}[I]$ be the free $\mathbb{Z}$-module generated by the elements of I. We write $\ell \geq 0$ for $\ell = \sum_{\tau \in I} \ell_\tau \cdot \tau \in \mathbb{Z}[I]$ if $\ell_\tau \geq 0$ for all τ, and $\ell > 0$ if $\ell_\tau > 0$ for some τ. We fix an ideal N of $\mathcal{O}_F$ and put

$$E = \{\varepsilon \in \mathcal{O}_F^\times \mid \varepsilon \in F_{\infty+}^\times\}$$

$$U = U_F(N) = \{u \in (\hat{\mathcal{O}}_F)^\times \mid u \equiv \varepsilon \bmod N\hat{\mathcal{O}}_F \text{ for some } \varepsilon \in E\},$$

$$C\ell = C\ell_F(N) = F^\times \backslash F_A^\times / F_{\infty+}^\times U_F(N).$$

Then $C\ell$ is a finite group. We can associate to each $\ell = \sum_{\tau \in I} \ell_\tau \cdot \tau \in \mathbb{Z}[I]$ a character : $F^\times \longrightarrow \overline{\mathbb{Q}}^\times$ defined by

$$x \longmapsto x^\ell = \prod_{\tau \in I} (x^\tau)^{\ell_\tau}.$$

This character can be extended by continuity to the characters :

$$\ell : F_p^\times \longrightarrow \Omega^\times \quad \text{and} \quad \ell : F_\infty^\times \longrightarrow \mathbb{C}^\times.$$

For each $\ell \in \mathbb{Z}[I]$, we shall define "the space of modular forms" of GL_1 for $U_F(N)$ of weight ℓ, which will be denoted by $m_\ell(N)$, by the space of functions $f : F_A^\times \longrightarrow \mathbb{C}$ satisfying the following condition :

(2.1) $f(\alpha x u) = f(x) u_\infty^{-\ell}$ <u>for all</u> $\alpha \in F^\times$ <u>and</u> $u \in U_F(N) \cdot F_{\infty+}^\times$.

For each $y \in F_0^\times$, we can define an operator on $m_\ell(N)$ associated with a class of y modulo U by

$$f|[yU](x) = f(xy).$$

Then the group ring $\mathbb{Z}[F_0^\times/U]$ acts on $m_\ell(N)$. The following fact is easy :

<u>Lemma 2.1.</u> $m_\ell(N) \neq 0$ if and only if $\varepsilon^\ell = 1$ for all $\varepsilon \in E$. The space $m_\ell(N)$ is generated by Hecke characters $\lambda : F_A^\times \to \mathbb{C}$ such that the restriction of λ to $F_{\infty+}^\times$ coincides with the character : $x \mapsto x^{-\ell}$. If $m_\ell(N) \neq 0$ and λ is as above, then the correspondence : $m_0(N) \ni \mu \mapsto \lambda \cdot \mu \in m_\ell(N)$ gives an isomorphism : $m_0(N) \simeq m_\ell(N)$. Furthermore, it satisfies the relation : $(\lambda\cdot\mu)|[xU] = \lambda(x)(\mu|[xU])$.

For each $x \in F_0^\times$, write $x_{\mathfrak{p}} O_{F_{\mathfrak{p}}} = \mathfrak{p}^m$ and put $xO_F = \prod_{\mathfrak{p}} \mathfrak{p}^m$ as an ideal of O_F. Let $I(N) = I_F(N)$ be the set of ideals of O_F prime to N. For each $\mathfrak{a} \in I(N)$, we can find $x \in F_0^\times$ such that $x_{\mathfrak{p}} = 1$ if $\mathfrak{p}|N$ and $\mathfrak{a} = xO_F$. The correspondence : $\mathfrak{a} \mapsto x$ induces a map : $I(N) \to F_0^\times/U$, and the image of $I(N)$ covers $Cl_F(N)$. We may thus consider each element λ of $m_\ell(N)$ as a function on $I(N)$ such that

(2.2a) $\lambda(\alpha\mathfrak{a}) = \alpha^\ell \lambda(\mathfrak{a})$ <u>for any</u> $\alpha \in F_+^\times$ <u>with</u> $\alpha \equiv 1 \bmod^\times N$.

where $F_+^\times = \{\alpha \in F^\times | \alpha \in F_{\infty+}^\times\}$. Put $W(N) = \varprojlim_n Cl_F(Np^n)$.

Then, the set $I(Np)$ is a dense subset of the topological group $W(N)$.
If $\lambda \in m_\ell(Np^r)$ (for $r = 0,1,\ldots$) has values in $\overline{\mathbb{Q}}$ on $I(Np)$, we know
(according to Weil [15]) from (2.2a) that λ is continuous with respect
to the topology on $W(N)$ as a function on $I(Np)$ with values in Ω. Then,
By continuity, we thus have a continuous function $\hat\lambda : W(N) \to \Omega$.
we see easily that

(2.2b) $\hat\lambda(ux) = u_p^{-\ell}\hat\lambda(x)$ <u>for</u> $u \in U(Np^r)$ <u>if</u> $\lambda \in m_\ell(Np^r)$.

When λ is a Hecke character, then $\hat\lambda$ is a continuous character on $W(N)$ with values in Ω. For each $\mathfrak{a} \in I(N)$, we take a corresponding

element $xU \in F_0^\times/U$ and define the (Hecke) operator $T(\mathfrak{a})$ on $m_\ell(N)$ by the action of xU. Naturally the absolute Galois group $\mathfrak{G}_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $\mathbb{Z}[I_F]$ from the right. For each $\ell \in \mathbb{Z}[I]$, we put

$$\mathfrak{G}_\ell = \{\sigma \in \mathfrak{G}_{\mathbb{Q}} | \ell \cdot \sigma = \ell\}.$$

Then the fixed field $\Phi(\ell)$ of $\mathfrak{G}_\ell$ in $\overline{\mathbb{Q}}$ is a finite extension of $\mathbb{Q}$. Let $O_{\Phi(\ell)}$ be the integer ring of $\Phi(\ell)$. Then the character : $x \mapsto x^\ell$ has values in $\Phi(\ell)$ on F. For each $O_{\Phi(\ell)}$-subalgebra A of $\mathbb{C}$, put

$$m_\ell(N;A) = \{\lambda \in m_\ell(N) | \lambda(\mathfrak{a}) \in A \text{ for all } \mathfrak{a} \in I(N)\}.$$

From (2.2a), the following assertion is obvious :

<u>Lemma 2.2</u>. Suppose that $\ell \geq 0$. If A and A' are $O_{\Phi(\ell)}$-algebras in $\mathbb{C}$ such that $A' \supset A$, then $m_\ell(N;A) \otimes_A A' \subset m_\ell(N;A')$. Moreover, we have the equality : $m_\ell(N;A') = m_\ell(N;A) \otimes_A A'$ if one of the following assertions is satisfied :

(i) A' is A-free of finite rank;

(ii) A' is a localization of A;

(iii) A' is a field.

Let $g_\ell^F(N;O_{\Phi(\ell)})$ be the subalgebra of the endomorphism algebra of $m_\ell(N;\Phi(\ell))$ generated over $O_{\Phi(\ell)}$ by $T(\mathfrak{a})$ for all $\mathfrak{a} \in I(N)$. For each $O_{\Phi(\ell)}$-algebra A (not necessarily contained in $\mathbb{C}$), we put $g_\ell(N;A) = g_\ell^F(N;A) = g_\ell(N;O_{\Phi(\ell)}) \otimes_{O_{\Phi(\ell)}} A$. For any $O_{\Phi(\ell)}$-algebra A in Ω, we put $m_\ell(N;A) = m_\ell(N;O_{\Phi(\ell)}) \otimes_{O_{\Phi(\ell)}} A$. Then, for any $O_{\Phi(\ell)}$-algebra A in $\mathbb{C}$ or Ω, we may consider $g_\ell(N;A)$ as a subalgebra of $\mathrm{End}_A(m_\ell(N;A))$.

<u>Lemma 2.3</u>. Let A be an $O_{\Phi(\ell)}$-subalgebra of Ω or $\mathbb{C}$. Let λ be a Hecke character in $m_\ell(N;A)$ for $\ell \geq 0$. Then, we have an isomorphism of A-algebras associated with λ :

$$g_\ell^F(N;A) \simeq A[Cl_F(N)]$$

<u>Proof</u>. Let A_0 be the subalgebra of $\mathbb{C}$ generated over $O_{\Phi(\ell)}$ by $\lambda(\mathfrak{a})$

for all $\mathfrak{a} \in I(N)$. Since $g_\ell(N;A) = g_\ell(N;A_0) \otimes_{A_0} A$, we may suppose that $A = A_0$. By definition, we know that $g_0(N;A) \simeq A[C\ell]$. The multiplication of λ yields a commutative diagram :

$$\begin{array}{ccc} m_0(N;A) & \xrightarrow{\ \sim\ } & m_\ell(N;A) \\ \downarrow{\scriptstyle \lambda(\mathfrak{a})T(\mathfrak{a})} & & \downarrow{\scriptstyle T(\mathfrak{a})} \\ m_0(N;A) & \xrightarrow{\ \sim\ } & m_\ell(N;A). \end{array}$$

For each $\mathfrak{a} \in I(N)$, let $[\mathfrak{a}]$ be the class of $\mathfrak{a}$ in $C\ell$. Then, the correspondence

$$g_\ell(N;A) \ni T(\mathfrak{a}) \longmapsto \lambda(\mathfrak{a})[\mathfrak{a}] \in A[C\ell]$$

yields an embedding : $g_\ell(N;A) \longrightarrow A[C\ell]$. Obviously, the two algebras coincide after tensoring the quotient field K of A. Thus $g_\ell(N;A)$ is a lattice in $K[C\ell]$. For any prime number q of $\mathbb{Z}$, one can choose a complete representative set $\{\mathfrak{a}_1,\dots,\mathfrak{a}_h\}$ of $C\ell$ in $I(N)$ so that $\lambda(\mathfrak{a}_i)$ is prime to q for $1 \le i \le h$. Then over $A_q = A \otimes_{\mathbb{Z}} \mathbb{Z}_q$, $g_\ell(N;A_q) = A_q[C\ell]$. Since q is arbitrary and A is finite flat over $\mathbb{Z}$, we obtain the result.

Let A be an $O_{\Phi(\ell)}$-subalgebra in $\mathbb{C}$ or Ω. Define a pairing $\langle\ ,\ \rangle : g_\ell(N;A) \times m_\ell(N;A) \longrightarrow A$ by $\langle h,\lambda\rangle = (\lambda|h)(1)$.

Lemma 2.4. Suppose that A is a field or a valuation ring. Then the pairing as above is perfect.

Proof. Note that $\langle T(\mathfrak{a}),\lambda\rangle = \lambda(\mathfrak{a})$ for $\mathfrak{a} \in I(N)$. If A is a field, this shows the non-degeneracy of the pairing, hence its perfectness. Now we suppose that A is a valuation ring. Let K be the quotient field of A. If $\phi \in \mathrm{Hom}_A(g_\ell(N;A),A)$, then we find $\lambda \in m_\ell(N;K)$ such that $\langle h,\lambda\rangle = \phi(h)$ for all $h \in g_\ell(N;A)$. Since $\lambda(\mathfrak{a}) = \langle T(\mathfrak{a}),\lambda\rangle = \phi(T(\mathfrak{a})) \in A$, $\lambda \in m_\ell(N;A)$. This finishes the proof.

For $\lambda,\mu \in m_\ell(N)$, consider the function

$$\lambda * \mu : F_A^\times \longrightarrow \mathbb{C} \quad \text{defined by} \quad \lambda * \mu(x) = \lambda(x)\mu(x^{-1}).$$

From (2.2a) or (2.1), $\lambda * \mu \in m_0(N)$. Thus $\lambda * \mu(\mathfrak{a})$ depends only on the class of $\mathfrak{a}$ in $C\ell$. One sees easily from (2.2a) that if $\lambda \in m_\ell(N;A)$,

$\lambda(\mathfrak{a}^{-1})$ is contained in the quotient field of A. Let K be the quotient field of A. We shall define a pairing :

$$\langle\ ,\ \rangle : m_\ell(N;A) \times m_\ell(N;A) \longrightarrow K$$

by $\langle \lambda,\mu\rangle = \sum_{[\mathfrak{a}] \in C\ell} \lambda * \mu(\mathfrak{a})$.

Lemma 2.5. Suppose that A is a valuation ring in Ω finite over $\mathbb{Z}_p$ (containing $O_{\Phi(\ell)}$). Then the above pairing has values in A and is perfect. For each $\mathfrak{a} \in I(N)$, we have that

$$\langle \lambda|T(\mathfrak{a}),\mu\rangle = \langle \lambda,\mu|T(\mathfrak{a})\rangle.$$

Furthermore, we have an isomorphism of $g_\ell(N;A)$-modules :

$$g_\ell(N;A) \simeq \mathrm{Hom}_A(g_\ell(N;A),A).$$

Proof. We shall choose a representative set $\{\mathfrak{a}_1,\dots,\mathfrak{a}_h\}$ for $C\ell_F(N)$ in $I(Np)$. Then by (2.2a), we can find $\alpha \in O_F$ $(\alpha \ne 0)$ so that $|\alpha^\ell|_p = 1$, $\alpha \equiv 1 \bmod^\times N$ and $\alpha\mathfrak{a}_i^{-1} \in I(N)$. Then we see that $\alpha^\ell\mu(\mathfrak{a}_i^{-1}) = \mu(\alpha\mathfrak{a}_i^{-1}) \in A$ and thus $\mu(\mathfrak{a}_i^{-1}) \in A$ since $\alpha^\ell \in A^\times$. This shows that the pairing has values in A. Any $\mathfrak{a} \in I(N)$ can be written as $\mathfrak{a} = \alpha\mathfrak{a}_i$ and $\mathfrak{a} = \alpha'\mathfrak{a}_i^{-1}$ with $\alpha \equiv 1 \bmod^\times N$, $\alpha' \equiv 1 \bmod^\times N$, $\alpha \in \mathfrak{a}_i^{-1}$ and $\alpha' \in \mathfrak{a}_i$. Thus $|\alpha|_p \le 1$ and $|\alpha'|_p \le 1$, and we can define two basis $\{\lambda_i\}_{i=1,\dots,h}$, $\{\mu_i\}_{i=1,\dots,h}$ of $m_\ell(N;A)$ so that $\lambda_i(\mathfrak{a}_j) = \delta_{ij}$ and $\mu_i(\mathfrak{a}_j^{-1}) = \delta_{ij}$. Since $\lambda_j(\mathfrak{a}) = \alpha^\ell\lambda_j(\mathfrak{a}_i) \in A$ and $\mu_j(\mathfrak{a}) = \alpha'^\ell\mu_j(\mathfrak{a}_i^{-1}) \in A$, they are well defined. Then, we have that $\langle \lambda_i,\mu_j\rangle = \delta_{ij}$ and thus, the pairing is perfect. Since $\{\mathfrak{a}\mathfrak{a}_i\}_{i=1,\dots,h}$ for each $\mathfrak{a} \in I(N)$ forms a complete representative set for $C\ell_F(N)$ in $I(N)$, we see that $\langle \lambda|T(\mathfrak{a}),\mu\rangle = \sum_i \lambda(\mathfrak{a}\mathfrak{a}_i)\mu(\mathfrak{a}_i^{-1}) = \sum_i \lambda(\mathfrak{a}\mathfrak{a}_i)\mu(\mathfrak{a}\mathfrak{a}^{-1}\mathfrak{a}_i^{-1}) = \langle \lambda,\mu|T(\mathfrak{a})\rangle$. The last assertion follows from this and Lemma 2.4.

Let $\lambda \in m_\ell(N)$ be a Hecke character and let A be an $O_{\Phi(\ell)}$-algebra in $\mathbb{C}$ or Ω. If $\lambda \in m_\ell(N;A)$ $(\ell \ge 0)$, then λ induces an A-algebra homomorphism $\lambda_a : g_\ell(N;A) \longrightarrow A$ by $\lambda_a(T(\mathfrak{a})) = \lambda(\mathfrak{a})$.

Theorem 2.6. Let A be an $O_{\Phi(\ell)}$-algebra contained in $\mathbb{C}$ or Ω and λ be a Hecke character in $m_\ell(N;A)$ for $\ell \ge 0$. Then, the modules C_0 and C_1 associated with the A-algebra homomorphism $\lambda_a : g_\ell(N;A) \longrightarrow A$ is explicitly given by

$$C_1(\lambda_a;A) \simeq A \otimes_{\mathbb{Z}} C\ell_F(N), \quad C_0(\lambda_a;A) \simeq A/|C\ell_F(N)|A.$$

Furthermore, the Fitting ideals of these modules coincide, and any one of them is generated by the class number $|C\ell_F(N)|$.

This follows from Lemma 2.3 and Lemma 1.9.

Let L be a finite extension of F and let $N_{L/F} : L_A^\times \longrightarrow F_A^\times$ be the norm map. Write $U_L = U_L(N)$ for an ideal N of $\mathcal{O}_F$ and put $U_F = U_F(N)$. Then, $N_{L/F}$ induces a morphism of abelian groups $N_{L/F} : C\ell_L(N) \longrightarrow C\ell_F(N)$. We have also natural linear maps

$$\mathrm{res} : \mathbb{Z}[I_L] \longrightarrow \mathbb{Z}[I_F], \quad \inf : \mathbb{Z}[I_F] \longrightarrow \mathbb{Z}[I_L]$$

defined by $\mathrm{res}(\tau) = \tau|_F$ for $\tau \in I_L$ and $\inf(\tau) = \sum_{\mathrm{res}(\sigma)=\tau} \sigma$ for $\tau \in I_F$. Then the lifting map $\mathrm{Inf} : m_\ell(N;A) \longrightarrow m_{\inf(\ell)}(N;A)$ can be defined by $\mathrm{Inf}(\lambda) = \lambda \circ N_{L/F}$. Then we have a commutative diagram for each ideal $\mathfrak{a} \in I_L(N)$:

$$\begin{array}{ccc} \mathrm{Inf} : m_\ell(N;A) & \longrightarrow & m_{\inf(\ell)}(N;A) \\ \downarrow T(N_{L/F}(\mathfrak{a})) & & \downarrow T(\mathfrak{a}) \\ \mathrm{Inf} : m_\ell(N;A) & \longrightarrow & m_{\inf(\ell)}(N;A) . \end{array}$$

Thus the correspondence between the operator $T(\mathfrak{a})$ on $m_{\inf(\ell)}(N;A)$ and $T(N_{L/F}(\mathfrak{a}))$ on $m_\ell(N;A)$ gives an A-algebra homomorphism :

$$N^{\mathfrak{a}}_{L/F} : g_{\inf(\ell)}(N;A) \longrightarrow g_\ell(N;A).$$

One verifies the following formulae :

$(\lambda \circ N_{L/F})_{\mathfrak{a}} = \lambda_{\mathfrak{a}} \circ N^{\mathfrak{a}}_{L/F}$,

$\langle N^{\mathfrak{a}}_{L/F}(h), \lambda \rangle = \langle h, \lambda \circ N_{L/F} \rangle$ for the pairing $\langle\ ,\ \rangle$ defined in Lemma 2.4.

<u>Corollary 2.7</u>. The norm map $N^{\mathfrak{a}}_{L/F}$ induces natural morphisms :

$$N^0_{L/F} : C_0((\lambda \circ N_{L/F})_{\mathfrak{a}};A) \longrightarrow C_0(\lambda_{\mathfrak{a}};A),$$

$$N^1_{L/F} : C_1((\lambda \circ N_{L/F})_{\mathfrak{a}};A) \longrightarrow C_1(\lambda_{\mathfrak{a}};A).$$

Moreover, we have a commutative diagram :

$$\begin{array}{ccc} C_1((\lambda \circ N_{L/F})_{\mathfrak{a}};A) & \simeq & A \otimes_{\mathbb{Z}} C\ell_L(N) \\ \downarrow N^1_{L/F} & & \downarrow 1 \otimes N_{L/F} \\ C_1(\lambda_{\mathfrak{a}};A) & \simeq & A \otimes_{\mathbb{Z}} C\ell_F(N). \end{array}$$

§ 3. <u>Hecke algebras for</u> GL_2.

Hereafter always, we suppose that

(3.1) F <u>is totally real</u>.

We shall give in this section the definitions of Hecke algebras for the spaces of modular forms for $GL_2(F)$ and bring together without proof some results necessary for the comparison of modules of congruences for GL_1 and GL_2, which will be done in the next section. We shall indicate briefly their proof and the references if possible. A detailed proof will appear elsewhere. For the proofs for $F = \mathbb{Q}$ and for a generalization of the results in § 2 for $GL_2(\mathbb{Q})$, we refer to our previous papers [3], [4], [6] and [7]. We use the same notation for the field F as in § 2. Put

$$V = V_F(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\hat{\mathcal{O}}_F) \mid c \in N\hat{\mathcal{O}}_F,\ d \equiv \varepsilon \bmod N\hat{\mathcal{O}}_F \text{ for some } \varepsilon \in E\}$$

for the fixed integral ideal N of $\mathcal{O}_F$. Let $G_{\infty+}$ be the identity component of $GL_2(F_\infty)$, $G_A = GL_2(F_A)$ and G_0 be the finite part of G_A. Let $\mathfrak{Z}$ be the product $\mathfrak{H}^I$ of the upper half complex planes indexed by I_F, on which $G_{\infty+}$ acts via linear fractional transformation. Put $z_0 = (\sqrt{-1},\ldots,\sqrt{-1}) \in \mathfrak{Z}$ and $C_{\infty+} = \{g \in G_{\infty+} \mid g(z_0) = z_0\}$. We denote by $S_k(N) = S^F_k(N)$ for an integer $k \geq 0$ the space of functions $f : G_A \longrightarrow \mathbb{C}$ which satisfies the following conditions :

(3.2a) <u>Put, for</u> $\gamma = (\gamma_\tau)_{\tau\in I} = \left(\begin{pmatrix} a_\tau & b_\tau \\ c_\tau & d_\tau \end{pmatrix}\right)_{\tau\in I} \in G_{\infty+}$,

$$j_k(\gamma,z) = \prod_{\tau\in I} \det(\gamma_\tau)^{-1}(c_\tau z_\tau + d_\tau)^k \qquad (z \in \mathfrak{Z}).$$

<u>Then</u> $f(\alpha x u) = j_k(u_\infty, z_0)^{-1} f(x)$ <u>for all</u> $\alpha \in GL_2(F)$ <u>and</u> $u \in V_F(N)C_{\infty+}$

(3.2b) <u>For each</u> $x \in G_0$ <u>and</u> $z \in \mathfrak{Z}$, <u>put</u>

$$f_x(z) = j_k(u_\infty, z_0) f(xu_\infty) \quad \text{for} \quad u_\infty \in G_{\infty+} \quad \text{with} \quad z = u_\infty(z_0).$$

<u>Then</u> $\dfrac{\partial f_x}{\partial \bar{z}_\tau} = 0$ <u>for all</u> $\tau \in I$.

(3.2c) <u>For a Haar measure</u> du <u>on the additive group</u> F_A/F,

$$\int_{F_A/F} f(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} x)du = 0 \quad \underline{\text{for all}} \quad x \in G_A.$$

(If $F = \mathbb{Q}$, we have to add the holomorphy condition at each cusps).

In order to describe the Fourier expension of elements of $S_k(N)$, we define some notations : Put $e_F(x) = \exp(2\pi\sqrt{-1} \cdot \sum_{\tau \in I} x_\tau)$ for $x \in \mathbb{C}^I$. Especially, for $\xi \in F^\times$, $e_F(\xi x) = \exp(2\pi\sqrt{-1} \cdot \sum_{\tau \in I} \xi^\tau x_\tau)$. Let ψ be the standard additive character of F_A/F; i.e., ψ is the unique one satisfying $\psi(x_\infty) = e_F(x_\infty)$ for $x_\infty \in F_\infty$. Then $f \in S_k(N)$ has the following form of the Fourier expansion (e.g. [13, (2.18)]) :

(3.3)
$$f(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}) = N_{F/\mathbb{Q}}(y\mathcal{O}_F)^{-1} \sum_{\xi \in F^\times_+} a(\xi y \mathfrak{d}, f) y_\infty^t e_F(\sqrt{-1}\,\xi y_\infty)\psi(\xi x) \quad (y \in F_0^\times F_{\infty+}^\times, x \in F_A),$$

where $\mathfrak{d}$ denote the different of $F/\mathbb{Q}$ and $t = \sum_{\tau \in I} \tau \in \mathbb{Z}[I]$ and $F^\times_+ = \{\alpha \in F^\times \mid \alpha^\tau > 0 \text{ for all } \tau \in I\}$ (the exponent of y_∞ is usually $\frac{k}{2} \cdot t$, but in our case, f is not "unitarized" by (3.2a); so, it becomes t instead of $\frac{k}{2} \cdot t$). The function : $\mathfrak{a} \longmapsto a(\mathfrak{a}, f)$ in (3.3) is a function on the set of all fractional ideals of $\mathcal{O}_F$ such that $a(\mathfrak{a}, f) = 0$ unless $\mathfrak{a}$ is integral.

We shall now define Hecke operators. For each integral ideal $\mathfrak{a}$ of $\mathcal{O}_F$, we take an element $a \in F_0$ such that $a\mathcal{O}_F = \mathfrak{a}$. We decompose the double coset $V\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}V$ as a disjoint union of right cosets $\coprod_i x_i V$ and define an operator by

$$f|T(\mathfrak{a})(x) = \sum_i f(xx_i) \quad \text{for} \quad f \in S_k(N).$$

Then, $T(\mathfrak{a})$ gives an operator on $S_k(N)$ into itself. Similarly, if $\mathfrak{a} \in I_F(N)$, we can define an action of $C\ell_F(N)$ on $S_k(N)$ by

$$(f|\mathfrak{a})(x) = f(xa)N_{F/\mathbb{Q}}(\mathfrak{a})^{2-k}.$$

The effect of the Hecke operator $T(\mathfrak{n})$ on the coefficients $a(\mathfrak{m}, f)$ is given by

$$(3.4) \qquad a(\mathfrak{m}, f|T(\mathfrak{n})) = \sum_{\substack{\mathfrak{m}+\mathfrak{n} \subset \mathfrak{a} \\ (\mathfrak{a}, N)=1}} N_{F/\mathbb{Q}}(\mathfrak{a})^{k-1} a(\mathfrak{a}^{-2}\mathfrak{m}\mathfrak{n}, f|\mathfrak{a}).$$

(cf. [13, (2.20)]). When $F = \mathbb{Q}$, if we define $f_1(z)$ as in (3.2b), f_1 is an element of $S_k(\Gamma_1(N))$ in the usual sense for $\Gamma_1(N) = SL_2(\mathbb{Z}) \cap V_{\mathbb{Q}}(N)$, and $f_1(z) = \sum_{n=1}^{\infty} a(n, f)e_{\mathbb{Q}}(nz) \quad (z \in \mathfrak{H})$. Thus, by this correspondence, we know that $S_k^{\mathbb{Q}}(N) \simeq S_k(\Gamma_1(N))$, and the definition of the Hecke operator $T(n)$ coincides with the classical one. For any subring A of $\mathbb{C}$, we put

$$S_k(N;A) = S_k^F(N;A) = \{f \in S_k^F(N) \mid a(\mathfrak{a}, f) \in A \text{ for all } \mathfrak{a} \in I_F(A)\}.$$

Then it is known by [13, Prop. 1.7] or [8, (1.2.15,16)] that if A is finite over $\mathbb{Z}$ or a field

$$(3.5) \qquad S_k(N;\mathbb{Z}) \otimes_{\mathbb{Z}} A = S_k(N;A).$$

For any subalgebra A of Ω, we put

$$S_k(N;A) = S_k(N;\mathbb{Z}) \otimes_{\mathbb{Z}} A.$$

Furthermore, $S_k(N;A)$ is stable under the action of $T(\mathfrak{a})$ and $C\ell_F(N)$. This result follows from [13, Th. 1.5] when A is a $\mathbb{Q}$-algebra. For an arbitrary A, we need the q-expansion principle in [8, (1.2.15-16)], which guarantees the stableness of $S_k(N;A)$ under $C\ell_F(N)$. Then, from (3.4), we see the stableness under $T(\mathfrak{a})$. Let $h_k(N;\mathbb{Z}) = h_k^F(N;\mathbb{Z})$ be a subalgebra of $\mathrm{End}_{\mathbb{Q}}(S_k(N;\mathbb{Q}))$ generated over $\mathbb{Z}$ by $T(\mathfrak{a})$ for all $\mathfrak{a} \in I_F(1)$. One can show that the action of $C\ell_F(N)$ gives a character of $C\ell_F(N)$ with values in $h_k(N;\mathbb{Z})$. Put

$$h_k(N;A) = h_k^F(N;A) = h_k^F(N;\mathbb{Z}) \otimes_{\mathbb{Z}} A$$

for each commutative algebra A. If A is a subring of Ω or $\mathbb{C}$, then $h_k(N;A)$ may be considered as a subalgebra of $\mathrm{End}_A(S_k(N;A))$. We shall define a pairing

$$(3.6) \quad \langle\, , \rangle : h_k(N;A) \times S_k(N;A) \longrightarrow A \quad \text{by} \quad \langle h, f \rangle = a(1, f|h).$$

Then, as in [6, Prop. 2.1], we have

Proposition 3.1. When A is a field or a discrete valuation ring in Ω or $\mathbb{C}$, the pairing (3.6) is perfect.

Now we shall assume that $F=\mathbb{Q}$ and take a primitive form f in $S_k^{\mathbb{Q}}(N)$. Write $L(s,f)=\sum_{n=1}^{\infty} a(n,f)n^{-s}=\prod_q((1-\alpha_q q^{-s})(1-\beta_q q^{-s}))^{-1}$. Let ψ be the character of $(\mathbb{Z}/N\mathbb{Z})^{\times}$ defined by

$$f|n=\psi(n)f \quad \text{for} \quad 0<n\in\mathbb{Z} \quad \text{prime to } N.$$

Let ψ_0 be the primitive Dirichlet character which induces ψ and put

$$D(s,f)=\prod_q((1-\overline{\psi}_0(q)\alpha_q^2 q^{-s})(1-\overline{\psi}_0(q)\alpha_q\beta_q q^{-s})(1-\overline{\psi}_0(q)\beta_q^2 q^{-s}))^{-1}.$$

We suppose that $p\geq 5$ and that f is ordinary at p; i.e. $|a(p,f)|_p=1$. Let $\mathbb{Q}_p(f)$ be the subfield of Ω generated over $\mathbb{Q}_p$ by $a(n,f)$ for all n, and let $\mathcal{O}_p(f)$ be the p-adic integer ring of $\mathbb{Q}_p(f)$. We can associate an $\mathcal{O}_p(f)$-algebra homomorphism f_a of $h_k^{\mathbb{Q}}(N;\mathcal{O}_p(f))$ onto $\mathcal{O}_p(f)$ by $f_a(T(n))=a(n,f)$. On the other hand, one can define a canonical transcendental factor $U_\infty(f)$ as in [7, (10.8b)] of $D(k,f)$. Then, we restate here [7, Th. 10.5] for our later use :

Theorem 3.2. Let R be the local ring of $h_k(N;\mathcal{O}_p(f))$ through which f_a factors through. Suppose that $\mathrm{Hom}_{\mathcal{O}_p(f)}(R,\mathcal{O}_p(f))\simeq R$ as R-modules, and further assume one of the following conditions :

(3.7a) N is prime to p and $k\not\equiv 2 \bmod p-1$;

(3.7b) $k=2$ and the tame part of the restriction of ψ to $\mathbb{Z}_p^{\times}$ is non-trivial. Then there exists a p-adic unit $U_p(f)$ in Ω such that

$$C(f)=D(k,f)/U_\infty(f)U_p(f)\in\mathcal{O}_p(f) \quad \text{and} \quad \phi_{\mathcal{O}_p(f)}(C_0(f_a;\mathcal{O}_p(f)))$$

is generated by $C(f)$ as an ideal of $\mathcal{O}_p(f)$.

§ 4. Functoriality between GL_1 and GL_2.

Let M be a totally imaginary quadratic extension of F (i.e. CM-field). The result here may be generalized to any quadratic extension of F, but we shall content ourselves with CM-fields. Let D be the relative discriminant of M/F. We fix a CM-type J of M; i.e., J is a subset of I_M such that $J\ni\tau\mapsto\tau|_F\in I_F$ gives an isomorphism : $J\simeq I_F$. We write $\varphi=\sum_{\tau\in J}\tau$. Let A be an $\mathcal{O}_{\Phi(\varphi)}$-algebra in Ω or $\mathbb{C}$. Then, it is known after Hecke (e.g. [13, § 5] or [2, 5.2]) that there is an A-linear map for each $0<j\in\mathbb{Z}$ and each ideal $\mathfrak{c}$ of $\mathcal{O}_M$:

(4.1a) $$\theta : m_{j.\varphi}^M(\mathfrak{c};A)\longrightarrow S_k^F(N_{M/F}(\mathfrak{c})D;A) \quad \text{for} \quad k=j+1$$

such that for $\lambda\in m_{j.\varphi}^M(\mathfrak{c};A)$,

$$a(\mathfrak{a};\theta(\lambda))=\sum_{\substack{\mathfrak{b}\in I_M(\mathfrak{c})\\ N_{M/F}(\mathfrak{b})=\mathfrak{a}}}\lambda(\mathfrak{b}) \quad \text{for each} \quad \mathfrak{a}\in I_F(1).$$

By definition, θ satisfies the following relation for each prime ideal $\mathfrak{p}$ of $\mathcal{O}_F$:

(4.16) $$\theta(\lambda)|T(\mathfrak{p})=\begin{cases}\theta(\lambda|(T(\mathfrak{P})+T(\overline{\mathfrak{P}}))) & \text{if } \mathfrak{p}\mathcal{O}_M=\mathfrak{P}\overline{\mathfrak{P}},\ \mathfrak{P}\neq\overline{\mathfrak{P}} \text{ and } \mathfrak{p}\in I_M(\mathfrak{c})\\ 0 & \text{if } \mathfrak{p}\mathcal{O}_M \text{ is a prime ideal of } \mathcal{O}_M\\ \theta(\lambda|T(\mathfrak{P})) & \text{if } \mathfrak{p}\mathcal{O}_M=\mathfrak{P}^2, \text{ or } \mathfrak{p}\mathcal{O}_M=\mathfrak{P}\overline{\mathfrak{P}} \text{ with } \overline{\mathfrak{P}}|\mathfrak{c} \\ & \text{but } \mathfrak{P}\nmid\mathfrak{c}.\end{cases}$$

Thus, this induces an A-algebra homomorphism :

$$\theta^* : h_k^F(N_{M/F}(\mathfrak{c})D;A)\longrightarrow g_{j.\varphi}^M(\mathfrak{c};A) \quad \text{for} \quad k=j+1,$$

which satisfies $\langle\theta^*(h),\lambda\rangle_g=\langle h,\theta(\lambda)\rangle_h$ where $\langle\ ,\ \rangle_h$ is the pairing in (3.6) and $\langle\ \rangle_g$ is that of Lemma 2.4.

Suppose the following conditions (e.g. [8, p. 204]) :

(4.2a) Every prime ideal of $\mathcal{O}_F$ over p splits in $\mathcal{O}_M$; namely, if $\mathfrak{p}$ is a prime ideal of $\mathcal{O}_F$ over p, then $\mathfrak{p}\mathcal{O}_M=\mathfrak{P}\overline{\mathfrak{P}}$ with $\mathfrak{P}\neq\overline{\mathfrak{P}}$.

(4.2b) J is ordinary; namely, whenever $\sigma\in J$ and $\tau\in I_M-J$, then the pull back of the norm of Ω by σ and τ on M is different.

Let P be the product of prime ideals $\mathfrak{P}$ of $\mathcal{O}_M$ over p such that $\mathfrak{P}^{\varphi}=\prod_{\tau\in J}\mathfrak{P}^{\tau}$ is contained in $\{\alpha\in\mathcal{O}_{\Phi(\varphi)}\ |\ |\alpha|_p<1\}$. By the assumption (4.2a,b), P and P^{ρ} for complex conjugation ρ are

mutually prime and p divides $(PP^{\rho})^m$ for sufficiently large m. Let A be a discrete valuation ring finite over $\mathbb{Z}_p$ (in Ω) containing $\mathcal{O}_{\Phi(\varphi)}$. Let K be the quotient field of A. Put

$$m^M_{j.\varphi}(N;K/A) = m^M_{j.\varphi}(N;A) \otimes_A K/A,$$

$$S^F_k(N;K/A) = S^F_k(N;A) \otimes_A K/A.$$

Proposition 4.1. Suppose that $\mathfrak{c}$ is prime to p, and write $P_F = P \cap F$ and $N = N_{M/F}(\mathfrak{c})D$. Let $\theta : m^M_{j.\varphi}(\mathfrak{c}P^r;K/A) \longrightarrow S^F_k(NP^r_F;K/A)$ be the map induced from (4.1a). Put $W = \prod_{\mathfrak{p}|P} \mathcal{O}^{\times}_{F_{\mathfrak{p}}}$. Then, via $\hat{\lambda}(x) \longmapsto \hat{\lambda}(wx)$, W acts on $m^M_{j.\varphi}(\mathfrak{c}P^r;K/A)$. Futhermore, $\mathrm{Ker}(\theta)$ is contained in $H^0(W, m^M_{j.\varphi}(\mathfrak{c}P^r;K/A))$.

Proof. Put $X = \varprojlim_r C\ell_M(\mathfrak{c}P^r)$, $X^{\rho} = \varprojlim_r C\ell_M(\mathfrak{c}P^{\rho r})$ and $Y = \varprojlim_r C\ell_M(\mathfrak{c}p^r)$. As seen in paragraph 2, one can regard each element of $m^M_{j.\varphi}(\mathfrak{c}P^r;K/A)$ as a continuous function on Y with values in the discrete module K/A. Thus W acts on $m^M_{j.\varphi}(\mathfrak{c}P^r;K/A)$ via translation. By the assumption (4.2a,b), (2.2b) shows that the function in $m^M_{j.\varphi}(\mathfrak{c}P^r;K/A)$ factors through X. Let $C(X;K/A)$ be the space of continuous functions on X with values in K/A. Thus $m_{j.\varphi}(\mathfrak{c}P^r;K/A) \subset C(X;K/A)$. For any $\phi \in C(X;K/A)$ and $\mathfrak{a} \in I_F(1)$, we put $a(\mathfrak{a},\theta(\phi)) = \sum_{\substack{\mathfrak{b}\in I_M(\mathfrak{c}P) \\ N_{M/F}(\mathfrak{b})=\mathfrak{a}}} \phi(\mathfrak{b})$. There are projection maps : $Y \longrightarrow X$ and $Y \longrightarrow X^{\rho}$ and thus, there is a diagonal map $\Delta : Y \longrightarrow X \times X^{\rho}$. Put $W^{\rho} = \prod_{\mathfrak{p}|P^{\rho}} \mathcal{O}^{\times}_{M_{\mathfrak{p}}}$. We have a commutative diagram :

$$\begin{array}{ccc} W \times W^{\rho} & \xrightarrow{\ \delta\ } & X \times X^{\rho} \\ \downarrow & & \| \\ Y & \xrightarrow{\ \Delta\ } & X \times X^{\rho}, \end{array}$$

where all the maps are natural ones. Thus the image $\Delta(Y)$ contains the image $\delta(W \times W^{\rho})$. Let $S = \{\mathfrak{Q} \in I_M(\mathfrak{c}p) | \mathfrak{Q}$ is totally split over $\mathbb{Q}\}$. Then, by the Tchebotarev density theorem, S is dense in Y and thus in $\Delta(Y)$. For $\phi \in C(X;K/A)$, define $\hat{\phi} \in C(X \times X^{\rho};K/A)$ by $\hat{\phi}(x,y) = \phi(x) + \phi(y^{\rho})$. Then we see that for $\mathfrak{q} = N_{M/F}(\mathfrak{Q})$

$$\hat{\phi}(\mathfrak{Q},\mathfrak{Q}) = a(\mathfrak{q},\theta(\phi)) \quad \text{for} \quad \mathfrak{Q} \in S.$$

If $a(\mathfrak{q},\theta(\phi)) = 0$ for all $\mathfrak{Q} \in S$, then $\hat{\phi} = 0$ on $\Delta(Y)$. Since $\Delta(Y) \supset \delta(W \times W^{\rho})$, $\phi(tx) = -\phi(t'y)$ for all $x, y \in W$ and $(t,t') \in \Delta(Y)$. This shows that ϕ is invariant under the action of W, which finishes the proof.

For any integer $q > 0$, put $\mu_q = \{\zeta \in \Omega \mid \zeta^q = 1\}$.

Write $P = \mathfrak{P}_1 \ldots \mathfrak{P}_r$ and put $\mathfrak{p}_i = \mathfrak{P}_i \cap F$. Let e_i be the ramification index of $\mathfrak{p}_i$ over $\mathbb{Q}$. Then $\mu = \mu_{q_1} \times \ldots \times \mu_{q_r} \subset W$ for $q_i = N_{F/\mathbb{Q}}(\mathfrak{p}_i) - 1$. For $\zeta = (\zeta_1,\ldots,\zeta_r) \in \mu$, we see from (2.2b) that

(4.3) $\quad \hat{\lambda}|\zeta = \prod_{i=1}^{r} N_{M_i/\mathbb{Q}_p}(\zeta_i)^{-1}\hat{\lambda}$ for $M_i = M_{\mathfrak{P}_i}$ if $\lambda \in m^M_{j.\varphi}(\mathfrak{c};K/A)$.

Let ψ be a character of μ into A, and put

$$m_{j.\varphi}(\mathfrak{c}P^r;\psi,K/A) = \{\lambda \in m_{j.\varphi}(\mathfrak{c}P^r;K/A) \mid \hat{\lambda}|\zeta = \psi(\zeta)\hat{\lambda}\}$$

$$g_{j.\varphi}(\mathfrak{c}P^r;\psi;A) = \{h \in g_{j.\varphi}(\mathfrak{c}P^r;A) \mid h|\zeta = \psi(\zeta)h\},$$

where on the Hecke algebra, μ acts through the adjoint action under the pairing in Lemma 2.4. By the assumption (4.2a), we may identify W with $\prod_{\mathfrak{p}|p} \mathcal{O}^{\times}_{F_{\mathfrak{p}}}$ and regard it as a subgroup of $F^{\times}_A$. Now we denote the class of $w \in W$ in $C\ell_F(NP^m_F)$ by $[w]$, then we have

(4.4) $$\theta(\lambda)|[w] = N_{F_p/\mathbb{Q}_p}(w)^j\, \theta(\lambda|w).$$

Considering μ as a subgroup of $C\ell_F(NP^m_F)$ via $\zeta \longmapsto [\zeta]$, we put

$$S^F_k(NP^m_F,\psi;K/A) = \{f \in S^F_k(NP^m_F,K/A) \mid f|\zeta = \psi(\zeta)f \text{ for } \zeta \in \mu\}$$

$$h^F_k(NP^m_F,\psi;A) = \{h \in h^F_k(NP^m_F,A) \mid h|\zeta = \psi(\zeta)h \text{ for } \zeta \in \mu\}.$$

By (4.4), θ induces morphisms :

(4.5) $$\theta : m^M_{j.\varphi}(\mathfrak{c}P^m,\psi';K/A) \longrightarrow S^F_k(NP^m_F,\psi;K/A) \quad (k = j+1)$$

$$\theta^* : h^F_k(NP^m_F,\psi;K/A) \longrightarrow g^M_{j.\varphi}(\mathfrak{c}P^m,\psi';A),$$

where $\psi'(\zeta) = N_{F_p/\mathbb{Q}_p}(\zeta)^{-j}\psi(\zeta)$.

Corollary 4.2. If ψ' as above is non-trivial, then θ is injective and θ^* is surjective. Especially, if $m=0$ and $je_i \not\equiv 0 \bmod p-1$ for one of $i=1,\dots,r$, θ is injective and θ^* is surjective.

Proof. The pairing in Lemma 2.4 (resp. (3.6)) induces the Pontryagin duality between $g_{j.\varphi}(\mathfrak{c}P^m,\psi';A)$ and $m_{j.\varphi}(\mathfrak{c}P^m,\psi';K/A)$ (resp. $h_k(NP_F^m,\psi;A)$ and $S_k(NP_F^m,\psi;K/A)$). Thus the surjectivity of θ^* follows from the injectivity of θ, which is a consequence of Proposition 4.1. When $m=0$, μ acts on $m_{j.\varphi}(\mathfrak{c};K/A)$ by the character

$$\zeta \longmapsto N_{T_i/\mathbb{Q}_p}(\zeta)^{-e_i j} \quad \text{for } \zeta\in\mu_{q_i},$$

where T_i is the maximal unramified extension of $\mathbb{Q}_p$ in F_i.
Since $N_{T_i/\mathbb{Q}_p}:\mu_{q_i}\longrightarrow\mu_{p-1}$ is surjective, this character is non-trivial if and only if $je_i\not\equiv 0 \bmod p-1$. This shows the last assertion.

Theorem 4.3. Let $\mathfrak{C}=\mathfrak{c}\mathfrak{P}_1^{i_1}\dots\mathfrak{P}_r^{i_r}$ with $\mathfrak{c}$ prime to p, and let λ be a primitive Hecke character in $m_{j.\varphi}(\mathfrak{C};A)$ with $j>0$. Let $R(\lambda)$ (resp. $R(\theta(\lambda))$) be the local ring of $g^M_{j.\varphi}(\mathfrak{C};A)$ (resp. $h^F_k(DN_{M/F}(\mathfrak{C});A)$) through which λ_a (resp. $\lambda_a\circ\theta^*$) factors. Then we have

(i) θ^* is of finite cokernel and induces morphisms of A-modules :

$$\theta_0 : C_0(\lambda_a\circ\theta^*;A)\longrightarrow C_0(\lambda_a;A)\simeq A/|C\ell_M(\mathfrak{C})|A,$$

$$\theta_1 : C_1(\lambda_a\circ\theta^*;A)\longrightarrow C_1(\lambda_a;A)\simeq A\otimes_{\mathbb{Z}} C\ell_M(\mathfrak{C}) .$$

Especially θ_0 is surjective.

(ii) Let ψ be the restriction of λ to μ. If ψ is non-trivial, then $\theta^* : R(\theta(\lambda))\longrightarrow R(\lambda)$ is surjective, and we have a natural exact sequence :

$$C_1(\theta^*;R(\lambda))\otimes_{R(\lambda)}A \longrightarrow C_1(\lambda_a\circ\theta^*;A)\xrightarrow{\theta_1} C_1(\lambda_a;A)\longrightarrow 0.$$

Proof. The fact that θ^* has finite cokernel follows from Proposition 4.1 since $j>0$. It is known that if λ is primitive modulo $\mathfrak{C}$, then $\theta(\lambda)$ is also primitive of conductor $N_{M/F}(\mathfrak{C})D$. Then we can decompose

$$R(\lambda)\otimes_A K\simeq K\oplus D(\lambda) \quad\text{and}\quad R(\theta(\lambda))\otimes_A K\simeq K\oplus D(\theta(\lambda))$$

as algebra direct sum so that the projections into K coincides with λ_a and $\lambda_a\circ\theta^*$, respectively. By definition, θ^* induces a surjection : $C_0(\lambda_a\circ\theta^*;A)\longrightarrow A/K\cap\theta^*(R(\theta(\lambda)))$. Evidently,

$$\theta^*(R(\theta(\lambda)))\cap K\subset R(\lambda)\cap K \quad\text{in}\quad K\oplus D(\lambda).$$

Since $C_0(\lambda_a;A)\simeq A/K\cap R(\lambda)$, θ_0 is surjective. The other assertions in (i) follows from Theorem 2.6. The surjectivity of θ^*, when ψ is non-trivial, follows from Corollary 4.2. Thus, we have surjective morphisms of A-algebras :

$$\theta^*:R(\theta(\lambda))\longrightarrow R(\lambda),\ \lambda_a:R(\lambda)\longrightarrow A \ \text{and}\ \lambda_a\circ\theta^*:R(\theta(\lambda))\longrightarrow A.$$

This implies that the following sequence is exact :

$$0\longrightarrow \mathrm{Ker}(\theta^*)\longrightarrow \mathrm{Ker}(\lambda_a\circ\theta^*)\xrightarrow{\theta^*}\mathrm{Ker}(\lambda_a)\longrightarrow 0.$$

This induces another exact sequence :

$$\begin{array}{ccccc} \mathrm{Ker}(\theta^*)\otimes_{R(\theta(\lambda))}A & \to & \mathrm{Ker}(\lambda_a\circ\theta^*)\otimes_{R(\theta(\lambda))}A & \to & \mathrm{Ker}(\lambda_a)\otimes_{R(\theta(\lambda))}A\to 0 \\ \cong & & \cong & & \cong \end{array}$$

$$C_1(\theta^*;R(\lambda))\otimes_{R(\theta(\lambda))}A=\mathrm{Ker}(\theta^*)\otimes_{R(\theta(\lambda))}R(\lambda)\otimes_{R(\theta(\lambda))}A\to C_1(\lambda_a\circ\theta^*;A)\to C_1(\lambda_a;A)\to 0.$$

This shows the assertion (ii).

§ 5. Relation to the Iwasawa theory over imaginary quadratic fields.

Hereafter, we suppose that $F=\mathbb{Q}$ and that M is an imaginary quadratic field such that

(5.1) the prime p splits in M $(p>2)$.

We write $p=\mathfrak{P}\bar{\mathfrak{P}}$ for $\mathfrak{P}=\{x\in O_M \mid |x|_p<1\}$. Put

$$W=\varprojlim_n C\ell_M(\mathfrak{P}^n).$$

Then, W is a product of torsion subgroup W_t and a p-profinite group W_0 isomorphic to $\mathbb{Z}_p$. Let F_∞ be the unique $\mathbb{Z}_p$-extension of $\mathbb{Q}$ unramified outside p. Put $\Gamma_r=1+p^{r=1}\mathbb{Z}_p\subset\mathbb{Z}_p^\times$ and identify Γ $(=\Gamma_0)$

with $\mathrm{Gal}(F_\infty/\mathbb{Q})$ naturally. Write F_n for the fixed field of Γ_n in F_∞. Put $M_n = F_n \cdot M$. Then, M_∞ is a $\mathbb{Z}_p$-extension over M. Let $\mathfrak{P}_n$ (resp. $\bar{\mathfrak{P}}_n$) the unique prime of $\mathcal{O}_{M_n}$ over $\mathfrak{P}$ (resp. $\bar{\mathfrak{P}}$). Put

$$^nW = \varprojlim_m C\ell_{M_n}(\mathfrak{P}_n^m).$$

There is the norm map $N_n^m : {}^mW \longrightarrow {}^nW$ for each $m \geq n$. We take a CM-type of M_n defined by

$$J = J_n = \{\tau \in I_{M_n} \mid \tau|_M \text{ is the identity}\}.$$

Then, J satisfies (4.2a,b). Put $\varphi = \varphi_n = \sum_{\tau \in J_n} \tau$. Then, we see easily that $\mathcal{O}_{\Phi(\varphi_n)} = \mathcal{O}_M$. Let A be a valuation ring in Ω finite over $\mathbb{Z}_p$. Then, by (5.1), A automatically contains $\mathcal{O}_{\Phi(\varphi_n)}$. Let λ be a Hecke character in $m_{j.\varphi_0}^M(1;A)$ for $j>0$. Put $\lambda_n = \lambda \circ N_0^n \in m_{j.\varphi_n}^{M_n}(1;A)$. Then, by Lemma 2.3, we know that

$$g_{j.\varphi_n}(\mathfrak{P}_n^r;A) \simeq A[C\ell_{M_n}(\mathfrak{P}_n^r)]$$

by the isomorphism corresponding to λ_n. On the other hand, the restriction of operators in $g_{j.\varphi_n}(\mathfrak{P}_n^r;A)$ to $m_{j.\varphi_n}(\mathfrak{P}_n^s;A)$ for $s \leq r$ gives a surjective homomorphism of A-algebras : $g_{j.\varphi_n}(\mathfrak{P}_n^r;A) \longrightarrow g_{j.\varphi_n}(\mathfrak{P}_n^s;A)$. We shall take a projective limit :

$$g_{j.\varphi_n}^{*}(A) = \varprojlim_r g_{j.\varphi_n}(\mathfrak{P}_n^r;A),$$

which is a compact topoligical ring. By Lemma 2.3, we have

Proposition 5.1. $\quad g_{j.\varphi_n}(A) \simeq A[[{}^nW]]$.

Thus $g_{j.\varphi_n}(A)$ is independent of j.

Let X_n be the maximal abelian extension of M_n unramified outside $\mathfrak{P}_n$. Then, by class field theory, we have the following commutative diagram : for $m > n$,

$$\begin{array}{ccc} {}^mW & \xrightarrow{N_n^m} & {}^nW \\ \wr\| & & \uparrow \\ \mathrm{Gal}(X_m/M_n) & \xrightarrow{\text{res}} & \mathrm{Gal}(X_n/M_m \cap X_n). \end{array}$$

Note that ${}^nW \simeq \mathrm{Gal}(X_n/M_n)$ and $M_n \cap X_n = M_n$, because M_m is totally ramified at $\bar{\mathfrak{P}}_n$. Since $X_m \supset X_n$, the restriction map of the Galois groups is surjective, and thus, the map $N_n^m : {}^mW \longrightarrow {}^nW$ is surjective. It is known by Coates and Wiles (e.g. [5, § 1]) that ${}^nW = {}^nW_0 \times {}^nW_t$ with a finite torsion subgroup ${}^nW_t$ and ${}^nW_0 \simeq \mathbb{Z}_p$. Naturally, $\mathrm{Gal}(M_n/M)$ acts on ${}^nW$, and we may assume that $\mathrm{Gal}(M_n/M)$ acts on ${}^nW_0$ trivially, and N_n^m induces an isomorphism : ${}^mW_0 \simeq {}^nW_0$. We shall identify all ${}^nW_0$ with W_0 by N_0^n; thus, $g_{j.\varphi_n}(A)$ becomes an $A[[W_0]]$-algebra. Let $\hat{\lambda}_n$ denote the continuous character of ${}^nW$ into $A^\times (\subset \Omega^\times)$ associated with λ_n introduced in § 2. Since $g_{j.\varphi_n}(A) \simeq A[[{}^nW]]$ (this isomorphism only depends on the choice of $\lambda \in m_{j.\varphi_0}^M(1;A)$), the restriction of $\hat{\lambda}_n$ to ${}^nW_t$ gives a homomorphism of $A[[W_0]]$-algebras :

$$\Lambda_n : g_{j.\varphi_n}(A) \longrightarrow A[[W_0]];$$

namely, by regarding $g_{j.\varphi_n}(A)$ as the group algebra of ${}^nW_t$ over $A[[W_0]]$, $\Lambda_n(w_t) = \hat{\lambda}_n(w_t) \in A$ for all $w_t \in {}^nW_t$. Thus, we know from the results in § 1 :

Theorem 5.2. We have that

$$C_0(\Lambda_n;A[[W_0]]) \simeq A[[W_0]]/|{}^nW_t|A[[W_0]],$$

$$C_1(\Lambda_n;A[[W_0]]) \simeq A[[W_0]] \otimes_{\mathbb{Z}} {}^nW_t,$$

$$\phi_{A[[W_0]]}(C_0(\Lambda_n;A[[W_0]])) = \phi_{A[[W_0]]}(C_1(\Lambda_n;A[[W_0]])).$$

The Galois group $\Gamma/\Gamma_n = \mathrm{Gal}(M_n/M)$ acts naturally on $g_{j.\varphi_n}(A)$ and ${}^nW_t$, and its action is trivial on ${}^nW_0$. Thus we have

Corollary 5.3. The Galois group $\mathrm{Gal}(M_n/M)$ acts trivially on $C_0(\Lambda_n;A[[W_0]])$ and acts on $C_1(\Lambda_n;A[[W_0]])$ through the natural action on ${}^{n}W_t$.

We shall write $h_k^n(N;A)$ for $h_k^F(N;A)$ with $F = F_n$, and put

$$h_k^n(Np^\infty;A) = \varprojlim_r h_k^n(Np^r;A),$$

and let $h^{ord}(N;A) = h^{ord}_{F_n}(N;A)$ be the ordinary part of $h_k^n(Np^\infty;A)$, which is the largest algebra direct summand of $h_k^n(Np^\infty;A)$ in which the image of $T(p)$ is a unit (cf. [7, § 1]). Put

$${}^{n}V(N) = \varprojlim_r C\ell_{F_n}(Np^r).$$

Then, by the action of $C\ell_{F_n}(Np^r)$ on $h_k^n(Np^r;A)$ defined in paragraph 3 (in § 3, we have defined its action on $S_k(Np^r;A)$, but via the adjoint action under the pairing (3.6), it acts on the Hecke algebra), the algebra $h_k^n(Np^\infty;A)$ as well as $h^{ord}_{F_n}(N;A)$ becomes an $A[[{}^{n}V(N)]]$-algebra. Since Γ is naturally a subgroup of ${}^{n}V(N)$ through $\mathbb{Q}_p \longrightarrow F_p$, $h^{ord}_{F_n}(N;A)$ is a Λ_A-algebra for $\Lambda_A = A[[\Gamma]]$. The following fact will be proved in our subsequent paper :

<u>Theorem 5.4</u>. $h^{ord}_{F_n}(N;A)$ is independent of k if $k \geq 2$ and is finite over Λ_A.

Let D_n be the relative discriminant of M_n/F_n. Then, there is a natural morphism of A-algebra :

$$\theta^* : h^{ord}_{F_n}(D_n;A) \longrightarrow g_{j.\varphi_n}(A),$$

which is the projective limit of θ^* at each level defined in § 4 (the fact that θ^* factors through the ordinary part follows from the assumption (4.2a,b), which is verified easily in our present case). Let $R(\Lambda_n)$ (resp. $R(\theta(\lambda_n))$) be the local ring of $g_{j.\varphi_n}(A)$ (resp. $h^{ord}_{F_n}(D_n;A)$) through which Λ_n (resp. $\Lambda_n \circ \theta^*$) factors. By Theorem 5.4, we have reasonably defined modules : $C_1(\Lambda_n \circ \theta^*;A[[W_0]])$, $C_1(\Lambda_n;A[[W_0]])$ and $C_1(\theta^*;R(\Lambda_n))$. Then we know from Theorem 4.3 the following fact :

<u>Theorem 5.5</u>. If $j \not\equiv 0 \bmod p\text{-}1$ and $j > 0$, we have an exact sequence of $A[[W_0]]$-modules :

$$C_1(\theta^*;R(\Lambda_n)) \otimes_{R(\Lambda_n)} A[[W_0]] \longrightarrow C_1(\Lambda_n \circ \theta^*;A[[W_0]]) \xrightarrow{\theta_1} C_1(\Lambda_n;A[[W_0]]) \longrightarrow 0.$$

<u>Remark 5.6</u>. By the base change lift (Langlands [9]) relative to F_m/F_n $(m > n)$, there is a natural algebra homomorphism of $h^{ord}(D_m;A)$ to $h^{ord}(D_n;A)$, which is compatible with the morphism of $g_{j.\varphi_m}(A)$ to $g_{j.\varphi_n}(A)$ induced by N_n^m (defined above Corollary 2.7). Thus we may take a natural projective limit

$$C_1(\Lambda_\infty \circ \theta^*;A[[W_0]]) = \varprojlim_n C_1(\Lambda_n \circ \theta^*;A[[W_0]]),$$

$$C_1(\Lambda_\infty;A[[W_0]]) = \varprojlim_n C_1(\Lambda_n;A[[W_0]]),$$

$${}^{\infty}W_t = \varprojlim_n {}^{n}W_t.$$

These become $A[[W_0 \times \Gamma]]$-modules through the action of $\Gamma = \mathrm{Gal}(F_\infty/\mathbb{Q})$. In fact, $\mathrm{Gal}(F_n/\mathbb{Q})$ acts on $h^{ord}(D_n;A)$ via $T(a) \longmapsto T(a^\sigma)$ for $a \in I_{F_n}(A)$, and this induces the action of Γ on these modules. Although the module $C_1(\Lambda_n \circ \theta^*;A[[W_0]])$ is a finitely generated torsion $A[[W_0]]$-module at every layer F_n, it is still unknown that the module $C_1(\Lambda_\infty \circ \theta^*;A[[W_0]])$ is a finitely generated torsion module over $A[[W_0 \times \Gamma]]$ or not. However, by the following reason, we are inclined to think that $C_1(\Lambda_\infty \circ \theta^*;A[[W_0]])$ is in fact finitely generated and torsion over $A[[W_0 \times \Gamma]]$: The group $W_0 \times \Gamma$ may be regarded as a subgroup of the maximal abelian extension L of M unramified outside p. Let Y_∞ be the maximal p-abelian extension of L_∞ unramified outside $\mathfrak{P}$. Then $\mathrm{Gal}(Y_\infty/L_\infty)$ is known to be a torsion $\mathbb{Z}_p[[W_0 \times \Gamma]]$-module (cf. [11, § 2]). Let K_∞ be the fixed field of ${}^{0}W_t$ in X_0. Then naturally, $\mathrm{Gal}(K_\infty/M) \simeq W_0$ and $\mathrm{Gal}(X_\infty/K_\infty \cdot M_\infty) \simeq \mathbb{Z}_p \hat{\otimes}_{\mathbb{Z}} {}^{\infty}W_t = \varprojlim_n \mathbb{Z}_p \otimes_{\mathbb{Z}} {}^{n}W_t$, which is a finite torsion module over $\mathbb{Z}_p[[\Gamma]]$. Thus $C_1(\Lambda_\infty;A[[W_0]])$ which is isomorphic to $A[[W_0]] \hat{\otimes}_{\mathbb{Z}} {}^{\infty}W_t$ is finitely generated and torsion over $A[[W_0 \times \Gamma]]$. Let H_∞ be the unique anti-cyclotomic extension of M inside L_∞; i.e., H_∞ is the fixed field of the centralizer of complex conjugation in $\mathrm{Gal}(L_\infty/M)$, and let Z_n be the maximal p-abelian extension of $M_n H_\infty$ unramified outside $\mathfrak{P}_n$. If one can construct a surjective morphism of $\mathrm{Gal}(Z_n/H_\infty K_\infty)$ onto $C_1(\theta^*;R(\Lambda_n)) \otimes_{R(\Lambda_n)} A[[W_0]]$ for every layer M_n, $C_1(\theta^*;R(\Lambda_\infty)) \otimes_{R(\Lambda_\infty)} A[[W_0]]$ becomes a quotient of

$Gal(Z_\infty/H_\infty K_\infty)$, which is a finite torsion module over $A[[W_0 \times \Gamma]]$ (in fact, $Gal(Z_\infty/H_\infty K_\infty)$ is the quotient of $Gal(Y_\infty/L_\infty)$). By the exact sequence of Theorem 5.5 :

$$C_1(\theta^*;R(\Lambda_\infty))\otimes_{R(\Lambda_\infty)}A[[W_0]] \longrightarrow C_1(\Lambda_\infty\circ\theta^*;A[[W_0]]) \longrightarrow C_1(\Lambda_\infty;A[[W_0]]) \longrightarrow 0,$$

we may conclude that $C_1(\Lambda_\infty\circ\theta^*;A[[W_0]])$ is finitely generated and torsion over $A[[W_0\times\Gamma]]$ if one knows the existence of the morphisms in question. A result on the affirmative side of this question has been obtained as Kummer's criterion in [5] when $n=0$.

§ 6. Gorenstein condition for $R(\theta(\lambda))$.

In this section, for an ideal $\mathfrak{c}$ of $\mathcal{O}_M$, we shall prove that $R(\theta(\lambda))$ is a Gorenstein algebra under certain assumptions, when $F=\mathbb{Q}$. We use the same notation as in § 5. Let $\mathfrak{c}$ be an ideal of $\mathcal{O}_M$ prime to p. Suppose that $p\geq 5$ and

(6.1) p decomposes in $\mathcal{O}_M$ into the product of two different prime ideals.

Let λ be a primitive quasi character in $m^M_{j.\varphi_0}(\mathfrak{c}\mathfrak{P})$ for $j>0$. Since $\mathcal{O}_{\Phi(\varphi_0)}=\mathcal{O}_M$ and $\mathbb{Z}_p\supset\mathcal{O}_M$ in Ω, we can consider the algebra $g_{j.\varphi_0}(\mathfrak{c}\mathfrak{P};\mathbb{Z}_p)$. Let $\lambda_a : g_{j.\varphi_0}(\mathfrak{c}\mathfrak{P};\mathbb{Z}_p)\longrightarrow\Omega$ be the algebra homomorphism associated with λ, and let $R(\theta(\lambda))$ (resp. $R(\lambda)$) be the local ring of $h_k(Np;\mathbb{Z}_p)$ ($k=j+1$, $N=N_{M/\mathbb{Q}}(\mathfrak{c})D$ for the discriminant D of M over $\mathbb{Q}$) (resp. $g_{j.\varphi_0}(\mathfrak{c}\mathfrak{P};\mathbb{Z}_p)$) through which $\lambda_a\circ\theta^*$(resp. λ_a) factors.

Lemma 6.1. Suppose the following conditions :

(i) $j=1$,

(ii) the restriction of $\hat\lambda$ to μ_{p-1} is non-trivial, and

(iii) λ is primitive modulo $\mathfrak{c}\mathfrak{P}$. Then we have that

$$Hom_{\mathbb{Z}_p}(R(\theta(\lambda)),\mathbb{Z}_p)\simeq R(\theta(\lambda)) \text{ as } R(\theta(\lambda))\text{-modules.}$$

Proof. If we write $\hat\lambda(\zeta)=\zeta^a$ with $0\leq a<p-1$ for $\zeta\in\mu_{p-1}$, then $a\neq p-2$ and $a\neq 0$ by the assumptions (i), (ii) and (iii). Let $J_1(Np)/\mathbb{Q}$ be the jacobian variety of the modular curve $X_1(Np)/\mathbb{Q}$ (for the precise definition of $X_1(Np)$, see [5, § 3]). Let J_p be the p-divisible group of p-power torsion points of $J_1(Np)$. Let e be the idempotent attached to $T(p)$ in $h_2(Np;\mathbb{Z}_p)$ (cf. [7, § 1]). Note that the image of e in $R(\theta(\lambda))$ is the identity element of $R(\theta(\lambda))$. Put

$$J_p(R)=\sum_{\alpha\in R(\theta(\lambda))}\alpha J_p\subset J_p.$$

Then $J_p(R)$ is known to be contained in an abelian subvariety of $J_1(Np)$ defined over $\mathbb{Q}$ which acquires good reduction over $\mathbb{Z}_p[\zeta_p]$ for a primitive p-th root of unity ζ_p (e.g. [5 § 3]). Thus $J_p(R)$ has a structure of p-divisible group over $\mathbb{Z}_p[\zeta_p]$. Let C (resp. E) be the connected component (resp. maximal etale quotient) of $J_p(R)$ over $\mathbb{Z}_p[\zeta_p]$. As seen in [5, Prop. 3.1], we know that as $R(\theta(\lambda))$-modules

$$C\simeq R(\theta(\lambda))\otimes_{\mathbb{Z}_p}\mathbb{Q}_p/\mathbb{Z}_p \quad\text{and}\quad E\simeq Hom(R(\theta(\lambda)),\mathbb{Q}_p/\mathbb{Z}_p).$$

Since C and E are topologically defined, the continuous automorphism of $\mathbb{Z}_p[\zeta_p]$ acts on C and E (see the argument in [5, p. 446-7]). Thus $Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ acts on C and E through characters χ_C and χ_E with values in $R(\lambda)$ (the action of $R(\lambda)$ commutes with the Galois action). If θ^* induces an isomorphism : $R(\theta(\lambda))\simeq R(\lambda)$, then the assertion follows from Lemma 2.5. Since θ^* is surjective onto $R(\lambda)$ by Corollary 4.2, we may assume that θ^* has non-trivial kernel. Since $\theta(\lambda)$ is primitive, we can decompose

$$R(\theta(\lambda))\otimes_{\mathbb{Z}_p}\mathbb{Q}_p = A\oplus B$$

such that A can be embedded into Ω and the projection to A coincides with $\lambda_a\circ\theta^*$. Let $\mathfrak{a}=A\cap R(\theta(\lambda))$ and $\mathfrak{b}=B\cap R(\theta(\lambda))$. Put

$$J_p(\lambda)=\mathfrak{a}J_p(R)$$

Then, $J_p(\lambda)$ is a p-divisible subgroup of the abelian variety V attached to $\theta(\lambda)$ (in the sense of [14, Th. 7.14]). By Shimura [12], V has complex multiplication under M, the characters χ_C and χ_E modulo $\mathfrak{b}$ coincide with $\hat\lambda$ and $\hat\lambda_\rho$ (cf. [5, § 5]), where

$$\hat{\lambda}_\rho(\sigma) = \hat{\lambda}(\rho\sigma\rho^{-1}) \text{ for complex conjugation } \rho.$$

Let $\mathfrak{m}$ be the maximal ideal of $R(\theta(\lambda))$, and put for each $R(\theta(\lambda))$-module X, $X[\mathfrak{m}] = \{x \in X \mid mx = 0 \text{ for all } m \in \mathfrak{m}\}$. Since V has complex multiplication and $\hat{\lambda} \bmod \mathfrak{m} \neq \hat{\lambda}_\rho \bmod \mathfrak{m}$ by (ii), one can decompose $V[\mathfrak{m}] = X \oplus Y$ so that on X (resp. Y), $\mathrm{Gal}(\bar{\mathbb{Q}}/M)$ acts via $\hat{\lambda} \bmod \mathfrak{m}$ (resp. $\hat{\lambda}_\rho \bmod \mathfrak{m}$). Since the representation of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on the successive quotients of the Jordan-Hölder series of the Galois module $V[\mathfrak{m}]$ is isomorphic to the induced representation of $\hat{\lambda} \bmod \mathfrak{m}$ from $\mathrm{Gal}(\bar{\mathbb{Q}}/M)$ to $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, which is irreducible by (ii). Thus $Y \neq 0$ and $X \neq 0$.
If on $C[\mathfrak{m}]$, $\mathrm{Gal}(\bar{\mathbb{Q}}/M)$ acts via $\hat{\lambda} \bmod \mathfrak{m}$, then $C[\mathfrak{m}] \cap Y = 0$.
If on $C[\mathfrak{m}]$, $\mathrm{Gal}(\bar{\mathbb{Q}}/M)$ acts via $\hat{\lambda}_\rho \bmod \mathfrak{m}$, then $C[\mathfrak{m}] \cap X = 0$.
On the other hand, $E[\mathfrak{m}] \simeq R(\theta(\lambda))/\mathfrak{m}$ and hence, we have an exact sequence :

$$0 \longrightarrow C[\mathfrak{m}] \longrightarrow J_p(R)[\mathfrak{m}] \longrightarrow E[\mathfrak{m}] \longrightarrow 0,$$

which splits as $\mathrm{Gal}(\bar{\mathbb{Q}}/M)$-module. Thus we may identify

$$J_p(R)[\mathfrak{m}] = C[\mathfrak{m}] \oplus E[\mathfrak{m}] \text{ as } \mathrm{Gal}(\bar{\mathbb{Q}}/M)\text{-module.}$$

Since $\hat{\lambda}_\rho(\sigma) = \hat{\lambda}(\rho\sigma\rho^{-1})$, the action of ρ on $J_p(R)[\mathfrak{m}]$ interchanges $C[\mathfrak{m}]$ and $E[\mathfrak{m}]$. This shows that for $R = R(\theta(\lambda))$

$$R/\mathfrak{m} \simeq E[\mathfrak{m}] \simeq C[\mathfrak{m}] \simeq (R/pR)[\mathfrak{m}].$$

This shows that $R/pR \simeq \mathrm{Hom}(R/pR, \mathbb{Z}/p\mathbb{Z})$ as R-module and hence, the assertion follows.

<u>Theorem 6.2</u>. Let λ be a Hecke character in $m_{j.\varphi_0}(\mathfrak{c}\mathfrak{P})$, whose conductor is divisible by $\mathfrak{c}$. Suppose that if $\hat{\lambda}(\zeta) = \zeta^a$ for $\zeta \in \mu_{p-1}$ with $0 \leq a < p-1$, a is neither 0 nor $p-2$ (i.e. the tame part of the p-part of the character of $\theta(\lambda)$ is given by $\zeta \mapsto \zeta^a$ for $a \not\equiv -j, 1-j \bmod p-1$). Then we have that

$$\mathrm{Hom}_{\mathbb{Z}_p}(R(\theta(\lambda)), \mathbb{Z}_p) \simeq R(\theta(\lambda)) \text{ as } R(\theta(\lambda))\text{-module;}$$

namely $R(\theta(\lambda))$ is a Gorenstein algebra.

<u>Proof</u>. Let $h^{ord}(N; \mathbb{Z}_p)$ be as in Theorem 5.4 for $N = N_{M/F}(\mathfrak{c})D$. Then it is known by [6, Th. 3.1] that $h^{ord}(N; \mathbb{Z}_p)$ is finite flat over

$\Lambda = \mathbb{Z}_p[[\Gamma]]$. For each integer k, let $\chi_k : \Lambda \longrightarrow \mathbb{Z}_p$ be the algebra homomorphism such that $\chi_k(\gamma) = \gamma^k \in \mathbb{Z}_p$ for $\gamma \in \Gamma$. Then it is known (cf. [6, Cor. 3.2]) that, for the kernel $P_k = \mathrm{Ker}(\chi_k)$ there is a unique local ring R of $h^{ord}(N; \mathbb{Z}_p)$ such that

$$R \otimes_\Lambda \Lambda/P_k \simeq R(\theta(\lambda)) \text{ as } R\text{-module.}$$

Then, by [6, Th. 7.1] and [7, Prop. 2.3], there is a Hecke character $\lambda' \in m^M_{\varphi_0}(\mathfrak{c}\mathfrak{P})$ which satisfies the conditions of Lemma 6.1 and $R/P_2R \simeq R(\theta(\lambda'))$ as R-module. Then, by Lemma 6.1, we know that $\mathrm{Hom}_{\mathbb{Z}_p}(R/P_2R, \mathbb{Z}_p) \simeq R/P_2R$ as R-module. Since R is flat over Λ, this implies that

$$\mathrm{Hom}_\Lambda(R,\Lambda) \simeq R \text{ as } R\text{-module.}$$

Thus we conclude that, as R-module,

$$\mathrm{Hom}_{\mathbb{Z}_p}(R(\theta(\lambda)), \mathbb{Z}_p) \simeq \mathrm{Hom}_\Lambda(R,\Lambda) \otimes_\Lambda \Lambda/P_k\Lambda \simeq R/P_kR \simeq R(\theta(\lambda)).$$

<u>Corollary 6.3</u>. Under the assumption of Theorem 6.2, we have the vanishing of the module of defect N_S for the primitive irreducible component of $h^{ord}(N; \mathbb{Z}_p)$ to which $\theta(\lambda)$ belongs. Especially, the assumptions of Theorem 6.1 are satisfied if λ is a primitive Hecke character in $m_{j.\varphi_0}(\mathfrak{c})$ with $j \not\equiv 0, 1 \bmod p-1$ and $j > 1$ (i.e. $k \not\equiv 1, 2 \bmod p-1$ and $k > 2$).

This follows from Lemma 2.5, Theorem 6.2 and [6, Prop: 3.9].

<u>Corollary 6.4</u>. Let h be the class number of M and let $\lambda \in m_{j.\varphi_0}(1)$. Suppose that $j \not\equiv 0, 1 \bmod p-1$. Then the number $D(k,\theta(\lambda))/U_\infty(\theta(\lambda))$ is divisible by h in the p-adic integer ring of Ω.

This follows from Theorem 3.2, Theorem 6.2, Theorem 2.6 and Theorem 4.3 (i).

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