§ 0. Introduction.

In our previous papers [6] and [7], we have defined the module of congruences of the Hecke algebra of the space of modular forms of \( \text{GL}_2(\mathbb{Q}) \) and studied its arithmetic relation with the special values of a certain \( L \)-function of modular forms. In the present paper, we shall propose the study of the module of relative \( 1 \)-differentials of the Hecke algebra (over a base ring) as well as its module of congruences. These modules have the same Fitting ideal under certain assumptions (Lemmas 1.8, 1.9 and 1.11). The precise definition of these modules will be given in § 1. Then, we shall define in § 2 the Hecke algebra for \( \text{GL}_1 \) over arbitrary algebraic number fields \( F \) and show that if the level of this algebra is equal to \( 1 \), the module of \( 1 \)-differentials over \( \mathbb{Z}_p \) is isomorphic to the \( p \)-part of the ideal class group of \( F \) and the module of congruences is isomorphic to \( \mathbb{Z}_p/\mathfrak{h}\mathbb{Z}_p \) for the class number \( \mathfrak{h} \) of \( F \) (Theorem 2.6). In § 3, we shall define the Hecke algebras for \( \text{GL}_2(F) \) for an arbitrary totally real field \( F \) and list some of their properties without proof. When \( M \) is a CM-field quadratic over \( F \), there is a natural algebra homomorphism of the Hecke algebra for \( \text{GL}_2(F) \) to that of \( \text{GL}_1(M) \) (4.5). Under a certain assumption (4.2), we shall show that these modules for \( \text{GL}_1(M) \) are obtained as a quotient of the corresponding one for \( \text{GL}_2(F) \) (Theorem 4.3). In § 5, we shall investigate the relation between our theory and the Iwasawa theory of the natural \( \mathbb{Z}_p \times \mathbb{Z}_p \)-extension of imaginary quadratic fields. To explain this, we shall introduce some symbols for \( \mathbb{Z}_p \)-extensions. Let \( M \) be an imaginary quadratic field with class number one. We fix an odd prime \( p \) which decomposes in \( M \) into the product of two different primes \( \mathfrak{p} \) and \( \mathfrak{p} \). Let \( F_\infty \) (resp. \( K_\infty \)) be the unique \( \mathbb{Z}_p \)-extension of \( \mathbb{Q} \) (resp. \( M \)) unramified outside \( p \) (resp. \( p \)), and put \( M_\infty = M \cdot F_\infty \). Let \( L_\infty \) be the maximal abelian exten-
sion of \( M \) unramified outside \( p \) and \( Y_\infty \) be the maximal \( p \)-abelian extension of \( L_\infty \) unramified outside \( p \). Let \( C \) be the centralizer of complex conjugation in \( \text{Gal}(L_\infty/M) \), and let \( H_\infty \) denote the fixed field of \( C \) in \( L_\infty \) (i.e. the anti-cyclotomic part of \( L_\infty \)). Finally, let \( Z_n \) (resp. \( X_n \)) be the maximal \( p \)-abelian extension of \( H_\infty \cdot M_n \) (resp. \( M_n \)) for the \( n \)-th layer \( M_n \) of \( M_\infty \) unramified outside \( p \). There are restriction morphisms: \( \text{Gal}(Z_n/L_n) \to \text{Gal}(X_n/M_n \cdot k) \) and \( \text{Gal}(Y_n/L_n) \to \text{Gal}(Z_n/M_n \cdot k) \) for \( n = 1, 2, \ldots, \infty \). Our Theorem 5.5 asserts that the finite module \( \text{Gal}(X_n/M_n \cdot k) \) can be obtained as a quotient of the module of differentials of an appropriate Hecke algebra of \( GL_2(F_n) \). After taking the projective limit relative to \( n \), the (one variable) Iwasawa module \( \text{Gal}(X_\infty/M_\infty \cdot k) \) is again a quotient of the limit map of these modules for \( GL_2(F_n) \) with respect to the base change lifting maps for \( GL_2(F_n) \) ([9]). On the other hand, the (two variable) Iwasawa module \( \text{Gal}(Z_\infty/M_\infty \cdot k) \) is expected to be related to the kernel of this quotient map. In fact, in the case of \( n = 0 \) (i.e. the case of the module \( \text{Gal}(Z_\infty/M_\infty \cdot k) \)), the result in [5] supports this expectation (for details, see Remark 5.6). Finally in § 6, we shall prove that the local ring of the Hecke algebra of \( GL_2(Q) \) obtained from \( M \) is a Gorenstein ring under a certain irreducibility assumption of the Galois action. Thus the module of defect \( N_\infty \) defined in [6] vanishes as conjectured there when it corresponds to \( M \).

Originally, the author had a formulation of the results presented here by using certain modules expressed as the first “torsion” (or extension) modules over the Hecke algebra. The use of the module of 1-differentials or the (co)-tangent space was suggested to the author by B. Mazur. Anyway, the two definitions coincide in almost all cases (Lemma 1.1).

In the seminar talk, the author gave an account of the construction of Galois representations into \( GL_2 \) over the algebra \( \mathbb{Z}_p[[X]] \). A detailed proof of this has already appeared in [7]. We thus take this opportunity to show that the ideas in [7] developed for \( GL_2 \) do reveal some new aspect even for the theory of \( GL_1 \).

§ 1. Some lemmas and definitions.

Let \( A \) be a noetherian integral domain of characteristic 0 and \( R \) and \( B \) be \( A \)-algebras finite over \( A \). We suppose that there is a surjective \( A \)-algebra homomorphism \( \lambda : R \to B \). Let \( \Omega^1_{R/A} \) be the module of relative 1-differentials of \( R \) over \( A \) (cf. [1, § 20]). We recall its definition: Let \( I \) be the kernel of the natural \( A \)-algebra homomorphism: \( R \to R \) given by \( ab \mapsto a \cdot b \cdot r \). Then, the \( R \)-module \( I/I^2 \) is by definition isomorphic to \( \Omega^1_{R/A} \). The map \( d : R \to \Omega^1_{R/A} \) given by \( d(a) = a \cdot 1 - 1 \cdot a \cdot \mod I^2 \) gives a derivation trivial over \( A \). For each \( R \)-module \( M \), the module \( \Omega^1_{R/A} \) is the cotangent space of \( R \) over \( A \). We shall define a \( B \)-module by

\[
C_1(\lambda ; B) = \text{Tor}^R_1(B, B),
\]

where we regard \( B \) as an \( R \)-module through \( \lambda \).

**Lemma 1.1.** Let \( a \equiv 0 \mod I \). Then we have that

\[
C_1(\lambda ; B) \cong a/a^2 \cong a \cdot \Omega^1_{R/A} B.
\]

If \( B \) is etale over \( A \), then \( C_1(\lambda ; B) \cong \Omega^1_{R/A} B \).

**Proof.** If \( B/A \) is smooth, there is a natural exact sequence (cf. [1, 20, 5.14]).

\[
0 \to a/a^2 \to \Omega^1_{R/A} B \to \Omega^1_{B/A} \to 0.
\]

If \( B \) is etale over \( A \), then \( \Omega^1_{B/A} = 0 \) and thus \( a/a^2 \cong \Omega^1_{R/A} B \). We have by definition the exact sequence of \( R \)-modules:

\[
0 \to a \to R \to B \to 0,
\]

which yields a long exact sequence of the torsion modules:

\[
0 = \text{Tor}^R_1(R, B) \to \text{Tor}^R_1(B, B) \to a \cdot \Omega^1_{R/A} B \to R \cdot \Omega^1_{R/A} B \to B \cdot \Omega^1_{R/A} B \to 0.
\]

This shows that \( \text{Tor}^R_1(B, B) \cong a \cdot B \cong a \cdot R/a = a/a^2 \).

The following fact is easy:

**Lemma 1.2.** Let \( A \to A' \) be a flat \( A \)-algebra morphism and write \( \lambda_{A'} : R \to A' \to B \) for the base change morphism of \( \lambda \). Then, we
have that $C_1(\lambda_A;B \otimes A') = C_1(\lambda;B) \otimes A'$.

**Lemma 1.3.** Let $\lambda : R \rightarrow B$ and $\lambda' : R' \rightarrow B'$ be surjective $A$-algebra morphisms in the category of finite flat $A$-algebras. Then, we have that $C_1(\lambda \otimes_A \lambda';B \otimes A') = (C_1(\lambda;B) \otimes B') \otimes (C_1(\lambda';B') \otimes A')$.

**Proof.** Put $a = \text{Ker}(\lambda)$ and $a' = \text{Ker}(\lambda')$. By Lemma 1.2, we may assume that $A$ is a local ring. Then we have by the flatness of $B$ over $A$ decompositions of $A$-modules : $R = a \otimes B$ and $R' = a' \otimes B'$. The kernel of $\lambda \otimes \lambda'$ is given by $A = (a \otimes a') \otimes (a \otimes B') \otimes (B \otimes a') = a \otimes R' + R \otimes a'$ in $R \otimes A'$. We see easily that

$$A^2 = a^2 \otimes R' + R \otimes a' = (a \otimes a') \otimes (a^2 \otimes B') \otimes (B \otimes a'(a')^2).$$

Thus we know that $A/A^2 = ((a/a^2) \otimes A' B') \otimes ((a'/a')^2) \otimes A B$, which finishes the proof.

Hereafter we suppose that

(1.1a) $R$ and $B$ are flat over $A$.

Let $K$ be the quotient field of $A$. We now suppose that we can decompose $R \otimes_A K$ into the $K$-algebra direct sum : $R \otimes_A K = B \otimes A K \oplus D'$ with a $K$-algebra $D'$

so that $\lambda$ induces the projection of $R$ into $B \otimes A K$. Let $\rho$ be the projection of $R$ into $D'$ and put $D = \rho(R)$. We shall define another $B$-module :

$$C_0(\lambda;B) = (B \otimes D)/R.$$

This is the module of congruences between $B$ and $D$. The following result is easy : $C_1(\lambda;B) = B/b \otimes D/a = R/a \otimes b$ as $R$-modules.

For each finite torsion $B$-module $M$, let $\phi_B(M)$ be the fitting ideal of $M$ over $B$ (Northcott [10, Chap. 3]). We recall the definition: we take a finite presentation : $B^a \rightarrow B^b \rightarrow M \rightarrow 0$ (exact). Note that $a \geq b$ since $M$ is $B$-torsion. We write matricially $\gamma$ as $C \in \mathcal{M}_{a,b}(B)$ and let $\phi_B(M)$ be the ideal of $B$ generated by all $b \times b$-minors of $C$. Then $\phi_B(M)$ depends only on $M$. The following facts are known :

(1.2a) $\phi_B(M_1 \otimes M_2) = \phi_B(M_1) \cdot \phi_B(M_2)$

(1.2b) for any multiplicative subset $S$ of $B$,

$$\phi_S(M \otimes S) = \phi_S(M)S,$$

where $S$ is the localization of $B$.

(1.2c) If $M = B/b$ with an ideal $b$ of $B$, we have $\phi_B(M) = b$.

**Corollary 1.5.** $C_1(\lambda;B)$ is annihilated by $\phi_B(C_0(\lambda;B))$.

This follows from (1.2c) and Lemma 1.4.

**Lemma 1.6.** Suppose that $\text{Hom}_A(R,A) = R$ and $\text{Hom}_A(B,A) = B$ as $R$-modules. By identifying $\text{Hom}_A(R,A)$ with $R$ (resp. $\text{Hom}_A(B,A)$ with $B$) let $\lambda^*$ be the adjoint map of $\lambda : R \rightarrow B$. Then $\lambda^* : B \rightarrow R$ is a $B$-linear map and $\phi_B(C_0(\lambda;B))$ is generated over $B$ by $\lambda^* \lambda^* \in \text{Hom}_B(B,B) = B$.

**Proof.** We may assume by (1.2b) that $A$ is a local ring. Let $\langle,\rangle_R$ (resp. $\langle,\rangle_B$) be the pairing : $R \times R \rightarrow A$ (resp. $B \times B \rightarrow A$) which induces the isomorphism $\text{Hom}_A(R,A) = R$ (resp. $\text{Hom}_A(B,A) = B$). Since $A$ is local, $R$ and $B$ are $A$-free by (1.1a). Then we see easily that $a = \text{Ker}(\lambda) = \text{Ann}(\lambda^*(B))$, and thus, for $r \in R$, $b, c \in B$, and $b' \in R$ with $b(b') = b$,

$$\langle r, \lambda^*(b) \rangle_R = \langle b\lambda(r), c \rangle_B = \langle \lambda(b'), c \rangle_B = \langle r, b' \lambda^*(c) \rangle_R = \langle r, b \lambda^*(c) \rangle_R.$$

This shows the $B$-linearity of $\lambda^*$. For any $R$-module $M$, we write $M^* = \text{Hom}_A(M,A)$. Since $a$ is $A$-free, we have an exact sequence :

$$0 \rightarrow B^* \rightarrow \lambda^* \rightarrow a^* \rightarrow 0.$$

Thus $a^* \otimes D$ as $R$-module. This shows that $\lambda^*(B) = \text{Ker}(\rho)$, and $\phi_B(C_0(\lambda;B)) = \lambda^* \lambda^*(B)$ by (1.2c).

**Lemma 1.7.** Let $\lambda_i : R_i \rightarrow A$ be surjective homomorphisms satisfying (1.1a,b) for $i = 1, 2, \ldots, h$. Put $\lambda = \lambda_1 \otimes \cdots \otimes \lambda_h : R = R_1 \otimes \cdots \otimes R_h \rightarrow A$. Then
Then \( \lambda \) satisfies (1.1a,b) and if \( \text{Hom}_A(R_i,A) \cong R_i \) as \( R_i \)-modules for all \( i \), then \( \phi_A(C_0(\lambda;A)) = \phi_A(C_0(\lambda_1;A)) \cdots \phi_A(C_0(\lambda_h;A)) \) and it is a principal ideal.

**Proof.** The first assertion is obvious. We write \( \langle , \rangle \) for the pairing of \( R_i \) over \( A \) by which we identify \( R_i \) with \( \text{Hom}_A(R_i,A) \). Then the pairing \( \langle , \rangle : R \times R \rightarrow A \) defined by

\[
\langle a_1, \ldots, a_h, b_1, \ldots, b_h \rangle = \prod_{i=1}^h \langle a_i, b_i \rangle_i
\]

gives an isomorphism:

\[ R \cong \text{Hom}_A(R_i,A) \] as \( R \)-modules. Then we see that \( \lambda^* = \lambda_1^* \cdots \lambda_h^* \) and \( \lambda \circ \lambda^* = \prod_{i=1}^h \lambda_i \circ \lambda_i^* \). This combined with Lemma 1.6 shows the result.

**Lemma 1.8.** Under the same assumption as in Lemma 1.7, decompose

\[ R_i \otimes_A K = D_1 \otimes D_i \] as in (1.1b) and let \( D_i \) be the image of \( R_i \) in \( D_i \).

Suppose that \( \text{Hom}_A(D_i,A) \) = \( D_i \) as \( R_i \)-modules for \( i = 1, 2, \ldots, h \). Then, we have that for \( \lambda = \lambda_1 \otimes \cdots \otimes \lambda_h \),

\[
C_1(\lambda;A) = C_0(\lambda_1;A) \otimes \cdots \otimes C_0(\lambda_h;A)
\]

\[
\phi_A(C_1(\lambda;A)) = \phi_A(C_0(\lambda_1;A)) \cdots \phi_A(C_0(\lambda_h;A)).
\]

**Proof.** By Lemmas 1.7 and 1.3, we may suppose that \( h = 1 \).

Write \( R = R_1 \) and take \( \rho: R \rightarrow D \) as in (1.1b). Since \( D^* = \text{Hom}_A(D,A) = D \) as \( R \)-module, we know, in the same manner as in the proof of Lemma 1.6 that for \( a = \text{Ker}(\lambda) \),

\[ D = a^* \] as \( R \)-module

and thus \( a \otimes D^* \otimes D \) as \( R \)-module. Thus \( a \) is generated by \( a \otimes \rho \otimes \rho^* \in \text{Hom}_R(D, D) = D \), which is not a zero divisor. Thus, we know that

\[
C_1(\lambda;A) = a^2 = aD/aD = D/aD = D/a \cong C_0(\lambda;A).
\]

This shows the result.

**Lemma 1.9.** Let \( G \) be a commutative finite group, and put \( R = A[G] \) (the group algebra). For each character \( \chi: G \rightarrow A' \), let \( x_\chi: R \rightarrow A \) be the \( A \)-algebra homomorphism such that \( x_\chi(\sigma) = \chi(\sigma) \) for all \( \sigma \in G \).

Then, we have that

\[
C_0(x_\chi;A) = A/|G|A \quad \text{and} \quad C_1(x_\chi;A) = A \otimes_{\mathbb{Z}} G,
\]

where \( |G| \) denotes the cardinality of \( G \) and we regard \( G \) naturally as a \( \mathbb{Z} \)-module. In particular, we have that

\[
\phi_A(C_0(x_\chi;A)) = \phi_A(C_1(x_\chi;A)) = |G|A.
\]

**Proof.** Evidently, \( x_\chi(1) = |G|^{-1} \sum_{\sigma \in G} \chi^{-1}(\sigma) \sigma \) in \( x_\chi(A) = A \). This shows that \( b = R_nA \) is generated by \( \sum_{\sigma \in G} \chi^{-1}(\sigma) \sigma \), and thus,

\[
C_0(\lambda;A) = A/b \cong A/|G|A \quad \text{and} \quad \phi_A(C_0(\lambda;A)) = |G|A.
\]

Write \( G = G_1 \times \cdots \times G_n \), where each \( G_i \) is a cyclic group. Then we see that

\[
R = A[G_1] \otimes A[G_n] \quad \text{and} \quad x_\chi = x_1 \otimes \cdots \otimes x_n,
\]

where \( x_i \) is the restriction of \( \chi \) to \( G_i \). Thus, by Lemma 1.3 and Lemma 1.8, we may assume that \( G \) is cyclic of order \( N \). Then \( A[G] \cong A[X]/(X^N-1) \). Let \( F(X) \) be a monic polynomial of degree \( d \) in \( A[X] \). Then \( R' = A[X]/(F(X)) \) has a basis \( 1, x, \ldots, x^{d-1} \) for the image \( x \) of \( X \) in \( R' \). For any \( \xi \in R' \), write

\[
\xi = a_1(\xi) + a_2(\xi) x + \cdots + a_d(\xi) x^{d-1}
\]

with \( a_i(\xi) \in A \).

Define a pairing \( \langle , , \rangle : R' \times R' \rightarrow A \) by

\[
\langle \xi, \eta \rangle = a_d(\xi \cdot \eta).
\]

This gives an isomorphism \( \text{Hom}_R(R',A) \cong R' \) as \( R' \)-modules (in fact, \( \langle x, x^d-1-j \rangle = a_d(j) \) if \( i \leq j \)). By this fact, we know that \( \text{Hom}_A(R,A) \cong R \) and \( \text{Hom}_A(D,A) = D \) (because \( D \) is also generated by one element over \( A \)). Then the result follows from Lemma 1.7.

**Remark 1.10.** We use the same notation as in Lemma 1.9. Then the automorphism group \( \text{Aut}(G) \) naturally acts on \( A \)-algebra \( R = A[G] \). Let \( x_\chi = \sigma \circ x_{\chi(\sigma)} \). Then \( C_0(x_\chi;A) \) and \( C_1(x_\chi;A) \) can naturally be regarded as \( G \)-modules. By definition, \( G \) acts naturally on \( C_0(x_\chi;A) \), and the isomorphism

\[
\epsilon_i(x_\chi;A) \cong A \otimes_{\mathbb{Z}} G
\]

gives an isomorphism of \( G \)-modules. In fact, we see easily that \( G \) is generated by \( x_\sigma = \sigma \cdot x_{\chi(\sigma)} \) for \( \sigma \in G \) and \( a^2 \) is generated by...
Let $W$ be an abelian $p$-profinite group, and write $W_t$ for the torsion part of $W$. Put $W_0 = W/W_t$. Suppose that
\[(1.3) \quad W_0 = Z_p \quad \text{and} \quad W_t \quad \text{is a finite group.}\]

Let $\Gamma$ be a subgroup of $W_0$ of finite index. We may decompose (non-canonically) as $W = W_0 \times W_\infty$. Let $A$ be a valuation ring finite flat over $\mathbb{Z}_p$, and write $A[[\Gamma]]$, $A[[W_0]]$ and $A[[W]]$ for the $p$-adically continuous group ring of $\Gamma$, $W_0$ and $W$. We write simply $A$ for $A[[\Gamma]]$.

**Lemma 1.11.** Let $\chi$ be a character of $\mathcal{W}_T$ with values in $A$. We denote by $\chi^\Gamma_A$ for the $A[[W]]$-algebra homomorphism
\[
\chi^\Gamma_A : A[[W]] \otimes A[[W_0]] \rightarrow A[[W_0]]
\]
defined as follows: for $w = (w_0, w_\infty) \in W_0 \times W_\infty$ and $a \in A[[W_0]]$,
\[
\chi^\Gamma_A(w \cdot a) = \chi(w_0)w_0 \cdot a \in A[[W_0]].
\]
Then we have that
\[
C_{1}^{\chi}(x^\Gamma_A; A[[W_0]]) = A[[W_0]] \otimes A[[W_0]]
\]
and
\[
C_{0}^{\chi}(x^\Gamma_A; A[[W_0]]) = A[[W_0]] \otimes A[[W_0]] / A[[W_0]] / A[[W_0]].
\]

**Proof.** Let $u$ be a topological generator of $W_0$. Then we see that $u^p \in \Gamma$ for $p^r = [W_0 : \Gamma]$, and thus $A[[W_0]] = A[[X]] / (X^{p^r} - u^{p^r})$. Write $W = G_1 \times \cdots \times G_r$ for cyclic groups $G_i$ and $A$ for $A[[W_0]]$. Then we see that
\[
A[[W]] \otimes A = (A[[X]] / (X^{p^r} - u^{p^r})) \otimes A[G_1] \otimes \cdots \otimes A[G_r].
\]

Since we know that
\[
\frac{d}{dx} (x^{p^r} - u^{p^r}) = p^r x^{p^r - 1},
\]
we know that $C_1$ for $A[[X]] / (X^{p^r} - u^{p^r})$ is isomorphic to $A/p^r A \otimes A[[W_0 / \Gamma]]$.

by Lemma 1.1. Then the result follows from Lemma 1.8.

§ 2. The case of $\text{GL}_1$.

Let $K$ be an arbitrary number field of finite degree over $Q$ and $O_K$ be the ring of integers of $K$. We take the algebraic closure $\bar{Q}$ of $Q$ inside of $C$ and always consider algebraic number fields in $\bar{Q}$. We fix a prime number $p$ in $Z$, and let $\mathfrak{p}$ be the $p$-adic completion of an algebraic closure of $Q_p$. The normalized norm of $\mathfrak{p}$ is written as $|\cdot|_p$. We fix once and for all an embedding of $\bar{Q}$ into $K$ and consider $\bar{Q}$ also a subfield of $K$. Put $F_\infty = F \otimes \bar{Q}$ and $F_p = F \otimes \bar{Q}_p$, and let $F_\infty^*$ be the idele group of $F$. Let $F_\infty^*$ be the finite part of $F_\infty^*$, i.e. $F_\infty^* = F_\infty \setminus F_\infty$. For each prime $\mathfrak{p}$ of $F$, let $F_\mathfrak{p}$ be the $\mathfrak{p}$-adic completion of $F$ and $O_F^\mathfrak{p}$ be its $\mathfrak{p}$-adic integer ring. Put $\mathfrak{p}_F^\mathfrak{p} = F_\mathfrak{p} = F_\infty^* \setminus F_\infty$. For each prime $\mathfrak{p}$ of $F$, let $F_\mathfrak{p}$ be the $\mathfrak{p}$-adic completion of $F$ and $O_F^\mathfrak{p}$ be its $\mathfrak{p}$-adic integer ring. Put $\mathfrak{p}_F^\mathfrak{p} = F_\mathfrak{p} = F_\infty^* \setminus F_\infty$. Let $I = I_F^\mathfrak{p}$ be the set of all embeddings of $F$ into $\bar{Q}$ and $Z[I]$ be the free $Z$-module generated by the elements of $I$. We write $\xi_I > 0$ for $\xi = \Sigma \xi_I \cdot I \in Z[I]$ if $\xi_I \geq 0$ for all $I$, and $\xi_I > 0$ if $\xi_I > 0$ for some $I$. We fix an ideal $N$ of $O_F^\mathfrak{p}$ and put
\[
E = \{ \epsilon \in O_F^\mathfrak{p} | \epsilon \in E_{\infty} \}
\]
\[
U = U_F(N) = \{ u \in (O_F^\mathfrak{p})^X | u \equiv c \mod N_0^\mathfrak{p} \text{ for some } c \in E \},
\]
\[
C_{\xi} = C_{\xi}^E(N) = F_\mathfrak{p} \setminus F_\mathfrak{p} / E_{\infty} U_F(N).
\]

Then $C_{\xi}$ is a finite group. We can associate to each $
\xi = \Sigma \xi_I \cdot I \in Z[I]$ a character : $F_\mathfrak{p}^* \rightarrow \bar{Q}^*$ defined by
\[
\chi \mapsto \chi^\xi = \prod_{I \in I} \chi_I^{\xi_I}.
\]

This character can be extended by continuity to the characters :
\[
\xi : F_\mathfrak{p}^* \rightarrow \bar{Q}^*, \quad \xi : F_\infty^* \rightarrow \bar{Q}^*.
\]

For each $\xi \in Z[I]$, we shall define "the space of modular forms" of $GL_1$ for $U_F(N)$ of weight $\xi$, which will be denoted by $m_\xi(N)$, by the space of functions $f : F_\mathfrak{p}^* \rightarrow \bar{Q}^*$ satisfying the following condition :
element \( xU \in F_0^+ / U \) and define the (Hecke) operator \( T(a) \) on \( m_\xi(N) \) by the action of \( xU \). Naturally the absolute Galois group \( G_\xi = \text{Gal}(\bar{\mathbb{Q}} / \mathbb{Q}) \) acts on \( \mathbb{Z}[I_\xi] \) from the right. For each \( \xi \in \mathbb{Z}[I_\xi] \), we put

\[
G_\xi = \{ \sigma \in G_\xi | \lambda \cdot \sigma = \lambda \}.
\]

Then the fixed field \( \Phi(\xi) \) of \( G_\xi \) in \( \bar{\mathbb{Q}} \) is a finite extension of \( \mathbb{Q} \). Let \( O_\Phi(\xi) \) be the integer ring of \( \Phi(\xi) \). Then the character \( x \mapsto x^\xi \) has values in \( \Phi(\xi) \) on \( F \). For each \( O_\Phi(\xi) \)-subalgebra \( A \) of \( \mathbb{C} \), put

\[
m_\xi(N; A) = (\lambda \in m_\xi(N) | \lambda(a) \in A \text{ for all } a \in I(N)).
\]

From (2.2a), the following assertion is obvious:

**Lemma 2.2.** Suppose that \( \xi \geq 0 \). If \( A \) and \( A' \) are \( O_\Phi(\xi) \)-algebras in \( \mathbb{C} \) such that \( A' \simeq A \), then \( m_\xi(N; A) \circ A' = m_\xi(N; A') \). Moreover, we have the equality: \( m_\xi(N; A') = m_\xi(N; A) \circ A' \) if one of the following assertions is satisfied:

(i) \( A' \) is \( A \)-free of finite rank;

(ii) \( A' \) is a localization of \( A \);

(iii) \( A' \) is a field.

Let \( \mathcal{O}_\xi(\xi; N) \) be the subalgebra of the endomorphism algebra of \( m_\xi(N; \Phi(\xi)) \) generated over \( O_\Phi(\xi) \) by \( T(a) \) for all \( a \in I(N) \). For each \( O_\Phi(\xi) \)-algebra \( A \) (not necessarily contained in \( \mathbb{C} \)), we put

\[
g_\xi(N; A) = \mathcal{O}_\xi(\xi; N) \circ O_\Phi(\xi) A.
\]

For any \( O_\Phi(\xi) \)-algebra \( A \) in \( \mathbb{C} \), we may consider \( g_\xi(N; A) \) as a subalgebra of \( \text{End}_A(m_\xi(N; A)) \).

**Lemma 2.3.** Let \( A \) be an \( O_\Phi(\xi) \)-subalgebra of \( \mathbb{C} \) or \( \mathbb{Q} \). Let \( \lambda \) be a Hecke character in \( m_\xi(N; A) \) for \( \xi \geq 0 \). Then, we have an isomorphism of \( A \)-algebras associated with \( \lambda \):

\[
\mathcal{O}_\xi(N; A) \simeq A[\mathcal{C}_F(N)]
\]

**Proof.** Let \( A_0 \) be the subalgebra of \( \mathbb{C} \) generated over \( O_\Phi(\xi) \) by \( \lambda(a) \).
Lemma 2.5. Suppose that \( A \) is a valuation ring in \( \Omega \) and is finite over \( \mathbb{Z} \). Then the above pairing has values in \( A \) and is perfect. For each \( a \in \Omega \), we have that
\[
\langle \lambda(a), \mu \rangle = \frac{1}{|A|} \sum_{a \in \Omega} \lambda(a) \mu(a).
\]
Furthermore, we have an isomorphism of \( g_1(N; A) \)-modules:
\[
\langle \lambda(a), \mu \rangle = \frac{1}{|A|} \sum_{a \in \Omega} \lambda(a) \mu(a).
\]

Proof. We shall choose a representative set \( \{a_0, \ldots, a_n\} \) for \( g_1(N; A) \)-modules.

For each \( a \in \Omega \), let \( \{a_0, \ldots, a_n\} \) be the class of \( a \) in \( \mathbb{C} \). Then, the correspondence
\[
g_1(N; A) \ni \lambda \mapsto \lambda(a)(a_0, \ldots, a_n) \in \mathbb{C}
\]

yields an embedding: \( g_1(N; A) \rightarrow \mathbb{C}(A) \). Obviously, the two algebras coincide after tensoring with \( \Lambda \). Thus, \( g_1(N; A) \) is a complete representation set \( \{a_0, \ldots, a_n\} \) of \( \mathbb{C}(A) \) in \( \mathbb{C}(A) \), so that
\[
g_1(N; A) \rightarrow \mathbb{C}(A) \text{ injective.}
\]

Let \( A \) be an \( \mathbb{Q}(q) \)-subalgebra in \( \mathbb{Q}(q) \). Define a pairing
\[
\langle \lambda, \mu \rangle = \frac{1}{|A|} \sum_{a \in \Omega} \lambda(a) \mu(a).
\]

For each \( a \in \Omega \), consider the function
\[
\lambda : A \rightarrow A, \quad \lambda(a) = \lambda(a)(a_0, \ldots, a_n)\text{,}
\]
defined by \( \lambda(a)(a_0, \ldots, a_n) = (A(a_0, \ldots, a_n))^{-1} \). This finishes the proof.

Theorem 2.6. Let \( A \) be an \( \mathbb{Q}(q) \)-algebra in \( \mathbb{Q}(q) \). Then \( \lambda \) induces an \( A \)-algebra homomorphism
\[
\lambda : g_1(N; A) \rightarrow A.
\]

From (2.2a) or (2.2b), \( \lambda_* \mathcal{M}(N) \). Thus, \( \lambda_* \mathcal{M}(N) \) depends only on the class of \( a \) in \( \mathbb{C} \). One sees easily from (2.2a) that if
Furthermore, the Fitting ideals of these modules coincide, and any one of them is generated by the class number \(|C_L(F)|\).

This follows from Lemma 2.3 and Lemma 1.9.

Let \(L\) be a finite extension of \(F\) and let \(N_{L/F} : L^* \rightarrow F^*\) be the norm map. Write \(U_L = U_L(N)\) for an ideal \(N\) of \(O_F\) and put \(U_F = U_F(N)\). Then, \(N_{L/F}\) induces a morphism of abelian groups \(N_{L/F} : C_L(N) \rightarrow C_L(F)\). We have also natural linear maps

\[
\text{res} : \mathbb{Z}[I_L] \rightarrow \mathbb{Z}[I_F], \quad \text{inf} : \mathbb{Z}[I_F] \rightarrow \mathbb{Z}[I_L]
\]

defined by \(\text{res}(\tau) = \tau|_F\) for \(\tau \in I_L\) and \(\text{inf}(\sigma) = \sum_{\tau \in I_F} \text{res}(\sigma \circ \tau)\) for \(\tau \in I_F\).

Then the lifting map \(\text{Inf} : m_{\chi}(N;A) \rightarrow m_{\text{inf}(\lambda)}(N;A)\) can be defined by \(\text{Inf}(\lambda) = \lambda \circ N_{L/F}\). Then we have a commutative diagram for each ideal \(a \in I_L(N)\):

\[
\begin{array}{ccc}
\text{Inf} : m_{\chi}(N;A) & \rightarrow & m_{\text{inf}(\lambda)}(N;A) \\
\downarrow T(N_{L/F}(a)) & & \downarrow T(a) \\
\text{Inf} : m_{\chi}(N;A) & \rightarrow & m_{\text{inf}(\lambda)}(N;A)
\end{array}
\]

Thus the correspondence between the operator \(T(a)\) on \(m_{\text{inf}(\lambda)}(N;A)\) and \(T(N_{L/F}(a))\) on \(m_{\chi}(N;A)\) gives an \(A\)-algebra homomorphism:

\[
N_{L/F}^A : g_{\text{inf}(\lambda)}(N;A) \rightarrow g_{\chi}(N;A).
\]

One verifies the following formulae:

\[
(\lambda \circ N_{L/F})_a = \lambda_a \circ N_{L/F},
\]

\[
\langle h, \lambda \circ N_{L/F} \rangle = \langle h, \lambda \rangle \quad \text{for the pairing } \langle , \rangle \text{ defined in Lemma 2.4}.
\]

Corollary 2.7. The norm map \(N_{L/F}^0\) induces natural morphisms:

\[
N_{L/F}^0 : C_0((\lambda \circ N_{L/F})_a;A) \rightarrow C_0(\lambda_a;A),
\]

\[
N_{L/F}^1 : C_1((\lambda \circ N_{L/F})_a;A) \rightarrow C_1(\lambda_a;A).
\]

Moreover, we have a commutative diagram:

\[
\begin{array}{ccc}
C_1((\lambda \circ N_{L/F})_a;A) & \rightarrow & A \circ Z \cdot C_L(N) \\
\downarrow N_{L/F}^1 & & \downarrow 1 \circ N_{L/F} \\
C_1(\lambda_a;A) & \rightarrow & A \circ Z \cdot C_L(F).
\end{array}
\]

§ 3. Hecke algebras for \(GL_2\).

Hereafter always, we suppose that \(F\) is totally real.

We shall give in this section the definitions of Hecke algebras for the spaces of modular forms for \(GL_2(F)\) and bring together without proof some results necessary for the comparison of modules of congruences for \(GL_1\) and \(GL_2\), which will be done in the next section. We shall indicate briefly their proof and the references if possible. A detailed proof will appear elsewhere. For the proofs for \(F = \mathbb{Q}\) and for a generalization of the results in § 2 for \(GL_2(\mathbb{Q})\), we refer to our previous papers [3], [4], [6] and [7]. We use the same notation for the field \(F\) as in § 2.

Put

\[
V = V_F(N) = \{(a b \over c d) \in GL_2(F) | c \in \mathbb{N}_0, d = \text{mod } \mathbb{N}_0, \text{ for some } e \in E\}
\]

for the fixed integral ideal \(N\) of \(O_F\). Let \(G_{\infty}^+\) be the identity component of \(GL_2(F)\), \(G_A = GL_2(F_A)\) and \(G_0\) be the finite part of \(G_A\). Let \(S^+\) be the product \(H^+\) of the upper half complex planes indexed by \(I_F\), on which \(G_{\infty}^+\) acts via linear fractional transformation. Put \(z_0 = (\sqrt{-1}, \ldots, \sqrt{-1}) \in E\) and \(C_{\infty} = \{g \in G_{\infty}^+ | g(z_0) = z_0\}\). We denote by \(S_k(N) = S_k^+(N)\) for an integer \(k > 0\) the space of functions \(f : G_A \rightarrow \mathbb{E}\) which satisfies the following conditions:

\[
(3.2a) \quad \text{Put, for } y = (y_t)_{t \in E} = \left((c_t d_t)_{t \in E}\right),
\]

\[
\text{Then } f(axu) = j_k(u, z_0)^{-1} f(x) \text{ for all } a \in GL_2(F) \text{ and } u \in V_F(N)c_{\infty}.
\]

(3.2b) For each \(x \in G_0\) and \(z \in \mathbb{E}\), put

\[
\]
\[ f(x) = f_k(\mu, z_0) \cdot f(\mu x) \text{ for } \mu, \zeta, \tau, \sigma \in G \text{ with } z = \mu z_0. \]

Then \( f \) is a \( \tau \)-invariant function \( f \in \text{AF} \).

For a Haar measure \( \mu \) on the additive group \( F_\tau \cdot F \),
\[
\int_{F_\tau \cdot F} f(\mu x) \mu(x) \text{ d}x = 0 \quad \text{for all } x \in G_A.
\]

(3.2c)

If \( F = \mathbb{Q} \), we have to add the holomorphy condition at each cusps.

In order to describe the Fourier expansion of elements of \( S_k(N) \), we define some notations: Put \( \epsilon_F(x) = \exp(2\pi \sqrt{-1} \cdot \tau) \cdot x \) \( x \in G \). Especially, for \( \tau \in F \), \( \epsilon_F(x_\tau) = \exp(2\pi \sqrt{-1} \cdot \tau \cdot x_\tau) \). Let \( \psi \) be the standard additive character of \( F_\tau \cdot F \); i.e., \( \psi \) is the unique one satisfying \( \psi(x_\tau) = \epsilon_F(x_\tau) \) \( x_\tau \in F_\tau \). Then \( f \in S_k(N) \) has the following form of the Fourier expansion (e.g., [13, (2.18)]):

\[ f(t^1) = \sum_{t \in F/\mathbb{Q}} \sum_{\psi \in \mathcal{A}} a(t \psi \tau) \psi(x_\tau) \epsilon_F(t \cdot x_\tau) \psi(x_\tau), \]

where \( t \) denotes the different of \( F/\mathbb{Q} \) and \( \tau = \sum_{\tau \in \mathbb{Z}[1]} \).

(3.3)

Furthermore, \( S_k(N) \) is stable under the action of \( T(a) \) and \( C_k(N) \). This result follows from [13, Th. 1.5] when \( A \) is a \( \mathbb{Q} \)-algebra. For an arbitrary \( A \), we need the \( \mathbb{Q} \)-expansion principle in [8, (1.2.15, 16)], which guarantees the stability of \( S_k(N) \) under \( C_k(N) \). Then, from (3.4), we see the stability under \( T(a) \). Let \( h_k(N; \mathbb{Z}) = h_k^F(N; \mathbb{Z}) \) be a subalgebra of \( \text{End}_A(S_k(N; \mathbb{A})) \) generated over \( \mathbb{Z} \) by \( T(a) \) for all \( a \in I(1) \).

(3.5)

We shall now define Hecke operators. For each integral ideal \( \mathfrak{a} \) of \( \mathbb{Q} \), we take an element \( a \in F_0 \) such that \( a\mathfrak{a} = \mathfrak{a} \). We decompose the double coset \( N(0) \cdot V \) as a disjoint union of right cosets \( N(0) \cdot x_1 \cdot V \)

and define an operator by

\[ f|T(a)(x) = f(x x_1) \text{ for } f \in S_k(N). \]

Then, \( T(a) \) gives an operator on \( S_k(N) \) into itself. Similarly, if \( \mathfrak{a} \in I(F) \), we can define an action of \( C_k(F) \) on \( S_k(N) \) by

\[ (f|a)(x) = f(xa)N(F)(a)^{-2k}. \]

The effect of the Hecke operator \( T(n) \) on the coefficients \( a(m, f) \) is given by

\[ a(m, f|T(n)) = \sum_{a(m, n|\mathfrak{a}) = 1} a(m, f|a)^{-1} a(m, n, f|a). \]

(3.4)

(cf. [13, (2.20)]). When \( F = \mathbb{Q} \), if we define \( f_1(z) \) as in (3.2b), \( f_1 \) is an element of \( S_k(T_1(N)) \) in the usual sense for \( T_1(N) = SL_2(\mathbb{Z}) \cdot \mathbb{Z}[N] \), and \( f_1(z) = \sum_{n=1}^{\infty} a(n, f)e_{q}(nz) \) \( z \in \mathbb{H} \). Thus, by this correspondence, we know that \( S_k^T(N) = S_k(T_1(N)) \), and the definition of the Hecke operator \( T(n) \) coincides with the classical one. For any subring \( A \) of \( \mathbb{F} \), we put

\[ S_k(N; A) = S_k^F(N; A) = \{ f \in S_k^F(N; A) \mid a(f) \in A \text{ for all } a \in I(F) \}. \]

Then it is known by [13, Prop. 1.7] or [8, (1.2.15, 16)] that if \( A \) is finite over \( \mathbb{Z} \) or a field

\[ S_k(N; \mathbb{Z}) \oslash \mathbb{Z} = S_k(N; \mathbb{A}). \]

For any subalgebra \( A \) of \( \mathbb{A} \), we put

\[ S_k(N; A) = S_k(N; \mathbb{Z}) \oslash \mathbb{Z} \oslash A. \]

Furthermore, \( S_k(N; A) \) is stable under the action of \( T(a) \) and \( C_k(N) \). This result follows from [13, Th. 1.5] when \( A \) is a \( \mathbb{Q} \)-algebra. For an arbitrary \( A \), we need the \( \mathbb{Q} \)-expansion principle in [8, (1.2.15, 16)], which guarantees the stability of \( S_k(N; A) \) under \( C_k(N) \). Then, from (3.4), we see the stability under \( T(a) \). Let \( h_k(N; \mathbb{Z}) = h_k^F(N; \mathbb{Z}) \) be a subalgebra of \( \text{End}_A(S_k(N; \mathbb{A})) \) generated over \( \mathbb{Z} \) by \( T(a) \) for all \( a \in I(1) \). One can show that the action of \( C_k(F) \) gives a character of \( C_k(N) \) with values in \( h_k(N; \mathbb{Z}) \). Put

\[ h_k(N; A) = h_k^F(N; A) = h_k^F(N; \mathbb{Z}) \oslash \mathbb{Z} \oslash A \]

for each commutative algebra \( A \). If \( A \) is a subring of \( \mathbb{A} \) or \( \mathbb{F} \), then \( h_k(N; A) \) may be considered as a subalgebra of \( \text{End}_A(S_k(N; A)) \). We shall define a pairing

\[ \langle \cdot, \cdot \rangle : h_k(N; A) \times h_k(N; A) \rightarrow A \text{ by } \langle h, f \rangle = a(1, f|h). \]

Then, as in [6, Prop. 2.1], we have
Proposition 3.1. When $A$ is a field or a discrete valuation ring in $\Omega$ or $\mathfrak{c}$, the pairing (3.6) is perfect.

Now we shall assume that $F=\mathbb{Q}$ and take a primitive form $f$ in $S_{k}^0(\mathbb{Q})$. Write $L(s,f)=\sum_{n=1}^{\infty} a(n,n)f(n)/n^s = \prod_{q}(1-\omega_0(q^k)|1-\omega_0(q^k)|1-\omega_0(q^kbler)^{-1}. Let $\psi$ be the character of $(\mathbb{Z}/\mathbb{Z})^\times$ defined by

$$f|n=\psi(n)f$$

for $0<n<\mathbb{Z}$ prime to $N$.

Let $\psi_0$ be the primitive Dirichlet character which induces $\psi$ and put

$$D(s,f)=\prod_{q}(1-\omega_0(q^k)|1-\omega_0(q^k)|1-\omega_0(q^k)|1-\omega_0(q^kbler)^{-1}. We suppose that $p\geq 5$ and that $f$ is ordinary at $p$, i.e. $|a(p^n)|=1$. Let $\mathbb{Q}_p(f)$ be the subfield of $\Omega$ generated over $\mathbb{Q}_p$ by $a(n,n)f$ for all $n$, and let $\mathcal{O}_p(f)$ be the $p$-adic integer ring of $\mathbb{Q}_p(f)$. We can associate an $\mathcal{O}_p(f)$-algebra homomorphism $f_\mathfrak{a}$ of

$$h_k^0(N;\mathcal{O}_p(f))$$

onto $\mathcal{O}_p(f)$ by $f_\mathfrak{a}(I(n))=a(n,n)f$. On the other hand, one can define a canonical transcendental factor $U_\infty(f)$ as in (7, (10.8b)) of $D(f,k)$. Then, we restate here [7, Th. 10.5] for our later use:

Theorem 3.2. Let $R$ be the local ring of $h_k^0(N;\mathcal{O}_p(f))$ through which $f_\mathfrak{a}$ factors through. Suppose that $\text{Hom}_{\mathcal{O}_p(f)}(R,\mathcal{O}_p(f))=R$ as $R$-modules, and further assume one of the following conditions:

(3.7a) $N$ is prime to $p$ and $k\not\equiv 2$ mod $p$;

(3.7b) $k=2$ and the tame part of the restriction of $\psi$ to $\mathbb{Z}^\times$ is non-trivial. Then there exists a $p$-adic unit $U_\infty(f)$ in $\Omega$ such that

$$C(f)=D(f,k)/U_\infty(f)$$

and $f_\mathfrak{a}(c_0)=f_\mathfrak{a}(f_0;\mathcal{O}_p(f))$ is generated by $C(f)$ as an ideal of $\mathcal{O}_p(f)$.

§ 4. Functoriality between $\text{GL}_1$ and $\text{GL}_2$.

Let $\mathcal{M}$ be a totally imaginary quadratic extension of $F$ (i.e. CM-field). The result here may be generalized to any quadratic extension of $F$, but we shall content ourselves with CM-fields. Let $D$ be the relative discriminant of $\mathcal{M}/F$. We fix a CM-type $J$ of $\mathcal{M}$; i.e., $J$ is a subset of $\mathcal{M}$ such that $J\cap \mathfrak{c}^1 = \mathfrak{c}$ gives an isomorphism $J\cong \mathfrak{c}$. We write $\psi=\Sigma\tau$. Let $A$ be an $\mathcal{O}_\mathfrak{c}$-algebra in $\Omega$ or $\mathfrak{c}$.

Then, it is known after Hecke (e.g. [13, § 5]) or [2, 5.21]) that there is an $A$-linear map $\theta$ for each $0<\mathfrak{c}<\mathfrak{c}$ and each ideal $\mathfrak{c}$ of $\mathcal{O}_\mathfrak{c}$:

(4.1a) $\theta: m^\mathcal{M}_j,\mathfrak{c}(\mathfrak{c};A) \rightarrow S_k^F(N_{\mathcal{M}/F}(\mathfrak{c};D;A)$ for $k=j+1$

such that for $\lambda \in \mathcal{M}_j,\mathfrak{c}(\mathfrak{c};A)$,

$$a(\alpha;\theta(\lambda))=\Sigma_{b\in I^\mathcal{M}(\mathfrak{c})} \lambda(b)$$

for each $\alpha \in I^\mathcal{M}(1)$.

By definition, $\theta$ satisfies the following relation for each prime ideal $\mathfrak{p}$ of $\mathcal{O}_\mathfrak{c}$:

(4.16) $\theta(\lambda)\mid T(\mathfrak{p}) = \begin{cases} 0 & \text{if } \mathfrak{p} \mathcal{O}_\mathfrak{c} \text{ is a prime ideal of } \mathcal{O}_\mathfrak{c} \\ \theta(\lambda)\mid T(\mathfrak{p}) & \text{if } \mathfrak{p} \mathcal{O}_\mathfrak{c} \text{ is a prime ideal of } \mathcal{O}_\mathfrak{c} \end{cases}$

Thus, this induces an $A$-algebra homomorphism:

$$\theta^* : h_k^0(N_{\mathcal{M}/F}(\mathfrak{c};D;A) \rightarrow \mathcal{M}_j,\mathfrak{c}(\mathfrak{c};A)$$

for $k=j+1$.

which satisfies $\langle \theta^*(\mathfrak{c}),\lambda \rangle = \langle \mathfrak{c},\theta(\lambda) \rangle$ where $\langle , \rangle$ is the pairing in (3.6) and $\langle , \rangle$ is that of Lemma 2.4.

Suppose the following conditions (e.g. [8, p. 204]):

(4.2a) Every prime ideal of $\mathcal{O}_\mathfrak{c}$ over $\mathfrak{p}$ splits in $\mathcal{O}_\mathfrak{c}$; namely, if $\mathfrak{p}$ is a prime ideal of $\mathcal{O}_\mathfrak{c}$ over $\mathfrak{p}$, then $\mathfrak{p}^2 = \mathfrak{p} \mathcal{O}_\mathfrak{c}$ and $\mathfrak{p} \mathcal{O}_\mathfrak{c}$ with $\mathfrak{p} \mathcal{O}_\mathfrak{c}$.

(4.2b) $J$ is ordinary; namely, whenever $\sigma \in J$ and $\tau \in I_{\mathcal{M}} - J$, then the pull back of the norm of $\mathfrak{c}$ by $\sigma$ and $\tau$ on $\mathcal{M}$ is different.

Let $\mathfrak{p}$ be the product of prime ideals $\mathfrak{p}$ of $\mathcal{O}_\mathfrak{c}$ over $\mathfrak{p}$ such that $\mathfrak{p}^2 = \mathfrak{p} \mathcal{O}_\mathfrak{c}$ is contained in $\{\alpha \in \mathcal{O}_\mathfrak{c} | |\alpha|_p < 1 \}$.

By the assumption (4.2a,b), $\mathfrak{p}$ and $\mathfrak{p}^2$ for complex conjugation $\rho$ are
mutually prime and \( p \) divides \((PP^\mathbb{P})^m\) for sufficiently large \( m \). Let \( A \) be a discrete valuation ring finite over \( \mathbb{Z}_p \) (in \( \Omega \)) containing \( O_p(\psi) \). Let \( K \) be the quotient field of \( A \). Put

\[
\begin{align*}
\mathcal{M}_{j,\psi}^M(N;K/A) &= \mathcal{M}_{j,\psi}^M(N;A)_{\theta_A K/A}, \\
S_k^F(N;K/A) &= S_k^F(N;A)_{\theta_A K/A}.
\end{align*}
\]

Proposition 4.1. Suppose that \( c \) is prime to \( p \), and write \( P_F = P \cap F \) and \( N = N_{M/F}(c) \). Let \( \theta : m_{j,\psi}^M(cP^F;K/A) \rightarrow S_k^F(N;P_F;K/A) \) be the map induced from (4.1a). Put \( W = \prod_{P \mid \mathcal{F}_F} \mathcal{O}_F^x \). Then, via \( \hat{\lambda}(x) \mapsto \hat{\lambda}(wx), \) \( W \) acts on \( m_{j,\psi}^M(cP^F;K/A) \). Furthermore, \( \text{Ker}(\theta) \) is contained in \( H_0(W,m_{j,\psi}^M(cP^F;K/A)) \).

**Proof.** Put \( X = \lim_{\rightarrow} C^M_{cP^F}(cP^F), \ Z^0 = \lim_{\rightarrow} C^M_{cP^F}(cP^F) \) and \( Y = \lim_{\rightarrow} C^M_{cP^F}(cP^F) \).

As seen in paragraph 2, one can regard each element of \( m_{j,\psi}^M(cP^F;K/A) \) as a continuous function on \( Y \) with values in the discrete module \( K/A \). Thus \( W \) acts on \( m_{j,\psi}^M(cP^F;K/A) \) via translation. By the assumption (4.2a,b), (2.2b) shows that the function in \( m_{j,\psi}^M(cP^F;K/A) \) factors through \( X \). Let \( C(X;K/A) \) be the space of continuous functions on \( X \) with values in \( K/A \). Thus \( m_{j,\psi}^M(cP^F;K/A) \subset C(X;K/A) \). For any \( \phi \in C(X;K/A) \) and \( a \in \mathbb{I}_F(1) \), we put \( a(\phi) = \sum_{\phi \in C(cP^F)} a(\phi)b \). There are projection maps \( N_{M/F}(h) = a \rightarrow X \) and \( Y \rightarrow X \times X^0 \). Put \( W^0 = \prod_{P \mid \mathcal{F}_F} \mathcal{O}_F^x \). We have a commutative diagram:

\[
\begin{array}{ccc}
W \times W^0 & \rightarrow & X \times X^0 \\
\downarrow & & \downarrow \\
Y & \rightarrow & X \times X^0,
\end{array}
\]

where all the maps are natural ones. Thus the image \( \Delta(Y) \) contains the image \( \Delta(W \times W^0) \). Let \( S = \{Q \in M(cP)|Q \text{ is totally split over } Q\} \).

Then, by the Tchebotarev density theorem, \( S \) is dense in \( Y \) and thus in \( \Delta(Y) \). For \( \phi \in C(X;K/A) \), define \( \hat{\phi} \in C(X;X^0;K/A) \) by \( \hat{\phi}(x,y) = \phi(x)(y^0) \). Then we see that for \( q = N_{M/F}(Q) \)

\[
\hat{\phi}(q,q) = a(q,\theta(\phi)) \quad \text{for } q \in S.
\]

If \( a(q,\theta(\phi)) = 0 \) for all \( Q \in S \), then \( \hat{\phi} = 0 \) on \( \Delta(Y) \). Since \( \Delta(Y) = \Delta(W \times W^0) \), \( \phi(tx) = -\phi(t'x) \) for all \( x, y \in W \) and \( (t,t') \in \Delta(Y) \). This shows that \( \phi \) is invariant under the action of \( W \), which finishes the proof.

For any integer \( q > 0 \), put \( \mu_q = (\zeta \in \mathbb{C}| \zeta^q = 1) \).

Write \( P = P_j \cdots P_k \) and put \( P_i = P_j \cap F \). Let \( e_i \) be the ramification index of \( P_i \) over \( Q \). Then \( \mu = \mu_{q_1} \times \cdots \times \mu_{q_n} \in W \) for \( q = N_{F/Q}(P_i) \).

For \( \zeta = (\zeta_1, \ldots, \zeta_n) \in \mu \), we see from (2.2b) that

\[
\hat{\lambda}(\zeta) = \prod_{i=1}^n N_{M/F}(cP^F_{i}) \phi(\zeta_i)^{-1} \lambda
\]

for \( \lambda \in m_{j,\psi}^M(cP^F;K/A) \).

Let \( \psi \) be a character of \( M \) into \( A \), and put

\[
m_{j,\psi}^M(cP^F;K/A) = \{ \lambda \in m_{j,\psi}^M(cP^F;K/A) | \hat{\lambda}(\zeta) = \psi(\zeta) \lambda \}
\]

\[
g_{j,\psi}^F(cP^F;K/A) = \{ h \in g_{j,\psi}^F(cP^F;K/A) | h(\zeta) = \psi(\zeta) h \},
\]

where on the Hecke algebra, \( \mu \) acts through the adjoint action under the pairing in Lemma 2.4. By the assumption (4.2a), we may identify \( W \) with \( \mathcal{O}_F^x \) and regard it as a subgroup of \( F^x \). Now we denote the class of \( w \in W \) in \( C^M_{cP^F}(N;P^M) \) by \( [w] \), then we have

\[
(4.4) \quad \theta(\lambda)[w] = N_{F/Q} \theta(\lambda)[w].
\]

Considering \( \mu \) as a subgroup of \( C^M_{cP^F}(N;P^M) \) via \( \iota \mapsto \{ \iota \} \), we put

\[
S_k^F(N;P^M;K/A) = \{ f \in S_k^F(N;P^M;K/A) | f|\zeta = \psi(\zeta)f \text{ for } \zeta \in \mu \}
\]

\[
h_k^F(N;P^M;K/A) = \{ h \in h_k^F(N;P^M;K/A) | h|\zeta = \psi(\zeta)h \text{ for } \zeta \in \mu \}.
\]

By (4.4), \( \theta \) induces morphisms:

\[
(4.5) \quad \theta : m_{j,\psi}^M(cP^M;K/A) \rightarrow S_k^F(N;P^M;K/A) \quad (k = j+1)
\]

\[
\theta^* : h_k^F(N;P^M;K/A) \rightarrow g_{j,\psi}^F(cP^M;K/A),
\]

where \( \psi(\zeta) = N_{F/Q} \theta(\zeta)^{-1} \psi(\zeta) \).
Corollary 4.2. If $\varphi$ as above is non-trivial, then $0$ is injective and $\Theta^*$ is surjective. Especially, if $m = 0$ and $j \equiv 0 \mod p-1$ for one of $i = 1, \ldots, r$, $\Theta$ is injective and $\Theta^*$ is surjective.

Proof. The pairing in Lemma 2.4 (resp. (3.6)) induces the Pontryagin duality between $g_j,F(c,c';A)$ and $m_j,\psi(c,c';A)$ (resp. $h_k,F(c,c';A)$ and $s_k,F(c,c';A)$). Thus the surjectivity of $\Theta^*$ follows from the injectivity of $\Theta$, which is a consequence of Proposition 4.1. When $m = 0$, $u$ acts on $m_j,\psi(c,c';A)$ by the character

$$
\zeta \mapsto N_{T_i/F}(c)^{-e_i} \quad \text{for} \quad c \in \mathbb{F}_q,
$$

where $T_i$ is the maximal unramified extension of $\mathbb{F}_p$ in $F_i$.

Since $N_{T_i/F}(c)^{-e_i} \rightarrow \mu_{p-1}$ is surjective, this character is non-trivial if and only if $j \equiv 0 \mod p-1$. This shows the last assertion.

Theorem 4.3. Let $C = c_{p_1} \cdots c_{p_r}$ with $c$ prime to $p$, and let $\lambda$ be a primitive Hecke character in $m_j,\psi(c,c';A)$ with $j > 0$. Let $R(\lambda)$ (resp. $R(\Theta(\lambda))$) be the local ring of $g_j,F(c,c';A)$ (resp. $h_k,F(c,c';A)$) through which $\lambda_\varphi$ (resp. $\lambda_\varphi \Theta^*$) factors. Then we have

(i) $\Theta^*$ is of finite cokernel and induces morphisms of $A$-modules:

$$
\begin{align*}
0 \hookrightarrow C_0(\lambda_\varphi \Theta^*;A) \rightarrow C_0(\lambda_\varphi;A) \rightarrow A/C_0M(c) | A, \\
0 \hookrightarrow C_1(\lambda_\varphi \Theta^*;A) \rightarrow C_1(\lambda_\varphi;A) \rightarrow A \otimes \mathbb{Z}/C_0M(c).
\end{align*}
$$

Especially $0_0$ is surjective.

(ii) Let $\psi$ be the restriction of $\lambda$ to $\mu$. If $\psi$ is non-trivial, then $\Theta^* : R(\Theta(\lambda)) \rightarrow R(\lambda)$ is surjective, and we have a natural exact sequence:

$$
C_1(\Theta^*;R(\lambda)) \otimes R(\lambda) \rightarrow C_1(\lambda_\varphi \Theta^*;A) \rightarrow C_1(\lambda_\varphi;A) \rightarrow 0.
$$

Proof. The fact that $\Theta^*$ has finite cokernel follows from Proposition 4.1 since $j > 0$. It is known that if $\lambda$ is primitive modulo $c$, then $\Theta(\lambda)$ is also primitive of conductor $M/c$. Then we can decompose

$$
R(\lambda) \otimes A \cong K \otimes D(\lambda) \quad \text{and} \quad R(\Theta(\lambda)) \otimes A \cong K \otimes D(\Theta(\lambda)).
$$

Theorem 4.3 follows from Theorem 2.6. The surjectivity of $\Theta^*$, when $\psi$ is non-trivial, follows from Corollary 4.2. Thus we have surjective morphisms of $A$-algebras:

$$
\Theta^* : R(\Theta(\lambda)) \rightarrow R(\lambda), \quad \lambda_\varphi : R(\lambda) \rightarrow A \quad \text{and} \quad \lambda_\varphi \Theta^* : R(\Theta(\lambda)) \rightarrow A.
$$

This implies that the following sequence is exact:

$$
0 \rightarrow \ker(\Theta^*) \rightarrow \ker(\lambda_\varphi \Theta^*) \rightarrow \ker(\lambda_\varphi) \rightarrow 0.
$$

This induces another exact sequence:

$$
\begin{align*}
\ker(\Theta^* \otimes R(\Theta(\lambda))) &\rightarrow \ker(\lambda_\varphi \Theta^* \otimes R(\Theta(\lambda))) \rightarrow \ker(\lambda_\varphi \otimes R(\Theta(\lambda))) \rightarrow 0, \\
\ker(\Theta^* \otimes R(\lambda)) &\rightarrow \ker(\lambda_\varphi \Theta^* \otimes R(\lambda)) \rightarrow \ker(\lambda_\varphi \otimes R(\lambda)) \rightarrow 0.
\end{align*}
$$

This shows the assertion (ii).

§ 5. Relation to the Iwasawa theory over imaginary quadratic fields.

Hereafter, we suppose that $F = \mathbb{Q}$ and that $M$ is an imaginary quadratic field such that

$$
(5.1) \quad \text{the prime } p \text{ splits in } M \ (p > 2).
$$

We write $p = \mathbb{Z}[\mathbb{Q} \cap |x|_p < 1]$. Put

$$
W = \lim_{\rightarrow \mathbb{Z}_p}\mathbb{Z}_p/(\mathbb{Z}_p^n).
$$

Then, $W$ is a product of torsion subgroup $W_p$ and a $p$-profinite group $W_0$ isomorphic to $\mathbb{Z}_p$. Let $F_p$ be the unique $\mathbb{Z}_p$-extension of $\mathbb{Q}$ unramified outside $p$. Put $\Gamma = 1 + p^{-1}\mathbb{Z}_p \subset \mathbb{Z}_p^\times$ and identify $\Gamma = \mathbb{F}_p$.
with \( \text{Gal}(F_n/Q) \) naturally. Write \( F_n \) for the fixed field of \( \Gamma_n \) in \( F_\infty \). Put \( M_n = F_n \cdot \mathbb{P} \). Then, \( M_n \) is a \( \mathbb{Z}_p \)-extension over \( M \). Let \( \mathfrak{p} \) (resp. \( \mathfrak{p}_n \)) the unique prime of \( \mathcal{O}_M \) over \( p \) (resp. \( p_n \)). Put

\[
\mathfrak{m}_W = \lim_{m \to \infty} \mathcal{O}_M(p_m).
\]

There is the norm map \( N^M_m : \mathfrak{m}_W \to \mathfrak{m}_W \) for each \( m \geq n \).

We take a CM-type of \( M_n \) defined by

\[
J_n = J = \{ \pi \in J_n \mid \pi \mid M \text{ is the identity} \}.
\]

Then, \( J \) satisfies (4.2a,b). Put \( \psi = \psi_n = \sum_{\tau \in J_n} \psi \). Then, we see easily that \( \mathcal{O}_\psi(p_n) = \mathcal{O}_M \). Let \( A \) be a valuation ring in \( \Omega \) finite over \( \mathbb{Z}_p \).

Then, by (5.1), \( A \) automatically contains \( \mathcal{O}_{\psi(p_n)} \). Let \( \lambda \) be a Hecke character in \( m_{j,0}^M(1;A) \) for \( j > 0 \). Put \( \lambda_n = \lambda = \mathcal{O}_{\psi(p_n)}(1;A) \). Then, by Lemma 2.3, we know that

\[
g_{j,\psi_n}(p_n^r:A) \simeq A[\mathcal{C}_M(p_n^r)]
\]

by the isomorphism corresponding to \( \lambda_n \). On the other hand, the restriction of operators in \( g_{j,\psi_n}(p_n^r:A) \) to \( m_{j,0}^M(p_n^r:A) \) for \( s \leq r \) gives a surjective homomorphism of \( A \)-algebras:

\[
g_{j,\psi_n}(p_n^r:A) \to g_{j,\psi_n}(p_n^r:A).
\]

We shall take a projective limit:

\[
g_{j,\psi_n}(A) = \lim_{r \to \infty} g_{j,\psi_n}(p_n^r:A),
\]

which is a compact topological ring. By Lemma 2.3, we have

Proposition 5.1. \( g_{j,\psi_n}(A) \simeq A[[\mathfrak{m}_W]] \).

Thus \( g_{j,\psi_n}(A) \) is independent of \( j \).

Let \( X_n \) be the maximal abelian extension of \( M_n \) unramified outside \( \mathfrak{p}_n \). Then, by class field theory, we have the following commutative diagram:

\[
\begin{array}{ccc}
\mathfrak{m}_W & \longrightarrow & \mathfrak{m}_W \\
\mathfrak{m}_W \downarrow & & \downarrow \mathfrak{m}_W \\
\mathfrak{m}_W \downarrow & & \downarrow \\
\mathfrak{m}_W & \to & \mathfrak{m}_W \\
\end{array}
\]

\[
\text{Gal}(X_n/M_n) \to \text{Gal}(X_n/M_n \cap X_n).
\]

Note that \( \mathfrak{m}_W \simeq \text{Gal}(X_n/M_n) \) and \( M_n \cap X_n = M_n \), because \( M_n \) is totally ramified at \( \mathfrak{p}_n \). Since \( \lambda_n \simeq X_n \), the restriction map of the Galois groups is surjective, and thus, the map \( N^M_m : \mathfrak{m}_W \to \mathfrak{m}_W \) is surjective. It is known by Coates and Wiles (e.g., [5, §1]) that \( \mathfrak{m}_W = \mathfrak{m}_W_t \otimes^\mathbb{Z}_p \mathfrak{m}_t \) with a finite torsion subgroup \( \mathfrak{m}_t \) and \( \mathfrak{m}_W_t \otimes^\mathbb{Z}_p \mathfrak{m}_t \). Naturally, \( \text{Gal}(M_n/M) \) acts on \( \mathfrak{m}_W \), and we may assume that \( \text{Gal}(M_n/M) \) acts on \( \mathfrak{m}_W_t \) trivially, and \( \mathfrak{m}_W \) induces an isomorphism: \( \mathfrak{m}_W_t \simeq \mathfrak{m}_W \). We shall identify all \( \mathfrak{m}_W \) with \( \mathfrak{m}_W_t \) by \( \mathfrak{m}_W_t \); thus, \( g_{j,\psi_n}(A) \) becomes an \( \mathbb{A}[\mathfrak{m}_W_t] \)-algebra. Let \( \lambda_n \) denote the continuous character of \( \mathfrak{m}_W \) into \( \mathbb{A}^* = \mathbb{A}^* \) associated with \( \lambda_n \) introduced in §2. Since \( g_{j,\psi_n}(A) \simeq A[[\mathfrak{m}_W]] \) (this isomorphism only depends on the choice of \( \lambda_n \in m_{j,0}^M(1;A) \)), the restriction of \( \lambda_n \) to \( \mathfrak{m}_W_t \) gives a homomorphism of \( \mathbb{A}[\mathfrak{m}_W_t] \)-algebras:

\[
\Lambda_n : g_{j,\psi_n}(A) \to A[[\mathfrak{m}_W_t]]; \quad \Lambda_n(w) = \lambda_n(w) \in A \text{ for all } w \in \mathfrak{m}_W_t.
\]

namely, by regarding \( g_{j,\psi_n}(A) \) as the group algebra of \( \mathfrak{m}_W_t \) over \( A[[\mathfrak{m}_W]] \), \( \Lambda_n(w_t) = \lambda_n(w_t) \in A \) for all \( w_t \in \mathfrak{m}_W_t \). Thus, we know from the results in §1:

Theorem 5.2. We have that

\[
C_0(\Lambda_n;A[[\mathfrak{m}_W]]) \simeq A[[\mathfrak{m}_W]]/\mathfrak{m}_W_t|A[[\mathfrak{m}_W]],
\]

\[
C_1(\Lambda_n;A[[\mathfrak{m}_W]]) \simeq A[[\mathfrak{m}_W]]/\mathbb{Z}_p^\mathfrak{m}_W_t,
\]

\[
\phi(\mathbb{A}[\mathfrak{m}_W])C_0(\Lambda_n;A[[\mathfrak{m}_W]]) = \phi(A[[\mathfrak{m}_W]])C_1(\Lambda_n;A[[\mathfrak{m}_W]]).
\]

The Galois group \( \Gamma_n = \text{Gal}(M_n/M) \) acts naturally on \( g_{j,\psi_n}(A) \) and \( \mathfrak{m}_W_t \), and its action is trivial on \( \mathfrak{m}_W_t \). Thus we have

Corollary 5.3. The Galois group \( \text{Gal}(M_n/M) \) acts trivially on \( C_0(\Lambda_n;A[[\mathfrak{m}_W]]) \) and acts on \( C_1(\Lambda_n;A[[\mathfrak{m}_W]]) \) through the natural action
on $\mathcal{W}_t$.

We shall write $h_k^0(N;A)$ for $h_k^F(N;A)$ with $F = F_n$, and put

$$
h_k^0(Np_n;A) = \lim_{r \to \infty} h_k^0(Np_r;A),
$$

and let $h_k^{\text{ord}}(N;A) = h_k^F(N;A)$ be the ordinary part of $h_k^0(Np_n;A)$, which is the largest algebra direct summand of $h_k^0(Np_n;A)$ in which the image of $T(p)$ is a unit (cf. [7, § 1]). Put

$$
\Gamma(N) = \lim_{r \to \infty} \gg_{F_n}(Np_r).
$$

Then, by the action of $\gg_{F_n}(Np_r)$ on $h_k^0(Np_r;A)$ defined in paragraph 3 (in § 3, we have defined its action on $S_k(Np_r;A)$, but via the adjoint action under the pairing (3.6), it acts on the Hecke algebra), the algebra $h_k^0(Np_n;A)$ as well as $h_k^{\text{ord}}(N;A)$ becomes an $A[\Gamma(N)]$-algebra.

Since $\Gamma$ is naturally a subgroup of $\Gamma(N)$ through $\theta_p : \mathbb{F}_p$, $h_k^{\text{ord}}(N;A)$ is a $\Lambda_{\text{A}}$-algebra for $\Lambda_{\text{A}} = A[\Gamma(N)]$. The following fact will be proved in our subsequent paper:

**Theorem 5.4.** $h_k^{\text{ord}}(N;A)$ is independent of $k$ if $k \geq 2$ and is finite over $\Lambda_{\text{A}}$.

Let $D_n$ be the relative discriminant of $M_n/F_n$. Then, there is a natural morphism of $A$-algebra:

$$
\theta^* : h_k^{\text{ord}}(D_n;A) \longrightarrow g_j \cdot \varphi(A),
$$

which is the projective limit of $\theta^*$ at each level defined in § 4 (the fact that $\theta^*$ factors through the ordinary part follows from the assumption (4.2a,b), which is verified easily in our present case). Let $R(A_n)$ (resp. $R\theta(A_n)$) be the local ring of $g_j \cdot \varphi(A)$ (resp. $h_k^{\text{ord}}(D_n;A)$) through which $A_n$ (resp. $A_n \cdot \theta^*$) factors. By Theorem 5.4, we have reasonably defined modules: $C_1(A_n \cdot \theta^*;A[[W_0]])$, $C_1(A_n;A[[W_0]])$ and $C_1(\theta^*;R(A_n))$. Then we know from Theorem 4.3 the following fact:

**Theorem 5.5.** If $j \not\equiv 0 \mod p-1$ and $j > 0$, we have an exact sequence of $A[[W_0]]$-modules:

$$
C_1(\theta^*;R(A_n)) \cdot R(A_n;A[[W_0]]) \longrightarrow C_1(A_n \cdot \theta^*;A[[W_0]])) \longrightarrow C_1(A_n;A[[W_0]]) \longrightarrow 0.
$$

**Remark 5.6.** By the base change lift (Langlands [9]) relative to $F_m/F_n$ ($n > m$), there is a natural algebra homomorphism of $h_k^{\text{ord}}(D_n;A)$ to $h_k^{\text{ord}}(D_n;A)$, which is compatible with the morphism of $g_j \cdot \varphi(A)$ to $g_j \cdot \varphi(A)$ induced by $h_k^{\text{ord}}(A)$ (defined above Corollary 2.7). Thus we may take a natural projective limit

$$
C_1(A_n \cdot \theta^*;A[[W_0]]) = \lim_{\longrightarrow} C_1(A_n \cdot \theta^*;A[[W_0]]),
$$

$$
C_1(A_n;A[[W_0]]) = \lim_{\longrightarrow} C_1(A_n;A[[W_0]]),
$$

$$
\gamma_{W_	ext{t}} = \lim_{\longrightarrow} \gamma_{W_	ext{t}}.
$$

These become $A[[W_0 \times \Gamma]]$-modules through the action of $\Gamma = \text{Gal}(\mathbb{F}_m/Q)$. In fact, $\text{Gal}(F_n/Q)$ acts on $h_k^{\text{ord}}(D_n;A)$ via $T(a) \longrightarrow T(a')$ for $a \in F_n$, and this induces the action of $\Gamma$ on these modules. Although the module $C_1(A_n \cdot \theta^*;A[[W_0]])$ is a finitely generated torsion $A[[W_0]]$-module at every layer $F_n$, it is still unknown that the module $C_1(A_n \cdot \theta^*;A[[W_0]])$ is a finitely generated torsion module over $A[[W_0 \times \Gamma]]$ and not. However, by the following reason, we are inclined to think that $C_1(A_n \cdot \theta^*;A[[W_0]])$ is in fact finitely generated and torsion over $A[[W_0 \times \Gamma]]$: The group $W_0 \times \Gamma$ may be regarded as a subgroup of the maximal abelian extension $L$ of $M$ unramified outside $p$. Let $W_0$ be the maximal $p$-abelian extension of $L$ unramified outside $p$. Then $\text{Gal}(L'/W)$ is known to be a torsion $\mathbb{Z}_p[[L]]$-module (cf. [11, § 2]). Let $W_{n}$ be the fixed field of $W_{n}$ in $L_n$. Then, $\text{Gal}(L'/W_{n}) \approx W_0$ and $\text{Gal}(L'/W_{n}) \approx W_0$. Let $W_{n}$ be the unique anti-cyclotomic extension of $M$ inside $L'$; i.e., $W_{n}$ is the fixed field of the centralizer of complex conjugation in $\text{Gal}(L'/W_0)$, and let $W$ be the maximal $p$-abelian extension of $M: W_{n}$ unramified outside $F_n$. If one can construct a surjective morphism of $\text{Gal}(\mathbb{Z}_p/H)$ onto $C_1(\theta^*;R(A_n)) \cdot R(A_n;A[[W_0]])$ for every layer $M_n$, $C_1(\theta^*;R(A_n)) \cdot R(A_n;A[[W_0]])$ becomes a quotient of
Gal(Z_w/H_wK_w), which is a finite torsion module over A[[W_0 \times \Gamma]] (in fact, Gal(Z_w/H_wK_w) is the quotient of Gal(Y_w/L_w)). By the exact sequence of Theorem 5.5:

\[ C_1(\Theta^*; R(A_w)) \to R(A_w) A[[W_0]] \to C_1(\Lambda_0 \circ \Theta^*; A[[W_0]]) \to C_1(\Lambda_0; A[[W_0]]) \to 0, \]

we may conclude that C_1(\Lambda_0 \circ \Theta^*; A[[W_0]]) is finitely generated and torsion over A[[W_0 \times \Gamma]] if one knows the existence of the morphisms in question. A result on the affirmative side of this question has been obtained as Kummer’s criterion in [5] when n = 0.

§ 6. Gorenstein condition for R(\theta(\lambda)).

In this section, for an ideal c of O_M, we shall prove that R(\theta(\lambda)) is a Gorenstein algebra under certain assumptions, when F = \mathbb{Q}. We use the same notation as in § 5. Let \epsilon be an ideal of O_M prime to p. Suppose that p > 5 and

\[(6.1) p \text{ decomposes in } O_M \text{ into the product of two different prime ideals.} \]

Let \lambda be a primitive quasi character in \sigma_{M,j,w_0}(cF) for j > 0.

Since \sigma_{M,j,w_0}(cF) = O_M and Z_p \supseteq O_M in \Omega, we can consider the algebra \sigma_{j,w_0}(cF; Z_p). Let \lambda_a : \sigma_{j,w_0}(cF; Z_p) \to \Omega be the algebra homomorphism associated with \lambda, and let R(\theta(\lambda)) (resp. R(\lambda)) be the local ring of H_{t}(N_{p}\Omega; Z_p) (k = j+1, N = N_{M/\Omega}(c)D for the discriminant D of M over \Omega) (resp. \sigma_{j,w_0}(cF; Z_p)) through which \lambda_a = \Theta^*(\text{resp. } \lambda_a) factors.

Lemma 6.1. Suppose the following conditions:

(i) j = 1,

(ii) the restriction of \hat{\lambda} to \mu_{p-1} is non-trivial, and

(iii) \lambda is primitive modulo cF. Then we have that

\[ \text{Hom}_{Z_p}(R(\theta(\lambda)), Z_p) \cong R(\theta(\lambda)) \text{ as } R(\theta(\lambda))\text{-modules.} \]

Proof. If we write \hat{\lambda}(\xi) = \xi^a with 0 \leq a < p-1 for \xi \in \mu_{p-1}, then a \not\equiv 0 \pmod{p-2} and a \not\equiv 0 \pmod{p-2} by the assumptions (i), (ii) and (iii). Let J_{\eta}(N_{p}/\mathbb{Q}) be the Jacobian variety of the modular curve X_{1}(N_{p}/\mathbb{Q}) (for the precise definition of X_{1}(N_{p}), see [5, § 31]). Let J_{p} be the p-divisible group of p-power torsion points of J_{1}(N_{p}). Let e be the idempotent attached to \mathfrak{I}(p) in \text{Hom}(N_{p}/\mathbb{Q}; Z_p) (cf. [7, § 11]). Note that the image of e in R(\theta(\lambda)) is the identity element of R(\theta(\lambda)). Put

\[ J_{p}(R) = E = \sum_{\alpha \in R(\theta(\lambda))} \alpha J_{p} \subset J_{p}. \]

Then J_{p}(R) is known to be contained in an abelian subvariety of J_{1}(N_{p}) defined over \mathbb{Q} which acquires good reduction over \mathbb{Z}衰退[p^e]\ ] for a primitive p-th root of unity \zeta_{p} (e.g. [5 § 31]). Thus J_{p}(R) has a structure of p-divisible group over \mathbb{Z}_{p}(\zeta_{p}). Let C (resp. E) be the connected component (resp. maximal étale quotient) of J_{p}(R) over \mathbb{Z}_{p}(\zeta_{p}). As seen in [5, Prop. 3.1], we know that as \text{R}(\theta(\lambda))-modules

\[ C = R(\theta(\lambda)) \otimes^{\mathbb{Z}_{p}} \mathbb{Q}_{p}/\mathbb{Z}_{p} \text{ and } E = \text{Hom}(R(\theta(\lambda)), \mathbb{Q}_{p}/\mathbb{Z}_{p}). \]

Since C and E are topologically defined, the continuous automorphism of \mathbb{Z}_{p}(\zeta_{p}) acts on C and E (see the argument in [5, p. 446-7]). Thus Gal(\mathbb{Q}_{p}/\mathbb{Q}) acts on C and E through characters \chi_{C} and \chi_{E} with values in R(\theta(\lambda)) (the action of R(\lambda) commutes with the Galois action). If \Theta^* induces an isomorphism: R(\theta(\lambda)) \cong R(\lambda), then the assertion follows from Lemma 2.5. Since \Theta^* is surjective onto R(\lambda) by Corollary 4.2, we may assume that \Theta^* has non-trivial kernel. Since \theta(\lambda) is primitive, we can decompose

\[ R(\theta(\lambda)) \otimes^{\mathbb{Z}_{p}} \mathbb{Q}_{p} = A \otimes \Theta^{*} \]

such that A can be embedded into \Omega and the projection to A coincides with \lambda \circ \Theta^*. Let a = A \cap R(\theta(\lambda)) and b = B \cap R(\theta(\lambda)). Put

\[ J_{p}(\lambda) = a J_{p}(R) \]

Then, J_{p}(\lambda) is a p-divisible subgroup of the abelian variety V attached to \theta(\lambda) (in the sense of [14, Th. 7.14]). By Shimura [12], V has complex multiplication under M, the characters \chi_{C} and \chi_{E} modulo b coincide with \hat{\lambda} and \hat{\lambda}_{p} (cf. [5, § 5]), where
\[ \lambda_\rho(\sigma) = \lambda(\rho \sigma \rho^{-1}) \] for complex conjugation \( \rho \).

Let \( \mathfrak{m} \) be the maximal ideal of \( R(\theta(\lambda)) \), and put for each \( R(\theta(\lambda)) \)-module \( X \), \( X[\mathfrak{m}] = \{ x \in X | mx = 0 \text{ for all } m \in \mathfrak{m} \} \). Since \( V \) has complex multiplication and \( \lambda \mod \mathfrak{m} \mapsto \lambda \mod \mathfrak{m} \) by (ii), one can decompose \( V[\mathfrak{m}] = X \otimes Y \) so that on \( X \) (resp. \( Y \)), \( \text{Gal}(\overline{\mathbb{Q}}/M) \) acts via \( \lambda \mod \mathfrak{m} \) (resp. \( \rho \mod \mathfrak{m} \)). Since the representation of \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) on the successive quotients of the Jordan-Hölder series of the Galois module \( V[\mathfrak{m}] \) is isomorphic to the induced representation of \( \lambda \mod \mathfrak{m} \) from \( \text{Gal}(\overline{\mathbb{Q}}/M) \) to \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \), which is irreducible by (ii). Thus \( Y \not\cong 0 \) and \( X \not\cong 0 \).

If on \( C[\mathfrak{m}] \), \( \text{Gal}(\overline{\mathbb{Q}}/M) \) acts via \( \lambda \mod \mathfrak{m} \), then \( C[\mathfrak{m}] = Y \). If on \( C[\mathfrak{m}] \), \( \text{Gal}(\overline{\mathbb{Q}}/M) \) acts via \( \rho \mod \mathfrak{m} \), then \( C[\mathfrak{m}] = X \).

On the other hand, \( E[\mathfrak{m}] = R(\theta(\lambda))/\mathfrak{m} \) and hence, we have an exact sequence:

\[
0 \longrightarrow C[\mathfrak{m}] \longrightarrow J_p(R)[\mathfrak{m}] \longrightarrow E[\mathfrak{m}] \longrightarrow 0,
\]

which splits as \( \text{Gal}(\overline{\mathbb{Q}}/M) \)-module. Thus we may identify

\[
J_p(R)[\mathfrak{m}] = C[\mathfrak{m}] \oplus E[\mathfrak{m}] = \text{Gal}(\overline{\mathbb{Q}}/M) \)-module.
\]

Since \( \lambda_\rho(\sigma) = \lambda(\rho \sigma \rho^{-1}) \), the action of \( \rho \) on \( J_p(R)[\mathfrak{m}] \) interchanges \( C[\mathfrak{m}] \) and \( E[\mathfrak{m}] \). This shows that for \( R = R(\theta(\lambda)) \)

\[
R/\mathfrak{m} = E[\mathfrak{m}] \cong C[\mathfrak{m}] \cong (R/pR)[\mathfrak{m}] = R/pR[\mathfrak{m}].
\]

This shows that \( R/pR \cong \text{Hom}(R/pR, \mathbb{Z}/p\mathbb{Z}) \) as \( R \)-module and hence, the assertion follows.

Theorem 6.2. Let \( \lambda \) be a Hecke character in \( m_j \varphi_0(c \mathfrak{p}) \), whose conductor is divisible by \( c \). Suppose that if \( \lambda(c) = \zeta^a \) for \( \zeta \in \mathbb{U}_{p-1} \) with \( 0 \leq a < p-1 \), \( a \) is neither 0 nor \( p-2 \) (i.e. the tame part of the \( p \)-part of the character of \( \theta(\lambda) \) is given by \( \zeta \mapsto \zeta^a \) for \( a + j \equiv 1 \mod p-1 \)). Then we have that

\[
\text{Hom}_{\mathbb{Z}_p}(R(\theta(\lambda)), \mathbb{Z}_p) \cong R(\theta(\lambda)) \text{-module;}
\]

namely \( R(\theta(\lambda)) \) is a Gorenstein algebra.

Proof. Let \( h^{ord}(N;\mathbb{Z}_p) \) be as in Theorem 5.4 for \( N = \mathcal{N}_M(c)D \). Then it is known by [6, Th. 3.1] that \( h^{ord}(N;\mathbb{Z}_p) \) is finite flat over

\( \Lambda = \mathbb{Z}_p[[\mathfrak{p}]] \). For each integer \( k \), let \( x_k : \Lambda \longrightarrow \mathbb{Z}_p \) be the algebra homomorphism such that \( x_k(\gamma) = \gamma^k \in \mathbb{Z}_p \) for \( \gamma \in \Gamma \). Then it is known (cf. [6, Cor. 3.2]) that for the kernel \( \mathbb{P}_k = \text{Ker}(x_k) \) there is a unique local ring \( R \) of \( h^{ord}(N;\mathbb{Z}_p) \) such that

\[
R \otimes_{\Lambda} \mathbb{P}_k = R(\theta(\lambda)) \text{ as } \mathbb{Z}_p \text{-module.}
\]

Then, by [6, Th. 7.1] and [7, Prop. 2.3], there is a Hecke character \( \lambda' \in \mathbb{M}_p(c \mathfrak{p}) \) which satisfies the conditions of Lemma 6.1 and \( R/\mathfrak{p}R \cong R(\theta(\lambda')) \) as \( \mathbb{Z}_p \)-module. Then, by Lemma 6.1, we know that

\[
\text{Hom}_{\mathbb{Z}_p}(R(\mathbb{P}_k), \mathbb{Z}_p) \cong R/\mathfrak{p}R \text{ as } \mathbb{Z}_p \text{-module. Since } R \text{ is flat over } \Lambda, \text{ this implies that}
\]

\[
\text{Hom}_{\Lambda}(R, \Lambda) = R \text{ as } \mathbb{Z}_p \text{-module.}
\]

Thus we conclude that, as \( \mathbb{Z}_p \)-module,

\[
\text{Hom}_{\mathbb{Z}_p}(R(\theta(\lambda)), \mathbb{Z}_p) = \text{Hom}_{\Lambda}(R, \Lambda) \otimes_{\Lambda} \mathbb{P}_k = R/\mathfrak{p}R = R(\theta(\lambda)).
\]

Corollary 6.3. Under the assumption of Theorem 6.2, we have the vanishing of the module of defect \( N_s \) for the primitive irreducible component of \( h^{ord}(N;\mathbb{Z}_p) \) to which \( \theta(\lambda) \) belongs. Especially, the assumptions of Theorem 6.1 are satisfied if \( \lambda \) is a primitive Hecke character in \( m_j \varphi_0(c \mathfrak{p}) \) with \( j \not\equiv 0 \mod p-1 \) and \( j > 1 \) (i.e. \( k \equiv 1,2 \mod p-1 \) and \( k > 2 \)).

This follows from Lemma 2.5, Theorem 6.2 and [6, Prop. 3.9].

Corollary 6.4. Let \( h \) be the class number of \( M \) and let \( \lambda \in m_j \varphi_0(1) \).

Suppose that \( j \not\equiv 0 \mod p-1 \). Then the number \( D(\mathbb{Q}(\mathbb{Q}(\theta(\lambda))/U_{\infty}(\theta(\lambda))) \)

is divisible by \( h \) in the \( p \)-adic integer ring of \( \mathbb{Q} \).

This follows from Theorem 3.2, Theorem 6.2, Theorem 2.6 and Theorem 4.3 (i).
BIBLIOGRAPHIE


