Siegel-Weil Formulas

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Contents

1 Review of Algebraic Geometry

We recall here definitions and result necessary to read Weil's paper [We2], which is the purpose of this series of lectures. All rings A we consider have the identity element 1_A, and we denote by 0_A the zero element of A.

1.1 Affine schemes

Let A be a base ring, which is always assumed to be noetherian. Let B be a noetherian A–algebra. The *affine scheme* $S = S_B$ associated to B is a function of A–algebras R given by $S(R) = \mathrm{Hom}_{A\text{-alg}}(B, R)$. The set $S_B(R)$ is called the set of R–rational points (or R–integral points) of S_B. An *A–morphism* (or a morphism defined over A) $\phi : S_B \to S_C$ is given by $\phi(P) = P \circ \underline{\phi}$ for an

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underlying A–algebra homomorphism $\underline{\phi} : C \to B$; in other words, we have the following commutative diagram:

$$\begin{array}{ccc} B & \xrightarrow{P} & R \\ {\scriptstyle \underline{\phi}}\uparrow & & \uparrow{\scriptstyle P\circ\underline{\phi}} \\ C & = & C \end{array}$$

By definition, we have the following properties of the function S_B:

(F1) If $R \xrightarrow{f} R' \xrightarrow{g} R''$ are A–algebra homomorphisms, then we have maps $f_* : S_B(R) \to S_B(R')$ and $g_* : S_B(R') \to S_B(R'')$ by $f_*(P) = f \circ P$ and $g_*(Q) = g \circ Q$ and $(g \circ f)_* = g_* \circ f_*$.

(F2) If $R' = R''$ and g as above is the identity map $i_{R'} : R' \to R'$, we have $i_{R',*} \circ f_* = f_*$. If $R = R'$ and f as above is the identity map $i_R : R \to R$, we have $g_* \circ i_{R,*} = g_*$.

(F3) For the identity map $i_R : R \to R$, $i_{R,*} : S_B(R) \to S_B(R)$ is the identity map of the set $S_B(R)$.

Thus $R \mapsto S_B(R)$ is a covariant functor of A–algebras into sets (see [GME] 1.4 for functors and categories). For two affine schemes S and T over A, a morphism $\phi : S \to T$ is a family of maps $\phi_R : S(R) \to T(R)$ indexed by A–algebras R such that for any A–algebra homomorphism $\alpha : R \to R'$, the following diagram commutes:

$$\begin{array}{ccc} S(R) & \xrightarrow{\phi_R} & T(R) \\ {\scriptstyle \alpha_*}\downarrow & & \downarrow{\scriptstyle \alpha_*} \\ S(R') & \xrightarrow[\phi_{R'}]{} & T(R'). \end{array}$$

We write $\mathrm{Hom}_A(S, T)$ for the set of all morphisms from S into T.

The following fact is trivial but useful:

(U1) By definition, if $R = \bigcup_i R_i$ for A–algebras R_i, we have

$$S_B(R) = \bigcup_i S_B(R_i),$$

because $B \mapsto \mathrm{Hom}_{A\text{-alg}}(B, R)$ satisfies this property. This can be applied in the following way. The adele ring $\mathbb{A}$ can be written as:

$$\mathbb{A} = \bigcup_{\Sigma} \mathbb{A}_\Sigma \quad \text{for} \quad \mathbb{A}_\Sigma = (\widehat{\mathbb{Z}}^{(\Sigma)} \prod_{p \in \Sigma} \mathbb{Q}_p) \times \mathbb{R},$$

where Σ runs over all finite set of primes of $\mathbb{Q}$ containing a given finite set Σ_0. Here $\widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$ and $\widehat{\mathbb{Z}}^{(\Sigma)} = \{(x_p) \in \widehat{\mathbb{Z}} | x_p = 1 \text{ for } p \in \Sigma\}$. If an affine scheme is therefore defined over

$$\mathbb{Z}_{(\Sigma_0)} = \left\{ \frac{a}{b} \middle| a \in \mathbb{Z} \text{ and } b \text{ is a product of primes in } \Sigma_0 \right\},$$

we have $S(\mathbb{A}) = \bigcup_\Sigma S(\mathbb{A}_\Sigma)$. Since $\mathbb{A}_\Sigma$ is made of ideles integral outside a finite set Σ, we have

(U2) If an affine scheme S is defined over the Σ_0–integer ring $\mathbb{Z}_{(\Sigma_0)}$, then $S(\mathbb{A})$ is a union of points almost everywhere integral.

(U3) We have

$$S(\mathbb{A}_\Sigma) = S(\prod_{p\in\Sigma} \mathbb{Q}_p) \times S(\widehat{\mathbb{Z}}^{(\Sigma)}) \times S(\mathbb{R}) = (\prod_{p\in\Sigma} S(\mathbb{Q}_p)) \times (\prod_{p\notin\Sigma} S(\mathbb{Z}_p)) \times S(\mathbb{R}).$$

Thus if we have a measure $d\omega_p$ on the p–adic analytic space $S(\mathbb{Q}_p)$ and $d\omega_\infty$ on the archimedean analytic space $S(\mathbb{R})$ and if $\int_{S(\widehat{\mathbb{Z}}^{(\Sigma)})} d\omega$ given by $\prod_{p\notin\Sigma}\int_{S(\mathbb{Z}_p)} d\omega_p$ is finite, we can think of the product measure $d\omega$ on $S(\mathbb{A})$ whose integral of $\phi : S(\mathbb{A}) \to \mathbb{C}$ with $\phi((x_v)) = \phi_\infty(x_\infty)\prod_p \phi_p(x_p)$ for functions $\phi_v : S(\mathbb{Q}_v) \to \mathbb{C}$ is given by the product: $\prod_v \int_{S(\mathbb{Q}_v)} \phi_v(x_v) d\omega_v$.

By definition, we also have the following properties of affine schemes:

(cf1) If $B \xrightarrow{\phi} C \xrightarrow{\psi} D$ are A–algebra homomorphisms, then we have morphisms of schemes $S_D \xrightarrow{\underline{\psi}} S_C \xrightarrow{\underline{\phi}} S_B$ such that $\phi\circ\psi$ is associated to $\underline{\psi}\circ\underline{\phi}$.

(cf2) If $B = C$ and ϕ in (cf1) is the identity map i_B of B, we have $\underline{i_B}\circ\underline{\psi} = \underline{\psi}$. If $C = D$ and ψ in (cf1) is the identity map i_C of C, we have $\underline{\phi}\circ\underline{i_C} = \underline{\phi}$.

(cf3) For the identity map $i_B : B \to B$, $\underline{i_B} : S_B(R) \to S_B(R)$ is the identity map for all A–algebras R.

Thus the function $B \mapsto S_B$ is a contravariant functor from A–algebras into affine schemes. One of the most basic fact in functorial algebraic geometry is (e.g. [GME] 1.4.3):

$$\mathrm{Hom}_{A\text{-alg}}(B', B) \cong \mathrm{Hom}_A(S_B, S_{B'}) \quad \text{via } \alpha \leftrightarrow \underline{\alpha}. \tag{1.1}$$

Here are some examples of affine schemes:

Example 1.1. Take $f(X,Y,Z) = X^p + Y^p - Z^p$ for a prime p, and let $A = \mathbb{Z}$. Then consider $B = \mathbb{Z}[X,Y,Z]/(f(X,Y,Z))$. For each algebra R, we claim

$$S_B(R) \cong \{(x,y,z) \in R^3 | x^p + y^p = z^p\}.$$

Indeed, for each solution $P = (x,y,z)$ of the Fermat's equation in R, we define an algebra homomorphism $\phi : A[X,Y,Z] \to R$ by sending polynomials $\Phi(X,Y,Z)$ to its value $\Phi(x,y,z) = \phi(\Phi) \in R$. Since $\Phi \in (f(X,Y,Z)) \Leftrightarrow \Phi = \Psi f$, we find that $\phi(\Phi f) = \Psi(x,y,z)f(x,y,z) = 0$; so, ϕ factors through getting $\phi \in S_B(R)$. In this way, we get an injection from the right-hand-side to $S_B(R)$. If we start from $\phi : B \to R$ in $S_B(R)$, we find

$$0 = \phi(0) = \phi(X^p + Y^p - Z^p) = \phi(X)^p + \phi(Y)^p - \phi(Z)^p.$$

Thus $(x, y, z) = (\phi(X), \phi(Y), \phi(Z))$ is an element in the right-hand-side, getting the isomorphism. By Fermat's last theorem, we have

$$S_B(\mathbb{Z}) \cong \{(a, 0, a), (0, b, b), (c, -c, 0) | a, b, c \in \mathbb{Z}\} \quad \text{if } p \text{ is a prime} > 3.$$

There is a simpler example: We have

$$S_{\mathbb{Z}[X_1,\ldots,X_n]}(R) = R^n \quad \text{via} \quad \phi \mapsto (\phi(X_1), \ldots, \phi(X_n)).$$

Thus often $S_{\mathbb{Z}[X_1,\ldots,X_n]}$ is written as $\mathbb{G}_a^n$ and is called the affine space of dimension n. We have an algebra homomorphism $A[X, Y, Z] \to B$ for B in Example 1.1 sending Φ to $(\Phi \mod f(X))$. This in turn induces a morphism $i : S_B \to \mathbb{G}_a^3$, which is visibly injective.

When we have a morphism of affine schemes $\phi : S_B \to S_C$, and if $\underline{\phi} : C \to B$ is a surjective ring homomorphism, we call ϕ a *closed immersion*. Then ϕ is injective and we can identify $S_B \subset S_C$ all the time. In this case, S_B regarded as a subfunctor of S_C is called A*–closed* in S_C. As we will see in Exercise 2, if $S_i \subset S_C$ is closed for finite number of affine schemes S_i, the intersection $R \mapsto \bigcap_i S_i(R)$ is again closed. Thus we can give a topology on $S_C(R)$ for each R so that closed set is given by the empty set $\emptyset$ and those of $S_B(R)$ for closed immersion $S_B \hookrightarrow S_C$. This topology is called the *Zariski topology* of S_C.

If A' is an A–algebra, we may regard $B' = A' \otimes_A B$ as an A'–algebra by $a' \mapsto a \otimes 1$. Then we get a new scheme $S_{B'}$ over the ring A', which sometimes written as $S_{A'} \times_A S_B$ and is called the fiber product of S_B and $S_{A'}$ over A. If we have a point $\phi : S_B(R)$ for an A'–algebra R, we can extend $\phi : B \to R$ to $\phi' : B' = A' \otimes_A B \to R$ by $\phi'(a \otimes b) = a\phi(b)$. Thus $\phi \mapsto \phi'$ gives the natural map $S_B(R) \to S_{B'}(R)$ for all A'–algebras R. This map is an isomorphism, because for any given $\phi' \in S_{B'}(R)$, $\phi(b) = \phi(1 \otimes b)$ gives a point $\phi \in S_B(R)$ as long as R is an A'–algebra (Exercise 3). However an A'–closed subset of $S_{B'}$ may not be A–closed; so, the Zariski topology depends on the base ring A.

Exercises

1. Prove that a closed immersion $i : S_B(R) \hookrightarrow S_C(R)$ is an injection for any A–algebras R.

2. If $i : S_B \subset S_C$ and $j : S_D \subset S_C$ are closed, then $R \mapsto S_B(R) \cap S_D(R)$ is closed and is isomorphic to S_E for $E = B \otimes_C D$, where the tensor product is taken with respect to the associated algebra homomorphisms $\underline{i} : C \to B$ and $\underline{j} : C \to D$.

3. Prove $S_B(R) \cong S_{B'}(R)$ if $B' = A' \otimes_A B$ and R is an A'–algebra, where A' is another A–algebra.

4. For two A–algebras B and C, show that $S_{B \otimes_A C}(R) = S_B(R) \times S_C(R)$ for any A–algebra R. Hint: $\phi \in S_B(R)$ and $\psi \in S_C(R)$, we associate $\phi \otimes \psi \in S_{B \otimes_A C}(R)$ given by $(\phi \otimes \psi)(b \otimes c) = \phi(b)\psi(c)$. Thus a product of affine scheme is again an affine scheme.

1.2 Affine algebraic groups

Let G be an affine scheme over a ring A. Thus G is a covariant functor from A–algebras to sets. If the values $G(R)$ for all A–algebras are groups and $\phi_* : G(R) \to G(R')$ for any A–algebra homomorphism $\phi : R \to R'$ is a group homomorphism, G is called an *affine group scheme* or an affine algebraic group.

Example 1.2.

1. Let $B = \mathbb{Z}[X_1, \ldots, X_n]$. Then $S_B(R) = R^n$ (as already remarked), which is an additive group. Since

$$\phi_*(r_1, \ldots, r_n) = (\phi(r_1), \ldots, \phi(r_n))$$

for each algebra homomorphism $\phi : R \to R'$, ϕ_* is a homomorphism of additive groups. Thus $\mathbb{G}_a^n$ is an additive group scheme.

2. More generally, we can think of $C = \mathbb{Z}[X_{ij}]$ for n^2 variables. Then $S_C(R) = M_n(R)$ and S_C is not just a group scheme but is a ring scheme. This scheme is written often as M_n. As additive group schemes (ignoring ring structure), we have $M_n \cong \mathbb{G}_a^{n^2}$.

3. Consider the ring $D = \mathbb{Z}[X_{ij}, \frac{1}{\det(X)}]$ for n^2 variables X_{ij} and the variable matrix $X = (X_{ij})$. Then $S_D(R) = GL_n(R)$ and S_D is a multiplicative group scheme, which is a subscheme of S_C because $GL_n(R) \subset M_n(R)$ for all R. This scheme S_D is written as $GL(n)$. In particular, $S_{\mathbb{Z}[t,t^{-1}]} = GL(1)$ is called the multiplicative group and written as $\mathbb{G}_m$.

   More generally, for a given A–module X free of rank n, we define $X_R = X \otimes_A R$ (which is R–free of the same rank n) and

$$GL_X(R) = \left\{ \alpha \in \mathrm{End}_R(X_R) \middle| \exists \alpha^{-1} \in \mathrm{End}_R(X_R) \right\}.$$

   Then GL_X is isomorphic to $GL(n)_{/A}$ by choosing a coordinate system of X; so, GL_X is an affine group scheme defined over a ring A. We can generalize this to a locally free A–module X, but in such a case, it is a bit more difficult to prove that GL_X is an affine scheme.

4. We can then think of $E = C/(\det(X) - 1)$. Then

$$S_E(R) = \{x \in GL_n(R) | \det(x) = 1\}.$$

   This closed subscheme of M_n and also of $GL(n)$ is written as $SL(n)$ and is a multiplicative group scheme defined over $\mathbb{Z}$.

5. Let X is a free A–module of finite rank. We fix a bilinear form $S : X \times X \to A$. Then we consider

$$G(R) = \{\alpha \in GL_X(R) | S_R(x\alpha, y\alpha) = S_R(x, y) \text{ for all } x, y \in X_R\},$$

   where $S_R(r \otimes x, s \otimes y) = rsS(x, y)$ for $r, s \in R$ and $x, y \in X$.

   To see that this G is an affine algebraic group defined over A, we fix a base $x_1, \ldots, x_n$ of X over A and define a matrix S by $S = (S(x_i, x_j)) \in M_n(A)$.

Then every (ij) entry $s_{ij}(X)$ of the matrix $XS^tX - S$ $(X = (X_{ij}))$ is a quadratic polynomial with coefficients in A. Then we consider $L = A[X_{ij}, \det(X)^{-1}]/(s_{ij}(X))$. By definition,

$$S_L(R) = \left\{\alpha \in GL_X(R) \middle| \alpha S^t\alpha = S\right\} \cong G(R).$$

We find $\alpha S^t\alpha = S \Rightarrow S = \alpha^{-1}S^t\alpha^{-1}$; so, G is an affine algebraic group.

If $X = A^n$ and $S(x, y) = xS^ty$ for a non-degenerate symmetric matrix S, $G = O_{S/A}$ is called the orthogonal group of S. If $X = Y \times Y$ and S is non-degenerate skew symmetric of the form $S((y, y'), (z, z')) = T(y, z') - T(z, y')$ for a symmetric bilinear form $T : Y \times Y \to A$, we write $G = Sp_{T/A}$. In particular, if $S(x, y) = x \left(\begin{smallmatrix} 0 & -1_n \\ 1_n & 0 \end{smallmatrix}\right) {}^ty$, the group $G = Sp_{n/A}$ is called the symplectic group of genus n.

6. We consider a quadratic polynomial $f(T) = T^2 + aT + b \in \mathbb{Z}[T]$. Then define $S_f(R) = \mathbb{G}_a(R)[T]/(f(T))$. As a scheme $S_f \cong \mathbb{G}_a^2$ but its value is a ring all the time. If $\phi : R \to R'$ is an algebra homomorphism, $\phi_*(r+sT) = \phi(r) + \phi(s)T$; so, it is a ring homomorphism of $S_f(R) = R[T]/(f(T))$ into $S_f(R') = R'[T]/(f(T))$. Thus S_f is a ring scheme, and writing O for the order of the quadratic field $\mathbb{Q}[\sqrt{a^2 - 4b}]$ generated by the root of $f(T)$, we have $S_f(R) \cong R \otimes_{\mathbb{Z}} O$.

7. Since any given number field F is generated by one element, we know $F = \mathbb{Q}[T]/(f(T))$ for an irreducible monic polynomial $f(T)$. For any $\mathbb{Q}$–algebra R, define $S_f(R) = R[T]/(f(T))$. Then in the same way as above, S_f is a ring scheme defined over $\mathbb{Q}$ such that $S_f(R) = F \otimes_{\mathbb{Q}} R$.

8. Let G be an affine algebraic group defined over a number field F. Thus for an algebra $\mathcal{B}$, $G(R) = \mathrm{Hom}_{F\text{-alg}}(\mathcal{B}, R)$. Then we define a new functor G' defined over $\mathbb{Q}$–algebras R by $G'(R) = G(S_f(R)) = G(F \otimes_{\mathbb{Q}} R)$. We can prove that G' is an affine group scheme defined over $\mathbb{Q}$, which we write $G' = \mathrm{Res}_{F/\mathbb{Q}}G$ (see Exercise 4 and [AAG] 1.3).

9. Assume that f is a quadratic polynomial in $\mathbb{Q}[T]$. Then we have $S_f(\mathbb{Q}) = F$ is a quadratic extension with $\mathrm{Gal}(F/\mathbb{Q}) = \{1, \sigma\}$. Let X be a finite dimensional vector space over $\mathbb{Q}$ and let $\mathrm{Gal}(F/\mathbb{Q})$ act on $X_F = F \otimes_{\mathbb{Q}} X$ through F. We suppose to have a hermitian form $H : X_F \times X_F \to F$ such that $H(x, y) = \sigma(H(y, x))$. Then for $\mathbb{Q}$–algebra R

$$U_H(R) = \left\{\alpha \in GL_X(S_f(R)) \middle| H_{S_f(R)}(x\alpha, y\alpha) = H_{S_f(R)}(x, y)\right\}$$

is an affine algebraic group, which is called the unitary group of H. Note that U_H is defined over $\mathbb{Q}$ (not over F).

For two affine algebraic group G, G' defined over A, we write

$$\begin{aligned} &\mathrm{Hom}_{A\text{-alg gp}}(G, G') \\ &\quad = \left\{\phi \in \mathrm{Hom}_A(G, G') \middle| \phi_R \text{ is a group homomorphism for all } R\right\}. \quad (1.2) \end{aligned}$$

Exercises

1. Let F be a number field with the integer ring O. Is there any affine ring scheme S defined over $\mathbb{Z}$ such that $S(R) = O \otimes_{\mathbb{Z}} R$?

2. Give a detailed proof of the construction of the algebraic group $\mathrm{Res}_{F/\mathbb{Q}}(G)$.

3. Let $S : X \times X \to A$ is a bilinear form for an A–free module X, and suppose that $X \cong \mathrm{Hom}_A(X, A)$ by S. Then the matrix of S is in $GL_n(A)$ for any choice of a base of X over A.

4. For an affine algebraic group G over a number field F (that is, a finite extension of $\mathbb{Q}$), prove that $\mathrm{Res}_{F/\mathbb{Q}}G$ is an affine algebraic group defined over $\mathbb{Q}$.

5. Show that the unitary group U_H as above is an affine algebraic group.

1.3 Gauge forms

Let G be an affine algebraic group over $\mathbb{Q}$. We write its affine ring as B; so, we have $G(A) = \mathrm{Hom}_{\mathbb{Q}\text{-alg}}(B, A)$. We suppose that B is noetherian, does not have non-trivial nilpotents (so, B is reduced), and $\mathbb{Q}$ is integrally closed in B (so G is defined over $\mathbb{Q}$ in the sense of Weil). If we decompose $B = \bigoplus_{j=0}^{h} B_j$ as an algebra direct sum for integral domains B_j, we have $G(A) = \bigsqcup_j S_j(A)$ for $S_j(A) = \mathrm{Hom}_{\mathbb{Q}\text{-alg}}(B_j, A)$. Since G is a group, the action of G permutes S_j, and hence all the B_j's are isomorphic. In particular, the one $G_0 = S_0$ of the components among S_j in which 1_G sits is a normal subgroup. We call G_0 the connected component of G.

Assume that G is of dimension m over $\mathbb{Q}$. Thus the field of fractions of B_0 is algebraic over $\mathbb{Q}(x_1, \ldots, x_m)$ for variables x_j, and the Krull dimension of B_0 is m.

We may thus assume that $B_0 \supset \mathbb{Q}[x_1, \ldots, x_m]$. We can think of $b \in B$ as the algebraic function on G by $b(P) = P(b)$ for $P \in G(A)$. If $\phi : G \to G$ is any automorphism, $\underline{\phi} : B \to B$ is a $\mathbb{Q}$–algebra homomorphism. In other words, $\underline{\phi}(x_j) = x_j \circ \phi$, and $\phi(x) = (\underline{\phi}_j(x))$.

For example, if $G = GL(n)$, we take X_{ij} to be the coordinate x_j. Thus $B = \mathbb{Z}[X_{ij}, \det(X)^{-1}]$ and any algebra homomorphism $\underline{\phi} : B \to B$ corresponds to $\phi : GL(n) \to GL(n)$ given by $\phi(X) = (\underline{\phi}(X_{ij}))$.

Formally, we write

$$dx_j \circ \phi = \sum_i \frac{\partial \phi(x)}{\partial x_i} dx_i.$$

Thus we can think of the transport

$$\phi^* \omega = f(\phi(x)) dx_1 \circ \phi \wedge \cdots \wedge dx_m \circ \phi$$

of differential form $\omega = f(x) dx_1 \wedge \cdots \wedge dx_m$ by ϕ.

Since G is a group, any $g \in G(A)$ induces multiplication $g : G \to G$; so, we can think of $g^*\omega$. A non-zero differential firm ω of degree $m = \dim_{\mathbb{Q}} X$ is called a *gauge form* if $g^*\omega = \omega$ for all $g \in G$. If two gauge form exists, the ratio ω/ω' are invariant under G; so, constant.

Example 1.3. 1. Suppose $G = GL(n)$. Then we find, by linear algebra,

$$g^*(dx_1 \wedge \cdots \wedge dx_n) = \det(g)(dx_1 \wedge \cdots \wedge dx_n)$$

for a column vector $x = {}^t(x_1, \ldots, x_n) \in A^n$ and $g \in GL_n(A)$. Thus for $\omega' = \bigwedge_{ij} dX_{ij}$ is the wedge product of n such $\omega_i = dx_{1i} \wedge \cdots \wedge dx_{ni}$, and hence $g^*\omega' = \det(g)^n\omega'$. Thus $\omega = \det(X_{ij})^{-n}\omega'$ is a gauge form for $GL(n)$.

2. Let S be a $n \times n$ non-degenerate symmetric or skew symmetric matrix in $M_n(\mathbb{Q})$. Then we define $x^\iota = S^t x S^{-1}$, which is an involution of M_n. Then $O_S(A) = \{x \in GL_n(A) | xx^\iota = 1\}$. We consider $\mathfrak{s}_\pm = \{x \in M_n | x^\iota = \mp x\}$ ($\mathfrak{s}_+$ is the Lie algebra of O_S). We have $M_n = \mathfrak{s}_+ \oplus \mathfrak{s}_-$. Since ω' as above satisfies $\omega'(axb) = \det(a)^n \det(b)^n \omega'(x)$ for $a, b \in GL(n)$, we can split $\omega' = \omega_+ \wedge \omega_-$ according to the linear splitting $M_n = \mathfrak{s}_+ \oplus \mathfrak{s}_-$. Then ω_+ restricted to $O_S \subset M_n$ gives a gauge form on the connected component of O_S.

Let ω be a differential m–form on a $\mathbb{Q}$–scheme of dimension m. If $S(\mathbb{R}) \neq \emptyset$, writing $\omega = f(x)dx_1 \wedge \cdots \wedge dx_n$ choosing a coordinates, we define a measure $|\omega|_\infty$ by

$$\int_{S(\mathbb{R})} \phi(x) d|\omega|_\infty = \int_{S(\mathbb{R})} \phi |f(x)|_\infty dx_1 dx_2 \cdots dx_m$$

for the Lebesgue measure dx_j.

Since we have a canonical measure dx on $\mathbb{Q}_p$, we can imitate the above procedure to get a measure $|\omega|_p$ on $S(\mathbb{Q}_p)$:

$$\int_{S(\mathbb{Q}_p)} \phi(x) d|\omega|_\infty = \int_{S(\mathbb{Q}_p)} \phi |f(x)|_p dx_1 dx_2 \cdots dx_m$$

for the canonical measure dx_j on $\mathbb{Q}_p$.

If $S(\mathbb{A}) \neq \emptyset$, we can define the adelic measure $|\omega|_{\mathbb{A}}$ by $\bigotimes_v |\omega|_v$ by Fubini's theorem (this is all right because $S(\mathbb{A}) = \bigcup_\Sigma S(\mathbb{A}_\Sigma)$ as already seen). If $\phi : S' \to S$ is an isomorphism, by definition,

$$\int_{S(A)} f d|\omega| = \int_{S'(A)} f \circ \phi d|\phi^*\omega|.$$

Note that $|\xi\omega|_{\mathbb{A}} = |\xi|_{\mathbb{A}}|\omega|_{\mathbb{A}}$ for a constant $\xi \in \mathbb{Q}$. By the product formula, we know that $|\xi|_{\mathbb{A}} = 1$; so, $|\omega|_{\mathbb{A}}$ depends only on $\omega \mod \mathbb{Q}^\times$. In particular, for an affine algebraic group G, if ω is a gauge form, $|\omega|_{\mathbb{A}}$ gives, if it exists, a canonical Haar measure on $G(\mathbb{A})$. This measure is called the Tamagawa measure of $G(\mathbb{A})$.

Since $GL_n(\mathbb{Q}) \subset GL_n(\mathbb{A})$ is a discrete subgroup, by embedding G into $GL(n)$, we find that $G(\mathbb{Q})$ is a discrete subgroup. Taking a fundamental domain

Φ of $G(\mathbb{Q})\backslash G(\mathbb{A})$, we define the Tamagawa measure on $G(\mathbb{Q})\backslash G(\mathbb{A})$ just by the integration on Φ. Of course this definition does not depends on the choice of Φ. If $\int_{G(\mathbb{Q})\backslash G(\mathbb{A})} dg < \infty$ for the Tamagawa measure on the quotient, this number is called the Tamagawa number of G and written as $\tau(G)$.

For a closed subgroup $H \subset G$, if we have a differential form ω on $H\backslash G$ right-invariant under G, in the same manner as in the case of the group, $|\omega|_{\mathbb{A}}$ is a unique right invariant measure of $H\backslash G$, which is also called the Tamagawa measure of $H\backslash G$. We can similarly define the Tamagawa measure on G/H.

If G acts on a scheme S so that its action induces $G(\mathbb{A})/H(\mathbb{A}) \cong S(\mathbb{A})$, by the above argument, we can think of the Tamagawa measure on S, which is unique.

More details on Tamagawa measures, see [AAG].

Exercise

1. Find a gauge form on $SL(n)$.

1.4 Noetherian schemes

A (noetherian) scheme defined over a ring A is a covariant functor T from A–algebras into sets satisfying the following conditions:

1. There are finitely many noetherian affine schemes S_i which are subfunctors of S (so, $S_i(R) \subset T(R)$ for all A–algebras R, and the inclusion is functorial).

2. If S is an affine scheme over A and if $\phi : S \to T$ is a morphism of functors (so it is a family of maps $\phi_R : S(R) \to T(R)$ making the square diagram between (F3) and (U1) commutative), for any $x \in S(R)$ with an integral domain R, we can find an open neighborhood $U \subset S$ of x such that ϕ induces a map $U \to S_i$ for some i. In particular, if R is a field, then $T(R) = \bigcup_i S_i(R)$.

3. For any finite set J of indices, $\bigcap_{j\in J} S_j$ is open in S_i for all $i \in J$ under the Zariski topology.

Example 1.4. 1. The n–dimensional projective space $\mathbf{P}^n_{/A}$ is given by

$$\mathbf{P}^n(R) = \left\{ V \subset R^{n+1} \,\middle|\, \begin{array}{l} V \text{ is an } R\text{–submodule locally free of rank 1} \\ \text{with locally free quotient } R^{n+1}/V \end{array} \right\}.$$

If R is local (that is, having only one maximal ideal; so, a field is local), all locally free modules are free; so, $V \in \mathbf{P}^n(R)$ is free of rank 1 having a generator $x = (x_0, x_1, \ldots, x_n)$, and $R^{n+1} = V \oplus Y$ for a complementary direct summand Y isomorphic to R^{n+1}/V. By tensoring $F = R/\mathfrak{m}_R$ for the maximal ideal $\mathfrak{m}_R$, we find that $V/\mathfrak{m}V \neq 0$; so, one of $x_i \not\equiv 0 \mod \mathfrak{m}_R$. In other words, $x_i \in R^\times$. Thus by dividing x_i, we can normalize $x_i = 1$. In other words, defining $D_i(R) = \{(x_0, \ldots, \overset{i}{1}, \ldots, x_n) \in R^{n+1}\}$, we find

$D_i \cong \mathbb{G}_a^n$ and $\mathbf{P}^n(R) = \bigcup_i D_i(R)$. This is enough to prove that $\mathbf{P}^n$ is a scheme, because for any point x of an affine scheme $S_B(R)$ with an integral domain R, the localization $U = S_{B_x}$ is the intersection of all open neighborhoods of x (see [GME] 1.2.2). In particular, for a local ring (including a field) R,

$$\mathbf{P}^n(R) \cong \left\{(x_0, \ldots, x_n) \in R^{n+1} \middle| \exists i \text{ with } x_i \in R^\times\right\}/R^\times,$$

where $R^\times$ acts by scalar multiplication. This scheme is called the projective scheme of dimension n.

Since $\mathbb{Z}_p$ is p–adically compact, $\mathbb{G}_a^n(\mathbb{Z}_p) = \mathbb{Z}_p^n$ is p–adically compact. Therefore $\mathbf{P}^n(\mathbb{Z}_p)$ is also p–adically compact. If $x = (x_0, \ldots, x_n) \in \mathbf{P}^n(\mathbb{Q}_p)$, by multiplying a suitable scalar in $\mathbb{Q}_p^\times$ (taking off the denominators in the x_i's), we find $x \in \mathbf{P}^n(\mathbb{Z}_p)$; so, we find a funny fact: $\mathbf{P}^n(\mathbb{Q}_p) = \mathbf{P}^n(\mathbb{Z}_p)$. Thus $\mathbf{P}^n(\mathbb{Q}_p)$ is compact. It is well known that $\mathbf{P}^n(\mathbb{R})$ and $\mathbf{P}^n(\mathbb{C})$ are also compact under archimedean topology (Exercise 1).

2. Let X be an A–free module of rank m. For an integer n with $m > n > 0$, we write $\mathbf{G}_X^n(R)$ for the set of R–submodules V of X such that X/V is locally free of rank $m - n$. Thus $\mathbf{G}^1_{R^{n+1}} = \mathbf{P}^n$. One can show that $\mathbf{G}$ is again a scheme, which is called the Grassmannian of index (m, n).

3. Let X be as above. For a partition $\underline{m} : m_1 + m_2 + \cdots + m_r = m$ by positive integers m_i, a sequence of locally free R–submodules $0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_{r-1} \subset V_r = X_R$ is called an R–flag of index $\underline{m}$ if V_j/V_{j-1} is locally free R–module of rank m_j for all $j = 1, 2, \ldots, r$. The functor $\mathcal{F}_X^{\underline{m}}(R)$ sending R to the set of all R–flags of X_R of index $\underline{m}$ is known to be a scheme. The scheme $\mathcal{F}_X^{\underline{m}}$ is called the flag scheme of index $\underline{m}$. When $\underline{m} : n + (m - n) = m$, then $\mathcal{F}_X^{\underline{m}} = \mathbf{G}_X^n$.

Since a scheme is a covariant functor from A–algebras into sets, each A–algebra homomorphism $\alpha : R \to R'$ induces a map $\alpha_* : S(R) \to S(R')$. Moreover $(\alpha \circ \beta)_* = \alpha_* \circ \beta_*$. For two schemes S and T defined over A, $\phi \in \mathrm{Hom}_A(S, T)$ is a family of maps $\phi_R : S(R) \to T(R)$ which makes the following diagram commutative for any algebra homomorphism $\alpha : R \to R'$:

$$\begin{array}{ccc} S(R) & \xrightarrow{\phi_R} & T(R) \\ {\scriptstyle \alpha_*}\downarrow & & \downarrow{\scriptstyle \alpha_*} \\ S(R') & \xrightarrow[\phi_{R'}]{} & T(R'). \end{array}$$

A scheme $T_{/A}$ is called a closed subscheme of $S_{/A}$ if the following two conditions are satisfied:

(C1) We have an inclusion $T \hookrightarrow S$, that is, $T(R) \subset S(R)$ for all A–algebras R;

(C2) Covering S by affine schemes S_i, $S_i \cap T$ is a closed affine subscheme of S_i for all i.

A morphism $\phi : S \to T$ is called a closed immersion if we have a closed subscheme $S' \subset T$ such that ϕ induces an isomorphism: $S \cong S'$. A scheme S over A is called *projective* over A if we have a closed immersion of S into a projective space $\mathbf{P}^n_{/A}$.

Since a Zariski closed set is closed in p–adic or archimedean topology, if S is projective over $\mathbb{Q}$, $S(\mathbb{Q}_p)$ under the p–adic topology and $S(\mathbb{R})$ under the archimedean topology are compact spaces.

If X is a vector space over $\mathbb{Q}$ of dimension m, we can define a duality pairing $S : X \times X \to \mathbb{Q}$. For any $\mathbb{Q}$–algebra R, the induced pairing $S_R : X_R \times X_R \to R$ induces $X_R \cong \mathrm{Hom}_R(X_R, R)$. In particular, if $V \in \mathbf{G}^n_X(R)$, its orthogonal complement $V^\perp$ gives an element in $\mathbf{G}^{m-n}_X(R)$. Thus $\mathbf{G}^n_X \cong \mathbf{G}^{m-n}_X$ over $\mathbb{Q}$; so, to study the Grassmannian, we may assume its index n is less than or equal to $\frac{m}{2}$ for $m = \dim_{\mathbb{Q}} X$. Then the exterior power $\mathcal{X} = \bigwedge^n X$ is a $\mathbb{Q}$–vector space of dimension $d = \binom{m}{n}$ and $\bigwedge^n V$ for $V \in \mathbf{G}^n_X(R)$ is an element of $\mathbf{G}^1_{\mathcal{X}}(R)$. Since $\mathbf{G}^1_{\mathcal{X}} \cong \mathbf{P}^{d-1}$, we have a morphism $i : \mathbf{G}^n_{X/\mathbb{Q}} \to \mathbf{P}^{d-1}_{/\mathbb{Q}}$ given by $V \mapsto \bigwedge^n V$. This map (called the Plücker coordinate) is a closed immersion; so, the Grassmannian is projective. In particular, $\mathbf{P}^m_X \times \mathbf{P}^n_Y$ can be embedded into $\mathbf{G}^2_{X\oplus Y}$ by sending $(V, W) \in \mathbf{P}^m_X(R) \times \mathbf{P}^m_Y(R)$ to $V \oplus W$, we find that a product of projective spaces is projective; so, a finite (fiber) product of projective schemes is projective.

For each member V_j of an R–flag of index $\underline{m}$ is an element of $\mathbf{G}^{\widetilde{m}_j}_X$ for $\widetilde{m}_j = m_1 + m_2 + \cdots + m_j$. In this way, we can embed $\mathcal{F}^{\underline{m}}_X$ into the product $\prod_j \mathbf{G}^{\widetilde{m}_j}_X$, which is actually a closed immersion. Thus flag variety is projective.

Let B be the subgroup of $GL(n)$ made of upper triangular matrices. Then obviously $B \subset GL(n)$ is a closed algebraic subgroup. We write $\underline{1}$ for the maximal partition of $n = \overbrace{1 + 1 + \cdots + 1}^{n}$. Since B is the stabilizer of the maximal flag: $\underline{O} : R \subset R^2 \subset R^3 \subset \cdots \subset R^n$, regarding $R^m \subset R^n$ taking $x \in R^m$ to ${}^t(x, \overbrace{0, \ldots, 0}^{n-m}) \in R^n$. Moreover any other maximal flag is of the form $\alpha\underline{O} : \alpha R \subset \alpha R^2 \subset \cdots \subset \alpha R^n$ at least when R is a local ring (in particular a field), we find $GL_n(R)/B(R) \cong \mathcal{F}^{\underline{1}}_{R^n}(R)$ if R is a local ring. Thus we define the quotient variety $GL(n)/B$ to be the flag variety $\mathcal{F}^{\underline{1}}_{R^n}$. More generally, let $G_{/\mathbb{Q}}$ be a reduced affine closed subgroup of $GL_{X/\mathbb{Q}}$. For a closed subgroup $P \subset G$, we call P parabolic if G/P is projective (but for this definition, we need to define the quotient G/P for any closed subgroup (following Mumford); see for example [GME] 1.8.3). If P is parabolic, then it is a stabilizer of a flag of some index $\underline{n}$ (see [AGD] Chapter I).

For $G = GL(n)_{/F}$ and $O_{S/F}$ with anti-diagonal S with all anti-diagonal entries equal to 1 over a field F of characteristic different from 2, any closed connected subgroup containing $B \cap G$ is parabolic. For $Sp(n)_{/F}$, the above statement is correct if we replace B by

$$\left\{ \begin{pmatrix} b & u \\ 0 & {}^t b^{-1} \end{pmatrix} \in Sp(n) \middle| b \in B \subset GL(n) \right\},$$

which we write B for $Sp(n)$. The subgroup B is called the standard Borel subgroup (Borel subgroup is any maximal connected soluble closed subgroup).

Any other Borel subgroups in the above example are conjugate to the standard one. If a parabolic subgroup contains the standard Borel subgroup, we call it a standard parabolic subgroup. There are finitely many standard parabolic subgroups, and all other parabolic subgroup are conjugate to standard ones (see [AGD] Chapter I).

Exercises

1. Show that $\mathbf{P}^n(\mathbb{R})$ is a compact real manifold.
2. Show that a closed subscheme of an affine scheme is affine. (See [GME] (Af1-2) and (Mf1) in Subsection 1.2.2-3).
3. Give an example of duality pairing $S : X \times X \to \mathbb{Q}$.
4. Prove that B is a closed algebraic subgroup of $GL(n)_{/\mathbb{Q}}$.

2 Theta series and Eisenstein series

In this section, we shall give a sketch of one of the two proofs (due to Weil) of the Siegel-Weil formula, restricting ourselves to $SL(2)$ over $\mathbb{Q}$. At the end, we shall discuss briefly more general cases.

2.1 Orthogonal groups and theta series

Let V be a vector space over $\mathbb{Q}$ of finite dimension n with a quadratic form $\phi : V \to \mathbb{Q}$. We define the associated symmetric bilinear form $S : V \times V \to \mathbb{Q}$ by

$$S(x, y) = \phi(x + y) - \phi(x) - \phi(y).$$

Then for any $\mathbb{Q}$–algebra A, S induces a bilinear pairing $S_A : V_A \times V_A \to A$ by $S(x \otimes a, y \otimes b) = abS(x, y)$ for $a, b \in A$ and $x, y \in V$, where $V_A = V \otimes_{\mathbb{Q}} A$ (in other words, $A \mapsto V_A$ is the affine space of dimension d defined over $\mathbb{Q}$).

Suppose that S is non-degenerate. Then the orthogonal group $O = O_{S/\mathbb{Q}}$ of S, as an algebraic group defined over $\mathbb{Q}$, is given by

$$\begin{aligned} O(A) &= \left\{\alpha \in GL_V(A) \middle| \phi \circ \alpha = \phi\right\} \\ &= \left\{\alpha \in GL_V(A) \middle| S(x\alpha, y\alpha) = S(x, y)\ \forall x, y \in V_A\right\}. \end{aligned} \tag{2.1}$$

We identify V with $\mathrm{Hom}_{\mathbb{Q}}(V, \mathbb{Q})$ by S. Then by $(x, y)_? = \mathbf{e}_?(S_?(x, y))$, we have $V_? \cong V_?^*$ for $? = \infty, p, \mathbb{A}$. If $? = \infty$, for a lattice $L \subset V$, $L^\perp$ is the dual lattice:

$$L^\perp = \{\ell \in V | (L, \ell) = 1\} = \{\ell \in V | S(L, \ell) \subset \mathbb{Z}\}.$$

If $? = \mathbb{A}$, we take $L = V \subset V_{\mathbb{A}}$, and then

$$L^\perp = V^\perp = V.$$

For a Schwartz-Bruhat function Φ on $V_{\mathbb{A}}$ or $V_\infty = V_{\mathbb{R}}$, we defined in [M] 2.6 a theta series on the metaplectic group $Mp(V_?)$ for $? = \mathbb{A}$ or ∞:

$$\Theta(\Phi)(s) = \int_L (s\Phi)(\ell) = \sum_{\ell \in L} (s\Phi)(\ell), \tag{2.2}$$

where $L \subset V_?$ is a discrete subgroup as chosen above (in [M], the lattice L is written as Γ). Let

$$\Gamma_L = \left\{ \sigma \in Sp(V_?) \middle| (L \times L^\perp)\sigma = (L \times L^\perp) \right\},$$

which is a discrete subgroup of $Sp(V_?)$ written as $Sp_\Gamma(G)$ in [M] (2.10). Thus $\Gamma_L = Sp_S(\mathbb{Q}) \cong Sp_n(\mathbb{Q})$ for $n = \dim_{\mathbb{Q}} V$ if $? = \mathbb{A}$, where Sp_S is the symplectic group with respect to the skew symmetric form:

$$\widetilde{S}((x, x^*), (y, y^*)) = S(x, y^*) - S(y, x^*).$$

Using partial Fourier transform with respect to L, we defined the embedding $\mathbf{r}_L : \Gamma_L \hookrightarrow Mp(V_?)$ in [M] (2.11) following [We1] no.18. We just identify Γ_L with a subgroup of $Mp(V_?)$ by this embedding. By [M] Theorem 2.3 (which is a reformulation of [We1] Théorème 4), we have

$$\Theta(\Phi)(\gamma s) = \Theta(\Phi)(s) \quad \text{for all } \gamma \in \Gamma_L.$$

Thus $\Theta(\Phi)$ is a modular from on $Sp(V_?)$ in a wider sense. We made explicit the theta series in [M] 2.6 when $L = \mathbb{Z}^n$ and $S(x, y) = x \cdot {}^t y$, and the function was found to be a classical theta function.

Since $O_S(A)$ acts on V_A, we consider its diagonal action on $V_A \times V_A$. Then we define its commutant in $Sp(V_A)$:

$$Z_O(A) = \left\{ \sigma \in Sp(V_A) \middle| \sigma g = g\sigma \quad \text{for all } g \in O_S(A) \right\}.$$

We here regard O_S and $A \mapsto Sp(V_A)$ as algebraic groups over $\mathbb{Q}$. Obviously $SL_2(A) \subset Z_O(A)$, where $SL_2(A)$ is embedded into $Sp_n(A)$ by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} a1_V & b1_V \\ c1_V & d1_V \end{smallmatrix}\right)$ for the identity map 1_V of V. Note here that Weil's metaplectic group $Mp(V_A)$ is only defined for $A = \mathbb{R}$, $\mathbb{Q}_p$ or $\mathbb{A}$; so, it is not an algebraic group. Nevertheless, we write $Mp_S(A)$ for $Mp(V_A)$ for $A = \mathbb{R}$, $\mathbb{Q}_p$ and $\mathbb{A}$. We have a canonical projection $\pi : Mp_S(A) \twoheadrightarrow Sp_S(A)$ with kernel $\mathbb{T}$ (see [M] 2.4). Then we define

$$Mp_{S,1}(A) = \left\{ g \in Mp_S(A) \middle| \pi(g) \in SL_2(A) \right\}, \tag{2.3}$$

regarding $SL_2(A) \subset Sp_S(A)$ as above.

We now show that the action of $O_S(A)$ on $\mathcal{S}(V_A)$ given by $\Phi(x) \mapsto \Phi(xg)$ commutes with the action of $Mp_{S,1}(A)$. Let $P \subset Sp_S$ be the parabolic subgroup made of σ with $c_\sigma = 0$, where we have written $\sigma = \left(\begin{smallmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{smallmatrix}\right)$. Since $Mp_{S,1}(A)$ is generated by elements in $P_1 = Mp_{S,1} \cap P$ and $J = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}$, we only need to check this for $\mathbf{r}(\sigma)$ for $\sigma = J$ or $\sigma \in P_1$. We first note that for $g \in O_S(A)$

$$\langle xg, yg \rangle = \mathbf{e}_A(S(xg, yg)) = \mathbf{e}_A(S(x, y)) = \langle x, y \rangle$$

For $\sigma = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, by [M] 2.4, we have

$$g(\mathbf{r}(\sigma)\Phi)(x) = \langle xg, 2^{-1}xgb \rangle \Phi(xg) = \langle xg, 2^{-1}xbg \rangle \Phi(xg) = \mathbf{r}(\sigma)(g\Phi)(x).$$

Now take $\sigma = \begin{pmatrix} a & 0 \\ 0 & a^{-*} \end{pmatrix}$. By definition, $\langle xa, y \rangle = \langle x, ya^* \rangle$. If we identify V with A^n, then $a^* = S^t a S^{-1}$ writing S for the matrix of S. Thus we can write

$$O_S(A) = \left\{ g \in \mathrm{End}_A(V_A) \middle| gg^* = 1_{V_A} \right\}.$$

Again by [M] 2.4, we have

$$g(\mathbf{r}(\sigma)\Phi)(x) = \sqrt{|a|}\Phi(xga) = \sqrt{|a|}\Phi(xag) = \mathbf{r}(\sigma)(g\Phi)(x).$$

Finally for $\sigma = J$, we have

$$\begin{aligned} g(\mathbf{r}(J)\Phi)(x) &= \mathcal{F}(\Phi)(-xg) = \int_{V_A} \Phi(y)\langle y, -xg \rangle dy \\ &= \int_{V_A} \Phi(y)\langle yg^{-1}, -x \rangle dy \overset{yg^{-1} \mapsto y}{=} \int_{V_A} \Phi(yg)\langle y, -x \rangle dy = \mathbf{r}(J)(g\Phi)(x). \end{aligned}$$

Here we have used the fact that $|g| = 1$ for $g \in O_S(A)$, because $1 = |gS^t gS^{-1}| = |\det(g)|^2$.

Thus out of the theta series $\Theta(\Phi)$, we can create two variable modular forms of $(s, g) \in Mp_{S,1}(A) \times O_S(A)$ $(A = \mathbb{R}, \mathbb{A})$ in the following manner:

$$\Theta(\Phi)(s, g) = \sum_{\ell \in L} s\Phi(\ell g). \tag{2.4}$$

As already seen, for $\gamma \in \Gamma_{S,1} = \Gamma_L \cap Mp_{S,1}(A)$, we have $\Theta(\Phi)(\gamma s, g) = \Theta(\Phi)(s, g)$. Since the above sum is averaging over all $\ell \in L$, defining a discrete subgroup

$$\Gamma_{L,O} = \left\{ \delta \in O_S(A) \middle| L\delta = L \right\},$$

we have

$$\Theta(\Phi)(s, \delta g) = \Theta(\Phi)(s, g).$$

When $A = \mathbb{A}$, $L = V$ and hence, $\Gamma_{L,O} = O_S(\mathbb{Q})$. When $A = \mathbb{R}$, $\Gamma_{L,O}$ is a discrete subgroup of $O_S(\mathbb{R})$.

The Siegel-Weil formula in this setting is to write down the averaging integral $\int_{\Gamma_{L,O} \backslash O_S(A)} \Theta(\Phi)(s, g)dg$ over the orthogonal group as an Eisenstein series $E(\Phi)$ on $Mp_{S,1}(V_A)$ (associated to Φ):

$$\int_{\Gamma_{L,O} \backslash O_S(A)} \Theta(\Phi)(s, g)dg = E(\Phi)(s).$$

Weil calls $E(\Phi)$ Siegel-Eisenstein series. This was first proven by Siegel, and in this formulation, it was later proven by Weil [We2] Théorème 5 in a far more general setting which we will revisit later. To guarantee the absolute convergence of the Eisenstein series, we need to suppose $n > 4$ in the present setting.

2.2 Siegel's theta series

We study in more detail the theta series when $A = \mathbb{R}$ in this subsection. We will have formulas involving $\mathbf{e}(\frac{1}{2}x) = \exp(\pi i x)$ in many places; so, to make things simple, we write $e(x) = \exp(\pi i x)$ (only in this and the following subsection).

We fix a base $\{v_i\}_i$ of a lattice $L \subset V$ and write $S = (S(v_i, v_j))$ which is an $n \times n$ symmetric matrix. A positive definite symmetric matrix $P \in M_n(\mathbb{R})$ (or the symmetric bilinear form on $V_\mathbb{R}$ associated to P) is called a positive majorant of S if

$$PS^{-1} = SP^{-1} \ (\Leftrightarrow S^{-1}P = P^{-1}S). \tag{2.5}$$

Here are some examples:

Example 2.1. 1. If S is diagonal: $S = \mathrm{diag}[a_1, \dots, a_n]$, then the diagonal matrix $P = \mathrm{diag}[|a_1|, \dots, |a_n|]$ is a positive majorant, although as we will see, there are lots others. More generally, if $S = B\,\mathrm{diag}[a_1, \dots, a_n]^tB$ for $B \in GL_n(\mathbb{R})$, then $P = B\,\mathrm{diag}[|a_1|, \dots, |a_n|]^tB$ is a positive majorant.

2. Suppose that S is signature (λ, μ) (so $n = \lambda + \mu$). Then for any decomposition $V_\mathbb{R} = W \oplus W^\perp$ for a subspace W with $\dim_\mathbb{R} W = \lambda$ on which S is positive definite, then $P_W(x, y) = S(x_W, y_W) - S(x', y')$ is a positive majorant of S, where $x = x_W + x'$ and $y = y_W + y'$ for $x_W, y_W \in W$ and $x', y' \in W^\perp$. Here $W^\perp$ is the orthogonal complement of W in $V_\mathbb{R}$. By Witt's theorem (cf. [EPE] 1.2), if $V_\mathbb{R} = W' \oplus W'^\perp$ is another decomposition as above, then we find $g \in O_S(\mathbb{R})$ such that $W' = Wg$ and hence $W'^\perp = W^\perp g$.

3. By (2.5), we find $(P - S)(P^{-1} + S^{-1}) = 0$. Defining $W = \mathrm{Ker}(P - S)$, we find that P is given by P_W.

Define

$$\mathcal{Y} = \left\{W \in \mathbf{G}_V^\lambda(\mathbb{R}) \middle| S|_W > 0\right\} \cong \left\{P \middle| \text{P:positive majorants of } S\right\},$$

where $\mathbf{G}_V^\lambda$ is the Grassmannian variety of index λ and the last isomorphism is given by $W \mapsto P_W$. Then $O_S(\mathbb{R})$ acts on $\mathcal{Y}$ by $W \mapsto Wg$. By Example 2.1 (2-3), $O_S(\mathbb{R})$ acts transitively on $\mathcal{Y}$. It is easy to check that the action of $g \in O_S(\mathbb{R})$ on positive majorants are $P \mapsto gP \cdot {}^tg$. If we fix one positive majorant P_0, we find that for the maximal compact subgroup $C = O_S(\mathbb{R}) \cap O_{P_0}(\mathbb{R})$,

$$\mathcal{Y} \cong C \backslash O_S(\mathbb{R}).$$

So $\mathcal{Y}$ is the realization as a real manifold of the symmetric space of $O_S(\mathbb{R})$.

Then for $z = x + iy \in \mathfrak{H} = \{z \in \mathbb{C} | y = \mathrm{Im}(z) > 0\}$, we consider $\Phi_P(v; z) = e(S[v]x + P[v]iy)$, where $S[v] = S(v, v)$ and $P[v] = P(v, v)$. Then as a function of $v \in V_\mathbb{R}$, $\Phi_P(v; z)$ is a Schwartz function on $V_\mathbb{R}$ (Exercise 2). Thus for any homogeneous polynomial $Q(v)$ of degree k, we can think of the theta series:

$$\Theta(Q\Phi_P)(s) = \sum_{l \in L} s(Q\Phi_P)(\ell).$$

Let $\alpha(x,y) = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \sqrt{y}^{-1} \end{pmatrix} \in SL_2(\mathbb{R})$. Then $\alpha(x,y)(i) = x + iy = z$. By the formula of $\mathbf{r}(\sigma)$, we have

$$\mathrm{diag}[\sqrt{y}, \sqrt{y}^{-1}]\Phi(v) = |y|^{n/4}\Phi(\sqrt{y}v) \text{ and } \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\Phi(v) = e(S[v]x)\Phi(v).$$

Thus we see

$$\alpha(x,y)(Q(v)\Phi_P(v;i)) = |y|^{(n+2k)/4}Q(v)\Phi_P(v;z).$$

We also see that

$$\Phi_P(gv;z) = \Phi_{gP}(v;z),$$

where $gP = gP \cdot {}^t g$ Thus $\Theta(Q(v)\Phi_P(v;i))(\alpha(x,y),g) = |y|^{(n+2k)/4}\theta(z,gP;Q\circ g)$, where

$$\theta(z,P;Q) = \sum_{\ell\in L} Q(\ell)\Phi_P(\ell;z).$$

Since we know that $\Theta(\Phi)(s)$ is a modular form on $Mp_{S,1}(\mathbb{R})$, we would like to determine its level.

Actually, it is better to do it adelically. A Schwartz-Bruhat function Φ on $V_{\mathbb{A}^{(\infty)}}$ is a function vanishing outside $\widehat{L}'$ for a big lattice L' and factoring through $\widehat{L}'/\widehat{L}$ for a smaller lattice L, the adelic theta series $\Theta(\Phi)$ restricted to $Mp_{S,1}(\mathbb{R})$ is a linear combination of

$$|y|^{(n+2k)/4}\theta(z,P;v,Q,L) = \sum_{\ell\in v+L} Q(\ell)\Phi_P(\ell;z).$$

That is, for $\Phi_g = \Phi^{(\infty)}(Q\circ g)\Phi_P$ defined on $V_{\mathbb{A}}$, we have $\Theta(\Phi)(\alpha(x,y),g) = |y|^{(n+2k)/4}\theta(z,gP;\Phi)$, where

$$\theta(z,P;\Phi) = \sum_{v\in V} Q(v)e(S[v]x + P[v]iy) = \sum_{v\in L'/L} \Phi^{(\infty)}(v)\theta(z,gP;v,Q\circ g,L).$$

Thus we study the level of $\theta(z,P;v,Q,L)$ carefully.

It is well known that any homogeneous polynomial can be written uniquely as $Q(v) = \sum_{j=0}^{[k/2]} S[v]^j\eta_{k-2j}(v)$ for a spherical polynomial η_j of degree j (see [H1] Section 5). A function η is spherical if $\Delta\eta = 0$ for $\Delta = \sum_{i,j} s_{ij}\frac{\partial^2}{\partial x_i \partial x_j}$, where $S^{-1} = (s_{ij})$ and we write $v = x_1v_1 + \cdots + x_nv_n$ for the base v_i of L. By differentiation of θ with respect to the variable x, we get $S[v]$; so, we forget about $S[v]^j$. Thus we may assume that Q is a spherical function. Any spherical function is a linear combination of the following type of functions Q: Choose $w_\pm \in V_{\mathbb{C}}$ such that $S(v,w_\pm) = \pm P(v,w_\pm)$ for all $v\in V_{\mathbb{C}}$ and $S[w_\pm] = 0$ (Exercise 5). Then $Q(v) = S(v,w_+)^\ell S(v,w_-)^m$. When w_+ does not exist, we just suppose $\ell = 0$ and ignore the factor $S(v,w_+)$. We take this convention also for w_-.

We thus fix $Q(v) = S(v,w_+)^\ell S(v,w_-)^m$ for non-negative integers ℓ and m. For simplicity, we write $\theta(z;v,L)$ for $\theta(z,P;v,Q,L)$. Let $L^* = \{v\in V | S(v,L) \subset$

$\mathbb{Z}\}$ (the dual lattice). We assume that $S(L,L) \subset \mathbb{Z}$ (replacing L by a smaller lattice if necessary); so, $L^* \supset L$. Here is an easy lemma whose proof is left to the reader (Exercise 4):

Lemma 2.1. *Let $0 \neq c \in \mathbb{Z}$. Then we have*

1. *If $v \in L^*$ and $a \in c^{-1}\mathbb{Z}$, $\theta(z+a; v, cL) = e(aS[v])\theta(z; v, cL)$.*
2. *If $v \in L^*$, $\theta(z; v, L) = \sum_{w \in (v+L)/cL} \theta(z; w, cL)$.*
3. *$\theta(c^2 z; v, L) = c^{-\ell-m}\theta(z; cv, c^2 L)$.*

The Poisson summation formula yields:

Proposition 2.2. *For $0 \neq c \in \mathbb{Z}$, we have*

$$\theta\left(-\frac{1}{z}; w, cL\right) = (-1)^{\ell+m}(\sqrt{-1})^{(\mu-\lambda)/2}|c|^{-n}|D|^{-1/2} \times z^{\ell+(\lambda/2)}\overline{z}^{m+(\mu/2)} \sum_{v \in c^{-1}L^*/cL} \mathbf{e}(S(w,v))\theta(z; v, cL), \quad (2.6)$$

where $z^s = |z|^s \exp(i\sigma s)$ writing $z = |z|\exp(i\sigma)$ with $-\pi < \sigma < \pi$ and $D =$ $\det(S)$.

Writing $\psi_Q(z; v) = Q(v)e(S[v]x + P[v]iy)$, the idea of the proof is classical that we compute its Fourier transform and apply the Poisson summation formula to θ. The computation follows Hecke's technique in his Werke No.23 ([H] and [Sh1] Proposition 2.1).

Proof. We start computing Fourier transform of ψ_Q. Here ψ_1 indicates that we take $Q = 1$. Here is a well known formula. For $z \in \mathfrak{H}' = \{z \in \mathbb{C} | \operatorname{Re}(z) > 0\}$ and $a \in \mathbb{R}^\times$:

$$\int_{-\infty}^{\infty} \exp(-\pi|a|zv^2)\mathbf{e}(awv)dv = \sqrt{|a|}^{-1} z^{-1/2}\exp\left(-\frac{\pi|a|w^2}{z}\right). \quad (2.7)$$

We can always find $B \in GL_n(\mathbb{R})$ such that $BS \cdot {}^tB = \operatorname{diag}[a_1, \dots, a_n]$ and $BP \cdot {}^tB = \operatorname{diag}[|a_1|, \dots, |a_n|]$ (Exercise 7). We write $w_i = \sum_j b_{ij}v_j$ (so, for the real base, w_i, S and P are diagonal). Then writing $v = \alpha_1 w_1 + \cdots + \alpha_n w_n$ (so, $\alpha(v) = (\alpha_1, \dots, \alpha_n)$ is the coordinate of v with respect to the base $\{w_i\}$). Then

$$\psi_1(z; v) = \exp(-\pi \sum_i (\alpha_i^2 |a_i| y - \alpha_i^2 a_i \sqrt{-1}x)).$$

We write β for the coordinate with respect to $\{v_i\}$; so, $v = \sum_i \beta_i v_i$. Then we write $dv = \underline{d\beta_1 \cdots d\beta_n}$ (a Haar measure on $V_\mathbb{R}$). Then $dv = Cd\alpha_1 d\alpha_2 \cdots d\alpha_n$ for $C = \sqrt{(|a_1 \cdots a_n|D^{-1})}$. Writing $w = \sum_i \gamma_i w_i$ ($\alpha(w) = \gamma$), we find

$$\psi^*(z; w) = C\int_{V_\mathbb{R}} \exp(-\pi \sum_i (y|a_i|\alpha_i^2 - \sqrt{-1}xa_i\alpha_i^2))\mathbf{e}(\sum_i a_i\alpha_i\gamma_i)d\alpha_1 d\alpha_2 \cdots d\alpha_n.$$

By applying (2.7)

$$\psi_1^*(z;w) = |D|^{-1/2}(-\sqrt{-1}z)^{\lambda/2}(\sqrt{-1}\overline{z})^{\mu/2}\psi_1(-\frac{1}{z};w). \tag{2.8}$$

In order to compute ψ_Q^*, we write $\beta(w_+) = r = (r_1, \ldots, r_n)$ and $\beta(w_-) = s$ and define

$$\partial_+ = \sum_i r_i \frac{\partial}{\partial \beta_i} \quad \text{and} \quad \partial_- = \sum_i s_i \frac{\partial}{\partial \beta_i}.$$

Then by a simple computation, we get

$$\partial_\pm S[v] = 2S(v,w_\pm),\ \partial_\pm P[v] = 2P(v,w_\pm) = \pm 2S(v,w_\pm), \\ \partial_\pm S(u,v) = S(u,w_\pm). \tag{2.9}$$

From this, we get

$$\partial_\pm Q(v) = \\ \ell S(v,w_+)^{\ell-1}S(w_+,w_\pm)S(v,w_-)^m + mS(v,w_-)^{m-1}S(w_-,w_\pm)S(v,w_+)^\ell.$$

Since $S[w_\pm] = 0$ and $S(w_\pm, w_\mp) = 0$, we have

$$\partial_\pm Q(v) = 0. \tag{2.10}$$

From this, we have

$$\partial_+^\ell \partial_-^m \psi_1(z;w) = (2\pi i)^{\ell+m}\psi_Q(z;w)z^\ell \overline{z}^m.$$

Then, using the fact: $\partial_+^\ell \partial_-^m \mathbf{e}(S(v,w)) = (2\pi i)^{\ell+m}Q(v)\mathbf{e}(S(v,w))$, applying the differential operator $\partial_+^\ell \partial_-^m$ to the formula (2.8), we get

$$\psi_Q^*(z;w) = C_{\ell,m} z^{-\ell-\lambda/2}\overline{z}^{-m-\mu/2}\psi_Q(-\frac{1}{z};w) \tag{2.11}$$

for $C_{\ell,m} = (\sqrt{-1})^{(\lambda-\mu)/2}(-1)^{m+\ell}|D|^{-1/2}$. Then by the Poisson summation formula:

$$\sum_{l\in L} f(v+l) = \sum_{l^*\in L^*} f^*(l^*)\mathbf{e}(S(-v,l^*)),$$

we get

$$\theta(z,w;L) = \sum_{v\in L}\psi_Q(z;w+v) = \sum_{v\in L^*}\psi_Q^*(z;v)\mathbf{e}(S(-w,v)) \\ = C_{\ell,m}z^{-\ell-\lambda/2}\overline{z}^{-m-\mu/2}\sum_{v\in L^*}\mathbf{e}(S(-w,v))\psi_Q(-\frac{1}{z};v),$$

because $\mathbf{e}(S(-w,v))$ only depends on $v \mod L$ if $w \in L^*$. Now we make a variable change $z \mapsto -\frac{1}{z}$ and use Lemma 2.1 (3), we get the desired formula for $c = 1$. For general cL, we just replace L by cL and do the same argument. □

Applying Proposition 2.2 and Lemma 2.1 to:

$$\frac{az+b}{cz+d} = \frac{a}{c} - \frac{1}{c(cz+d)}$$

for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ with $c \neq 0$, we find

$$\theta(\gamma(z); v, L) = C_{\ell,m} c^{-(n/2)} (c/|c|)^{\lambda} (cz+d)^{\ell+(\lambda/2)} (c\overline{z}+d)^{m+(\mu/2)} \times \sum_{u \in L^*/c^2L} \sum_{w \in (v+L)/L} \varphi(v,u) \theta(z; u, c^2L), \quad (2.12)$$

where

$$\varphi(v,u) = \sum_{w \in (v+L)/cL} e(\frac{1}{c}(aS[w] + 2S(w,u) + dS[u]))$$

for $v \in L^*/L$ and $u \in L^*/c^2L$. Since up to scalar, any element on $\gamma \in SL_2(F)$ can be written as a product of an element in $SL_2(\mathbb{Z})$ and an upper-triangular matrix in $GL_2(F)$, we can compute the effect of $z \mapsto \gamma(z)$ by using this formula and Lemma 2.1. In particular, $s\Phi(0) = 0$ for all $s \in Mp_{S,1}(\mathbb{R})$ if Q is not a constant, because $SL_2(\mathbb{Q})$ is dense in $SL_2(\mathbb{R})$. The same is again true for $\Phi \in \mathcal{S}(V_{\mathbb{A}})$ and $s \in Mp_{S,1}(\mathbb{A})$ if Φ_∞ involves non-trivial homogeneous Q, because the action of $Mp_{S,1}(\mathbb{A}^{(\infty)})$ does not affect its action at ∞.

Lemma 2.3. *Let the notation and the assumption be as above. Then*

$$\varphi(v,u) = e(-b(dS[u] + 2S(v,u)))\varphi(v+du, 0).$$

In particular, $\varphi(v,u)$ depends only on $(u,v) \in (L^*/L)^2$.

Proof. We have

$$\varphi(v+du, 0) = \sum_{w \in (v+du+L)/cL} e(\frac{a}{c}S[w]) = \sum_{w \in (v+L)/cL} e(\frac{a}{c}S[w+du]),$$

which is a Gauss sum. However,

$$\begin{aligned} aS[w+du] &= aS[u] + 2adS(w,u) + ad^2S[u] \\ &= (aS[w] + 2S(w,u) + dS[u]) + c(2bS(w,u) + dbS[u]), \end{aligned}$$

where we used the fact: $ad - bc = 1$ to show the last equality. Since $w \in v + L$, $\mathbf{e}(bS(w,u)) = \mathbf{e}(bS(v,u))$, we have

$$e(\frac{a}{c}S[w+du]) = e(\frac{1}{c}(aS[w] + 2S(w,u) + dS[u]))e(2bS(v,u) + dbS[u]).$$

This shows:

$$\begin{aligned} \varphi(v+du, 0) = e(2bS(v,u) + dbS[u]) \sum_{w \in (v+L)/cL} e(\frac{1}{c}(aS[w] + 2S(w,u) + dS[u])) \\ = e(b(2S(v,u) + dS[u]))\varphi(v,u), \end{aligned}$$

which shows the formula. ∎

Combining (2.12) and Lemma 2.3, we get

Proposition 2.4. *Let* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$*. Then we have*

1. *If* $c \neq 0$*, then*

$$\theta(\gamma(z); v, L) = C_{\ell,m} c^{-n/2} (c/|c|)^{\lambda} (cz + d^{\ell+(\lambda/2)} (c\overline{z} + d)^{m+(\mu/2)} \times \sum_{u \in L^*/L} \varphi(v, u) \theta(z; u, L),$$

2. *If* $c = 0$*,* $\theta(\gamma(z); v, L) = e(dbS[v])\theta(z; v, L)$*.*

Exercises

1. Prove all the statements in Example 2.1 in details.
2. Prove that $v \mapsto \Phi_P(v; z)$ is a Schwartz function on $V_{\mathbb{R}}$ (that is, it is of C^∞–class and all its derivative multiplied by a polynomial on $V_{\mathbb{R}}$ decreases when $v \to \infty$).
3. Prove that $\theta(z; v, L)$ is absolutely and locally uniformly convergent for $z \in \mathfrak{H}$ and $P \in \mathcal{Y}$.
4. Prove Lemma 2.1.
5. Find $w_\pm \in V_{\mathbb{C}}$ such that $S[w_\pm] = 0$ and $S(v, w_\pm) = \pm P(v, w_\pm)$ for a positive majorant P, and specify a necessary and sufficient condition for the existence of non-zero $w_\pm$.
6. Prove that if η is spherical with respect to P, $\eta \circ g$ is spherical with respect to gP for $g \in O_S(\mathbb{R})$.
7. Explain why we can find $B \in GL_n(\mathbb{R})$ as in the proof of Proposition 2.2.

2.3 Transformation formula of theta series

Let M be the least integer such that $MS[L^*] \subset 2\mathbb{Z}$. If $M = 1$, then $L^* = L$, and by the theorem of Weil ([M] Theorem 2.3) applied to $V_{\mathbb{R}}$, $\theta(z; L)$ is invariant under $SL_2(\mathbb{Z})$. Thus we assume that $M > 1$. We are going to show

Theorem 2.5. *Let* $\gamma \in \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(M)$ *with* $d > 0$ *(by changing* γ *by* $-\gamma$ *if necessary). Then we have, for* $v \in L^*$*,*

$$\theta(\gamma(z); v, L) = \chi(d) e(abS[v]) (cz + d)^{\ell+(\lambda/2)} (c\overline{z} + d)^{m+(\mu/2)} \theta(z; av, L),$$

where

$$\chi(d) = \begin{cases} \left(\frac{(-1)^m D}{d}\right) & \text{if } n = 2m \text{ with } m \in \mathbb{Z}, \\ \varepsilon_d^n \left(\frac{-2c}{d}\right) \left(\frac{D}{d}\right) & \text{if } n = 2m + 1 \text{ for } m \in \mathbb{Z}. \end{cases}$$

Here if $c = 0$, *the Legendre symbol* $\left(\frac{-2c}{d}\right)$ *is assumed to be equal to* 1, *and*

$$\varepsilon_d = \begin{cases} 1 & \text{if } d \equiv 1 \mod 4, \\ \sqrt{-1} & \text{if } d \equiv 3 \mod 4. \end{cases}$$

We shall give a sketch of a proof of the theorem.

Proof. We start with $\gamma \in SL_2(\mathbb{Z})$. We may assume that $c \neq 0$ (otherwise the formula is obvious and follows from Proposition 2.4 (2)). We see easily that $ML^* \subset L$ (Exercise 1). By replacing z by $-\frac{1}{z}$ in the formula of Proposition 2.4 (1) and then applying Proposition 2.2, we get

$$\theta\left(\frac{bz-a}{dz-c}; v, L\right) = |c|^{-n/2}|D|^{-1}(dz-c)^{\ell+(\lambda/2)}(d\overline{z}-c)^{m+(\mu/2)} \times \sum_{t \in L^*/L}\left(\sum_{u \in L^*/L}\varphi(v,u)\mathbf{e}(S(u,t))\right)\theta(z; t, L).$$

In this computation, we assumed that $c < 0$ when n is odd (which results at the end the assumption that $d > 0$). Note that

$$\varphi(v,u) = e(-b(dS[u] + 2S(v,u)) \sum_{w \in (v+du+L)/cL} e(\frac{a}{c}S[w]).$$

We now suppose $\gamma = \left(\begin{smallmatrix} b & -a \\ d & -c \end{smallmatrix}\right) \in \Gamma_0(M)$; so, $d \equiv 0 \mod M$. Since $MS[L^*] \subset 2\mathbb{Z}$, we find $\frac{1}{2}dS[u] \in \mathbb{Z}$, and $e(-bdS[u]) = 1$. Since $ML^* \subset L$, $du \in L$ and hence $v \equiv v + du \mod L$. Thus we have

$$\varphi(v,u) = \mathbf{e}(-bS(v,u)) \sum_{w \in (v+L)/cL} e(\frac{a}{c}S[w]).$$

We put $\varphi(v) = \sum_{w \in (v+L)/cL} e(\frac{a}{c}S[w])$ for the part of the Gauss sum. By this maneuver, we reach the following expression:

$$\theta\left(\frac{bz-a}{dz-c}; v, L\right) = |c|^{-n/2}|D|^{-1}\varphi(v)(dz-c)^{\ell+(\lambda/2)}(d\overline{z}-c)^{m+(\mu/2)} \times \sum_{t \in L^*/L}\left(\sum_{u \in L^*/L}\mathbf{e}(S(u, t-bv))\right)\theta(z; t, L).$$

Now putting $\psi(u) = \mathbf{e}(S(u, t-bv))$ for $u \in L^*/L$, ψ is an additive character of the additive group L^*/L. By the orthogonality relation of characters, we have

$$\sum_{u \in L^*/L}\psi(u) = \begin{cases} [L^* : L] = |D| & \text{if } \psi \text{ is trivial,} \\ 0 & \text{otherwise.} \end{cases}$$

Thus the term which survives is those $t - bv \in (L^*)^* = L$. This yields

$$\theta\left(\frac{bz-a}{dz-c}; v, L\right) = \varphi(v)|c|^{-n/2}(dz-c)^{\ell+(\lambda/2)}(d\overline{z}-c)^{m+(\mu/2)}\theta(z; bv, L).$$

Thus we need to compute the Gauss sum $\varphi(v)$.

Hereafter we rewrite $\gamma = \left(\begin{smallmatrix} b & -a \\ d & -c \end{smallmatrix}\right)$ as $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(M)$. Thus the assumption $c < 0$ becomes $d > 0$. The above formula then states:

$$\theta(\gamma(z); v, L) = \varphi(v)|d|^{-n/2}(cz+d)^{\ell+(\lambda/2)}(c\overline{z}+d)^{m+(\mu/2)}\theta(z; av, L)$$

under $d > 0$ and $c \equiv 0 \mod M$, where $\varphi(v) = \sum_{w \in (v+L)/dL} e(\frac{b}{d}S[w])$. We modify $\varphi(v)$ slightly. Since $ad - bc = 1$, $ad \equiv 1 \mod M$ and $(ad-1)v \in L$ for all $v \in L^*$; so, $adv \equiv v \mod L$. Thus $w \in (v+L)/dL$ satisfies $w \equiv adv \mod L$ and hence $w = adv + u$ with $u \in L/dL$. Thus

$$\begin{aligned}\varphi(v) &= \sum_{u \in L/dL} e(\frac{b}{d}S[adv+u]) = \sum_u e(ba^2dS[v] + 2abS(v,u) + \frac{b}{d}S[u]) \\ &\overset{bad^2 \equiv ab \mod M}{=} e(abS[v]) \sum_{u \in L/dL} e(\frac{b}{d}S[u]).\end{aligned}$$

We write $W(b,d) = |d|^{-n/2}\sum_{u \in L/dL} e(\frac{b}{d}S[u])$. Thus the formula we are dealing with is:

$$\theta(\gamma(z); v, L) = e(abS[v])W(b,d)(cz+d)^{\ell+(\lambda/2)}(c\overline{z}+d)^{m+(\mu/2)}\theta(z; av, L).$$

Then it is standard from the time of Hecke that $W(b,d) = \chi(d)$ (see [H] or the proof of [Sh1] Proposition 2.1). □

Exercises

1. Prove $ML^* \subset L$.
2. Prove $W(b,d) = \chi(d)$.

2.4 Siegel-Eisenstein series

As before, we write Φ for a Schwartz-Bruhat function in $\mathcal{S}(V_A)$ for $A = \mathbb{A}$ and $\mathbb{R}$. Let $L = V \subset V_{\mathbb{A}}$ if $A = \mathbb{A}$ and $L \subset V$ is a lattice if $A = \mathbb{R}$. For any algebraic subgroup $H \subset Sp_S$, we write $H_L = \{h \in Sp_S(A) | Lh = L\}$. Then by [M] 2.4 and Theorem 2.2, we have

$$\mathbf{r}_L \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-*} \end{smallmatrix}\right) \Phi(v) = \sqrt{|a|}\Phi(va) \quad \text{and} \quad \mathbf{r}_L \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \Phi(v) = \mathbf{e}_A(\frac{1}{2}S(v, vb))\Phi(v).$$

Since the above type of element generate $P_{1,L}$, we have an exact description of the metaplectic action of $p \in P_{1,L}$ on Φ. If $A = \mathbb{R}$ and $\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-*} \end{smallmatrix}\right) \in P_{1,L}$, then $La = L$; so, a is unimodular, that is, $a = \pm 1$. Thus $(\mathbf{r}\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-*} \end{smallmatrix}\right)\Phi)(0) = \Phi(0)$.

When $A = \mathbb{A}$, we see $P_{1,L} = B(\mathbb{Q})$ for the Borel subgroup B of $SL(2)$. Thus if $\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-*} \end{smallmatrix}\right) \in P_{1,L} = B(\mathbb{Q})$, then $a \in \mathbb{Q}^\times$ and by the product formula, $|a| = |a|_{\mathbb{A}}^{n/2} = 1$. Thus again we have $(\mathbf{r}\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-*} \end{smallmatrix}\right)\Phi)(0) = \Phi(0)$. By the above description of the action, we immediately have $(\mathbf{r}_L\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)\Phi)(0) = \Phi(0)$. Therefore, for every $p \in P_{1,L}$, we have

Lemma 2.6. *The function $s \mapsto (s\Phi)(0)$ defined on $Mp_{S,1}(A)$ for $A = \mathbb{A}$ and $\mathbb{R}$ is left invariant under $P_{1,L}$.*

We then define the Eisenstein series, formally, by

$$E(\Phi)(s) = \sum_{\gamma \in P_{1,L} \backslash Sp_{S,1,L}} (\gamma s \Phi)(0). \tag{E}$$

To see the convergence, we first treat the case $A = \mathbb{R}$. In this case, we may assume that the Schwartz function is a linear combination of the functions of the form: $Q(v)\mathbf{e}_{\mathbb{R}}\left(\frac{1}{2}\left(S[v]x + iyP[v]\right)\right)$ for a polynomial $Q : V \to \mathbb{C}$, where P is a positive majorant of S. We have already seen that if Q is non-trivial, $s\Phi$ has polynomial part non-trivial; so, $(s\Phi)(0) = 0$. Thus we may assume that $Q = 1$ to compute $(s\Phi)(0)$. Then as we have already seen in the previous section that, for a constant $C > 0$,

$$|(s\Phi)(0)| \leq C|h(s,z)|^{-n},$$

writing according to Shimura, as in [M] Corollary 2.5

$$Mp_{S,1}(\mathbb{R}) = \left\{(s, h(s,z)) \middle| s \in SL_2(\mathbb{R})\right\},$$

where $h(s,z)$ for the variable z on the upper half complex plane $\mathfrak{H}$ is the automorphic factor of half integral weight: $h(s,z)^2 = t(cz+d)$ for $t \in \mathbb{T}$.

Note here that

$$P_{1,L} = \left\{\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \middle| b \in r\mathbb{Z}\right\}$$

for a rational number r and $Sp_{S,1,L}$ is a conjugate of $SL_2(\mathbb{Z})$ in $SL_2(\mathbb{Q})$. Then it is well known from the time of Hecke that the series

$$\sum_{\gamma \in P_{1,L} \backslash Sp_{S,1,L}} |h(\gamma, z)|^{-n}$$

is absolutely convergent if $n > 4$ (note that $h(s,z)$ is the automorphic factor of half integral weight).

We put the topology on $\mathcal{S}(V_{\mathbb{R}})$ of local uniform convergence of all derivatives. Then, as is easily seen, $\Phi \mapsto E(\Phi)(s)$ for a fixed s is a continuous linear map from $\mathcal{S}(V_{\mathbb{R}})$ into $\mathbb{C}$ (a tempered measure).

We now reduce the case when $A = \mathbb{A}$ to the case when $A = \mathbb{R}$. Then $P_{1,L} = B(\mathbb{Q})$ and $Sp_{S,1,L} = SL_2(\mathbb{Q})$. Thus we have:

$$E(\Phi)(s) = \sum_{\gamma \in B(\mathbb{Q}) \backslash SL_2(\mathbb{Q})} (\gamma s \Phi)(0).$$

By the strong approximation theorem (e.g. [MFG] 3.1.2), we have

$$SL_2(\mathbb{A}) = SL_2(\mathbb{Q})U \cdot SL_2(\mathbb{R})$$

for each open subgroup U of $SL_2(\widehat{\mathbb{Z}})$. Since we have seen in the previous section that there exists an open subgroup U of $SL_2(\widehat{\mathbb{Z}})$ such that $s\Phi = \Phi$ for all $s \in U$. Thus $E(\Phi)$ is left invariant under $SL_2(\mathbb{Q})$ and right invariant under U. Thus $E(\Phi)$ is determined by its restriction to $SL_2(\mathbb{R})$. Thus we may assume that $s = 1$ to see the convergence. We write $E(\Phi) = E(\Phi)(1)$. We consider the function: $s \mapsto |(s\Phi)(0)|$. We know that

$$|(\left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right) s\Phi)(0)| = |a|_{\mathbb{A}}^{n/2}|(s\Phi)(0)|.$$

Since $B(\mathbb{A})\backslash SL_2(\mathbb{A}) = \mathbf{P}^1(\mathbb{A})$ is compact (cf. Example 1.4) and $SL_2(\mathbb{A}) = B(\mathbb{A})SL_2(\widehat{\mathbb{Z}})SO_2(\mathbb{R})$ (the Iwasawa decomposition), defining $\varepsilon : SL_2(\mathbb{A}) \to \mathbb{R}_+$ by $\varepsilon(s) = |a|_{\mathbb{A}}^{n/2}$ if $s = \left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right) u$ for $u \in SL_2(\widehat{\mathbb{Z}})SO_2(\mathbb{R})$, we find a positive constant C such that

$$|(s\Phi)(0)| \le C\varepsilon(s).$$

Thus we have

$$|E(\Phi)(s)| \le C \sum_{\gamma \in B(\mathbb{Q})\backslash SL_2(\mathbb{Q})} \varepsilon(\gamma s).$$

Write the right-hand-side for $s = 1$ of the above equations $E(\varepsilon)$. For $\gamma = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Q})$, we decompose $\gamma = bu$ for $b \in B(\mathbb{A})$ and $u \in SL_2(\widehat{\mathbb{Z}})SO_2(\mathbb{R})$. Since $u_\infty \in SO_2(\mathbb{R})$ fixes $i = \sqrt{-1} \in \mathfrak{H}$ and $\mathrm{Im}\,(\left(\begin{smallmatrix} a & * \\ 0 & a^{-1} \end{smallmatrix}\right)(i)) = |a|_\infty^2$, we find

$$\varepsilon(\gamma_\infty) = \mathrm{Im}(\gamma(i))^{n/4} = |ci + d|^{-n/2}.$$

If the lower right corner of b is a^{-1}, we have $c\mathbb{Z} + d\mathbb{Z} = a^{-1}\mathbb{Z}$. We may assume that $\alpha = a^{(\infty)} \in \mathbb{Q}^\times$. Now regard $\alpha \in \mathbb{Q} \subset \mathbb{A}$, and changing γ by $\mathrm{diag}[\alpha^{-1}, \alpha]\gamma$, we may assume that $c, d \in \mathbb{Z}$ and $(c, d) = 1$. Thus we find

$$E(\varepsilon) = \frac{1}{2} \sum_{(c,d) \in \mathbb{Z}^2, (c,d)=1} |ci + d|^{-n/2},$$

which is absolutely convergent if $n > 4$. This finishes the proof of absolute convergence if $n > 4$.

2.5 The formula

In this subsection, we always assume that $A = \mathbb{A}$, because the case of $A = \mathbb{R}$ follows from the adelic case. We compute $s\Phi(0)$ to relate the Siegel Eisenstein series with the theta series.

As we have seen in [M] 2.4, if $c \neq 0$

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & ac^{-1} \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 0 & -c^{-1} \\ c & 0 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & c^{-1}d \\ 0 & 1 \end{smallmatrix}\right).$$

Thus $SL_2(\mathbb{Q}) = B(\mathbb{Q}) \sqcup B(\mathbb{Q})JB(\mathbb{Q})$ for $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. In other words, noting the fact:

$$B(A) \cap J^{-1}B(A)J = T(A) = \{\mathrm{diag}[a, a^{-1}] | a \in A^\times\},$$
$$T(A)\backslash B(A) = U(A) = \left\{\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \middle| b \in A\right\},$$

we have

$$E(\Phi) = \Phi(0) + \sum_{\gamma \in B(\mathbb{Q})\backslash B(\mathbb{Q})JB(\mathbb{Q})} (\gamma\Phi)(0)$$
$$= \Phi(0) + \sum_{\gamma \in U(\mathbb{Q})} (J\gamma\Phi)(0).$$

We know that

$$J\Phi(v) = \int_{V_{\mathbb{A}}} \Phi(w)\mathbf{e}_{\mathbb{A}}(S(w, -v))dw$$
$$J \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \Phi(v) = \int_{V_{\mathbb{A}}} \Phi(w)\mathbf{e}_{\mathbb{A}}(S(w, -v) + \frac{1}{2}S(w, wb))dw.$$

This shows

$$\left(J \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \Phi\right)(0) = \int_{V_{\mathbb{A}}} \Phi(w)\mathbf{e}_{\mathbb{A}}(\frac{1}{2}S(w, wb))dw. \tag{2.13}$$

Thus we get, for $\phi(v) = \frac{1}{2}S[v]$,

$$E(\Phi) = \Phi(0) + \sum_{\xi \in \mathbb{Q}} \int_{V_{\mathbb{A}}} \Phi(v)\mathbf{e}_{\mathbb{A}}(\phi(x)\xi)dv.$$

We therefore consider the function $\varphi : x \mapsto \int_{V_p} \Phi(v)\mathbf{e}_p(\phi(v)x)dv$. Identify $V = \mathbb{Q}^n$ and write the coordinates as $(x_1, \ldots, x_n)$. Thus $V_A = \mathbb{G}_a^n(A)$ is an affine space of dimension n; in this way, we consider V as an affine group scheme sending A to V_A. We then consider the Tamagawa measure $|\omega|_p$ on V_p given by $\omega = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$. The quadratic form $\phi(v) = \frac{1}{2}S[v]$ is a morphism $\phi : V \to \mathbb{G}_a$ of schemes over $\mathbb{Q}$. Writing $V_A[x] = \phi^{-1}(x)$ for $x \in \mathbb{G}_a(A)$, we know from Witt's theorem (cf. [EPE] 1.2), if $x \in \mathbb{G}_m(\mathbb{Q}_p)$, $V_p[x] - \{0\} \cong O_\xi(\mathbb{Q}_p)\backslash O_S(\mathbb{Q}_p)$ for the stabilizer of ξ with $\phi(\xi) = x$. It is known that if $x = \phi(\xi), y = \phi(\eta) \in \mathbb{G}_m(\mathbb{Q}_p)$, we can find $\alpha \in GL_V(\mathbb{Q}_p)$ such that $S(v\alpha, w\alpha) = \phi(\xi)^{-1}\phi(\eta)S(v, w)$ for all $v, w \in V(\mathbb{Q}_p)$. In other words, $\alpha^{-1}O_\xi\alpha = O_{\xi\alpha}$. Since $\phi(\xi\alpha) = \phi(\eta)$, we know that

$$V_p[x] \cong O_\xi(\mathbb{Q}_p)\backslash O_S(\mathbb{Q}_p) \cong O_{\xi\alpha}(\mathbb{Q}_p)\backslash O_S(\mathbb{Q}_p) \cong V_p[y]$$

for $y = \psi(\eta)$. Thus $V_p - V_p[0] \cong V_p[x] \times \mathbb{G}_m$ and hence we can split $\omega = \omega_x \wedge dx$ for the variable x of $\mathbb{G}_a$, where ω_x is a gauge form on $O_x\backslash O_S$ (here one can of course choose $x \in V$; so, ω_x is $\mathbb{Q}$–rational). Actually this can be done also for

$V_p[0] - 0$ since ϕ has singularity only at $v = 0$. Thus we find a measure $|\omega_x|$ supported on $V_p[x]$ such that

$$\int_{V_p} \Phi(v)dv = \int_{\mathbb{Q}_p} \int_{V_p[x]} \Phi(v)d|\omega_x|dx.$$

If $\phi(v) = 0$ but $v \neq 0$, S restricted to $\mathbb{Q}_p v$ is trivial. Since S is non-degenerate, we find v' with $S(v, v') = \lambda \neq 0$. We have $S[v' - xv] = S[v'] - 2x\lambda$ for $x \in \mathbb{Q}_p$. Thus taking $x = (2\lambda)^{-1}S[v'-xv]$ and replacing v' by $v'-xv$, we may assume that $S[v'] = 0$. By dividing v' by λ, we may assume that $\lambda = 1$. Then taking a base v_j of $W = (\mathbb{Q}_p v+\mathbb{Q}_p v')^{\perp}$, the matrix form of S with respect to $v, v_1, \ldots, v_{n-2}, v'$ is of the form

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & S' & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{2.14}$$

for a symmetric matrix S' of degree $n-2$. We can now write v'' with $S[v''] = 0$ as follows: Writing $v'' = av+w+bv'$ for $a, b \in \mathbb{Q}_p$ and $w \in W$, we find $S[v''] = 2ab+S'[w] = 0$. Thus $ab = -\frac{1}{2}S'[w]$. In other words, $v'' = -(2a)^{-1}S'[w]v + w + av'$ for $a \in \mathbb{Q}_p^{\times}$ and for any $w \in W$ with $S'[w] \neq 0$ or $v'' \in \mathbb{Q}_p v \cup \mathbb{Q}_p v'$. This shows that $i : (W-W[0])\times\mathbb{G}_m \subset V_p[0]-\{0\}$ sending (w, a) to $-(2a)^{-1}S'[w]v+w+av'$ brings $W - W[0]$ into a Zariski open dense subset of $V_p[0]$. Since the Tamagawa measure $|\omega_0|$ has measure 0 on a proper Zariski closed subset, it is determined by its restriction to $W' = (W-W[0])\times\mathbb{G}_m$. Since on W', $i^*\omega_0$ is a constant multiple of $dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{n-2} \wedge \frac{da}{a}$ (because it is translation invariant by $w' \in W$ and multiplication invariant for a:Exercise 2), we find $d|\omega_0|(sv) = |s|_p^{n-2}d|\omega_0|$ for $s \in \mathbb{Q}_p$. Anyway $d|\omega_0|$ is supported by $V_p[0] - \{0\}$.

Defining $F_\Phi(x) = \int_{V_p[x]} \Phi d|\omega_x|$, we find that

$$\varphi_p(x) = \int_{V_p} \Phi_p(v)\mathbf{e}_p(\phi(v)x)dv = \int_{\mathbb{Q}_p} F_\Phi(y)\mathbf{e}_p(yx)dx,$$

which is the Fourier transform F_Φ^* of F_Φ on $\mathbb{Q}_p$ with respect to dx and $\langle x, y\rangle = \mathbf{e}_p(xy)$.

We can carry out the same argument also for $\mathbb{R}$ and $\mathbb{A}$. In other words, we can find a measure $|\omega_x|$ on $V_{\mathbb{A}}[x]$ such that

$$\int_{V_{\mathbb{A}}} \Phi(v)dv = \int_{\mathbb{A}} \int_{V_{\mathbb{A}}[x]} \Phi(v)d|\omega_x|dx$$

for all $\Phi \in \mathcal{S}(V_{\mathbb{A}})$. We then define

$$F_\Phi(x) = \int_{V_{\mathbb{A}}[x]} \Phi d|\omega_x|$$

for $\Phi \in \mathcal{S}(V_{\mathbb{A}})$. Then $\varphi(x) = \int_{V_{\mathbb{A}}} \Phi(v)\mathbf{e}_{\mathbb{A}}(\phi(v)x)dv$ is the Fourier transform F_Φ^* of F_Φ with respect to $\langle x, y\rangle = \mathbf{e}_{\mathbb{A}}(xy)$.

By the Poisson summation formula, we find

$$\sum_{\eta\in\mathbb{Q}} F_\Phi(\eta) = \sum_{\eta\in\mathbb{Q}} F_\Phi^*(\eta) = \sum_{\eta\in\mathbb{Q}} \int_{V_{\mathbb{A}}} \Phi(v)\mathbf{e}_{\mathbb{A}}(\phi(v)\eta)dv.$$

This is exactly $E(\Phi)-\Phi(0)$. Weil verified that the Poisson summation formula is valid if $E(\Phi)$ converges absolutely (see [We2] Proposition 2).

On the other hand, we have

$$\sum_{\eta\in\mathbb{Q}} F_\Phi(\eta) = \sum_{\eta\in\mathbb{Q}} \int_{V_p[\eta]-\{0\}} \Phi d|\omega_x|,$$

or more generally, the Fourier expansion of the Eisenstein series is:

$$E(\Phi)\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) = \sum_{\eta\in\mathbb{Q}} \int_{V_p[\eta]-\{0\}} \Phi d|\omega_x| \mathbf{e}_{\mathbb{A}}(\eta x). \tag{2.15}$$

We choose the Tamagawa measure dg on $O_S(\mathbb{A})$. Then dg induces the Tamagawa measure $d|\omega_\eta|$ on $O_\xi(\mathbb{Q})\backslash O_\xi(\mathbb{A}) = V_{\mathbb{A}}[\eta]$ for $\xi \in V$ with $\phi(\xi)=\eta$ (by the uniqueness of the Tamagawa measure). We now compute the Fourier expansion of the theta series:

$$\int_{O_S(\mathbb{Q})\backslash O_S(\mathbb{A})} \Theta(\Phi)(s,g)dg.$$

Thus writing $\alpha(x) = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$, we compute $\int_{O_S(\mathbb{Q})\backslash O_S(\mathbb{A})} \Theta(\Phi)(\alpha(x),g)dg$. By definition,

$$\Theta(\Phi)(\alpha(x),g) = \sum_{\xi\in V} \alpha(x)\Phi(\xi g) = \sum_{\xi\in V} \Phi(\xi g)\mathbf{e}_{\mathbb{A}}(\phi[\xi g]x).$$

Then we have

$$\begin{aligned}\int_{O_S(\mathbb{Q})\backslash O_S(\mathbb{A})} \Theta(\Phi)(\alpha(x),g)dg &= \sum_{\xi\in V} \int_{O_S(\mathbb{Q})\backslash O_S(\mathbb{A})} \Phi(\xi g)\mathbf{e}_{\mathbb{A}}(\phi(\xi g)x)dg \\ &= \sum_{\eta\in\mathbb{Q}} \sum_{\xi:\phi(\xi)=\eta} \int_{O_S(\mathbb{Q})\backslash O_S(\mathbb{A})} \Phi(\xi g)dg\mathbf{e}_{\mathbb{A}}(\eta x).\end{aligned}$$

Suppose that $\phi(\xi)=\eta$. Then it is known by Witt's theorem (e.g.[EPE] 1.2) that for any $\mathbb{Q}$–algebra A,

$$O_\xi(\mathbb{Q})\backslash O_S(A) \cong V_A[\eta]-\{0\} \quad \text{via } g \mapsto \xi g, \tag{2.16}$$

where $\xi \in V$ is chosen so that $\phi(\xi)=\eta$, $V_A[\eta] = \{v \in V_A | \phi(v)=\eta\}$ and

$$O_\xi(A) = \left\{g \in O_S(A) \middle| \xi g = \xi\right\}.$$

Now note that

$$\begin{aligned}&\sum_{\xi:\phi(\xi)=\eta} \int_{O_S(\mathbb{Q})\backslash O_S(\mathbb{A})} \Phi(\xi g)\mathbf{e}_{\mathbb{A}}(\eta x)dg \\ &\quad= \sum_{\gamma\in O_\xi(\mathbb{Q})\backslash O_S(\mathbb{Q})} \int_{O_S(\mathbb{Q})\backslash O_S(\mathbb{A})} \Phi(\xi\gamma g)\mathbf{e}_{\mathbb{A}}(\eta x)dg \\ &\quad= \int_{O_\xi(\mathbb{Q})\backslash O_S(\mathbb{A})} \Phi(\xi g)\mathbf{e}_{\mathbb{A}}(\eta x)dg \\ &\quad= \tau(O_\xi)\int_{V_{\mathbb{A}}[\eta]} \Phi(v)d|\omega_\eta|\mathbf{e}_{\mathbb{A}}(\eta x),\end{aligned}$$

where $\tau(O_\xi) = \int_{O_\xi(\mathbb{Q})\backslash O_\xi(\mathbb{A})} dg_\eta$ (the Tamagawa number of O_ξ). This shows, if $\eta \neq 0$, writing $c(\eta, f)$ for the Fourier coefficients of $\mathbf{e}_{\mathbb{A}}(\eta x)$ of a function $f : \mathbb{A} \to \mathbb{C}$,

$$c(\eta; \int_{O_S(\mathbb{Q})\backslash O_S(\mathbb{A})} \Theta(\Phi)(\alpha(x), g)dg) = \tau(O_\xi) \int_{V_{\mathbb{A}}[\eta]} \Phi(v)d|\omega_\eta|. \tag{2.17}$$

Similarly, we get

$$c(0; \int_{O_S(\mathbb{Q})\backslash O_S(\mathbb{A})} \Theta(\Phi)(\alpha(x), g)dg) = \tau(G)\Phi(0) + \tau(O_\xi) \int_{V_{\mathbb{A}}[0]-\{0\}} \Phi(v)d|\omega_0| \tag{2.18}$$

for $\xi \neq 0$ with $\phi(\xi) = 0$. Thus, if $\tau(O_\xi) = \tau$ is independent of ξ if $\phi(\xi) \neq 0$, comparing the Fourier coefficients in (2.15) and (2.17), we get, for $\xi \neq 0$ with $\phi(\xi) = 0$,

$$\tau\Phi(0) + \tau \int_{V_{\mathbb{A}}[0]-\{0\}} \Phi(v)d|\omega_0| = \tau(G)\Phi(0) + \tau(O_\xi) \int_{V_{\mathbb{A}}[0]-\{0\}} \Phi(v)d|\omega_0|.$$

By replacing $\Phi(x)$ by $\Phi_t(x) = \Phi(tx)$ for $t \in \mathbb{A}^\times$, as we have seen already, $d|\omega_0(tx)| = |t|_{\mathbb{A}}^{n-2} d|\omega_0(x)|$; so, the second part of the above identity gets multiplied by $|t|_{\mathbb{A}}^{2-n}$ if we replace Φ by Φ_t while the first term is intact. Therefore we get

$$\tau(G) = \tau(O_\xi) = \tau$$

even if $\xi \neq 0$ with $\phi(\xi) = 0$.

If $\phi(\xi) = \eta \neq 0$, O_ξ is an orthogonal group O_{S_ξ} for S_ξ on a space of dimension $n-1$, because if $\alpha \in O_\xi$, α preserves $W_\xi = (\mathbb{Q}\xi)^\perp$ and hence $S_\xi = S|_{W_\xi}$. Thus $O_\xi \cong O_{S_\xi}$, and O_ξ is an orthogonal group of S_ξ which has dimension one less than $n = \dim V$. If we know $\tau = \tau(O_{S_\xi})$ is a constant independent of S_ξ of $n-1$ variables, we get the desired identity plus an extra information that $\tau(O_S) = \tau$ looking at the constant term. Thus by induction, we only need to prove the constancy: $\tau = \tau(O_S)$ when S has four variables.

Weil computed directly that $\tau(O_S) = 2$ if $\dim S = 3$ and 4. When $n = 4$, to show this, he uses the fact that O_S is either isomorphic to $(B^\times \times B^\times)/\{\pm(1,1)\}$ for a quaternion algebra $B/\mathbb{Q}$ or to $B^\times/\{\pm 1\}$ for a quaternion algebra B over a quadratic field over $\mathbb{Q}$ (see [AAG] Theorem 3.7.1). Thus the formula is valid for all S of dimension $n > 4$, and we get at the same time $\tau(O_S) = 2$, which actually follows also from [AAG] Theorem 4.5.1. Thus we have

Theorem 2.7. *Let dg be the invariant measure on $O_S(\mathbb{Q})\backslash O_S(\mathbb{A})$ with total volume 1 (so the half of the Tamagawa measure). Then we have, if $n > 4$,*

$$\int_{O_S(\mathbb{Q})\backslash O_S(\mathbb{A})} \Theta(\Phi)(s, g)dg = E(\Phi)$$

for all $\Phi \in \mathcal{S}(V_{\mathbb{A}})$.

The proof of this theorem given in [We2] VI is deeper than what we said, using a full force of measure theory, without making use of the known fact on the Tamagawa numbers; so, it gives another proof of the fact that $\tau(O_S) = 2$ (see Mars's papers in [AGD] Chapter II for a simplified account of thei Weil's second proof). Therefore, the determination of the Tamagawa numbers is basically equivalent to the identity as above. However the Siegel formula itself is deeper, because it also involves the explicit determination of Fourier coefficients of $E(\Phi)$, which contains more information than the Tamagawa number.

Exercises

1. Write S as in (2.14). Then show that $\alpha(w) = \begin{pmatrix} 1 & w & -S'[w]/2 \\ 0 & 1 & -S'{}^t w \\ 0 & 0 & 1 \end{pmatrix}$ is an element of $O_S(A)$ if $w \in A^{n-2}$.
2. Writing $V = \mathbb{Q}_p v \oplus W \oplus \mathbb{Q}_p v'$ with the notation in (2.14) and identifying $W = \mathbb{Q}_p^{n-2}$ using the base $\{v_j\}$ there, show that $i^*\omega_0$ is invariant under translation: $w \mapsto w + w'$ and $a \mapsto \lambda a$ for $\lambda \in \mathbb{Q}_p^\times$ and $w' \in W$. Hint: use the existence of $\alpha(w)$ and $\omega_0(gx) = \omega_0(x)$ for $g \in O_S(\mathbb{Q}_p)$.

2.6 General case

In this subsection, we briefly describe what Weil proved for more general symplectic groups Sp_S.

We keep the notation of the previous section, but we consider V^q for a positive integer q instead of V. Then we define $S[v] = S(v_i, v_j) \in M_q(\mathbb{Q})$ writing $v = (v_1, \dots, v_q)$ for $v_i \in V$. Then we have

$$Sp_q(A) \subset Z_O(A) = \left\{\sigma \in Sp(V_A^q)\middle|\sigma g = g\sigma \text{ for all } g \in O_S(A)\right\},$$

where $g \in O_S(A)$ acts on v diagonally as $vg = (v_1 g, \dots, v_q g)$. We put

$$Mp_q(A) = \left\{g \in Mp(V_A^q)\middle|\pi(g) \in Sp_q(A)\right\} \quad (A = \mathbb{R}, \mathbb{Q}_p, \mathbb{A})$$

and $P_q(A) = P(A) \cap Sp_q(A)$ inside $Sp_{V^q}(A)$. Then we can prove in the exactly the same way as in the case: $q = 1$ that the action of Mp_q commutes with the action of O_S; so, we have well define $\Theta(\Phi)(s, g)$. In this case, for $\Phi \in \mathcal{S}(V_{\mathbb{A}}^q)$,

$$E(\Phi)(s) = \sum_{\gamma \in P_q(\mathbb{Q}) \backslash Sp_q(\mathbb{Q})} (\gamma s \Phi)(0) \quad (s \in Mp_q(\mathbb{A}))$$

converges absolutely if $n > 2q + 2$ (the same proof as in the case of $q = 1$ works well), and the Siegel-Weil formula is valid in the same way under this assumption:

$$\int_{O_S(\mathbb{Q}) \backslash O_S(\mathbb{A})} \Theta(\Phi)(s, g)dg = E(\Phi)(s) \quad (s \in Sp_q(\mathbb{A})).$$

The computation itself is basically the same as in the case of $SL(2)$, but in this case, we need to take care of $\eta \in M_q(\mathbb{Q})$ such that $\operatorname{rank}\eta < q$. Modular forms on $SL(2)$ is determined by its Fourier coefficients of $\mathbf{e}_{\mathbb{A}}(\eta x)$ for $\eta \neq 0$; so, we have deduced the identity of $c(0;*)$ to those of $c(\eta;*)$ for $\eta \neq 0$. However this is no longer true in the general Siegel modular case; so, Weil used an induction argument on q a bit more technical.

In the above setting, the module V_A^q is considered to be a left module over $M_q(A)$, and it is also considered to be $\mathcal{A}_A = \operatorname{End}_A(V_A)$–module. In particular, $\mathcal{A}$ has involution $a \mapsto S^t a S^{-1}$ and $O_S(A) = \{x \in \mathcal{A}_A | xx^* = 1\}$. Thus we can further generalize this situation as follows: Take a simple algebra $\mathcal{A}$ over a field $\mathbb{Q}$ with involution $x \mapsto x^*$. Take a simple $\mathcal{A}$–module V, and put $\widehat{V} = \operatorname{Hom}_{\mathcal{A}}(V, \mathcal{A})$. We can identify $\widehat{V}_{\mathbb{A}}^q$ with $(V_{\mathbb{A}}^q)^*$ by $\langle x, y\rangle = \mathbf{e}(\operatorname{Tr}_{\mathcal{A}/\mathbb{Q}}(y(x)))$. By using these pairing, we construct $Sp(G_A)$ for $G_A = V_A^q \times (\widehat{V}_A^q)$ (A: any $\mathbb{Q}$–algebra), $Mp(G_A)$ ($A = \mathbb{R}, \mathbb{Q}_p, \mathbb{A}$) and $O_S(A) = \{x \in \mathcal{A}_A | xx^* = 1\}$. We then define

$$\begin{aligned} Sp_{\mathcal{A},q}(A) &= \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in Sp(G_A) \middle| a, b, c, d \in \operatorname{End}_{\mathcal{A}}(V_A^q) \right\} \\ Mp_{\mathcal{A},q}(\mathbb{A}) &= \left\{ x \in Mp(G_{\mathbb{A}}) \middle| \pi(x) \in Sp_{\mathcal{A},q}(\mathbb{A}) \right\}. \end{aligned} \tag{2.19}$$

We write P for the parabolic subgroup of $Sp_q(A)$ with lower left corner element $c = 0$. Then we can think of

$$\begin{aligned} E(\Phi)(s) &= \sum_{\gamma \in P(\mathbb{Q}) \backslash Sp_{\mathcal{A},q}(\mathbb{Q})} (\gamma s \Phi)(0) \quad (s \in Mp_{\mathcal{A},q}(\mathbb{A})) \\ \Theta(\Phi)(s, g) &= \sum_{\xi \in V^q} s\Phi(\xi g) \quad (s \in Mp_{\mathcal{A},q}(\mathbb{A}),\ g \in O_S(\mathbb{A})). \end{aligned} \tag{2.20}$$

Writing $\mathcal{A} = M_m(D)$ for a division algebra D with center F, then we can always write $x^* = S \cdot {}^t x^\iota S^{-1}$ for an involution $x \mapsto x^\iota$ of D and $S \in \mathcal{A}^\times$ with ${}^t S^\iota = \epsilon S$ ($\epsilon = \pm 1$). We now define

$$\delta = \dim_F D \quad \text{and} \quad \delta' = \dim_F \{\xi \in D | \xi^\iota = \epsilon \xi\}.$$

Then Weil proved that if $m > 2q + 4(\delta/\delta') - 2$, then $E(\Phi)$ is absolutely convergent. Under a splitting condition at one place (that is, a generalization of Witt's theorem holds there), Weil also proved

$$\int_{O_S(\mathbb{Q}) \backslash O_S(\mathbb{A})} \Theta(s, g) dg = E(\Phi)$$

for the measure dg on $O_S(\mathbb{Q}) \backslash O_S(\mathbb{A})$ of total volume 1 (see [We2] Théorème 5). The groups of type O_S cover almost all classical groups (another result of Siegel); so, the Siegel-Weil formula and the Tamagawa number are known for almost all such groups.

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