Metaplectic Groups

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1 Fourier Analysis

Let G be a locally compact abelian group; so, each $x \in G$ has an open neighborhood whose closure is compact, and the addition $(x, y) \mapsto x + y$ is a continuous map of $G \times G$ into G and $x \mapsto -x$ is a homeomorphism of G. We always assume that $x \mapsto 2x$ is an automorphism of topological group. We study Fourier analysis on G and its Pontryagin dual G^* in this section. In particular, we are going to prove the Plancherel formula:

$$\int_G |\Phi(x)|^2 dx = \int_{G^*} |\Phi^*(x)|^2 dx^*$$

as long as the Fourier transform Φ^* is well defined for a given measurable function Φ on G.

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1.1 Pontryagin duality

Let $\mathbb{T}$ be the multiplicative group of complex numbers of absolute value 1, and we often identify $\mathbb{T}$ with $\mathbb{R}/\mathbb{Z}$ by $\mathbb{R}/\mathbb{Z} \ni r \mapsto \exp(2\pi i r) \in \mathbb{T}$. We then define $G^* = \mathrm{Hom}_{cont}(G, \mathbb{T})$. We equip G^* with the uniform convergence topology over any compact subset of G. Then G^* again become a locally compact abelian group (cf. [IGA] and [TGP]). Thus the neighborhood of the trivial character $\mathbf{0}$ of G is given by $\{\phi \in G^* | \phi(B) \subset W\}$ for a compact subset $B \subset G$ and an open neighborhood W of 1 in $\mathbb{T}$. The character $\mathbf{0}$ satisfies $\mathbf{0}(x) = 1 \in \mathbb{T}$. We write $\langle x, x^* \rangle$ for $x^*(x) \in \mathbb{T}$, where $x^* \in G^*$ and $x \in G$. We can then define a homomorphism $G \to (G^*)^*$ by sending $x \in G$ to a character $x^{**} : G^* \to \mathbb{T}$ given by

$$x^{**}(x^*) = \langle x, x^* \rangle.$$

More generally, if $\phi : H \to G$ be a homomorphism of locally compact abelian groups (that is, a continuous homomorphism), we have a dual map $\phi^* : G^* \to H^*$ given by

$$\langle h, \phi^*(g^*) \rangle = \langle \phi(h), g^* \rangle.$$

This duality theory $G \mapsto G^*$ of locally compact abelian groups is a perfect duality called *Pontryagin duality* of locally compact abelian groups and was developed by Pontryagin in 1938 and Weil in 1940 independently. The perfectness of the duality implies

- $(G^*)^* \cong G$ by $x \mapsto x^{**}$;
- If $0 \to H \xrightarrow{\phi} G \xrightarrow{\psi} K \to 0$ is an exact sequence of locally compact abelian groups, then the dual sequence $0 \to K^* \xrightarrow{\psi^*} G^* \xrightarrow{\phi^*} H^* \to 0$ is also exact.

For all this type of results, see either the book of Pontryagin [TGP] or by Weil [IGA].

Many locally compact groups are isomorphic to their dual.

Example 1.1.

1. The pairing $\langle x, y \rangle = \exp(2\pi i a x y)$ for any non-zero real number $a \neq 0$ gives the self-duality of the additive group $\mathbb{R}$. We write $\mathbf{e}_\infty(x) = \exp(2\pi i x)$.
2. Similarly, expanding $x \in \mathbb{Q}_p$ into a p–adic expansion $x = \sum_{n \gg -\infty} c_n p^n$ with integers $0 \leq c_n < p$ and defining a rational number of p–power denominator $[x]_p = \sum_{n<0} c_n p^n \in \mathbb{Q}$, the pairing $\langle x, y \rangle = \exp(-2\pi i [axy]_p)$ for any non-zero p–adic number $a \in \mathbb{Q}_p$ gives a self duality of the additive group $\mathbb{Q}_p$. We write $\mathbf{e}_p(x) = \exp(2\pi i [x]_p)$.
3. For $x = (x_v), y = (y_v) \in \mathbb{A}$, we can define $\langle x, y \rangle = \langle x_\infty, y_\infty \rangle \prod_p \langle x_p, y_p \rangle$ gives a self duality of $\mathbb{A}$ if we choose a in $\mathbb{Q}$ in the examples (1) and (2). We write $\mathbf{e}_{\mathbb{A}}(x) = \prod_v \mathbf{e}_v(x_v)$ for $x = (x_v) \in \mathbb{A}$. Then $\mathbf{e}$ induces a character $\mathbf{e} : \mathbb{A}/\mathbb{Q} \to \mathbb{T}$.
4. For any semi-simple algebra B over $\mathbb{Q}$, $B_A = B \otimes_{\mathbb{Q}} A$ for $A = \mathbb{R}, \mathbb{Q}_p$ and $\mathbb{A}$ is a self dual additive group by $\langle x, y \rangle = \mathbf{e}_?(\mathrm{Tr}_{B/\mathbb{Q}}(axy))$ for $a \in B^\times$, where $? = p, \infty, \mathbb{A}$ according as $A = \mathbb{Q}_p, \mathbb{R}$ and $\mathbb{A}$.

5. If $X = F^n$ is a finite dimensional vector space over a number field F and if $(\ , \ )$ is either a non-degenerate symmetric or σ–hermitian form (with respect to $\sigma \in \mathrm{Aut}(F)$ of order 2) on X, $X_A = X \otimes_{\mathbb{Q}} A$ is a self dual locally compact abelian group by $\langle x, y \rangle = \mathbf{e}_?(\mathrm{Tr}_{F/\mathbb{Q}}(x, y))$.

Exercises

1. Show that G is compact $\Leftrightarrow G^*$ is discrete.
2. Give a proof of all the assertions in Example 1.1 (see [LFE] Section 8.3).
3. Show that $\mathbf{e}(\mathbb{Q}) = 1$ if we regard $\mathbb{Q}$ as a subfield of $\mathbb{A}$ diagonally.

1.2 Haar Measure

On any locally compact abelian group G, there exists a Harr measure dg with values in $\mathbb{R}$ (see [IGA]) satisfying the following conditions:

1. $\int_X dg$ is defined for subset X in a complete additive class containing all compact subsets of G (that is, a union of countably many compact subsets is measurable);
2. For all compact subsets $K \subset G$, we have $0 \leq \int_K dg < +\infty$;
3. We have, for all open subsets $U \subset G$, $\int_U dg = \mathrm{Sup}_{U \supset K:\mathrm{compact}} \int_K dg$ and for all measurable subsets X, $\int_X dg = \mathrm{Inf}_{U \supset X, U:\mathrm{open}} \int_U dg$;
4. $\int_{x+X} dg = \int_X dg$ for all measurable $X \subset G$ and $x \in G$.

Out of this measure, we can construct the Lebesgue measure dg associated to dg as above. In particular, we can think of integrable functions and square integrable functions on G. By (4) as above, if dg' is another Haar measure on G, we have $\int \phi dg = c \int \phi dg'$ for a positive constant c independent of ϕ. If $\alpha : G \to H$ is an isomorphism of locally compact abelian groups, then $\phi \mapsto \int_G \phi(g\alpha) dg$ for an integrable function ϕ on H gives a Haar measure $d(g\alpha^{-1})$ on H. Then for a chosen Haar measure dh on H, we have a positive constant $|\alpha|$ dependent only on α and the choice of dg on G and dh on H such that $d(g\alpha^{-1}) = |\alpha|^{-1} dh$. In other words,

$$\int_H \phi(h) dh = |\alpha| \int_G \phi(g\alpha) dg.$$

When $H = G$, we choose $dh = dg$, then $|\alpha|$ is determined independently of the choice of dg.

Example 1.2. 1. When $G = \mathbb{Z}_p$, any compact set is a disjoint union of subset of the form $a + p^n \mathbb{Z}_p$; so, we just define $\int_{a+p^n\mathbb{Z}_p} dg = p^{-n}$. Then $\int_X dg = \sum_a p^{-n(a)}$ for $X = \bigsqcup_a a + p^{n(a)} \mathbb{Z}_p$. Any continuous function ϕ can be written as $\phi = \lim_{n\to\infty} \phi_n$ for $\phi_n(x) = \phi(m)$ if $x \equiv m \mod p^n$ for an integer m with $) \le m < p^n$. Then we see

$$\int \phi dg = \lim_{n\to\infty} \left(\sum_{j=0}^{p^n-1} \phi(j) p^{-n} \right).$$

2. If $G = \mathbb{Q}_p$, $G = \bigcup_n p^n \mathbb{Z}_p$. By the above argument, we have a Haar measure dg on each $p^n\mathbb{Z}_p$ so that they coincide with the one given on $\mathbb{Z}_p$. Thus this measure gives a unique Haar measure dg on G such that $\int_{p^n\mathbb{Z}_p} = p^{-n}$.

3. If $G = \mathbb{Z}$, we just define that $\int_x dg = 1$ for any $x \in \mathbb{Z}$. Then for any compact subset $K \subset \mathbb{Z}$, K is a finite set and $\int_K dg = |K|$. If $\phi : \mathbb{Z} \to \mathbb{C}$ is a function, then $\int_K \phi dg = \sum_{n\in\mathbb{Z}} \phi(n)$.

4. If $G = \mathbb{R}$, we have the classical Lebesgue measure dg with $\int_0^1 dg = 1$.

5. For any product $G = G_1 \times G_2 \times \cdots \times G_r$ of the above groups, the product measure $dg = dg_1 dg_2 \cdots dg_r$ gives a Haar measure of G. In particular, finite dimensional vector space over $\mathbb{R}$ or $\mathbb{Q}_p$ has such a measure.

6. For the adele ring $G = \mathbb{A}$, we can define the measure dg so that if $\phi(x) = \prod_v \phi_v(x_v)$ for places v, we just define $\int \phi dg = \prod_v \int \phi_v dg_v$ for the measure dg_v on $\mathbb{Q}_p$ if $v = p$ and the Lebesgue measure dg_∞ on $\mathbb{R}$. In particular, $\int_X dg = 1$ for $X = \widehat{\mathbb{Z}} \times [0,1]$, where $\widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$.

1.3 Fourier Transform

Choose a Haar measure dg on G. We then define the Fourier transform $\mathcal{F}(\phi) = \phi^* : G^* \to \mathbb{C}$ of an integrable function $\phi : G \to \mathbb{C}$ by

$$\phi^*(g^*) = \mathcal{F}(\phi)(g^*) = \int_G \phi(g)\langle g, g^*\rangle dg.$$

See [IGA] Chapter 6 for a general theory of Fourier transform. We then choose a Haar measure dg^* on G^* and define the Fourier transform $\mathcal{F}^*$ on G^*. Then we have

Theorem 1.1. *Suppose that f is continuous, bounded and integrable on G and that f^* is integrable on G^*. Then we have*

$$\mathcal{F}^*(\mathcal{F}(f))(-g) = cf(g)$$

for a positive constant c independent of ϕ.

We shall give a sketch of a proof, supposing that either $G \cong G^*$ and G is a locally compact ring or G is finite. This is the case where we later deal with. For $h \in G$, we therefore have continuous multiplication $g \mapsto hg$. If h is invertible, this is an automorphism of the group G; so, $d(hg) = |h|dg$ for $|h| > 0$ by the uniqueness of the Haar measure. For simplicity, we assume to have a sequence of units ε_j converging to 0 in G and $\langle \varepsilon g, g^* \rangle = \langle g, \varepsilon g^* \rangle$ for a unit $\varepsilon \in G$. This fact is valid for $G = \mathbb{R}, \mathbb{Q}_p$ and $\mathbb{A}$ by the following reason. When $G = \mathbb{R}$, G is self dual by $\langle x, y \rangle = \exp(2\pi i xy)$ and therefore it is obvious. When $G = \mathbb{Q}_p$, for each $x \in \mathbb{Q}_p$, expand x into the p–adic expansion $x = \sum_{n \gg -\infty} c_n p^n$ for integers c_n with $0 \leq c_n < p$. Then define the fraction part $[x]_p = \sum_{n<0} c_n p^n \in \mathbb{Q}$. Then the self duality is given by $\langle x, y \rangle = \exp(-2\pi i [xy]_p)$, and again the assertion is obvious. For adeles $x, y \in A$, the pairing $\langle x, y \rangle = \prod_v \langle x_v, y_v \rangle$ (which is a finite product) does the job.

Proof. We formally compute

$$\begin{aligned}
\mathcal{F}^*(f^*)(-g) &= \int_{G^*} f^*(g^*)\langle g, g^* \rangle^{-1} dg^* \\
&= \int_{G^*} \int_G f(h)\langle h, g^* \rangle dh \langle g, g^* \rangle^{-1} dg^* \\
&= \int_{G^*} \int_G f(h)\langle h - g, g^* \rangle dh dg^* \\
&= \int_G f(h) \int_{G^*} \langle h - g, g^* \rangle dg^* dh.
\end{aligned} \tag{1.1}$$

When G is finite, $G^* \cong G$ (Exercise 1) and we may assume that

$$\int_G \phi(g) dg = |G|^{-1} \sum_{g \in G} \phi(g).$$

Then by the orthogonality relation of characters (cf. [LRG] Section 2.3), we have

$$\int_{G^*} \langle h - g, g^* \rangle dg^* = \begin{cases} 1 & \text{if } h = g, \\ 0 & \text{otherwise.} \end{cases}$$

From this, the assertion is clear, and $c = |G|^{-1}$. Since orthogonality relations hold for compact groups G (in this case, G^* is discrete), the same argument still works for compact and discrete groups (like $(G, G^*) = (\mathbb{Z}_p, \mathbb{Q}_p/\mathbb{Z}_p)$ and $(G, G^*) = (\mathbb{T}, \mathbb{Z})$).

We now assume that G is non-discrete and non-compact but a locally compact ring, like $G = \mathbb{R}, \mathbb{Q}_p, \mathbb{A}$. Then the last two integrals of (1.1) may not converge, because $|\langle x, y \rangle| = 1$ all the time. Thus we need to put a convergence factor $\varphi(g^*)$ in the integral:

$$\begin{aligned}
\int_{G^*} f^*(g^*)\varphi(g^*)\langle g, g^* \rangle dg^* &= \int_{G^*} \int_G f(h)\varphi(g^*)\langle h - g, g^* \rangle dh dg^* \\
&= \int_G f(g)\varphi^*(h - g) dh \\
&\overset{h-g \mapsto g}{=} \int_G f(g + h)\varphi^*(g) dh.
\end{aligned} \tag{1.2}$$

We choose a function $\varphi(g^*)$ so that φ and φ^* are integrable over G^* and G respectively (Exercise 2). Since f^* is bounded:

$$|f^*(g^*)| \leq |\int_G f(g)\langle g, g^*\rangle dg| \leq \int_G |f(g)|dg$$

for all g^*, we see that φf^* is also integrable. Then we put $\varphi_\varepsilon(g^*) = \varphi(\varepsilon g^*)$ for a unit $\varepsilon \in G^*$. Note that

$$\begin{aligned}(\varphi_\varepsilon)^*(g) &= \int_{G^*} \varphi(\varepsilon g^*)\langle g, g^*\rangle dg^* \\ &\overset{\varepsilon g^* \mapsto g^*}{=} |\varepsilon|^{-1} \int_{G^*} \varphi(g^*)\langle \varepsilon^{-1} g, g^*\rangle = |\varepsilon|^{-1}\varphi^*(\varepsilon^{-1}g).\end{aligned}$$

Replacing φ by φ_ε in (1.2), we get

$$\begin{aligned}\int_{G^*} f^*(g^*)\varphi(\varepsilon g^*)\langle g, g^*\rangle dg^* &= \int_G f(g+h)|\varepsilon|^{-1}\varphi^*(\varepsilon^{-1}h)dh \\ &\overset{\varepsilon^{-1}h \mapsto h}{=} \int_G f(g+\varepsilon h)\varphi^*(h)dh.\end{aligned}$$

Now we make $\varepsilon \to 0$, we get

$$\varphi(0)\int_{G^*} f^*(g^*)\langle g, g^*\rangle dg^* = f(g)\int_G \varphi^*(h)dh.$$

Choosing $\varphi(0) \neq 0 \neq \int_G \varphi^*(h)dh$ all positive real (for example, we may choose $\varphi(x) = \exp(-\pi g^2)$ when $G = \mathbb{R}$ and the characteristic function of $\mathbb{Z}_p$ when $G = \mathbb{Q}_p$, and product of these when $G = \mathbb{A}$), we get the desired constant $c > 0$. □

Now changing dg^* by $c^{-1}dg^*$, we may assume that the constant c is equal to 1. In this case, dg and dg^* are called dual each other. Further if $G \cong G^*$, first taking $dg = dg^*$ and changing dg by $\sqrt{c}^{-1}dg$, again we can make $c = 1$. In this case, dg is called self dual. The Lesbesgue measure dx on $\mathbb{R}$ is self dual. The measures described in 1.2 for $\mathbb{Q}_p$ and $\mathbb{A}$ are also self dual (Exercise 3).

Exercises

1. When G is finite, prove that $G \cong G^*$ (use the fundamental theorem of finite abelian groups).

2. When $G = \mathbb{R}, \mathbb{Q}_p, \mathbb{A}$, find a continuous function $\varphi(g^*)$ so that φ and φ^* are integrable over G^* and G, respectively.

3. Show that the measure described in Subsection 1.2 for $G = \mathbb{R}, \mathbb{Q}_p, \mathbb{A}$ is self dual.

4. Suppose that G is compact, and write dg for the Haar measure with $\int_G dg = 1$. Then show that its dual measure dg^* on G^* (which is discrete) is given by $\int_{G^*} \phi(g^*)dg^* = \sum_{g^* \in G^*} \phi(g^*)$.

5. For an isomorphism $\gamma : G^* \to G$, define the module $|\gamma|$ with respect to dg on G and the dual measure dg^* on G^*. Then show that the function $|\gamma|^{-1/2}\mathcal{F}(\phi)(-x\gamma^{-*})$ on G is determined independently of the choice of dg.

1.4 Plancherel Formula

Let dg and dg^* be dual Haar measure on G and G^*. We are going to prove the following theorem of Plancherel:

Theorem 1.2. *Let ϕ and f be continuous bounded integrable functions on G and ϕ^* and f^* are both integrable on G^*. If ϕ^* is continuous and bounded on G^*, we have*

$$\int_G f(g)\overline{\phi(g)}dg = \int_{G^*} f^*(g^*)\overline{\phi^*(g^*)}dg^*.$$

Therefore the Fourier transform keeps L^2–norm.

We shall give a sketch of a proof. Using boundedness of ϕ and ϕ^*, it is easy to show the integrals above are finite (Exercise 1).

Proof. By Fourier inversion formula, we have

$$\phi(g) = \mathcal{F}^*(\phi^*)(-g) = \int_{G^*} \phi^*(g^*)\langle -g, g^*\rangle dg^*.$$

Then we see

$$\begin{aligned}\int_G f(g)\overline{\phi(g)}dg &= \int_G f(g) \int_{G^*} \overline{\phi^*(g^*)}\langle g, g^*\rangle dg^* dg \\ &= \int_G \int_{G^*} f(g)\langle g, g^*\rangle dg\overline{\phi^*(g^*)}dg^* \\ &= \int_{G^*} f^*(g^*)\overline{\phi^*(g^*)}dg^*.\end{aligned}$$

This shows the desired formula. □

Consider the L^2–spaces $L^2(G)$ and $L^2(G^*)$. The functions satisfying the condition of Theorem 1.2 is dense in these Hilbert spaces (Exercise 2). Thus for each $f \in L^2(G)$, choosing a sequence f_n satisfying the conditions of Theorem 1.2 yet converging to f in the Hilbert space $L^2(G)$. Then by the theorem, $\mathcal{F}(f_n)$ converges to an element f' in $L^2(G^*)$. The function f' is well defined almost everywhere on G^* and is independent of the choice of the sequence f_n (Exercise 3). We then define $\mathcal{F}(f) = f'$. Then $\mathcal{F} : L^2(G) \cong L^2(G^*)$ gives an isometry of the two Hilbert spaces.

Exercises

1. Show the finiteness of the integrals in Theorem 1.2.
2. Show the density of functions in $L^2(G)$ satisfying the conditions of Theorem 1.2 for $G = \mathbb{R}$ and $G = \mathbb{Q}_p$.
3. Show the well-definedness of the Fourier transform as a bounded operator from the Hilbert space $L^2(G)$ onto $L^2(G^*)$.

2 Metaplectic Groups

First we construct a general metaplectic groups associated to (G, G^*) and then study in details when G is a free module of finite rank over $\mathbb{R}$, $\mathbb{Q}_p$ or $\mathbb{A}$.

2.1 Symmetric Maps

Let H and G be a locally compact abelian groups and $\rho : H \to G$ be a homomorphism (a continuous group homomorphism). Then $g^* \mapsto g^* \circ \rho$ induces a homomorphism $\rho^* : G^* \to H^*$ determined by $\langle h\rho, g^* \rangle = \langle h, g^*\rho^* \rangle$. In our convention, all $\rho \in \mathrm{Hom}(X, Y)$ (except for scalars) acts on X from the right: $x \mapsto x\rho$, which will be useful later. We call ρ^* the adjoint of ρ. If $\rho : G \to G^*$ is a homomorphism, then again $\rho^* : G \to G^*$ is a homomorphism; so, it makes sense to insist $\rho = \rho^*$. Such a homomorphism is called symmetric.

To each symmetric map $\rho : G \to G^*$, we can associate a multiplicative quadratic form (a character of second degree) $f_\rho : G \to \mathbb{T}$ by

$$f_\rho(x) = \langle x, 2^{-1}x\rho \rangle.$$

Then

$$\begin{aligned} &f_\rho(x+y)f_\rho(x)^{-1}f_\rho(y)^{-1} \\ &\qquad = \langle x+y, 2^{-1}(x\rho + y\rho)\rangle \langle x, -2^{-1}x\rho\rangle \langle y, -2^{-1}y\rho\rangle = \langle x, y\rho\rangle. \end{aligned} \tag{2.1}$$

Thus, under the assumption we made that $g \mapsto 2g$ is an automorphism of G, we have a bijection:

$$\{\text{symmetric homomorphisms}\} \\ \qquad \leftrightarrow \{\text{multiplicative homogeneous quadratic forms}\}$$

by $\rho \mapsto f_\rho$. Here the word "homogeneous" mean that f does not have linear terms, that is, f is of the form f_ρ for a symmetric map ρ.

Example 2.1. 1. Let F be a field of characteristic different from 2 and X be a finite dimensional vector space over F. A quadratic form $\phi : V \to F$ is a homogeneous polynomial on X of degree 2. Then $(x, y) = \phi(x + y) - \phi(x) - \phi(y)$ is a symmetric $\mathbb{Q}$–bilinear form on V. We call ϕ anisotropic if $\phi(x) = 0 \Leftrightarrow x = 0$. We call ϕ non-degenerate if $(x, V) = 0 \Rightarrow x = 0$. If ϕ is anisotropic, ϕ is non-degenerate.

2. Let $F = \mathbb{R}$. Then X is a locally compact abelian group isomorphic to $\mathbb{R}^n$ for $n > 0$. For a given quadratic form ϕ on X, $f(x) = \exp(2\pi i\phi(x))$ is a homogeneous multiplicative quadratic form. The set of all homogeneous multiplicative quadratic forms is in bijection with the set of all quadratic forms on X. Indeed, if $f : X \to \mathbb{T}$ is a multiplicative quadratic form, then on a small open neighborhood U of 0, $\phi(x) = (2\pi i)^{-1} \log(f(x))$ for $x \in U$.

3. Let $F = \mathbb{Q}_p$. Then $f(x) = \exp(-2\pi i[\phi(x)]_p)$ is a multiplicative quadratic form. In the same way as above, The set of all homogeneous multiplicative quadratic forms on X is in bijection with the set of all quadratic forms on X in this manner.

Exercise

1. Give a detailed proof of the assertions in the above examples.

2.2 Symplectic Groups

We now write $V = G \times G^*$. Then $V^* \cong V$ by $\eta : (x, x^*) \mapsto (-x^*, x)$. We can write an automorphism $\sigma : V \to V$ as a matrix:

$$(x, x^*) \overset{\sigma}{\mapsto} (x, x^*) \left(\begin{smallmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{smallmatrix}\right).$$

Here $a_\sigma \in \mathrm{End}(G)$, $d_\sigma \in \mathrm{End}(G^*)$, $b_\sigma \in \mathrm{Hom}(G, G^*)$ and $c_\sigma \in \mathrm{Hom}(G^*, G)$. We then define $J : V \times V \to \mathbb{T}$ by $J((x, x^*), (y, y^*)) = \langle x, y^*\rangle\langle -y, x^*\rangle$. We can write this equation symbolically:

$$(x, x^*) \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \left(\begin{smallmatrix} y \\ y^* \end{smallmatrix}\right) = \langle -y, x^*\rangle\langle x, y^*\rangle.$$

Then we define the group $Sp(G) \subset \mathrm{Aut}(V)$ by

$$Sp(G) = \{\sigma \in \mathrm{Aut}(V) | J(v\sigma, w\sigma) = J(v, w)\ \forall v, w \in V\}.$$

Since $Sp(G)$ is the stabilizer in $\mathrm{Aut}(V)$ of the multiplicative quadratic form J, it is a group, and we see easily that for $\sigma \in Sp(G)$,

$$\sigma^{-1} = \begin{pmatrix} d^*_\sigma & -b^*_\sigma \\ -c^*_\sigma & a^*_\sigma \end{pmatrix}. \tag{2.2}$$

Exercise

1. Prove (2.2).

2.3 Heisenberg Groups

For each $v = (x, x^*) \in V$, we define a unitary operator $U(v)$ on $L^2(G)$ by

$$U(v)\phi(g) = \Phi(g + x)\langle g, x^*\rangle.$$

Then by computation, we have for $v = (x, x^*)$ and $w = (y, y^*)$ both in V,

$$U(v)U(w) = \langle x, y^*\rangle U(v + w) = F(v, w)U(v + w), \tag{2.3}$$

for $F(v, w) = \langle x, y^*\rangle$. Thus $H(G) = \{tU(v)|v \in V,\ t \in \mathbb{T}\}$ is a subgroup of unitary operators acting on $L^2(G)$ with the identity operator given by $U(0)$. This group is sometimes called the *Heisenberg group* for G (and it is written as $\mathbf{A}(G)$ in [We1] no.4). Since $U(v)U(w) = U(w)U(v)$ implies $\langle x, y^*\rangle = \langle y, x^*\rangle$, if $U(v)$ commutes with $U(w)$ for all $w \in V$, we find $v = 0$. Indeed, taking $y = 0$, we find $\langle x, y^*\rangle = 0$ for any y^*, and non-degeneracy tells us $x = 0$, and similarly, taking $y^* = 0$, we find $x^* = 0$. Thus the center is given by $Z(H(G)) \cong \{tU(0)|t \in \mathbb{T}\} \cong \mathbb{T}$, and we have the following central extension:

$$1 \to \mathbb{T} \to H(G) \to V \to 0.$$

Thus any automorphism s of $H(G)$ induces an automorphism $\pi(s)$ of V and an automorphism of $\mathbb{T}$. Note that $\mathrm{Aut}(\mathbb{T}) \cong \{\pm 1\}$ with non-trivial one given by $t \mapsto \overline{t}$ (Exercise 1).

Let s be an automorphism of the Heisenberg group $H(G)$ and suppose that s induces the identity on $\mathbb{T}$. Then $s(U(v)) = f(v)U(v\sigma)$ for $\sigma = \pi(s)$ and $f(v) \in \mathbb{T}$. We write $s = (\sigma, f)$, which determines s. If $s = (\sigma, f)$ and $s' = (\sigma', f')$, then $s's(U(v)) = s'(f(v)U(v\sigma)) = f(v)f'(v\sigma)U(v\sigma\sigma')$, and thus we have

$$(\sigma', f'(v))(\sigma, f(v)) = (\sigma' \circ \sigma, f(v)f'(v\sigma)). \tag{2.4}$$

Since

$$\begin{aligned} f(v)f(w)F(v\sigma, w\sigma)U(v\sigma + w\sigma) &= f(v)U(v\sigma)f(w)U(w\sigma) \\ = s(U(v))s(U(w)) &= s(U(v)U(w)) = F(v, w)f(v + w)U(v\sigma + w\sigma) \end{aligned}$$

by (2.3), we find

$$f(v + w)f(v)^{-1}f(w)^{-1} = F(v\sigma, w\sigma)F(v, w)^{-1}. \tag{2.5}$$

Thus f is a multiplicative quadratic form of V, and there is a unique homogeneous quadratic form f_σ satisfying (2.5). Moreover for any given multiplicative quadratic form f satisfying (2.5), $s = (\sigma, f)$ gives an element in $B(G)$ (Exercise 1). Since the left-hand-side of the above formula is symmetric with respect to v and w, we find also

$$F(v\sigma, w\sigma)F(v, w)^{-1} = F(w\sigma, v\sigma)F(w, v)^{-1}.$$

Since $J(v, w) = F(v, w)F(w, v)^{-1}$, σ preserves the symplectic form J; so, $\pi(s) \in Sp(G)$. We write $B(G)$ for the automorphism group of $H(G)$ which induce the identity on $\mathbb{T}$. We have the projection $\pi : B(G) \to Sp(G)$ and $B(G) = Sp(G) \ltimes V^*$ by $\sigma \mapsto (\sigma, f_\sigma)$.

Exercises

1. Show that for any multiplicative quadratic form f satisfying (2.5), $U(v) \mapsto f(v)U(v\sigma)$ gives an automorphism of $H(G)$.

2. Give a detailed proof of the fact that $U(v)\phi \in L^2(G)$ if $\phi \in L^2(G)$. Also prove that for the L^2–norm $\| \phi \|^2 = \int_G |\phi(g)|^2 dg$, $\| U(v)\phi \| = \| \phi \|$.

3. When $\sigma = \left(\begin{smallmatrix} 1 & \rho \\ 0 & 1 \end{smallmatrix}\right)$ for a symmetric $\rho \in \mathrm{Hom}_{cont}(G, G^*)$, show that $f_\sigma = f_\rho$, where $f_\rho(v) = \langle g, 2^{-1}g\rho\rangle$ for $v = (g, g^*)$.

2.4 Metaplectic Cover

Here is an important theorem of A. Weil [We1] Theorem 1, which we do not prove, because the proof uses techniques from functional analysis and harmonic analysis on locally compact groups (that will not be used later).

Theorem 2.1. *Let $\mathbb{B}(G)$ be the normalizer of $H(G)$ in $\mathrm{Aut}(L^2(G))$. Then we have a canonical central exact sequence:*

$$1 \to \mathbb{T} \to \mathbb{B}(G) \xrightarrow{\mu} B(G) \to 1.$$

We now define the metaplectic group $Mp(G)$ by

$$Mp(G) = \{s \in \mathbb{B}(G) | \mu(s) = (\sigma, f_\sigma) \text{ for } \sigma \in Sp(G)\}. \tag{2.6}$$

By definition, $Mp(G)$ is a central extension of $Sp(G)$; so,

$$1 \to \mathbb{T} \to Mp(G) \xrightarrow{\pi} Sp(G) \to 1$$

is exact. For general G, the above extension is non-trivial. However over some subsets of $Sp(G)$, one can have a canonical section r of π. We now define some sections. Let

$$U(G) = \left\{ \left(\begin{smallmatrix} 1 & \rho \\ 0 & 1 \end{smallmatrix}\right) \in Sp(G) \middle| \rho \in \mathrm{Hom}_{cont}(G, G^*) \right\}.$$

Since $U(G)$ is a subgroup of $Sp(G)$, ρ is a symmetric homomorphism; so, we have the associated multiplicative quadratic form: $f_\rho(g) = \langle g, 2^{-1}g\rho\rangle$. Then we define a section $r : U(G) \to B(G)$ by

$$r\left(\left(\begin{smallmatrix} 1 & \rho \\ 0 & 1 \end{smallmatrix}\right)\right) = \left(\left(\begin{smallmatrix} 1 & \rho \\ 0 & 1 \end{smallmatrix}\right), f_\rho\right) \in B(G).$$

We extends this section to $\mathbf{r} : U(G) \to Mp(G)$ by

$$\left(\mathbf{r}\left(\left(\begin{smallmatrix} 1 & \rho \\ 0 & 1 \end{smallmatrix}\right)\right)\phi\right)(g) = \phi(g)f_\rho(g) \text{ for } \phi \in L^2(G).$$

We define another subgroup $L(G)$ of $Sp(G)$:

$$L(G) = \left\{ \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-*} \end{smallmatrix}\right) \middle| a \in \mathrm{Aut}(G) \right\}.$$

Then we define a section $r : L(G) \to B(G)$ by

$$r\left(\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-*} \end{smallmatrix}\right)\right) = \left(\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-*} \end{smallmatrix}\right), 1\right) \in B(G).$$

Again we extend this section to $\mathbf{r} : L(G) \to Mp(G)$ by

$$\left(\mathbf{r}\left(\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-*} \end{smallmatrix}\right)\right)\phi\right)(g) = \sqrt{|a|}\phi(ga).$$

Finally for any continuous isomorphism $c : G^* \cong G$, we define

$$\left(\mathbf{r}\left(\left(\begin{smallmatrix} 0 & -c^{-*} \\ c & 0 \end{smallmatrix}\right)\right)\phi\right)(g) = \sqrt{|c|}^{-1}\mathcal{F}(\phi)(-xc^{-*}).$$

Here we have fixed once and for all a Haar measure dg on G and $\mathcal{F}$ is the Fourier transform on $L^2(G)$. The module $|c|$ is defined with respect to dg and its dual measure dg^*.

Let $\Omega = \Omega(G)$ be the collection of all $\sigma = \left(\begin{smallmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{smallmatrix}\right) \in Sp(G)$ with $c_\sigma : G^* \cong G$. Since

$$\sigma = \left(\begin{smallmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & a_\sigma c_\sigma^{-1} \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & -c_\sigma^{-*} \\ c_\sigma & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & c_\sigma^{-1}d_\sigma \\ 0 & 1 \end{smallmatrix}\right),$$

we can extend $\mathbf{r}$ to $\mathbf{r} : \Omega \to Mp(G)$. In particular, we have

$$(\mathbf{r}(\sigma)\phi)(g) = |c_\sigma|^{1/2}\int_G \phi(ga_\sigma + g^*c_\sigma)f_\sigma(g, g^*)dg^*.$$

From this, it is easy to check that $\mathbf{r}(p)\mathbf{r}(\sigma)\mathbf{r}(p') = \mathbf{r}(p\sigma p')$ for $p, p' \in P(G)$ and $\sigma \in \Omega$ (cf. [Sh2] (1.3a,b,c)).

Exercises

1. Check that the sections r and $\mathbf{r}$ on $U(G)$ and $L(G)$ are group homomorphisms.

2. Let $P(G)$ be the subgroup of $Sp(G)$ generated by $U(G)$ and $L(G)$. Show that $U(G)$ is a normal subgroup of $P(G)$ and $P(G) = L(G) \ltimes U(G)$. Further show that $\mathbf{r}$ extends to a section $r : P(G) \to Mp(G)$ in an obvious manner, which is a group homomorphism.

3. Show that for each $p \in P(G)$ that $\| \mathbf{r}(p)\phi \| = \| \phi \|$ $(\phi \in L^2(G))$.

4. Give a detailed proof of $\mathbf{r}(p)\mathbf{r}(\sigma)\mathbf{r}(p') = \mathbf{r}(p\sigma p')$ for $p, p' \in P(G)$ and $\sigma \in \Omega$.

2.5 Sections over discrete and compact subgroups of $Sp(G)$

Let $\Gamma \subset G$ be a closed subgroup of G. Then G/Γ is again a locally compact abelian group under the quotient topology (Exercise 1). Then the exact sequence

$$0 \to \Gamma \to G \to G/\Gamma \to 0$$

yields, by the perfect Pontryagin duality, another exact sequence:

$$0 \to (G/\Gamma)^* \to G^* \to \Gamma^* \to 0.$$

Writing $\Gamma^\perp$ for the image of $(G/\Gamma)^*$, we thus obtain

$$\Gamma^\perp = \left\{\gamma^* \in G^* \middle| \langle \Gamma, \gamma^* \rangle = 0\right\}.$$

We suppose the following hypothesis:

(H1) Γ *is either compact or discrete;*

(H2) G/Γ *is discrete (resp. compact) if* Γ *is compact (resp. discrete).*

Once we start with $(G/\Gamma, \Gamma)$ as above, its dial $(G^*/\Gamma^\perp, \Gamma^\perp)$ is again the same type (Exercise 1 in Subsection 1.1).

Example 2.2. 1. If $G = X$ is a finite dimensional vector space over $\mathbb{R}$, a lattice L is a subgroup spanned by a base of X over $\mathbb{R}$. Then L is a discrete subgroup of X and X/L is compact. If we fix a dual pairing $(\ ,\ ) : X \times X \to \mathbb{R}$, the dual lattice $L^\perp$ with respect to $\langle x, y\rangle = \mathbf{e}_\infty((x, y))$ is the dual lattice $L^\perp = \{x \in X | (L, x) \subset \mathbb{Z}\}$.

2. If $G = X$ is a finite dimensional vector space over $\mathbb{Q}_p$, a lattice L of X is a $\mathbb{Z}_p$–submodule spanned by a base of X over $\mathbb{Q}_p$. Then L is compact, and X/L is discrete. Again fixing a non-degenerate bilinear pairing $(\ ,\ ) : X \times X \to \mathbb{Q}_p$ which gives rise to the self-duality of X: $\langle x, y\rangle = \mathbf{e}_p((x, y))$, the $\mathbb{Z}_p$–dual lattice gives $L^\perp$.

3. Let X be a finite dimensional vector space over a number field F (of finite degree). Then $X_\mathbb{A} = X \otimes_\mathbb{Q} \mathbb{A}$ for the adele ring $\mathbb{A}$ is a locally compact abelian group, and X is a discrete subgroup of $X_\mathbb{A}$ and $X_\mathbb{A}/X$ is compact. Fixing a non-degenerate F–bilinear pairing $(\ ,\ )$ on X and extend it $F_\mathbb{A}$–linearly to $X_\mathbb{A}$, we have the dual pairing $\langle x, y\rangle = \mathbf{e}_\mathbb{A}(\mathrm{Tr}_{F/\mathbb{Q}}(x, y))$. In this case, $X^\perp = X$.

We write $\dot{x}$ for the coset $x + \Gamma$ in G/Γ. Then we choose the canonical Haar measure $d\gamma$ on Γ so that $\int_\Gamma d\gamma = 1$ if Γ is compact and $\int_\Gamma \phi d\gamma = \sum_{\gamma\in\Gamma} \phi(\gamma)$. Similarly we choose the canonical Haar measure $d\dot{g}$ on G/Γ. Since the pair $(G^*/\Gamma^\perp, \Gamma^\perp)$ satisfies the same properties (H1-2), we have the Haar measures $d\gamma^\perp$ and $d\dot{g}^*$ on $(\Gamma^\perp, G^*/\Gamma^\perp)$. Then we consider the integration:

$$\int_G \phi(g) dg := \int_{G/\Gamma} \int_\Gamma \phi(\gamma + g) d\gamma d\dot{g}. \tag{2.7}$$

Obviously this integration is given by a Haar measure on G. Similarly we define a Haar measure dg^* on G^* by

$$\int_{G^*} \phi(g^*) dg^* := \int_{G^*/\Gamma^\perp} \int_{\Gamma^\perp} \phi(\gamma^\perp + g^*) d\gamma^\perp d\dot{g}^*. \tag{2.8}$$

We now define a partial Fourier transform $\Theta(\phi) : G \times G^* \to \mathbb{C}$ for $\phi \in L^1(G)$ by

$$\Theta(\phi)(g, g^*) = \int_\Gamma \phi(g+\gamma)\langle \gamma, g^* \rangle d\gamma. \tag{2.9}$$

By definition, we see

$$\Theta(\phi)(g+\gamma, g^* + \gamma^\perp) = \Theta(\phi)(g, g^*)\langle \gamma, g^* \rangle^{-1} \quad \text{for all } (\gamma, \gamma^\perp) \in \Gamma \times \Gamma^\perp$$

because $\langle \gamma, \gamma^\perp \rangle = 1$. We define $\mathcal{L}^2(G/\Gamma \times G^*/\Gamma^\perp)$ to be the space of functions $\Phi(g, g^*)$ on $G \times G^*$ such that

- $\Phi(g+\gamma, g^*+\gamma^\perp) = \Phi(g, g^*)\langle \gamma, g^* \rangle$ for all $(\gamma, \gamma^\perp) \in \Gamma \times \Gamma^\perp$,
- $|\Phi|$ is square integrable as a function on $G/\Gamma \times G^*/\Gamma^\perp$.

Then $\mathcal{L}^2(G/\Gamma \times G^*/\Gamma^\perp)$ is a Hilbert space under the norm given by

$$\| \Phi \|^2 = \int_{G/\Gamma \times G^*/\Gamma^\perp} |\Phi(\dot{g}, \dot{g}^*)|^2 d\dot{g} d\dot{g}^*.$$

Let $\phi_g(\gamma) = \phi(g+\gamma)$ as a function of Γ. Then by the Plancherel formula, we have, for a fixed $g \in G$

$$\| \phi \|^2 = \| \phi_g \|^2 = \int_{G/\Gamma} \int_\Gamma |\phi_g|^2 d\gamma d\dot{g} = \int_{G/\Gamma} \| \phi_g^* \|^2 \, d\dot{g} = \| \Theta(\phi)(g, \dot{g}^*) \|^2,$$

where $\| \Phi \|^2 = \int_{G/\Gamma \times G^*/\Gamma^\perp} |\Phi(\dot{g}, \dot{g}^*)|^2$. Thus $\phi \mapsto \Theta(\phi)$ preserves the mertic. Since $L^1(G) \cap L^2(G)$ is dense in $L^2(G)$, the linear map Θ extends to an isometry of $L^2(G)$ onto $L^2(G/\Gamma \times G^*/\Gamma^\perp)$ (surjectivity follows from the Fourier inversion formula). We thus have

$$\Theta : L^2(G) \cong \mathcal{L}^2(G/\Gamma \times G^*/\Gamma^\perp).$$

Since $Mp(G)$ and $\mathbb{B}(G)$ acts on $L^2(G)$ via the unitary representation we have constructed, these groups act at the same time on $L^2(G/\Gamma \times G^*/\Gamma^\perp)$ via the intertwining operator Θ.

On the other hand, if we write

$$Sp_\Gamma(G) = \left\{ \sigma \in Sp(G) \middle| (\Gamma \times \Gamma^\perp)\sigma = (\Gamma \times \Gamma^\perp) \right\}, \tag{2.10}$$

the group $Sp_\Gamma(G)$ acts on $\mathcal{L}^2(G/\Gamma \times G^*/\Gamma^\perp)$ naturally in the following manner:

$$\mathbf{r}_\Gamma(\sigma)\phi((g, g^*)) = \phi((g, g^*)\sigma) f_\sigma(g, g^*). \tag{2.11}$$

One can easily check using the fact:

$$f_\sigma(v+w) f_\sigma(v)^{-1} f_\sigma(w)^{-1} = F(w\sigma, v\sigma) F(w, v)^{-1}$$

that $\mathbf{r}_\Gamma(\sigma)\phi \in \mathcal{L}^2(G/\Gamma \times G^*/\Gamma^\perp)$ (Exercise 5). We would like to compare the two actions of $Sp_\Gamma(G)$.

Recall that $U(v)\phi(g,g^*) = \phi(g+x)\langle g, x^*\rangle$ for $v = (x,x^*) \in V = G \times G^*$. Then we see by computation, writing $w = (g,g^*)$

$$\begin{aligned}\Theta((U(v)\phi)(g,g^*)) &= \int_\Gamma \phi(g+\gamma+x)\langle g+\gamma, x^*\rangle\langle\gamma, g^*\rangle d\gamma \\ &= \langle g, x^*\rangle\Theta(\phi)(w+v) = F(w,v)\Theta(\phi)(w+v).\end{aligned}$$

Thus defining $U_\Theta(v)\Phi(w) = F(w,v)\Phi(w+v)$, we have the following commutative diagram:

$$\begin{array}{ccc} L^2(G) & \xrightarrow{\Theta} & \mathcal{L}^2(G/\Gamma \times G^*/\Gamma^\perp) \\ {\scriptstyle U(v)}\downarrow & & \downarrow{\scriptstyle U_\Theta(v)} \\ L^2(G) & \xrightarrow[\Theta]{} & \mathcal{L}^2(G/\Gamma \times G^*/\Gamma^\perp). \end{array}$$

By definition of the action of $U_\Theta(v)$ and $\mathbf{r}_\Gamma(\sigma)$, we see

$$\begin{aligned}U_\Theta(v)\mathbf{r}_\Gamma(\sigma)\phi(w) =& U_\Theta(v)\Phi(w\sigma)f_\sigma(w) \\ =& F(w,v)\Phi(w\sigma+v\sigma)f_\sigma(w+v) \\ \stackrel{(*)}{=}& \phi(w\sigma+v\sigma)f_\sigma(w)f_\sigma(v)F(w\sigma,v\sigma).\end{aligned}$$

where at the last equality $(*)$, we have used the following identity:

$$f_\sigma(w+v)f_\sigma(w)^{-1}f_\sigma(v)^{-1} = F(w\sigma,v\sigma)F(w,v)^{-1}.$$

We compute also:

$$\begin{aligned}f_\sigma(v)\mathbf{r}_\Gamma(\sigma)U(v\sigma)\phi(w) &= f_\sigma(v)\mathbf{r}_\Gamma(\sigma)(\phi(w+v\sigma)F(w,v\sigma)) \\ &= f_\sigma(v)f_\sigma(w)\phi(w\sigma+v\sigma)F(w\sigma,v\sigma).\end{aligned}$$

Thus we get

$$U_\Theta(v)\mathbf{r}_\Gamma(\sigma) = f_\sigma(v)\mathbf{r}_\Gamma(\sigma)U_\Theta(w\sigma). \tag{2.12}$$

From this, we conclude

Theorem 2.2. *For a subgroup $\Gamma \subset G$ satisfying* (H1-2)*, we have a section $\mathbf{r}_\Gamma : Sp_\Gamma(G) \to Mp(G)$ which coincides with $\mathbf{r}$ on $\Omega \cap \Gamma$.*

Exercises

1. Prove that the quotient of a locally compact abelian group by a closed subgroup is again locally compact.
2. Give a detailed proof of the assertions in Example 2.2.
3. Show that dg and dg^* defined by (2.7) and (2.8) are mutually dual (cf. Exercise 4 in Subsection 1.3).
4. Prove the integral (2.9) converges if ϕ is integrable on G.
5. Show $\mathbf{r}_\Gamma(\sigma)\phi \in \mathcal{L}^2(G/\Gamma \times G^*/\Gamma^\perp)$ if $\phi \in \mathcal{L}^2(G/\Gamma \times G^*/\Gamma^\perp)$.

2.6 Theta Series

Let X be a finite dimensional vector space over a number field F. We now assume that $G = X_p = X \otimes_{\mathbb{Q}} \mathbb{Q}_p$ or $X_\infty = X \otimes_{\mathbb{Q}} \mathbb{R}$ or $X_{\mathbb{A}} = X \otimes_{\mathbb{Q}} \mathbb{A}$. For X_∞, we define $\mathcal{S}(X_\infty)$ to be the Schwartz space of functions on X_∞. Thus $\mathcal{S}(X_\infty)$ is made of C^∞–class functions with all derivatives rapidly decreasing as Euclidean norm of $x \in X_\infty$ grows. In other words, $\phi \in \mathcal{S}(X_\infty)$ if and only if ϕ is of C^∞–class and for any polynomial $P(x)$ and any m–th derivative Φ of ϕ. $|P(x)\Phi(x)|$ goes to 0 as $|x| \to \infty$.

When $G = X_p$, we write $\mathcal{S}(X_p)$ for the space of Bruhat functions on X_p, which are locally constant with compact support. When $G = X_{\mathbb{A}}$, $\mathcal{S}(X_{\mathbb{A}})$ is the space of Schwartz-Bruhat functions on $X_{\mathbb{A}}$, which are spanned by the product $\phi((x_v)_v) = \prod_v \phi_v(x_v)$ with $\phi_v \in \mathcal{S}(X_v)$ and such that for almost all henselian p, ϕ_v is the characteristic function of a $\mathbb{Z}_p$–lattice $L_p = L \otimes_{\mathbb{Z}} \mathbb{Z}_p$ for a lattice $L \subset X$.

It is well know that the Fourier transform $\mathcal{F}$ gives an isomorphism $\mathcal{F} : \mathcal{S}(G) \cong \mathcal{S}(G^*)$ (cf. [IGA]). Then by definition of $\mathbf{r}(s)$, this operator preserves the space $\mathcal{S}(G)$ for $s \in \Omega$. Since Ω generates $Mp(G)$ for G as above, we know that the action of $Mp(X)$ preserves $\mathcal{S}(G) \overset{\text{dense}}{\subset} L^2(G)$ (see [We1] No.11-13 for more details).

We now prove the following generalized Poisson summation formula of Weil:

Theorem 2.3. *Let G be as above. Suppose $\Phi \in \mathcal{S}(G)$. Then we have*

$$\int_\Gamma \Phi(\gamma) d\gamma = \int_\Gamma (s\Phi)(\gamma) d\gamma$$

for all $s \in Sp_\Gamma(G)$.

Proof. We consider $\Theta(\Phi)$. Recall

$$\Theta(\Phi)(g, g^*) = \int_\Gamma \Phi(\gamma + g, g^*) \langle \gamma, g^* \rangle d\gamma.$$

Since $\mathbf{r}_\Gamma(\sigma)\Theta(\Phi)(v) = \Theta(\Phi)(v\sigma) f_\sigma(v)$, we find that

$$\int_\Gamma (\mathbf{r}_\Gamma(\sigma)\Phi)(\gamma) d\gamma = \mathbf{r}_\Gamma(\sigma)\Theta(\Phi)(0) = \Theta(\Phi)(0\sigma) f_\sigma(0) = \Theta(\Phi)(0) = \int_\Gamma \Phi(\gamma) d\gamma.$$

This shows the desired formula. The requirement $\Phi \in \mathcal{S}(G)$ is necessary to guarantee that $\Phi(\gamma)$ is well defined for all $\gamma \in \Gamma$. $\square$

We now assume that G is either X_∞ or $X_{\mathbb{A}}$. Thus Γ is discrete and is a lattice $L \subset X$ or $X \subset X_{\mathbb{A}}$. We consider the function:

$$\Theta(\Phi)(s) = \int_\Gamma (s\Phi)(\gamma) d\gamma = \sum_{\gamma \in \Gamma} (s\Phi)(\gamma)$$

as a function of $s \in Mp(G)$. Then by the above theorem, we find for $\xi \in Sp_\Gamma(G)$, $\Theta(\Phi)(\xi s) = \Theta(\Phi)(s)$. Note that, by identifying X with $\mathbb{Q}^n$, we find $Sp(X_{\mathbb{A}}) =$

$Sp_{2n}(\mathbb{A})$ and $Sp_X(X_{\mathbb{A}}) = Sp_{2n}(\mathbb{Q})$. Moreover, for $\sigma = (\sigma_v) \in Sp_{2n}(\mathbb{A})$, we see by definition, if $\Phi = \prod_v \Phi_v$ with $\Phi_v \in \mathcal{S}(X_v)$,

$$\mathbf{r}((\sigma_v))\Phi = \prod_v \mathbf{r}(\sigma_v)\Phi_v$$

as long as $\sigma_v \in P(X_v)$ or $\Omega(X_v)$. From this fact, we can easily conclude that for an open compact subgroup S of $Sp_{2n}(\mathbb{A}^{(\infty)})$, $\Theta(\Phi)(su) = \Theta(\Phi)(s)$ for $u \in \mathbf{r}_S(S)$. Thus $s \mapsto \Theta(\Phi)(s)$ is an automorphic form in a broad sense that they are functions on $Sp_{2n}(\mathbb{Q})\backslash \mathbf{M}_{\mathbb{A}}/S$ if $G = X_{\mathbb{A}}$ or $G = X_\infty$. We shall show in the following subsection that $\Theta(\Phi)$ gives basically all known theta series as automorphic forms on the metaplectic group $\mathbf{M}_{\mathbb{A}} = Mp(X_{\mathbb{A}})$ or $\mathbf{M}_\infty = Mp(X_\infty)$.

We are going to make explicit the form of $\Theta(\Phi)$. Suppose first that $X = \mathbb{Q}_n$ (row vector space of dimension n), and identify X with its $\mathbb{Q}$–dual by $(w, v) = w^t v$. Let $H = H_n = \{z \in \mathbb{C}_n^n | {}^t z = z,\ z = x + iy,\ y > 0\}$ (Siegel upper half space). We consider the Schwartz function $\varphi(v; (z, u)) = \exp(\pi i v z^t v + 2\pi i v u)$ defined on $v \in X_\infty$, $z \in H$ and $u \in \mathbb{C}^n$ (column vector space). We note $\mathcal{F}(\varphi(v; (i1_n, 0)) = \varphi(v; (i1_n, 0))$, and hence, writing $y = (y^{1/2})^2$ for a positive symmetric matrix, we have

$$\begin{aligned}
\mathcal{F}(\varphi) &= \left(\mathbf{r}\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\right)\varphi\right)(v; (iy, u)) \\
&= \int_{X_\infty} \exp(-\pi w y^t w) \exp(2\pi i w({}^t v + u)) dw \\
&\overset{wy^{1/2} \mapsto w}{=} \det(y)^{-1/2} \int_{X_\infty} \exp(-\pi w^t w) \exp(2\pi i y^{-1/2} w({}^t v + u)) dw \\
&= \det(y)^{-1/2} \exp(-\pi(v + {}^t u) y^{-1}({}^t v + u)) \\
&= \det(y)^{-1/2} \varphi(v; (iy^{-1}, iy^{-1}u)) \exp(-\pi {}^t u y^{-1} u)
\end{aligned}$$

For $\sigma = \left(\begin{smallmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{smallmatrix}\right) \in Sp_{2n}(\mathbb{R})$, following [Sh1] (1.7) and (1.11):

$$\begin{aligned}
\sigma(z, u) &= ((a_\sigma z + b_\sigma)(c_\sigma z + d_\sigma)^{-1}, {}^t(c_\sigma + d_\sigma)^{-1} u) \\
\zeta_\sigma(z, u) &= \exp(\pi i \cdot {}^t u (c_\sigma z + d_\sigma)^{-1} c_\sigma u).
\end{aligned} \tag{2.13}$$

Using this notation, the above computation yields for $z = iy$

$$\begin{aligned}
\mathcal{F}(\varphi)(v; (z, u)) &= \left(\mathbf{r}\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\right)\varphi\right)(v; (z, u)) \\
&= \det(-iz)^{-1/2} \zeta_\eta(z, u)^{-1} \varphi(v; \eta(z, u)),
\end{aligned} \tag{2.14}$$

where $\eta = \left(\begin{smallmatrix} 0 & -1_n \\ 1_n & 0 \end{smallmatrix}\right)$. By definition, $\mathcal{F}(\varphi)(v; (z, u))$ is a holomorphic function of (z, u). Since $\{iy | y \in \mathbb{R}_n^n,\ {}^t y = y > 0\}$ is a Zariski dense subset of H_n (Exercise 1), the above identity (2.14) has to be true for all $(z, u) \in H_n \times \mathbb{C}^n$. Similarly, we can verify for $p = \left(\begin{smallmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{smallmatrix}\right)$,

$$(\mathbf{r}(p)\varphi)(v; (z, u)) = |\det(a)|^{1/2} \varphi(v; p(z, u))$$

and for $\alpha = \left(\begin{smallmatrix} 1_n & b \\ 0 & 1_n \end{smallmatrix}\right)$

$$(\mathbf{r}(\alpha)\varphi)\,(v;(z,u)) = \exp(\pi v b \cdot {}^t v)\varphi(v;(z,u)) = \varphi(v;\alpha(z,u)).$$

Since $Mp(X_\infty)$ is generated by $\mathbb{T}$ and matrices of the form: η, p and α, we find a holomorphic function $h(s,z) : Mp(X_\infty) \times H_n \to \mathbb{C}$ with the following property (see [Sh1] Proposition 3.1):

Proposition 2.4. *Let $\sigma = \pi(s) \in Sp_{2n}(\mathbb{R})$.*

1. $s\varphi(v;z,u) = h(s,z)^{-1}\zeta_\sigma(z,u)^{-1}\varphi(v;\sigma(z,u))$*;*
2. $h(st,z) = h(s,t(z))h(t,z)$ *for all* $s,t \in Mp(X_\infty)$*;*
3. $h(s,z)^2 = t \cdot \det(c_\sigma z + d_\sigma)$ *for* $t \in \mathbb{T}$*;*
4. $h(s,z)^4 = (-1)^n \cdot \det(c_\sigma z + d_\sigma)^2$ *if* $s = \mathbf{r}(\sigma)$ *for* $\sigma \in \Omega$*.*

Proof. All the assertions except for (2) has already been proven. Thus we need to show the automorphic property: $\zeta_{\sigma\tau}(z,u) = \zeta_\sigma(\tau(z,u))\zeta_\tau(z,u)$ for $\sigma,\tau \in Sp_{2n}(\mathbb{R})$. We define $g(z,u) = \exp(\pi i {}^t u (z-\overline{z})^{-1} u)$. Since

$$\sigma \left(\begin{smallmatrix} z & \overline{z} \\ 1_n & 1_n \end{smallmatrix}\right) = \begin{pmatrix} \sigma(z) & \sigma(\overline{z}) \\ 1_n & 1_n \end{pmatrix} \left(\begin{smallmatrix} c_\sigma z + d_\sigma & 0 \\ 0 & c_\sigma \overline{z} + d_\sigma \end{smallmatrix}\right)$$

and ${}^t\sigma\eta\sigma = \eta$, we have for $T = \left(\begin{smallmatrix} z & \overline{z} \\ 1_n & 1_n \end{smallmatrix}\right)$

$$\begin{aligned} \left(\begin{smallmatrix} 0 & z-\overline{z} \\ \overline{z}-z & 0 \end{smallmatrix}\right) &= {}^tT\eta T = {}^t(\sigma T)\eta\sigma T \\ &= \begin{pmatrix} 0 & {}^t(c_\sigma z + d_\sigma)(z-\overline{z})(c_\sigma \overline{z} + d_\sigma) \\ {}^t(c_\sigma \overline{z} + d_\sigma)(z-\overline{z})(c_\sigma z + d_\sigma) & 0 \end{pmatrix}. \end{aligned}$$

From this, we find $g(\sigma(z,u)) = \zeta_\sigma(z,u)^{-1}g(z,u)$, and hence we get the desired assertion. ☐

Let $\Gamma = \mathbb{Z}^n \subset X$. It is now an easy exercise to see

$$\int_\Gamma \varphi(\gamma;(z,u))d\gamma = \sum_{m \in \mathbb{Z}^n} \varphi(m;(z,u)) = \sum_m \exp(\pi i m z {}^t m + 2\pi i m u) = \theta(z,u)$$

is the standard Siegel modular theta function, ant the generalized Poisson summation formula of Weil includes as a special case the transformation formula of this theta function.

We consider the set $\mathcal{G}$ made up of pairs $(\sigma, j_\sigma(z))$ with $\sigma \in Sp_{2n}(\mathbb{R})$ and a holomorphic functions $j_\sigma : H_n \to \mathbb{C}$ such that $j_\sigma^2 = t \cdot \det(c_\sigma z + d_\sigma)$ for $t \in \mathbb{T}$. We make $\mathcal{G}$ into a group by the multiplication (cf. [Sh1] (1.5)):

$$(\sigma, j_\sigma)(\tau, j_\tau) = (\sigma\tau, j_\sigma(\tau(z))j_\tau(z)).$$

Then we have the following exact sequence:

$$1 \to \mathbb{T} \xrightarrow{t \mapsto (1,t)} \mathcal{G} \xrightarrow{(\sigma, j_\sigma) \mapsto \sigma} Sp_{2n}(\mathbb{R}) \to 1.$$

Corollary 2.5. *The map $\iota : Mp(X_\infty) \to \mathcal{G}$ given by $s \mapsto (\pi(s), h(s,z))$ gives an isomorphism of groups.*

Proof. By Proposition 2.4 (1) and (2), ι is a homomorphism sending isomorphically $\mathbb{T} \subset Mp(X_\infty)$ onto $\mathbb{T} \subset \mathcal{G}$ and inducing an isomorphism to the quotient $Sp_{2n}(\mathbb{R}) = Mp(X_\infty)/\mathbb{T} = \mathcal{G}/\mathbb{T}$; so, it is an isomorphism. $\square$

Exercises

1. Show that $f = 0$ if a meromorphic function $f : H_n \to \mathbf{P}^1_{/\mathbb{C}}$ vanishes on $Y = \{iy | y \in \mathbb{R}^n_n,\ {}^t y = y > 0\} \subset H_n$.
2. Give a detailed proof of Proposition 2.4.
3. Check that the multiplicatin given above makes $\mathcal{G}$ into a group.
4. Show that $\varphi(v; (z,u)) \in \mathcal{S}(X_\infty)$ as a function of v for a fixed $(z,u) \in H_n \times \mathbb{C}^n$.

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