

Metaplectic Groups

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1 Fourier Analysis

Let G be a locally compact abelian group; so, each $x \in G$ has an open neighborhood whose closure is compact, and the addition $(x, y) \mapsto x + y$ is a continuous map of $G \times G$ into G and $x \mapsto -x$ is a homeomorphism of G . We always assume that $x \mapsto 2x$ is an automorphism of topological group. We study Fourier analysis on G and its Pontryagin dual G^* in this section. In particular, we are going to prove the Plancherel formula:

$$\int_G |\Phi(x)|^2 dx = \int_{G^*} |\Phi^*(x)|^2 dx^*$$

as long as the Fourier transform Φ^* is well defined for a given measurable function Φ on G .

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1.1 Pontryagin duality

Let \mathbb{T} be the multiplicative group of complex numbers of absolute value 1, and we often identify \mathbb{T} with \mathbb{R}/\mathbb{Z} by $\mathbb{R}/\mathbb{Z} \ni r \mapsto \exp(2\pi ir) \in \mathbb{T}$. We then define $G^* = \text{Hom}_{\text{cont}}(G, \mathbb{T})$. We equip G^* with the uniform convergence topology over any compact subset of G . Then G^* again become a locally compact abelian group (cf. [IGA] and [TGP]). Thus the neighborhood of the trivial character $\mathbf{0}$ of G is given by $\{\phi \in G^* | \phi(B) \subset W\}$ for a compact subset $B \subset G$ and an open neighborhood W of 1 in \mathbb{T} . The character $\mathbf{0}$ satisfies $\mathbf{0}(x) = 1 \in \mathbb{T}$. We write $\langle x, x^* \rangle$ for $x^*(x) \in \mathbb{T}$, where $x^* \in G^*$ and $x \in G$. We can then define a homomorphism $G \rightarrow (G^*)^*$ by sending $x \in G$ to a character $x^{**} : G^* \rightarrow \mathbb{T}$ given by

$$x^{**}(x^*) = \langle x, x^* \rangle.$$

More generally, if $\phi : H \rightarrow G$ be a homomorphism of locally compact abelian groups (that is, a continuous homomorphism), we have a dual map $\phi^* : G^* \rightarrow H^*$ given by

$$\langle h, \phi^*(g^*) \rangle = \langle \phi(h), g^* \rangle.$$

This duality theory $G \mapsto G^*$ of locally compact abelian groups is a perfect duality called *Pontryagin duality* of locally compact abelian groups and was developed by Pontryagin in 1938 and Weil in 1940 independently. The perfectness of the duality implies

- $(G^*)^* \cong G$ by $x \mapsto x^{**}$;
- If $0 \rightarrow H \xrightarrow{\phi} G \xrightarrow{\psi} K \rightarrow 0$ is an exact sequence of locally compact abelian groups, then the dual sequence $0 \rightarrow K^* \xrightarrow{\psi^*} G^* \xrightarrow{\phi^*} H^* \rightarrow 0$ is also exact.

For all this type of results, see either the book of Pontryagin [TGP] or by Weil [IGA].

Many locally compact groups are isomorphic to their dual.

Example 1.1. 1. The pairing $\langle x, y \rangle = \exp(2\pi i axy)$ for any non-zero real number $a \neq 0$ gives the self-duality of the additive group \mathbb{R} . We write $\mathbf{e}_\infty(x) = \exp(2\pi ix)$.

2. Similarly, expanding $x \in \mathbb{Q}_p$ into a p -adic expansion $x = \sum_{n \gg -\infty} c_n p^n$ with integers $0 \leq c_n < p$ and defining a rational number of p -power denominator $[x]_p = \sum_{n < 0} c_n p^n \in \mathbb{Q}$, the pairing $\langle x, y \rangle = \exp(-2\pi i [axy]_p)$ for any non-zero p -adic number $a \in \mathbb{Q}_p$ gives a self duality of the additive group \mathbb{Q}_p . We write $\mathbf{e}_p(x) = \exp(2\pi i [x]_p)$.

3. For $x = (x_v), y = (y_v) \in \mathbb{A}$, we can define $\langle x, y \rangle = \langle x_\infty, y_\infty \rangle \prod_p \langle x_p, y_p \rangle$ gives a self duality of \mathbb{A} if we choose a in \mathbb{Q} in the examples (1) and (2). We write $\mathbf{e}_\mathbb{A}(x) = \prod_v \mathbf{e}_v(x_v)$ for $x = (x_v) \in \mathbb{A}$. Then \mathbf{e} induces a character $\mathbf{e} : \mathbb{A}/\mathbb{Q} \rightarrow \mathbb{T}$.

4. For any semi-simple algebra B over \mathbb{Q} , $B_A = B \otimes_{\mathbb{Q}} A$ for $A = \mathbb{R}, \mathbb{Q}_p$ and \mathbb{A} is a self dual additive group by $\langle x, y \rangle = \mathbf{e}_?(\text{Tr}_{B/\mathbb{Q}}(axy))$ for $a \in B^\times$, where $? = p, \infty, \mathbb{A}$ according as $A = \mathbb{Q}_p, \mathbb{R}$ and \mathbb{A} .

5. If $X = F^n$ is a finite dimensional vector space over a number field F and if $(\ , \)$ is either a non-degenerate symmetric or σ -hermitian form (with respect to $\sigma \in \text{Aut}(F)$ of order 2) on X , $X_A = X \otimes_{\mathbb{Q}} A$ is a self dual locally compact abelian group by $\langle x, y \rangle = \mathbf{e}_?(\text{Tr}_{F/\mathbb{Q}}(x, y))$.

Exercises

1. Show that G is compact $\Leftrightarrow G^*$ is discrete.
2. Give a proof of all the assertions in Example 1.1 (see [LFE] Section 8.3).
3. Show that $\mathbf{e}(\mathbb{Q}) = 1$ if we regard \mathbb{Q} as a subfield of \mathbb{A} diagonally.

1.2 Haar Measure

On any locally compact abelian group G , there exists a Harr measure dg with values in \mathbb{R} (see [IGA]) satisfying the following conditions:

1. $\int_X dg$ is defined for subset X in a complete additive class containing all compact subsets of G (that is, a union of countably many compact subsets is measurable);
2. For all compact subsets $K \subset G$, we have $0 \leq \int_K dg < +\infty$;
3. We have, for all open subsets $U \subset G$, $\int_U dg = \text{Sup}_{U \supset K: \text{compact}} \int_K dg$ and for all measurable subsets X , $\int_X dg = \text{Inf}_{U \supset X, U: \text{open}} \int_U dg$;
4. $\int_{x+X} dg = \int_X dg$ for all measurable $X \subset G$ and $x \in G$.

Out of this measure, we can construct the Lebesgue measure dg associated to dg as above. In particular, we can think of integrable functions and square integrable functions on G . By (4) as above, if dg' is another Haar measure on G , we have $\int \phi dg = c \int \phi dg'$ for a positive constant c independent of ϕ . If $\alpha : G \rightarrow H$ is an isomorphism of locally compact abelian groups, then $\phi \mapsto \int_G \phi(g\alpha) dg$ for an integrable function ϕ on H gives a Haar measure $d(g\alpha^{-1})$ on H . Then for a chosen Haar measure dh on H , we have a positive constant $|\alpha|$ dependent only on α and the choice of dg on G and dh on H such that $d(g\alpha^{-1}) = |\alpha|^{-1} dh$. In other words,

$$\int_H \phi(h) dh = |\alpha| \int_G \phi(g\alpha) dg.$$

When $H = G$, we choose $dh = dg$, then $|\alpha|$ is determined independently of the choice of dg .

Example 1.2. 1. When $G = \mathbb{Z}_p$, any compact set is a disjoint union of subset of the form $a + p^n \mathbb{Z}_p$; so, we just define $\int_{a+p^n \mathbb{Z}_p} dg = p^{-n}$. Then $\int_X dg = \sum_a p^{-n(a)}$ for $X = \bigsqcup_a a + p^{n(a)} \mathbb{Z}_p$. Any continuous function ϕ can be

written as $\phi = \lim_{n \rightarrow \infty} \phi_n$ for $\phi_n(x) = \phi(m)$ if $x \equiv m \pmod{p^n}$ for an integer m with $0 \leq m < p^n$. Then we see

$$\int \phi dg = \lim_{n \rightarrow \infty} \left(\sum_{j=0}^{p^n-1} \phi(j) p^{-n} \right).$$

2. If $G = \mathbb{Q}_p$, $G = \bigcup_n p^n \mathbb{Z}_p$. By the above argument, we have a Haar measure dg on each $p^n \mathbb{Z}_p$ so that they coincide with the one given on \mathbb{Z}_p . Thus this measure gives a unique Haar measure dg on G such that $\int_{p^n \mathbb{Z}_p} dg = p^{-n}$.
3. If $G = \mathbb{Z}$, we just define that $\int_x dg = 1$ for any $x \in \mathbb{Z}$. Then for any compact subset $K \subset \mathbb{Z}$, K is a finite set and $\int_K dg = |K|$. If $\phi : \mathbb{Z} \rightarrow \mathbb{C}$ is a function, then $\int_K \phi dg = \sum_{n \in K} \phi(n)$.
4. If $G = \mathbb{R}$, we have the classical Lebesgue measure dg with $\int_0^1 dg = 1$.
5. For any product $G = G_1 \times G_2 \times \cdots \times G_r$ of the above groups, the product measure $dg = dg_1 dg_2 \cdots dg_r$ gives a Haar measure of G . In particular, finite dimensional vector space over \mathbb{R} or \mathbb{Q}_p has such a measure.
6. For the adèle ring $G = \mathbb{A}$, we can define the measure dg so that if $\phi(x) = \prod_v \phi_v(x_v)$ for places v , we just define $\int \phi dg = \prod_v \int \phi_v dg_v$ for the measure dg_v on \mathbb{Q}_p if $v = p$ and the Lebesgue measure dg_∞ on \mathbb{R} . In particular, $\int_X dg = 1$ for $X = \widehat{\mathbb{Z}} \times [0, 1]$, where $\widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$.

1.3 Fourier Transform

Choose a Haar measure dg on G . We then define the Fourier transform $\mathcal{F}(\phi) = \phi^* : G^* \rightarrow \mathbb{C}$ of an integrable function $\phi : G \rightarrow \mathbb{C}$ by

$$\phi^*(g^*) = \mathcal{F}(\phi)(g^*) = \int_G \phi(g) \langle g, g^* \rangle dg.$$

See [IGA] Chapter 6 for a general theory of Fourier transform. We then choose a Haar measure dg^* on G^* and define the Fourier transform \mathcal{F}^* on G^* . Then we have

Theorem 1.1. *Suppose that f is continuous, bounded and integrable on G and that f^* is integrable on G^* . Then we have*

$$\mathcal{F}^*(\mathcal{F}(f))(-g) = cf(g)$$

for a positive constant independent of ϕ .

We shall give a sketch of a proof, supposing that either $G \cong G^*$ and G is a locally compact ring or G is finite. This is the case where we later deal with. For $h \in G$, we therefore have continuous multiplication $g \mapsto hg$. If h is invertible,

this is an automorphism of the group G ; so, $d(hg) = |h|dg$ for $|h| > 0$ by the uniqueness of the Haar measure. For simplicity, we assume to have a sequence of units ε_j converging to 0 in G and $\langle \varepsilon_j, g^* \rangle = \langle g, \varepsilon_j^* \rangle$ for a unit $\varepsilon \in G$. This fact is valid for $G = \mathbb{R}, \mathbb{Q}_p$ and \mathbb{A} by the following reason. When $G = \mathbb{R}$, G is self dual by $\langle x, y \rangle = \exp(2\pi ixy)$ and therefore it is obvious. When $G = \mathbb{Q}_p$, for each $x \in \mathbb{Q}_p$, expand x into the p -adic expansion $x = \sum_{n \geq -\infty} c_n p^n$ for integers c_n with $0 \leq c_n < p$. Then define the fraction part $[x]_p = \sum_{n < 0} c_n p^n \in \mathbb{Q}$. Then the self duality is given by $\langle x, y \rangle = \exp(-2\pi i[x]_p y)$, and again the assertion is obvious. For adeles $x, y \in A$, the pairing $\langle x, y \rangle = \prod_v \langle x_v, y_v \rangle$ (which is a finite product) does the job.

Proof. We formally compute

$$\begin{aligned}
\mathcal{F}^*(f^*)(-g) &= \int_{G^*} f^*(g^*) \langle g, g^* \rangle^{-1} dg^* \\
&= \int_{G^*} \int_G f(h) \langle h, g^* \rangle dh \langle g, g^* \rangle^{-1} dg^* \\
&= \int_{G^*} \int_G f(h) \langle h - g, g^* \rangle dh dg^* \\
&= \int_G f(h) \int_{G^*} \langle h - g, g^* \rangle dg^* dh.
\end{aligned} \tag{1.1}$$

When G is finite, $G^* \cong G$ (Exercise 1) and we may assume that

$$\int_G \phi(g) dg = |G|^{-1} \sum_{g \in G} \phi(g).$$

Then by the orthogonality relation of characters (cf. [LRG] Section 2.3), we have

$$\int_{G^*} \langle h - g, g^* \rangle dg^* = \begin{cases} 1 & \text{if } h = g, \\ 0 & \text{otherwise.} \end{cases}$$

From this, the assertion is clear, and $c = |G|^{-1}$. Since orthogonality relations hold for compact groups G (in this case, G^* is discrete), the same argument still works for compact and discrete groups (like $(G, G^*) = (\mathbb{Z}_p, \mathbb{Q}_p/\mathbb{Z}_p)$ and $(G, G^*) = (\mathbb{T}, \mathbb{Z})$).

We now assume that G is non-discrete and non-compact but a locally compact ring, like $G = \mathbb{R}, \mathbb{Q}_p, \mathbb{A}$. Then the last two integrals of (1.1) may not converge, because $|\langle x, y \rangle| = 1$ all the time. Thus we need to put a convergence factor $\varphi(g^*)$ in the integral:

$$\begin{aligned}
\int_{G^*} f^*(g^*) \varphi(g^*) \langle g, g^* \rangle dg^* &= \int_{G^*} \int_G f(h) \varphi(g^*) \langle h - g, g^* \rangle dh dg^* \\
&= \int_G f(g) \varphi^*(h - g) dh \\
&\stackrel{h \xrightarrow{g^*} g}{=} \int_G f(g + h) \varphi^*(g) dh.
\end{aligned} \tag{1.2}$$

We choose a function $\varphi(g^*)$ so that φ and φ^* are integrable over G^* and G respectively (Exercise 2). Since $|f^*(g^*)| \leq |\int_G f(g)\langle g, g^* \rangle dg| \leq \int_G |f(g)| dg$ for all g^* , we see that φf^* is also integrable. Then we put $\varphi_\varepsilon(g^*) = \varphi(\varepsilon g^*)$ for a unit $\varepsilon \in G^*$. Note that

$$\begin{aligned} (\varphi_\varepsilon)^*(g) &= \int_{G^*} \varphi(\varepsilon g^*) \langle g, g^* \rangle dg^* \\ &\stackrel{\varepsilon g^* \mapsto g^*}{=} |\varepsilon|^{-1} \int_{G^*} \varphi(g^*) \langle \varepsilon^{-1} g, g^* \rangle = |\varepsilon|^{-1} \varphi^*(\varepsilon^{-1} g). \end{aligned}$$

Replacing φ by φ_ε in (1.2), we get

$$\begin{aligned} \int_{G^*} f^*(g^*) \varphi(\varepsilon g^*) \langle g, g^* \rangle dg^* &= \int_G f(g+h) |\varepsilon|^{-1} \varphi^*(\varepsilon^{-1} h) dh \\ &\stackrel{\varepsilon^{-1} h \mapsto h}{=} \int_G f(g+\varepsilon h) \varphi^*(h) dh. \end{aligned}$$

Now we make $\varepsilon \rightarrow 0$, we get

$$\varphi(0) \int_{G^*} f^*(g^*) \langle g, g^* \rangle dg^* = f(g) \int_G \varphi^*(h) dh.$$

Choosing $\varphi(0) \neq 0 \neq \int_G \varphi^*(h) dh$ all positive real (for example, we may choose $\varphi(x) = \exp(-\pi x^2)$ when $G = \mathbb{R}$ and the characteristic function of \mathbb{Z}_p when $G = \mathbb{Q}_p$, and product of these when $G = \mathbb{A}$), we get the desired constant $c > 0$. \square

Now changing dg^* by $c^{-1} dg^*$, we may assume that the constant c is equal to 1. In this case, dg and dg^* are called dual each other. Further if $G \cong G^*$, first taking $dg = dg^*$ and changing dg by $\sqrt{c}^{-1} dg$, again we can make $c = 1$. In this case, dg is called self dual. The Lebesgue measure dx on \mathbb{R} is self dual. The measures described in 1.2 for \mathbb{Q}_p and \mathbb{A} are also self dual (Exercise 3).

Exercises

1. When G is finite, prove that $G \cong G^*$ (use the fundamental theorem of finite abelian groups).
2. When $G = \mathbb{R}, \mathbb{Q}_p, \mathbb{A}$, find a continuous function $\varphi(g^*)$ so that φ and φ^* are integrable over G^* and G , respectively.
3. Show that the measure described in Subsection 1.2 for $G = \mathbb{R}, \mathbb{Q}_p, \mathbb{A}$ is self dual.
4. Suppose that G is compact, and write dg for the Haar measure with $\int_G dg = 1$. Then show that its dual measure dg^* on G^* (which is discrete) is given by $\int_{G^*} \phi(g^*) dg^* = \sum_{g^* \in G^*} \phi(g^*)$.
5. For an isomorphism $\gamma : G^* \rightarrow G$, define the module $|\gamma|$ with respect to dg on G and the dual measure dg^* on G^* . Then show that the function $|\gamma|^{-1/2} \mathcal{F}(\phi)(-x\gamma^{-*})$ on G is determined independently of the choice of dg .

1.4 Plancherel Formula

Let dg and dg^* be dual Haar measure on G and G^* . We are going to prove the following theorem of Plancherel:

Theorem 1.2. *Let ϕ and f be continuous bounded integrable functions on G and ϕ^* and f^* are both integrable on G^* . If ϕ^* is continuous and bounded on G^* , we have*

$$\int_G f(g)\overline{\phi(g)}dg = \int_{G^*} f^*(g^*)\overline{\phi^*(g^*)}dg^*.$$

Therefore the Fourier transform keeps L^2 -norm.

We shall give a sketch of a proof. Using boundedness of ϕ and ϕ^* , it is easy to show the integrals above are finite (Exercise 1).

Proof. By Fourier inversion formula, we have

$$\phi(g) = \mathcal{F}^*(\phi^*)(-g) = \int_{G^*} \phi^*(g^*)\langle -g, g^* \rangle dg^*.$$

Then we see

$$\begin{aligned} \int_G f(g)\overline{\phi(g)}dg &= \int_G f(g) \int_{G^*} \overline{\phi^*(g^*)}\langle g, g^* \rangle dg^* dg \\ &= \int_G \int_{G^*} f(g)\langle g, g^* \rangle dg \overline{\phi^*(g^*)} dg^* \\ &= \int_{G^*} f^*(g^*)\overline{\phi^*(g^*)} dg^*. \end{aligned}$$

This shows the desired formula. \square

Consider the L^2 -spaces $L^2(G)$ and $L^2(G^*)$. The functions satisfying the condition of Theorem 1.2 is dense in these Hilbert spaces (Exercise 2). Thus for each $f \in L^2(G)$, choosing a sequence f_n satisfying the conditions of Theorem 1.2 yet converging to f in the Hilbert space $L^2(G)$. Then by the theorem, $\mathcal{F}(f_n)$ converges to an element f' in $L^2(G^*)$. The function f' is well defined almost everywhere on G^* and is independent of the choice of the sequence f_n (Exercise 3). We then define $\mathcal{F}(f) = f'$. Then $\mathcal{F} : L^2(G) \cong L^2(G^*)$ gives an isometry of the two Hilbert spaces.

Exercises

1. Show the finiteness of the integrals in Theorem 1.2.
2. Show the density of functions in $L^2(G)$ satisfying the conditions of Theorem 1.2 for $G = \mathbb{R}$ and $G = \mathbb{Q}_p$.
3. Show the well-definedness of the Fourier transform as a bounded operator from the Hilbert space $L^2(G)$ onto $L^2(G^*)$.

2 Metaplectic Groups

First we construct a general metaplectic groups associated to (G, G^*) and then study in details when G is a free module of finite rank over \mathbb{R} , \mathbb{Q}_p or \mathbb{A} .

2.1 Symmetric Maps

Let H and G be a locally compact abelian groups and $\rho : H \rightarrow G$ be a homomorphism (a continuous group homomorphism). Then $g^* \mapsto g^* \circ \rho$ induces a homomorphism $\rho^* : G^* \rightarrow H^*$ determined by $\langle h\rho, g^* \rangle = \langle h, g^* \rho^* \rangle$. In our convention, all $\rho \in \text{Hom}(X, Y)$ (except for scalars) acts on X from the right: $x \mapsto x\rho$, which will be useful later. We call ρ^* the adjoint of ρ . If $\rho : G \rightarrow G^*$ is a homomorphism, then again $\rho^* : G \rightarrow G^*$ is a homomorphism; so, it makes sense to insist $\rho = \rho^*$. Such a homomorphism is called symmetric.

To each symmetric map $\rho : G \rightarrow G^*$, we can associate a multiplicative quadratic form (a character of second degree) $f_\rho : G \rightarrow \mathbb{T}$ by

$$f_\rho(x) = \langle x, 2^{-1}x\rho \rangle.$$

Then

$$\begin{aligned} f_\rho(x+y)f_\rho(x)^{-1}f_\rho(y)^{-1} \\ = \langle x+y, 2^{-1}(x\rho+y\rho) \rangle \langle x, -2^{-1}x\rho \rangle \langle y, -2^{-1}y\rho \rangle = \langle x, y\rho \rangle. \end{aligned} \quad (2.1)$$

Thus, under the assumption we made that $g \mapsto 2g$ is an automorphism of G , we have a bijection:

$$\begin{aligned} \{\text{symmetric homomorphisms}\} \\ \leftrightarrow \{\text{multiplicative homogeneous quadratic forms}\} \end{aligned}$$

by $\rho \mapsto f_\rho$. Here the word ‘‘homogeneous’’ mean that f does not have linear terms, that is, f is of the form f_ρ for a symmetric map ρ .

Example 2.1. 1. Let F be a field of characteristic different from 2 and X be a finite dimensional vector space over F . A quadratic form $\phi : V \rightarrow F$ is a homogeneous polynomial on X of degree 2. Then $(x, y) = \phi(x+y) - \phi(x) - \phi(y)$ is a symmetric \mathbb{Q} -bilinear form on V . We call ϕ anisotropic if $\phi(x) = 0 \Leftrightarrow x = 0$. We call ϕ non-degenerate if $(x, V) = 0 \Rightarrow x = 0$. If ϕ is anisotropic, ϕ is non-degenerate.

2. Let $F = \mathbb{R}$. Then X is a locally compact abelian group isomorphic to \mathbb{R}^n for $n > 0$. For a given quadratic form ϕ on X , $f(x) = \exp(2\pi i\phi(x))$ is a homogeneous multiplicative quadratic form. The set of all homogeneous multiplicative quadratic forms is in bijection with the set of all quadratic forms on X . Indeed, if $f : X \rightarrow \mathbb{T}$ is a multiplicative quadratic form, then on a small open neighborhood U of 0, $\phi(x) = (2\pi i)^{-1} \log(f(x))$ for $x \in U$.

3. Let $F = \mathbb{Q}_p$. Then $f(x) = \exp(-2\pi i[\phi(x)]_p)$ is a multiplicative quadratic form. In the same way as above, The set of all homogeneous multiplicative quadratic forms on X is in bijection with the set of all quadratic forms on X in this manner.

Exercise

1. Give a detailed proof of the assertions in the above examples.

2.2 Symplectic Groups

We now write $V = G \times G^*$. Then $V^* \cong V$ by $\eta : (x, x^*) \mapsto (-x^*, x)$. We can write an automorphism $\sigma : V \rightarrow V$ as a matrix:

$$(x, x^*) \xrightarrow{\sigma} (x, x^*) \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix}.$$

Here $a_\sigma \in \text{End}(G)$, $d_\sigma \in \text{End}(G^*)$, $b_\sigma \in \text{Hom}(G, G^*)$ and $c_\sigma \in \text{Hom}(G^*, G)$. We then define $J : V \times V \rightarrow \mathbb{T}$ by $J((x, x^*), (y, y^*)) = \langle x, y^* \rangle \langle -y, x^* \rangle$. We can write this equation symbolically:

$$(x, x^*) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y \\ y^* \end{pmatrix} = \langle -y, x^* \rangle \langle x, y^* \rangle.$$

Then we define the group $Sp(G) \subset \text{Aut}(V)$ by

$$Sp(G) = \{\sigma \in \text{Aut}(V) \mid J(v\sigma, w\sigma) = J(v, w) \forall v, w \in V\}.$$

Since $Sp(G)$ is the stabilizer in $\text{Aut}(V)$ of the multiplicative quadratic form J , it is a group, and we see easily that for $\sigma \in Sp(G)$,

$$\sigma^{-1} = \begin{pmatrix} d_\sigma^* & -b_\sigma^* \\ -c_\sigma^* & a_\sigma^* \end{pmatrix}. \quad (2.2)$$

Exercise

1. Prove (2.2).

2.3 Heisenberg Groups

For each $v = (x, x^*) \in V$, we define a unitary operator $U(v)$ on $L^2(G)$ by

$$U(v)\phi(g) = \Phi(g+x)\langle g, x^* \rangle.$$

Then by computation, we have for $v = (x, x^*)$ and $w = (y, y^*)$ both in V ,

$$U(v)U(w) = \langle x, y^* \rangle U(v+w) = F(v, w)U(v+w), \quad (2.3)$$

for $F(v, w) = \langle x, y^* \rangle$. Thus $H(G) = \{tU(v) \mid v \in V, t \in \mathbb{T}\}$ is a subgroup of unitary operators acting on $L^2(G)$ with the identity operator given by $U(0)$.

This group is sometimes called the *Heisenberg group* for G (and it is written as $\mathbf{A}(G)$ in [We1] no.4). Since $U(v)U(w) = U(w)U(v)$ implies $\langle x, y^* \rangle = \langle y, x^* \rangle$, if $U(v)$ commutes with $U(w)$ for all $w \in V$, we find $v = 0$. Thus the center is given by $Z(H(G)) \cong \{tU(0) | t \in \mathbb{T}\} \cong \mathbb{T}$, and we have the following central extension:

$$1 \rightarrow \mathbb{T} \rightarrow H(G) \rightarrow V \rightarrow 0.$$

Thus any automorphism s of $H(G)$ induces an automorphism $\pi(s)$ of V and an automorphism of \mathbb{T} . Note that $\text{Aut}(\mathbb{T}) \cong \{\pm 1\}$ with non-trivial one given by $t \mapsto \bar{t}$ (Exercise 1).

Let s be an automorphism of the Heisenberg group $H(G)$ and suppose that s induces the identity on \mathbb{T} . Then $s(U(v)) = f(v)U(v\sigma)$ for $\sigma = \pi(s)$ and $f(v) \in \mathbb{T}$. We write $s = (\sigma, f)$, which determines s . If $s = (\sigma, f)$ and $s' = (\sigma', f')$, then $s's(U(v)) = s'(f(v)U(v\sigma)) = f(v)f'(v\sigma)U(v\sigma\sigma')$, and thus we have

$$(\sigma', f'(v))(\sigma, f(v)) = (\sigma' \circ \sigma, f(v)f'(v\sigma)). \quad (2.4)$$

Since

$$\begin{aligned} f(v)f(w)F(v\sigma, w\sigma)U(v\sigma + w\sigma) &= f(v)U(v\sigma)f(w)U(w\sigma) \\ &= s(U(v))s(U(w)) = s(U(v)U(w)) = F(v, w)f(v + w)U(v\sigma + w\sigma) \end{aligned}$$

by (2.3), we find

$$f(v + w)f(v)^{-1}f(w)^{-1} = F(v\sigma, w\sigma)F(v, w)^{-1}. \quad (2.5)$$

Thus f is a multiplicative quadratic form of V , and there is a unique homogeneous quadratic form f_σ satisfying (2.5). Moreover for any given multiplicative quadratic form f satisfying (2.5), $s = (\sigma, f)$ gives an element in $B(G)$ (Exercise 1). Since the left-hand-side of the above formula is symmetric with respect to v and w , we find also

$$F(v\sigma, w\sigma)F(v, w)^{-1} = F(w\sigma, v\sigma)F(w, v)^{-1}.$$

Since $J(v, w) = F(v, w)F(w, v)^{-1}$, σ preserves the symplectic form J ; so, $\pi(s) \in Sp(G)$. We write $B(G)$ for the automorphism group of $H(G)$ which induce the identity on \mathbb{T} . We have the projection $\pi : B(G) \rightarrow Sp(G)$ and $B(G) = Sp(G) \ltimes V^*$ by $\sigma \mapsto (\sigma, f_\sigma)$.

Exercises

1. Show that for any multiplicative quadratic form f satisfying (2.5), $U(v) \mapsto f(v)U(v\sigma)$ gives an automorphism of $H(G)$.
2. Give a detailed proof of the fact that $U(v)\phi \in L^2(G)$ if $\phi \in L^2(G)$. Also prove that for the L^2 -norm $\|\phi\|^2 = \int_G |\phi(g)|^2 dg$, $\|U(v)\phi\| = \|\phi\|$.
3. When $\sigma = \begin{pmatrix} 1 & \rho \\ 0 & 1 \end{pmatrix}$ for a symmetric $\rho \in \text{Hom}_{cont}(G, G^*)$, show that $f_\sigma = f_\rho$, where $f_\rho(v) = \langle g, 2^{-1}g\rho \rangle$ for $v = (g, g^*)$.

2.4 Metaplectic Cover

Here is an important theorem of A. Weil [We1] Theorem 1, which we do not prove, because the proof uses techniques from functional analysis and harmonic analysis on locally compact groups (that will not be used later).

Theorem 2.1. *Let $\mathbb{B}(G)$ be the normalizer of $H(G)$ in $\text{Aut}(L^2(G))$. Then we have a canonical central exact sequence:*

$$1 \rightarrow \mathbb{T} \rightarrow \mathbb{B}(G) \xrightarrow{\mu} B(G) \rightarrow 1.$$

We now define the metaplectic group $Mp(G)$ by

$$Mp(G) = \{s \in \mathbb{B}(G) \mid \mu(s) = (\sigma, f_\sigma) \text{ for } \sigma \in Sp(G)\}. \quad (2.6)$$

By definition, $Mp(G)$ is a central extension of $Sp(G)$; so,

$$1 \rightarrow \mathbb{T} \rightarrow Mp(G) \xrightarrow{\pi} Sp(G) \rightarrow 1$$

is exact. For general G , the above extension is non-trivial. However over some subsets of $Sp(G)$, one can have a canonical section r of π . We now define some sections. Let

$$U(G) = \left\{ \begin{pmatrix} 1 & \rho \\ 0 & 1 \end{pmatrix} \in Sp(G) \mid \rho \in \text{Hom}_{cont}(G, G^*) \right\}.$$

Since $U(G)$ is a subgroup of $Sp(G)$, ρ is a symmetric homomorphism; so, we have the associated multiplicative quadratic form: $f_\rho(g) = \langle g, 2^{-1}g\rho \rangle$. Then we define a section $r : U(G) \rightarrow B(G)$ by

$$r \left(\begin{pmatrix} 1 & \rho \\ 0 & 1 \end{pmatrix} \right) = \left(\begin{pmatrix} 1 & \rho \\ 0 & 1 \end{pmatrix}, f_\rho \right) \in B(G).$$

We extends this section to $\mathbf{r} : U(G) \rightarrow Mp(G)$ by

$$\left(\mathbf{r} \left(\begin{pmatrix} 1 & \rho \\ 0 & 1 \end{pmatrix} \right) \phi \right) (g) = \phi(g) f_\rho(g) \text{ for } \phi \in L^2(G).$$

We define another subgroup $L(G)$ of $Sp(G)$:

$$L(G) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-*} \end{pmatrix} \mid a \in \text{Aut}(G) \right\}.$$

Then we define a section $r : L(G) \rightarrow B(G)$ by

$$r \left(\begin{pmatrix} a & 0 \\ 0 & a^{-*} \end{pmatrix} \right) = \left(\begin{pmatrix} a & 0 \\ 0 & a^{-*} \end{pmatrix}, 1 \right) \in B(G).$$

Again we extend this section to $\mathbf{r} : L(G) \rightarrow Mp(G)$ by

$$\left(\mathbf{r} \left(\begin{pmatrix} a & 0 \\ 0 & a^{-*} \end{pmatrix} \right) \phi \right) (g) = \sqrt{|a|} \phi(ga).$$

Finally for any continuous isomorphism $c : G^* \cong G$, we define

$$\left(\mathbf{r} \left(\begin{pmatrix} 0 & -c^{-*} \\ c & 0 \end{pmatrix} \right) \phi \right) (g) = \sqrt{|c|}^{-1} \mathcal{F}(\phi)(-xc^{-*}).$$

Here we have fixed once and for all a Haar measure dg on G and \mathcal{F} is the Fourier transform on $L^2(G)$. The module $|c|$ is defined with respect to dg and its dual measure dg^* .

Let $\Omega = \Omega(G)$ be the collection of all $\sigma = \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix} \in Sp(G)$ with $c_\sigma : G^* \cong G$. Since

$$\sigma = \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix} = \begin{pmatrix} 1 & a_\sigma c_\sigma^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -c_\sigma^{-*} \\ c_\sigma & 0 \end{pmatrix} \begin{pmatrix} 1 & c_\sigma^{-1} d_\sigma \\ 0 & 1 \end{pmatrix},$$

we can extend \mathbf{r} to $\mathbf{r} : \Omega \rightarrow Mp(G)$. In particular, we have

$$(\mathbf{r}(\sigma)\phi)(g) = |c_\sigma|^{1/2} \int_G \phi(ga_\sigma + g^*c_\sigma) f_\sigma(g, g^*) dg^*.$$

From this, it is easy to check that $\mathbf{r}(p)\mathbf{r}(\sigma)\mathbf{r}(p') = \mathbf{r}(p\sigma p')$ for $p, p' \in P(G)$ and $\sigma \in \Omega$ (cf. [Sh2] (1.3a,b,c)).

Exercises

1. Check that the sections r and \mathbf{r} on $U(G)$ and $L(G)$ are group homomorphisms.
2. Let $P(G)$ be the subgroup of $Sp(G)$ generated by $U(G)$ and $L(G)$. Show that $U(G)$ is a normal subgroup of $P(G)$ and $P(G) = L(G) \times U(G)$. Further show that \mathbf{r} extends to a section $r : P(G) \rightarrow Mp(G)$ in an obvious manner, which is a group homomorphism.
3. Show that for each $p \in P(G)$ that $\|\mathbf{r}(p)\phi\| = \|\phi\|$ ($\phi \in L^2(G)$).
4. Give a detailed proof of $\mathbf{r}(p)\mathbf{r}(\sigma)\mathbf{r}(p') = \mathbf{r}(p\sigma p')$ for $p, p' \in P(G)$ and $\sigma \in \Omega$.

2.5 Sections over discrete and compact subgroups of $Sp(G)$

Let $\Gamma \subset G$ be a closed subgroup of G . Then G/Γ is again a locally compact abelian group under the quotient topology (Exercise 1). Then the exact sequence

$$0 \rightarrow \Gamma \rightarrow G \rightarrow G/\Gamma \rightarrow 0$$

yields, by the perfect Pontryagin duality, another exact sequence:

$$0 \rightarrow (G/\Gamma)^* \rightarrow G^* \rightarrow \Gamma^* \rightarrow 0.$$

Writing Γ^\perp for the image of $(G/\Gamma)^*$, we thus obtain

$$\Gamma^\perp = \{\gamma^* \in G^* \mid \langle \Gamma, \gamma^* \rangle = 0\}.$$

We suppose the following hypothesis:

(H1) Γ is either compact or discrete;

(H2) G/Γ is discrete (resp. compact) if Γ is compact (resp. discrete).

Once we start with $(G/\Gamma, \Gamma)$ as above, its dual $(G^*/\Gamma^\perp, \Gamma^\perp)$ is again the same type (Exercise 1 in Subsection 1.1).

Example 2.2. 1. If $G = X$ is a finite dimensional vector space over \mathbb{R} , a lattice L is a subgroup spanned by a base of X over \mathbb{R} . Then L is a discrete subgroup of X and X/L is compact. If we fix a dual pairing $(\cdot, \cdot) : X \times X \rightarrow \mathbb{R}$, the dual lattice L^\perp with respect to $\langle x, y \rangle = \mathbf{e}_\infty((x, y))$ is the dual lattice $L^\perp = \{x \in X \mid (L, x) \subset \mathbb{Z}\}$.

2. If $G = X$ is a finite dimensional vector space over \mathbb{Q}_p , a lattice L of X is a \mathbb{Z}_p -submodule spanned by a base of X over \mathbb{Q}_p . Then L is compact, and X/L is discrete. Again fixing a non-degenerate bilinear pairing $(\cdot, \cdot) : X \times X \rightarrow \mathbb{Q}_p$ which gives rise to the self-duality of X : $\langle x, y \rangle = \mathbf{e}_p((x, y))$, the \mathbb{Z}_p -dual lattice gives L^\perp .

3. Let X be a finite dimensional vector space over a number field F (of finite degree). Then $X_\mathbb{A} = X \otimes_{\mathbb{Q}} \mathbb{A}$ for the adèle ring \mathbb{A} is a locally compact abelian group, and X is a discrete subgroup of $X_\mathbb{A}$ and $X_\mathbb{A}/X$ is compact. Fixing a non-degenerate F -bilinear pairing (\cdot, \cdot) on X and extend it $F_\mathbb{A}$ -linearly to $X_\mathbb{A}$, we have the dual pairing $\langle x, y \rangle = \mathbf{e}_\mathbb{A}(\text{Tr}_{F/\mathbb{Q}}(x, y))$. In this case, $X^\perp = X$.

We write \dot{x} for the coset $x + \Gamma$ in G/Γ . Then we choose the canonical Haar measure $d\gamma$ on Γ so that $\int_\Gamma d\gamma = 1$ if Γ is compact and $\int_\Gamma \phi d\gamma = \sum_{\gamma \in \Gamma} \phi(\gamma)$. Similarly we choose the canonical Haar measure $d\dot{g}$ on G/Γ . Since the pair $(G^*/\Gamma^\perp, \Gamma^\perp)$ satisfies the same properties (H1-2), we have the Haar measures $d\gamma^\perp$ and $d\dot{g}^*$ on $(\Gamma^\perp, G^*/\Gamma^\perp)$. Then we consider the integration:

$$\int_G \phi(g) dg := \int_{G/\Gamma} \int_\Gamma \phi(\gamma + g) d\gamma d\dot{g}. \quad (2.7)$$

Obviously this integration is given by a Haar measure on G . Similarly we define a Haar measure dg^* on G^* by

$$\int_{G^*} \phi(g^*) dg^* := \int_{G^*/\Gamma^\perp} \int_{\Gamma^\perp} \phi(\gamma^\perp + g^*) d\gamma^\perp d\dot{g}^*. \quad (2.8)$$

We now define a partial Fourier transform $\Theta(\phi) : G \times G^* \rightarrow \mathbb{C}$ for $\phi \in L^1(G)$ by

$$\Theta(\phi)(g, g^*) = \int_\Gamma \phi(g + \gamma) \langle \gamma, g^* \rangle d\gamma. \quad (2.9)$$

By definition, we see

$$\Theta(\phi)(g + \gamma, g^* + \gamma^\perp) = \Theta(\phi)(g, g^*) \langle \gamma, g^* \rangle^{-1} \quad \text{for all } (\gamma, \gamma^\perp) \in \Gamma \times \Gamma^\perp$$

because $\langle \gamma, \gamma^\perp \rangle = 1$. We define $\mathcal{L}^2(G/\Gamma \times G^*/\Gamma^\perp)$ to be the space of functions $\Phi(g, g^*)$ on $G \times G^*$ such that

$$\bullet \Phi(g + \gamma, g^* + \gamma^\perp) = \Phi(g, g^*) \langle \gamma, g^* \rangle \quad \text{for all } (\gamma, \gamma^\perp) \in \Gamma \times \Gamma^\perp,$$

- $|\Phi|$ is square integrable as a function on $G/\Gamma \times G^*/\Gamma^\perp$.

Then $\mathcal{L}^2(G/\Gamma \times G^*/\Gamma^\perp)$ is a Hilbert space under the norm given by

$$\|\Phi\|^2 = \int_{G/\Gamma \times G^*/\Gamma^\perp} |\Phi(\dot{g}, \dot{g}^*)|^2 d\dot{g}d\dot{g}^*.$$

Let $\phi_g(\gamma) = \phi(g + \gamma)$ as a function of Γ . Then by the Plancherel formula, we have, for a fixed $g \in G$

$$\|\phi\|^2 = \|\phi_g\|^2 = \int_{G/\Gamma} \int_{\Gamma} |\phi_g|^2 d\gamma d\dot{g} = \int_{G/\Gamma} \|\phi_g^*\|^2 d\dot{g} = \|\Theta(\phi)(g, \dot{g}^*)\|^2,$$

where $\|\Phi\|^2 = \int_{G/\Gamma \times G^*/\Gamma^\perp} |\Phi(\dot{g}, \dot{g}^*)|^2$. Thus $\phi \mapsto \Theta(\phi)$ preserves the metric. Since $L^1(G) \cap L^2(G)$ is dense in $L^2(G)$, the linear map Θ extends to an isometry of $L^2(G)$ onto $L^2(G/\Gamma \times G^*/\Gamma^\perp)$ (surjectivity follows from the Fourier inversion formula). We thus have

$$\Theta : L^2(G) \cong \mathcal{L}^2(G/\Gamma \times G^*/\Gamma^\perp).$$

Since $Mp(G)$ and $\mathbb{B}(G)$ acts on $L^2(G)$ via the unitary representation we have constructed, these groups act at the same time on $L^2(G/\Gamma \times G^*/\Gamma^\perp)$ via the intertwining operator Θ .

On the other hand, if we write

$$Sp_\Gamma(G) = \{\sigma \in Sp(G) \mid (\Gamma \times \Gamma^\perp)\sigma = (\Gamma \times \Gamma^\perp)\}, \quad (2.10)$$

the group $Sp_\Gamma(G)$ acts on $\mathcal{L}^2(G/\Gamma \times G^*/\Gamma^\perp)$ naturally in the following manner:

$$\mathbf{r}_\Gamma(\sigma)\phi((g, g^*)) = \phi((g, g^*)\sigma)f_\sigma(g, g^*). \quad (2.11)$$

One can easily check using the fact:

$$f_\sigma(v+w)f_\sigma(v)^{-1}f_\sigma(w)^{-1} = F(w\sigma, v\sigma)F(w, v)^{-1}$$

that $\mathbf{r}_\Gamma(\sigma)\phi \in \mathcal{L}^2(G/\Gamma \times G^*/\Gamma^\perp)$ (Exercise 5). We would like to compare the two actions of $Sp_\Gamma(G)$.

Recall that $U(v)\phi(g, g^*) = \phi(g+x)\langle g, x^* \rangle$ for $v = (x, x^*) \in V = G \times G^*$. Then we see by computation, writing $w = (g, g^*)$

$$\begin{aligned} \Theta((U(v)\phi)(g, g^*)) &= \int_{\Gamma} \phi(g+\gamma+x)\langle g+\gamma, x^* \rangle \langle \gamma, g^* \rangle d\gamma \\ &= \langle g, x^* \rangle \Theta(\phi)(w+v) = F(w, v)\Theta(\phi)(w+v). \end{aligned}$$

Thus defining $U_\Theta(v)\Phi(w) = F(w, v)\Phi(w+v)$, we have the following commutative diagram:

$$\begin{array}{ccc} L^2(G) & \xrightarrow{\Theta} & \mathcal{L}^2(G/\Gamma \times G^*/\Gamma^\perp) \\ U(v) \downarrow & & \downarrow U_\Theta(v) \\ L^2(G) & \xrightarrow{\Theta} & \mathcal{L}^2(G/\Gamma \times G^*/\Gamma^\perp). \end{array}$$

By definition of the action of $U_\Theta(v)$ and $\mathbf{r}_\Gamma(\sigma)$, we see

$$\begin{aligned} U_\Theta(v)\mathbf{r}_\Gamma(\sigma)\phi(w) &= U_\Theta(v)\Phi(w\sigma)f_\sigma(w) \\ &= F(w, v)\Phi(w\sigma + v\sigma)f_\sigma(w + v) \\ &\stackrel{(*)}{=} \phi(w\sigma + v\sigma)f_\sigma(w)f_\sigma(v)F(w\sigma, v\sigma). \end{aligned}$$

where at the last equality (*), we have used the following identity:

$$f_\sigma(w + v)f_\sigma(w)^{-1}f_\sigma(v)^{-1} = F(w\sigma, v\sigma)F(w, v)^{-1}.$$

We compute also:

$$\begin{aligned} f_\sigma(v)\mathbf{r}_\Gamma(\sigma)U(v\sigma)\phi(w) &= f_\sigma(v)\mathbf{r}_\Gamma(\sigma)(\phi(w + v\sigma)F(w, v\sigma)) \\ &= f_\sigma(v)f_\sigma(w)\phi(w\sigma + v\sigma)F(w\sigma, v\sigma). \end{aligned}$$

Thus we get

$$U_\Theta(v)\mathbf{r}_\Gamma(\sigma) = f_\sigma(v)\mathbf{r}_\Gamma(\sigma)U_\Theta(w\sigma). \quad (2.12)$$

From this, we conclude

Theorem 2.2. *For a subgroup $\Gamma \subset G$ satisfying (H1-2), we have a section $\mathbf{r}_\Gamma : Sp_\Gamma(G) \rightarrow Mp(G)$ which coincides with \mathbf{r} on $\Omega \cap \Gamma$.*

Exercises

1. Prove that the quotient of a locally compact abelian group by a closed subgroup is again locally compact.
2. Give a detailed proof of the assertions in Example 2.2.
3. Show that dg and dg^* defined by (2.7) and (2.8) are mutually dual (cf. Exercise 4 in Subsection 1.3).
4. Prove the integral (2.9) converges if ϕ is integrable on G .
5. Show $\mathbf{r}_\Gamma(\sigma)\phi \in \mathcal{L}^2(G/\Gamma \times G^*/\Gamma^\perp)$ if $\phi \in \mathcal{L}^2(G/\Gamma \times G^*/\Gamma^\perp)$.

2.6 Theta Series

Let X be a finite dimensional vector space over a number field F . We now assume that $G = X_p = X \otimes_{\mathbb{Q}} \mathbb{Q}_p$ or $X_\infty = X \otimes_{\mathbb{Q}} \mathbb{R}$ or $X_{\mathbb{A}} = X \otimes_{\mathbb{Q}} \mathbb{A}$. For X_∞ , we define $\mathcal{S}(X_\infty)$ to be the Schwartz space of functions on X_∞ . Thus $\mathcal{S}(X_\infty)$ is made of C^∞ -class functions with all derivatives rapidly decreasing as Euclidean norm of $x \in X_\infty$ grows. In other words, $\phi \in \mathcal{S}(X_\infty)$ if and only if ϕ is of C^∞ -class and for any polynomial $P(x)$ and any m -th derivative Φ of ϕ , $|P(x)\Phi(x)|$ goes to 0 as $|x| \rightarrow \infty$.

When $G = X_p$, we write $\mathcal{S}(X_p)$ for the space of Bruhat functions on X_p , which are locally constant with compact support. When $G = X_{\mathbb{A}}$, $\mathcal{S}(X_{\mathbb{A}})$ is the

space of Schwartz-Bruhat functions on $X_{\mathbb{A}}$, which are spanned by the product $\phi((x_v)_v) = \prod_v \phi_v(x_v)$ with $\phi_v \in \mathcal{S}(X_v)$ and such that for almost all henselian p , ϕ_v is the characteristic function of a \mathbb{Z}_p -lattice $L_p = L \otimes_{\mathbb{Z}} \mathbb{Z}_p$ for a lattice $L \subset X$.

It is well known that the Fourier transform \mathcal{F} gives an isomorphism $\mathcal{F} : \mathcal{S}(G) \cong \mathcal{S}(G^*)$ (cf. [IGA]). Then by definition of $\mathbf{r}(s)$, this operator preserves the space $\mathcal{S}(G)$ for $s \in \Omega$. Since Ω generates $Mp(G)$ for G as above, we know that the action of $Mp(X)$ preserves $\mathcal{S}(G) \stackrel{\text{dense}}{\subset} L^2(G)$ (see [We1] No.11-13 for more details).

We now prove the following generalized Poisson summation formula of Weil:

Theorem 2.3. *Let G be as above. Suppose $\Phi \in \mathcal{S}(G)$. Then we have*

$$\int_{\Gamma} \Phi(\gamma) d\gamma = \int_{\Gamma} (s\Phi)(\gamma) d\gamma$$

for all $s \in Sp_{\Gamma}(G)$.

Proof. We consider $\Theta(\Phi)$. Recall

$$\Theta(\Phi)(g, g^*) = \int_{\Gamma} \Phi(\gamma + g, g^*) \langle \gamma, g^* \rangle d\gamma.$$

Since $\mathbf{r}_{\Gamma}(\sigma)\Theta(\Phi)(v) = \Theta(\Phi)(v\sigma)f_{\sigma}(v)$, we find that

$$\int_{\Gamma} (\mathbf{r}_{\Gamma}(\sigma)\Phi)(\gamma) d\gamma = \mathbf{r}_{\Gamma}(\sigma)\Theta(\Phi)(0) = \Theta(\Phi)(0\sigma)f_{\sigma}(0) = \Theta(\Phi)(0) = \int_{\Gamma} \Phi(\gamma) d\gamma.$$

This shows the desired formula. The requirement $\Phi \in \mathcal{S}(G)$ is necessary to guarantee that $\Phi(\gamma)$ is well defined for all $\gamma \in \Gamma$. \square

We now assume that G is either X_{∞} or $X_{\mathbb{A}}$. Thus Γ is discrete and is a lattice $L \subset X$ or $X \subset X_{\mathbb{A}}$. We consider the function:

$$\Theta(\Phi)(s) = \int_{\Gamma} (s\Phi)(\gamma) d\gamma = \sum_{\gamma \in \Gamma} (s\Phi)(\gamma)$$

as a function of $s \in Mp(G)$. Then by the above theorem, we find for $\xi \in Sp_{\Gamma}(G)$, $\Theta(\Phi)(\xi s) = \Theta(\Phi)(s)$. Note that, by identifying X with \mathbb{Q}^n , we find $Sp(X_{\mathbb{A}}) = Sp_{2n}(\mathbb{A})$ and $Sp_X(X_{\mathbb{A}}) = Sp_{2n}(\mathbb{Q})$. Moreover, for $\sigma = (\sigma_v) \in Sp_{2n}(\mathbb{A})$, we see by definition, if $\Phi = \prod_v \Phi_v$ with $\Phi_v \in \mathcal{S}(X_v)$,

$$\mathbf{r}((\sigma_v))\Phi = \prod_v \mathbf{r}(\sigma_v)\Phi_v$$

as long as $\sigma_v \in P(X_v)$ or $\Omega(X_v)$. From this fact, we can easily conclude that for an open compact subgroup S of $Sp_{2n}(\mathbb{A}^{(\infty)})$, $\Theta(\Phi)(su) = \Theta(\Phi)(s)$ for $u \in \mathbf{r}_S(S)$. Thus $s \mapsto \Theta(\Phi)(s)$ is an automorphic form in a broad sense that they are functions on $Sp_{2n}(\mathbb{Q}) \backslash \mathbf{M}_{\mathbb{A}}/S$ if $G = X_{\mathbb{A}}$ or $G = X_{\infty}$. We shall show

in the following subsection that $\Theta(\Phi)$ gives basically all known theta series as automorphic forms on the metaplectic group $\mathbf{M}_{\mathbb{A}} = Mp(X_{\mathbb{A}})$ or $\mathbf{M}_{\infty} = Mp(X_{\infty})$.

We are going to make explicit the form of $\Theta(\Phi)$. Suppose first that $X = \mathbb{Q}_n$ (row vector space of dimension n), and identify X with its \mathbb{Q} -dual by $(w, v) = w^t v$. Let $H = H_n = \{z \in \mathbb{C}_n^n \mid z = x + iy, y > 0\}$ (Siegel upper half space). We consider the Schwartz function $\varphi(v; (z, u)) = \exp(\pi i v z^t v + 2\pi i v u)$ defined on $v \in X_{\infty}$, $z \in H$ and $u \in \mathbb{C}^n$ (column vector space). We note $\mathcal{F}(\varphi(v; (i1_n, 0))) = \varphi(v; (i1_n, 0))$, and hence, writing $y = (y^{1/2})^2$ for a positive symmetric matrix, we have

$$\begin{aligned} \mathcal{F}(\varphi) &= \left(\mathbf{r} \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) \varphi \right) (v; (iy, u)) \\ &= \int_{X_{\infty}} \exp(-\pi w y^t w) \exp(2\pi i w ({}^t v + u)) dw \\ &\stackrel{wy^{1/2} \mapsto w}{=} \det(y)^{-1/2} \int_{X_{\infty}} \exp(-\pi w^t w) \exp(2\pi i y^{-1/2} w ({}^t v + u)) dw \\ &= \det(y)^{-1/2} \exp(-\pi (v + {}^t u) y^{-1} ({}^t v + u)) \\ &= \det(y)^{-1/2} \varphi(v; (iy^{-1}, iy^{-1}u)) \exp(-\pi {}^t u y^{-1} u) \end{aligned}$$

For $\sigma = \begin{pmatrix} a_{\sigma} & b_{\sigma} \\ c_{\sigma} & d_{\sigma} \end{pmatrix} \in Sp_{2n}(\mathbb{R})$, following [Sh1] (1.7) and (1.11):

$$\begin{aligned} \sigma(z, u) &= ((a_{\sigma} z + b_{\sigma})(c_{\sigma} z + d_{\sigma})^{-1}, {}^t(c_{\sigma} z + d_{\sigma})^{-1} u) \\ \zeta_{\sigma}(z, u) &= \exp(\pi i \cdot {}^t u (c_{\sigma} z + d_{\sigma})^{-1} c_{\sigma} u). \end{aligned} \tag{2.13}$$

Using this notation, the above computation yields for $z = iy$

$$\begin{aligned} \mathcal{F}(\varphi)(v; (z, u)) &= \left(\mathbf{r} \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) \varphi \right) (v; (z, u)) \\ &= \det(-iz)^{-1/2} \zeta_{\eta}(z, u)^{-1} \varphi(v; \eta(z, u)), \end{aligned} \tag{2.14}$$

where $\eta = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}$. By definition, $\mathcal{F}(\varphi)(v; (z, u))$ is a holomorphic function of (z, u) . Since $\{iy \mid y \in \mathbb{R}_n^n, {}^t y = y > 0\}$ is a Zariski dense subset of H_n (Exercise 1), the above identity (2.14) has to be true for all $(z, u) \in H_n \times \mathbb{C}^n$. Similarly, we can verify for $p = \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix}$,

$$(\mathbf{r}(p)\varphi)(v; (z, u)) = |\det(a)|^{1/2} \varphi(v; p(z, u))$$

and for $\alpha = \begin{pmatrix} 1_n & b \\ 0 & 1_n \end{pmatrix}$

$$(\mathbf{r}(\alpha)\varphi)(v; (z, u)) = \exp(\pi v b \cdot {}^t v) \varphi(v; (z, u)) = \varphi(v; \alpha(z, u)).$$

Since $Mp(X_{\infty})$ is generated by \mathbb{T} and matrices of the form: η , p and α , we find a holomorphic function $h(s, z) : Mp(X_{\infty}) \times H_n \rightarrow \mathbb{C}$ with the following property (see [Sh1] Proposition 3.1):

Proposition 2.4. *Let $\sigma = \pi(s) \in Sp_{2n}(\mathbb{R})$.*

1. $s\varphi(v; z, u) = h(s, z)^{-1}\zeta_\sigma(z, u)^{-1}\varphi(v; \sigma(z, u))$;
2. $h(st, z) = h(s, t(z))h(t, z)$ for all $s, t \in Mp(X_\infty)$;
3. $h(s, z)^2 = t \cdot \det(c_\sigma z + d_\sigma)$ for $t \in \mathbb{T}$;
4. $h(s, z)^4 = (-1)^n \cdot \det(c_\sigma z + d_\sigma)^2$ if $s = \mathbf{r}(\sigma)$ for $\sigma \in \Omega$.

Proof. All the assertions except for (2) has already been proven. Thus we need to show the automorphic property: $\zeta_{\sigma\tau}(z, u) = \zeta_\sigma(\tau(z, u))\zeta_\tau(z, u)$ for $\sigma, \tau \in Sp_{2n}(\mathbb{R})$. We define $g(z, u) = \exp(\pi i^t u(z - \bar{z})^{-1}u)$. Since

$$\sigma \begin{pmatrix} z & \bar{z} \\ 1_n & 1_n \end{pmatrix} = \begin{pmatrix} \sigma(z) & \sigma(\bar{z}) \\ 1_n & 1_n \end{pmatrix} \begin{pmatrix} c_\sigma z + d_\sigma & 0 \\ 0 & c_\sigma \bar{z} + d_\sigma \end{pmatrix}$$

and ${}^t\sigma\eta\sigma = \eta$, we have for $T = \begin{pmatrix} z & \bar{z} \\ 1_n & 1_n \end{pmatrix}$

$$\begin{aligned} \begin{pmatrix} 0 & z - \bar{z} \\ \bar{z} - z & 0 \end{pmatrix} &= {}^tT\eta T = {}^t(\sigma T)\eta\sigma T \\ &= \begin{pmatrix} 0 & {}^t(c_\sigma z + d_\sigma)(z - \bar{z})(c_\sigma \bar{z} + d_\sigma) \\ {}^t(c_\sigma \bar{z} + d_\sigma)(z - \bar{z})(c_\sigma z + d_\sigma) & 0 \end{pmatrix}. \end{aligned}$$

From this, we find $g(\sigma(z, u)) = \zeta_\sigma(z, u)^{-1}g(z, u)$, and hence we get the desired assertion. \square

Let $\Gamma = \mathbb{Z}^n \subset X$. It is now an easy exercise to see

$$\int_\Gamma \varphi(\gamma; (z, u))d\gamma = \sum_{m \in \mathbb{Z}^n} \varphi(m; (z, u)) = \sum_m \exp(\pi i m z^t m + 2\pi i m u) = \theta(z, u)$$

is the standard Siegel modular theta function, and the generalized Poisson summation formula of Weil includes as a special case the transformation formula of this theta function.

We consider the set \mathcal{G} made up of pairs $(\sigma, j_\sigma(z))$ with $\sigma \in Sp_{2n}(\mathbb{R})$ and a holomorphic functions $j_\sigma : H_n \rightarrow \mathbb{C}$ such that $j_\sigma^2 = t \cdot \det(c_\sigma z + d_\sigma)$ for $t \in \mathbb{T}$. We make \mathcal{G} into a group by the multiplication (cf. [Sh1] (1.5)):

$$(\sigma, j_\sigma)(\tau, j_\tau) = (\sigma\tau, j_\sigma(\tau(z))j_\tau(z)).$$

Then we have the following exact sequence:

$$1 \rightarrow \mathbb{T} \xrightarrow{t \mapsto (1, t)} \mathcal{G} \xrightarrow{(\sigma, j_\sigma) \mapsto \sigma} Sp_{2n}(\mathbb{R}) \rightarrow 1.$$

Corollary 2.5. *The map $\iota : Mp(X_\infty) \rightarrow \mathcal{G}$ given by $s \mapsto (\pi(s), h(s, z))$ gives an isomorphism of groups.*

Proof. By Proposition 2.4 (1) and (2), ι is a homomorphism sending isomorphically $\mathbb{T} \subset Mp(X_\infty)$ onto $\mathbb{T} \subset \mathcal{G}$ and inducing an isomorphism to the quotient $Sp_{2n}(\mathbb{R}) = Mp(X_\infty)/\mathbb{T} = \mathcal{G}/\mathbb{T}$; so, it is an isomorphism. \square

Exercises

1. Show that $f = 0$ if a meromorphic function $f : H_n \rightarrow \mathbf{P}_{\mathbb{C}}^1$ vanishes on $Y = \{iy | y \in \mathbb{R}_n^+, {}^t y = y > 0\} \subset H_n$.
2. Give a detailed proof of Proposition 2.4.
3. Check that the multiplication given above makes \mathcal{G} into a group.
4. Show that $\varphi(v; (z, u)) \in \mathcal{S}(X_\infty)$ as a function of v for a fixed $(z, u) \in H_n \times \mathbb{C}^n$.

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