

# $\mathcal{L}$ -INVARIANT OF THE SYMMETRIC POWERS OF TATE CURVES

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In my earlier paper [H07] and in my talk at the workshop on “Arithmetic Algebraic Geometry” at RIMS in September 2006, we made explicit a conjectural formula of the  $\mathcal{L}$ -invariant of symmetric powers of a Tate curve over a totally real field (generalizing the conjecture of Mazur-Tate-Teitelbaum, which is now a theorem of Greenberg-Stevens). In this paper, we prove the formula for Greenberg's  $\mathcal{L}$ -invariant when the symmetric power is of adjoint type, assuming a standard conjecture (see Conjecture 0.1) on the ring structure of a Galois deformation ring of the symmetric powers.

Let  $p$  be an odd prime and  $F$  be a totally real field of degree  $d < \infty$  with integer ring  $O$ . Order all the prime factors of  $p$  in  $O$  as  $\mathfrak{p}_1, \dots, \mathfrak{p}_e$ . Throughout this paper, we study an elliptic curve  $E/F$  over  $O$  with split multiplicative reduction at  $\mathfrak{p}_j|p$  for  $j = 1, 2, \dots, b$  and ordinary good reduction at  $\mathfrak{p}_j|p$  for  $j > b$ . Write  $F_j = F_{\mathfrak{p}_j}$  for the  $\mathfrak{p}_j$ -adic completion of  $F$  and  $q_j \in F_j^\times$  with  $j \leq b$  for the Tate period of  $E/F_j$ . Put  $Q_j = N_{F_{\mathfrak{p}_j}/\mathbb{Q}_p}(q_j)$ . When  $b = 0$ , as a convention, we assume that  $E/F$  has good ordinary reduction at every  $p$ -adic place of  $F$ . We assume throughout the paper that  $E$  does not have complex multiplication, and for simplicity, we also assume that  $E$  is semi-stable over  $O$ . Some cases of complex multiplication are treated in [HMI] Section 5.3.3. Take an algebraic closure  $\overline{F}$  of  $F$ . Writing  $\rho_E : \text{Gal}(\overline{F}/F) \rightarrow GL_2(\mathbb{Q}_p)$  for the Galois representation on  $T_p E \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$  for the Tate module  $T_p E = \varprojlim_n E[p^n]$ , at each prime factor  $\mathfrak{p}|p$ , we have  $\rho_E|_{\text{Gal}(\overline{F}_{\mathfrak{p}}/F_{\mathfrak{p}})} \sim \begin{pmatrix} \beta_{\mathfrak{p}} & * \\ 0 & \alpha_{\mathfrak{p}} \end{pmatrix}$  for an unramified character  $\alpha_{\mathfrak{p}}$ . Since  $\beta_{\mathfrak{p}}$  restricted to the inertia subgroup  $I_{\mathfrak{p}} \subset \text{Gal}(\overline{F}_{\mathfrak{p}}/F_{\mathfrak{p}})$  is equal to the  $p$ -adic cyclotomic character  $\mathcal{N}$ , we have  $\alpha_{\mathfrak{p}}^i \neq \beta_{\mathfrak{p}}^j$  for any pair of integers  $(i, j)$  except for  $i = j = 0$ . Write  $\rho_{n,0}$  for the symmetric  $n$ -th tensor power of  $\rho_E$ ,

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which is an  $(n + 1)$ -dimensional Galois representation semi-stable over  $O$ . More generally, we write  $\rho_{n,m}$  for  $\rho_{n,0} \otimes \mathcal{N}^{-m} : \text{Gal}(\overline{F}/F) \rightarrow G_n(\mathbb{Q}_p)$ , where  $\mathcal{N}$  is the  $p$ -adic cyclotomic character. By semi-stability, the sets of ramification primes for  $\rho_E$  and  $\rho_{n,m}$  are equal.

Consider  $J_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ . We then define  $J_n = \text{Sym}^{\otimes n}(J_1)$ . Since  ${}^t\alpha J_1 \alpha = \det(\alpha) J_1$  for  $\alpha \in GL(2)$ , we have  ${}^t\rho_{n,0}(\sigma) J_n \rho_{n,0}(\sigma) = \mathcal{N}^n(\sigma) J_n$ . Define an algebraic group  $G_n$  over  $\mathbb{Z}_p$  by

$$G_n(A) = \{ \alpha \in GL_{n+1}(A) \mid {}^t\alpha J_n \alpha = \nu(\alpha) J_n \}$$

with the similitude homomorphism  $\nu : G_n \rightarrow \mathbb{G}_m$ . Then  $G_n$  is a quasi-split orthogonal or symplectic group according as  $n$  is even or odd. The representation  $\rho_{n,0}$  of  $\text{Gal}(\overline{F}/F)$  actually factors through  $G_n(\mathbb{Q}_p) \subset GL_{n+1}(\mathbb{Q}_p)$ . Two representations  $\rho$  and  $\rho' : G \rightarrow G_n(A)$  for a group  $G$  are isomorphic if  $\rho(g) = x\rho'(g)x^{-1}$  for  $x \in G_n(A)$  independent of  $g \in G$ . If  $\rho$  is isomorphic to  $\rho'$ , we write  $\rho \cong \rho'$ .

Let  $S$  be the set of prime ideals of  $O$  prime to  $p$  where  $E$  has bad reduction (and by semi-stability,  $S \sqcup \{\mathfrak{p}|p\} \sqcup \{\infty\}$  gives the set of ramified primes for  $\rho_{n,0}$ ). Let  $K/\mathbb{Q}_p$  be a finite extension with  $p$ -adic integer ring  $W$ . We may take  $K = \mathbb{Q}_p$ , but it is useful to formulate the result allowing other choices of  $K$ . Start with  $\rho_{n,0}$  and consider the deformation ring  $(R_n, \rho_n)$  which is universal among the following deformations: Galois representations  $\rho_A : \text{Gal}(\overline{F}/F) \rightarrow G_n(A)$  for Artinian local  $K$ -algebras  $A$  with residue field  $K = A/\mathfrak{m}_A$  such that

- (K<sub>n</sub>1) unramified outside  $S, \infty$  and  $p$ ;
- (K<sub>n</sub>2)  $\rho_A|_{\text{Gal}(\overline{F}_p/F_p)} \cong \begin{pmatrix} \alpha_{0,A,p} & * & \cdots & * \\ 0 & \alpha_{1,A,p} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_{n,A,p} \end{pmatrix}$  with  $\alpha_{j,A,p} \equiv \beta_p^{n-j} \alpha_p^j \pmod{\mathfrak{m}_A}$   
with  $\alpha_{j,A,p}|_{I_p}$  ( $j = 0, 1, \dots, n$ ) factoring through  $\text{Gal}(F_p^{ur}[\mu_{p^\infty}]/F_p^{ur})$  for the maximal unramified extension  $F_p^{ur}/F_p$  for all prime factors  $\mathfrak{p}$  of  $p$ ;
- (K<sub>n</sub>3)  $\nu \circ \rho_A = \mathcal{N}^n$  for the  $p$ -adic cyclotomic character  $\mathcal{N}$ ;
- (K<sub>n</sub>4)  $\rho_A \equiv \rho_{n,0} \pmod{\mathfrak{m}_A}$ .

Since  $\rho_{n,0}$  is absolutely irreducible as long as  $E$  does not have complex multiplication (because  $\text{Im}(\rho_E)$  is open in  $GL_2(\mathbb{Z}_p)$  by a result of Serre) and all  $\alpha_p^i \beta_p^{n-i}$  for  $i = 0, 1, \dots, n$  are distinct, the deformation problem specified by (K<sub>n</sub>1-4) is representable by a universal couple  $(R_n, \rho_n)$  (see [Ti]). In other words, for any  $\rho_A$  as above, there exists a unique  $K$ -algebra homomorphism  $\varphi : R_n \rightarrow A$  such that  $\varphi \circ \rho_n \cong \rho_A$ .

Write now

$$\rho_n|_{\text{Gal}(\overline{F}_p/F_p)} \cong \begin{pmatrix} \delta_{0,p} & * & \cdots & * \\ 0 & \delta_{1,p} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \delta_{n,p} \end{pmatrix}$$

with  $\delta_{j,p} \equiv \beta_p^{n-j} \alpha_p^j \pmod{\mathfrak{m}_n}$  (for  $\mathfrak{m}_n = \mathfrak{m}_{R_n}$ ).

Let  $\Gamma_p$  be the maximal torsion-free quotient of  $\text{Gal}(F_p^{ur}[\mu_{p^\infty}]/F_p^{ur})$ . Then the character  $\widehat{\delta}_{j,p} = \delta_{j,p}(\beta_p^{n-j} \alpha_p^j)^{-1}$  restricted to  $I_p$  factors through  $\Gamma_p$ , giving rise to an algebra structure of  $R_n$  over  $W[[\Gamma_p]]$ . Take the product  $\Gamma = \prod_{\mathfrak{p}|p} \Gamma_p^{n+1}$  of  $n + 1$  copies of  $\Gamma_p$  over all prime factors  $\mathfrak{p}$  of  $p$  in  $F$ . We write general elements of  $\Gamma$  as  $x = (x_{j,\mathfrak{p}})_{j,\mathfrak{p}}$  with  $x_{j,\mathfrak{p}}$  in the  $j$ -th component  $\Gamma_p$  in  $\Gamma$  ( $j = 0, 1, \dots, n$ ). Consider the character  $\widehat{\delta} : \Gamma \rightarrow R_n^\times$  given by  $\widehat{\delta}(x) = \prod_{j=0}^n \prod_{\mathfrak{p}|p} \widehat{\delta}_{j,\mathfrak{p}}(x_{j,\mathfrak{p}})$ . Choosing a generator  $\gamma_i = \gamma_{\mathfrak{p} = \mathfrak{p}_i}$  (for  $\mathfrak{p} = \mathfrak{p}_i$ ) of the topologically cyclic group  $\Gamma_p$ , we identify  $W[[\Gamma]]$

with a power series ring  $W[[X_{j,\mathfrak{p}}]]_{j,\mathfrak{p}}$  by associating the generator  $\gamma_{\mathfrak{p}}$  of the  $j$ -th component:  $\Gamma_{\mathfrak{p}}$  of  $\Gamma$  with  $1+X_{j,\mathfrak{p}}$ . The character  $\widehat{\delta} : W[[\Gamma]] \rightarrow R_n$  extends uniquely to an algebra homomorphism  $\widehat{\delta} : W[[X_{j,\mathfrak{p}}]]_{j,\mathfrak{p}} \rightarrow R_n$  by the universality of the (continuous) group ring  $W[[\Gamma]]$ . Thus  $R_n$  is naturally an algebra over  $K[[X_{j,\mathfrak{p}}]]_{j,\mathfrak{p}}$ . This algebra structure of  $R_n$  over the local Iwasawa algebra  $W[[\Gamma]]$  is a standard one which has been studied for long (about 20 years) in many places (for example, [Ti] Chapter 8 and [MFG] 5.2.2). The  $(n+1)e$  variables  $X_{j,\mathfrak{p}}$  may not be independent in  $R_n$ , and we expect that only a half of them survives. More precisely, we have the following conjectural statement:

**Conjecture 0.1.** *Suppose that  $n$  is odd. Then  $R_n$  is isomorphic to the power series ring  $K[[X_{j,\mathfrak{p}}]]_{\mathfrak{p}|p,j:\text{odd}}$  of  $e\frac{n+1}{2}$  variables.*

When  $n = 1$ , we write  $\beta_i = \delta_{0,\mathfrak{p}_i}$ ,  $\alpha_i = \delta_{1,\mathfrak{p}_i}$  and  $T_i = X_{1,\mathfrak{p}_i}$ . If  $n = 1$  and  $F = \mathbb{Q}$ , via the solution of the Shimura-Taniyama conjecture, this conjecture follows from Kisin's work (generalizing earlier works of Wiles, Taylor-Wiles and Skinner-Wiles). Assuming potential modularity of  $\rho_E$  (see [Ta]) with additional assumptions that  $\text{Im}(\bar{\rho})$  is nonsoluble and that the semi-simplification of  $\bar{\rho}|_{\text{Gal}(\bar{F}_{\mathfrak{p}}/F_{\mathfrak{p}})}$  is non-scalar for all prime  $\mathfrak{p}|p$  in  $F$ , we will prove this conjecture for  $n = 1$  in this paper (see Proposition 2.1). Assuming Hilbert modularity over  $F$  of  $E$  and the following two conditions:

- (ai) The  $\mathbb{F}_p$ -linear Galois representation  $\bar{\rho} = (T_p E \pmod{p})$  is absolutely irreducible over  $\text{Gal}(\bar{F}/F[\mu_p])$ .
- (ds)  $\bar{\rho}^{ss}$  has a non-scalar value over  $\text{Gal}(\bar{F}_{\mathfrak{p}}/F_{\mathfrak{p}})$  for all prime factors  $\mathfrak{p}|p$ ,

the conjecture for  $n = 1$  follows from a result of Fujiwara (see [F] and [F1]) and Skinner-Wiles [SW1] as described in [HMI] Theorem 3.65 and Proposition 3.78.

In the special case of rational elliptic curve  $E/\mathbb{Q}$  with multiplicative reduction at  $p$ , the following conjecture (generalizing the one by Mazur-Tate-Teitelbaum in [MTT]) was proven by R. Greenberg for his  $\mathcal{L}$ -invariant of symmetric powers of  $E$ . His proof is described in his remark in page 170 of [Gr]. Although his proof might also be generalized to our setting, our point of view is different from [Gr], relating the following conjecture to Conjecture 0.1, and indeed, if one can generalize Greenberg's proof to cover the following conjecture, it might supply us with a proof of Conjecture 0.1 (we hope to discuss this point in our future work).

**Conjecture 0.2.** *Let the notation and the assumption be as in Theorem 0.3. Suppose that the  $n$ -th symmetric power motive  $\text{Sym}^{\otimes n}(H_1(E))(-m)$  with Tate twist by an integer  $m$  is critical at 1. Then if  $\text{Ind}_F^{\mathbb{Q}}(\text{Sym}^{\otimes n}(\rho_E)(-m))$  has an exceptional zero at  $s = 1$ , we have*

$$\begin{aligned} & \mathcal{L}(\text{Ind}_F^{\mathbb{Q}} \rho_{n,m}) \\ &= \begin{cases} \left( \prod_{i=1}^b \frac{\log_p(Q_i)}{\text{ord}_p(Q_i)} \right) \mathcal{L}(m) & \text{for a constant } \mathcal{L}(m) \in \mathbb{Q}_p^{\times} \text{ if } n = 2m \text{ with odd } m, \\ \prod_{i=1}^b \frac{\log_p(Q_i)}{\text{ord}_p(Q_i)} & \text{if } n \neq 2m. \end{cases} \end{aligned}$$

We have  $\mathcal{L}(m) = 1$  if  $b = e$ , and the value  $\mathcal{L}(1)$  when  $b < e$  is given by

$$\mathcal{L}(1) = \det \left( \frac{\partial \delta_i([p, F_i])}{\partial X_j} \right)_{i>b, j>b} \Big|_{X_1=X_2=\dots=X_e=0} \prod_{i>b} \frac{\log_p(\gamma_i)}{[F_i : \mathbb{Q}_p] \alpha_i([p, F_i])}$$

for the local Artin symbol  $[p, F_i]$ , where  $\gamma_{\mathfrak{p}}$  is the generator of  $\mathcal{N}(\text{Gal}(F_{\mathfrak{p}}[\mu_{p^\infty}]/F_{\mathfrak{p}}))$  by which we identify the group algebra  $W[[\Gamma_{\mathfrak{p}}]]$  with  $W[[X_{\mathfrak{p}}]]$ .

The analytic  $\mathcal{L}$ -invariant of  $p$ -adic analytic  $L$ -functions (when  $n = 1$ ) is studied by C.-P. Mok [M] following the method of [GS], and his result confirms the conjecture in some special cases (see a remark in [H07] after Conjecture 1.3).

The motive  $\text{Sym}^{\otimes n}(H_1(E))(-m)$  is critical at 1 if and only if the following two conditions are satisfied:

- $0 \leq m < n$ ;
- either  $n$  is odd or  $n = 2m$  with odd  $m$ .

We will specify  $\mathcal{L}(m)$  in Definition 1.11 assuming Conjecture 0.1. There is a wild guess that  $\mathcal{L}(m)$  might be independent of  $m$  only depending on  $E$ . We hope to discuss this matter in our future work.

We will prove in this paper (for Greenberg's  $\mathcal{L}$ -invariant of  $\rho_{2n,n}$ ) that Conjecture 0.1 implies the above conjecture for  $\rho_{2m,m}$ . Here are some additional remarks about the conjecture:

- (1) When  $n = 2m$  with even  $m$ , the motive associated to  $\text{Sym}^{\otimes n}(\rho_E)(-m)$  is not critical at  $s = 1$ ; so, the situation is drastically different (and in such a case, we do not make any conjecture; see [H00] Examples 2.7 and 2.8).
- (2) The above conjecture applies to arithmetic and analytic  $p$ -adic  $L$ -functions.

We let  $\sigma \in \text{Gal}(\overline{F}/F)$  act on the Lie algebra of  $G_n/K$

$$\mathfrak{s}_n(K) = \{x \in M_{n+1}(K) \mid \text{Tr}(x) = 0 \text{ and } {}^t x J_n + J_n x = 0\}$$

by conjugation:  $x \mapsto \sigma x = \rho_{n,0}(\sigma)x\rho_{n,0}(\sigma)^{-1}$ . This representation  $\text{Ad}(\rho_{n,0})$  is isomorphic to  $\bigoplus_{0 < j \leq n, j: \text{odd}} \rho_{2j,j}$  and is called the adjoint square representation of  $\rho_{n,0}$ . By using a canonical isomorphism between the tangent space of  $\text{Spf}(R_n)$  and a certain Selmer group of  $\text{Ad}(\rho_{n,0})$ , we get

**Theorem 0.3.** *Let  $m$  be an odd positive integer. Assume Conjecture 0.1 for all odd integers  $n$  with  $0 < n \leq m$ . Then Conjecture 0.2 holds for Greenberg's  $\mathcal{L}$ -invariant of  $\rho_{2m,m}$ .*

All the assumptions in [Gr] (particularly,  $\text{Sel}_F(\rho_{2m,m}) = 0$ : Lemma 1.2) made to define the invariant can be verified under Conjecture 0.1 for  $\rho_{2m,m}$ . The assumption in the theorem that  $E$  has split (multiplicative) reduction at  $\mathfrak{p}_j$  with  $j \leq b$  is inessential, because  $\text{Ad}(\rho_{n,0}) \cong \text{Ad}(\text{Sym}^{\otimes n}(\rho_E \otimes \chi))$  (for a  $K^\times$ -valued Galois character  $\chi$ ) and we can bring any elliptic curve with multiplicative reduction at  $\mathfrak{p}_j$  to an elliptic curve with split multiplicative reduction at  $\mathfrak{p}_j$  by a quadratic twist. We will prove this theorem as Theorem 1.14 later.

Conjecture 0.1 and Conjecture 0.2 are logically close. Since  $\rho_{2m,m}$  is self dual, the complex  $L$ -function  $L(s, \rho_{2m,m})$  has functional equation of the form  $s \leftrightarrow 1 - s$ , and the complex  $L$ -value  $L(1, \rho_{2m,m})$  should not vanish at  $s = 1$  (the abscissa of convergence). Conjecturally, this should imply  $\text{Sel}_F(\rho_{2m,m}) = 0$ , since  $\rho_{2m,m}$  with odd  $m$  is critical at 1. This vanishing is essential for Greenberg's definition of his  $\mathcal{L}$ -invariant to work (especially in his definition of the subspace  $\tilde{\mathbf{T}} \subset H^1(\text{Gal}(\overline{F}/F), \rho_{2m,m})$  ( $\tilde{\mathbf{T}}$  is written later as  $\mathbf{H}_F$  in this paper; see [Gr] page 163–4). Conjecture 0.1 for an integer  $n \geq m$  implies  $\text{Sel}_F(\rho_{2m,m}) = 0$  for odd  $m > 0$  (see Lemma 1.2). Indeed, at least in appearance, a much weaker infinitesimal version than Conjecture 0.1 asserting that  $R_n$  shares the tangent space with

$K[[X_{j,\mathfrak{p}}]]_{\mathfrak{p}|p, 0 < j \leq n, j: \text{odd}}$  (that is,  $K[[X_{j,\mathfrak{p}}]]/(X_{j,\mathfrak{p}})^2 \cong R_n/\mathfrak{m}_n^2$ ) is sufficient for this vanishing  $\text{Sel}_F(\rho_{2m,m}) = 0$  and to prove Conjecture 0.2. However, for example, if  $m = 1$  and  $n = 1$ , any characteristic 0  $p$ -adic (motivic) Galois deformation  $\rho$  over  $\mathbb{Z}_p$  (not over  $\mathbb{Q}_p$  in Conjecture 0.1) of  $\bar{\rho} := (\rho_E \bmod p)$  has its  $p$ -adic  $L$ -function  $L_p(s, \rho_{2,1})$  with an exceptional zero at  $s = 1$ . Thus the weaker infinitesimal statement at each  $\rho$  should actually imply the stronger statement as in Conjecture 0.1 (if we admit the “ $R = T$ ” theorem as in [MFG] Theorem 5.29 for  $F = \mathbb{Q}$  or [HMI] Theorem 3.50 for general  $F$  for nearly ordinary deformations). In this sense, the two conjectures are almost equivalent if we include motivic deformations  $\rho$  of  $\bar{\rho}$  in the scope of Conjecture 0.2 not limiting ourselves to elliptic curves. This point will be discussed in more details in our future work.

### 1. SYMMETRIC TENSOR $\mathcal{L}$ -INVARIANT

We recall briefly an  $F$ -version (given in [HMI] Definition 3.85) of Greenberg’s formula of the  $\mathcal{L}$ -invariant for a general  $p$ -adic totally  $p$ -ordinary Galois representation  $V$  (of  $\text{Gal}(\bar{F}/F)$ ) with an exceptional zero. This definition is equivalent to the one in [Gr] if we apply it to  $\text{Ind}_F^{\mathbb{Q}} V$  as proved in [HMI] (in Definition 3.85). When  $V = \rho_{2m,m}$  with odd  $m$ , the definition can be outlined as follows. Under some hypothesis, he found a unique subspace  $\mathbf{H} \subset H^1(\mathbb{Q}, \text{Ind}_F^{\mathbb{Q}} \rho_{2m,m})$  of dimension  $e$ . By Shapiro’s lemma,  $H^1(\mathbb{Q}, \text{Ind}_F^{\mathbb{Q}} \rho_{2m,m}) \cong H^1(F, \rho_{2m,m})$ , and one can give a definition of the image  $\mathbf{H}_F$  of  $\mathbf{H}$  in  $H^1(F, \rho_{2m,m})$  without reference to the induction  $\text{Ind}_F^{\mathbb{Q}} \rho_{2m,m}$  ([HMI] Definition 3.85) as we recall the precise definition later (see Lemma 1.7). The space  $\mathbf{H}_F$  is represented by cocycles  $c : \text{Gal}(\bar{F}/F) \rightarrow \rho_{2m,m}$  such that

- (1)  $c$  is unramified outside  $p$ ;
- (2)  $c$  restricted to the decomposition subgroup  $\text{Gal}(\bar{F}_{\mathfrak{p}}/F_{\mathfrak{p}}) \cong D_{\mathfrak{p}} \subset \text{Gal}(\bar{F}/F)$  at each  $\mathfrak{p}|p$  has values in  $\mathcal{F}_{\mathfrak{p}}^- \rho_{2m,m}$  and  $c|_{D_{\mathfrak{p}}}$  modulo  $\mathcal{F}_{\mathfrak{p}}^+ \rho_{2m,m}$  becomes unramified over  $F_{\mathfrak{p}}[\mu_{p^\infty}]$  for all  $\mathfrak{p}|p$ .

Here  $\mathcal{F}_{\mathfrak{p}}^- \rho_{2m,m} = \mathcal{F}_{\mathfrak{p}}^0 \rho_{2m,m}$ ,  $\mathcal{F}_{\mathfrak{p}}^+ \rho_{2m,m} = \mathcal{F}_{\mathfrak{p}}^1 \rho_{2m,m}$ , and  $\mathcal{F}^j \rho_{2m,m}$  is the decreasing filtration on  $\rho_{2m,m}$  such that  $I_{\mathfrak{p}}$  acts by  $\mathcal{N}^j$  on  $\mathcal{F}_{\mathfrak{p}}^j \rho_{2m,m} / \mathcal{F}_{\mathfrak{p}}^{j+1} \rho_{2m,m}$ .

Let  $\mathbb{Q}_\infty/\mathbb{Q}$  be the cyclotomic  $\mathbb{Z}_p$ -extension, and put  $F_\infty/F$  for the composite of  $F$  and  $\mathbb{Q}_\infty$ . By the condition (2),  $(c|_{D_{\mathfrak{p}'}} \bmod \mathcal{F}_{\mathfrak{p}'}^+ \rho_{2m,m})$  with a prime  $\mathfrak{p}'|p$  may be regarded as a homomorphism  $a : D_{\mathfrak{p}'} \rightarrow K$  because  $\mathcal{F}_{\mathfrak{p}'}^- \rho_{2m,m} / \mathcal{F}_{\mathfrak{p}'}^+ \rho_{2m,m}$  is isomorphic to the trivial  $D_{\mathfrak{p}'}$ -module  $K$ . Hence  $a$  becomes unramified everywhere over the cyclotomic  $\mathbb{Z}_p$ -extension  $F_\infty/F$ . In other words, the cohomology class  $[c]$  is in  $\text{Sel}_{F_\infty}(\rho_{2m,m})$  but not in  $\text{Sel}_F(\rho_{2m,m})$ . In other words, we have

$$\mathbf{H}_F \cong \text{Sel}_F^{cyc}(\rho_{2m,m}) := \text{Res}^{-1}(\text{Sel}_{F_\infty}(\rho_{2m,m}))$$

for the restriction map  $\text{Res} : H^1(F, \rho_{2m,m}) \rightarrow H^1(F_\infty, \rho_{2m,m})$  (see the definition of various Selmer groups given in the following section).

Take a basis  $\{c_{\mathfrak{p}}\}_{\mathfrak{p}|p}$  of  $\mathbf{H}_F$  over  $K$ . Write  $a_{\mathfrak{p}} : D_{\mathfrak{p}'} \rightarrow K$  for  $c_{\mathfrak{p}} \bmod \mathcal{F}_{\mathfrak{p}'}^+ \rho_{2m,m}$  regarded as a homomorphism (identifying  $\mathcal{F}_{\mathfrak{p}'}^- \rho_{2m,m} / \mathcal{F}_{\mathfrak{p}'}^+ \rho_{2m,m}$  with  $K$ ). We now have two  $e \times e$  matrices with coefficients in  $K$ :  $A = (a_{\mathfrak{p}}([p, F_{\mathfrak{p}'}}]))_{\mathfrak{p}, \mathfrak{p}'|p}$  and  $B = (\log_p(\gamma_{\mathfrak{p}'})^{-1} a_{\mathfrak{p}}([\gamma_{\mathfrak{p}'}, F_{\mathfrak{p}'}}]))_{\mathfrak{p}, \mathfrak{p}'|p}$ . Under Conjecture 0.1 for  $\rho_{n,0}$  for all odd  $n \leq m$ , we can show that  $B$  is invertible. Then Greenberg’s  $\mathcal{L}$ -invariant is defined by

$$(1.1) \quad \mathcal{L}(\text{Ind}_F^{\mathbb{Q}} \rho_{2m,m}) = \det(AB^{-1}).$$

The determinant  $\det(AB^{-1})$  is independent of the choice of the basis  $\{c_{\mathfrak{p}}\}_{\mathfrak{p}}$ . Though  $L(s, \text{Ind}_F^{\mathbb{Q}} \rho) = L(s, \rho)$  for a Galois representation  $\rho : \text{Gal}(\overline{\mathbb{Q}}/F) \rightarrow GL_n(K)$  in a compatible system, the (nonvanishing) modification Euler  $p$ -factors  $\mathcal{E}^+(\rho)$  and  $\mathcal{E}^+(\text{Ind}_F^{\mathbb{Q}} \rho)$  (cf. [Gr] (6)) to define the corresponding  $p$ -adic  $L$ -functions could be different (see [H07] (1.1)). Thus the  $\mathcal{L}(\rho)$  and  $\mathcal{L}(\text{Ind}_F^{\mathbb{Q}} \rho)$  could be slightly different. As in [H07] (1.1), we have the following relation

$$(1.2) \quad \mathcal{L}(\rho_{2m,m}) = \left( \prod_{\mathfrak{p}|p} f_{\mathfrak{p}} \right) \mathcal{L}(\text{Ind}_F^{\mathbb{Q}} \rho_{2m,m}),$$

where  $f_{\mathfrak{p}} = [O/\mathfrak{p} : \mathbb{F}_p]$ .

Choose a generator  $\gamma$  of  $\mathcal{N}(\text{Gal}(F_{\infty}/F)) \subset \mathbb{Z}_p^{\times}$  for the  $p$ -adic cyclotomic character  $\mathcal{N}$ , and identify  $\Lambda = W[[\text{Gal}(F_{\infty}/F)]]$  with  $W[[T]]$  by  $\gamma \mapsto 1+T$ . The Selmer group  $\text{Sel}_{F_{\infty}}(\rho_{2m,m}^*) := \text{Sel}_{F_{\infty}}(\text{Sym}^{\otimes 2m}(T_p E)(-m) \otimes (\mathbb{Q}_p/\mathbb{Z}_p))$  has its Pontryagin dual which is a  $\Lambda$ -module of finite type. Choose a characteristic power series  $\Phi^{\text{arith}}(T) \in \Lambda$  of the Pontryagin dual. Put  $L_p^{\text{arith}}(s, \rho_{2m,m}) = \Phi^{\text{arith}}(\gamma^{1-s} - 1)$ . We consider the following condition stronger than (ds):

(ds<sub>m</sub>)  $\overline{\rho}_{m,0}^{ss}$  (for  $\overline{\rho}_{m,0} = \text{Sym}^{\otimes m}(\overline{\rho})$ ) is a direct sum of  $m+1$  distinct characters of  $D_{\mathfrak{p}}$  for all prime factors  $\mathfrak{p}|p$ .

For the known cases of the following conjecture, see [Gr] Proposition 4 and [H07] Theorem 5.3.

**Conjecture 1.1** (Greenberg). *Suppose (ds<sub>m</sub>) and that  $\overline{\rho}_{m,0}$  is absolutely irreducible. Then  $L_p^{\text{arith}}(s, \rho_{2m,m})$  has zero of order equal to  $e = |\{\mathfrak{p}|p\}|$  and for the constant  $\mathcal{L}(\rho_{2m,m}) \in K^{\times}$  given in (1.1) and (1.2), we have*

$$\lim_{s \rightarrow 1} \frac{L_p^{\text{arith}}(s, \rho_{2m,m})}{(s-1)^d} = \mathcal{L}(\rho_{2m,m}) \|\text{Sel}_F(\rho_{2m,m}^*)\|_p^{-1/[K:\mathbb{Q}_p]}$$

up to units.

This conjecture has been proven by Greenberg (see [Gr] Proposition 4) for more general ordinary Galois representation than  $\rho_{2m,m}$  under some (mild, believable but possibly restrictive) assumptions. Especially the assumption (5) in [Gr] proposition 4 is difficult to verify just by assuming (ds<sub>m</sub>) and absolute irreducibility of  $\overline{\rho}_{n,0}$  and could be far deeper (even for those of adjoint type like  $\rho_{2m,m}$ ) than the modularity statement like Conjecture 0.1; so, unfortunately, the above statement remains to be a conjecture.

In the above conjecture, the modifying Euler factor at the  $p$ -adic places  $\mathfrak{p}_j$  of good reduction ( $j > b$ ):

$$\mathcal{E}^+(\rho_{2m,m}) = \prod_{j>b} \left( \prod_{i=1}^m (1 - \alpha_j^{-2i} N(\mathfrak{p}_i)^{i-1})(1 - \alpha_j^{-2i} N(\mathfrak{p}_i)^i) \right)$$

does not appear, where  $\alpha_j = \alpha_j(\text{Frob}_{\mathfrak{p}_j})$ . However, if we replace Greenberg's Selmer group  $\text{Sel}_F(\rho_{2m,m}^*)$  by the Bloch-Kato Selmer group  $S_F(\rho_{2m,m}^*)$  over  $F$  (crystalline at  $\mathfrak{p}_j$  for  $j > b$ ), we expect to have the relation

$$\|\text{Sel}_F(\rho_{2m,m}^*)\|_p^{-1/[K:\mathbb{Q}_p]} = \mathcal{E}^+(\rho_{2m,m}) \|S_F(\rho_{2m,m}^*)\|_p^{-1/[K:\mathbb{Q}_p]}$$

up to  $p$ -adic units (as described in [MFG] page 284 for  $\rho_{2,1}$ ). Thus if one uses the formulation of Bloch-Kato, we should have the modifying Euler factor in the

formula, and the size of the Bloch-Kato Selmer group is expected to be equal to the primitive archimedean  $L$ -values (divided by a suitable period; see Greenberg's Conjecture 0.1 in [H06]).

**1.1. Selmer Groups.** First we recall Greenberg's definition of Selmer groups. Write  $F^{(S)}/F$  for the maximal extension unramified outside  $S$ ,  $p$  and  $\infty$ . Put  $\mathfrak{G} = \text{Gal}(F^{(S)}/F)$  and  $\mathfrak{G}_M = \text{Gal}(F^{(S)}/M)$ . Let  $V$  be a potentially ordinary representation of  $\mathfrak{G}$  on a  $K$ -vector space  $V$ . Thus  $V$  has decreasing filtration  $\mathcal{F}_p^i V$  such that an open subgroup of  $I_p$  (for each prime factor  $\mathfrak{p}|p$ ) acts on  $\mathcal{F}_p^i V/\mathcal{F}_p^{i+1} V$  by the  $i$ -th power  $\mathcal{N}^i$  of the  $p$ -adic cyclotomic character  $\mathcal{N}$ . We fix a  $W$ -lattice  $T$  in  $V$  stable under  $\mathfrak{G}$ .

Put  $\mathcal{F}_p^+ V = \mathcal{F}_p^1 V$  and  $\mathcal{F}_p^- V = \mathcal{F}_p^0 V$ . Writing  $\mathcal{F}_p^\bullet T = T \cap \mathcal{F}_p^\bullet V$  and  $\mathcal{F}_p^\bullet V/T = \mathcal{F}_p^\bullet V/\mathcal{F}_p^\bullet T$ , we have a 3-step filtration for  $A = V, T$  or  $V/T$ :

$$(\text{ord}) \quad A \supset \mathcal{F}_p^- A \supset \mathcal{F}_p^+ A \supset \{0\}.$$

Its dual  $V^*(1) = \text{Hom}_K(V, K) \otimes \mathcal{N}$  again satisfies (ord).

Let  $M/F$  be a subfield of  $F^{(S)}$ , and put  $\mathfrak{G}_M = \text{Gal}(F^{(S)}/M)$ . We write  $\mathfrak{p}$  for a prime of  $M$  over  $p$  and  $\mathfrak{q}$  for general primes outside  $p$  of  $M$ . We write  $I_p$  and  $I_q$  for the inertia subgroup in  $\mathfrak{G}_M$  at  $\mathfrak{p}$  and  $\mathfrak{q}$ , respectively. We put

$$L_p(A) = \text{Ker}(\text{Res} : H^1(M_p, A) \rightarrow H^1(I_p, \frac{A}{\mathcal{F}_p^+(A)})),$$

and

$$L_q(A) = \text{Ker}(\text{Res} : H^1(M_q, A) \rightarrow H^1(I_q, A)).$$

Then we define the Selmer submodule in  $H^1(M, A)$  by

$$(1.3) \quad \text{Sel}_M(A) = \text{Ker}(H^1(\mathfrak{G}_M, A) \rightarrow \prod_{\mathfrak{q}} \frac{H^1(M_q, A)}{L_q(A)} \times \prod_{\mathfrak{p}} \frac{H^1(M_p, A)}{L_p(A)}))$$

for  $A = V, V/T$ . The classical Selmer group of  $V$  is given by  $\text{Sel}_M(V/T)$ , equipped with discrete topology. We define the "minus", the "locally cyclotomic" and the "strict" Selmer groups  $\text{Sel}_M^-(A)$ ,  $\text{Sel}_M^{\text{cyc}}(A)$  and  $\text{Sel}_M^{\text{st}}(A)$ , respectively, replacing  $L_p(A)$  by

$$L_p^-(A) = \text{Ker}(\text{Res} : H^1(M_p, V) \rightarrow H^1(I_p, \frac{V}{\mathcal{F}_p^-(A)})) \supset L_p(A)$$

$$L_p^{\text{cyc}}(A) = \text{Ker}(\text{Res} : L_p^-(A) \rightarrow H^1(I_{p,\infty}, \frac{V}{\mathcal{F}_p^+(A)})) \subset L_p^-(A)$$

$$L_p^{\text{st}}(A) = \text{Ker}(\text{Res} : L_p^-(A) \rightarrow H^1(M_p, \frac{V}{\mathcal{F}_p^+(A)})) \subset L_p(A),$$

where  $I_{p,\infty}$  is the inertia group of  $\text{Gal}(\overline{M}_p/M_p[\mu_{p^\infty}])$ . Then we have

$$\text{Sel}_F^{\text{cyc}}(A) = \text{Res}_{F_\infty/F}^{-1}(\text{Sel}_{F_\infty}(A)).$$

**Lemma 1.2.** *We have*

$$\text{Sel}_F^{\text{cyc}}(\text{Ad}(\rho_{n,0})) \cong \bigoplus_{0 < m \leq n, m: \text{odd}} \text{Sel}_F^{\text{cyc}}(\rho_{2m,m}) \cong \text{Hom}_K(\mathfrak{m}_n/\mathfrak{m}_n^2, K),$$

where  $\mathfrak{m}_n$  is the maximal ideal of  $R_n$ . If we suppose Conjecture 0.1 for odd  $n > 0$ , we have  $\text{Sel}_F(\rho_{2m,m}) = 0$  for all odd  $m$  with  $0 < m \leq n$ .

*Proof.* Let  $V = Ad(\rho_{n,0})$ . Then we have the filtration:

$$V \supset \mathcal{F}_p^- V \supset \mathcal{F}_p^+ V \supset \{0\},$$

where taking a basis so that the semi-simplification of  $\rho_{n,0}|_{D_p}$  is diagonal with diagonal character  $\beta_p^n, \beta_p^{n-1}\alpha_p, \dots, \alpha_p^n$  in this order from top to bottom,  $\mathcal{F}_p^- V$  is made up of upper triangular matrices and  $\mathcal{F}_p^+ V$  is made up of upper nilpotent matrices, and on  $\mathcal{F}_p^- V/\mathcal{F}_p^+ V$ ,  $D_p$  acts trivially (getting eigenvalue 1 for  $Frob_p$ ). We consider the space  $Der_K(R_n, K)$  of continuous  $K$ -derivations of  $R_n$ . Let  $K[\varepsilon] = K[t]/(t^2)$  for the dual number  $\varepsilon = (t \bmod t^2)$ . Then writing each  $K$ -algebra homomorphism  $\phi : R_n \rightarrow K[\varepsilon]$  as  $\phi(r) = \phi_0(r) + \partial_\phi(r)\varepsilon$  and sending  $\phi$  to  $\partial_\phi \in Der_K(R_n, K)$ , we have  $\text{Hom}_{K\text{-alg}}(R_n, K[\varepsilon]) \cong Der_K(R_n, K) = \text{Hom}_K(\mathfrak{m}_n/\mathfrak{m}_n^2, K)$ . By the universality of  $(R_n, \rho_n)$ , we have

$$\text{Hom}_{K\text{-alg}}(R_n, K[\varepsilon]) \cong \frac{\{\rho : \text{Gal}(\overline{F}/F) \rightarrow G_n(K[\varepsilon]) \mid \rho \text{ satisfies (K}_n\text{1-4)}\}}{\cong}$$

by  $\text{Hom}_{K\text{-alg}}(R_n, K[\varepsilon]) \ni \phi \mapsto \rho_\phi = \phi \circ \rho_n = \rho_{n,0} + \varepsilon \partial_\phi \rho_n$ . Pick  $\rho = \rho_\phi$  as above. Write  $\rho(\sigma) = \rho_0(\sigma) + \rho_1(\sigma)\varepsilon$  with  $\rho_1(\sigma) = \frac{\partial \rho}{\partial t} = \partial_\phi \rho_n(\sigma)$ . Then  $c_\rho = (\partial_\phi \rho_n) \rho_{n,0}^{-1}$  can be easily checked to be an inhomogeneous 1-cocycle having values in  $M_{n+1}(K) \supset V$ . Here  $\sigma \in \text{Gal}(\overline{F}/F)$  acts on  $x \in M_{n+1}(K)$  by  $x \mapsto \rho_{n,0}(\sigma)x\rho_{n,0}(\sigma)^{-1}$ .

Since  $\nu \circ \rho = \nu \circ \rho_{n,0}$  by (K<sub>n</sub>3), we have  $\det(\rho) = \det(\rho_{n,0})$ , which implies  $\text{Tr}(c_\rho) = 0$ ; so,  $c_\rho$  has values in  $\mathfrak{sl}_{n+1}(K)$ . For  $\partial \in Der_K(R_n, K)$  and  $X \in GL_{n+1}(R_n)$  with  ${}^t X J_n X = J_n$ , writing  $\overline{X} = (X \bmod \mathfrak{m}_n) \in GL_{n+1}(K)$

$$0 = \partial(X^{-1}X) = \overline{X}^{-1} \partial X + (\partial X^{-1}) \overline{X}.$$

Since  ${}^t \rho_n J_n \rho_n = \mathcal{N}^n J_n = {}^t \rho_{n,0} J_n \rho_{n,0}$ , we have  ${}^t \rho_{n,0}^{-1} {}^t \rho_n J_n \rho_n \rho_{n,0}^{-1} = J_n$ . Let  $X = \rho_n \rho_{n,0}^{-1}$ . Differentiating the identity:  ${}^t X J_n X = J_n$  by  $\partial$ , we have  $({}^t \partial X J_n) \overline{X} + {}^t \overline{X} (J_n \partial X) = 0$ , which is equivalent to  $c_\rho(\sigma) \in \mathfrak{s}_n(K) = V$ . By the reducibility condition (K<sub>n</sub>2),  $[c_\rho]$  vanishes in  $\frac{H^1(M_p, V)}{L_p^-(V)}$ . By the local cyclotomy condition in

(K<sub>n</sub>2),  $[c_\rho]$  vanishes in  $\frac{H^1(M_p, V)}{L_p^{cyc}(V)}$ . If  $E$  has multiplicative reduction at  $\mathfrak{q}$  (so,  $\mathfrak{q} \in S$ ), the unramifiedness of  $c_\rho$  follows from the following lemma. Thus the cohomology class  $[c_\rho]$  of  $c_\rho$  is in  $\text{Sel}_F^{cyc}(V)$ . We see easily that  $\rho \cong \rho' \Leftrightarrow [c_\rho] = [c_{\rho'}]$ .

We can reverse the above argument starting with a cocycle  $c$  giving an element of  $\text{Sel}_F^{cyc}(V)$  to construct a deformation  $\rho_c = \rho_{n,0} + \varepsilon(c\rho_{n,0})$  with values in  $G_n(K[\varepsilon])$ . Thus we have

$$\frac{\{\rho : \text{Gal}(\overline{F}/F) \rightarrow G_n(K[\varepsilon]) \mid \rho \text{ satisfies the conditions (K}_n\text{1-4)}\}}{\cong} \cong \text{Sel}_F^{cyc}(V).$$

Recall that the isomorphism  $Der_K(R_n, K) \cong \text{Sel}_F^{cyc}(V)$  is given by

$$Der_K(R_n, K) \ni \partial \mapsto [c_\partial] \in \text{Sel}_F^{cyc}(V)$$

for the cocycle  $c_\partial = c_\rho = (\partial \rho_n) \rho_{n,0}^{-1}$ , where  $\rho = \rho_{n,0} + \varepsilon(\partial \rho_n)$ .

Suppose Conjecture 0.1. Since the algebra structure of  $R_n$  over  $W[[X_{j,p}]]_{\mathfrak{p}|p}$  is given by  $\delta_{j,p}(\beta_p^{n-j}\alpha_p^j)^{-1}$  and  $\delta_{n-j,p}\delta_{j,p} = \mathcal{N}^n$ , the  $K$ -derivation  $\partial = \partial_\phi : R_n \rightarrow K$  corresponding to a  $K[\varepsilon]$ -deformation  $\rho$  is a  $W[[X_{j,p}]]$ -derivation for odd  $j$  if and only if  $\partial \rho_n|_{I_p}$  is upper nilpotent, which is equivalent to  $[c_\partial] \in \text{Sel}_F(V)$ . Thus we have  $\text{Sel}_F(V) \cong Der_{W[[X_p]]}(R_n, K) = 0$ . Since  $V \cong \bigoplus_{0 < m \leq n, m: \text{odd}} \rho_{2m, m}$  as global Galois modules, we have  $\text{Sel}_F(V) \cong \bigoplus_{0 < m \leq n, m: \text{odd}} \text{Sel}_F(\rho_{2m, m})$ , and we conclude  $\text{Sel}_F(\rho_{2m, m}) = 0$ .  $\square$

**Lemma 1.3.** *Let  $\mathfrak{q}$  be a prime outside  $p$  at which  $E$  has potentially multiplicative reduction. Then for a deformation  $\rho$  of  $\rho_{n,0}$  satisfying (K<sub>n</sub>1–4), the cocycle  $c_\rho$  (defined in the above proof) is unramified at  $\mathfrak{q}$ .*

*Proof.* Since  $Ad((\rho_E \otimes \eta)_{n,0}) \cong Ad(\rho_{n,0})$  twisting by a character  $\eta$ , we may assume that the restriction of  $\rho_E$  to the inertia group  $I_{\mathfrak{q}}$  has values in the upper unipotent subgroup having the form  $\begin{pmatrix} 1 & \xi_{\mathfrak{q}}(\sigma) \\ 0 & 1 \end{pmatrix}$  for  $\sigma \in I_{\mathfrak{q}}$  up to conjugation. Thus we may assume

$$\rho_{n,0}|_{I_{\mathfrak{q}}} = \begin{pmatrix} 1 & n\xi_{\mathfrak{q}} & \binom{n}{2}\xi_{\mathfrak{q}}^2 & \cdots & \xi_{\mathfrak{q}}^n \\ 0 & 1 & (n-1)\xi_{\mathfrak{q}} & \cdots & \xi_{\mathfrak{q}}^{n-1} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & \xi_{\mathfrak{q}} \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix}.$$

Since  $I_{\mathfrak{q}} \ni \sigma \mapsto \log(\rho_{n,0}(\sigma))$  is a homomorphism of  $I_{\mathfrak{q}}$  into the Lie algebra  $\mathfrak{u}_n$  of the unipotent radical of the Borel subgroup of  $G_n$  containing the image of  $I_{\mathfrak{q}}$ , it factors through the tame inertia group  $\cong \widehat{\mathbb{Z}}^{(a)}(1)$ . By the theory of Tate curves,  $\rho_{n,0}$  ramifies at  $\mathfrak{q}$  and hence  $\xi_{\mathfrak{q}}$  is nontrivial. The  $p$ -factor of  $\widehat{\mathbb{Z}}^{(a)}$  is of rank 1 isomorphic to  $\mathbb{Z}_p(1)$ . Then  $\rho(I_{\mathfrak{q}})$  is cyclic, and therefore  $\dim_K \rho(I_{\mathfrak{q}}) = 1 = \dim_K \rho_{n,0}(I_{\mathfrak{q}})$ . Thus the deformation  $\rho$  is constant over the inertia subgroup, and hence  $c_\rho$  restricted to  $I_{\mathfrak{q}}$  is trivial.  $\square$

**Corollary 1.4.** *Let  $n$  be an odd positive integer. Suppose Conjecture 0.1 for all odd integers  $m$  with  $0 < m \leq n$ . Then we have  $\dim_K \text{Sel}_F^{cyc}(\rho_{2n,n}) = e$ .*

*Proof.* Let  $V = \rho_{2n,n}$ . By Lemma 1.2, we have  $\dim_K \text{Sel}_F^{cyc}(Ad(\rho_{m,0})) = e \cdot \frac{m+1}{2}$ . Since

$$\text{Sel}_F^{cyc}(Ad(\rho_{n,0})) = \text{Sel}_F^{cyc}(Ad(\rho_{n-2,0})) \oplus \text{Sel}_F^{cyc}(V),$$

we find that  $\dim_K \text{Sel}_F^{cyc}(V) = e$ .  $\square$

Let  $\rho_{n,m} = \text{Sym}^{\otimes n}(\rho_E)(-m)$ , and write  $V$  for either the representation space of  $\rho_{n,m}$  or that of  $Ad(\rho_{n,0})$ . For each prime  $\mathfrak{q} \in S \cup \{\mathfrak{p}|p\}$ , we put

$$\overline{L}_{\mathfrak{q}}(V) = \begin{cases} \text{Ker}(H^1(F_{\mathfrak{q}}, V) \rightarrow H^1(F_{\mathfrak{q}}, \frac{V}{\mathcal{F}_{\mathfrak{p}_j^+}(V)})) \subset L_{\mathfrak{p}_j}(V) & \text{if } \mathfrak{q} = \mathfrak{p}_j \text{ with } j \leq b, \\ L_{\mathfrak{q}}(V) & \text{otherwise} \end{cases}$$

Once  $\overline{L}_{\mathfrak{q}}(V)$  is defined, we define  $\overline{L}_{\mathfrak{q}}(V^*(1)) = \overline{L}_{\mathfrak{q}}(V)^\perp$  under the local Tate duality between  $H^1(F_{\mathfrak{q}}, V)$  and  $H^1(F_{\mathfrak{q}}, V^*(1))$ , where  $V^*(1) = \text{Hom}_K(V, \mathbb{Q}_p(1))$  as Galois modules. Then we define the balanced Selmer group  $\overline{\text{Sel}}_F(V)$  (resp.  $\overline{\text{Sel}}_F(V^*(1))$ ) by the same formula as in (1.3) replacing  $L_{\mathfrak{p}}(V)$  (resp.  $L_{\mathfrak{p}}(V^*(1))$ ) by  $\overline{L}_{\mathfrak{p}}(V)$  (resp.  $\overline{L}_{\mathfrak{p}}(V^*(1))$ ). By definition,  $\overline{\text{Sel}}_F(V) \subset \text{Sel}_F(V)$ . We will show in Lemma 1.6,  $\overline{L}_{\mathfrak{p}}(V) = L_{\mathfrak{p}}(V)$  for  $V = Ad(\rho_{n,0})$  and  $\rho_{2n,n}$  for odd  $n$ , and we actually have  $\text{Sel}_F(V) = \overline{\text{Sel}}_F(V)$ .

**Lemma 1.5.** *Let  $V$  be  $Ad(\rho_{n,0})$  or  $\rho_{n,m}$ . If  $V$  is critical at  $s = 1$ ,*

$$(V) \quad \text{Sel}_F(V) = 0 \Rightarrow H^1(\mathfrak{G}, V) \cong \prod_{\mathfrak{q} \in S} \frac{H^1(F_{\mathfrak{q}}, V)}{L_{\mathfrak{q}}(V)} \times \prod_{\mathfrak{p}|p} \frac{H^1(F_{\mathfrak{p}}, V)}{\overline{L}_{\mathfrak{p}}(V)}.$$

*Proof.* Since  $\overline{\text{Sel}}_F(V) \subset \text{Sel}_F(V)$ , the assumption implies  $\overline{\text{Sel}}_F(V) = 0$ . Then the Poitou-Tate exact sequence tells us the exactness of the following sequence:

$$\overline{\text{Sel}}_F(V) \rightarrow H^1(\mathfrak{G}, V) \rightarrow \prod_{l \in S \sqcup \{p\}} \frac{H^1(F_l, V)}{\overline{L}_l(V)} \rightarrow \overline{\text{Sel}}_F(V^*(1))^*.$$

It is an old theorem of Greenberg (which assumes criticality at  $s = 1$ ) that

$$\dim \overline{\text{Sel}}_F(V) = \dim \overline{\text{Sel}}_F(V^*(1))^*$$

(see [Gr] Proposition 2 or [HMI] Proposition 3.82); so, we have the assertion (V). In [HMI], Proposition 3.82 is formulated in terms of  $\overline{\text{Sel}}_{\mathbb{Q}}(\text{Ind}_F^{\mathbb{Q}} V)$  and  $\overline{\text{Sel}}_{\mathbb{Q}}(\text{Ind}_F^{\mathbb{Q}} V^*(1))$  defined in [HMI] (3.4.11), but this does not matter because we can easily verify  $\overline{\text{Sel}}_{\mathbb{Q}}(\text{Ind}_F^{\mathbb{Q}}?) \cong \overline{\text{Sel}}_F(?)$  (similarly to [HMI] Corollary 3.81).  $\square$

**1.2. Greenberg's  $\mathcal{L}$ -invariant.** In this subsection, we let  $V = \rho_{2n,n}$  or  $Ad(\rho_{n,0})$  for odd  $n$  (so,  $V$  is critical at  $s = 1$ ). Write  $t(\mathfrak{p})$  for  $\dim \mathcal{F}_{\mathfrak{p}}^- V / \mathcal{F}_{\mathfrak{p}}^+ V$  (thus,  $t(\mathfrak{p}) = 1$  or  $\frac{n+1}{2}$  according as  $V = \rho_{2n,n}$  or  $Ad(\rho_{n,0})$ ). We recall a little more detail of the  $F$ -version of Greenberg's definition of  $\mathcal{L}(\text{Ind}_F^{\mathbb{Q}} V)$  (which is equivalent to the one given in [Gr] if we apply Greenberg's definition to  $\text{Ind}_F^{\mathbb{Q}} V$  as explained in [HMI] 3.4.4 without assuming the simplifying condition). Let  $F_{\mathfrak{p}}^{gal}$  be the Galois closure of  $F_{\mathfrak{p}}/\mathbb{Q}_p$  in  $\overline{\mathbb{Q}_p}$ . Write  $D_p = \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ ,  $D_{\mathfrak{p}} = \text{Gal}(\overline{\mathbb{Q}_p}/F_{\mathfrak{p}})$  and  $D_p^{gal} = \text{Gal}(\overline{\mathbb{Q}_p}/F_{\mathfrak{p}}^{gal})$ . Write  $D_L = \text{Gal}(\overline{\mathbb{Q}_p}/L)$  for an intermediate field  $L$  of  $F_{\mathfrak{p}}^{gal}/\mathbb{Q}_p$ . For a  $D_L$ -module  $M$  (which is a  $K$ -vector space), the group  $D_L$  acts on  $H^{\bullet}(F_{\mathfrak{p}}^{gal}, M)$  naturally through the finite quotient  $\text{Gal}(F_{\mathfrak{p}}^{gal}/L)$ . Since, for  $q > 0$ ,

$$H^q(\text{Gal}(F_{\mathfrak{p}}^{gal}/L), H^0(D_p^{gal}, M)) = 0,$$

by the inflation-restriction sequence, taking  $L = \mathbb{Q}_p$  and  $L = F_{\mathfrak{p}}$ , we verify that  $H^1(F_{\mathfrak{p}}^{gal}, M)^{D_p}$  is canonically isomorphic to a subspace of  $H^1(F_{\mathfrak{p}}, M)$  even if  $F_{\mathfrak{p}}/\mathbb{Q}_p$  is not a normal extension. We regard  $H^1(F_{\mathfrak{p}}^{gal}, M)^{D_p}$  as a subspace of  $H^1(F_{\mathfrak{p}}, M)$ .

The long exact sequence associated to the short one  $\mathcal{F}_{\mathfrak{p}}^- V / \mathcal{F}_{\mathfrak{p}}^+ V \hookrightarrow V / \mathcal{F}_{\mathfrak{p}}^+ V \twoheadrightarrow V / \mathcal{F}_{\mathfrak{p}}^- V$  gives a homomorphism

$$H^1(F_{\mathfrak{p}}^{gal}, \frac{\mathcal{F}_{\mathfrak{p}}^- V}{\mathcal{F}_{\mathfrak{p}}^+ V})^{D_p} = \text{Hom}((D_p^{gal})^{ab}, \frac{\mathcal{F}_{\mathfrak{p}}^- V}{\mathcal{F}_{\mathfrak{p}}^+ V})^{D_p} \xrightarrow{\iota_{\mathfrak{p}}} H^1(F_{\mathfrak{p}}^{gal}, V) / \overline{L}_{\mathfrak{p}}(V),$$

where  $D_p$  acts on  $H^1(F_{\mathfrak{p}}^{gal}, \frac{\mathcal{F}_{\mathfrak{p}}^- V}{\mathcal{F}_{\mathfrak{p}}^+ V})$  regarding  $\frac{\mathcal{F}_{\mathfrak{p}}^- V}{\mathcal{F}_{\mathfrak{p}}^+ V}$  as the trivial  $D_p$ -module; so, its action on  $\phi \in \text{Hom}((D_p^{gal})^{ab}, \frac{\mathcal{F}_{\mathfrak{p}}^- V}{\mathcal{F}_{\mathfrak{p}}^+ V})$  is given by  $\phi \mapsto \tau \cdot \phi(\sigma) = \phi(\tau\sigma\tau^{-1})$ . Note that canonically

$$\begin{aligned} H^1(F_{\mathfrak{p}}^{gal}, \frac{\mathcal{F}_{\mathfrak{p}}^- V}{\mathcal{F}_{\mathfrak{p}}^+ V})^{D_p} &\xleftarrow{\sim}_{\text{Res}} \text{Hom}(D_p^{ab}, \frac{\mathcal{F}_{\mathfrak{p}}^- V}{\mathcal{F}_{\mathfrak{p}}^+ V}) \\ &\cong \text{Hom}(\mathbb{Q}_p^{\times}, \frac{\mathcal{F}_{\mathfrak{p}}^- V}{\mathcal{F}_{\mathfrak{p}}^+ V}) \cong (\mathcal{F}_{\mathfrak{p}}^- V / \mathcal{F}_{\mathfrak{p}}^+ V)^2 \cong K^{2t(\mathfrak{p})} \end{aligned}$$

by  $\phi \mapsto (\frac{\phi([\gamma, F_{\mathfrak{p}}]}{\log_p(\gamma)}, \phi([p, F_{\mathfrak{p}}]))$ . Here, as before,  $[x, F_{\mathfrak{p}}]$  is the local Artin symbol. Identifying  $H^1(F_{\mathfrak{p}}^{gal}, \frac{\mathcal{F}_{\mathfrak{p}}^- V}{\mathcal{F}_{\mathfrak{p}}^+ V})^{D_p}$  with  $\text{Hom}(D_p^{ab}, \frac{\mathcal{F}_{\mathfrak{p}}^- V}{\mathcal{F}_{\mathfrak{p}}^+ V})$ , a homomorphism  $\phi : D_p^{ab} \rightarrow \frac{\mathcal{F}_{\mathfrak{p}}^- V}{\mathcal{F}_{\mathfrak{p}}^+ V}$

in  $\text{Ker}(\iota_{\mathfrak{p}})$  is unramified if  $\mathfrak{p} = \mathfrak{p}_i$  with  $i > b$ ; so, the image of  $\iota_{\mathfrak{p}}$  is one-dimensional (those ramified classes modulo unramified ones). In other words, the image of  $\iota_{\mathfrak{p}}$  is isomorphic to  $\mathcal{F}_{\mathfrak{p}}^- V / \mathcal{F}_{\mathfrak{p}}^+ V \cong K^{t(\mathfrak{p})}$ . Even if  $\mathfrak{p} = \mathfrak{p}_j$  with  $j \leq b$ , if  $\overline{L}_{\mathfrak{p}_j}(V) = L_{\mathfrak{p}_j}(V)$ , by the same argument, the image of  $\iota_{\mathfrak{p}}$  is isomorphic to  $\mathcal{F}_{\mathfrak{p}}^- V / \mathcal{F}_{\mathfrak{p}}^+ V \cong K^{t(\mathfrak{p})}$ . The fact  $\overline{L}_{\mathfrak{p}_j}(V) = L_{\mathfrak{p}_j}(V)$  follows from the following  $F$ -version of the argument in [Gr] page 160:

**Lemma 1.6.** *Let  $V = \rho_{2n,n}$  or  $Ad(\rho_{n,0})$  for odd  $n$ . Then we have  $\overline{L}_{\mathfrak{p}}(V) = L_{\mathfrak{p}}(V)$ .*

Thus for  $K$ -vector space  $V$  with Galois action, we have  $\overline{\text{Sel}}_F(V) = \text{Sel}_F(V)$ .

*Proof.* Since we have  $\overline{L}_{\mathfrak{p}}(V) = L_{\mathfrak{p}}(V)$  by definition if  $\mathfrak{p} = \mathfrak{p}_j$  with  $j > b$ ; so, we may assume that  $j \leq b$ . Write  $H^\bullet(M)$  for  $H^\bullet(F_{\mathfrak{p}}, M)$  for  $\text{Gal}(\overline{F}_{\mathfrak{p}}/F_{\mathfrak{p}})$ -modules  $M$ . We need to show the image  $\overline{L}_{\mathfrak{p}}(V)$  of  $H^1(\mathcal{F}_{\mathfrak{p}}^+ V)$  in  $H^1(V)$  is equal to  $L_{\mathfrak{p}}(V) := \text{Ker}(r : H^1(V) \rightarrow H^1(I_{\mathfrak{p}}, \overline{V}))$  for  $\overline{V} = V/\mathcal{F}^+ V$ . We can factor the map  $r$  as  $r = \text{Res} \circ \gamma$  for  $\gamma : H^1(V) \rightarrow H^1(\overline{V})$  and  $\text{Res} : H^1(\overline{V}) \rightarrow H^1(I_{\mathfrak{p}}, \overline{V})$ . Since  $\text{Ker}(\gamma) = \overline{L}_{\mathfrak{p}}(V)$ , we need to show that  $\text{Im}(\gamma) \cap \text{Ker}(\text{Res}) = 0$ .

Writing  $Y = \mathcal{F}_{\mathfrak{p}}^- V / \mathcal{F}_{\mathfrak{p}}^2 V$  and  $\overline{Y} = \mathcal{F}_{\mathfrak{p}}^- V / \mathcal{F}_{\mathfrak{p}}^+ V$ , we have exact sequences of  $D_{\mathfrak{p}}$ -modules:  $Y \hookrightarrow V/\mathcal{F}_{\mathfrak{p}}^2 V \twoheadrightarrow V/\mathcal{F}_{\mathfrak{p}}^- V$  and  $\overline{Y} \hookrightarrow \overline{V} \twoheadrightarrow V/\mathcal{F}_{\mathfrak{p}}^- V$ . Since  $H^0(V/\mathcal{F}_{\mathfrak{p}}^- V) = 0$ , by the long exact sequences of the above two short exact sequences, we find that the natural maps  $H^1(Y) \rightarrow H^1(V/\mathcal{F}_{\mathfrak{p}}^2 V)$  and  $H^1(\overline{Y}) \rightarrow H^1(\overline{V})$  are injective. Identify  $H^1(\overline{Y})$  with its image in  $H^1(\overline{V})$ . We have

$$\text{Im}(\gamma) = \text{Im}(\overline{\gamma} : H^1(Y) \rightarrow H^1(\overline{Y})) \subset H^1(\overline{V}).$$

By the inflation-restriction sequence,

$$\text{Ker}(\text{Res}) = H^1(D_{\mathfrak{p}}/I_{\mathfrak{p}}, \overline{V}^{I_{\mathfrak{p}}}) = \overline{V}^{I_{\mathfrak{p}}} / (Frob_{\mathfrak{p}} - 1)\overline{V}^{I_{\mathfrak{p}}} = \mathcal{F}_{\mathfrak{p}}^- V / \mathcal{F}_{\mathfrak{p}}^+ V.$$

Similarly

$$\begin{aligned} \text{Ker}(\text{Res}_Y : H^1(\overline{Y}) \rightarrow H^1(I_{\mathfrak{p}}, \overline{Y})) &= H^1(D_{\mathfrak{p}}/I_{\mathfrak{p}}, \overline{Y}^{I_{\mathfrak{p}}}) \\ &= \overline{Y}^{I_{\mathfrak{p}}} / (Frob_{\mathfrak{p}} - 1)\overline{Y}^{I_{\mathfrak{p}}} = \mathcal{F}_{\mathfrak{p}}^- Y / \mathcal{F}_{\mathfrak{p}}^+ Y = \mathcal{F}_{\mathfrak{p}}^- V / \mathcal{F}_{\mathfrak{p}}^+ V. \end{aligned}$$

Thus inside  $H^1(\overline{V})$ ,  $\text{Ker}(\text{Res}) = \text{Ker}(\text{Res}_Y)$ , and we may replace  $V$  by  $Y$  in our argument. We therefore need to show that

$$\text{Im}(\overline{\gamma} : H^1(Y) \rightarrow H^1(\overline{Y})) \cap \text{Ker}(\text{Res} : H^1(\overline{Y}) \rightarrow H^1(I_{\mathfrak{p}}, \overline{Y})) = 0.$$

We have the long exact sequence attached to the short one  $\mathcal{F}_{\mathfrak{p}}^+ Y \hookrightarrow Y \rightarrow \overline{Y}$ :

$$0 \rightarrow \overline{Y} = H^0(\overline{Y}) \rightarrow H^1(\mathcal{F}_{\mathfrak{p}}^+ Y) \rightarrow H^1(Y) \xrightarrow{\overline{\gamma}} H^1(\overline{Y}) \rightarrow H^2(\mathcal{F}_{\mathfrak{p}}^+ Y) \rightarrow H^2(Y) = 0.$$

By the non-splitting of the short sequence,  $H^0(\overline{Y})$  injects into  $H^1(\mathcal{F}_{\mathfrak{p}}^+ Y)$ . By the local Tate duality,

$$\dim_K H^2(Y) = \dim_K H^0(\text{Hom}_K(Y, K(1))) = 0 \text{ and } \dim_K H^2(\mathcal{F}_{\mathfrak{p}}^+ Y) = t(\mathfrak{p}).$$

This shows that  $\dim_K H^1(Y) = 2t(\mathfrak{p})d$  and  $\dim_K \text{Im}(\overline{\gamma}) = t(\mathfrak{p})d$ , because by Kummer's theory

$$H^1(K(1)) = K \otimes_{\mathbb{Z}_p} \varprojlim_n F_{\mathfrak{p}}^\times / (F_{\mathfrak{p}}^\times)^{p^n} \cong K^{d+1}$$

and  $H^1(K) \cong \text{Hom}((F_{\mathfrak{p}})^\times, K) \cong K^{d+1}$  for  $d = [F_{\mathfrak{p}}, \mathbb{Q}_p]$ . By the inflation-restriction sequence, we have

$$L_{\mathfrak{p}}(\overline{Y}) := \text{Ker}(H^1(\overline{Y}) \rightarrow H^1(I_{\mathfrak{p}}, \overline{Y})) \cong H^1(D_{\mathfrak{p}}/I_{\mathfrak{p}}, \overline{Y}^{I_{\mathfrak{p}}}) \cong \overline{Y}.$$

Thus  $\dim L_{\mathfrak{p}}(\overline{Y}) + \dim \text{Im}(\overline{\gamma}) = \dim H^1(I_{\mathfrak{p}}, \overline{Y})$ . Thus we need to show  $L_{\mathfrak{p}}(\overline{Y}) + \text{Im}(\overline{\gamma}) = H^1(I_{\mathfrak{p}}, \overline{Y})$ . By the local Tate duality, noting  $Y^*(1) \cong Y$ , this statement is equivalent to

$$\text{Ker}(\delta : H^1(\mathcal{F}_{\mathfrak{p}}^+ Y) \rightarrow H^1(Y)) \cap L_{\mathfrak{p}}(\overline{Y})^{\perp} = 0.$$

Here  $L_{\mathfrak{p}}(\overline{Y})^{\perp} = H_{fl}^1(\mathcal{F}_{\mathfrak{p}}^+ Y) = \overline{Y} \otimes_{\mathbb{Z}_p} \varprojlim_n O_{\mathfrak{p}}^{\times} / (O_{\mathfrak{p}}^{\times})^{p^n} \subset H^1(\overline{Y}(1))$ , because  $\overline{Y}^*(1) = \overline{Y}(1) = K(1)^{t(\mathfrak{p})}$ . Since  $\text{Ker}(\delta)$  gives rise to the subspace spanned by extension class of  $K(1)^{t(\mathfrak{p})} = \mathcal{F}_{\mathfrak{p}}^+ Y \hookrightarrow Y \twoheadrightarrow \overline{Y} \cong K^{t(\mathfrak{p})}$ , it is given by the cocycles in  $\xi_q \otimes \overline{Y}$  for the Tate period  $q$  of  $E$  at  $\mathfrak{p} = \mathfrak{p}_j$  (where  $\xi_q$  is as in the proof of Lemma 1.3). Defining  $\xi_n : D_{\mathfrak{p}} \rightarrow \mu_{p^n}$  by  $\xi_n(\sigma) = (q^{1/p^n})^{\sigma-1}$ , the map  $\xi_q = \varprojlim_n \xi_n$  having values in  $\mathbb{Z}_p(1) \subset K(1)$  is an explicit form of the cocycle  $\xi_q$  (see [H07] Section 4). In particular,  $(\overline{Y} \otimes \xi_q) \cap H_{fl}^1(\mathcal{F}_{\mathfrak{p}}^+ Y)$  is given by

$$(q \otimes \overline{Y}) \cap (\overline{Y} \otimes_{\mathbb{Z}_p} \varprojlim_n O_{\mathfrak{p}}^{\times} / (O_{\mathfrak{p}}^{\times})^{p^n})$$

inside  $\overline{Y} \otimes_{\mathbb{Z}_p} \varprojlim_n F_{\mathfrak{p}}^{\times} / (F_{\mathfrak{p}}^{\times})^{p^n}$ , which is trivial (because  $q$  is a nonunit).  $\square$

Suppose  $R_n \cong K[[X_{\mathfrak{p}}]]_{\mathfrak{p}|p}$ . Then by (V) in Lemma 1.5 (and Lemma 1.2), we have a unique subspace  $\mathbf{H}_F$  of  $H^1(\mathfrak{G}, V)$  projecting down onto

$$\prod_{\mathfrak{p}} \text{Im}(\iota_{\mathfrak{p}}) \hookrightarrow \prod_{\mathfrak{p}} \frac{H^1(F_{\mathfrak{p}}, V)}{\overline{L}_{\mathfrak{p}}(V)}.$$

Then by the restriction,  $\mathbf{H}_F$  gives rise to a subspace  $L = L_V$  of

$$\prod_{\mathfrak{p}} \text{Hom}((D_{\mathfrak{p}}^{gal})^{ab}, \mathcal{F}_{\mathfrak{p}}^- V / \mathcal{F}_{\mathfrak{p}}^+ V)^{D_{\mathfrak{p}}} \cong \prod_{\mathfrak{p}} \text{Hom}(D_{\mathfrak{p}}^{ab}, \mathcal{F}_{\mathfrak{p}}^- V / \mathcal{F}_{\mathfrak{p}}^+ V) \cong \prod_{\mathfrak{p}} (\mathcal{F}_{\mathfrak{p}}^- V / \mathcal{F}_{\mathfrak{p}}^+ V)^2$$

isomorphic to  $\prod_{\mathfrak{p}} (\mathcal{F}_{\mathfrak{p}}^- V / \mathcal{F}_{\mathfrak{p}}^+ V)$ . If a cocycle  $c$  representing an element in  $\mathbf{H}_F$  is unramified, it gives rise to an element in  $\text{Sel}_F(V)$ . By the vanishing of  $\text{Sel}_F(V)$  (Lemma 1.2), this implies  $c = 0$ ; so, the projection of  $L$  to the first factor  $\prod_{\mathfrak{p}} \frac{\mathcal{F}_{\mathfrak{p}}^- V}{\mathcal{F}_{\mathfrak{p}}^+ V}$  (via  $\phi \mapsto (\phi([\gamma, F_{\mathfrak{p}}^{gal}]) / \log_p(\gamma))_{\mathfrak{p}}$ ) is surjective. Thus this subspace  $L$  is a graph of a  $K$ -linear map

$$(1.5) \quad \mathcal{L} : \prod_{\mathfrak{p}} \mathcal{F}_{\mathfrak{p}}^- V / \mathcal{F}_{\mathfrak{p}}^+ V \rightarrow \prod_{\mathfrak{p}} \mathcal{F}_{\mathfrak{p}}^- V / \mathcal{F}_{\mathfrak{p}}^+ V.$$

We then define  $\mathcal{L}(\text{Ind}_F^{\mathbb{Q}} V) = \det(\mathcal{L}) \in K$ . This is a description of the direct construction of  $\mathbf{H}_F$ . In the following lemma, we verify the equivalence between the earlier definition and this direct one:

**Lemma 1.7.** *Let  $V = \text{Ad}(\rho_{n,0})$  or  $\rho_{2m,m}$  for an odd  $m > 0$ , and assume that  $\text{Sel}_F(V) = 0$ . The space  $\mathbf{H}_F$  defined above consists of cohomology classes of 1-cocycles  $c : \text{Gal}(\overline{F}/F) \rightarrow V$  such that*

- (1)  $c$  is unramified outside  $p$ ;
- (2)  $c$  restricted to the decomposition subgroup  $\text{Gal}(\overline{F}_{\mathfrak{p}}/F_{\mathfrak{p}}) \cong D_{\mathfrak{p}} \subset \text{Gal}(\overline{F}/F)$  at each  $\mathfrak{p}|p$  has values in  $\mathcal{F}_{\mathfrak{p}}^- V$  and  $c|_{D_{\mathfrak{p}}}$  modulo  $\mathcal{F}_{\mathfrak{p}}^+ V$  becomes unramified over  $F_{\mathfrak{p}}[\mu_{p^\infty}]$  for all  $\mathfrak{p}|p$ .

We here give a sketch of the proof, assuming  $F_{\mathfrak{p}} = F_{\mathfrak{p}}^{gal}$  (leaving the general case to the attentive reader).

*Proof.* Since  $Ad(\rho_{n,0}) \cong \bigoplus_{0 < j \leq n, j: \text{odd}} \rho_{2j,j}$ , we may assume that  $V = Ad(\rho_{n,0})$ . Recall the decomposition groups  $D_p \supset D_{\mathfrak{p}}$  in  $\text{Gal}(\overline{F}/\mathbb{Q})$  at  $p$ , and write  $I_p \supset I_{\mathfrak{p}}$  for the corresponding inertia groups. Let  $\mathbf{H}'_F \subset H^1(\text{Gal}(\overline{F}/F), V)$  be the subspace spanned by the cohomology classes satisfying (1) and (2). Take a cocycle  $c$  satisfying (1) and (2). Note that for any  $\sigma \in D_p$ ,  $\sigma(F_{\mathfrak{p}}) = F_{\mathfrak{p}}$  by our simplifying assumption. Since  $\mathbb{Q}_p[\mu_{p^\infty}]/\mathbb{Q}_p$  is abelian, we have  $\sigma\gamma\sigma^{-1} = \gamma$  for any  $\gamma \in \text{Gal}(F_{\mathfrak{p}}[\mu_{p^\infty}]/F_{\mathfrak{p}})$ . Since  $(c|_{I_{\mathfrak{p}}} \bmod \mathcal{F}_{\mathfrak{p}}^+ V) : I_{\mathfrak{p}} \rightarrow \mathcal{F}_{\mathfrak{p}}^- V/\mathcal{F}_{\mathfrak{p}}^+ V$  factors through  $\text{Gal}(F_{\mathfrak{p}}[\mu_{p^\infty}]/F_{\mathfrak{p}})$ , for any  $\sigma \in D_p$ ,  $c(\sigma\gamma\sigma^{-1}) = c(\gamma)$  for any  $\gamma \in I_{\mathfrak{p}}$ . Since  $D_p = \phi^{\mathbb{Z}} \rtimes I_p$  for a Frobenius element  $\phi = \text{Frob}_p$ , the cocycle  $(c|_{D_p} \bmod \mathcal{F}_{\mathfrak{p}}^+ V)$  is actually  $D_p$ -invariant. Thus  $c|_{D_p} \bmod \mathcal{F}_{\mathfrak{p}}^+ V$  is in  $H^1(F_{\mathfrak{p}}, \mathcal{F}_{\mathfrak{p}}^+ V/\mathcal{F}_{\mathfrak{p}}^- V)^{D_p}$ . For  $\mathfrak{q} \in S$ ,  $c|_{D_{\mathfrak{q}}}$  is unramified and vanishes on  $I_{\mathfrak{q}}$ ; so, the restriction map in Lemma 1.5

$$\text{Res} : H^1(\text{Gal}(\overline{F}/F), V) \rightarrow \prod_{\mathfrak{q} \in S} \frac{H^1(F_{\mathfrak{q}}, V)}{L_{\mathfrak{q}}(V)} \times \prod_{\mathfrak{p}|p} \frac{H^1(F_{\mathfrak{p}}, V)}{L_{\mathfrak{p}}(V)}$$

brings  $c$  into  $\prod_{\mathfrak{p}|p} \text{Im}(\iota_{\mathfrak{p}})$ . Note here  $L_{\mathfrak{p}}(V) = \overline{L}_{\mathfrak{p}}(V)$  by the above lemma, and hence the above map  $\text{Res}$  is the map in Lemma 1.5. Thus we conclude  $\mathbf{H}'_F \subset \text{Res}^{-1}(\prod_{\mathfrak{p}|p} \text{Im}(\iota_{\mathfrak{p}}))$ .

Conversely, we suppose that the class  $[(c|_{D_p} \bmod \mathcal{F}_{\mathfrak{p}}^+ V)]$  falls in  $\text{Im}(\iota_{\mathfrak{p}})$ . Thus the homomorphism  $(c|_{D_p} \bmod \mathcal{F}_{\mathfrak{p}}^+ V) : D_p \rightarrow \mathcal{F}_{\mathfrak{p}}^- V/\mathcal{F}_{\mathfrak{p}}^+ V$  is  $D_p$ -invariant. Then it extends to a homomorphism  $\tilde{c}_{\mathfrak{p}} : D_p \rightarrow \mathcal{F}_{\mathfrak{p}}^- V/\mathcal{F}_{\mathfrak{p}}^+ V$ . Indeed, for any two groups  $G \triangleright H$  with finite index and a torsion-free divisible abelian group  $X$ , every  $G$ -invariant homomorphism  $\phi : H \rightarrow X$  extends to a homomorphism  $\tilde{\phi} : G \rightarrow X$  by Schur's theory of multipliers (e.g. [MFG] 4.3.5), because the obstruction lies in  $H^2(G/H, X)$  which vanishes by the finiteness of  $G/H$  and divisibility of  $X$ . Then  $\tilde{c}_{\mathfrak{p}}$  has to factor through  $\text{Gal}(\mathbb{Q}_p^{ab}/\mathbb{Q}_p)$  for the maximal abelian extension  $\mathbb{Q}_p^{ab}/\mathbb{Q}_p$ , which is equal to  $\mathbb{Q}_p^{ur}[\mu_{p^\infty}]$  for the maximal unramified extension  $\mathbb{Q}_p^{ur}/\mathbb{Q}_p$  (by local class field theory); so,  $(c|_{I_p} \bmod \mathcal{F}_{\mathfrak{p}}^+ V)$  factors through  $\text{Gal}(F_{\mathfrak{p}}[\mu_{p^\infty}]/F_{\mathfrak{p}})$  and  $c$  satisfies (2). The condition (1) for  $c|_{D_{\mathfrak{q}}}$  ( $\mathfrak{q} \nmid p$ ) is equivalent to the vanishing of  $i_{\mathfrak{q}}(c|_{D_{\mathfrak{q}}})$  in  $\frac{H^1(F_{\mathfrak{q}}, V)}{L_{\mathfrak{q}}(V)}$ . Then we get the reverse inclusion. Since  $\text{Res}$  is an isomorphism if  $\text{Sel}_F(V) = 0$  by Lemma 1.5,  $\mathbf{H}'_F \xrightarrow{\text{Res}} \prod_{\mathfrak{p}|p} \text{Im}(\iota_{\mathfrak{p}})$  is a surjective isomorphism, and hence  $\mathbf{H}_F = \mathbf{H}'_F$ .  $\square$

If one restricts  $c \in \mathbf{H}_F$  to  $\mathfrak{G}_{\infty} = \text{Gal}(F^{(S)}/F_{\infty})$ , its ramification is exhausted by  $\Gamma = \text{Gal}(F_{\infty}/F)$  (because of the definition of  $\text{Sel}_F^{cyc}(\rho_{2n,n})$  and  $\mathbf{H}_F$ ) giving rise to a class  $[c] \in \text{Sel}_{F_{\infty}}(V)$ . The kernel of the restriction map:  $H^1(\mathfrak{G}, V) \rightarrow H^1(\mathfrak{G}_{\infty}, V)$  is given by  $H^1(\Gamma, H^0(\mathfrak{G}_{\infty}, V)) = 0$  because  $H^0(\mathfrak{G}_{\infty}, V) = 0$ . Thus the image of  $\mathbf{H}_F$  in  $\text{Sel}_{F_{\infty}}(V/T)$  gives rise to the order  $e$  exceptional zero of  $L^{arith}(s, \rho_{2n,n})$  at  $s = 1$ . We have reproved the first half of the following result in [Gr] Proposition 1.

**Proposition 1.8.** *Let  $n$  be an odd positive integer. Suppose Conjecture 0.1 for all odd  $m \leq n$ . Then for the number  $e$  of prime factors of  $p$  in  $F$ , we have*

$$\text{ord}_{s=1} L_p^{arith}(s, \rho_{2n,n}) \geq e.$$

*Further we have  $\mathcal{L}(\rho_{2n,n}) = 0 \iff \text{ord}_{s=1} L_p^{arith}(s, \rho_{2n,n}) > e$ .*

The last assertion follows from [Gr] Proposition 3. In [Gr] Proposition 3, Conjecture 0.1 is not assumed. However, in the very definition of Greenberg's  $\mathcal{L}$ -invariant,

the condition (V) in Lemma 1.5 is necessary as explicitly pointed out in pages 163–4 of [Gr]. As is clear from Lemma 1.2, Conjecture 0.1 supplies us the vanishing  $\text{Sel}_F(V) = 0$  (which is equivalent to the finiteness of Greenberg’s Selmer group  $S_A(\mathbb{Q})$  in [Gr]).

**1.3. Factorization of  $\mathcal{L}$ -invariants.** In this section, we factorize  $\mathcal{L}(\text{Ind}_F^{\mathbb{Q}} \rho_{2n,n})$  and  $\mathcal{L}(\text{Ind}_F^{\mathbb{Q}} \text{Ad}(\rho_{n,0}))$  for odd  $n$  into the product over multiplicative places and the contribution of the good reduction part. This good reduction part gives  $\mathcal{L}(n)$  for  $\mathcal{L}(\text{Ind}_F^{\mathbb{Q}} \rho_{2n,n})$  in Conjecture 0.2. We keep notation introduced in the previous section; so,  $V$  is either  $\rho_{2n,n}$  or  $\text{Ad}(\rho_{n,0})$ .

**Proposition 1.9.** *Let  $V$  be either  $\rho_{2n,n}$  or  $\text{Ad}(\rho_{n,0})$ . Suppose  $b > 0$ , and fix an index  $k$  with  $1 \leq k \leq b$ . Let  $a \in \prod_{i=1}^e \text{Hom}(D_{\mathfrak{p}_i}^{gal}, \mathcal{F}_{\mathfrak{p}_i}^- V / \mathcal{F}_{\mathfrak{p}_i}^+ V)^{D_p}$  be induced by  $c \in \mathbf{H}_F$  such that  $c \in \mathbf{H}_F$  restricts down trivially to  $\frac{H^1(F_i, V)}{L_{\mathfrak{p}_i}(V)}$  for all  $i \neq k$ . Then we have  $a([\gamma_i, F_i]) = 0$  for all  $i \neq k$  and  $a([p, F_{k'}]) = 0$  for all  $k' \neq k$  with  $k' \leq b$ .*

*Proof.* For the index  $k \leq b$ ,  $\overline{L}_{\mathfrak{p}}(V)$  is exactly  $\mathcal{F}_{\mathfrak{p}_k}^+ H^1(F_k, V)$ . Take a cocycle  $c \in \mathbf{H}_F$  restricting down to  $\frac{H^1(F_k, V)}{L_{\mathfrak{p}_k}(V)}$  trivially to  $\frac{H^1(F_i, V)}{L_{\mathfrak{p}_i}(V)}$  for all  $i \neq k$ . Since  $\mathbf{H}_F \cong \prod_{i=1}^e \text{Im}(\iota_{\mathfrak{p}_i})$  by the restriction map (Lemmas 1.2 and 1.5), such cocycles  $c$  form a direct summand of  $\mathbf{H}_F$  isomorphic to  $\text{Im}(\iota_{\mathfrak{p}_k})$ .

If  $i > b$ ,  $L_{\mathfrak{p}_i}(V)$  is made of classes of cocycles becoming unramified modulo those with values in  $\mathcal{F}_{\mathfrak{p}_i}^+ V$ ; so, even if  $c|_{D_{\mathfrak{p}_i}}$  vanishes in  $\frac{H^1(F_i, V)}{L_{\mathfrak{p}_i}(V)}$  (that is,  $c|_{D_{\mathfrak{p}_i}} \in L_{\mathfrak{p}_i}(V)$ ), we cannot pull out much information on the value  $a([p, F_i])$  because of the ambiguity modulo unramified cocycles with values in  $\mathcal{F}_{\mathfrak{p}_i}^- V / \mathcal{F}_{\mathfrak{p}_i}^+ V$ . Anyway,  $a([\gamma_i, F_i]) = 0$  because  $[\gamma_i, F_i] \in I_{\mathfrak{p}_i}$ .

For  $i \leq b$  with  $i \neq k$ ,  $\overline{L}_{\mathfrak{p}_i}(V)$  is made of cocycles of  $D_{\mathfrak{p}_i}$  with values in  $\mathcal{F}_{\mathfrak{p}_i}^+ V$ , and the condition that  $c|_{D_{\mathfrak{p}_i}} \in \overline{L}_{\mathfrak{p}_i}(V)$  implies the vanishing of  $a(\sigma) = c(\sigma) \bmod \mathcal{F}_{\mathfrak{p}_i}^+ V$  for all  $\sigma \in D_{\mathfrak{p}_i}$ . This shows the last assertion:  $a([p, F_{k'}]) = 0$ .  $\square$

By the above lemma, we get immediately the following fact.

**Corollary 1.10.** *Let the notation be as in Proposition 1.9. Then the linear operator  $\mathcal{L}$  acting on  $\prod_{\mathfrak{p}} \mathcal{F}_{\mathfrak{p}}^- V / \mathcal{F}_{\mathfrak{p}}^+ V$  preserves the following exact sequence:*

$$0 \rightarrow \prod_{i>b} \mathcal{F}_{\mathfrak{p}_i}^- V / \mathcal{F}_{\mathfrak{p}_i}^+ V \rightarrow \prod_{\mathfrak{p}} \mathcal{F}_{\mathfrak{p}}^- V / \mathcal{F}_{\mathfrak{p}}^+ V \rightarrow \prod_{k \leq b} \mathcal{F}_{\mathfrak{p}_k}^- V / \mathcal{F}_{\mathfrak{p}_k}^+ V \rightarrow 0,$$

and  $\mathcal{L}$  acting on the quotient  $\prod_{k \leq b} \mathcal{F}_{\mathfrak{p}_k}^- V / \mathcal{F}_{\mathfrak{p}_k}^+ V$  sends  $\mathcal{F}_{\mathfrak{p}_k}^- V / \mathcal{F}_{\mathfrak{p}_k}^+ V$  into itself for each  $k \leq b$ .

**Definition 1.11.** *Define  $\mathcal{L}(n)$  (resp.  $\mathcal{L}_k(V)$ ) by*

$$\det \left( \mathcal{L}|_{\prod_{i>b} \mathcal{F}_{\mathfrak{p}_i}^- V / \mathcal{F}_{\mathfrak{p}_i}^+ V} \right) \in \mathbb{Q}_p$$

for  $V = \rho_{2n,n}$  (resp. the determinant of the linear operator induced by  $\mathcal{L}$  on  $\prod_{\mathfrak{p}} \mathcal{F}_{\mathfrak{p}}^- V / \mathcal{F}_{\mathfrak{p}}^+ V / \prod_{i \neq k} \mathcal{F}_{\mathfrak{p}_i}^- V / \mathcal{F}_{\mathfrak{p}_i}^+ V$  for  $V = \rho_{2n,n}$  and  $V = \text{Ad}(\rho_{n,0})$ ).

**Corollary 1.12.** *Let the notation be as above. Then we have*

$$\mathcal{L}(\text{Ind}_F^{\mathbb{Q}} \rho_{2n,n}) = \mathcal{L}(n) \prod_{k=1}^b \mathcal{L}_k(\rho_{2n,n})$$

for odd  $n \geq 1$ .

**Proposition 1.13.** *Suppose  $n = 1$ . Then for  $k \leq b$ , we have  $\mathcal{L}_k(\rho_{2,1}) = \frac{\log_p(Q_k)}{\text{ord}_p(Q_k)}$ , where  $Q_k = N_{F_k/\mathbb{Q}_p}(q_k)$  for the Tate period  $q_k$  of  $E/F_k$ .*

This follows from [H07] Theorem 5.3. In [H07], the above corollary is proved by automorphic means in Section 3 of [H07], but replacing the result of [H07] Section 3 by the above factorization result, the same argument proving Theorem 5.3 there proves the above proposition.

We now generalize Proposition 1.13 to arbitrary odd  $n > 1$ .

**Theorem 1.14.** *Let  $n$  be an odd positive integer, and assume  $V = \rho_{2n,n}$ . Suppose Conjecture 0.1 for all odd positive  $m \leq n$ . Then  $\mathcal{L}_k(V) = \frac{\log_p(Q_k)}{\text{ord}_p(Q_k)}$  for  $k \leq b$ , where  $Q_k = N_{F_k/\mathbb{Q}_p}(q_k)$  for the Tate period  $q_k$  of  $E$ .*

*Proof.* Fix  $k \leq b$ , and write  $\mathfrak{p} = \mathfrak{p}_k$ . Write  $X_i = X_{i,\mathfrak{p}_j}$  if  $i$  is odd. Define  $\mathfrak{M}_\ell$  be the ideal generated by  $X_i$  for odd  $i \neq \ell$  and  $X_\ell^2$ . We fix an odd  $\ell$  with  $0 < \ell \leq n$ , and write  $\mathfrak{M}$  for  $\mathfrak{M}_\ell$  and  $\tilde{K} = R_n/\mathfrak{M} \cong K[\varepsilon]$  with  $\varepsilon^2 = 0$  by  $X_\ell \mapsto \varepsilon$ . Let  $\bar{\rho} = (\rho_n \bmod \mathfrak{M})$ , and write  $\bar{\delta}_i$  for  $\delta_{i,\mathfrak{p}} \bmod \mathfrak{M}$ . We consider the exact sequence of  $\tilde{K}[D_{\mathfrak{p}}]$ -modules:

$$0 \rightarrow \frac{\mathcal{F}_{\mathfrak{p}}^{i+1}\bar{\rho}}{\mathcal{F}_{\mathfrak{p}}^{i+2}\bar{\rho}} \rightarrow \frac{\mathcal{F}_{\mathfrak{p}}^i\bar{\rho}}{\mathcal{F}_{\mathfrak{p}}^{i+2}\bar{\rho}} \rightarrow \frac{\mathcal{F}_{\mathfrak{p}}^i\bar{\rho}}{\mathcal{F}_{\mathfrak{p}}^{i+1}\bar{\rho}} \rightarrow 0.$$

Writing  $\tilde{K}(\psi)$  for the rank one free  $\tilde{K}$ -module on which  $D_{\mathfrak{p}}$  acts by a character  $\psi : D_{\mathfrak{p}} \rightarrow \tilde{K}^\times$ , this exact sequence gives the following exact sequence

$$0 \rightarrow \tilde{K}(\bar{\delta}_{i+1}) \rightarrow \frac{\mathcal{F}_{\mathfrak{p}}^i\bar{\rho}}{\mathcal{F}_{\mathfrak{p}}^{i+2}\bar{\rho}} \rightarrow \tilde{K}(\bar{\delta}_i) \rightarrow 0.$$

Twisting by  $\bar{\delta}_{i+1}^{-1}\mathcal{N}$ , we get another exact sequence of  $\tilde{K}[D_{\mathfrak{p}}]$ -modules:

$$0 \rightarrow \tilde{K}(\mathcal{N}) \rightarrow M \rightarrow \tilde{K}(\bar{\delta}_i\bar{\delta}_{i+1}^{-1}\mathcal{N}) \rightarrow 0.$$

By [H07] Lemma 5.1, this sequence gives the top row of the following commutative diagram of  $D_{\mathfrak{p}}$ -modules with exact rows:

$$\begin{array}{ccccc} \tilde{K}(\mathcal{N}) & \hookrightarrow & M & \twoheadrightarrow & \tilde{K}(\bar{\delta}_i\bar{\delta}_{i+1}^{-1}\mathcal{N}) \\ \text{mod } X_\ell \downarrow & & \text{mod } X_\ell \downarrow & & \downarrow \text{mod } X_\ell \\ K(\mathcal{N}) & \hookrightarrow & T_p E \otimes_{\mathbb{Z}_p} K & \twoheadrightarrow & K. \end{array}$$

Then by taking the induction from  $\text{Gal}(\overline{F}_{\mathfrak{p}}/F_{\mathfrak{p}})$  to  $\text{Gal}(\overline{F}_{\mathfrak{p}}/\mathbb{Q}_p)$ , we get the following new commutative diagram with exact rows:

$$\begin{array}{ccccc} \text{Ind}_{F_{\mathfrak{p}}}^{\mathbb{Q}_p} \tilde{K}(\mathcal{N}) & \hookrightarrow & \text{Ind}_{F_{\mathfrak{p}}}^{\mathbb{Q}_p} M & \twoheadrightarrow & \text{Ind}_{F_{\mathfrak{p}}}^{\mathbb{Q}_p} \tilde{K}(\bar{\delta}_i\bar{\delta}_{i+1}^{-1}\mathcal{N}) \\ \text{mod } X_\ell \downarrow & & \text{mod } X_\ell \downarrow & & \downarrow \text{mod } X_\ell \\ \text{Ind}_{F_{\mathfrak{p}}}^{\mathbb{Q}_p} K(\mathcal{N}) & \hookrightarrow & \text{Ind}_{F_{\mathfrak{p}}}^{\mathbb{Q}_p} T_p E \otimes_{\mathbb{Z}_p} K & \twoheadrightarrow & \text{Ind}_{F_{\mathfrak{p}}}^{\mathbb{Q}_p} K. \end{array}$$

By [H07] Lemma 4.8, we have a unique extension  $\tilde{\delta}_j$  of  $\bar{\delta}_j$  to  $\text{Gal}(\overline{F}_{\mathfrak{p}}/\mathbb{Q}_p)$  with  $\tilde{\delta}_j \equiv \mathcal{N}^{n-j} \bmod \mathfrak{m}_n$ . We write this extension as  $\tilde{\delta}_j$ . For any potentially ordinary  $\text{Gal}(\overline{F}_{\mathfrak{p}}/\mathbb{Q}_p)$ -module  $X$ , write the maximal quotient of  $\mathcal{F}^+X$  on which  $\text{Gal}(\overline{F}_{\mathfrak{p}}/\mathbb{Q}_p)$  acts by  $\mathcal{N}$  as  $\mathcal{F}^+X/\mathcal{F}^{11}X$ . Similarly, we define  $\mathcal{F}^+X \subset \mathcal{F}^{00}X \subset X$  by

$\mathcal{F}^{00}X/\mathcal{F}^+X = H^0(\text{Gal}(\overline{F}_p/\mathbb{Q}_p), X/\mathcal{F}^+X)$ . Then the above commutative diagram yields another commutative diagram with exact rows:

$$\begin{array}{ccccc} \widetilde{K}(\mathcal{N}) & \xrightarrow{\hookrightarrow} & \mathcal{F}^{00} \text{Ind}_{F_p}^{\mathbb{Q}_p} M / \mathcal{F}^{11} \text{Ind}_{F_p}^{\mathbb{Q}_p} M & \xrightarrow{\twoheadrightarrow} & \widetilde{K}(\widetilde{\delta}_i \widetilde{\delta}_{i+1}^{-1} \mathcal{N}) \\ \text{mod } X_\ell \downarrow & & \text{mod } X_\ell \downarrow & & \downarrow \text{mod } X_\ell \\ K(\mathcal{N}) & \xrightarrow{\hookrightarrow} & \frac{\mathcal{F}^{00} \text{Ind}_{F_p}^{\mathbb{Q}_p} T_p E \otimes_{\mathbb{Z}_p} K}{\mathcal{F}^{11} \text{Ind}_{F_p}^{\mathbb{Q}_p} T_p E \otimes_{\mathbb{Z}_p} K} & \xrightarrow{\twoheadrightarrow} & K. \end{array}$$

By Theorem 4.7 of [H07], this implies

$$\frac{\partial \widetilde{\delta}_i \widetilde{\delta}_{i+1}^{-1} \mathcal{N}}{\partial X_\ell}([Q_k, \mathbb{Q}_p]) = 0.$$

Since  $\mathcal{N}([Q_k, \mathbb{Q}_p])$  is constant in  $\mathbb{Q}_p^\times$ , we get

$$\frac{\partial \widetilde{\delta}_i \widetilde{\delta}_{i+1}^{-1}}{\partial X_\ell}([Q_k, \mathbb{Q}_p]) = 0$$

which yields by the Leibnitz formula

$$\left( \delta_i^{-1} \frac{\partial \widetilde{\delta}_i}{\partial X_\ell} - \delta_{i+1}^{-1} \frac{\partial \widetilde{\delta}_{i+1}}{\partial X_\ell} \right) ([Q_k, \mathbb{Q}_p]) = 0,$$

where  $\delta_i = (\widetilde{\delta}_i \text{ mod } \mathfrak{m}_n) = \mathcal{N}^{n-i}$ . Since this holds for  $i = 0, 1, \dots, n$ , we get

$$\left( \delta_0^{-1} \frac{\partial \widetilde{\delta}_0}{\partial X_\ell} - \delta_n^{-1} \frac{\partial \widetilde{\delta}_n}{\partial X_\ell} \right) ([Q_k, \mathbb{Q}_p]) = 0.$$

Since  $\widetilde{\delta}_0 \widetilde{\delta}_n = \mathcal{N}$  which is the unique extension of  $\overline{\delta}_0 \overline{\delta}_n = \mathcal{N}$  to  $\text{Gal}(\overline{F}_p/\mathbb{Q}_p)$  congruent to  $\mathcal{N}$  modulo  $\mathfrak{m}_n$  (see Lemma 4.8 of [H07]), we have

$$\delta_0^{-1} \frac{\partial \widetilde{\delta}_0}{\partial X_\ell} = -\delta_n^{-1} \frac{\partial \widetilde{\delta}_n}{\partial X_\ell},$$

and hence

$$\delta_n^{-1} \frac{\partial \widetilde{\delta}_n}{\partial X_\ell}([Q_k, \mathbb{Q}_p]) = 0.$$

This in turn yields

$$\delta_i^{-1} \frac{\partial \widetilde{\delta}_i}{\partial X_\ell}([Q_k, \mathbb{Q}_p]) = 0$$

for all  $i = 0, 1, \dots, n$ .

Write  $Q_k = p^a u$  for  $a = \text{ord}_p(Q_k)$  and  $u \in \mathbb{Z}_p^\times$ . Then  $\log_p(u) = \log_p(Q_k)$ . Write  $d_k = [F_k : \mathbb{Q}_p]$  and  $N_k = N_{F_k/\mathbb{Q}_p} : F_k^\times \rightarrow \mathbb{Q}_p^\times$  for the norm map. Since  $[p, \mathbb{Q}_p]^{d_k} = [N_k(p), \mathbb{Q}_p] = [p, F_k]_{\mathbb{Q}_p^{ab}}$  and  $[u, \mathbb{Q}_p]^{d_k} = [N_k(u), \mathbb{Q}_p] = [u, F_k]_{\mathbb{Q}_p^{ab}}$ , for odd  $i$ , we have

$$\begin{aligned} \widetilde{\delta}_i([N(q_k), \mathbb{Q}_p]^{d_k}) &\equiv \delta_i([p, F_k])^a \delta_i([u, F_k]) \\ &\equiv \delta_i([p, F_k])^a (1 + X_i)^{-d_j \log_p(u) / \log_p(\gamma_j)} \pmod{\mathfrak{M}} \end{aligned}$$

(because  $\mathcal{N}([u, F_{\mathfrak{p}}]) = u^{-d_{\mathfrak{p}}}$  for  $d_{\mathfrak{p}} = [F_{\mathfrak{p}} : \mathbb{Q}_p]$ ). Differentiating this identity with respect to  $X_{\ell}$ , we get from  $\delta_i([p, F_k]) = \mathcal{N}^{n-i}([p, F_k]) = 1$

$$a \frac{\partial \delta_{\ell}}{\partial X_{\ell}} \Big|_{X=0} ([p, F_k]) - \frac{d_k \log_p(u)}{\log_p(\gamma_k)} = 0$$

and

$$a \frac{\partial \delta_i}{\partial X_{\ell}} \Big|_{X=0} ([p, F_k]) = 0 \quad \text{if odd } i \neq \ell.$$

Since  $a \neq 0$ , we have

$$\frac{\partial \delta_i}{\partial X_{\ell}} \Big|_{X=0} ([p, F_k]) = 0 \quad \text{if odd } i \neq \ell,$$

and

$$\frac{\partial \delta_{\ell}}{\partial X_{\ell}} \Big|_{X=0} ([p, F_k]) d_k^{-1} \log_p(\gamma_k) = \frac{\log_p(Q_k)}{\text{ord}_p(Q_k)}.$$

Since  $\frac{\partial \rho_{\mathfrak{p}}}{\partial X_{\ell}} \Big|_{X=0} \rho_{n,0}^{-1}$  for odd  $\ell$  with  $0 < \ell \leq n$  gives a basis of the  $\mathfrak{p}$ -part of  $\mathbf{H}_F$  isomorphic to  $\text{Im}(\iota_{\mathfrak{p}})$ , we find that  $\mathcal{L}_k(\text{Ad}(\rho_{n,0})) = \left( \frac{\log_p(Q_k)}{\text{ord}_p(Q_k)} \right)^{(n+1)/2}$ . Since  $\text{Sel}_F^{\text{cyc}}(\text{Ad}(\rho_{n,0})) \cong \bigoplus_{0 < m \leq n, m: \text{odd}} \text{Sel}_F^{\text{cyc}}(\rho_{2m,m})$ , we find

$$\mathcal{L}_k(\text{Ad}(\rho_{n,0})) = \prod_{0 < m \leq n, m: \text{odd}} \mathcal{L}_k(\rho_{2m,m}) = \left( \frac{\log_p(Q_k)}{\text{ord}_p(Q_k)} \right)^{(n+1)/2}.$$

By induction on  $m$  starting with the case  $m = 1$  treated in Proposition 1.13, we find  $\mathcal{L}_k(\rho_{2n,n}) = \frac{\log_p(Q_k)}{\text{ord}_p(Q_k)}$  as desired.  $\square$

## 2. PROOF OF CONJECTURE 0.1 UNDER POTENTIAL MODULARITY WHEN $n = 1$

We suppose that

- (NS)  $\bar{\rho} = E[p]$  has non-soluble image in  $GL_2(\mathbb{F}_p)$ ;
- (DS) the semi-simplification of  $\bar{\rho}$  restricted to  $D_{\mathfrak{p}}$  is non-scalar.

We now give a sketch of a proof of Conjecture 0.1 under these two conditions:

**Proposition 2.1.** *Suppose (NS) and (DS). If there exists a totally real Galois extension  $L/F$  totally split at  $p$  such that  $\bar{\rho}_L = \bar{\rho}|_{\text{Gal}(\bar{F}/L)}$  is associated to a Hilbert modular form, then we have  $R_1 \cong K[[X_{1,\mathfrak{p}}]]_{\mathfrak{p}|p}$ .*

By the result of [Ta] and [Ta1], the Galois representation  $\bar{\rho}$  is potentially modular in the sense that there exists a totally real Galois extension  $L/F$  in which  $p$  totally split and  $\bar{\rho}_L$  is associated to a Hilbert cusp form of weight 2. Actually, in the above paper of Taylor, details of the proof is given for  $F = \mathbb{Q}$ , but we should be able to adjust his argument to prove the result for general  $F$  (see [V] Theorem 2.2).

*Proof.* To indicate the dependence of  $R_n$  on the base-field  $L$ , we write  $(R_{n/L}, \rho_{n/L})$  if we consider the universal couple of  $\rho_E|_{\text{Gal}(\bar{L}/F)}$  (under (K<sub>n</sub>1–4)). By the potential modularity assumption,  $\bar{\rho}_L$  is modular. By further making a soluble base-change, by the potential level-lowering done by [SW], we may assume that  $\bar{\rho}_L$  is associated to a Hilbert modular cusp form of weight 2 of level  $\Gamma_0(\mathfrak{N}p)$  satisfying the conditions (h1–4) of [HMI] page 185 for the prime-to- $p$  Artin conductor  $\mathfrak{N}$  of  $\bar{\rho}_L$ . Then by [HMI] Corollary 3.77 and Proposition 3.78, we have  $R_{1/L} \cong K[[X_{1,\mathfrak{P}}]]_{\mathfrak{P}|p}$ , where  $\mathfrak{P}$  runs over all prime factors of  $p$  in  $L$ . For  $\sigma \in \text{Gal}(L/F)$ , we take a lift  $\tilde{\sigma} \in$

$\text{Gal}(\overline{F}/L)$  inducing  $\sigma$  on  $L$ , for any deformation  $\rho$  of  $\rho_E$  over  $L$ , we can define  $\rho^\sigma(g) = \rho(\tilde{\sigma}g\tilde{\sigma}^{-1})$ . The isomorphism class of  $\rho^\sigma$  is determined independently of the choice of the lift  $\tilde{\sigma}$  and depends only on  $\sigma$ . Since  $E$  is defined over  $F$ ,  $\rho_E^\sigma \cong \rho_E$ ,  $\rho_{n/L}^\sigma$  is another deformation of  $\rho_E$  over  $L$  satisfying (K<sub>n</sub>1–4). Thus we have a unique ring automorphism  $[\sigma] \in \text{Aut}(R_{n/L})$  such that  $\rho_{n/L}^\sigma \cong [\sigma] \circ \rho_{n/L}$ . In this way,  $\Delta := \text{Gal}(L/F)$  acts on  $R_{n/L}$ . Since  $\delta_{1,\mathfrak{p}}^\sigma(g) = \delta_{1,\mathfrak{p}}(\tilde{\sigma}g\tilde{\sigma}^{-1})$  coincides with  $\delta_{1,\mathfrak{p}}^\sigma$ , we have  $[\sigma](X_{1,\mathfrak{p}}) = X_{1,\mathfrak{p}}^\sigma$ . By the  $K$ -deformation version of Theorem 5.42 in [MFG], we have  $R_{1/F} \cong R_{1/L} / \sum_{\sigma \in \Delta} R_{1/L}([\sigma] - 1)R_{1/L}$ , where  $\sum_{\sigma \in \Delta} R_{1/L}([\sigma] - 1)R_{1/L}$  is the ideal of  $R_{1/L}$  generated by  $[\sigma](r) - r$  for all  $r \in R_{1/L}$ . Then it is clear that  $R_{1/F} \cong K[[X_{1,\mathfrak{p}}]]_{\mathfrak{p}|p}$ .  $\square$

*Remark 2.1.* Since the potential modularity for  $\rho_{n,0}$  is proven in [Ta2] under mild assumptions, we expect that the above argument (or a modified version) would prove Conjecture 0.1 for general  $n$  in near future.

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