p–ADIC AUTOMORPHIC FORMS ON REDUCTIVE GROUPS

HARUZO HIDA

Abstract. In these lecture notes, we will prove vertical control theorems for ordinary p-adic automorphic forms and irreducibility of the Igusa tower over untary and symplectic Shimura varieties.

Contents

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1. Introduction

Let p be a prime. What I would like to present in this series of lectures is the theory of families of p–ordinary p–adic (cohomological) automorphic forms on reductive groups. After going through basics of the theory of p–adic automorphic forms, we would like to study

1. Vertical Control Theorem (VCT: construction of p-adic families);
2. p–adic L–functions (in Symplectic and Unitary cases);
3. Galois representations;
4. the Iwasawa theoretic significance of p–adic L–functions.

1.1. **Automorphic forms on classical groups.**

Let $G_{/\mathbb{Z}}$ be an affine group scheme whose fiber over $\mathbb{Z}_p$ is a classical Chevalley group; so, unitary groups are included (dependent on the choice of p). Take a Borel subgroup B and its torus T. When G is split over $\mathbb{Q}$, we may embed G into $GL(n)_{/\mathbb{Q}}$. Let B be the Borel subgroup (we can take it to be the group of upper triangular matrices in G). Let T be the group of diagonal matrices. We have a splitting $B = T \ltimes U$ for the unipotent radical U of B. On the quotient variety G/U (which is a T–torsor over the projective flag variety G/B), T acts by $gUt = gtU$, and hence T acts on the structure sheaf $\mathcal{O}_{G/U}$ by $t\phi(gU) = \phi(gtU)$. This action gives rise to an order on $X(T) = \mathrm{Hom}(T_{/\overline{\mathbb{Q}}}, \mathbb{G}_m)$ so that the positive cone in $X(T)$ is made of $\kappa \in X(T)$ such that the κ–eigenspace $L(\kappa)$ on the global sections of $\mathcal{O}_{G/U}$ is non-trivial. We then have a representation $L(\kappa; A) = L_G(\kappa; A)$ on $L(\kappa)$ given by $\phi(gU) \mapsto \phi(h^{-1}gU)$ for $h \in G(A)$, as long as T is split over a ring A. When $G = SL(2)$, $T \cong \mathbb{G}_m$, $X(T) \cong \mathbb{Z}$ by $\kappa \leftrightarrow n$ if $\kappa(x) = x^n$, and $L(\kappa; A)$ is the symmetric κ–th tensor representation of $SL(2)$, which can be realized on the space of homogeneous polynomials of degree n so that $\alpha \in SL(2)$ acts on a polynomial $P(X, Y)$ by $P(X, Y) \mapsto P((X, Y)^t\alpha^{-1})$.

There are two ways of associating a weight to automorphic forms on G: One is to consider the cohomology group $H^d(\Gamma, L(\kappa; A))$ of an appropriate degree d for a given arithmetic subgroup $\Gamma \subset G(\mathbb{Q})$, and we call harmonic automorphic forms spanning $H^d(\Gamma, L(\kappa; \mathbb{C}))$ automorphic forms of (topological) weight κ. This way works well for any classical (or more general reductive) groups.

When the symmetric space of G is isomorphic to a (bounded) hermitian domain $\mathcal{H}$ with origin $\mathbf{0}$, like (the restriction of scalar to $\mathbb{Q}$ of) F–forms of Sp or $SU(m, n)$ over totally real fields F, we have another way to associate a weight to holomorphic automorphic forms. In this case, we have $\mathcal{H} \cong G(\mathbb{R})/C_{\mathbf{0}}$ for the stabilizer $C_{\mathbf{0}}$ of $\mathbf{0}$, which is a maximal compact subgroup of $G(\mathbb{R})$. In the simplest case of $SL(2)_{/\mathbb{Q}} = Sp(2)_{/\mathbb{Q}}$, $C_{\mathbf{0}} = SO_2(\mathbb{R})$ and $\mathcal{H} = H = \{z \in \mathbb{C} | \operatorname{Im}(z) > 0\}$ with $G(\mathbb{R})/SO_2(\mathbb{R}) \cong H$ by $g \mapsto g(\sqrt{-1})$. As is well known that H is holomorphically equivalent to the open unit disk in $\mathbb{C}$ by $z \mapsto \frac{z-\sqrt{-1}}{z+\sqrt{-1}}$.

The group $C_{\mathbf{0}}$ can be regarded as a group of real points with respect to a twisted complex conjugation in the complexification C of $C_{\mathbf{0}}$. In the case of $SL(2)_{/\mathbb{Q}}$, $SO_2(\mathbb{R})$ can be regarded as S^1 in $\mathbb{G}_m(\mathbb{C})$ by $\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \mapsto c\sqrt{-1} + d \in S^1$, and S^1 is the set of fixed points of the twisted "complex conjugation": $x \mapsto \overline{x}^{-1}$ in $\mathbb{G}_m(\mathbb{C}) = \mathbb{C}^\times$. Generalizing this example, we see that the compact group $U(n)$ is the subgroup of $GL_n(\mathbb{C})$ fixed by the complex conjugation: $x \mapsto {}^t\overline{x}^{-1}$. Any holomorphic representation $\rho : C \to GL(V(\rho))$ gives rise to a holomorphic complex vector bundle $\widetilde{V} = (G(\mathbb{R}) \times V)/C_{\mathbf{0}}$ by the action $(g, v) \mapsto (gu, u^{-1}v)$ for $u \in C_{\mathbf{0}}$. Since $\mathcal{H}$ is simply connected, we can split $\widetilde{V} \cong \mathcal{H} \times V$ as holomorphic vector bundles; so, we have a linear map $J_\rho(g,z) : V_z \to V_{g(z)}$ for each given $g \in G(\mathbb{R})$ which identifies the fibers V_z and $V_{g(z)}$ of $\widetilde{V}$. Thus we have a function $J_\rho : G(\mathbb{R}) \times \mathcal{H} \to GL(V)$ satisfying

1. (Cocycle Relation) $J_\rho(gh,z) = J_\rho(g,h(z))J_\rho(h,z)$ for $g,h \in G(\mathbb{R})$;
2. (Holomorphy) $J_\rho(g,z)$ is holomorphic in z.

When $G = SL(2)$, then $C_{\mathbf{0}} = SO_2(\mathbb{R}) \subset C = \mathbb{C}^\times$ whose irreducible complex representation is given by

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \mapsto \rho \begin{pmatrix} * & * \\ c & d \end{pmatrix} = (ci+d)^k = e^{ik\theta}.$$

In this case, $J_\rho(g,z) = (cz+d)^k$. This goes as follows: Split $GL_2(\mathbb{R}) = PC_{\mathbf{0}}$ for P made of upper triangular matrices with right lower corner 1. For $z = x+iy$, define $p_z = \left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right)$. Then for $g \in SL_2(\mathbb{R})$, write $gp_z = p_{g(z)}u$ with $u \in C_{\mathbf{0}}$, and we have $\rho(u) = \rho(p_{g(z)}^{-1} g p_z) = (cz+d)^k$ by computation. Indeed, $J(g,z)$ sends (v,p_z) to $(uv, p_{g(z)}) \sim (v, gp_z) = (v, p_{g(z)}u)$.

One can view the complexification C as a real algebraic group; let T_C be a maximal real torus of C. To any character κ of T_C, we can attach a rational representation $L_C(\kappa;\mathbb{C})$ $(= \rho_\kappa)$ of C. Let $V(\kappa) = L_C(\kappa;\mathbb{C})$. For an arithmetic discrete subgroup $\Gamma \subset G(\mathbb{Q})$, a holomorphic automorphic form of (coherent) weight κ is a holomorphic function $f : \mathcal{H} \to L_C(\kappa;\mathbb{C})$ satisfying $f(\gamma(z)) = J_\rho(\gamma,z)f(z)$ for all $\gamma \in \Gamma$ (with some additional growth condition if $\Gamma\backslash\mathcal{H}$ is not compact). Again the space of holomorphic automorphic forms is trivial unless the weight κ is positive (with respect to a fixed Borel subgroup B).

Often the complex manifold $\Gamma\backslash\mathcal{H}$ is canonically algebraizable, giving rise to an algebraic variety (or a scheme) X_Γ, called canonical models or *Shimura varieties*, defined over a valuation ring $\mathcal{W}$ in a number field with residual characteristic p. At the same time, we can algebraize the vector bundle $\widetilde{V}(\kappa)$ associated to $V(\kappa)$. Thus we often have a coherent sheaf $\underline{\omega}^\kappa$ on X_Γ giving rise to $\widetilde{V}(\kappa)$ after extending scalar to $\mathbb{C}$. The global sections of $H^0(X_\Gamma, \underline{\omega}^\kappa_{/A})$ for $\mathcal{W}$–algebra A are called A–integral automorphic forms of weight κ. Note that, T_C is isomorphic to T, because they are maximal tori in the same group G. Thus we can and will identify T and T_C (with compatible choice of Borel subgroups B and $B_C = B \cap C$). On X_Γ, we may regard the Γ–module $L_G(\kappa;A)$ as a locally constant sheaf associating to an open subset $U \subset X_\Gamma$ sections over U of the covering space $\widetilde{L}_G(\kappa;A) = \Gamma\backslash(\mathcal{D} \times L_G(\kappa;A))$ over X_Γ. Here the quotient $\Gamma\backslash(\mathcal{D} \times L_G(\kappa;A))$ is taken through the diagonal action. Thus each positive weight $\kappa \in X(T)$ gives two spaces of automorphic forms:

$$H^d(X_\Gamma, L_G(\kappa;A)), \quad H^0(X_\Gamma, \underline{\omega}^\kappa_{/A}) = G_\kappa(\Gamma;A).$$

There is (at least conjecturally) a correspondence $\kappa \mapsto \kappa^*$ such that

$$H^0(X_\Gamma, \underline{\omega}^\kappa) \hookrightarrow H^d(X_\Gamma, L_G(\kappa^*;\mathbb{C}))$$

by a "generalized Eichler-Shimura isomorphism" which is supposed to be equivariant under Hecke operators. If such equivariance holds, we say that the two modules: the source and the image are equivalent as *Hecke modules*. In the example of $SL(2)_{/\mathbb{Q}}$, we have $\kappa \in X(T) = X(\mathbb{G}_m) = \mathbb{Z}$ and $\kappa^* = \kappa - 2$ with:

$$G_\kappa(\Gamma;\mathbb{C}) \hookrightarrow H^1(X_\Gamma, L_{SL(2)}(\kappa-2;\mathbb{C})) \quad (\Gamma \subset SL_2(\mathbb{Z}))$$

via $f \mapsto$ the cohomology class of $[f(z)(X-zY)^{\kappa-2}dz]$. This is valid if $\kappa \geq 2$.

1.2. **p–Adic interpolation of automorphic forms.** We would like to interpolate these two sets of spaces $\{H^0(X_\Gamma, \underline{\omega}^\kappa)\}_\kappa$ and $\{H^d(X_\Gamma, L_G(\kappa; \mathcal{W}))\}_\kappa$ when the weights κ vary continuously in $\mathrm{Hom}_{top-gp}(T(\mathbb{Z}_p), \mathbb{Z}_p^\times)$. On these two spaces, there is a natural action of Hecke operators; so, we want this interpolation to take into account the Hecke operators. To describe our idea of how to interpolate automorphic forms, we write W for the p–adic completion of $\mathcal{W}$. What we would like to do in the two cases is:

(1) (Universality) Construct a (big) space V which is a compact module over $W[[T(\mathbb{Z}_p)]]$ such that the κ-eigenspace $V[\kappa]$ contains canonically the space $H^d(X_\Gamma, L_G(\kappa; W))$ in the topological case, resp. $H^0(X_{\Gamma/W}, \underline{\omega}^\kappa)$ in the coherent case as $W[[T(\mathbb{Z}_p)]]$–modules.
(2) (Hecke operators) Establish a natural action of Hecke operators on V, and show the inclusion in (1) is Hecke equivariant.
(3) (VCT) Find an appropriate $W[[T(\mathbb{Z}_p)]]$–submodule $X \subset V$ of co-finite type ($\Leftrightarrow$ W–dual is of finite type) such that X is stable under Hecke operators and $X[\kappa]$ is canonically isomorphic, as Hecke modules, to a well-described subspace of automorphic forms of weight κ if $\kappa \gg 0$.

The item (3) is called a *vertical control theorem* of the subspace X. Examples of the VCT are given as Theorem 3.2 for elliptic modular forms, Theorem 3.3 for p–adic family of elliptic modular forms, Theorem 8.5 for automorphic forms on unitary groups, Theorem 9.1 for Hilbert modular forms and Corollary 9.3 for Hilbert modular Hecke algebras. A more general result on VCT can be found in [H02] and [PAF]. In [H02] page 37 and [GME] 3.2.3, Hecke operators T are defined for a given (geometric) modular form f as a sum $f|T(A_{/S}) = \sum_\alpha f(A_{\alpha/S})$ of the values of f at abelian schemes A_α with a specific isogeny $\alpha : A \to A_\alpha$ of a given degree. This is perfectly fine if the degree is invertible on the base scheme S, but otherwise if S is of characteristic p and the degree is p, one has to replace the sum by the trace from the (possibly purely inseparable) extension of S over which the isogeny is defined (as was originally done for elliptic modular forms in Katz's definition in [K3] 3.11). Thus the argument proving the control theorem in these works has to be modified slightly. This adjustment will be described in the present lecture notes in 3.1.3, 7.1.6 and 8.2.2. The author is grateful to Eric Urban for his pointing out this error in the above cited works (except for [PAF]) of the author.

We will mainly deal with the coherent case where G admits Shimura varieties which are given as moduli of abelian varieties with PEL structure. However at some point, we need to use some results obtained in the topological case; so, a couple of lectures will be devoted also to describe the situation in topological cases. In any case, I will often suppose for simplicity that G to be $U(m, n)$ or its F–inner forms over a totally real field F, although we also give expositions for $GSp(2g)$ from time to time.

In the coherent case, we shall define V to be the space of formal functions on an formal pro-scheme, called the Igusa tower, classifying abelian schemes with a level p^∞ structure in addition to a PEL structure outside p. We will prove the vertical control for the space $X = V^{ord}$ of nearly p–ordinary automorphic forms and prove that its W–dual $\mathrm{Hom}_W(V^{ord}, W)$ is $W[[T(\mathbb{Z}_p)]]$–projective of finite type.

Actually, we have for any classical group a good definition of nearly p–ordinary cusp forms, that is, a cusp form is called nearly p–ordinary if it has the property that the Newton polygon of the Hecke polynomial at p is equal to the hypothetical

Hodge polygon mechanically constructed out of the weight κ (of the motive attached to the cusp form). We can prove that the Newton polygon is always on or above the Hodge polygon (without recourse to hypothetical motives); so, a nearly p–ordinary form has minimal possible Newton polygon (see Section 4).

One would expect that $\mathrm{Hom}_W(V^{ord}, W)$ should be $W[[T(\mathbb{Z}_p)]]$–projective of finite rank if G is associated to a bounded hermitian domain. Contrary to this, when we deal with the group like $GL(n)$ $(n > 2)$, the module $\mathrm{Hom}_W(V^{ord}, W)$ is of finite type over $W[[T(\mathbb{Z}_p)]]$, but it is known to be of torsion. Natural questions are:

(Q1) *When can one expect that the space V^{ord} is $W[[T(\mathbb{Z}_p)]]$–coprojective (that is, its W–dual is projective)? What is the (expected) minimal value of κ at which the vertical control holds? What happens if one specializes to a very low weight? If V^{ord} is co-torsion, what is the Krull dimension of the $W[[T(\mathbb{Z}_p)]]$–module* $\mathrm{Hom}_W(V^{ord}, W)$*? What is its characteristic power series if* $\mathrm{codim}(V^{ord})^* = 1$ *in Spec$(W[[T(\mathbb{Z}_p)]])$?*

It turns out that all these questions are quite arithmetic, as we will see it in the course. In the elliptic modular case, the lowest weight where VCT holds is 2. However, as Buzzard and Taylor studied, there is a good criterion via Galois representations to guarantee the limit at weight 1 to be a true modular form (not just p–adic), which played an important role in their proof of the Artin conjecture for some icosahedral cases.

In the simplest example of $SL(2)_{/\mathbb{Q}}$, we take an arbitrary p–adically complete W–algebra $A = \varprojlim_n A/p^n A$. We consider a test object $(E, \phi_p, \phi_N)_{/A}$ made of an elliptic curve E, a level p^∞–structure $\phi_p : \mu_{p^\infty} \hookrightarrow E$ (that is a closed immersion of ind-group schemes) and a level N–structure ϕ_N, like a point of order N (here, an inclusion of $\mathbb{Z}/N\mathbb{Z}$ into the set of N–torsion elements $E[N]$ in E), all these data being defined over A. A p–adic modular form f is a functorial rule associating an element of A to a test object $(E, \phi_p, \phi_N)_{/A}$. Thus we have $f(E, \phi_p, \phi_N) \in A$, and for each p–adically continuous W–algebra homomorphism $A \xrightarrow{\rho} B$,

$$f((E, \phi_p, \phi_N)_{/A,\rho} \times B) = \rho(f(E, \phi_p, \phi_N)).$$

A p–ordinary modular form which is an eigenform of $T(p)$ has by definition a p–adic unit eigenvalue for $T(p)$. In general, p–ordinary modular forms are linear combinations of such p–ordinary eigenforms (we will give a more conceptual definition in the text). The evaluation of f at the Tate curve $\mathrm{Tate}(q)_{/\mathbb{Z}((q))}$ at the cusp infinity yields the q–expansion:

$$f(q) = f(\mathrm{Tate}(q)_\infty, \phi_p^{can}, \phi_N^{can}) = \sum_{n=0}^{\infty} a(n, f) q^n.$$

We can deduce from the irreducibility of the Igusa tower that V is isomorphic to the p–adic completion of

$$\mathcal{W}[[q]] \bigcap \left(\sum_{k=0}^{\infty} \sum_{\alpha} G_k(\Gamma_1(Np^\alpha)) \right).$$

Here we have embedded $G_k(\Gamma_1(N))$ into $\mathbb{C}[[q]]$ by the Fourier expansion, writing $q = \exp(2\pi i z)$.

In the topological case of $SL(2)$, V is given by $H^1(\Gamma', \mathcal{C}(\widehat{\Gamma}'/U(\mathbb{Z}_p), W))$, where $\mathcal{C}(\widehat{\Gamma}'/U(\mathbb{Z}_p), W)$ is the space of continuous functions on $\widehat{\Gamma}'/U(\mathbb{Z}_p)$, $\Gamma' = \Gamma_1(N) \cap$

$\Gamma_0(p)$ and $\widehat{\Gamma}'$ is the closure of $\Gamma_0(p)$ in $SL_2(\mathbb{Z}_p)$. Then $L(\kappa;W) \hookrightarrow \mathcal{C}(\widehat{\Gamma}'/U(\mathbb{Z}_p);W)$ induces a map $H^1(\Gamma', L(\kappa;W))$ into V.

For any $W[[T(\mathbb{Z}_p)]]$–submodule $X \subset V$ satisfying (VCT), the eigenvalue $\lambda(t)$ of a Hecke operator t on X is algebraic over $W[[T(\mathbb{Z}_p)]]$. In fact, the Hecke algebra $\mathbf{h}$ in $\mathrm{End}_{W[[T(\mathbb{Z}_p)]]}(X)$ generated by (appropriate) Hecke operators are an algebra over $W[[T(\mathbb{Z}_p)]]$ of finite (generic) rank (or even of torsion). Take an irreducible component $\mathit{Spec}(\mathbb{I})$ of $\mathit{Spec}(\mathbf{h})$. The operator t projected to $\mathbb{I}$, written as $\lambda(t)$ (that is, $\lambda : \mathbf{h} \twoheadrightarrow \mathbb{I}$ is the projection), can be considered to be an algebraic function (that is, global section of the structure sheaf) on $\mathit{Spec}(\mathbb{I})$. In particular, if $P \in \mathit{Spec}(\mathbb{I})(W) = \mathrm{Hom}_{W-alg}(\mathbb{I}, W)$ with $P|_{W[[T(\mathbb{Z}_p)]]} = \kappa$ for $\kappa \gg 0$, $\lambda(t)(P) = P(\lambda(t))$ is the eigenvalue of t occurring in either $H^0(X_\Gamma, \underline{\omega}^\kappa)$ or $H^d(X_\Gamma, L(\kappa;W))$. In the simplest case of $SL(2)$, we have $T(\mathbb{Z}_p) = \mathbb{Z}_p^\times = u^{\mathbb{Z}_p} \times \Delta$ for a finite group Δ. Thus $W[[T(\mathbb{Z}_p)]] = \Lambda[\Delta]$ for $\Lambda = W[[u^{\mathbb{Z}_p}]] \cong W[[X]]$ (a formal power series ring) via $u^s \mapsto (1+X)^s = \sum_{n=0}^\infty \binom{n}{s} X^n$. Note that

$$\kappa((1+X)^s) = \kappa(u^s) = u^{\kappa s} = (1+X)^s|_{X=u^\kappa-1}.$$

The algebra homomorphism $\kappa : \Lambda \to W$ is the "evaluation" at $X = u^\kappa - 1$! Thus if $\mathbb{I} = \Lambda$, $\lambda(T(n))(\kappa) = \lambda(T(n))(u^\kappa - 1)$ (viewing $\lambda(T(n))$ as a power series) gives a p–adic analytic interpolation of Hecke eigenvalues. In general, we get the p–adic interpolation of Hecke operators parameterized by $\mathit{Spec}(\mathbb{I})$.

1.3. p–**Adic Automorphic** L–**function.** Since the specialization $\lambda(T(n))(u^\kappa - 1)$ is the Hecke eigenvalue occurring in the space of cusp forms, it can be considered as a complex number uniquely (by fixing embeddings $i_\infty : \overline{\mathbb{Q}} \to \mathbb{C}$ and $i_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$). Thus we can think of automorphic L–functions $L(s, \lambda(\kappa))$ made out of such eigenvalues; for example, the modular Hecke L–function of $GL(2)$:

$$L(s, \lambda(\kappa)) = \sum_{n=1}^\infty i(\lambda(T(n))(u^\kappa - 1))n^{-s},$$

writing $i = i_\infty \circ i_p^{-1}$. Supposing that $L(m, \lambda(\kappa))$ for a fixed integer m has rationality (up to a transcendental factor or a period $\Omega(\kappa)$), a natural question we then ask is:

(Q2) *Is it possible to interpolate p–adically the value $\frac{L(m,\lambda(\kappa))}{\Omega(\kappa)}$? Is it possible to find $L_\lambda \in \mathbb{I}$ such that $L_\lambda(\kappa) = L(m, \lambda(\kappa))$ for $\kappa \gg 0$?*

This problem of course involves a subtle question of how to normalize the factor $\{\Omega(\kappa)\}_\kappa$ in the aggregate (varying $\kappa \in X(T)$) to get an "optimal" integrality; so, it is more involved than proving rationality (see Section 9 for some examples and [H96] for a general theory). Once we are successful in constructing canonical p–adic L–functions, we could ask more specifically

(Q3) *When is the p–adic L–function analytic? Where could it have singularity? If there is a singularity, what is the residue?*

See [H96] for some examples and conjectural discussions on these questions.

1.4. **Galois Representations.** Once an irreducible component $\mathbb{I}$ of the Hecke algebra is given, one would expect to have a Galois representation $\rho_\lambda : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to GL_n(\mathbb{I})$ such that the Hecke polynomial at $\ell \neq p$ gives rise to the characteristic polynomial of the Frobenius element. We could partially and conjecturally answer the question (Q3) that the associated p–adic L–function has singularity at P if the Galois representation specialized at P, that is, $\rho_\lambda \mod P$ contains the trivial representation (a p–adic Artin conjecture, See [H96]). We then further ask

(Q4) *For a given representation ρ_λ as above, is their any good way to associate a Selmer group* $\mathrm{Sel}(\rho_\lambda)$ *so that the characteristic element in* $\mathbb{I}$ *of the Selmer group should be equal to the p–adic L–function or its numerator?*

See [MFG] Chapter 5 for a general description of Selmer groups. If this is affirmative, then this would describe the zero-set of the p–adic L–function. Related to this, VCT is often useful to identify the nearly p–ordinary Hecke algebra with the nearly p–ordinary universal deformation ring if at one weight the deformation ring with the given weight is identified with the Hecke algebra of the specific weight (see for example, [HM] Section 4.3). The argument proving VCT often yields another type of control theorem: so-called the horizontal control theorem (HCT), giving a precise description of the behavior of a Hecke algebra if one add primes outside p to the level of the Hecke algebra. This horizontal control (HCT) is used in the case of $GL(2)$ to construct the Taylor-Wiles systems, which in turn proves the identification of the Hecke algebra of a specific weight with the deformation ring. See [MFG] Chapter 3 and [GME] Chapter 3 for these topics.

1.5. **Plan of the lectures.** I will try to answer some of these questions in the lecture in some specific cases in a concrete way and in some other cases conjecturally. Here is a plan:

1. In a first few lectures in April, 2000 (Sections 1-3), I will recall the theory in the elliptic modular case with some proofs as a prototype of the theory and basic properties of nearly ordinary automorphic forms on general groups.
2. Lectures in May, 2000 (Sections 4-8) will be devoted to prove the VCT for unitary groups. I will describe the proof in the cocompact case in details (and touch briefly the non-cocompact case taking Hilbert modular varieties as an example: Section 9).
3. Lectures in June, 2000 would first discuss applications of VCT and the q–expansion principle in the Hilbert and the elliptic modular cases (Section 9), and in Section 10, we shall give a sketch of a proof of the q–expansion principle of p–adic automorphic forms for split symplectic groups and quasi-split unitary groups (acting on a tube domain).

Some of the papers and preprints of mine related to these subjects can be downloaded from my web site: `www.math.ucla.edu/~hida`.

Although we have tried to give details of the proofs of the material described above in these notes, many results have to be taken for granted here in these notes. The book [PAF] covers similar materials with more details and contains a proof (different from the one presented in Section 10 of these notes) of the irreducibility of the Igusa tower over the mod p canonical models (in a more general setting).

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2. Elliptic Curves

In this lecture, I try to sketch a proof of the VCT in the elliptic modular case. There are several different approaches:

(1) Through the moduli theory of elliptic curves; this is what we do ([H86a] and [GME] Chapter 3).
(2) Through studying of topological cohomology groups and jacobians of modular curves. This way has an advantage of producing at the same time Galois representations into $GL_2(\mathbb{I})$, where $\mathbb{I}$ is a big ring (which is finite and often flat over $W[[X]]$; [H86b]).
(3) Through the theory of p-adic Eisenstein measures and p–adic Rankin convolution theory. This method was found by A. Wiles and explained in the elliptic modular case in my book: [LFE] Chapter 7.
(4) As an application of the identification of Hecke algebras and universal Galois deformation rings at many different weights (done by Wiles and Taylor). This method is exposed in my book [MFG].

We follow the first method. A shorter proof than the original in 1986 can be found in my forthcoming book [PAF] Chapter 3 and also in my Tata lecture notes: Control Theorems and Applications, which can be downloaded from my web site. Also Chapter 3 of the book [GME] contains a more down-to-earth description of the proof.

2.1. **Basics of Elliptic Curves.** We shall give a brief description of the theory of the moduli of elliptic curves. Chapter 2 of the book [GME] contains a thorough exposition of the theory.

2.1.1. *Definition of Elliptic Curves.* For a given scheme S, a proper smooth curve $f : E \to S$ is called an *elliptic curve* if it satisfies the following conditions:

(E1) E has a section $\mathbf{0} = \mathbf{0}_E \in E(S)$ (thus $f \circ \mathbf{0} = 1_S$);
(E2) $\dim_S E = 1$, and E is geometrically connected (this means that each geometric fiber of E over a geometric point is connected and of dimension 1);
(E3) $f_*\Omega_{E/S}$ (equivalently $R^1 f_*\mathcal{O}_E$ by Grothendieck-Serre duality) is locally free of rank 1 (genus = 1).

There is no harm to assume that S is connected, as we do from now on. For any S–scheme $\phi : T \to S$, the fiber product $E_T = E \times_S T$ is again an elliptic curve with the zero section $\mathbf{0}_T = \mathbf{0}_E \times 1_T$. For two elliptic curves E and E' over S, an S–morphism $h : E \to E'$ is always supposed to take $\mathbf{0}_E$ to $\mathbf{0}_{E'}$.

2.1.2. *Cartier Divisors.* A closed subscheme $D \subset E$ is called an effective Cartier divisor (relative to S) on E if $f_*\mathcal{O}_D = f_*(\mathcal{O}_E/I(D))$ given by an invertible sheaf of ideals $I(D)$ is S–flat (so locally free). We define $\mathcal{L}(D) = I(D)^{-1}$ and put $\deg(D) = \deg(\mathcal{L}(D)) = \mathrm{rank}_S f_*(\mathcal{O}_D)$. In particular, the $\mathbf{0}$ section gives rise to a divisor $[\mathbf{0}]$ of degree 1 given by $\mathcal{O}_{[\mathbf{0}]} \overset{\mathbf{0}^\#}{\cong} \mathcal{O}_S$. We then think of $I(m[\mathbf{0}]) = I([\mathbf{0}])^m$ and $\mathcal{L}(m[\mathbf{0}]) = I(m[\mathbf{0}])^{-1}$ for $m \in \mathbb{Z}$. The line bundle $\mathcal{L}(m[\mathbf{0}])$ can be regarded as the sheaf of meromorphic functions on E with sole singularity at $\mathbf{0}$ having pole of order equal to or less than m at $\mathbf{0}$.

Write $\mathrm{Div}^r(E/S)$ for the set of all degree r effective divisors relative to S. The association $T \mapsto \mathrm{Div}^r(E_T/T)$ is a contravariant functor by pull-back of divisors $D_{/E} \mapsto D_{/E_T} = D \times_E E_T$.

If $S = Spec(k)$ for an algebraically closed field k, k–rational effective divisors can be identified with positive linear combinations of points on $E(k)$. We have $\deg(\sum_P m_P[P]) = \sum_P m_P$. We can thus consider the group $\mathrm{Div}(E/k)$ of all formal linear combinations (including negative coefficients) of points on E. Then $\deg : \mathrm{Div}(E/k) \to \mathbb{Z}$ is a well defined homomorphism given by the above formula.

2.1.3. *Picard Schemes.* For any scheme X, we define $\mathrm{Pic}(X)$ as the set of all isomorphism classes of invertible sheaves on X. The association $X \mapsto \mathrm{Pic}(X)$ is a contravariant functor by the pull-back of invertible sheaves, and $\mathrm{Pic}(X)$ is a group by tensor product. We define, for each S–scheme $\phi : T \to S$

$$\mathrm{Pic}_{E/S}(T) = \mathrm{Pic}(E_T)/\phi^* \mathrm{Pic}(T).$$

We can extend the degree map to $\deg : \mathrm{Pic}_{E/S}(T) \to \mathbb{Z}^{\pi_0(T)}$ for the set $\pi_0(T)$ of connected components. Indeed, for any algebraically closed field k and a geometric point $s : Spec(k) \hookrightarrow T$, the fiber $E(s) = E \times_{S,\phi\circ s} s = E_T \times_T s$ is an elliptic curve over the field k and $\deg(\mathcal{L}) = \deg(\mathcal{L}(s))$ for the pull back $\mathcal{L}(s)$ at s, which is well defined. By this fact, we can define

$$\mathrm{Pic}^r_{E/S}(T) = \left\{ \mathcal{L} \in \mathrm{Pic}_{E/S}(T) \middle| \deg(\mathcal{L}) = r \text{ for all connected component of } T \right\}.$$

Here is Abel's theorem (e.g. [GME] 2.2.2):

$$\text{(Abel)} \qquad \mathrm{Pic}^r_{E/S}(T) \cong E(T) = \mathrm{Hom}_S(T, E) \quad \text{by } \mathcal{L}([P]) \otimes \mathcal{L}([\mathbf{0}])^{r-1} \leftrightarrow P$$

Thus an elliptic curve is a group scheme with the identity $\mathbf{0}$. If $\phi : C \to C'$ is a non-constant S–morphism of two smooth geometrically connected curves, $|\phi^{-1}(s)|$ is constant for geometric points s of C', that is, $\phi_*\mathcal{O}_C$ is locally free of finite rank. We write this number as $\deg(\phi)$. Thus $\bigwedge^{\deg(\phi)} \phi_*\mathcal{O}_C$ is an invertible sheaf on C'. If $\phi : E \to E'$ is an S–morphism of elliptic curves, by our convention, ϕ takes $\mathbf{0}_E$ to $\mathbf{0}_{E'}$, and hence, at the side of the Picard scheme, it is just $\mathcal{L} \mapsto \bigwedge^{\deg(\phi)} \phi_*\mathcal{L}$; so, obviously ϕ is a homomorphism of group schemes.

2.1.4. *Invariant Differentials.* By (E3), for a dense affine open subset $Spec(A)$ of S, $H^0(E, \Omega_{E/A}) = A\omega$ for a 1–differential ω. For each point $P \in E(S)$, $T_P : x \mapsto x + P$ gives an automorphism on E. Since we can therefore bring any given cotangent vector at $\mathbf{0}$ to P isomorphically to a cotangent vector at P, each cotangent vector at $\mathbf{0}$ extends to a global section of $\Omega_{E/S}$. Thus $T_P^*\omega = \omega$ (cf. [GME] 2.2.3).

2.1.5. *Classification Functors.* An important fact from functorial algebraic geometry is: we can associate to each S–scheme X, a contravariant functor $\underline{X} : S\text{–}SCH \to SETS$ such that $\underline{X}(T) = \mathrm{Hom}_S(T, X)$. This association is fully faithful; in other words, writing CTF for the category of contravariant functors from S–schemes to $SETS$, we have $\mathrm{Hom}_S(X, Y) \cong \mathrm{Hom}_{CTF}(\underline{X}, \underline{Y})$ by $X \xrightarrow{\phi} Y \mapsto \phi(T) : \underline{X}(T) \to \underline{Y}(T)$ given by $\phi(T)(T \xrightarrow{f} X) = \phi \circ f$ (e.g. [GME] Lemma 1.4.1). This is intuitively clear because an algebraic variety is just a function associating to each ring R its R–integral points $X(R) = \underline{X}(Spec(R))$. I leave the verification of this to the reader as an exercise (the inverse is given by $\mathrm{Hom}_{CTF}(\underline{X}, \underline{Y}) \ni F \mapsto F(X)(1_X)$ where $F(X) : \underline{X}(X) \to \underline{Y}(X) = \mathrm{Hom}_S(X, Y)$).

Here is an example of how to use the faithfulness: Let N be a positive integer. Since $E(T)$ is a group, $x \mapsto Nx$ gives a functorial map $N(T) : E(T) \to E(T)$; so, an endomorphism of elliptic curves $N : E \to E$. We define its kernel $E[N] =$

$E \times_{E,N,\mathbf{0}} S$:

$$\begin{array}{ccc} E[N] & \longrightarrow & E \\ \Big\downarrow & & \Big\downarrow{\scriptstyle N} \\ S & \underset{\mathbf{0}}{\longrightarrow} & E. \end{array}$$

It is clear that $E[N](T) = \mathrm{Ker}(N(T))$. It is known that $\deg N = N^2$ and if N is invertible over S, $E[N](k) \cong (\mathbb{Z}/N\mathbb{Z})^2$ for all algebraically closed fields k with $Spec(k) \hookrightarrow S$.

We consider the following functor:

$$\mathcal{P}'_{\Gamma_1(N)}(A) = \left[(E, P, \omega)_{/A}\right]$$

from the category ALG of $\mathbb{Z}$–algebras into $SETS$, where ω is a nowhere vanishing invariant differential, P is a point of order exactly N, that is, $m \mapsto mP$ induces an isomorphism $\mathbb{Z}/N\mathbb{Z}_{/A} \hookrightarrow E$ of group schemes defined over A and $[\cdot] = \{\cdot\}/\cong$ is the set of all isomorphism classes of the objects inside the brackets. Here $\mathbb{Z}/N\mathbb{Z}$ as a group functor associates with T the constant group $(\mathbb{Z}/N\mathbb{Z})^{\pi_0(T)}$.

Therefore $\mathcal{O}_{(\mathbb{Z}/N\mathbb{Z})/S} = \bigoplus_{\mathbb{Z}/N\mathbb{Z}} \mathcal{O}_S$; so, the structure sheaf of $\mathbb{Z}/N\mathbb{Z}$ is free of finite rank N. Such a group scheme is called a locally free group scheme (of rank N). There is another example: Start with the multiplicative group $\mathbb{G}_m$ (as a functor $\mathbb{G}_m(A) = A^\times$ and as a scheme $Spec(\mathbb{Z}[t, t^{-1}])$, we consider the kernel μ_N of $N : x \mapsto x^N$ as a functor $\mu_N(A) = \{\zeta \in A | \zeta^N = 1\}$ and as a scheme

$$\mu_N = Spec(\mathbb{Z}[t]/(t^N - 1)) = Spec(\mathbb{Z}[(\mathbb{Z}/N\mathbb{Z})]).$$

Then μ_N is a locally free group scheme of rank N. If $N > 1$, it is not isomorphic to $(\mathbb{Z}/N\mathbb{Z})$, since for any prime p, $\mu_p(\mathbb{F}_p) = \{1\}$ but $(\mathbb{Z}/p\mathbb{Z})(\mathbb{F}_p) = \mathbb{Z}/p\mathbb{Z}$ for a prime p. We consider a version of the functor $\mathcal{P}'_{\Gamma_1(N)}$ defined as follows:

$$\mathcal{P}_{\Gamma_1(N)}(A) = \left[(E, \phi_N : \mu_N \hookrightarrow E[N], \omega)_{/A}\right]$$

2.1.6. *Cartier Duality.* The two functors $\mathcal{P}_{\Gamma_1(N)}$ and $\mathcal{P}'_{\Gamma_1(N)}$ are isomorphic by the following theory of Cartier duality: If G is a locally free group scheme of rank N over S, there exists a group scheme $\widehat{G}_{/S}$ such that $\widehat{G}(T) = \mathrm{Hom}_T(G_T, \mathbb{G}_{m/T}) = \mathrm{Hom}_T(G_T, \mu_{N/T})$, where $\mathbb{G}_{m/S} = \mathbb{G}_m \times S$ and $\mu_{N/S} = \mu_N \times S$ over $Spec(\mathbb{Z})$. We have $\widehat{\widehat{G}} \cong G$ in an obvious manner, and $\widehat{\mathbb{Z}/N\mathbb{Z}} = \mu_N$ by $\zeta(m) = \zeta^m$ for $\zeta \in \mu_N(A)$ and $m \in (\mathbb{Z}/N\mathbb{Z})(A)$.

Let $E_{/S}$ be an elliptic curve. The section $\mathbf{0} : S \to E$ induces a section of $f^* : \mathrm{Pic}(S) \to \mathrm{Pic}(E)$; so, we have a splitting:

$$\mathrm{Pic}(E_T) = f_T^* \mathrm{Pic}(T) \oplus \mathrm{Ker}(\mathbf{0}_T^*) \quad \text{and} \quad \mathrm{Ker}(\mathbf{0}_T^*) = \mathrm{Pic}_{E/S}(T),$$

regarding $\mathrm{Pic}_{E/S}(T)$ as a set of isomorphism classes of invertible sheaves whose restriction to $\mathbf{0}$ is trivial, that is, $\mathbf{0}^*\mathcal{L}$ is isomorphic to $\mathcal{O}_S$.

Let $P \in \mathrm{Ker}(\pi)$ for a non-constant S–morphism $\pi : E \to E'$. Then

$$\pi^* : E' \cong \mathrm{Pic}_{E'/S} \to \mathrm{Pic}_{E/S} = E$$

is an S–homomorphism. We are going to show that $\mathrm{Ker}(\pi^*) = \widehat{\mathrm{Ker}(\pi)}$. Take $\mathcal{L} \in \mathrm{Ker}(\pi^*)$, and take an open covering $E' = \bigcup_i U_i$ such that $\mathcal{L}|_{U_i} = f_i^{-1}\mathcal{O}_{U_i}$. Since $\mathbf{0}^*\mathcal{L} = \mathcal{O}_S$, one can assume $f_i \circ \mathbf{0}_{E'} = f_j \circ \mathbf{0}_{E'}$ for all $i \neq j$ on $U_i \cap U_j$. Let

$h_i = f_i \circ \pi$; we have $\pi^*\mathcal{L}|_{V_i} = h_i^{-1}\mathcal{O}_{V_i}$ for $V_i = \pi^{-1}(U_i)$. Let $P \in (\mathrm{Ker}\,\pi)(T)$; then on $P^{-1}(V_i) \cap P^{-1}(V_j)$, we have

$$h_i \circ P = f_i \circ \pi \circ P = f_i \circ \mathbf{0}_{E'} = f_j \circ \mathbf{0}_{E'} = h_j \circ P.$$

This implies that $h_i \circ P$'s glue to give a global section $h \circ P \in \Gamma(T, \mathcal{O}_T^\times) = \mathbb{G}_m(T)$, getting a homomorphism $\mathrm{Ker}(\pi^*) \to \widehat{\mathrm{Ker}(\pi)}$, which can be easily verified to be an isomorphism (because twice this operation yields an identity map of $\mathrm{Ker}(\pi^*)$). Since $N^* = N$ as we can see easily, we get $\widehat{E[N]} = E[N]$. Writing the pairing as $\langle\ ,\ \rangle : E[N] \times_S E[N] \to \mu_{N/S}$, we get $\langle \phi(P), Q\rangle = \langle P, \phi^*(Q)\rangle$; so, $\phi \mapsto \phi^*$ is an involution with $\phi^* \circ \phi = \deg(\phi) \geq 0$ (a positive involution).

For a given additive level N–structure $\phi_N : \mathbb{Z}/N\mathbb{Z} \hookrightarrow E[N]$, by duality, we get $\pi_N : E[N] \twoheadrightarrow \mu_N$ which has a section ϕ'_N well determined modulo $C = \phi_N(\mathbb{Z}/N\mathbb{Z})$. Thus $(E/C, \phi'_N : \mu_N \hookrightarrow (E/C)[N], \omega')$ is well defined as an element of $\mathcal{P}_{\Gamma_1(N)}(A)$, where ω' coincides with ω at the identity (because the projection $E \to E/C$ is a local isomorphism; that is, an étale morphism). The inverse: $\mathcal{P} \to \mathcal{P}'$ is given by

$$(E', \phi'_N : \mu_N \hookrightarrow E'[N], \omega') \mapsto (E'' = E'/\mathrm{Im}(\phi'_N), \phi_N : \mathbb{Z}/N\mathbb{Z} \hookrightarrow E''[N], \omega'')$$

similarly. Since $(E/C)/\phi'_N(\mu_N) = E/E[N] \cong E$, we have $\mathcal{P}' \cong \mathcal{P} \cong \mathcal{P}'$ and hence equivalence.

2.2. **Moduli of Ordinary Elliptic Curves and the Igusa Tower.** We now study the scheme $Y_1(N)$ representing $\mathcal{P}_{\Gamma_1(N)}$ over $\mathbb{Z}[\frac{1}{N}]$–algebras. This eventually leads us to the vertical control theorems in the elliptic modular cases.

2.2.1. *Moduli of level* 1 *over* $\mathbb{Z}[\frac{1}{6}]$. Hereafter, we assume until Section 3 (for simplicity) that 6 is invertible in any algebra we consider. Let $(E, \omega)_{/A}$ be a couple of an elliptic curve and a nowhere vanishing differential. We choose a parameter T at $\mathbf{0}$ so that

$$\omega = (1 + \text{higher terms of } T)dT.$$

By the Riemann-Roch theorem, $\dim H^0(E, \mathcal{L}(m[\mathbf{0}])) = m$ if $m > 0$. We have two morphisms $x, y : E \to \mathbf{P}^1$ such that

1. x has a pole of order 2 at $\mathbf{0}$ with the leading term T^{-2} in its Taylor expansion in T (removing constant term by translation);
2. y has a pole of order 3 with leading term $-T^{-3}$.

Out of these functions, we can create bases of $H^0(E, \mathcal{L}(m[\mathbf{0}]))$:

- $H^0(E, \mathcal{L}(2[\mathbf{0}])) = A + Ax$, $H^0(E, \mathcal{L}(3[\mathbf{0}])) = A + Ax + Ay$. This implies that x has a pole of order 2 at $\mathbf{0}$ and y has order 3 at $\mathbf{0}$. They are regular outside $\mathbf{0}$;
- Out of these functions $1, x, y$, we create functions with pole of order n at $\mathbf{0}$ as follows:

$$\begin{aligned} n \leq 4 &: 1, x, y, x^2 \ (\dim = 4) \\ n \leq 5 &: 1, x, y, x^2, xy \ (\dim = 5) \\ n \leq 6 &: 1, x, y, x^2, xy, x^3, y^2 \ (\dim = 6). \end{aligned}$$

Comparing the leading term of T^{-6}, one sees that the seven sections

$$1, x, y, x^2, xy, x^3, y^2$$

of $H^0(E, \mathcal{L}(6[\mathbf{0}]))$ have to satisfy the following relation:

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6.$$

We can kill in a unique way the terms involving xy and y by a variable change $y \mapsto y + ax + b$. Indeed, by the variable change $y \mapsto y - \frac{a_1}{2}x - \frac{a_3}{2}$, we get the simplified equation:

$$y^2 = x^3 + b_2 x^2 + b_4 x + b_6.$$

Again a variable change: $x \mapsto x - \frac{b_3}{3}$ simplifies the equation to

$$y^2 = x^3 + c_2 x + c_3.$$

Since $\mathcal{L}(3[\mathbf{0}])$ is very ample $(\deg(\mathcal{L}(3[\mathbf{0}]) = 3 \geq 2g + 1)$, finally making a variable change $2y \mapsto y$ (so now the T–expansion of y begins with $-2T^{-3}$), we get a unique equation out of $(E, \omega)_{/A}$:

$$y^2 = 4x^3 - g_2(E, \omega)x - g_3(E, \omega) \ \text{ for } g_2(E, \omega), g_3(E, \omega) \in A.$$

In other words, $E \subset \mathbf{P}^2_{/A}$ is given by

$$Proj(A[X, Y, Z]/(ZY^2 - 4X^3 + g_2(E, \omega)XZ^2 + g_3(E, \omega)Z^3)).$$

It is easy to see that this equation gives a smooth curve of genus 1 having $\mathbf{0} = \infty = (0, 1, 0)$ in $\mathbf{P}^2$ if $\Delta = \Delta(E, \omega) = g_2^3 - 27(g_3)^2 \in A^\times$. We recover the differential ω by $\frac{dx}{y}$. This shows that, writing $R = \mathbb{Z}[\frac{1}{6}, g_2, g_3, \frac{1}{\Delta}]$ for variables g_2 and g_3,

$$\mathcal{P}_{\Gamma_1(1)}(A) \cong \mathrm{Hom}_{\mathbb{Z}[\frac{1}{6}]-alg}(R, A) = \mathcal{M}_1(A),$$

where $\mathcal{M}_1 = Spec(R)$ for $R = \mathbb{Z}[\frac{1}{6}, g_2, g_3, \frac{1}{\Delta}]$. We have the universal elliptic curve and the universal differential $\boldsymbol{\omega}$ given by

$$(\mathbf{E}, \boldsymbol{\omega})_{/\mathcal{M}_1} = \left(Proj(R[X, Y, Z]/(ZY^2 - 4X^3 + g_2 XZ^2 + g_3 Z^3)), \frac{dx}{y} \right).$$

For each couple $(E, \omega)_{/A}$, we have a unique $\varphi \in \mathcal{M}_1(A) = \mathrm{Hom}_S(Spec(A), \mathcal{M}_1)$ $(S = Spec(\mathbb{Z}[\frac{1}{6}]))$ such that

$$(E, \omega)_{/A} \cong \varphi^*(\mathbf{E}, \boldsymbol{\omega}) = (\mathbf{E}, \boldsymbol{\omega}) \times_{\mathcal{M}_1} Spec(A).$$

If we change ω by $\lambda\omega$ for $\lambda \in A^\times = \mathbb{G}_m(A)$, the parameter T will be changed to λT and hence (x, y) is changed to $(\lambda^{-2}x, \lambda^{-3}y)$. Thus $(E, \lambda\omega)_{/A}$ will be defined by

$$(\lambda^{-3}y)^2 = 4(\lambda^{-2}x)^3 - g_2(E, \lambda\omega)(\lambda^{-2}x) - g_2(E, \lambda\omega).$$

This has to be equivalent to the original equation by the uniqueness of the Weierstrass equation, and we have

$$g_j(E, \lambda\omega) = \lambda^{-2j} g_j(E, \omega).$$

Again by the uniqueness of the Weierstrass equation, we find that

$$\mathrm{Aut}((E, \omega)_{/A}) = \{1_E\}$$

as long as 6 is invertible in A.

2.2.2. *Moduli of $\mathcal{P}_{\Gamma_1(N)}$.* Consider (E, P, ω) for a point $P \in E[\ell](A)$ of order ℓ for a prime ℓ. We have a unique $\varphi \in \mathcal{M}_1(A)$ such that

$$\varphi_E : (E, \omega)_{/A} \cong \varphi^*(\mathbf{E}, \boldsymbol{\omega}) = (\mathbf{E}, \boldsymbol{\omega}) \times_{\mathcal{M}_1} Spec(A).$$

We thus have a commutative diagram:

$$\begin{array}{ccccc} & & \mathbf{E} & \rightarrow & \mathcal{M}_1 \\ & & {\scriptstyle\varphi_E}\uparrow & & \uparrow{\scriptstyle\varphi} \\ Spec(A) & \xrightarrow{P} & E & \rightarrow & Spec(A). \end{array}$$

Then P induces a unique morphism $\varphi_P = \varphi_E \circ P : Spec(A) \to (\mathbf{E}[\ell] - \{0\})(A)$. This shows that, over $\mathbb{Z}[\frac{1}{6\ell}]$,

$$\mathcal{P}_{\Gamma_1(\ell)}(A) \cong \mathcal{P}'_{\Gamma_1(\ell)}(A) \cong (\mathbf{E}[\ell] - \{0\})(A).$$

Similarly, over $\mathbb{Z}[\frac{1}{6N}]$

$$\mathcal{P}_{\Gamma_1(N)}(A) \cong \mathcal{P}'_{\Gamma_1(p)}(A) \cong \left(\mathbf{E}[N] - \bigcup_{N>d|N} \mathbf{E}[d]\right)(A).$$

We put $\mathcal{M}_{\Gamma_1(N)} = \mathbf{E}[N] - \bigcup_{N>d|N} \mathbf{E}[d]$. Thus we have proven

Theorem 2.1. *There is an affine scheme $\mathcal{M}_{\Gamma_1(N)} = Spec(R_{\Gamma_1(N)})$ defined over $\mathbb{Z}[\frac{1}{6N}]$ such that*

$$\mathcal{P}_{\Gamma_1(N)}(A) \cong \mathcal{P}'_{\Gamma_1(N)}(A) \cong \mathrm{Hom}_{\mathbb{Z}[\frac{1}{6N}]-alg}(R_{\Gamma_1(N)}, A) = \mathcal{M}_{\Gamma_1(N)}(A)$$

for all $\mathbb{Z}[\frac{1}{6N}]$–algebras A. The scheme $\mathcal{M}_{\Gamma_1(N)}/\mathcal{M}_1$ is an étale covering of degree $\varphi(N)$ for the Euler function φ.

The fact that the covering is étale finite follows from the same fact for $\mathbf{E}[N]$ over $\mathbb{Z}[\frac{1}{6N}]$ since $\mathbf{E}[N](k) \cong (\mathbb{Z}/N\mathbb{Z})^2$ for all algebraically closed fields k with characteristic not dividing N. Since $\mathcal{M}_1$ is affine, any finite covering of $\mathcal{M}_1$ is affine.

2.2.3. *Action of $\mathbb{G}_m$.* The group scheme $\mathbb{G}_m$ acts on the functor $\mathcal{P}_{\Gamma_1(N)}$ in the following way: $(E, \phi_N, \omega)_{/A} \mapsto (E, \phi_N, \lambda\omega)_{/A}$ for $\lambda \in \mathbb{G}_m(A)$. This induces an action of $\mathbb{G}_m$ on $\mathcal{M}_{\Gamma_1(N)}$ and hence on $R_{\Gamma_1(N)}$.

Here is a general fact on the action of $\mathbb{G}_m$. Let X be an A–module. Regard X as a functor from A–ALG to the category of A–modules A–MOD by $\underline{X}(B) = X \otimes_A B$. If a group scheme $G_{/A}$ has a functorial action: $\underline{G} \times \underline{X} \to \underline{X}$, we call X a schematic representation of G. It is known (e.g. [GME] 1.6.5) that if X has a schematic action of $\mathbb{G}_{m/A}$, then

$$X = \bigoplus_{\kappa \in \mathbb{Z}} X[\kappa]$$

such that $X[\kappa](B) = \{x \in X | \lambda \cdot x = \lambda^\kappa x\}$, that is, $X[\kappa]$ is the eigenspace for the character $\mathbb{G}_m(B) \to B^\times$ taking $z \in \mathbb{G}_m(B) = B^\times$ to z^κ.

The action of $\mathbb{G}_{m/A}$ on $\mathcal{M}_{\Gamma_1(N)}$ gives rise to a schematic action on $R_{\Gamma_1(N)}$ (because it was defined by functorial action). Thus we can split

$$R_{\Gamma_1(N)/A} = \bigoplus_{\kappa \in \mathbb{Z}} R_\kappa(\Gamma_1(N); A),$$

where on $f \in R_\kappa(\Gamma_1(N); A)$, $\mathbb{G}_m$ acts by the character $-\kappa$.

Since $f \in R_\kappa(\Gamma_1(N);A)$ is a functorial morphism:

$$\underline{\mathcal{M}}_{\Gamma_1(N)}(B) = \mathcal{P}_1(B) \to \mathbf{A}^1(B) = B,$$

we may regard f as a function of $(E,\phi_N,\omega)_{/B}$ with $f((E,\phi_N,\omega)_{/B}) \in B$ satisfying

(G0) $f((E,\phi_N,\lambda\omega)_{/B}) = \lambda^{-\kappa} f((E,\phi_N,\omega)_{/B})$ for $\lambda \in B^\times = \mathbb{G}_m(B)$;

(G1) If $(E,\phi_N,\omega)_{/B} \cong (E',\phi'_N,\omega')_{/B}$, then $f((E,\phi_N,\omega)_{/B}) = f((E',\phi'_N,\omega')_{/B})$;

(G2) If $\rho : B \to B'$ is a morphism of A–algebras, then $f((E,\phi_N,\omega)_{/B} \times_B B') = \rho(f((E,\phi_N,\omega)_{/B}))$.

If a graded ring $\mathcal{A} = \bigoplus_j \mathcal{A}_j$ has a unit u of degree 1, $\mathcal{A} = \mathcal{A}_0 \otimes_{\mathbb{Z}} \mathbb{Z}[u,u^{-1}]$ and $Spec(\mathcal{A}) = Spec(\mathcal{A}_0) \times \mathbb{G}_m$ by definition; so, $Proj(\mathcal{A}) = Spec(\mathcal{A})/\mathbb{G}_m = Spec(\mathcal{A}_0)$. If $\mathcal{A}$ has a unit of degree $n > 0$, then $Proj(\mathcal{A}) = Proj(\mathcal{A}^{(n)}) = Spec(\mathcal{A}_0)$ for $\mathcal{A}^{(n)} = \bigoplus_j \mathcal{A}_{nj}$. Since $\Delta^{-1} \in R \subset R_{\Gamma_1(N)}$, the graded ring $R_{\Gamma_1(N)}$ has a unit of degree 12, and hence, we have

$$\mathbb{G}_m \backslash \mathcal{M}_{\Gamma_1(N)} = Proj(R_{\Gamma_1(N)/A}) \cong Spec(R_0(\Gamma_1(N);A)) =: Y_1(N)_{/A}.$$

We consider the functor defined over $\mathbb{Z}[\frac{1}{N}]$–ALG given by

$$\mathcal{E}_{\Gamma_1(N)}(A) = \left[(E,\phi_N : \mu_N \hookrightarrow E[N])_{/A}\right].$$

By definition, $\mathcal{E}_{\Gamma_1(N)} = \mathbb{G}_m \backslash \mathcal{P}_{\Gamma_1(N)}$. Since $Proj(R_{\Gamma_1(N)})$ gives the quotient by $\mathbb{G}_m$ of $Spec(R_{\Gamma_1(N)})$ (see [GME] Theorem 1.8.2), we conclude

Theorem 2.2 (Shimura, Igusa)**.** *We have an affine curve*

$$Y_1(N) = Proj(R_{\Gamma_1(N)}) = \mathbb{G}_m \backslash \mathcal{M}_{\Gamma_1(N)}$$

defined over $\mathbb{Z}[\frac{1}{6N}]$, which is locally free of finite rank over $M_1 = Proj(R) = \mathbf{P}^1(J) - \{\infty\}$. For all geometric point $Spec(k)$ of $Spec(\mathbb{Z}[\frac{1}{6N}])$, we have $Y_1(N)(k) = \left[(E,\phi_N)_{/k}\right]$. The above assertion holds for any $\mathbb{Z}[\frac{1}{6N}]$–algebra A in place of k if $N \geq 4$.

Here a "geometric point" means that k is an algebraically closed field. It is well known that $\Gamma_1(N)\backslash\mathfrak{H}$ classifies all elliptic curves with a point of order N over $\mathbb{C}$ for $\mathfrak{H} = \{z \in \mathbb{C} |\, \mathrm{Im}(z) > 0\}$; so, we conclude

$$Y_1(N)(\mathbb{C}) = \Gamma_1(N)\backslash\mathfrak{H}.$$

Thus $Y_1(N)(\mathbb{C})$ is an open Riemann surface.

2.2.4. *Compactification.* For any $\mathbb{Z}[\frac{1}{6}]$–algebra A, we put

$$G(A) = A[g_2,g_3] = \mathbb{Z}\left[\frac{1}{6}, g_2, g_3\right] \otimes A.$$

Let $G_{\Gamma_1(N)}(\mathbb{Z}[\frac{1}{6N}])$ be the integral closure of $G(\mathbb{Z})$ in the graded ring $R_{\Gamma_1(N)/\mathbb{Z}[\frac{1}{6N}]}$.

To see that $G_{\Gamma_1(N)}(\mathbb{Z}[\frac{1}{6N}])$ is a graded ring, we write $\widetilde{r}$ for the non-trivial homogeneous projection of highest degree of $r \in R_{\Gamma_1(N)/\mathbb{Z}[\frac{1}{6N}]}$. If $r \in R_{\Gamma_1(N)/\mathbb{Z}[\frac{1}{6N}]}$ is integral over $G(\mathbb{Z})$, r satisfies an equation $P(X) = X^n + a_1 X^{n-1} + \cdots + a_n = 0$ with $a_j \in G(\mathbb{Z})$. Then $\widetilde{r}$ satisfies $\widetilde{P}(X) = X^n + \widetilde{a}_1 X^{n-1} + \cdots + \widetilde{a}_n = 0$, and $\widetilde{r}$ is integral over $G(\mathbb{Z})$. Then by induction of the degree of $\widetilde{r}$, we see that $G_{\Gamma_1(N)}(\mathbb{Z}[\frac{1}{6N}])$ is graded (cf. [BCM] V.1.8).

We put for any $\mathbb{Z}[\frac{1}{6N}]$–algebra A

$$G_{\Gamma_1(N)}(A) = G_{\Gamma_1(N)}\left(\mathbb{Z}\left[\frac{1}{6N}\right]\right) \otimes A = \bigoplus_{k=0}^{\infty} G_k(\Gamma_1(N);A).$$

We then define $X_1(N)_{/A} = Proj(G_{\Gamma_1(N)}(A))$. By definition, $X_1(N)$ is the normalization of $Proj(G) = Proj(G^{(12)}) = \mathbf{P}^1(J)$ $(J = \frac{(12g_2)^3}{\Delta})$ for $G^{(12)} = \bigoplus_{k=0}^{\infty} G_{12k}$ in $Y_1(N)$. As classically known, J^{-1} has q–expansion starting with q, that is, $J^{-1} \in q\mathbb{Z}[[q]]$ (see [IAT] (4.6.1)). Thus the completion of the local ring of $\mathbf{P}^1(J)$ at the cusp ∞ is isomorphic to $\mathbb{Z}[\frac{1}{6}][[q]]$. Moreover we have the Tate curve (e.g. [GME] 2.5):

$$\mathrm{Tate}(q) = Proj(\mathbb{Z}[[q]][\frac{1}{6}][X,Y,Z]/(ZY^2 - 4X^3 + g_2(q)XZ^2 + g_3(q)Z^3)),$$

which extends the universal curve over $\mathbf{P}^1(J) - \{\infty\}$ to $\mathbf{P}^1(J)$ locally at the cusp ∞.

Since $\mathrm{Tate}(q)(A[[q]]) \supset (A[[q]]^\times)/q^{\mathbb{Z}}$ (see [GME] Theorem 2.5.1 (2)), we may think $\mathrm{Tate}(q)$ to be a "quotient" $\mathbb{G}_{m/\mathbb{Z}[[q]]}/q^{\mathbb{Z}}$ of $\mathbb{G}_m$; so, it has a canonical level structure $\phi_N^{can} : \mu_N \hookrightarrow \mathbb{G}_m \twoheadrightarrow \mathrm{Tate}(q)$. The Tate couple $(\mathrm{Tate}(q), \phi_N^{can})_{/\mathbb{Z}[[q]]}$ is a test object over $\mathbb{Z}[[q]][q^{-1}]$; so, by the universality of $Y_1(N)$, we have a morphism

$$\iota_\infty : Spec(\mathbb{Z}[\frac{1}{N}][[q]][q^{-1}]) \to Y_1(N).$$

Since we may regard the Tate curve as a universal formal deformation of a stable curve of genus 1 (with the level structure ϕ_N^{can}) centered at the $\mathbb{Z}[\frac{1}{N}]$–point represented by an ideal (q) of $\mathbb{Z}[\frac{1}{N}][[q]]$ ([GME] 2.5.2-3), the morphism ι_∞ is an infinitesimal isomorphism centered at the cusp ∞ (by the universality of the $Y_1(N)$ and the universality of the Tate curve). Since $X_1(N)$ is the normalization of $\mathbf{P}^1(J)$ in $Y_1(N)$, we conclude that the formal completion along the cusp ∞ on $X_1(N)$ is canonically identified with $A[[q]]$ by ι_∞. Replacing the level structure ϕ_N^{can} by $\phi_N^{can} \circ \alpha$ for $\alpha \in SL_2(\mathbb{Z}/N\mathbb{Z})$, basically by the same argument, the local ring at the cusp $\alpha(\infty)$ of $X_1(N)_{/A}$ is given by $A[\mu_d][[q^{1/d}]]$ for a suitable divisor $d|N$. We need to extend scalar to $A[\mu_d][[q^{1/d}]]$ because the Tate curve $(\mathrm{Tate}(q), \phi_N^{can} \circ \alpha)$ is only defined over $A[\mu_d][[q^{1/d}]]$ for a suitable divisor $d|N$ dependent on the choice of α. This point is a bit technical, and we refer the reader to a more detailed account, which can be found in [AME] Chapter 10 and [GME] 3.1.1. Thus $X_1(N)$ is smooth at the cusps, and moreover $f \in G_k(\Gamma_1(N); A)$ is a function of (E, ϕ_N, ω) satisfying (G0-2) and

(G3) $f(\mathrm{Tate}(q), \phi_N, \omega) \in A[\zeta_N][[q^{1/N}]]$ for any choice of ϕ_N and ω.

Since $\Gamma_1(N)\backslash(\mathfrak{H} \cup \mathbf{P}^1(\mathbb{Q}))$ is a smooth compact Riemann surface and is the normalization of $\mathbf{P}^1(J)$ in $Y_1(N)(\mathbb{C})$, we conclude

$$X_1(\mathbb{C}) = \Gamma_1(N)\backslash(\mathfrak{H} \cup \mathbf{P}^1(\mathbb{Q})).$$

The space $G_k(\Gamma_1(N); \mathbb{C})$ is the classical space of modular forms on $\Gamma_1(N)$ of weight k. Since $\mathrm{Tate}(q)$ is the "quotient" $\mathbb{G}_{m/\mathbb{Z}[[q]]}/q^{\mathbb{Z}}$, it has a canonical differential ω_{can} induced by $\frac{dt}{t}$ identifying $\mathbb{G}_m = Spec(\mathbb{Z}[t, t^{-1}])$. In particular,

$$f(q) = f(\mathrm{Tate}(q), \phi_N^{can}, \omega_{can}) = \sum_{n=0}^{\infty} a(n; f)q^n \quad \text{with } q = \exp(2\pi i z)$$

coincides with the Fourier expansion of f at the infinity if $f \in G_k(\Gamma_1(N); \mathbb{C})$.

2.2.5. *Hasse Invariant.* Let A be a ring of characteristic p and (E,ω) be an elliptic curve over $S = Spec(A)$. On each affine open subset $U = Spec(\Gamma(U,\mathcal{O}_E))$ in E, the Frobenius endomorphism $x \mapsto x^p$ induces a morphism $F_{abs} : U \to U$. These glue each other to the absolute Frobenius endomorphism $F_{abs} : E_{/A} \to E_{/A}$. Note here that F_{abs} acts non-trivially on the coefficient ring A. We can define the relative Frobenius map: $E \to E^{(p)} = E \times_{S,F_{abs}} S$ by $F_{abs} \times_S f$ for the structure morphism $f : E \to S$. This relative Frobenius is the classical map taking homogeneous coordinates of E to their p–powers.

Let $\mathcal{T}_{E/S}$ be the relative tangent bundle; so, its global section $H^0(E,\mathcal{T}_{E/S})$ is the A–dual of $H^0(E,\Omega_{E/S})$, and $H^0(E,\mathcal{T}_{E/S})$ is spanned by a dual base $\eta = \eta(\omega)$. One can identify $H^0(E,\mathcal{T}_{E/S})$ with the module of $\mathcal{O}_S$–derivations $Der_{\mathcal{O}_S}(\mathcal{O}_{E,\mathbf{0}},\mathcal{O}_S)$ (cf. [GME] 1.5.1). For each derivation D of $\mathcal{O}_{E,\mathbf{0}}$, by the Leibnitz formula, we have

$$D^p(xy) = \sum_{j=0}^{p} \binom{p}{j} D^{p-j}xD^jy = xD^py + yD^px.$$

Thus D^p is again a derivation. The association: $D \mapsto D^p$ induces an F_{abs}–linear endomorphism F^* of $\mathcal{T}_{E/S}$. Then we define $H(E,\omega) \in A$ by $F^*\eta = H(E,\omega)\eta$. Since $\eta(\lambda\omega) = \lambda^{-1}\eta(\omega)$, we see

$$\begin{aligned}&H(E,\lambda\omega)\eta(\lambda\omega) = F^*\eta(\lambda\omega) = F^*(\lambda^{-1}\eta(\omega))\\ &= \lambda^{-p}F^*\eta(\omega) = \lambda^{-p}H(E,\omega)\eta(\omega) = \lambda^{-p}H(E,\omega)\lambda\eta(\lambda\omega) = \lambda^{1-p}H(E,\omega)\eta(\lambda\omega).\end{aligned}$$

Thus we get

$$H(E,\lambda\omega) = \lambda^{1-p}H(E,\omega).$$

Then H is a modular form of weight $p-1$ defined over $\mathbb{F}_p$:

$$H(E,\omega) \in G_{p-1}(\Gamma_1(1),\mathbb{F}_p).$$

We compute $H(E_\infty, \frac{dw}{w})$. The dual of $\frac{dw}{w}$ is given by $D = w\frac{d}{dw}$. The action of F keeps D intact, because $D(w) = w$ (so $D^p(w) = w$). On the tangent space, F acts as identity, and hence $H(E_\infty,\omega) = 1$.

An important fact is:

$$H(E,\omega) = 0 \iff E \text{ is super singular.}$$

This is because:

1. If $E_{/\overline{\mathbb{F}}_p}$ is ordinary, then $E[p] \cong \mu_p \times (\mathbb{Z}/p\mathbb{Z})$ over $\overline{\mathbb{F}}_p$;
2. $\mu_p = Spec(\mathbb{F}_p[t]/(t^p-1))$ shares the tangent space with $\mathbb{G}_m$, because they are both of dimension 1 infinitesimally;
3. $F^2 = p$ up to units in the super singular case.

The zero locus of a section of a line bundle is a divisor; hence, on the moduli space, $X_1(N)$ for $p \nmid N$, the points in $X_0(N)(\overline{\mathbb{F}}_p)$ corresponding to super-singular elliptic curves are finitely many.

2.2.6. *Igusa Curves.* Let $W = \mathbb{Z}_p$ and $W_m = W/p^mW$. Fix N with $p \nmid N$. We have a lift of Hasse invariant in $G_{p-1}(\Gamma_1(1);\mathbb{Z}_p)$, which is the Eisenstein series E normalized so that $a(0,E) = 1$. By Von Staut theorem, the q–expansion $E(q)$ of E is congruent to 1 modulo p; so, $E \mod p$ coincides with H. Let $(\mathbf{E},\phi_N)_{/M}$ be the genus 1 semi-stable curve (completed by appropriate Tate curves at the cusps) over $M = X_1(N)_{/W}$. Let $M_m = X_1(N)_{/W_m} = X_1(N) \times_W W_m$. Define $S_m \subset M_m$ by the open subscheme of M_m on which E is invertible. The scheme S_m does not depends on the choice of the lift E, since $E \equiv E' \equiv H \mod p$ guarantees $M_m[\frac{1}{E}] = M_m[\frac{1}{E'}]$ for any other lift E' as long as p is nilpotent in the base ring. We write S_∞ for the formal completion $\varprojlim_m S_m$ of S along S_1.

Since we have defined $X_1(N)$ by $Proj(G_{\Gamma_1(N)})$, the invertible sheaf $\underline{\omega}^k$ $(k > 0)$ associated to the k–th graded piece is ample. To see for which k, $\underline{\omega}^k$ becomes very ample, we recall that an invertible sheaf of degree $\geq 2g+1$ over a curve of genus g is very ample by Riemann-Roch theorem (see [GME] Proposition 2.1.4). Computing the genus of $X_1(N)$ (e.g. [GME] Theorem 3.1.2), the invertible sheaf $\underline{\omega}^k_{/A}$ corresponding to $G_k(\Gamma_1(N); A)$ is very ample if $k \geq 2$ and $N \geq 4$ (or $k > 2$). Thus S_m is affine, and $S_m = Spec(V_{m,0})$ for a W_m–flat algebra $V_{m,0}$. We consider the functors

$$\mathcal{E}'^{ord}_\alpha(A) = \left[(E, P, \phi_N)_{/A}\right] \quad\text{and}\quad \mathcal{E}^{ord}_\alpha(A) = \left[(E, \mu_{p^\alpha} \hookrightarrow E[p^\alpha], \phi_N)_{/A}\right],$$

where P is a point of order p^α. Then we see that

$$\mathcal{E}^{ord}(A) \cong \mathcal{E}'^{ord}(A) = \left(\mathbf{E}[p^\alpha]^{et} - \mathbf{E}[p^{\alpha-1}]^{et}\right)_{/S_m}(A)$$

for all W_m–algebras A. We write $T_{m,\alpha/S_m} = \left(\mathbf{E}[p^\alpha]^{et} - \mathbf{E}[p^{\alpha-1}]^{et}\right)_{/S_m}$, which is an étale covering of degree $p^{\alpha-1}(p-1)$. It is a classical result of Igusa that $T_{m,\alpha}$ is irreducible (and hence connected; see [GME] 2.9.3), although we do not need this irreducibility here. We will come back to the proof of the irreducibility of the Igusa tower over more general Shimura varieties later in Lecture 10. Since S_m is affine, $T_{m,\alpha}$ is also affine. We write $T_{m,\alpha} = Spec(V_{m,\alpha})$. We have a tower of W_m–flat algebras:

$$V_{m,0} \subset V_{m,1} \subset \cdots V_{m,\alpha} \subset \cdots.$$

These algebras are étale over $V_{m,0}$ and $\mathrm{Gal}(V_{m,\alpha}/V_{m,0}) \cong (\mathbb{Z}/p^\alpha\mathbb{Z})^\times$. Over $V_{m,\alpha}$, we have a canonical isomorphism

$$I_{can} = \boldsymbol{\phi}_{p^\alpha} : \mathbb{Z}/p^\alpha\mathbb{Z} \cong P_\alpha = \mathbf{E}[p^\alpha]^{et}.$$

We then define $V_{m,\infty} = \bigcup_\alpha V_{m,\alpha}$ and

$$\mathcal{V} = \mathcal{V}_{\Gamma_1(N)} = \varinjlim_m V_{m,\infty} \quad\text{and}\quad V = V_{\Gamma_1(N)} = \varprojlim_m V_{m,\infty}.$$

The space $V_{\Gamma_1(N)}$ is the space of p–adic modular forms on $\Gamma_1(N)$. By taking the Cartier dual of $\mathbb{Z}/p^\alpha\mathbb{Z} \hookrightarrow E[p^\alpha]$, we may regard $f \in V_{m,\alpha}$ as a rule associating an element of A to $(E, \phi_p : \mu_{p^\alpha} \hookrightarrow E[p^\alpha], \phi_N)_{/A}$ satisfying the conditions similar to (G0-3). Each element $f \in V_{\Gamma_1(N)}\widehat{\otimes}_W A$ for a W–algebra $A = \varprojlim_m A/p^m A$ is a function of (E, ϕ_p, ϕ_N) satisfying the conditions similar to (G0-3) (see [GME] (G_p1-3) in page 230).

3. Vertical Control for Elliptic Modular Forms

3.1. **Vertical Control Theorem.** We have a p–divisible module $\mathcal{V}_{\Gamma_1(N)}$ on which $\mathrm{Gal}(V_{m,\infty}/V_{m,0}) = \mathbb{Z}_p^\times = T(\mathbb{Z}_p)$ acts continuously. Here $T = \mathbb{G}_m$. We shall construct a projector e acting on $\mathcal{V}$ out of the Hecke operator $U(p)$ commuting with the action of $\mathbb{Z}_p^\times = \mathrm{Gal}(V_{m,\infty}/V_{m,0})$. The important features of e are

- $e = \lim_{n\to\infty} U(p)^{n!}$;
- $\mathcal{V}^{ord} = e\mathcal{V}$ has Pontryagin dual which is projective over $W[[\mathbb{G}_m(\mathbb{Z}_p)]]$;

- For any $k \geq 3$, there is a canonical isomorphism

$$\mathcal{V}^{ord}[-k] \cong eH^0(S, \underline{\omega}^k \otimes \mathbb{T}_p) = eH^0(M, \underline{\omega}^k \otimes \mathbb{T}_p) \ (\mathbb{T}_p = \mathbb{Q}_p/\mathbb{Z}_p),$$

where $\mathcal{V}[-k] = \{f \in \mathcal{V} | zf = z^{-k}f \ \forall z \in \mathbb{Z}_p^\times\}$. We hereafter write H^q_{ord} for eH^q and G^{ord}_k for eG_k.

3.1.1. *Axiomatic treatment.* Let $\underline{\omega}^k = \widetilde{G_{\Gamma_1(N)}}(k) = \mathcal{O}(k)$ for the embedding of $X_1(N) = Proj(G_{\Gamma_1(N)})$ into the projective space. Then $\underline{\omega}^k = \underline{\omega}^{\otimes k}$. Computing the genus of $X_1(N)$, the Riemann-Roch theorem tell us that $\underline{\omega}^k$ is very ample if $k \geq 3$ (see [GME] Proposition 2.1.4 and Theorem 3.1.2). Therefore $\underline{\omega}^k$ is the pull back of $\mathcal{O}(k)$ of the target projective space. Let $(\mathbf{E}, \boldsymbol{\phi}_N, \boldsymbol{\omega})$ be the universal elliptic curve over $Y_1(N)$. For each triple (E, ϕ_N, ω) defined over A (called a test object), we have a unique $\iota : Spec(A) \to Y_1(N)$ such that $\iota^*(\mathbf{E}, \boldsymbol{\phi}_N, \boldsymbol{\omega}) = (E, \phi_N, \omega)$. For each section $f \in H^0(Y_1(N), \underline{\omega}^k)$, we define

$$\iota^* f = f(E, \phi_N, \omega)\omega^{\otimes k}.$$

The function $(E, \phi_N, \omega) \mapsto f(E, \phi_N, \omega)$ satisfies (G0-2). The condition (G3) assures that f extends to $X_1(N)$. This shows

$$H^0(X_1(N)_{/A}, \underline{\omega}^k) = G_k(\Gamma_1(N); A)$$

for all $\mathbb{Z}[\frac{1}{6N}]$–algebra A.

Let $(\mathbf{E}, \boldsymbol{\phi}_p, \boldsymbol{\phi}_N)$ be the universal elliptic curve over S_m. Pick a section $f \in H^0(S_m, \underline{\omega}^k)$. Since μ_{p^∞} carries a canonical differential $\omega_{can} = \frac{dt}{t}$, writing $\mu_{p^\alpha} = Spec(\mathbb{Z}[t]/(t^{p^\alpha} - 1))$, we may regard f as a function of (E, ϕ_p, ϕ_N) by $f(E, \phi_p, \phi_N) = f(E, \phi_N, \phi_{p,*}\omega_{can})$. For each $(E, \phi_p, \phi_N) \in \mathcal{E}^{ord}_\infty(A)$ for a W_m–algebra A, we have a unique morphism $\iota : Spec(A) \to T_{m,\infty}$ such that $(E, \phi_p, \phi_N) = \iota^*(\mathbf{E}, \boldsymbol{\phi}_p, \boldsymbol{\phi}_N)$. Then $\iota^* f$ is just a function of $f(E, \phi_p, \phi_N)$ such that $f(E, z^{-1}\phi_p, \phi_N) = z^k f(E, \phi_p, \phi_N)$ for $z \in \mathrm{Gal}(V_{m,\infty}/V_{m,0}) = \mathbb{Z}_p^\times$. This shows that

$$V_{m,\infty}[k] = H^0(S_m, \underline{\omega}^k) \ \text{ and } \ \mathcal{V}[k] = H^0(S_{/W}, \underline{\omega}^k \otimes \mathbb{T}_p) = H^0(S_{/W}, \underline{\omega}^k) \otimes \mathbb{T}_p,$$

where $\mathbb{T}_p = \mathbb{Q}_p/\mathbb{Z}_p$. The last identity follows, since S is affine. This shows that $\mathcal{V}[k]$ is p–divisible, and its direct summand $e\mathcal{V}[k]$ is also p–divisible.

We consider the following condition:

(F) $\quad \mathrm{corank}_W \, e\mathcal{V}[k] = \mathrm{rank}_W \, \mathrm{Hom}(e\mathcal{V}[k], \mathbb{T}_p)$ *is finite for an integer* k.

In practice, this condition is often proven by showing

$$H^0_{ord}(S_{/W}, \underline{\omega}^k \otimes \mathbb{T}_p) = H^0_{ord}(X_1(N)_{/W}, \underline{\omega}^k \otimes \mathbb{T}_p) = G^{ord}_k(\Gamma_1(N), \mathbb{T}_p). \tag{3.1}$$

The left-hand-side (LHS) of (3.1) is p–divisible, since S is affine. The (RHS) is of finite corank since $X_1(N)$ is projective. Thus $e\mathcal{V}[k]$ is p–divisible of finite corank.

Decompose $\mathbb{Z}_p^\times = \Gamma_T \times \Delta$ for a p–profinite group Γ_T and a prime-to-p finite group Δ. For simplicity suppose that $p > 2$. Then Γ_T is isomorphic to $\mathbb{Z}_p$ and for its generator γ, we have $W[[\Gamma_T]] \cong W[[X]] = \Lambda$ via $\gamma \mapsto 1 + X$ (that is, $\gamma^s \mapsto (1+X)^s = \sum_{j=0}^\infty \binom{j}{s} X^j$), and $W[[\mathbb{Z}_p^\times]] = \Lambda[\Delta]$. Let V^{ord} be the Pontryagin dual module of $e\mathcal{V}$. If $e\mathcal{V}[k]$ is of finite corank for one k, then by duality, we have

$$\begin{aligned} V^{ord}[\chi]/(X + 1 - \gamma^k)V^{ord}[\chi] &= V^{ord} \otimes_{W[[T(\mathbb{Z}_p)]],k} W \\ &\cong \mathrm{Hom}_W(H^0_{ord}(X_1(N), \underline{\omega}^k), W) \cong \mathrm{Hom}_W(G^{ord}_k(\Gamma_1(N); W), W) \ \ (\chi = k|_\Delta). \end{aligned} \tag{3.2}$$

In the middle equality, we have assumed (3.1). Here the subscript or superscript "ord" implies the image of e. Decompose V^{ord} by the character of Δ as follows:

$$V^{ord} = \bigoplus_{\chi \in \widehat{\Delta}} V^{ord}[\chi].$$

If $z \mapsto z^k$ coincides with χ on Δ_T, then $V^{ord}[\chi] \otimes_{\Lambda,k} W = V^{ord} \otimes_{W[[T(\mathbb{Z}_p)]],k} W$. By Nakayama's lemma, we have a surjective homomorphism of Λ–modules:

$$\pi : \Lambda^{s(\chi)} \twoheadrightarrow V^{ord}[\chi],$$

where $s = s(\chi) = \mathrm{corank}_W\, e\mathcal{V}[k]$. If (F) holds for one k, it holds for all κ inducing χ, and π has to be an isomorphism by the following reason: The number s is the minimum number of generators of $V^{ord}[\chi] \otimes_{\Lambda,\kappa} W$ over κ. We know that this module is W–free, because its dual $\mathcal{V}[\kappa]$ is p–divisible; so, it is free of rank s. The morphism π induces an isomorphism modulo $(1+X) - \gamma^\kappa$ for all κ inducing χ. Then

$$\mathrm{Ker}(\pi) \subset \bigcap_\kappa \mathrm{Ker}(\pi \mod (1 + X - \gamma^\kappa)) = 0,$$

and we get

Theorem 3.1. *Suppose that* (F) *holds for one k. Write H^0_{ord} for eH^0 and G_k^{ord} for eG_k. Then $V^{ord}[\chi]$ is Λ–free of finite rank $s(\chi)$, and if* (3.1) *holds for k, then*

$$V^{ord} \otimes_{W[[\mathbb{Z}_p^\times]],k} W \cong \mathrm{Hom}_W(G_k^{ord}(\Gamma_1(N); W), W).$$

3.1.2. *Bounding the p–ordinary rank.* Since S_1 is affine, we have

$$H^0(S_1, \underline{\omega}^k) = H^0(S_{/W}, \underline{\omega}^k) \otimes_W W_1.$$

If $\overline{f}_1, \overline{f}_2, \ldots, \overline{f}_j$ is a sequence of linearly independent sections in $H^0_{ord}(S_1, \underline{\omega}^k)$, we can lift them to $f_i \in H^0(S_{/W}, \underline{\omega}^k)$ so that $\overline{f}_i = (f_i \mod p)$. Since $S = M[\frac{1}{E}]$, we have

$$H^0(S_{/W}, \underline{\omega}^k) = \varinjlim_n H^0(M_{/W}, \underline{\omega}^{k+n(p-1)})/E^n.$$

Thus $E^m f_i \in H^0(M_{/W}, \underline{\omega}^{k+m(p-1)})$ for all $i = 1, \ldots, j$ for sufficiently large m, and they are linearly independent. We now assume

(C) $$e(Ef) = E(ef) \text{ for all } f \in H^0(S_1, \underline{\omega}^k).$$

By this, $e(E^m f_i)$ are still linearly independent in $H^0_{ord}(M_{/K}, \underline{\omega}^{k+m(p-1)})$; so, we have

$$\dim_K H^0_{ord}(M_{/K}, \underline{\omega}^{k+m(p-1)}) \geq j.$$

If $\mathrm{rank}_{W_1} H^0_{ord}(S_1, \underline{\omega}^k)$ is infinite, the rank of $H^0_{ord}(M_{/K}, \underline{\omega}^{k+m(p-1)})$ grows as $m \to \infty$. The condition (F) for all k follows from

(F′) $\dim_K G_k^{ord}(\Gamma_1(N), K)$ is bounded independent of k ($K = \mathbb{Q}_p$).

Actually, the Eichler-Shimura isomorphism combined with a calculation of group cohomology $H^1_{ord}(\Gamma_1(N), L(k; K))$ proves much stronger

(E) If $k \geq 3$, $\dim_K G_k^{ord}(\Gamma_1(N), K)$ depends only on $k \mod p-1$ ([LFE] 7.2).

The projector e will be constructed in the following subsection.

3.1.3. *Construction of the projector.* Let $(E, \phi_p, \phi_N)_{/A}$ be a test object. Suppose that A is flat over $\mathbb{Z}_p$. Each subgroup C of order p outside the image of ϕ_p is étale over $A[\frac{1}{p}]$; so, we can think of the quotient $(E/C, \phi_p, \phi_N)$ defined over an étale finite extension B of $A[\frac{1}{p}]$. We define

$$f|U(p)(E, \phi_p, \phi_N) = \frac{1}{p}\sum_C f(E/C, \phi_p, \phi_N). \tag{U}$$

Computing q–expansion, we know

$$a(n, f|U(p)) = a(np, f).$$

So the operator preserves integral structure over A. The above construction of $U(p)$ works well for triples (E, ϕ_p, ϕ_N) over general scheme T as long as T is flat over $\mathbb{Z}_p$. Thus we have $U(p)$ operator well defined over $S[\frac{1}{p}]$.

We shall extend the definition of $U(p)$ to A with p–torsion following Katz [K3] 3.10. For the universal elliptic curve $\mathbf{E}$ over S, we have a non-split exact sequence

$$0 \to \mathbf{E}[p]^\circ \to \mathbf{E}[p] \to \mathbf{E}[p]^{et} \to 0.$$

To have an étale subgroups C in $\mathbf{E}[p]$, we need to split the above sequence via base-change from S to its finite flat covering S'. By the deformation theory of elliptic curves by Serre-Tate (which we will expose in Lecture 8), for each closed point $x \in S_1(\overline{\mathbb{F}}_p)$, we have a canonical identification of the formal completion $\widehat{S}_x$ of S along x with the formal multiplicative group $\widehat{G}_{m/\overline{W}}$ over the Witt ring $\overline{W}$ of $\overline{\mathbb{F}}_p$. Then the above extension is equivalent to

$$0 \to \mu_p \to T_p \to \mathbb{Z}/p\mathbb{Z} \to 0,$$

where the group scheme $T_{p/\mathbb{G}_m}$ is defined as follows (cf. [GME] Example 1.6.5 in page 43):

$$T_p = Spec(\prod_{i=0}^{p-1} \frac{\mathbb{Z}[t, t^{-1}][x]}{(x^p - t^i)}).$$

Thus T_p is a finite flat group scheme over $\mathbb{G}_m = Spec(\mathbb{Z}[t, t^{-1}])$, and $\mathbf{E}[p] \times_S \widehat{S}_x \cong T_p \times_{\mathbb{G}_m} \widehat{S}_x$. For any commutative ring R

$$\begin{aligned} T_p(R) = &\left\{(x, \frac{i}{p})\big| x^p = t^i,\ x \in \mathbb{G}_m(R),\ \frac{i}{p} \in p^{-1}\mathbb{Z}/\mathbb{Z}\right\} \\ &= \mathrm{Ker}(\mathbb{G}_m(R)/x^{\mathbb{Z}} \xrightarrow{t \mapsto t^p} \mathbb{G}_m(R)/x^{\mathbb{Z}}). \end{aligned}$$

This shows that $\widehat{\mathcal{O}}_{S',x}$ has to be isomorphic to the formal completion of the ring $\overline{W}[t^{1/p}, t^{-1/p}] = \frac{\overline{W}[t,t^{-1}][x]}{(x^p - t)}$ along $x = 1$. Thus S' is a finite flat covering of S radiciel (or purely inseparable) at the special fiber over p. In any case, we have the trace map $\mathrm{Tr}_\varphi : \mathcal{O}_{S'} \to \mathcal{O}_S$ and the inclusion $\iota : \mathcal{O}_S \hookrightarrow \mathcal{O}_{S'}$. We also have the Frobenius map $\varphi : \mathcal{O}_S \to \mathcal{O}_{S'}$. In other words, S' is the moduli of quadruples (E, ϕ_p, C, ϕ_N) for an étale subgroup $C \subset E$, and the Frobenius map φ for general base is induced by the correspondence:

$$(E, \phi_p, \phi_N) \mapsto (E^{(p)} = E/\phi_p(\mu_p), \phi_p' : \mu_p \cong E[p^2]^\circ/\phi_p(\mu_p), E[p]^{et}, \phi_N),$$

where ϕ_p' is induced by

$$\mu_p \xrightarrow{\phi_p} \phi_p(\mu_p) \xrightarrow{\zeta \mapsto \zeta^{1/p}} E[p^2]^\circ/\phi_p(\mu_p).$$

Then it is easy to check that the $U(p)$ operator coincides with $\frac{1}{p}\mathrm{Tr}_\varphi$ after inverting p. We thus use the formula (U) heuristically over general base A under the understanding that $\sum_C$ in (U) indicates Tr_φ if A has non-trivial p–torsion. In other words, in $\sum_C$, the étale subgroups C is counted with multiplicity p if A has p–torsion; so, $U(p)$ is divisible by p. In particular, $p \cdot T(p) = p \cdot U(p) + \varphi$ is the p–adic lift of the congruence relation of Eichler and Shimura as given in [K3] 3.11.3.3.

Since $E \equiv 1 \mod p$, we confirm (C). Let

$$G_{\Gamma_1(N)}(A) = \bigoplus_{k=0}^{\infty} G_k(\Gamma_1(N); A).$$

One can prove the p–adic density of $G_{\Gamma_1(N)}(W)[\frac{1}{p}] \cap V$ in V ([GME] Corollary 3.2.4 and Theorem 3.2.10). Using this fact, we can show that $\lim_{n\to\infty} U(p)^{n!}$ exists. The final result is as follows:

Theorem 3.2 (VCT). *For all $k \geq 3$, we have*

$$V^{ord} \otimes_{W[[\mathbb{Z}_p^\times]],k} W \cong \mathrm{Hom}_W(G_k^{ord}(\Gamma_1(N); W), W).$$

Similarly, if we write $\mathcal{V}^{ord}_{cusp}$ for the subspace of cusp forms in $\mathcal{V}^{ord}$ and write V^{ord}_{cusp} for its Pontryagin dual (that is the cuspidal quotient of V^{ord}), the above result holds for spaces of cusp forms replacing V^{ord} and G_k^{ord} by V^{ord}_{cusp} and the subspace S_k^{ord} of cusp forms in G_k^{ord}.

3.1.4. *Families of p–ordinary modular forms.* Let $a(n) : \mathcal{V}^{ord} \to \mathbb{T}_p$ be the linear map associating f its coefficient of q^n in the q–expansion; so, $a(n)$ is in the dual V^{ord}. We now consider

$$G(\chi; \Lambda) = \mathrm{Hom}_\Lambda(V^{ord}[\chi], \Lambda).$$

With each $\phi \in G(\chi; \Lambda)$, we associate its q–expansion

$$\phi(q) = \sum_{n=0}^{\infty} \phi(a(n)) q^n \in \Lambda[[q]].$$

Theorem 3.3. *For each $k \geq 2$, we have*

1. *$G(\chi; \Lambda)$ is Λ-free of finite rank;*
2. *$G(\chi; \Lambda) \otimes_{\Lambda,k} W \cong G_k^{ord}(\Gamma_1(N) \cap \Gamma_0(p), \chi\omega^{-k}; W)$;*
3. *the above identification is induced by $\phi \mapsto \sum_n a(n, \phi(\gamma^k - 1)) q^n \in W[[q]]$.*

Proof. The Λ–freeness follows from the freeness of $V^{ord}[\chi]$. We only prove the assertion when k induces χ on Δ; so, $\chi\omega^{-k}$ is trivial. We have

$$\begin{aligned} G(\chi; \Lambda) \otimes_{\Lambda,k} W &\cong \mathrm{Hom}_\Lambda(V^{ord}[\chi], \Lambda) \otimes_{\Lambda,k} W \\ &\cong \mathrm{Hom}_W(V^{ord}[\chi] \otimes_{\Lambda,k} W, W) \cong \mathrm{Hom}_W(\mathrm{Hom}_W(G_k^{ord}(\Gamma_1(N); W); W)) \\ &= G_k^{ord}(\Gamma_1(N); W) = G_k^{ord}(\Gamma_1(N) \cap \Gamma_0(p); W). \end{aligned}$$

We leave the verification of the specialization of q–expansion to the audience. ☐

There is a version of this type of results for $\Gamma_0(N)$ and also for cusp forms, which is valid for all weights $k \geq 2$ (see [GME] Chapter 3 in particular Theorem 3.2.17).

4. Hecke Equivariance of the Eichler-Shimura Map

Let $G_{/\mathbb{Z}_{(p)}}$ ($\mathbb{Z}_{(p)} = \mathbb{Q} \cap \mathbb{Z}_p$) be a connected reductive group (split over $\mathbb{Z}_p$). For simplicity, we often assume that G has trivial center. We shall prove semi-simplicity of the commutative Hecke algebra acting on the nearly ordinary cohomology group $H^q_{n.ord}(X(U), L) \subset H^q_!(X(U), L)$ for a modular variety $X(U)$ associated to an arbitrary p–power level open compact subgroup U of $G(\mathbb{A}^\infty)$. Here the locally constant or coherent sheaf L on $X(U)$ is associated to a rational representation of G twisted by a finite order character. Although we have assumed that G is split over $\mathbb{Z}_p$, the argument works equally well for a connected reductive group $G_{/\mathcal{W}}$ split over W (see [PAF] Chapter 5). Here $\mathcal{W}$ is a valuation ring over $\mathbb{Z}_{(p)}$ of a number field, and $W = \varprojlim_n \mathcal{W}/p^n\mathcal{W}$ is the p–adic completion of $\mathcal{W}$.

4.1. **Semi-simplicity of Hecke Algebras.** We shall describe the semi-simplicity of the Hecke algebra acting on topological cohomology groups. Later we relate the topological and the coherent cohomology groups by the generalized Eichler-Shimura map, which shows the semi-simplicity of the Hecke algebra acting on the (degree 0) coherent cohomology.

4.1.1. *Jacquet Modules.* Let π be an admissible semi-simple representation of $G(\mathbb{Q}_p)$ on a vector space $V = V(\pi)$ over a field K of characteristic 0 (in this lecture, K is just a characteristic 0 field no more no less). Contrary to the tradition, I always suppose that V is a right $G(\mathbb{Q}_p)$–module. Let B be a Borel subgroup with split torus $T = B/N$ for the unipotent radical N.

We have a Haar measure du of $N(\mathbb{Q}_p)$ with $\int_{N(\mathbb{Z}_p)} du = 1$. We then define

$$V(B) = V(B, \pi) = \left\{ v - v\pi(n) \in V(\pi) \middle| v \in V(\pi) \ n \in N(\mathbb{Q}_p) \right\},$$

and put $V_B = V_B(\pi) = V/V(B)$, which is called the Jacquet module. We take a sufficiently large open compact subgroup $U_w \subset N(\mathbb{Q}_p)$ for each $w = v - v\pi(n) \in V(B)$ so that $n \in U_w$. Then we see that $\int_U v\pi(u)du = 0$ for every open subgroup U with $U_w \subset U \subset N(\mathbb{Q}_p)$. By this fact, we can conclude that the association $V \mapsto V_B$ is an exact functor. Later we shall give a canonical splitting $V^N \cong V_B \oplus V(B)^N$ as Hecke modules, where $V^N = H^0(N(\mathbb{Z}_p), V)$ (Bernstein-Casselman).

Let V' be a K–vector space. A function $f : G(\mathbb{Q}_p) \to V'$ is called *smooth* if it is locally constant (uniformly under the left translation). In other words, there exists an open compact subgroup $C_f \subset G(\mathbb{Q}_p)$ such that $f(kg) = f(g)$ for all $g \in G(\mathbb{Q}_p)$ and $k \in C_f$. For each admissible $T(\mathbb{Q}_p)$–module V', we define $\mathrm{Ind}_B^G V'$ to be the space of smooth functions on $G(\mathbb{Q}_p)$ such that $f(gb) = f(g)\overline{b}$ for all $b \in B(\mathbb{Q}_p)$, where $\overline{b}$ is the projection of b in $T(\mathbb{Q}_p)$. Then we let $G(\mathbb{Q}_p)$ act on $\mathrm{Ind}_B^G V'$ by $f(g)g' = f(g'g)$ for $g \in G(\mathbb{Q}_p)$. This representation $\mathrm{Ind}_B^G V'$ is the smooth induction of V' from B to G. In this definition, we may replace B by a parabolic subgroup of G and T by the reductive part of P. Hereafter all representations of G, B and T are assumed to be smooth admissible.

Since the smooth induction preserves admissibility ([BZ] 2.3), $\mathrm{Ind}_B^G V'$ has composition series, and hence its semi-simplification $(\mathrm{Ind}_B^G V')^{ss}$ is well defined. The beauty of the theory of admissible representations is its purely algebraic nature; so, we do not need to assume any analytic assumptions; in particular, our representations are often not unitary as is clear from our main result Theorem 4.2.

The following results are due to Jacquet and Bernstein-Zelevinsky and are well known [BZ]:

1. (Frobenius reciprocity) $\mathrm{Hom}_B(V_B, V') \cong \mathrm{Hom}_G(V, \mathrm{Ind}_B^G V')$;
2. If π is absolutely irreducible, then $\dim_K V_B \le |W|$, where W is the Weyl group of T in G (Bernstein-Zelevinsky);
3. If π is absolutely irreducible and $V_B \neq 0$, then $\mathrm{Ind}_B^G \lambda \twoheadrightarrow V$ for a character $\lambda : T(\mathbb{Q}_p) \to K^\times$ (Jacquet);
4. $\left(\mathrm{Ind}_B^G \widetilde{\lambda}\right)^{ss} \cong \left(\mathrm{Ind}_B^G \widetilde{\lambda^w}\right)^{ss}$ for all $w \in W$ (Bernstein-Zelevinsky),

where "ss" indicates semi-simplification, $\lambda^w(t) = \lambda(wtw^{-1})$ and $\widetilde{\lambda} = \delta_B^{1/2}\lambda$ for the right module character δ_B of B: $\int_{N(\mathbb{Q}_p)} \phi(u)du = \delta_B(b)\int_{N(\mathbb{Q}_p)} \phi(b^{-1}xb)du$ $(\forall\phi)$. We have the following corollary of the above facts:

Corollary 4.1. *Suppose that π is irreducible and that $V_B[\widetilde{\lambda}] \neq 0$, where $\widetilde{\lambda} = \delta_B^{1/2}\lambda$ for the module character δ_B on B. Then π is a quotient of $\mathrm{Ind}_B^G \widetilde{\lambda}$. If $\widetilde{\lambda^w}(t)$ for $w \in W$ are all distinct, $V_B \subset \bigoplus_{w\in W} \widetilde{\lambda^w}$ as $T(\mathbb{Q}_p)$–modules.*

Proof. Since the algebra in $\mathrm{End}_K(V_B)$ generated by the action of T is a finite dimensional commutative algebra, the $\widetilde{\lambda}$–eigenspace is non-trivial if and only if the maximal $\widetilde{\lambda}$–quotient is non-trivial. Thus, we have a morphism of T–modules: $V_B \twoheadrightarrow V(\widetilde{\lambda})$. Since we have $(\mathrm{Ind}_B^G \widetilde{\lambda})^{ss} \cong (\mathrm{Ind}_B^G \widetilde{\lambda^w})^{ss}$, by Frobenius reciprocity, all eigenvalues $\widetilde{\lambda^w}$ can show up as a quotient of $(\mathrm{Ind}_B^G \widetilde{\lambda})_B$ whose dimension is bounded by $|W|$. Thus if all characters $\widetilde{\lambda^w}$ are distinct, we have $V_B \subset (\mathrm{Ind}_B^G \widetilde{\lambda})_B \cong \bigoplus_{w\in W} \widetilde{\lambda^w}$. Since $V \mapsto V_B$ is exact, this is enough to conclude the assertion. □

4.1.2. *Double Coset Algebras.* We consider the double coset algebra made of formal linear combinations of double cosets of a subgroup in a semi-group. This type of algebra is considered in [IAT] 3.1 and often called a Hecke ring. We shall use the terminology "double coset algebra" to avoid confusion with Hecke algebras later we shall study.

Let
$$D = \left\{x \in T(\mathbb{Q}_p) \middle| xN_B(\mathbb{Z}_p)x^{-1} \supset N_B(\mathbb{Z}_p)\right\}$$
which is called the *expanding semi-group* in $T(\mathbb{Q}_p)$. Write $B = B(\mathbb{Z}_p)$ and $N = N(\mathbb{Z}_p)$ for simplicity. Define so-called Iwahori subgroups by

$$\begin{aligned} U_0(r) &= \left\{u \in G(\mathbb{Z}_p) \middle| u \mod p^r \in B(\mathbb{Z}/p^r\mathbb{Z})\right\} \\ U_1(r) &= \left\{u \in G(\mathbb{Z}_p) \middle| u \mod p^r \in N(\mathbb{Z}/p^r\mathbb{Z})\right\}. \end{aligned} \tag{4.1}$$

These subgroups S have the Iwahori decomposition: $S = N'T'N \cong N' \times T' \times N$ for open compact subgroups $T' \subset T(\mathbb{Z}_p)$ and N' in the opposite unipotent ${}^tN = {}^tN(\mathbb{Z}_p)$. Each $x \in D$ shrinks ${}^tN$: $x{}^tNx^{-1} \subset {}^tN$. Then we have

$$\begin{aligned} N\xi N &= \bigsqcup_{u\in\xi^{-1}N\xi\backslash N} N\xi u = \bigsqcup_{u\in N\backslash\xi N\xi^{-1}} Nu\xi, \\ &B\xi B = \bigsqcup_{u\in N\backslash\xi N\xi^{-1}} Bu\xi \text{ and } S\xi S = \bigsqcup_{u\in N\backslash\xi N\xi^{-1}} Su\xi, \end{aligned} \tag{4.2}$$

where S is an Iwahori subgroup. By this fact, $\Delta_N = NDN$, $\Delta = \Delta_B = BDB$ and $\Delta_S = \Delta_p = SDS$ are sub-semigroups of $G(\mathbb{Q}_p)$, and the double coset algebras generated additively over $\mathbb{Z}$ by double cosets of the group in the semigroup are all isomorphic as algebras:

$$R = R(N, \Delta_N) \cong R(B, \Delta_B) \cong R(S, \Delta_S).$$

Further these algebras are commutative: $T(\xi)T(\eta) = T(\xi\eta)$ for $T(\xi) = N\xi N$ and $\xi, \eta \in D$ (cf. [IAT] Chapter 3 and [H95] Section 2). We let R act on $v \in V^N = H^0(N(\mathbb{Z}_p), V)$ by

$$v|T(\xi) = v|[N\xi N] = \sum_{u \in \xi^{-1}N\xi\backslash N} v\pi(\xi u) = \int_{\xi N \xi^{-1}} v\pi(u)\pi(\xi)du, \tag{4.3}$$

and similarly for $\overline{v} \in V_B$ in place of $v \in V^N$; then the projection: $V^N \to V_B$ is R–linear. Here the Haar measure du is normalized so that $\int_{N(\mathbb{Z}_p)} du = 1$.

Let Σ be the set of maximal (proper) parabolic subgroups $P \supset B$. Decomposing $P = M_P N_P$ for reductive part $M_P \supset T$ and the unipotent radical N_P, we can identify the set Σ with the following set of co-characters:

$$\left\{\alpha_P : \mathbb{G}_m \to G \middle| \alpha_P(p) \text{ generates } Z(M_P)(\mathbb{Q}_p) \cap D \text{ modulo } Z(M_P)(\mathbb{Z}_p)\right\},$$

where $Z(M_P)$ is the center of M_P. Then $\{\xi_\alpha = \alpha(p)\}_{\alpha\in\Sigma}$ generate $D/T(\mathbb{Z}_p)$, and $R \cong \mathbb{Z}[T(\xi_\alpha)]$ if the center of G is trivial. If $G = GL(n)$,

$$\Sigma = \{\alpha_j | \alpha_j(p) = \mathrm{diag}[1_j, p1_{n-j}]\}.$$

For $\xi = \prod_{\alpha\in\Sigma} \xi_\alpha$, we have

$$N(\mathbb{Q}_p) = \bigcup_{j=0}^{\infty} \xi^j N \xi^{-j}.$$

We still denote by $T(\xi)$ the action of $N\xi N$ on V^N. The formula (4.3) defines as well an action of $T(\xi)$ on V_B. We see easily from (4.3) that $T(\xi^j) = T(\xi)^j$ and for each finite dimensional subspace $X \subset V(B)$, $T(\xi)|_{X^N}$ is nilpotent on X^N by (4.3).

For any R–eigenvector $v \in V^N$ with $\overline{v}t = \lambda(t)\overline{v}$ ($t \in T(\mathbb{Q}_p)$, $\overline{v} = v \mod V(B)$), we get

$$v|[NxN] = [N : x^{-1}Nx]\lambda(x)v = |\det(Ad_N(x))|_p \lambda(x)v, \tag{4.4}$$

where "$|\ |_p$" is the standard absolute value of $\mathbb{Q}_p$ such that $|p|_p^{-1} = p$ and Ad is the adjoint representation of T on the Lie algebra of N.

4.1.3. *Rational representations of G.* Let us first define a canonical splitting:

$$V^N = V_B \oplus V(B)^N \quad \text{as } R\text{–modules}.$$

We have by definition, $V^N = V^{N(\mathbb{Z}_p)} = \bigcup_r V^{U_1(r)}$. The subspace $V_r = V^{U_1(r)}$ is finite dimensional and is stable under R. By Jordan decomposition applied to $T(\xi)$ ($\xi = \prod_{\alpha\in\Sigma} \xi_\alpha$), we can decompose uniquely that $V_r = V_r^\circ \oplus V^{nil}$ so that $T(\xi)$ is an automorphism on V_r° and is nilpotent on V^{nil}. We may replace $T(\xi)$ by $T(\xi^a) = T(\xi)^a$ for any positive a in the definition of the above splitting. Since $T(\xi)$ is nilpotent on any finite dimensional subspace of $V(B)$, V_r° injects into V_B; so, $\dim V_r^\circ$ is bounded by $\dim V_B \le |W|$. For any T–eigenvector $\overline{v} \in V_B$, lift it to $v \in V$. Then for sufficiently large j, $v\pi(\xi^{-j})$ is in V^N. Since $\overline{v}\pi(\xi^{-j})$ is a constant multiple of $\overline{v}$, we may replace $\overline{v}$ and v by $\overline{v}\pi(\xi^{-j})$ and $v\pi(\xi^{-j})$, respectively. Then for sufficiently large k, $w = vT(\xi^k) \in V_r^\circ$. Then $\overline{w}T(\xi)^{-k}$ is equal to $\overline{v}$ for the image $\overline{w}$ in V_B. This shows the splitting: $V^N \cong V_B \oplus V(B)^N$ as R–modules when the action of T on V_B is semi-simple. In general, taking a sufficiently large r so that V_r surjects down to V_B. We apply the above argument to the semi-simplification of V_r under the action of the Hecke algebra. Thus $V^\circ = \bigcup_r V_r^\circ \cong V_B$, and this concludes the proof.

Let $G(\widehat{\mathbb{Z}}) \subset G(\mathbb{A}^\infty)$ denote a maximal compact subgroup hyperspecial everywhere (by abusing notation; see [Tt] for hyperspecial compact subgroups). We assume that the p–component of $G(\widehat{\mathbb{Z}})$ is given by $G(\mathbb{Z}_p)$. We now assume K to be a finite extension over $\mathbb{Q}_p$. Let $\mathcal{O}$ be the p–adic integer ring of K. We write $U = U_0(r)$ for $r > 0$. Recall the Iwahori decomposition $U = N'T(\mathbb{Z}_p)N$. We consider the space of continuous functions: $\mathcal{C}(A) = \{\phi : U/N(\mathbb{Z}_p) \to A\}$ for $A = \mathcal{O}$ and K. We would like to make $\mathcal{C}$ a left Δ_p^{-1}–module for the opposite semi-group Δ_p^{-1} of $\Delta_p = \Delta_U$. For that, we first define a left action of Δ_p on $Y_U = U/N(\mathbb{Z}_p)$. Since U acts on $Y_U = U/N(\mathbb{Z}_p)$ by left multiplication, we only need to define a left action of D. Pick $yN(\mathbb{Z}_p) \in Y_U$ and by the Iwahori decomposition, we may assume that $y \in N'T(\mathbb{Z}_p) \subset U$ and consider yN. Then for $d \in D$, $dyNd^{-1} = dyd^{-1}dNd^{-1} \subset dyd^{-1}N(\mathbb{Q}_p)$ and $dyd^{-1}N(\mathbb{Q}_p)$ is well defined in $G(\mathbb{Q}_p)/N(\mathbb{Q}_p)$. Since conjugation by $d \in D$ expands $N(\mathbb{Z}_p)$ and shrinks N', $dud^{-1} \in U$, and the coset $dyd^{-1}N(\mathbb{Q}_p) \cap U = dyd^{-1}U$ is a well defined single coset of N, which we designate to be the image of the action of $d \in D$. We now let Δ_p^{-1} act on $\mathcal{C}$ by $d\phi(y) = \phi(d^{-1}y)$. In this way, $\mathcal{C}$ becomes a Δ_p^{-1}–module.

We consider the algebro-geometric induction module:

$$(4.5)\qquad L(\kappa; K) = \left\{\phi : G/N \to K \in H^0(G/N, \mathcal{O}_{G/N}) \middle| \phi(yt) = \kappa(t)\phi(y)\ \forall t \in T\right\},$$

where $\mathcal{O}_{G/N}$ is the structure sheaf of the scheme G/N. We let G act on $L(\kappa; K)$ by $g\phi(y) = \phi(g^{-1}y)$. Then $L(\kappa; K) = \mathrm{ind}_B^G \kappa^{-1}$ (following the normalization of induction as in [RAG] I.3.3), which is the induction in the category of scheme theoretic representations (that is, polynomial representations). We call κ *dominant* if $L(\kappa; K) \neq 0$. We write this representation as $\rho_\kappa = \rho_\kappa^G : G \to GL(L(\kappa; K))$.

We restrict functions in $L(\kappa; K)$ to $Y_U = U/N(\mathbb{Z}_p)$ and regard $L(\kappa; K) \subset \mathcal{C}(K)$. Then multiply $L(\kappa; K)$ by a character $\varepsilon : T(\mathbb{Z}/p^r\mathbb{Z}) = U_0(r)/U_1(r) \to \mathcal{O}^\times$ (regarding it as a function on $\mathcal{C}(\mathcal{O})$). Since $\mathbb{Q}_p^\times = \mathbb{Z}_p^\times \times p^{\mathbb{Z}}$, we can decompose $T(\mathbb{Q}_p) = T(\mathbb{Z}_p) \times \left(p^{\mathbb{Z}}\right)^r$ for the rank r of T, and we can extend ε to $T(\mathbb{Q}_p)$ requiring it to have constant value 1 on $\left(p^{\mathbb{Z}}\right)^r$. In this way, we get the twisted Δ_p^{-1}–module $L(\kappa\varepsilon; K) = \varepsilon L(\kappa; K) \subset \mathcal{C}$. The pull-back Δ_p^{-1}–action preserves $L(\kappa\varepsilon; \mathcal{O}) = L(\kappa\varepsilon; K) \cap \mathcal{C}(\mathcal{O})$ but original ρ_κ may not be. Then for $\xi \in D$,

$$(4.6)\qquad \textit{the action of } \xi^{-1} \in \Delta_p^{-1} \textit{ is given by } \kappa(\xi)^{-1}\varepsilon(\xi)\rho_\kappa(\xi^{-1}).$$

Since the action of ξ on Y_U is conjugation: $x \mapsto \xi x \xi^{-1}$, the front $\kappa(\xi)^{-1}$ comes from the definition of $L(\kappa; K)$ in (4.5): $\phi(\xi x \xi^{-1}) = \kappa(\xi^{-1})\rho_\kappa(\xi^{-1})\phi(x)$. By definition, the new action is optimally integral.

Example 4.1. To illustrate our integral modification of the action, let us give an example in the simplest non-trivial case: Let $L(\kappa; K)$ be the space of homogeneous polynomial of two variable (X, Y) of degree $n > 0$. Then we let $G = GL(2)$ act on $\phi(X, Y) \in L(\kappa; K)$ by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^{-1} \phi(X, Y) = (ad - bc)^v \phi(dX - bY, -cX + aY)$ for an integer $v \in \mathbb{Z}$. Then $L(\kappa; K) = \mathrm{ind}_B^G \kappa^{-1}$ for $\kappa : \mathrm{diag}[a, d] \mapsto (ad)^v a^n$ for the upper triangular Borel subgroup $B \subset GL(2)$. If the integer v is negative, the lattice $L(\kappa;\mathcal{O})$ is obviously not stable under the action of the diagonal matrices

$$D = \{z\,\mathrm{diag}[1,d] | 0 \neq d \in \mathbb{Z}_p,\ z \in \mathbb{Q}_p^\times\}.$$

The modified (integral) action defined above is just

$$\left(\begin{smallmatrix} 1 & 0 \\ 0 & d \end{smallmatrix}\right)^{-1} \circ \phi(X,Y) = \phi(dX,Y) = d^{-v}\left(\begin{smallmatrix} 1 & 0 \\ 0 & d \end{smallmatrix}\right)^{-1}\phi(X,Y) = \kappa\left(\begin{smallmatrix} 1 & 0 \\ 0 & d \end{smallmatrix}\right)^{-1}\left(\begin{smallmatrix} 1 & 0 \\ 0 & d \end{smallmatrix}\right)^{-1}\phi(X,Y).$$

4.1.4. *Nearly p–Ordinary Representations.* Hereafter we assume that κ is an element κ_0 of $X(T)$ up to finite order character of $T(\mathbb{Z}_p)$. Let U be an open subgroup of $G(\widehat{\mathbb{Z}})$. We consider the associated modular variety:

$$X(U) = G(\mathbb{Q})\backslash G(\mathbb{A})/UC_{\infty+} \cong \bigsqcup_{\Gamma} X_\Gamma,$$

where $C_{\infty+}$ is the identity connected component of the maximal compact subgroup of the Lie group $G(\mathbb{R})$ and Γ runs over the following finite set

$$\left\{G(\mathbb{Q}) \cap tUG_+(\mathbb{R})t^{-1} \middle| t \in (G(\mathbb{Q})\backslash G(\mathbb{A})/UG(\mathbb{R}))\right\}$$

where $G_+(\mathbb{R})$ is the identity connected component of $G(\mathbb{R})$. For the symmetric space $X = G_+(\mathbb{R})/C_{\mathbf{0}}$, we have written $X_\Gamma = \Gamma\backslash X$. For any $\mathcal{O}$–module A, we define a right action of $u \in UC_{\infty+}$ on $L(\kappa;A)$ by $\phi|u = \rho_\kappa(u_p^{-1})\phi$ if $U_p \subset U_0(r)$ for some $r > 0$.

We define the covering space $\mathcal{X}(U)$ of $X(U)$ by

$$\mathcal{X}(U) = G(\mathbb{Q})\backslash(G(\mathbb{A}) \times L(\kappa;A))/UC_{\infty+} \cong \bigsqcup_{\Gamma} \mathcal{X}_\Gamma, \tag{4.7}$$

where $\gamma(x,\phi)u = (\gamma xu, \phi|u)$ for $\gamma \in G(\mathbb{Q})$ and $u \in UC_{\infty+}$, and we define $\mathcal{X}_\Gamma = \Gamma\backslash(X \times L(\kappa;A))$ by the diagonal action. We use the same symbol $L(\kappa;A)$ for the sheaf of locally constant sections of $\mathcal{X}(U)$ over $X(U)$.

We consider the limit, shrinking S,

$$\mathcal{L}(A) = \mathcal{L}^q(\kappa;A) = \varinjlim_S H_!^q(X(S), L(\kappa;A)). \tag{4.8}$$

Here $H_!^q(X(S), L(\kappa;A))$ ($A = K$ or $\mathcal{O}$) is the image of the compactly supported cohomology group $H_c^q(X(S), L(\kappa;A))$ in $H^q(X(S), L(\kappa;K))$. On the space $\mathcal{L}(K)$, the group $G(\mathbb{A}^\infty)$ acts from the right via a smooth representation, which is completely reducible. Thus in particular, we have an action on $H^0(U, \mathcal{L}^q(\kappa;K)) = \mathcal{L}^q(\kappa;K)^U$ of the double coset algebra

$$R_U = R(U, G(\mathbb{A}^{p\infty}) \times \Delta_p) \cong R(U^{(p)}, G(\mathbb{A}^{p\infty})) \otimes R$$

of double cosets UxU with $x \in G(\mathbb{A}^{p\infty}) \times \Delta_p$, where $U = U_p \times U^{(p)}$ and we have assumed that $U_p = U_0(r)$.

We take $\xi \in D$ such that $N(\mathbb{Q}_p) = \bigcup_j \xi^j N(\mathbb{Z}_p)\xi^{-j}$. We may assume that $\xi = \prod_{\alpha\in\Sigma}\xi_\alpha$. Then $T(\xi)$ acts on $\mathcal{L}^q(\kappa;\mathcal{O})^N$ ($N = N(\mathbb{Z}_p)$) through the Δ_p^{-1}–module structure on $L(\kappa;\mathcal{O})$. We write this operator as $\mathbb{T}$. On the other hand, $T(\xi)$ acts on $\mathcal{L}^q(\kappa;K)^N$ through the action of $G(\mathbb{A}^\infty)$ via the rational representation ρ_κ. The corresponding operator will be written by the same symbol T. Since the action through $\rho_\kappa(\xi^{-1})$ and the modified integral action of $\xi^{-1} \in \Delta_p^{-1}$ differs by the scalar $\kappa(\xi)^{-1}$ (4.6), the two operators $\mathbb{T}$ and T are related on the image of $\mathcal{L}^q(\kappa;\mathcal{O})^N$ by

$$\mathbb{T}(\xi) = \kappa(\xi)^{-1}T(\xi). \tag{4.9}$$

When $\kappa = 0$ (the identity character), the action of the Hecke operator is (truly canonically) induced by the Hecke correspondence $T(\xi) \subset (X(U) \times X(U))$, and in this case, $\mathbb{T}(\xi) = T(\xi)$. If $\kappa > 0$, we may relate cohomology groups of the sheaf $L(\kappa; K)$ as a part of the cohomology group with constant coefficients of a certain self-product Z of copies of the universal abelian scheme over $X(U)$. Since the Hecke operator then has interpretation as an isogeny action on the universal abelian scheme, it can be regarded as the action induced by the Hecke correspondence in $Z \times Z$. The action of $\mathbb{T}(\xi)$ and $T(\xi)$ uses different action of Δ_p^{-1}. This action of Δ_p^{-1} determines the part of the cohomology group over Z identified with the cohomology group over $X(U)$ with locally constant (but non-constant) coefficients. Thus the motivic realization of the two operators $T(\xi)$ and $\mathbb{T}(\xi)$ could be actually different, and the operator $\mathbb{T}(\xi)$ may not even have motivic realization (as in the Hilbert modular case of non-parallel weight). For example, in Scholl's construction [Sc] of the Grothendieck motive associated to an elliptic Hecke eigenform f, if one changes the action of congruence subgroup Γ of $SL_2(\mathbb{Z})$ by a power of determinant character, the physical sheaf over $X_\Gamma(\mathbb{C})$ obtained is the same, but its rational structure (including the Galois action) different. In this way, we can construct the motive associated to the standard p–adic Galois representation ρ_f of f and its Tate twists $\rho_f(m)$ as the étale realization of motives directly realized over a self-product of the universal elliptic curve. For a Hilbert Hecke eigenform f, we could twist ρ_f locally at each p–adic place by a power of the p–adic cyclotomic character, but this twist may not extend to a global twist because the exponent of the cyclotomic character depends on the p–adic place. In particular, if f is of non-parallel weight, the process of defining $\mathbb{T}(\xi)$ corresponds to untwisting ρ_f to reach a $\mathfrak{p}$–ordinary Galois representation at each p–adic place $\mathfrak{p}$, which cannot be performed globally; so, the operator $\mathbb{T}(\xi)$ may not have a motivic interpretation.

For any $U = U_p \times U^{(p)}$ with $U_p \supset N = N(\mathbb{Z}_p)$, the limit $e = \lim_{n\to\infty} \mathbb{T}(\xi)^{n!}$ exists as an endomorphism of $H^q(X(U), L(\kappa; A))$ for $A = \mathcal{O}$ and K. Thus the limit e extends to an endomorphism of $\mathcal{L}^q(\kappa; A)^N$ for $A = \mathcal{O}$ and K. It is easy to see, if $U_p \supset N$,

$$H^0(U, e(\mathcal{L}^q(\kappa; K)^N)) = e\left(H^q_!(X(U), L(\kappa; K))\right). \tag{4.10}$$

We write $\mathcal{L}^q_{n.ord}(\kappa; A)$ for $e\left(\mathcal{L}^q(\kappa; A)^N\right)$. An irreducible representation π of $G(\mathbb{A}^{\infty})$, which is a subquotient of $\mathcal{L}^q(\kappa; K)$, is called *nearly ordinary* of p–type κ if $e(V(\pi)^N)$ does not vanish for the representation space $V(\pi)$ of π.

4.1.5. *Semi-simplicity of Interior Cohomology Groups.* Let π be a cohomological automorphic representation of p–type κ. Suppose π_p is a subquotient of $\mathrm{Ind}_B^G \widetilde{\lambda}$ (this is automatic if π is nearly p–ordinary). Then for its p–component π_p (acting on $V := V(\pi_p)$), we find a character $\lambda : T(\mathbb{Q}_p) \to K^\times$ with the above property such that $V_B[\widetilde{\lambda}] \neq 0$ and

$$|\det(Ad_N(x))^{-1}\widetilde{\lambda}(x)|_p = \left||\det(Ad_N(x))|_p\widetilde{\lambda}(x)\right|_p \le |\kappa(x)|_p.$$

The equality holds if and only if π is p–nearly ordinary (in this case, automatically $V_B \neq 0$ and $\mathrm{Ind}_B^G \widetilde{\lambda} \twoheadrightarrow \pi_p$ because $V^N \cong V_B \oplus V(B)^N$ as R–modules).

For the moment, suppose that $G(\mathbb{Q}_p) = GL_n(\mathbb{Q}_p)$ and write $\lambda(\mathrm{diag}[t_1, \dots, t_n]) = \prod_{i=1}^n \lambda_i(t_i)$. Define the Hecke polynomial (at p) by

$$H_\pi(T) = \prod_{i=1}^{n}(1 - \lambda_i(p)T),$$

and write Δ_N for the Newton polygon of $H_\pi(T)$. Define the Hodge polygon Δ_H of π to be the Newton polygon of $\prod_{i=1}^n (1 - (\kappa\rho)_i(p)T)$. Then the above inequality implies

$$\Delta_N \geq \Delta_H$$

and the two extreme ends of the two polygons match.

We return to a general group G and assume that π is nearly p–ordinary. By definition,

$$\int_{N(\mathbb{Q}_p)} \phi(u)du = \delta_B(b) \int_{N(\mathbb{Q}_p)} \phi(b^{-1}xb)du.$$

This shows that

$$\delta_B = |\det \circ Ad_N|_p^{-1}. \tag{4.11}$$

By definition, $2\rho = \det \circ Ad_N$ is a sum of positive roots, and ρ is a sum of fundamental weights with respect to B. This shows

$$|\lambda|_p = |\kappa\rho|_p. \tag{4.12}$$

Note that κ is non-negative with respect to B because κ is dominant. Since $\kappa \geq 0$, $\kappa\rho > 0$, that is, $\kappa\rho$ is in the interior of the Weyl chamber of B. This shows that if $w \neq 1$,

$$|\lambda^w(d)|_p < |\lambda(d)|_p \quad \text{for all } d \in D, \tag{4.13}$$

because W acts simply transitively on Weyl chambers and each element in the interior of the chamber of λ has the maximum p–adic absolute value in its conjugates under W. In particular, we get

Theorem 4.2. *Let π be an irreducible nearly ordinary representation of p–type κ. Then there exists a character $\lambda : T(\mathbb{Q}_p) \to K^\times$ such that $\widetilde{\lambda} \hookrightarrow V(\pi_p)_B \hookrightarrow \oplus_{w \in W} \widetilde{\lambda^w}$ and $|\lambda|_p = |\rho\kappa|_p$, where ρ is the sum of fundamental weight with respect to B and $|\ |_p$ is the absolute value on K. Moreover $eH^0(N(\mathbb{Z}_p), V(\pi_p))$ is one dimensional, on which $T(\xi) = U\xi U$ for $\xi \in D$ acts by scalar $|\rho(\xi)|_p \lambda(\xi)$.*

Now suppose that $U = U_p \times G(\widehat{\mathbb{Z}}^p)$ with $U_p = U_0(r)$ for $r > 0$. By the above theorem, we get the following semi-simplicity of the Hecke algebra (for cohomological nearly ordinary cusp forms of p–type κ) from the fact that the spherical irreducible representation of $G(\mathbb{Q}_\ell)$ has a unique vector fixed by (any given) maximal compact subgroup:

Corollary 4.3. *Let the notation and the assumption be as above. Then the Hecke module $eH^q_!(X(U), L(\kappa; K))$ is semi-simple.*

Note that the projector $e = e_p$ is actually defined over $\mathcal{L}^q(\kappa; \overline{\mathbb{Q}})^N$. Thus the above semi-simplicity remains true on $e_S H^q_!(X(U), L(\kappa; \overline{\mathbb{Q}}))$ for $e_S = \prod_{\ell \in S} e_\ell$ with a finite set of primes S (where G is split over $\mathbb{Z}_\ell$) and a subgroup U of level M, which is a product of powers of primes in S. For such nearly S–ordinary automorphic forms, semi-simplicity of the Hecke operator action is always true.

4.2. **The Eichler-Shimura Map.** Before starting detailed study of the nearly ordinary part of coherent cohomology groups, we shall make explicit a generalized Eichler-Shimura map for unitary groups and hence the association of the weight: $\kappa \mapsto \kappa^*$ so that $H^0_{cusp}(X_\Gamma, \underline{\omega}^\kappa) \hookrightarrow H^d(X_\Gamma, L(\kappa^*;\mathbb{C}))$. To construct the map, we briefly recall an explicit shape of the symmetric domain of unitary groups.

4.2.1. *Unitary groups.* Define the complex unitary group G by

$$G = U(m,n)(\mathbb{R}) = \left\{ g \in GL_{m+n}(\mathbb{C}) \middle| g I_{m,n} g^* = I_{m,n} \right\},$$

where $I_{m,n} = \mathrm{diag}[1_m, -1_n] = \left(\begin{smallmatrix} 1_m & 0 \\ 0 & -1_n \end{smallmatrix}\right)$.

We want to make explicit the quotient space $X = G/C_{\mathbf{0}}$ following [AAF] 3.2. We consider

$$\mathcal{Y} = \left\{ Y \in GL_{m+n}(\mathbb{C}) \middle| \begin{array}{l} Y^* I_{m,n} Y = \mathrm{diag}[T,S] \text{ with } 0 < T = T^* \in M_m(\mathbb{C}), \\ 0 > S = S^* \in M_n(\mathbb{C}) \end{array} \right\}.$$

Write $Y = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$. By a simple calculation, we have

$$Y^* I_{m,n} Y = \begin{pmatrix} A^*A - C^*C & A^*B - C^*D \\ B^*A - D^*C & B^*B - D^*D \end{pmatrix} = \mathrm{diag}[T,S].$$

Since $A^*A > C^*C \geq 0$, A is invertible. Similarly D is invertible. Put $z = BD^{-1}$. We then see

$$A^*B = C^*D \iff (CA^{-1})^* = A^{-*}C^* = BD^{-1} = z,$$
$$B^*B - D^*D = D^*(z^*z - 1_n)D < 0$$

and

$$Y = \begin{pmatrix} 1_m & z \\ z^* & 1_n \end{pmatrix} \mathrm{diag}[A,D]$$

with $z^*z < 1$ thus we get

$$\mathcal{D} \times GL_m(\mathbb{C}) \times GL_n(\mathbb{C}) \cong \mathcal{Y} \tag{4.14}$$

by $(z, A, D) \mapsto Y(z)\,\mathrm{diag}[A,D]$ for $Y(z) = \begin{pmatrix} 1_m & z \\ z^* & 1_n \end{pmatrix}$. Here

$$\mathcal{D} = \left\{ z \in M_{m,n}(\mathbb{C}) \middle| z^*z < 1 \right\}.$$

Since $Y \mapsto gY$ for $g \in G$ takes Y into itself isomorphically, we have

$$gY(z) = Y(g(z))\,\mathrm{diag}[\overline{h}(g,z), j(g,z)] \quad h(g,z) = \overline{a} + \overline{b}^t z \quad \text{and} \quad j(g,z) = cz + d$$

if $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$.

By these formulae, it is clear that for the zero matrix $\mathbf{0} \in \mathcal{D}$,

$$G/C_{\mathbf{0}} \cong \mathcal{D} \text{ via } g \mapsto g(\mathbf{0})$$

and $C_{\mathbf{0}} = U(m) \times U(n)$. Therefore the complexification C of $C_{\mathbf{0}}$ is $GL_m(\mathbb{C}) \times GL_n(\mathbb{C})$. The functions ${}^t h^{-1}$ and j correspond to the standard representation of GL_m and GL_n, respectively. Since

$$Y(w)^* I_{m,n} Y(z) = \begin{pmatrix} 1 - wz^* & z - w \\ w^* - z^* & w^*z - 1 \end{pmatrix},$$

replacing z and w by $z + \Delta z$ and z, we get

$$\begin{aligned} \mathrm{diag}[{}^t h(g,z), j(g,z)^*] & \begin{pmatrix} 1 - g(w)g(z)^* & \Delta g(z) \\ \Delta g(z)^* & g(w)^*g(z) - 1 \end{pmatrix} \mathrm{diag}[h(g,z), j(g,z)] = \\ & \mathrm{diag}[{}^t h(g,z), j(g,z)^*] Y(g(z)) I_{m,n} Y(g(z + \Delta z)\,\mathrm{diag}[h(g,z), j(g,z)] \\ & \qquad = Y(z)^* g^* I_{m,n} g Y(z + \Delta z) = \begin{pmatrix} 1 - wz^* & \Delta z \\ \Delta z^* & w^*z - 1 \end{pmatrix}. \end{aligned}$$

From this, we conclude

$$dg(z) = {}^t h(g,z)^{-1} dz j(g,z)^{-1}. \tag{4.15}$$

We can show (see Shimura's books: [EPE] (6.3.9) and [AAF] Section 3):

$$\det(h(g,z)) = \det(g)^{-1}\det(j(g,z)).$$

This can be shown also as follows: On the diagonal torus $T_C \subset U(m)\times U(n)$, for $g = \mathrm{diag}[t_1,\dots,t_m,t_{m+1},\dots,t_{m+n}]$, t_j satisfies $\overline{t}_j = t_j^{-1}$ and

$$j(g,z) = cz+d = \mathrm{diag}[t_{m+1},\dots,t_{m+n}] \text{ and } h(g,z) = \overline{a}+\overline{b}^t z = \mathrm{diag}[t_1^{-1},\dots,t_m^{-1}].$$

Then j (resp. h) corresponds therefore to (resp. the contragredient of) the standard representation of $GL_n(\mathbb{C})$ (resp. $GL_m(\mathbb{C})$); so, the corresponding highest weight character, after applying "det", is:

$$\mathrm{diag}[t_1,\dots,t_{m+n}] \mapsto \prod_{j=m+1}^{m+n} t_j \quad (\text{resp. } \prod_{j=1}^{m} t_j^{-1}).$$

This relation coincides with the above formula of Shimura. We thus embed the product $U(m)\times U(n)$ into $GL(m)\times GL(n)$ by $g \mapsto J(g) = ({}^t h(g,\mathbf{0})^{-1}, j(g,\mathbf{0}))$. We also write $J(g,z) = ({}^t h(g,z)^{-1}, j(g,z))$.

Writing $\mathbf{d}z = \bigwedge_{i,j} dz_{ij}$, we get

$$\mathbf{d}g(z) = \det(g)^n \det(j(g,z))^{-m-n}\mathbf{d}z.$$

Write $\mu_{m,n} \in X(T)$ for the character

$$\mu_{m,n}(\mathrm{diag}[t_1,t_2,\dots,t_{m+n}]) = (t_1\times t_2\times\cdots\times t_m)^{-n}\times(t_{m+1}\times\cdots\times t_{m+n})^m.$$

Suppose that $\kappa \geq \mu_{m,n}$, and write $\kappa^* = \kappa - \mu_{m,n}$. We try to find a non-zero polynomial function $p_{\kappa^*} : \mathcal{D} \to \mathrm{Hom}_{\mathbb{C}}(L_C(\kappa^*;\mathbb{C}), L_G(\kappa^*;\mathbb{C}))$ in z such that

$$p(\alpha(z))\circ\rho^C_{\kappa^*}(J(\alpha,z)) = \alpha p(z) \quad (\rho^C_{\kappa^*} = \mathrm{ind}^C_{B\cap C}\kappa^*)$$

for all $\alpha \in G$, where $C = GL(m)\times GL(n)$. Since $\mathcal{D} = G/C_{\mathbf{0}}$, if it exists, such a function is unique. By the above (hypothetical) formula, we could define p by

$$p(\alpha(\mathbf{0}))\circ\rho^C_{\kappa^*}(J(\alpha,\mathbf{0})) = \alpha p(\mathbf{0}),$$

if we find an appropriate map $p(\mathbf{0}) \in \mathrm{Hom}_{\mathbb{C}}(L_C(\kappa^*;\mathbb{C}), L_G(\kappa^*;\mathbb{C}))$. If we change α by αu for $u \in U(m)\times U(n)$, then we have

$$p(\alpha(\mathbf{0}))\circ\rho^C_{\kappa^*}(J(\alpha)J(u)) = \alpha u p(\mathbf{0})$$
$$\iff p(\alpha(\mathbf{0}))\circ\rho^C_{\kappa^*}(J(\alpha)) = \alpha u p(\mathbf{0})\circ\rho^C_{\kappa^*}(J(u))^{-1}.$$

Such a map $p(\mathbf{0})$ with $up(\mathbf{0})\circ\rho^C_{\kappa^*}(J(u))^{-1} = p(\mathbf{0})$ exists because $GL(m)\times GL(n)$ is identified with a subgroup of $GL(m+n)(\mathbb{C}) = U(m+n)(\mathbb{C})$ (thus it corresponds to the identity inclusion: $L_C(\kappa^*;\mathbb{C}) = \rho^C_{\kappa^*} \hookrightarrow \mathrm{ind}_P^{GL(m+n)}\rho^C_{\kappa^*} = L_G(\kappa^*;\mathbb{C}) = \rho^G_{\kappa^*}$ for $P = \mathrm{diag}[GL(m), GL(n)]B$). Take $\kappa^*|_{T_C}$ to be the highest weight ω_n associated to the standard representation of $GL(n)$. Then κ^* corresponds to the standard representation of $U(m,n)$, and we have $p_{\omega_n}(z)(x) = \left(\begin{smallmatrix} z \\ 1\end{smallmatrix}\right)x$ for $x \in \mathbb{C}^n$. We verify easily that $gp_{\omega_n}(z)(x) = p_{\omega_n}(z)(j(g,z)x)$. Thus $p(z)$ is a polynomial in z in this special case. Similarly to the above, if $\kappa^*|_{T_C} = \omega_m$ corresponds to the contragredient of the standard representation of $GL(m)$, then κ^* is associated to the complex conjugate of the standard representation of $U(m,n)$, and we have $p_{\omega_m}(z)(x) = \left(\begin{smallmatrix} 1_m \\ {}^t z\end{smallmatrix}\right)x$ for $x \in \mathbb{C}^m$. Again we verify that $\overline{g}p_{\omega_m}(z)(x) = p_{\omega_m}(z)(h(g,z)x)$, and $p_{\omega_m}(z)$

is a polynomial in z. For general κ, $L_C(\kappa^*;\mathbb{C})$ (resp. $L_G(\kappa^*;\mathbb{C})$) is a quotient of $L_C(\omega_n;\mathbb{C})^{\otimes t}\bigotimes L_C(\omega_m;\mathbb{C})^{\otimes s}$ (resp. $L_G(\omega_n;\mathbb{C})^{\otimes t}\bigotimes L_G(\omega_m;\mathbb{C})^{\otimes s}$). The general $p_{\kappa^*}(z)$ is a constant multiple of the projected image of the tensor product of copies of $p_{\omega_1}(z)$ and hence is a polynomial in z.

We define for $f\in H^0(X_\Gamma,\underline{\omega}^\kappa_{/\mathbb{C}})$ a holomorphic differential with values in $L(\kappa^*;\mathbb{C})$ by

$$\omega(f)=p_{\kappa^*}(z)(f)\mathbf{d}z.$$

Note that here $L_C(\kappa;\mathbb{C})=L_C(\kappa^*;\mathbb{C})\otimes L_C(\mu_{m,n};\mathbb{C})$ and that $L_C(\mu_{m,n};\mathbb{C})$ is one-dimensional; so, we can identify $L_C(\kappa;\mathbb{C})$ with $L_C(\kappa^*;\mathbb{C})$ canonically as vector space, and thus, the above definition is consistent. We can easily verify that $\alpha^*\omega(f)=\rho^G_{\kappa^*}(\alpha)\omega(f)$.

Theorem 4.4. *Assume that $\kappa\geq\mu_{m,n}$. Then the association: $f\mapsto[\omega(f)]\in H^d(X_\Gamma,L(\kappa^*;\mathbb{C}))$ for $d=\dim_{\mathbb{C}}\mathcal{D}$ induces the embedding:*

$$H^0_{cusp}(X_\Gamma,\underline{\omega}^\kappa)\hookrightarrow H^d(X_\Gamma,L_G(\kappa^*;\mathbb{C})),$$

where $[\omega(f)]$ is the de Rham cohomology class of $\omega(f)$.

As an exercise, compute κ^* when $G=GSp$.

4.2.2. *Hecke equivariance.* We are going to show that the Eichler-Shimura map is equivariant under Hecke operators and is compatible with normalization of Hecke operators.

We have normalized the Hecke operator on the topological cohomology group taking the action of $\xi\in\Delta_B$ normalized as $\widetilde{\rho}^G_\kappa(\xi^{-1})=\kappa^{-1}(\xi)\rho^G_\kappa(\xi^{-1})$. Note that here, for any algebraic character $\chi:G$ or $C\to\mathbb{G}_m$, $\widetilde{\rho}_\kappa=\widetilde{\rho_\kappa\otimes\chi}=\widetilde{\rho}_{\chi\kappa}$.

We normalize again in the same way the action on $\underline{\omega}^\kappa$ taking the action of $\widetilde{\rho}^C_\kappa(\xi^{-1})=\kappa^{-1}(\xi)\rho^C_\kappa(\xi^{-1})$ in addition to the division by $\mu(\xi)$ ($\mu=\mu_{m,n}$). Let $\mathbb{T}=\mathbb{T}(\xi)$ and write also coset representatives by ξ. Recalling $\kappa^*=\kappa\mu^{-1}$ and noting that $(\kappa^*)^{-1}(\xi)\rho_{\kappa^*}(\xi^{-1})=\kappa^{-1}(\xi)\rho_\kappa(\xi^{-1})$ for $\xi\in D$, we have

$$\begin{aligned}\omega(f)|\mathbb{T}_{top}&=\sum_\xi(\kappa\mu^{-1}(\xi))^{-1}\rho^G_{\kappa^*}(\xi^{-1})p(\xi(z))(f(\xi(z)))\mathbf{d}(\xi(z))\\&=\sum_\xi p(z)((\kappa\mu^{-1}(\xi))^{-1}\rho_{\kappa-\mu}(J(\xi,z))^{-1}f(\xi(z))\mu(\xi)^{-1}\mathbf{d}z\\&=p(z)\left(\mu(\xi)^{-1}\sum_\xi\kappa^{-1}(\xi)\rho_\kappa(J(\xi,z))^{-1}f(\xi(z))\right)\mathbf{d}z=\omega(f|\mathbb{T}_{coh}).\end{aligned}$$

In short, the extra modification of the action of the Hecke operator $T(\xi)$ by the character μ on the coherent cohomology is absorbed by $\mathbf{d}(\xi(z))=\mu(\xi)^{-1}\mathbf{d}z$ on the topological cohomology. Hence the normalization of Hecke operators at p is identical on the left-hand-side and the right-hand-side of the Eichler-Shimura map.

5. Moduli of Abelian Schemes

We recall the construction of moduli spaces of abelian schemes. The theory of moduli varieties of abelian varieties has been studied mainly by Shimura and Mumford in the years 1950's to 1960's. Shimura proved in the late 1950's to the early 1960's the existence of the moduli varieties over a canonically determined number field relative to a given endomorphism ring, a level N–structure and a polarization. This of course gives a moduli over the integer ring of the field with sufficiently large number of primes inverted.

Basically at the same time, Grothendieck studied the moduli of subschemes in a given projective scheme $X_{/S}$ (flat over S) and also that of the Picard functors. The existence of a moduli scheme, the Hilbert scheme $\mathrm{Hilb}_{X/S}$, of closed flat subschemes of $X_{/S}$ enabled Mumford, via his theory of geometric quotients of quasi-projective schemes ([GIT]), to construct moduli of abelian schemes with level N–structure over $\mathbb{Z}[\frac{1}{N}]$.

We recall here the construction of Grothendieck and Mumford briefly, limiting to the cases which we will need later. We will redo the construction of Shimura varieties with a canonical family of abelian varieties in the following lectures.

5.1. **Hilbert Schemes.** In this subsection, we describe the theory of the Hilbert scheme which classifies all closed S–flat subschemes of a given projective variety $X_{/S}$. This is a generalization of the earlier theory of Chow coordinates which classifies cycles on a projective variety. The theory is due to A. Grothendieck and main source of the exposition here is his Exposé 221 in Sém. Bourbaki 1960/61.

5.1.1. *Grassmannians.* Let $\underline{GL(n)} : ALG \to GP$ for the category of groups GP be the functor given by $\underline{GL(n)}(A) = GL_n(A)$. This functor is representable by a group scheme $GL(n) = Spec\left(\mathbb{Z}\left[t_{ij}, \frac{1}{\det(t_{ij})}\right]\right)$. We may extend the functor to the category of schemes SCH by $\underline{GL(n)}(S) = \mathrm{Aut}_{\mathcal{O}_S}(\mathcal{O}_S^n)$. We recall the notation $[*] = \{*\}/\cong$ introduced in 2.1.5 which implies the set of isomorphism classes of the objects: "$*$" in the bracket. Then we define a contravariant functor $\underline{\mathrm{Grass}}$: $SCH \to SETS$ by

$$\begin{aligned}&\underline{\mathrm{Grass}}_{\mathcal{O}^n,m}(S)\\ &\qquad = \left[\pi : \mathcal{O}_S^n \twoheadrightarrow \mathcal{F} \middle| \pi : \mathcal{O}_S\text{–linear surjective, } \mathcal{F} \text{ locally } \mathcal{O}_S\text{–free of rank } m\right].\end{aligned}$$

For each morphism $f : T \to S$, the pullback $f^*\pi : \mathcal{O}_T^n \twoheadrightarrow f^*\mathcal{F}$ gives contravariant functoriality. The quotient $\pi : \mathcal{O}_S^n \to \mathcal{F}$ is isomorphic to $\pi' : \mathcal{O}_S^n \to \mathcal{F}'$ if we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ker}(\pi) & \longrightarrow & \mathcal{O}_S^n & \xrightarrow{\pi} & \mathcal{F} & \longrightarrow & 0 \\ & & \| & & \| & & \wr\downarrow & & \\ 0 & \longrightarrow & \mathrm{Ker}(\pi') & \longrightarrow & \mathcal{O}_S^n & \xrightarrow{\pi'} & \mathcal{F}' & \longrightarrow & 0 \end{array}$$

with exact rows. The stabilizer of $\pi : \mathcal{O}_S^n \to \mathcal{O}_S^m$ can be identified with the maximal parabolic subgroup

$$P = \left\{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in GL(n) \middle| d \text{ is of size } m \times m\right\}.$$

As is well known, the quotient $\mathrm{Grass}_{\mathcal{O}^n,m} = GL(n)/P$ is a projective scheme defined over $\mathbb{Z}$ and represents the functor Grass_m, that is,

$$\mathrm{Hom}_{SCH}(S, \mathrm{Grass}_{\mathcal{O}^n,m}) \cong [\pi : \mathcal{O}_S^n \twoheadrightarrow \mathcal{F}]$$

functorially. Of course, if $m = n - 1$ or 1, we have $\mathrm{Grass}_{\mathcal{O}^n,m} = \mathbf{P}^{n-1}$.

We can generalize this construction slightly: Let S be a scheme. Let $\mathcal{E}_{/S}$ be a locally free sheaf on S of constant rank n. Then, for each S–scheme $S' \xrightarrow{f} S$, we define a contravariant functor from S–SCH to $SETS$ by

$$\begin{aligned}&\underline{\mathrm{Grass}}_{\mathcal{E},m}(S')\\&\quad=\left[\pi : f^*\mathcal{E} \twoheadrightarrow \mathcal{F}\middle|\pi : \mathcal{O}_T\text{–linear surjective, } \mathcal{F} \text{ locally } \mathcal{O}_{S'}\text{–free of rank } m\right].\end{aligned}$$

Then covering S by sufficiently small open subschemes U_i so that $\mathcal{E}_{U_i} \cong \mathcal{O}_{U_i}^n$, we have $\underline{\mathrm{Grass}}_{\mathcal{O}_{U_i}^n,m}$ represented by $\mathrm{Grass}_{m/U_i} = \mathrm{Grass}_m \times U_i$. The gluing data $g_{ij} : \mathcal{O}_{U_i \cap U_j} \cong \mathcal{E}_{U_i \cap U_j} \cong \mathcal{O}_{U_j \cap U_i}$ give rise to a Čech cocycle g_{ij} with values in $GL(n)$. This gluing datum induces a gluing datum of $\left\{\mathrm{Grass}_{m/U_i}\right\}$, giving rise to the scheme $\mathrm{Grass}_{\mathcal{E},m}$ over S which represents the above functor. One can find a detailed proof (from a slightly different view point) of what we have said here in [EGA] I.9.7.

5.1.2. *Flag Varieties.* We can further generalize our construction of the grassmannian to flag varieties. We follow [EGA] I.9.9. We consider the following functor from S–SCH to $SETS$:

$$\underline{\mathrm{Flag}}_B(S') = \left[\pi_j : \mathcal{E} \twoheadrightarrow \mathcal{E}_j \middle| \begin{matrix}\mathrm{Ker}(\pi_{j+1}) \subset \mathrm{Ker}(\pi_j), \text{ and } \mathcal{E}_j \text{ is locally}\\ \text{free of rank } n-j \ (1 \leq j \leq n-1)\end{matrix}\right].$$

Here the subscript B indicate a split Borel subgroup of $GL(n)$, since $\underline{\mathrm{Flag}}$ is represented by a projective scheme $\mathrm{Flag}_B = GL(n)/B$ if $\mathcal{E} = \mathcal{O}_S^n$. In general, we can show that

$$\mathrm{Flag}_B \hookrightarrow \mathrm{Grass}_1 \times_S \mathrm{Grass}_2 \times_S \cdots \times_S \mathrm{Grass}_{n-1}$$

given by $(\pi_j) \mapsto (\pi_{n-j} \in \mathrm{Grass}_j)_j$ is a closed immersion ([EGA] I.9.9.3). By Plücker coordinates ([EGA] I.9.8), we can embed Grass_m into the projective bundle of $\bigwedge^m \mathcal{E}$; so, Flag_B is projective.

The Flag variety is basically the quotient of $GL(n)$ by its upper triangular Borel subgroup B. We can generalize the construction to the quotient of $GL(n)$ by the unipotent radical of B. We consider the following functor:

$$\underline{\mathrm{Flag}}_U(S') = \left[(\pi_j, \phi_j)\middle|(\pi_j) \in \mathrm{Flag}_B(S') \ \text{ and } \ \phi_{j+1} : \mathrm{Ker}(\pi_j)/\mathrm{Ker}(\pi_{j+1}) \cong \mathcal{O}_{S'}\right].$$

Here we understand that $\mathrm{Ker}(\pi_0) = \mathcal{E}$, and j runs over all integers between 0 and $n-1$. If $\mathcal{E} \cong \mathcal{O}_S^n$ and S is affine, writing $\mathbf{1} = (\pi_j, \phi_j)$ for the standard flag $\pi_j : \mathcal{O}_S^n \to \mathcal{O}_S^{n-j}$, projecting column vectors down to lower $n-j$ coordinates, the upper unipotent subgroup U of $GL(n) = \mathrm{Aut}_{\mathcal{O}_S}(\mathcal{O}_S^n)$ is the stabilizer of $\mathbf{1}$. Therefore $\underline{\mathrm{Flag}}_U$ is represented by $\mathrm{Flag}_{U/S} = GL(n)/U$. In general, cover S by open affine schemes $S_i = Spec(A_i)$ so that we have an isomorphism $\mathcal{E}|_{S_i} \cong \mathcal{O}_{S_i}^n$. On $S_{ij} = S_i \cap S_j$, by the universality, $\mathrm{Flag}_{U/S_i} \times_S S_{ij}$ is canonically isomorphic to $\mathrm{Flag}_{U/S_j} \times_S S_{ij}$. Thus these schemes glue each other, giving rise to a scheme $\mathrm{Flag}_{U/S}$ representing $\underline{\mathrm{Flag}}_U$. Obviously Flag_U is a T–torsor over Flag_B for the maximal split torus $T \subset GL(n)$. Here the action of T on Flag_U is given by $(\pi_j, \phi_j) \mapsto (\pi_j, t_j\phi_j)$ for $(t_1, \ldots, t_n) \in T = \overbrace{\mathbb{G}_m \times \cdots \times \mathbb{G}_m}^{n}$. See [GME] 1.8.3 about torsors.

Let $\pi : \mathrm{Flag}_U \to \mathrm{Flag}_B$ be the projection: $(\pi_j, \phi_j) \mapsto (\pi_j)$. Then for a character κ of T, we define a sheaf $\mathcal{E}^\kappa(V) = H^0(\pi^{-1}(V), \mathcal{O}_{\mathrm{Flag}_U}[\kappa])$ for each open subset $V \subset \mathrm{Flag}_B$. Then $\mathcal{E}^\kappa$ is a locally free sheaf on Flag_B. Since $f : \mathrm{Flag}_B \to S$ is proper flat over S, we find that $f_*\mathcal{E}^\kappa$ (which we again write $\mathcal{E}^\kappa$) is a locally free sheaf on S. In this way, we can associate a κ-power $\mathcal{E}^\kappa$ of the original locally free sheaf $\mathcal{E}$, which is non-zero if and only if κ is dominant weight κ of $GL(n)$ with respect to (B,T).

5.1.3. *Flat Quotient Modules.* Let $f : X \to S$ be a flat projective scheme over a (separated) noetherian connected scheme S of relative dimension n. Here the word *"projective"* means that we have a closed immersion $\iota : X_{/S} \hookrightarrow \mathbf{P}^N_{/S}$. Thus X has a very ample invertible sheaf $\mathcal{O}_X(1) = \iota^*\mathcal{O}_{\mathbf{P}^N}(1)$. The sheaf of graded algebras $\mathcal{A} = \bigoplus_{n=0}^{\infty} f_*(\mathcal{O}_X(1)^n)$ determines X as $X = Proj_S(\mathcal{A})$.

For a given coherent sheaf $\mathcal{F}$ on X, we write $\mathcal{F}(k)$ for $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(k)$ and define a sheaf of graded modules $F = \bigoplus_{k \geq 0} F_k$ by $F_k = f_*\mathcal{F}(k)$. Then F is a graded $\mathcal{A}$–module of finite type, and we have $\mathcal{F} = \widetilde{F}$. Removing first finitely many graded pieces of F does not alter $\mathcal{F} = \widetilde{F}$. Thus defining $F(n) = \oplus_{k \geq 0} F(n)_k$ with $F(n)_k = F_{n+k}$, we have $\widetilde{F(n)} = \mathcal{F}(n)$.

We suppose that $\mathcal{F}$ is $\mathcal{O}_S$–*flat*. For each geometric point $s = Spec(k(s)) \in S$, there is a polynomial $P_{\mathcal{F}}(T)$ such that

$$\chi(\mathcal{F}(n)) = \sum_{j=0}^{\dim X(s)} (-1)^j \dim_{k(s)} H^j(X(s), \mathcal{F}(n) \otimes_{\mathcal{O}_S} k(s)) = P_{\mathcal{F}}(n).$$

For sufficiently large n, the ampleness of $\mathcal{O}(1)$ tells us that

$$H^j(X(s), \mathcal{F}(n) \otimes_{\mathcal{O}_S} k(s)) = 0 \quad \text{if } j > 0 \ .$$

Thus actually $P_{\mathcal{F}}(n)$ gives the dimension of $H^0(X(s), \mathcal{F}(n) \otimes_{\mathcal{O}_S} k(s))$, which is equal to the $\mathcal{O}_S$–rank of $f_*\mathcal{F}$ (by flatness of $\mathcal{F}$); so, $P_{\mathcal{F}}$ is independent of the choice of $s \in S$ (connectedness of S). When $S = Spec(A)$, then F is associated to a graded module, and $P_{\mathcal{F}}(n)$ is the Hilbert polynomial of this graded module (see [CRT] Section 13).

We consider the following contravariant functor for each locally noetherian S–scheme $\phi : T \to S$ (inducing $\phi_X : X_T = X \times_S T \to X$):

$$\underline{\mathrm{Quot}}_{\mathcal{F}/X/S}(T) = \left[\pi : \phi_X^*\mathcal{F} \twoheadrightarrow \mathcal{M} \middle| \mathcal{M} \text{ is a coherent } \mathcal{O}_{X_T}\text{–module flat over } \mathcal{O}_T \ \right].$$

The isomorphism between such π's are similarly defined as in the case of $\underline{\mathrm{Grass}}$.

For simplicity, we always assume that schemes T are noetherian. Each $\mathcal{M} \in \underline{\mathrm{Quot}}(T)$ has its Hilbert polynomial $P_{\mathcal{M}}$, and obviously for $g : T' \to T$, $g^*\mathcal{M}$ has the same Hilbert polynomial. Thus we can split the functor as

$$\underline{\mathrm{Quot}}_{\mathcal{F}/X/S} = \bigsqcup_P \underline{\mathrm{Quot}}^P_{\mathcal{F}/X/S},$$

where

$$\underline{\mathrm{Quot}}^P_{\mathcal{F}/X/S}(T) = \left[\pi : \phi_X^*\mathcal{F} \twoheadrightarrow \mathcal{M} \in \underline{\mathrm{Quot}}_{\mathcal{F}/X/S}(T) \middle| P_{\mathcal{M}} = P\right].$$

Here is a theorem of Grothendieck:

Theorem 5.1. *Let the notation be as above. Suppose that $X_{/S}$ is projective. Then the functor $\underline{\mathrm{Quot}}^P_{\mathcal{F}/X/S}$ is representable by a projective scheme $Q^P = \mathrm{Quot}^P_{\mathcal{F}/X/S}$ of finite type over S. Thus for any S–scheme T,*

$$\mathrm{Hom}_S(T, Q^P) \cong \left[\pi : f^*\mathcal{F} \twoheadrightarrow \mathcal{M} \in \underline{\mathrm{Quot}}_{\mathcal{F}/X/S}(T) \middle| P_{\mathcal{M}} = P\right]$$

functorially

We are going to give a sketch of the proof of this theorem. We recall $X = Proj_S(\mathcal{A})$ for a sheaf $\mathcal{A}$ of graded $\mathcal{O}_S$–algebras generated by $\mathcal{A}_1$. We cover $\underline{\mathrm{Quot}}^P$ by the subfunctor $\underline{Q}_j$ defined as follows for each positive integer j: $\underline{Q}_j(T)$ consists of isomorphism classes of $\pi : \mathcal{F}_{/X_T} \twoheadrightarrow \mathcal{M}_{/X_T}$ satisfying the following three conditions:

(a) $R^i f_{T,*}\mathcal{M}(n)_{/X_T} = 0$ for all $i > 0$ and $n \geq j$;
(b) $R^i f_{T,*}\mathcal{K}(n)_{/X_T} = 0$ for all $i > 0$ and $n \geq j$, where $\mathcal{K} = \mathrm{Ker}(\pi)$;
(c) $\mathcal{A}_k f_{T,*}(\mathcal{K}(j)) = f_{T,*}(\mathcal{K}(j+k))$ for all $k > 0$.

Write K (resp. M) for the graded $\phi^*\mathcal{A}$–module with $\widetilde{K} = \mathcal{K}$ (resp. $\mathcal{M} = \widetilde{M}$). Define $K(j)$ and $M(j)$ as above; so, $K(j) = \oplus_{k\geq 0} K(j)_k$ with $K(j)_k = K(j+k)$. First covering T by affine schemes $Spec(B_i)$ and writing B_i as a union of noetherian rings, we can reduce proofs to noetherian T; so, we may assume that T is noetherian as we remarked already. Then by a theorem of Serre ([EGA] III Section 2), for any coherent sheaf $\mathcal{G}_{/X_T}$, we have the vanishing: $R^i f_{T,*}\mathcal{G}(n)_{/X_T} = 0$ for $n \gg 0$. Thus (a) and (b) will be satisfied for a given π for a suitable j. Since $\mathcal{F}$ is coherent (and $X_{/S}$ is of finite type), it is finitely presented; so, K is finitely generated as $\phi^*\mathcal{A}$–modules, because M is finitely presented (cf. [CRT] Theorem 2.6). Thus $K(j)$ is generated by $K_j = K(j)_0$ for some j, and the last condition will be fulfilled again if $j \gg 0$. This shows that $\underline{\mathrm{Quot}}^P(T)$ is covered by $\underline{Q}_j(T)$ for each T.

The Euler characteristic is additive with respect to the exact sequence: $0 \to \mathcal{K} \to \mathcal{F} \to \mathcal{M} \to 0$: $\chi(\mathcal{F}) = \chi(\mathcal{K}) + \chi(\mathcal{M})$. Thus the conditions (a) and (b) tell us that the Hilbert polynomials $P_{\mathcal{K}}(n)$ and $P_{\mathcal{F}}(n)$ give exact $\mathcal{O}_T$–rank of $f_{T,*}\mathcal{K}$ and $f_{T,*}\mathcal{F}$. The vanishing of $R^1 f_{T,*}\mathcal{X}_{/X_T} = 0$ implies that $f_{T,*}\phi_X^*\mathcal{X} = \phi^* f_*\mathcal{X}_{/X}$ ([EGA] III, [ALG] III.12.10 or [GIT] 0.5); so, the conditions (a) and (b) are stable under base-change. The tensor product is a right exact functor; so, the surjectivity of $p_k : \mathcal{A}_k \otimes f_{T,*}\mathcal{K}(j) \to f_{T,*}\mathcal{K}(j+k)$ is also kept under base-change; so, $\underline{Q}_j$ is a well defined contravariant functor, and we have

$$\underline{\mathrm{Quot}}^P = \bigcup_j \underline{Q}_j.$$

By these three conditions (a-c), $\pi \in \underline{Q}_j(T)$ is determined by $M_j = f_{T,*}\mathcal{M}(j)$ as a flat quotient of rank $P(j)$ of $F_j = f_{T,*}\mathcal{F}(j)$. Thus $\pi \mapsto (\pi_j : F_j \twoheadrightarrow M_j)$ induces a functorial injection

$$\underline{Q}_j(T) \hookrightarrow \underline{\mathrm{Grass}}_{F_j, P(j)}(T).$$

If $\pi' : F_j \to M'$ is in the image of $\underline{Q}_j$, then π' satisfies:

(i) $F_{j+k}/\phi^*(\mathcal{A}_k)K'$ for $K' = \mathrm{Ker}(\pi')$ is locally $\mathcal{O}_T$–free of rank $P(j+k)$ for all $k \geq 0$. Here we consider $\phi^*(\mathcal{A}_k)K'$ in $\phi^*(\mathcal{A}) \cdot F$;
(ii) Define a graded module K'^* by $\phi^*(\mathcal{A}) \cdot K'$. We require the associated sheaf $\mathcal{K}' = \widetilde{K'}^*_{/X_T}$ on X_T to satisfy (b) and the quotient $\mathcal{M}' = \mathcal{F}/\mathcal{K}'$ to satisfy (a) (these (a) and (b) are open conditions).

For any graded $\phi^*(\mathcal{A})$–module $M = \bigoplus_k M_k$, defining $M^{(t)} = \bigoplus_{k\geq t} M_k$, we have $\widetilde{M^{(t)}} \cong \widetilde{M}$ as already remarked. By this fact, the condition (i) assures that $f_{T,*}\mathcal{M}$ is locally $\mathcal{O}_T$–free, and the image of $\underline{Q}_j(T)$ is characterized by (i) and (ii).

We are going to prove the representability of $\underline{Q}_j$, assuming that $j = 0$. The general case follows from the same argument replacing 0 by j everywhere. Let $\pi^{univ} : F_{0/G} \twoheadrightarrow M_0$ ($F_{0/G} = F_0 \otimes_{\mathcal{O}_S} \mathcal{O}_G$) be the universal object defined over

$G = \text{Grass}$. Here we have changed our notation and write M_0 for the universal quotient of $F_{0/G}$ (with rank $P(0)$). Thus for any morphism $\pi' : F_0 \to M_0'$ on T with M_0' locally-free of rank $P(0)$, we have a unique S–morphism $\phi : T \to \text{Grass}$ such that $\pi' = \phi^* \pi^{univ}$. Let $K = \mathrm{Ker}(\pi^{univ})$. Write $g : G = \text{Grass} \to S$ for the structure morphism. We consider the subset:

$$Z = \left\{ s \in G \middle| \begin{array}{l} \dim_{k(s)}(\mathcal{A}_k F_{0/G}/(\mathcal{A}_k K)) \otimes k(s) = P(k) \text{ for all } k \geq 0 \\ \text{and the stalk } (\mathcal{A}_k F_{0/G}/(\mathcal{A}_k K))_s \text{ is free for all } k \geq 0 \end{array} \right\}.$$

Write $M_k = \mathcal{A}_k F_{0/G}/(\mathcal{A}_k K)$ and put $M = \bigoplus_{k \geq 0} M_k$. Then $M_k = \mathcal{A}_k M_0$. The $\mathcal{O}_G$–module M is flat on a generic point of $Z_0 = Supp(M) = G$. Since flatness is an open condition, we find an open connected subscheme $V_0 \subset Z_0$ which is maximal among open subschemes V over which M is flat. Repeating this process, replacing M and Z_0 by $M \otimes_{\mathcal{O}_{Z_0}} \mathcal{O}_{Z_1}$ and $Z_1 = Z_0 - V_0$, we can split $Z_0 = \bigsqcup_i V_i$ into a finite disjoint union of connected subschemes V_i so that $M \otimes_{Z_0} \mathcal{O}_{V_i}$ is flat over V_j. Then we find a polynomial $Q_i(n)$ such that

$$\mathrm{rank}_{\mathcal{O}_{V_i}}(M_n \otimes_{Z_0} \mathcal{O}_{V_i}) = Q_i(n) \quad \text{if } n \geq n_i.$$

By this fact, the open subscheme

$$U_N = \left\{ s \in Z_0 \middle| \dim_{k(s)}(M_n \otimes_{\mathcal{O}_{Z_0}} k(s)) \leq P(n) \ 0 \leq \forall n \leq N \right\}$$

stabilizes as N grows. Therefore on an open (dense) subscheme $U = U_\infty$ of Z_0, if $n \geq 0$, we have $\dim_{k(s)}(M_n \otimes_{\mathcal{O}_{Z_0}} k(s)) \leq P(n)$ for all $s \in U$. Then we have an exact sequence:

$$\mathcal{O}_U^q \xrightarrow{p_k} \mathcal{O}_U^{P(k)} \longrightarrow M_k \longrightarrow 0,$$

and Z is the closed subscheme of U on which all matrix coefficients of p_k vanishes for all $k \geq 0$. Thus the image of Q_0 fall into $\underline{Z}$. The condition (ii) can be checked to be satisfied on an open subscheme of Z. Thus we have

Theorem 5.2. *The functor $\underline{Q}_j$ is represented by a quasi-projective scheme Q_j of finite type over S.*

Here the word *"quasi-projective"* means that the scheme has an open immersion into a projective scheme. Since Grass is projective, Q_j is quasi-projective.

The next step is to show that the increasing sequence of quasi-projective schemes $\{Q_j\}_j$ stabilizes after $j \geq N_0$; so, $\underline{\mathrm{Quot}}^P$ is represented by a quasi-projective scheme. The key point of the argument is to show that for any given set of coherent sheaves on X, each of whose members appear as a fiber of a coherent sheaf $\mathcal{L}$ on an extension X_T (for an S–scheme T), we can take T to be of finite type over S. This is an involved argument; so, we refer the audience to the first section of the paper of Grothendieck already quoted at the beginning. Once this is shown, $\bigcup_j Q_j$ has to be quasi-compact (covered by T as above), and the union is finite; so, Quot^P itself is of finite type and quasi-projective.

We can check this fact for $X = \mathbf{P}^n_{/S}$ and $\mathcal{F} = \mathcal{O}_X$ in a different manner. In this case, writing $X = Proj(\mathcal{O}_S[T_0, \ldots, T_n])$ and $D_i \subset X$ for $Spec\left(\mathcal{O}_S\left[\frac{T_0}{T_i}, \ldots, \frac{T_n}{T_i}\right]\right)$, then $R^i f_* \mathcal{O}(j) = 0$ for all $i > 0$ if $j > n+1$ by a computation of cohomology groups by Čech cohomology with respect to the covering $X = \bigcup_{j=0}^n D_j$ (see [ALG] III.5). A version of the argument of Grothendieck for $X = \mathbf{P}^n$ to prove (a) and (b) for sufficiently large j for all $\mathcal{M}$ and $\mathcal{K}$ is as follows: Since $P_{\mathcal{F}} = P_{\mathcal{K}} + P_{\mathcal{M}}$ with $P_{\mathcal{M}} = P$, $P_{\mathcal{K}}$ is determined by P. Choosing homogeneous generators $x_1, \ldots, x_r$ of degree $-p$ of $\mathcal{K}$, we have a surjection: $\mathcal{O}(p)^r \twoheadrightarrow \mathcal{K}$ taking $(a_1, \dots, a_r) \mapsto \sum_{i=1}^r a_i x_i$. Here r and p are determined by the first two leading terms of $P_{\mathcal{K}}$ and hence those of P. Let $\mathcal{K}_0 = \mathrm{Ker}(\pi)$. Then $rP_{\mathcal{O}(p)} = P_{\mathcal{K}_0} + P_{\mathcal{K}}$. Let $r_0 = r$ and $p = p_0$. The polynomial $P_{\mathcal{K}_0}$ is determined by $P_{\mathcal{K}}$. Thus the first two leading terms of $P_{\mathcal{K}_0}$ are bounded below and above independent of $\mathcal{K}$.

Repeating this argument, we find an integer $N_0 \gg 0$ such that for integers $p_i > -N_0$ $(i = 0, 1, \dots, n)$ we have the following exact sequences:

$$0 \to \mathcal{K}_i \to \mathcal{O}(p_i)^{r_i} \to \mathcal{K}_{i-1} \to 0$$

with $\mathcal{K}_{-1} = \mathcal{K}$. Then by the associated long exact sequence, if $j \geq N_0 + n + 1$, $R^{q+1} f_* \mathcal{K}_i(j) \cong R^q f_* \mathcal{K}_{i-1}(j)$. Since cohomological dimension of $\mathbf{P}^n$ is n (as easily checked by Čech cohomology), for $i > 0$,

$$0 = R^{n+i+1} f_* \mathcal{K}_n(j) = R^{n+i} f_* \mathcal{K}_{n-1}(j) = \cdots = R^i f_* \mathcal{K}(j).$$

By the same argument, $R^i f_* \mathcal{M}(j) = 0$ for all $i > 0$ and all $j > N_0 + n + 1$.

Since $D_i \cap X$ is affine for any projective scheme $X_{/S} \subset \mathbf{P}^n_{/S}$, the same argument works for X in place of $\mathbf{P}^n$, and $\underline{\mathrm{Quot}}^P_{\mathcal{O}_X/X/S}$ is represented by a quasi-projective scheme (see [PAF] pages 261-262). It is customary to identify $\pi \in \mathrm{Quot}_{\mathcal{O}_X/X/S}(T)$ with a closed immersion of $\mathit{Spec}_Q(\mathrm{Im}(\pi))$ into X; so, $\mathrm{Quot}^P_{\mathcal{O}_X/X/S}$ represents the following contravariant functor

$$\underline{\mathrm{Hilb}}^P_{X/S}(T) = \{\text{closed subschemes of } X_T \text{ flat over } T \text{ with Hilbert polynomial } P\}.$$

This scheme is called the *Hilbert scheme* of X for the polynomial P.

We now finish the proof of the following theorem.

Theorem 5.3 (Grothendieck)**.** *For each projective scheme $X_{/S}$ for a noetherian connected scheme S and a numerical polynomial $P(t) \in \mathbb{Q}[t]$, the functor $\underline{\mathrm{Hilb}}^P_{X/S}$ is represented by a projective scheme $\mathrm{Hilb}^P_{X/S}$ over S.*

Proof. We only need to prove the projectivity by the valuative criterion. Let $\pi : \mathcal{O}_{X_\eta} \to \mathcal{M}_{/\eta} \in \underline{\mathrm{Quot}}^P_{\mathcal{O}_X/X/S}(\eta)$ for $\eta = \mathit{Spec}(K)$ of the field K of fractions of a discrete valuation ring V. Then we define $\mathrm{Ker}(\pi)_{/S}$ for $T = \mathit{Spec}(V)$ by the largest subsheaf over T of $\mathcal{O}_{X_T}$ inducing $\mathrm{Ker}(\pi)$, that is, $\mathcal{O}_{X_T} \cap \mathrm{Ker}(\pi)$, which is a coherent sheaf with quotient $\mathcal{M}_{/T}$ locally free over X_T inducing $\mathcal{M}_{/\eta}$ after tensoring K, because V is a discrete valuation ring. Thus the point $\pi \in \underline{\mathrm{Quot}}^P_{\mathcal{O}_X/X/S}(\eta)$ extends to $\underline{\mathrm{Quot}}^P_{\mathcal{O}_X/X/S}(T)$. Since $\mathrm{Quot}^P = \bigcup_j Q_j$ is quasi-projective, it is separated; so, it is proper. Since $\mathrm{Quot}^P_{\mathcal{O}_X/X/S}$ is quasi-projective, it has to be projective. $\square$

5.1.4. *Morphisms between Schemes.* In this section, we first consider the contravariant functor $\underline{\mathrm{Sec}}_{X/Y/S}, \underline{\mathrm{Hom}}_S(X, Y) : S\text{–}\mathit{SCH} \to \mathit{SETS}$ given by

$$\underline{\mathrm{Sec}}_{X/Y/S}(T) = \mathrm{Hom}_{Y_T}(Y_T, X_T) \quad \text{and} \quad \underline{\mathrm{Hom}}_S(X, Y)(T) = \mathrm{Hom}_T(X_T, Y_T).$$

Here for $\underline{\mathrm{Sec}}$, X is supposed to be an S–scheme over Y. The latter is a special case of the former because

$$\mathrm{Hom}_T(X_T, Y_T) = \underline{\mathrm{Sec}}_{Y_X/X}(T) \quad (Y_X = Y \times_S X).$$

Each section $s : Y \to X$ defines a closed subscheme of $X_{/S}$ isomorphic to Y via the given projection $f : X \to Y$. Write $H = \mathrm{Hilb}_{X/S} = \bigsqcup_P \mathrm{Hilb}^P_{X/S}$. Then we have the universal closed subscheme Z of $X_H = X \times_S H$ satisfying the commutative diagram:

$$\begin{array}{ccc} Z & \xrightarrow{\subset} & X_H \\ \downarrow & & \downarrow \\ H & = & H \end{array}$$

such that for any S–scheme T and a closed subscheme $W \hookrightarrow X_T$ flat over T, we have a unique morphism $\phi_W : T \to H$ over S such that the pull back of the above square by ϕ_W is identical to

$$\begin{array}{ccc} W & \xrightarrow{\subset} & X_T \\ \downarrow & & \downarrow \\ T & = & T. \end{array}$$

We consider S–subschemes $U \subset H$ such that $\widetilde{f}_U : Z_U \subset X_U \xrightarrow{f_U} Y_U$ for a given $f : X \to Y$ induces an isomorphism $\widetilde{f}_U : Z_U \cong Y_U$. From this, it is easy to see that $\underline{\mathrm{Sec}}_{X/Y/S}$ is represented by the maximal subscheme U of $H_{/S}$ with this property $\widetilde{f}_U : Z_U \cong Y_U$. For each closed point $x \in H$, if $\widetilde{f}_x$ is an isomorphism, it is an isomorphism on an open neighborhood of x; so, U is an open subscheme of H. Since $\mathrm{Hilb}^P_{X/S}$ is projective over S, and $U \cap \mathrm{Hilb}^P_{X/S}$ is open, each connected component of U is an open-subscheme of the projective scheme $\mathrm{Hilb}^P_{X/S}$ for some P; so, each connected component of U is quasi-projective over S. Thus we get

Theorem 5.4. *Let $X_{/S}$ and $Y_{/S}$ be projective schemes over a connected noetherian scheme S. Then the functors $\underline{\mathrm{Sec}}_{X/Y/S}$ and $\underline{\mathrm{Hom}}_S(X, Y)$ are representable by schemes $\mathrm{Sec}_{X/S}$ and $\mathrm{Hom}_{X/S}$ over S, respectively. Each connected component of $\mathrm{Sec}_{X/S}$ and $\mathrm{Hom}_{X/S}$ is quasi-projective over S.*

By construction, the scheme representing these functors may not be of finite type over S, because $\mathrm{Hilb}_{X/S}$ could have infinitely many components. However each connected component of the scheme is of finite type over S.

Corollary 5.5. *Let the notation and the assumption be as in the theorem. Then the functor: $T_{/S} \mapsto \mathrm{Hom}_T(X_T, X_T)$ is represented by a scheme $E_{X/S}$ over S whose connected components are quasi-projective over S.*

If a section $s : S \hookrightarrow X$ is given, keeping representability, we can insist an endomorphism $\phi \in E_T(X_T)$ to take s_T to s_T. This goes as follows: Consider the functorial map: $\underline{E}_{X/S}(T) \ni \phi \mapsto \phi(s_T) \in \underline{X}(T)$ which induces a morphism $\sigma : E_{X/S} \to X$. Then writing the set of endomorphisms keeping s as $\underline{E}^s_T(X_T)$, the functor: $T \mapsto \underline{E}^s_T(X_T)$ is again representable by a scheme

$$E^s_{X/S} = E_{X/S} \times_{X,\sigma,s} S$$

over S.

5.1.5. *Abelian Schemes.* An abelian scheme $X_{/S}$ is a smooth geometrically connected group scheme proper over a separated locally noetherian base S.

We can drop "local noetherian" hypothesis, because a smooth geometrically connected and proper group scheme over any base is a base change of such a scheme over a locally noetherian base (cf. [DAV] I.1.2).

We actually suppose that S is a noetherian scheme for simplicity. Since X is a group, it has the identity section $\mathbf{0}: S \to X$. As in the elliptic curve case, any S–morphism $\phi: X \to X'$ of abelian schemes is a homomorphism if $\phi(\mathbf{0}_X) = \phi\circ\mathbf{0} = \mathbf{0}_{X'}$ (by Rigidity lemma: [ABV] Section 4, [GIT] 6.4 and [GME] 4.1.5). In particular, if X is an abelian scheme over S, every scheme endomorphism of $X_{/S}$ keeping the zero section is a homomorphism of group structure. Thus $E^{\mathbf{0}}_{X/S}$ is a ring scheme associated to the functor: $T \mapsto \underline{E}^{\mathbf{0}}_T(X_T)$ with values in the category of rings.

Assume that X is an abelian scheme over a connected noetherian base S. Take a connected component $E \subset E^{\mathbf{0}}_{X/S}$. Each connected component of $E_{X/S}$ is quasi-projective over S. Since S is noetherian, E is of finite type over S, because of our construction:

$$E^{\mathbf{0}}_{X/S} = E_{X/S} \times_{X,\sigma,\mathbf{0}} S.$$

Suppose we have a discrete valuation ring A with field of fractions K and a morphism $\eta: Spec(K) \to E$ which is over a morphism $i: Spec(A) \hookrightarrow S$. In other words, we have the following commutative diagram:

$$\begin{array}{ccc} Spec(K) & \xrightarrow{\eta} & E \\ \big\downarrow & & \big\downarrow \\ Spec(A) & \xrightarrow[i]{} & S. \end{array}$$

Then η gives rise to a section of $\underline{E}^{\mathbf{0}}_{X/S}(K)$. Since homomorphisms of abelian schemes are kept under specialization (which we call the rigidity of endomorphism; see [GME] Subsections 4.1.5-6 and [DAV] I.2.7), η extends to $Spec(A)$ uniquely. By the valuative criterion of properness, we find that E is projective over S. If ϕ is an endomorphism of the abelian scheme $X_{/S}$, $\mathrm{Ker}(\phi)$ is again a group scheme. If $\dim_S \mathrm{Ker}(\phi) = 0$, $\mathrm{Ker}(\phi)$ is a locally-free group scheme of finite rank; in this case, we call ϕ an *isogeny*. We define the degree $\deg(\phi)$ of ϕ by the rank of $\mathrm{Ker}(\phi)$ over S in this case. If $\dim_S \mathrm{Ker}(\phi) > 0$, we simply put $\deg(\phi) = 0$. If the connected component $E \subset E^{\mathbf{0}}_{X/S}$ contains an isogeny, the degree is independent of the point of E. As is well known, for any abelian variety over a field, the number of isogeny with a given positive degree is finite. Thus E is projective and quasi-finite; so, E is finite over S ([GME] Proposition 1.9.11).

Corollary 5.6. *Let $X_{/S}$ be an abelian scheme over a connected noetherian base S. Then the functor $T \mapsto End_T(X_T)$ is represented by a scheme $End_{X/S} = E^{\mathbf{0}}_{X/S}$ over S. Each connected component of $End_{X/S}$ is projective over S. If the connected component contains an isogeny, it is finite over S. Here $End_T(X_T)$ denotes endomorphisms of X compatible with group structure on X.*

The subscheme E corresponds to a section x_E of $\underline{\mathrm{End}}_{X/S}(X_E) = \mathrm{End}_E(X_E)$. We assume that $x_E: X_E \to X_E$ is an isogeny; so, $\mathrm{Ker}(x_E)$ is a locally-free group scheme over E, and E is finite over S. We consider the subalgebra $\mathcal{A}_E = \mathbb{Z}[x_E] \subset \underline{\mathrm{End}}(X_E)$. Since E is connected, $\mathrm{End}_E(X_E)$ is an algebra free of finite rank over $\mathbb{Z}$ (see [ABV] Section 19). We suppose that $\mathcal{B} = \mathcal{A}_E \otimes_{\mathbb{Z}} \mathbb{Q}$ is semi-simple and commutative. Thus it is a product of finitely many number fields, and hence the algebra automorphism group $\mathrm{Aut}(\mathcal{B})$ is finite.

Let us fix a commutative algebra A free of finite rank over $\mathbb{Z}$ with semi-simple $B = A \otimes_{\mathbb{Z}} \mathbb{Q}$. Suppose that $A = \mathbb{Z}[x]$ for a single generator x. Let $\mathbb{E}_x \subset \mathrm{End}_{X/S}$ be the collection of all connected components E such that $\theta_E : A \cong \mathcal{A}_E$ with $\theta(x) = x_E$. A priori, the scheme $\mathbb{E}_{x/S}$ may have infinitely many connected components, although we later see that up to inner automorphisms of $\mathrm{End}(X_{/S})$, the number of components are finite using the fact that $\mathrm{End}(X_{s/s}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a semi-simple algebra of bounded dimension, where $s \in S$ is a geometric point and X_s is the abelian variety fiber X_s over $s \in S$. Suppose that we have an embedding $\theta_T : A \hookrightarrow \mathrm{End}_T(X_T)$ for an S–scheme $T_{/S}$. Then by the rigidity of endomorphisms, $\theta_T(x)$ is a T–point of $\mathbb{E}_x$. In other words, $\mathbb{E}_{x/S}$ represents the following functor

$$\mathcal{F}_A(T) = \left\{\theta_T : A \hookrightarrow \mathrm{End}_T(X_T) \middle| \theta_T(1_A) = \mathrm{id}_{X_T}\right\}$$

from $SCH_{/S}$ into $SETS$. On $\mathcal{F}_A$, the finite group $\mathrm{Aut}(A)$ of algebra automorphisms acts by $\theta_T \mapsto \theta_T \circ \sigma$.

We can generalize the above argument to any algebra A free of finite rank over $\mathbb{Z}$ with semi-simple $A \otimes_{\mathbb{Z}} \mathbb{Q}$. We take a finite set of generators $\{x_1, \ldots, x_j\}$ and consider $\mathbb{E} = \mathbb{E}_{x_1} \times_S \mathbb{E}_{x_2} \times_S \cdots \times_S \mathbb{E}_{x_j}$. Then we define $\mathbb{E}_A$ to be the maximal subscheme of $\mathbb{E}$ such that we have an algebra embedding $\theta : A \hookrightarrow \mathrm{End}_{\mathbb{E}_A}(X_{\mathbb{E}_A})$ taking x_i to $x_{i,\mathbb{E}_A}$ for all i. Then we have

Corollary 5.7. *Let S be a noetherian scheme, and $X_{/S}$ be an abelian scheme over S. Let A be an algebra free of finite rank over $\mathbb{Z}$ with semi-simple $A \otimes_{\mathbb{Z}} \mathbb{Q}$. Then the functor*

$$T_{/S} \mapsto \left[(X_T, \theta : A \hookrightarrow End_T(X_T)) \middle| \theta(1_A) = \mathrm{id}_X\right]$$

is representable by a scheme $\mathbb{E}_A$ over S, and each connected component of $\mathbb{E}_A$ is finite over S.

A *semi-abelian scheme $X_{/S}$* is a smooth separated group scheme with geometrically connected fiber such that each geometric fiber is an extension of an abelian variety by a torus. The toric rank may depend on the fiber. We suppose that $X_{/S}$ is a semi-abelian scheme. It is known that any homomorphism of semi-abelian schemes: $X_U \to Y_U$ defined over an open dense subscheme $U \subset S$ extends uniquely to $X_{/S} \to Y_{/S}$ as long as S is normal (endomorphisms are kept under specialization; a result of M. Raynaud: [DAV] I.2.7). Thus if $X_{/S}$ is an abelian scheme a normal scheme S, we have a unique extension of the scheme $\mathrm{End}_{X_U/U}$ over a dense open subscheme U of S to the scheme $\mathrm{End}_{X/S}$ over S which represents the functor in the above corollary. Applying the valuative criterion using this rigidity of endomorphisms, we find that $\mathrm{End}_{X/S}$ has connected components each projective over S. Suppose that we have an embedding $\theta_s : A \hookrightarrow \mathrm{End}_{X(s)/s}(X(s))$ for a geometric point $s \in S$ with abelian variety fiber $X(s)$. Then by the rigidity of endomorphisms, the maximal connected subscheme $Z \subset S$ containing s such that θ_s extends to the embedding $\theta : A \hookrightarrow \mathrm{End}(X_{/Z})$ is a closed connected subscheme $Z \subset S$. Thus each connected component of $\mathrm{End}_{X/S}$ is projective over S. In the same manner as in the case of an abelian scheme $X_{/S}$, we can prove that each connected component of $\mathrm{End}_{X/S}$ is quasi-finite; so, it is finite over S. Thus we get (see [PAF] Corollary 6.10)

Corollary 5.8. *Let $X_{/S}$ be a semi-abelian scheme with abelian variety fiber over a dense open subset of S. If an abelian variety fiber $X(s)$ has an inclusion $A \hookrightarrow End_s(X(s))$, then the functor*

$$T_{/S} \mapsto [(X_T, \theta : A \hookrightarrow End_T(X_T))]$$

is represented by a scheme over S. Each connected component of this scheme is finite over S.

5.2. **Mumford Moduli.** We describe the Mumford construction of the moduli over $\mathbb{Z}$ of abelian schemes of dimension n with a given polarization of degree d^2.

5.2.1. *Dual Abelian Scheme and Polarization.* We consider the following Picard functor:

$$\mathrm{Pic}_{X/S}(T) = \mathrm{Pic}(X_T)/f_T^* \mathrm{Pic}(T) = \mathrm{Ker}(\mathbf{0}_T^*)$$

for $f : T \to S$. It is known that $\mathrm{Pic}_{X/S}$ is represented by a (locally noetherian) reduced group scheme (Grothendieck: Bourbaki Sem. Exp. 232, 1961/62; [ABV] Section 13, [GIT] 0.5 and [DAV] I.1). Let $\widehat{X}$ be the identity connected component $\mathrm{Pic}^\circ_{X/S}$ of the group scheme representing $\mathrm{Pic}_{X/S}$. Then $\widehat{X}_{/S}$ is an abelian scheme.

Let $\mathcal{L}^{univ} \in \widehat{X}(\widehat{X}) = \mathrm{Pic}^\circ_{X\times_S\widehat{X}/\widehat{X}} = \mathrm{Ker}(\mathbf{0}^*_{X\times_S\widehat{X}})$ corresponding to the identity. Then the sheaf $\mathcal{L}^{univ}$ is characterized by the following two properties:

- $\mathbf{0}^*_{X\times_S\widehat{X}}\mathcal{L}^{univ} = \mathcal{O}_{\widehat{X}}$;
- Let $T \to S$ be an S–scheme. For any invertible sheaf L on X_T algebraically equivalent to $\mathcal{O}_{X_T}$, there exists a unique morphism $\iota_L : T \to \widehat{X}$ such that $(\iota \times id_X)^*\mathcal{L}^{univ} = L$.

Let L be an invertible sheaf on X. For $x \in X$, we define the translation $T_x(y) = x+y$, which is an automorphism of X. Then $(T_x^*L)\otimes L^{-1}$ is an element in $\widehat{X}$, and we obtain a morphism $\Lambda(L) : X \to \widehat{X}$. This S–homomorphism is an *isogeny* (that is, surjective) if and only if L is *ample* (that is, choosing a base of $H^0(X, L^{\otimes n})$ for sufficiently large n, one can embed X into $\mathbf{P}^m_{/S}$ locally on S; see [ABV] Section 6). The degree of the polarization is defined to be the square root of the degree of the homomorphism $\Lambda(L)$.

There is another construction of $\Lambda(L)$. Consider $a^*(L) \otimes p_1^*(L) \otimes p_2^*(L)^{-1}$ as an invertible sheaf on $X_X = X \times_S X$, where $a : X \times X \to X$ is the addition on the group scheme X. Then this invertible sheaf induces an X–valued point of $\mathrm{Pic}_{X/S}(X)$, which factors through $\widehat{X}$, because at the identity, this sheaf specializes to the trivial invertible sheaf (so, the image is in the connected component of $\mathrm{Pic}_{X/S}$). We claim this X–valued point of $\widehat{X}$ is actually $\Lambda(L)$. By specializing this sheaf at $x : S \to X \in X(S)$, we get $T_x^*(L) \otimes L^{-1} \otimes x^*(L)$, which is equivalent in $\mathrm{Pic}(X)/f^*\mathrm{Pic}(S)$ to $T_x^*L \otimes L^{-1}$, as desired.

5.2.2. *Moduli Problem.* We fix three positive integers n, d and N. We consider the following functor over $Spec(\mathbb{Z}[\frac{1}{N}])$:

$$\mathcal{A}_{d,N}(S) = \left[(X, \phi_N : (\mathbb{Z}/N\mathbb{Z})^{2n} \cong X[N], \lambda)_{/S}\right]$$

where

1. $X_{/S}$ is an abelian scheme with $\dim_S X = n$,
2. ϕ_N is an isomorphism over S,
3. λ is a polarization, étale locally $\lambda = \Lambda(\mathcal{L})$ on S and $\deg(\lambda) = d^2$.

It is known that if λ is locally of the form $\Lambda(L)$, then 2λ is globally $\Lambda(L^\Delta(\lambda))$ for the invertible sheaf $L^\Delta(\lambda)$ given by $(1_X \times \lambda)^*(\mathcal{L}^{univ})$ (see [GIT] Proposition 6.10).

Here is a theorem of Mumford:

Theorem 5.9 (Mumford)**.** *There exists a quasi-projective scheme M_N over $\mathbb{Z}[\frac{1}{N}]$ such that*

1. *For any geometric point $s = Spec(k)$ of $Spec(\mathbb{Z}[\frac{1}{N}])$, $\mathcal{A}_{d,N}(k) \cong M_N(k)$;*

2. *If $N \geq 3$, there exists a universal object $(\mathbf{X}, \boldsymbol{\phi}_N, \boldsymbol{\lambda})_{/M_N}$ such that for each triple $(X, \phi_N, \lambda) \in \mathcal{A}_{d,N}(S)$ there exists a unique morphism $\iota : S \to M_N$ such that*
$$(X, \phi_N, \lambda) \cong (\mathbf{X}, \boldsymbol{\phi}, \boldsymbol{\lambda}) \times_{M_N} S = \iota^*(\mathbf{X}, \boldsymbol{\phi}, \boldsymbol{\lambda}).$$
3. *The above association: $(X, \phi_N, \lambda)_{/S} \mapsto \iota$ induces a functorial isomorphism: $\mathcal{A}_{d,N} \cong \underline{M}_N$, where $\underline{M}_N(S) = \mathrm{Hom}_{\mathbb{Z}[\frac{1}{N}]}(S, M_N)$.*

We are going to give a sketch of the proof of the above theorem. Let $X_{/A}$ be an abelian scheme over a ring A. The key idea is that for a given very ample invertible sheaf $L_{/X}$, the embedded image of X under L in $\mathbf{P}^m$ for $m+1 = \mathrm{rank}_A H^0(X, L)$ is determined just by the choice of basis b of $H^0(X, L)$. In other words, the images of the embeddings associated to different basis are transported each other by an element of $PGL_{m+1}(A)$. Since for an abelian scheme, by the generalized Riemann-Roch theorem (see [ABV] Section 16), we can compute the Hilbert polynomial P of L, the moduli functor of (X, b) is a subfunctor of $\underline{\mathrm{Hilb}}^P_{\mathbf{P}^m}$. Proving that the image is a quasi-projective subscheme H of $\mathrm{Hilb}^P_{\mathbf{P}^m}$, the moduli of X is constructed as $M_1 = PGL(m+1)\backslash H$. This an outline of what we are going to do.

5.2.3. *Abelian Scheme with Linear Rigidification.* Let $(X, \phi_N, \lambda)_{/S} \in \mathcal{A}_{d,N}(S)$, and write $f : X \to S$ for the structure morphism. We consider the invertible sheaf $L = f_* L^\Delta(\lambda)^3$ on S of rank $6^n d$. The sheaf $L^\Delta(\lambda)^3$ is very ample, because $\mathcal{L}^3$ is very ample if $\mathcal{L}$ is ample (see [ABV] Section 17) and ampleness of $L^\Delta(\lambda)$ follows from $\Lambda(L^\Delta(\lambda)) = 2\lambda$ as we already remarked.

Let $Sym(L)$ be the symmetric algebra: $\bigoplus_{k=0}^\infty L^k$ and put $\mathbf{P}(L) = Proj(Sym(L))$ which is a projective scheme over S locally isomorphic to $\mathbf{P}^m_{/S}$. A *linear rigidification* is an isomorphism $\iota : \mathbf{P}(L) \cong \mathbf{P}^m_{/S}$. Thus ι is determined by the choice of a base of L up to scalar multiplication.

Since the very ample sheaf $L^\Delta(\lambda)^3$ on an abelian scheme satisfies ([ABV] Section 16):

- $L = f_*(L^\Delta(\lambda)^3)$ is locally free of finite rank;
- $R^i f_*(L^\Delta(\lambda)^3) = 0$ if $i > 0$,

the formation of $f_*(L^\Delta(\lambda)^3)$ as above commutes with base change. Thus the association
$$\mathcal{A}'_{d,N}(S) = \left[(X, \phi_N, \lambda, \iota)_{/S}\right]$$
is a well defined contravariant functor.

The embedding $I : X_{/S} \hookrightarrow \mathbf{P}(L) \xrightarrow{\iota} \mathbf{P}^m_{/S}$ determines the sheaf $L^\Delta(\lambda)^3 = \mathcal{O}(1)$, which in turn determines λ because $\Lambda(L) = 6\lambda$ ($\mathrm{Pic}_{X/S}/\widehat{X}$ is torsion free). Having ϕ_N is equivalent to having $2n$ (linearly independent) sections $\sigma_j = \phi_N(e_j)$ of X over S for the standard base $\{e_1, \dots, e_{2n}\}$ of $(\mathbb{Z}/N\mathbb{Z})^{2n}$. We write $\sigma_0 = e$ for the identity section of X.

We record here what we have seen:

Proposition 5.10. *The data $(X, \phi_N, \lambda, \iota)$ is determined by the embedding*
$$(I : X_{/S} \hookrightarrow \mathbf{P}^m_{/S}, \sigma_0, \sigma_1, \dots, \sigma_{2n}).$$

In other words, defining a new functor by
$$\mathcal{H}_{d,N}(S) = \left[(I : X_{/S} \hookrightarrow \mathbf{P}^m_{/S}, \sigma_0, \sigma_1, \dots, \sigma_{2n})_{/S}\right],$$
we have an isomorphism of functors: $\mathcal{A}'_{d,N} \cong \mathcal{H}_{d,N}$.

5.2.4. *Embedding into the Hilbert Scheme.* For simplicity, we just write $\mathbf{P}$ for $\mathbf{P}^m_{\mathbb{Z}[\frac{1}{N}]}$. We write S_0 for $Spec(\mathbb{Z}[\frac{1}{N}])$. We consider the functor $\underline{\mathrm{Hilb}}^P_{\mathbf{P}}$ associating to each S the set of closed subschemes of $\mathbf{P}_{/S}$ flat over S with Hilbert polynomial P. As we have already seen, this functor is represented by a projective scheme $H = \mathrm{Hilb}^P_{\mathbf{P}}$ over $\mathbb{Z}$. Write $Z \to H$ for the universal flat family inside $\mathbf{P}_{/H}$ with Hilbert polynomial P. For each subscheme $V \subset \mathbf{P}_{/S}$ flat over S having Hilbert polynomial P, we have a unique morphism $h : S \to H$ such that V is given by to $S \times_H Z \subset \mathbf{P}_{/S}$ over S.

By the generalized Riemann-Roch theorem ([ABV] Section 16), the Hilbert polynomial of (X, L) (or of the image $I(X)$) is given by

$$P(T) = 6^n dT^n.$$

Thus the image $I(X)$ induces a unique morphism $h : S \to H$ such that $I(X) = S \times_H Z$ in $\mathbf{P}_{/S}$.

Let $H^k = \mathrm{Hilb}^{P,k}_{\mathbf{P}} = \overbrace{Z \times_H Z \times_H \cdots \times_H Z}^{k}$. Then by the very definition of the fiber product, we get

$$\mathrm{Hom}_{S_0}(S, H^k) = \left\{(h, s_1, \ldots, s_k)\middle| h \in \mathrm{Hom}_{S_0}(S, H),\ s_j \in \mathrm{Hom}_S(S, Z)\right\},$$

where $h : S \xrightarrow{s_j} Z \xrightarrow{p} H$ for the projection p of Z to H (so, h is determined by any of s_j). Thus H^k classifies all flat closed subschemes of $\mathbf{P}$ with Hilbert polynomial P having k sections over S. The universal scheme over H^k with k sections is given by $Z^{(k)} = Z \times_H H^k$. It has k sections:

$$\tau_i : H^k \ni z \mapsto (z_i, z) \in Z^{(k)}\ (z = (z_1, \ldots, z_k)).$$

This shows that $\mathcal{H}_{d,N} \subset \underline{H}^k$ for $k = 2n+1$. For simplicity, write H_0 for H^k. Since "smoothness" is an open condition (because it is local; in other words, smoothness at a point x of a morphism f follows from formal smoothness of the local ring at x over the local ring at $f(x)$), there is an open subscheme H_1 of H_0 over which Z is smooth. Then H_1 represents smooth closed subschemes in $\mathbf{P}$ with Hilbert polynomial P and k sections.

Now we use a result of Grothendieck. Abelian varieties have rigidity such that if in a smooth projective family $X \to S$ for connected locally noetherian S with a section $e : S \to X$, if one fiber is an abelian variety with the identity section induced by e, X itself is an abelian scheme ([GIT] Theorem 6.14). This shows that over H_1, H_1 has a closed subscheme H_2 over which $Z_2 = Z_{H_2}$ is an abelian scheme with the identity section e inducing τ_0 on X.

Let τ_j $(j = 1, \ldots, 2n)$ be the universal $2n$ sections of $Z^{(k)}$. We have a maximal closed subscheme $H_3 \subset H_2$ with $[N] \circ \tau_i = e$, where $[N]$ is the multiplication by the integer N.

The relation $\sum_{j=1}^{2n} a_j \tau_j = e$ for a given $a = (a_j) \in (\mathbb{Z}/N\mathbb{Z})^{2n} - \{0\}$ gives a closed subscheme H_a of H_3; so, we define $H_4 = H_3 - \bigcup_a H_a$. Thus the abelian scheme Z_4 over H_4 has $2n$ linearly independent sections of order N.

Since Z_4 is a subscheme of $\mathbf{P}_{/H_4}$, it has the line bundle $\mathcal{O}_{Z_4}(1)$ which is the restriction of $\mathcal{O}(1)_{/\mathbf{P}}$. Then we define H_5 to be the maximal subscheme of H_4 such that $p^*L \cong L^\Delta(\lambda)^3$ for a polarization $\lambda : Z_4 \to {}^tZ_4$, where $p : Z_5 = Z_4 \times_{H_4} H_5 \hookrightarrow Z_4$ is the inclusion. It is proved in [GIT] proposition 6.11 that the maximal subscheme H_5 with the above property exists and is closed in H_4.

5.2.5. *Conclusion.* By the argument in the previous proposition, the functor $\mathcal{H}_{d,N}$ is represented over S_0 by a quasi-projective scheme H_5 with the universal abelian scheme $\mathcal{X} = Z_5$ over H_5. The group $PGL(m+1)$ acts on H_5 by $\iota \mapsto \iota \circ g$ ($g \in PGL(m+1)$). Then Mumford verifies through his theory of geometric quotient that the quotient quasi-projective scheme exists ([GIT] Chapter 3 and Section 7.3):

$$M_{d,N} = PGL(m+1)\backslash H_5.$$

It is easy to check that if $PGL(m+1)$ has no fixed point, then H_5 is a $PGL(m+1)$–torsor over $M_{d,N}$. This is the case where the structure (X, ϕ_N, λ) does not have non-trivial automorphisms, which follows if $N \geq 3$ by a result of Serre (see [PAF] pages 281–282 for this point). In this case, $M_{d,N}$ represents the functor $\mathcal{A}_{d,N}$ over S_0. Otherwise, $M_{d,N}$ gives a coarse moduli scheme for the functor.

5.2.6. *Compactification.* Here we quote a result from Faltings-Chai [DAV] V.2 on the minimal compactification. Let $\mathbb{X} = (\mathbf{X}_{d,N}, \boldsymbol{\phi}_N, \boldsymbol{\lambda}) \xrightarrow{f} M_{d,N}$ be the universal abelian scheme of relative dimension n with level N–structure $\boldsymbol{\phi}_N$ and the polarization of degree d^2. We assume that $N \geq 3$.

Since we have already studied via Tate curves the compactification of $M = M_{d,N}$ when $n = 1$ (the moduli of elliptic curves), we assume here $n > 1$. We then define $\underline{\omega} = f_*\Omega_{\mathbb{X}/M}$. This is a locally free sheaf over M of rank n. We define $\det \underline{\omega} = \bigwedge^n \underline{\omega}$. In [DAV] IV, a smooth toroidal compactification $\overline{M} = \overline{M}_{d,N}$ over $\mathbb{Z}[\frac{1}{dN}]$ is made (actually, details are exposed there for $d = 1$ but the argument works for $d > 1$ over $\mathbb{Z}[\frac{1}{d}]$). We shall come back to this topic later with more details. They also proved that $\underline{\omega}$ extends to the compactification $\overline{M}$. Then we define a graded algebra

$$\mathcal{G} = \mathcal{G}_{d,N} = \bigoplus_{m=0}^{\infty} H^0\left(\overline{M}_{d,N}, \det(\underline{\omega})^{\otimes m}\right) = \bigoplus_{m=0}^{\infty} H^0\left(M_{d,N}, \det(\underline{\omega})^{\otimes m}\right),$$

where $\det(\underline{\omega})^{\otimes m}$ is the m–th power of the invertible sheaf $\det(\underline{\omega})$. The last identity in the above definition follows from Koecher's lemma ([DAV] V.1.5) if $n > 1$. It is proven in [DAV] V.2.5 that this graded algebra is finitely generated over $\mathbb{Z}[\zeta_N, \frac{1}{Nd}]$ for a fixed primitive N–th root ζ_N of unity, and by the first equality, the graded algebra is normal. Thus we may define $\mathcal{G}_{d,N}$ to be the normalization of $\mathcal{G}_{d,1}$ in the algebra $\mathcal{R}_{d,N}$ defined below. We define the minimal compactification by

$$M^*_{d,N} = Proj(\mathcal{G}_{d,N}).$$

It is called "minimal" because any smooth toroidal compactification $\overline{M}_{d,N}$ covers canonically $M^*_{d,N}$.

We can define a sheaf of graded algebras over $M_{d,N}$ by

$$\mathcal{R} = \mathcal{R}_{d,N} = \bigoplus_{m=-\infty}^{\infty} f_*\left(\det(\underline{\omega})^{\otimes m}\right).$$

Then $\mathcal{M}_{d,N} = Spec_M(\mathcal{R})$ represents the following functor:

$$\mathcal{P}_{d,N}(S) = \left[(X, \phi_N, \lambda, \omega)_{/S} \middle| (X, \phi_N, \lambda) \in \mathcal{A}_{d,N}(S), H^0(X, \det \Omega_{X/S}) = \mathcal{O}_S\omega\right],$$

and we have

$$M_{d,N} = \mathbb{G}_m\backslash \mathcal{M}_{d,N}$$

as a geometric quotient. Here $\mathcal{M}_{d,N}$ is the relative spectrum of $\mathcal{R}$ over M; so, $\mathcal{M}_{d,N}$ is affine over $M_{d,N}$. In particular, $\mathcal{M}_{d,N}$ is a $\mathbb{G}_m$–torsor over $M_{d,N}$ if $N \geq 3$. Here $a \in \mathbb{G}_m$ acts on the functor $\mathcal{P}_{d,N}$ by $(X, \phi_N, \lambda, \omega) \mapsto (X, \phi_N, \lambda, a\omega)$. The relation between the moduli scheme classifying abelian schemes with level structure and the one classifying with an extra information of nowhere vanishing n–differentials is exactly the same as in the elliptic modular case, which amuses me a bit, and it is also interesting that this is proven only after a hard work of compactifying smoothly the open moduli $M_{d,N}$.

6. Shimura Varieties

In this lecture, we sketch basic theory of Shimura varieties of PEL type following [Sh3], [D2] and [Ko].

Shimura originally constructed canonical models in the 1950's to 1960's as a tower of quasi-projective geometrically connected varieties (over a tower of canonical abelian extensions of the reflex field) with a specific reciprocity law at special algebraic points (in the case of Shimura varieties of PEL-type, they are called *CM points* carrying an abelian variety of CM type; [Sh3]). His theory includes interesting cases of canonical models of non PEL type (for example, Shimura curves over totally real fields different from $\mathbb{Q}$), but in this paper, we restrict ourselves to the case where we have a canonical family of abelian varieties over the canonical model (so, the construction of the models is easier, as was basically done in [Sh2]).

Deligne reformulated Shimura's tower as a projective limit of (possibly non-connected) models over the reflex field (incorporating theory of motives in its scope). We follow Deligne's treatment in order to avoid the definition of the canonical fields of definition of the connected components, although by doing this, we may lose some of finer information.

Kottwitz extended the Deligne's definition of Shimura varieties of PEL type to a projective limit of schemes over a valuation ring of mixed characteristic, when the level is prime to p. Since we are interested in formal completion at p of the Kottwitz model (and an analogue of the Igusa tower over the Kottwitz model), what we use most is Kottwitz's formulation.

6.1. **Shimura Varieties of PEL Type.** We construct the moduli of abelian schemes with specific endomorphism algebra.

6.1.1. *Endomorphisms.* Let B be a finite dimensional simple $\mathbb{Q}$–algebra with center F. Let S be a set of primes of F over p. We always assume

(unr) We have an isomorphism $B_p = B \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong \bigoplus_{\mathfrak{p}\in S} M_n(F_{\mathfrak{p}})$ and $F_{\mathfrak{p}}/\mathbb{Q}_p$ is unramified for all $\mathfrak{p} \in S$.

Let "$*$" be an involution on B which satisfies $\mathrm{Tr}(xx^*) > 0$ for all $0 \neq x \in B_\infty = B \otimes_{\mathbb{Q}} \mathbb{R}$. We call such an involution a *positive* involution. We fix a maximal order $O = O_B$ of B stable under $*$. We assume that the isomorphism in (unr) induces $O_p = O \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \bigoplus_{\mathfrak{p}\in S} M_n(O_{F,\mathfrak{p}})$.

We fix a left B–module V of finite type and assume that we have a non-degenerate alternating form $\langle\ ,\ \rangle : V \times V \to \mathbb{Q}$ such that $\langle bv, w\rangle = \langle v, b^*w\rangle$ for all $b \in B$. Write $V_p = V \otimes_{\mathbb{Q}} \mathbb{Q}_p$ and $V_\infty = V \otimes_{\mathbb{Q}} \mathbb{R}$. We also assume to have an O–submodule $L \subset V$ of finite type such that

(L1) $L \otimes_{\mathbb{Z}} \mathbb{Q} = V$;
(L2) $\langle\ ,\ \rangle$ induces $\mathrm{Hom}_{\mathbb{Z}_p}(L_p, \mathbb{Z}_p) \cong L_p$, where $L_p = L \otimes_{\mathbb{Z}} \mathbb{Z}_p$.

Put $C = \mathrm{End}_B(V)$, which is a semi-simple $\mathbb{Q}$–algebra with involution again denoted by "$*$" given by $\langle cv, w\rangle = \langle v, c^* w\rangle$. Then we define algebraic $\mathbb{Q}$–groups G and G_1 by

$$G(A) = \left\{x \in C \otimes_{\mathbb{Q}} A \middle| xx^* \in A^\times\right\};\ G_1(A) = \left\{x \in G(A) \middle| xx^* = 1\right\}. \tag{6.1}$$

We now take an $\mathbb{R}$–algebra homomorphism $h : \mathbb{C} \hookrightarrow C_\infty = C \otimes_{\mathbb{Q}} \mathbb{R}$ with $h\langle \overline{z}\rangle = h(z)^*$. We call such an algebra homomorphism $*$*–homomorphism*. Then $h(i)^* = -h(i)$ for $i = \sqrt{-1}$ and hence $x^\iota = h(i)^{-1} x^* h(i)$ is an involution of C_∞. We suppose

(pos) The symmetric real bilinear form $(v, w) \mapsto \langle v, h(i)w\rangle$ on V_∞ is positive definite.

The above condition implies that ι is a positive involution (e.g. [Ko] Lemma 2.2).

Since $h : \mathbb{C} \to C_\infty$ is an $\mathbb{R}$–algebra homomorphism, we can split $V_{\mathbb{C}} = V \otimes_{\mathbb{Q}} \mathbb{C}$ into the direct sum of eigenspaces $V_{\mathbb{C}} = V_1 \oplus V_2$ so that $h(z)$ acts on V_1 (resp. V_2)) through multiplication by z (resp. $\overline{z}$). Since $h(\mathbb{C}) \subset C_\infty$, $h(z)$ commutes with the action of B; so, V_j is stable under the action of $B_{\mathbb{C}} = B \otimes_{\mathbb{Q}} \mathbb{C}$. Thus we get the complex representation $\rho_1 : B \hookrightarrow \mathrm{End}_{\mathbb{C}}(V_1)$. We define E for the subfield of $\overline{\mathbb{Q}}$ fixed by

$$\left\{\sigma \in \mathrm{Aut}(\mathbb{C}) \middle| \rho_1^\sigma \cong \rho_1\right\}.$$

The field E is called the reflex field (of B). We write O_E for the integer ring of E. Let $\mathbb{Z}_{(p)} = \mathbb{Z}_p \cap \mathbb{Q}$ and put $O_{(p)} = O \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$.

Let $K^{(p)}$ be an open compact subgroup of $G(\mathbb{A}^{(p\infty)})$ for

$$\mathbb{A}^{(p\infty)} = \{x \in \mathbb{A} | x_p = x_\infty = 0\}.$$

Let $K_p = \{g \in G(\mathbb{Z}_p) | gL_p = L_p\}$, and put $K = K_p \times K^{(p)} \subset G(\mathbb{A}^{(\infty)})$. We call an open compact subgroup K of $G(\mathbb{A}^{(\infty)})$ of this type an open compact subgroup *maximal at* p.

We study classification problem of the following quadruples: $(X, \lambda, i, \overline{\eta}^{(p)})_{/S}$. Here X is a (projective) abelian scheme over a base S, $\widehat{X} = \mathrm{Pic}^0_{X/S}(X)$ is the dual abelian scheme of X, $\lambda : X \to \widehat{X}$ is an isogeny with degree prime to p (prime-to–p isogeny) geometrically fiber by fiber induced from an ample divisor (polarization), $i : O_{(p)} \hookrightarrow \mathrm{End}_S^{\mathbb{Z}_{(p)}}(X) = \mathrm{End}_S(X) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$, and $\overline{\eta}^{(p)}$ is the level $K^{(p)}$–structure (see below for the definition of the level structure). The base scheme S is assumed to be a scheme over $Spec(\mathbb{Z}_{(p)})$. We now explain the meaning of the level $K^{(p)}$–structure $\overline{\eta}^{(p)}$. We consider the Tate module:

$$T(X) = \varprojlim_n X[N],\ T^{(p)}(X) = T(X) \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^{(p)} \text{ and } \ V^{(p)}(X) = T(X) \otimes_{\mathbb{Z}} \mathbb{A}^{(p\infty)},$$

where N runs over all positive integers ordered by divisibility, and $\widehat{\mathbb{Z}}^{(p)} = \prod_{\ell \neq p} \mathbb{Z}_\ell$. This module is equipped with a non-degenerate skew hermitian form induced by the polarization λ. Fix a base (geometric) point $s \in S$ and write X_s for the fiber of X at s. Then the algebraic fundamental group $\pi_1(S, s)$ acts on $V^{(p)}(X_s)$ leaving stable the skew hermitian form up to scalar. Then $\eta^{(p)} : V(\mathbb{A}^{(p\infty)}) = V \otimes_{\mathbb{Q}} \mathbb{A}^{(p\infty)} \xrightarrow{\sim} V^{(p)}(X_s)$ is an isomorphism of skew hermitian B–modules. We write $\overline{\eta}^{(p)} = \eta^{(p)}$ mod $K^{(p)}$ and suppose that $\sigma \circ \overline{\eta}^{(p)} = \overline{\eta}^{(p)}$ for all $\sigma \in \pi_1(S, s)$ (this is a way of describing that the level structure $\eta^{(p)}$ is defined over S). Even if we change the point $s \in S$, everything will be conjugated by an isomorphism; so, the definition does not depend on the choice of s as long as S is connected (see [PAF] 6.4.1). When S is not connected, we choose one geometric point at each connected component.

As examples of $K^{(p)}$ and open compact subgroups K maximal at p of $G(\mathbb{A}^{(\infty)})$, we could offer the following subgroups:

$$\widehat{\Gamma} = \left\{x \in G(\mathbb{A}^{(\infty)}) \middle| xL = L\right\},\ \widehat{\Gamma}^{(p)} = \left\{x \in \widehat{\Gamma} \middle| x_p = 1\right\};$$
$$\widehat{\Gamma}^{(p)}(N) = \left\{x \in \widehat{\Gamma}^{(p)} \middle| x\ell \equiv \ell \mod NL \text{ for all } \ell \in L\right\}$$

for an integer $N > 0$ prime to p.

Since every maximal compact subgroup of $GL_B(V(\mathbb{A}^{(\infty)})) = \mathrm{Aut}_{B_{\mathbb{A}}}(V(\mathbb{A}^{(\infty)}))$ is the stabilizer of a lattice L stable under a maximal order, we find a lattice L with $L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^{(p)}$ stable under $K^{(p)}$, where $\widehat{\mathbb{Z}}^{(p)} = \prod_{\ell \neq p} \mathbb{Z}_\ell$ for ℓ running through all primes different from p. Changing L by a sublattice of p–power index if necessary, we may assume that L satisfies the conditions (L1-2). We call a quadruple $\underline{X}_{/S} = (X, \lambda, i, \overline{\eta}^{(p)})_{/S}$ isomorphic to $\underline{X}'_{/S} = (X', \lambda', i', \overline{\eta'}^{(p)})_{/S}$ if we have an isogeny $\phi : X \to X'$ defined over S such that $p \nmid \deg(\phi)$, $\widehat{\phi} \circ \lambda' \circ \phi = c\lambda$ with $c \in \mathbb{Z}_{(p)}^\times$, $\phi \circ i \circ \phi^{-1} = i'$ and $\overline{\eta'}^{(p)} = \phi \circ \overline{\eta}^{(p)}$. In this case, we write $\underline{X} \approx \underline{X}'$. We write $\underline{X} \cong \underline{X}'$ if the isogeny is an isomorphism of abelian schemes, that is, $\deg(\phi) = 1$.

Let $S_0 = Spec(O_E \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)})$. We take the fiber category $\mathcal{C} = \mathcal{C}_B$ of the quadruples $(X, \lambda, i, \overline{\eta}^{(p)})_{/S}$ over the category S_0–SCH of S_0–schemes and define

$$\text{(6.2)} \quad \mathrm{Hom}_{\mathcal{C}_{/S}}((X, \lambda, i, \overline{\eta}^{(p)})_{/S}, (X', \lambda', i', \overline{\eta'}^{(p)})_{/S}) = \left\{\phi \in \mathrm{Hom}_S(X, X') \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \middle| \begin{matrix} \widehat{\phi} \circ \lambda' \circ \phi = c\lambda \text{ with } 0 < c \in \mathbb{Z}_{(p)}^\times, \\ \phi \circ i = i' \circ \phi \text{ and } \overline{\eta'}^{(p)} = \phi \circ \overline{\eta}^{(p)} \end{matrix}\right\}.$$

We consider the functor $\mathcal{P}_K^{(p)} : S_0\text{–}SCH \to SETS$ given by

$$\mathcal{P}^{(p)}(S) = \mathcal{P}_K^{(p)}(S) = \left\{\underline{X}_{/S} = (X, \lambda, i, \overline{\eta}^{(p)})_{/S} \middle| X \text{ satisfies (det) below}\right\} / \approx.$$

This functor is representable by the Shimura variety $Sh_K^{(p)}$ defined over S_0 as we will see later. Here the determinant condition is given as follows: We fix a $\mathbb{Z}_{(p)}$–base $\{\alpha_j\}_{1 \leq j \leq t}$ of $O_{(p)}$ and consider a homogeneous polynomial

$$f(X_1, \ldots, X_t) = \det(\alpha_1 X_1 + \cdots + \alpha_t X_t|_{V_1}).$$

Then $f(X) \in O_E \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}[X_1, \ldots, X_t]$ and coefficients of $f(X)$ generates E over $\mathbb{Q}$. Here O_E is the integer ring of E. For a given quadruple $\underline{X} = (X, \lambda, i, \overline{\eta}^{(p)})_{/S}$, we have the Lie algebra $Lie(X)$ of X over $\mathcal{O}_S$, which is a $O_{(p)} \otimes_{\mathbb{Z}} \mathcal{O}_S$–module via i. Then we can think of $g(X_1, \ldots, X_t) = \det(\alpha_1 X_1 + \cdots + \alpha_t X_t|_{Lie(X)}) \in \mathcal{O}_S[X_1, \ldots, X_t]$. We impose

$$\text{(det)} \qquad j(f(X_1, \ldots, X_t)) = g(X_1, \ldots, X_t),$$

where $j : O_E \otimes \mathbb{Z}_{(p)} \to \mathcal{O}_S$ is the structure homomorphism. Over a field of characteristic 0, one can characterize representations of a semi-simple algebra by its trace, but over a general base, we need, by the Brauer-Nesbitt theorem, the entire characteristic polynomial to determine a given representation; so, the determinant has to be fixed as above.

Allowing any closed subgroup $K \subset G(\mathbb{A}^{(\infty)})$ (not necessarily maximal at p), replacing isogenies of degree prime to p by (any) isogenies and imposing one more condition (pol) below, we may consider the functor $\mathcal{P}_K : E\text{–}SCH \to SETS$ given by

$$\mathcal{P}(S) = \mathcal{P}_K(S) = \left\{ \underline{X}_{/S} = (X, \lambda, i, \overline{\eta})_{/S} \middle| \ X \text{ satisfies (det) and (pol)} \right\} / \sim,$$

where $\eta : V(\mathbb{A}^{(p)}) = V \otimes_{\mathbb{Q}} \mathbb{A}^{(\infty)} \cong V(X) = T(X) \otimes_{\mathbb{Z}} \mathbb{A}^{(\infty)}$, $\overline{\eta} = (\eta \mod K)$, K is any closed subgroup of $G(\mathbb{A}^{(\infty)})$, and $(X, \lambda, i, \overline{\eta})_{/S} \sim (X', \lambda', i', \overline{\eta}')_{/S}$ if the two quadruples are equivalent to each other under an isogeny (not necessarily of degree prime to p). Here is the condition (pol):

(pol) There exists an B-linear isomorphism $f : V \cong H_1(X, \mathbb{Q})$ such that $f^{-1} \circ h_X \circ f$ is a conjugate of h under $G(\mathbb{R})$, $(f \otimes 1_{\mathbb{A}^{(\infty)}}) \in (\eta \circ K)$ and $E_X(f(x), f(y)) = \alpha\langle x, y\rangle$ up to $\alpha \in \mathbb{Q}^\times$,

where E_X is the Riemann form on $H_1(X, \mathbb{Q})$ (see [ABV] Section 1) and $h_X : \mathbb{C} \hookrightarrow C_\infty$ is the $\mathbb{R}$–algebra homomorphism induced by the complex structure on $V_{\mathbb{R}} \cong H_1(X, \mathbb{R})$ induced by the complex structure of $X(\mathbb{C})$. This functor is representable by the Shimura variety Sh_K defined over E. The scheme $Sh_{K/E}$ is the model Deligne studied.

We have a canonical inclusion $i : Sh_{K/E} \hookrightarrow Sh_K^{(p)} \times_{S_0} E$ if K is maximal at p. The isomorphism class of G over each local field is determined by the level structure η, but G is not uniquely determined globally without the condition (pol). In other words, if G does not satisfy the Hasse principle, even if K is maximal at p, the inclusion i may not be an isomorphism. As verified by Kottwitz (see [Ko] Section 8 and [PAF] Theorem 7.5), if G is either an inner form of the symplectic group (type C groups) or an inner form of a quasi-split unitary group $U(n,n)$ (type A) or F is an imaginary quadratic field, we have $Sh_{K/E} \cong Sh_K^{(p)} \times_{S_0} E$. Otherwise, the situation is more subtle (see [Ko] Section 8).

6.1.2. *Construction of the Moduli.* Here is a brief outline of how to show the representability of the functor $\mathcal{P}_K^{(p)}$ for K maximal at p. If $K^{(p)}$ is sufficiently small so that $\mathrm{Aut}_S(X) = \{1_X\}$, the prime-to–$p$ isogeny giving the isomorphism $X \approx X'$ in the definition of $\mathcal{P}^{(p)}$ can be taken to be an isomorphism by changing X' in the isomorphism class under "$\approx$" (and insisting $\eta^{(p)}(L^{(p)}) = T^{(p)}(X)$; see the argument below and [D1] 4.10). Therefore we have $\mathcal{P}_K^{(p)}(S) \cong \mathcal{P}'_K(S)$, where

$$\mathcal{P}'_K(S) = \left[\underline{X}_{/S} \middle| \ \underline{X} \text{ with (det)}, \ \underline{X} \approx \exists \underline{X}' \in \mathcal{P}^{(p)}(S) \text{ and } \eta^{(p)}(\widehat{L}^{(p)}) = T^{(p)}(X) \right],$$

where $[\ \ ] = \{\ \ \}/\cong$. Under this setting, we change the morphism set of $\mathcal{C}_{/S}$ from $\mathrm{Hom}_{\mathcal{C}_{/S}}$ to $\mathrm{Isom}_{\mathcal{C}_{/S}}$:

$$\begin{aligned} (6.3)\quad & \mathrm{Isom}_{\mathcal{C}_{/S}}((X, \lambda, i, \overline{\eta}^{(p)})_{/S}, (X', \lambda', i', \overline{\eta'}^{(p)})_{/S}) \\ &= \left\{ \phi \in \mathrm{Isom}_S(X, X') \middle| \begin{matrix} \widehat{\phi} \circ \lambda' \circ \phi = c\lambda \ \text{ with } \ 0 < c \in \mathbb{Z}_{(p)}^\times, \\ \phi \circ i = i' \circ \phi \ \text{ and } \ \overline{\eta'}^{(p)} = \phi \circ \overline{\eta}^{(p)} \end{matrix} \right\}. \end{aligned}$$

Then we claim that the fiber category $\mathcal{C}$ is an algebraic stack for any given $K^{(p)}$ and is an algebraic space if $\mathrm{Aut}_{\mathcal{C}_{/S}}(X) = \{1_X\}$ for all objects X of $\mathcal{C}_{/S}$ (see [DM] for definition of stacks). By forgetting B–linearity of $\eta^{(p)}$ and restricting i to $\mathbb{Q}$, we have a functor from $\mathcal{C}_B$ into the fiber category $\mathcal{C}_{\mathbb{Q}}$ of $(X, \lambda, i, \overline{\eta}^{(p)})$ for $B = \mathbb{Q}$ for a suitable choice of an open compact subgroup $GSp(2d)(\mathbb{A}^{(p\infty)})$ ($d = \dim X = \frac{1}{2}\dim_{\mathbb{Q}} V$). This fiber category is proven to be an algebraic stack and is representable by a quasi projective scheme $M = M_{\mathbb{Q}/S_0}$ by Mumford (see Section 5, [GIT], [Sh2] and also [DAV], [CSM]).

We now supplement the above outline with details. We are going to show that we can replace "$\approx$" by "$\cong$" in the definition of the functor $\mathcal{P}_K^{(p)}$, imposing an additional condition. Let $\underline{A}_{/S} = (A, \lambda, i, \overline{\eta}^{(p)})_{/S}$. Then $\eta^{(p)}$ induces $V(\mathbb{A}^{(p\infty)}) = V \otimes_{\mathbb{Q}} \mathbb{A}^{(p\infty)} \cong V^{(p\infty)}(A)$, and therefore, we have $\eta^{(p)}(L^{(p)}) \subset V^{(p\infty)}(A)$ for $L^{(p)} = L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^{(p)}$.

If $\eta^{(p)}(L^{(p)}) \subset T^{(p)}(A)$, the cokernel is an étale group subscheme $C \subset A_{/S}$ locally-free over S of rank prime to p. Make the quotient abelian scheme $A' = A/C$ over S (see [ABV] Section 12 and [GME] Proposition 1.8.4), and then we have a prime-to-p isogeny $\phi : A \to A'$ with $\mathrm{Ker}(\phi) = C$. We then consider $\underline{A}' = (A', \lambda', i', \eta'^{(p)})_{/S}$ given by $\lambda = \widehat{\phi} \circ \lambda' \circ \phi$, $i'(\alpha) = \phi \circ i(\alpha) \circ \phi^{-1}$ and $\eta'^{(p)} = \phi \circ \eta^{(p)}$. Then $\underline{A}'$ satisfies $\eta'^{(p)}(L^{(p)}) = T^{(p)}(A)$.

If $\eta^{(p)}(L^{(p)}) \supset T^{(p)}(A)$, we can find a prime-to-p isogeny $\underline{A}' \to \underline{A}$ such that $\phi \circ \eta'^{(p)} = \eta^{(p)}$, $\lambda' = \widehat{\phi} \circ \lambda \circ \phi$, $i'(\alpha) = \phi^{-1} \circ i(\alpha) \circ \phi$ and $\eta'^{(p)}(L^{(p)}) = T^{(p)}(A)$. This fact follows from the canonical identification: $T(A) = \pi_1(A, \mathbf{0})$ for the origin $\mathbf{0}$ of A (see [ABV] Section 18).

If neither $\eta^{(p)}(L^{(p)}) \subset T^{(p)}(A)$ nor $\eta^{(p)}(L^{(p)}) \supset T^{(p)}(A)$, we can find two prime-to-p isogenies: $\underline{A} \xrightarrow{\phi'} \underline{A}''$ and $\underline{A}' \xrightarrow{\phi} \underline{A}''$ for $\underline{A}'' = (A'', \lambda'', i'', \eta''^{(p)})_{/S}$ so that $\eta''^{(p)}(L^{(p)}) = \eta^{(p)}(L^{(p)}) \cap T^{(p)}(A'')$ and $\eta'^{(p)}(L^{(p)}) = T^{(p)}(A')$. Thus always we can find in the prime-to-p isogeny class of a given $\underline{A}_{/S}$, a quadruple $\underline{A}'_{/S}$ with $\eta'^{(p)}(L^{(p)}) = T^{(p)}(A')$.

If $\phi : \underline{A}_{/S} \to \underline{A}'_{/S}$ is a prime-to-p isogeny with $\eta^{(p)}(L^{(p)}) = T^{(p)}(A)$ and $\eta'^{(p)}(L^{(p)}) = T^{(p)}(A')$, then $\deg(\phi) = 1$ and $\underline{A}_{/S} \cong \underline{A}'_{/S}$ by ϕ.

Thus insisting $\eta^{(p)}(L^{(p)}) = T^{(p)}(A)$, we can replace $\approx$ by $\cong$ in order to define the functor $\mathcal{P}_K^{(p)}$ (see [D1] 4.12). In other words,

$$\mathcal{P}_K^{(p)}(S) \cong \left\{ \underline{A}_{/S} \middle| (\det) \text{ and } \ \eta^{(p)}(L^{(p)}) = T^{(p)}(A) \right\} / \cong .$$

The functor defined in this way can be proven to be representable by an S_0–scheme $M(G_0, X_0)/K$ by works of Deligne, Mumford and Shimura (cf. [Ko] and [PAF] 7.1.2).

Since $\widehat{\Gamma}^{(p\infty)}(N)$ for N prime to p gives a fundamental system of neighborhoods of the identity in $G(\mathbb{A}^{(p\infty)})$, we may assume that $K = \widehat{\Gamma}(N) = \widehat{\Gamma}^{(p\infty)}(N) \times \widehat{\Gamma}_p$. We only need to show that $\mathcal{C}$ is relatively representable over the Mumford moduli $M_{\mathbb{Q}}$ given by Theorem 5.9. Let $\mathcal{P}'^{B}_{K}$ be the functor with respect to K and B. Write $2d = \dim_{\mathbb{Q}}(V)$. Then $d = \dim_S X$ for $\underline{X}_{/S} \in \mathcal{P}'^{B}_{K}(S)$, which is therefore independent of the choice of $\underline{X}_{/S}$ by (det). For simplicity, we assume that the polarization pairing $\langle\ ,\ \rangle$ in (L2) gives the self duality of L. Then we can identify the similitude group of $\langle\ ,\ \rangle$ acting on L with $GSp_{2d}(\mathbb{Z})$. In this way, we let $GSp_{2d}(A)$ act on $V \otimes_{\mathbb{Q}} A$. Write K_0 for the maximal compact subgroup of $GSp_{2d}(\mathbb{A}^{(\infty)})$ preserving L and principal level N structure. Then $K_0 \cap G(\mathbb{A}^{(\infty)}) = K$ and $K = \widehat{\Gamma}(N)$ with respect to $B = \mathbb{Q}$. As described in Theorem 5.9, $\mathcal{P}'^{\mathbb{Q}}_{K_0}$ is representable by a quasi-projective scheme $M = M_{\mathbb{Q}} = M_N$ defined over $\mathbb{Z}_{(p)}$. Let $\underline{\mathbb{X}}_{/M}$ be the universal quadruple over M and $\mathbf{A} \in \underline{\mathbb{X}}$ be the universal abelian scheme. We consider the functor from M–SCH into $SETS$:

$$T_{/S} \mapsto \left[(\mathbf{A}_T, i : O_B \hookrightarrow \mathrm{End}_T(\mathbf{A}_T)) \middle| i(1_B) = \mathrm{id}_{\mathbf{A}} \right]$$

This functor is representable by a scheme $M_{B/M}$ basically by Corollary 5.7 (see Corollary 6.11 in [PAF] for the version of Corollary 5.7 which is necessary to prove this fact). Since the level structure $\overline{\boldsymbol{\eta}}^{(p)}$ on $\mathbf{A}$ gives rise to a level structure $\overline{\eta}^{(p)}$ of $(\mathbf{A}_T, i)$, we have a triple $\underline{X} = (\mathbf{A}_T, i, \overline{\eta}^{(p)})_{/T}$. Thus $\mathcal{P}'^{B}_{K}$ is a subfunctor of the above functor. Again by the rigidity of endomorphisms under specialization, $\mathcal{P}'^{B}_{K}$ is represented by a closed subscheme $Sh^{(p)}_K$ of M_B whose connected components are (each) finite over $M_{\mathbb{Q}}$ (see [PAF] 7.1.2 for more details). We are going to show that $Sh^{(p)}_K$ is of finite type over $M_{\mathbb{Q}}$ (so it is projective and finite over $M_{\mathbb{Q}}$). Take a geometric point $x \in M_{\mathbb{Q}}$, suppose that we have $i : O_B \hookrightarrow \mathrm{End}_{\mathbf{A}}$ as above, which gives rise to a geometric point $y \in Sh^{(p)}_K$. For a given T, if T is connected, $\mathrm{End}(\mathbf{A}_T) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a semi-simple algebra of finite dimension with positive involution (see [ABV] IV). Thus the number of embedding $B \hookrightarrow \mathrm{End}(\mathbf{A}_T) \otimes_{\mathbb{Z}} \mathbb{Q}$ is finite up to inner automorphism. Moreover the number is bounded by a constant only depending on the dimension of $\mathbf{A}$, that is d, because $\dim(\mathrm{End}_T(\mathbf{A}_T) \otimes_{\mathbb{Z}} \mathbb{Q})$ is bounded by $4d^2$ (e.g. [GME] Theorem 4.1.19). If one changes i by an inner automorphism induced by $\alpha \in \mathrm{End}(\mathbf{A}_T)$ and if we suppose that $(\mathbf{A}_T, \alpha i \alpha^{-1}, \lambda, \overline{\eta}^{(p)})$ is still an element of $\mathcal{P}'^{B}_{K}(T)$, it is easy to show, by the condition that $\eta^{(p)}(T^{(p)}(X)) = L^{(p)}$ combined with (L1-2) that α has to be an automorphism of $\mathbf{A}_T$. Since automorphisms of an abelian variety preserving a given polarization are only finitely many by the positivity of polarization, there are only finitely many possibilities of having $i : O_B \hookrightarrow \mathrm{End}(\mathbf{A}_T)$ which gives rise to an element of $\mathcal{P}'^{B}_{K}(T)$. Thus $Sh^{(p)}_K \to M_{\mathbb{Q}}$ is quasi finite. Then the projectivity of each connected component of $Sh^{(p)}_K$ over $M_{\mathbb{Q}}$ implies the finiteness of the map: $Sh^{(p)}_K \to M_{\mathbb{Q}}$. Actually, one can show that the morphism: $Sh^{(p)}_K \to M_{\mathbb{Q}}$ is a closed immersion (over $\mathbb{Q}$) if N is sufficiently large (cf. [D1] 1.15 and [PAF] 8.4.2).

Again by the rigidity of endomorphism of abelian schemes (and semi-abelian schemes) over a normal base under specialization ([DAV] I.2.7), for N sufficiently large, $\mathcal{P}'^{B}_{K}$ is represented by the schematic closure of $Sh^{(p)}_{K/E}$ in $M_{\mathbb{Q}/S_0}$, and hence $\mathcal{C}_B$ for general B is a representable by a scheme $Sh^{(p)}_{K/S_0}$ projective over $M_{\mathbb{Q}/S_0}$ if $K^{(p)}$ is sufficiently small.

Although we assumed that L is self dual, replacing GSp_{2d} by its suitable conjugate in $GL(2d)$, we can easily generalize the above argument to a given polarization of degree prime to p.

In exactly the same way, we may conclude $\mathcal{P}_K \cong \mathcal{P}'^{B}_{K}$ over E (not over S_0) even if K is not maximal at p; so, we get the representability of $\mathcal{P}_K$ by the Shimura variety $Sh_{K/E}$ and the inclusion $Sh_{K/E} \hookrightarrow Sh^{(p)}_K \times_{S_0} E$ if K is maximal at p. Hereafter, if confusion is unlikely, we remove the superscript "(p)" from the notation $Sh^{(p)}_K$, and if we consider the Shimura variety Sh_K over S_0–scheme, we implicitly assume $Sh_{K/E} = Sh^{(p)}_K \times_{S_0} E$, that K is maximal at p and that the model is the integral Kottwitz model $Sh^{(p)}_K$. As we already remarked, $Sh_{K/E} = Sh^{(p)}_K \times_{S_0} E$ holds if G is a type C group or F is an imaginary quadratic field ([PAF] Theorem 7.5).

In the non-compact case, in [DAV], depending on the data at the cusps governing how toroidal compactification is done, a semi-abelian scheme $\mathbf{G}_{/\overline{M}_{\mathbb{Q}}}$ (universal under the data) is constructed. Then a similar argument using Corollary 5.8 (applied to $\mathbf{G}_{/\overline{M}_{\mathbb{Q}}}$ in place of $\mathbf{A}_{/M_{\mathbb{Q}}}$) gives a projective scheme over $\overline{M}_{\mathbb{Q}}$ for a toroidal compactification $\overline{M}_{\mathbb{Q}}$ of the Mumford moduli (by Chai and Faltings). Since the endomorphism algebra of an abelian variety $X_{/k}$ for an algebraically closed field k (after tensoring $\mathbb{Q}$) is semi-simple, there is only finitely many possibility of embedding B into $\mathrm{End}_k(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ up to conjugation. Thus the morphism $\overline{M}_B \to \overline{M}_{\mathbb{Q}}$ has finite geometric fiber everywhere, that is, the morphism is quasi-finite. Since the scheme $\overline{M}_B$ is proper over $\overline{M}_{\mathbb{Q}}$ (see Corollary 5.8), it has to be finite. Thus writing $\underline{\omega} = f_*\Omega_{\mathbf{A}/M_B}$ for $f : \mathbf{A} \to M_B$ and defining $\det(\underline{\omega})$ by its maximal exterior product, we can define a graded algebra:

$$\mathcal{G}^K = \mathcal{G}_B^K = \bigoplus_{n=0}^{\infty} H^0(\overline{M}_B, \det(\underline{\omega})^n).$$

Moreover, as seen in the last subsection of Section 5, $M_{\mathbb{Q}}^* = Proj(\mathcal{G}_{\mathbb{Q}}^K)$ and hence we have the minimal compactification of Sh_K defined by $Sh_K^* = Proj(\mathcal{G}_B^K)$, which is finite over the minimal compactification $M_{\mathbb{Q}}^*$ of the Mumford moduli.

If one shrinks enough the group K outside p, any endomorphism of the semi-abelian scheme sitting over the cusp of $\overline{M}_K$ extends uniquely to infinitesimal neighborhood of the image of the cusp of $\overline{M}_B$ in $\overline{M}_{\mathbb{Q}}$; so, $\overline{M}_B$ is étale around the cusp over the image of $\overline{M}_B$ in $\overline{M}_{\mathbb{Q}}$. The smoothness of $\overline{M}_B$ at cusps for a well chosen cuspidal datum was shown by Fujiwara for C of type A and C ([F]). If one choose the cuspidal data for $GSp(2d)$ and G so that they are compatible (in other words, so that the pull back of the semi-abelian scheme over $\overline{M}_{\mathbb{Q}}$ is the semi-abelian scheme over $\overline{M}_B$ associated to the cuspidal data for G), this guarantees that the q–expansion parameter is well defined over S_0 and projectivity for $\overline{M}_B$ of level prime to p, because it is finite over $\overline{M}_{\mathbb{Q}}$.

Even if K is not very small, we always have a coarse moduli scheme Sh_K representing the functor $\mathcal{P}_K^{(p)}$ or $\mathcal{P}_K$ over S_0 or E accordingly. The above arguments all work well. We write $\overline{Sh}_K$ for a toroidal compactification of Sh_K and Sh_K^* for the minimal compactification. Since the natural morphisms:

$$Sh_K^* \to M_{\mathbb{Q}}^* \text{ and } \overline{Sh}_K \to \overline{M}_{\mathbb{Q}}$$

are quasi-finite and projective, they are finite. Let V be the image of Sh_K^* in $M_{\mathbb{Q}}^*$. Then $V = Proj(\mathcal{G}^*)$ for a graded algebra $\mathcal{G}^*$ which is the quotient of $\mathcal{G}_{\mathbb{Q}}^K$. Then, assuming the existence of a smooth toroidal compactification of Sh_K, we have

$$\text{(Proj)} \qquad Sh_K^* = Proj(\mathcal{G}_B^K).$$

Here $\mathcal{G}_B^K$ is the integral closure of $\mathcal{G}_{\mathbb{Q}}^K$ for the Mumford moduli in the algebraic closure of the total quotient ring of $\mathcal{G}^*$ if K is sufficiently small. This follows from the fact that Sh_K^* is smooth outside cusps, and at the cusps, if K is sufficiently small, it is finite over $\overline{M}_{\mathbb{Q}}$ (and normal over V). The graded algebra $\mathcal{G}_B^K$ is the graded algebra of automorphic forms on G if $\dim Sh_K > 1$.

We have formulated the moduli problem for the similitude group G. But we can impose polarization λ without ambiguity modulo $\mathbb{Z}_{(p)} = \mathbb{Z}_p \cap \mathbb{Q}$. Then we automatically obtain the moduli problem for G_1; so, we do not describe the moduli problem and Shimura varieties for G_1, although our theorems are valid also for G_1 with some obvious modification.

6.2. **Shimura Variety of Unitary Similitude Groups.** We could think of the Shimura variety of the unitary similitude group given by

$$GU(A) = \left\{x \in C \otimes_{\mathbb{Q}} A \middle| xx^* \in (A \otimes_{\mathbb{Q}} F_0)^\times\right\}, \tag{6.4}$$

where F_0 is the subfield of F fixed by the involution "$*$". Thus we have $GU \supset G \supset G_1$.

To define the moduli problem of abelian schemes associated to $GU_{/\mathbb{Q}}$, we need to modify slightly the morphisms of the fiber category $\mathcal{C}_B$: We define the fiber category $\mathcal{A} = \mathcal{A}_B$ over $SCH_{/S_0}$ to be the category of quadruples $\underline{X}_{/S} = (X, \lambda, i, \overline{\eta}^{(p)})$ for $\overline{\eta}^{(p)} = \eta^{(p)} \mod K$, where $K \subset GU(\mathbb{A}^{(\infty)})$ is a closed subgroup maximal at p. Write O_0 for the integer ring of F_0. Then we define

$$\begin{aligned}&\mathrm{Hom}_{\mathcal{A}_{/S}}(\underline{X}_{/S}, \underline{X}'_{/S}) = \\ &\left\{\phi \in \mathrm{Hom}(X, X') \otimes \mathbb{Z}_{(p)} \middle| \begin{matrix}\widehat{\phi} \circ \lambda' \circ \phi = \lambda \circ i(a) \text{ with } a \in (O_0 \otimes \mathbb{Z}_{(p)})_+ \\ \phi \circ i = i' \circ \phi \text{ and } \overline{\eta'}^{(p)} = \phi \circ \overline{\eta}^{(p)}\end{matrix}\right\},\end{aligned} \tag{6.5}$$

where $(O_0 \otimes \mathbb{Z}_{(p)})_+$ indicates the group of totally positive units in $(O_0 \otimes \mathbb{Z}_{(p)})^\times$. We then consider the functor

$$\mathcal{P}_K^{(p),\mathcal{A}}(S) = \left\{\underline{X}_{/S} \middle| X \text{ satisfies (det)}\right\} / \approx,$$

where "$\approx$" indicates isomorphism classes in $\mathcal{A}_{B/S}$.

The above functor can be proved to be representable if K is sufficiently small by the same argument as in the case of G (see [PAF] 7.1.3), and its generic fiber gives the Shimura variety over E (defined adding a requirement analogous to (pol); see [PAF] Theorem 7.5). The compactification of the moduli space M_{K/S_0} can be also done as described above. The only point we need to make explicit is that if the class $\overline{\lambda}$ of polarizations modulo multiplication by totally positive element in $(O_0 \otimes \mathbb{Z}_{(p)})^\times$ is defined over S, we can always find a representative λ defined over S. Indeed, picking one symmetric polarization λ, the pull back by $1 \times \lambda$ of the universal line bundle over $X \times_S {}^tX/X$ (the Poincaré bundle) is always ample and is equal to 2λ (see [GIT] Proposition 6.10); so, in the class $\overline{\lambda}$, we can always find a polarization globally defined over S.

6.2.1. *Classification of* G. Let F_0 be the subfield of F fixed by the involution "$*$". We define for F_0–algebras A,

$$G_0(A) = \left\{x \in C \otimes_{F_0} A \middle| xx^* = 1\right\}.$$

Then we have $G_1 = \mathrm{Res}_{F_0/\mathbb{Q}} G_0$. The involution "$*$" either induces a non-trivial involution on F (a positive involution of the second kind) or the identity map on F (a positive involution of the first kind). If "$*$" is of second kind, F is a totally imaginary quadratic extension over a totally real field F_0 (a CM field over F_0), "$*$" coincides on F the unique non-trivial automorphism over F_0 (complex conjugation; see [Sh1] and [ABV] Section 21). Then G_0 is an inner form of a quasi split unitary group over F_0. We call this case Case A and call the group G type A.

When "$*$" induces the identity map on F, then $F = F_0$ is totally real, and the group G_0 is an inner form of either the symplectic group (Case C and the group of type C) or an orthogonal group of even variable (Case D and the group of type D). We have

$$C_\infty \cong \begin{cases} M_n(\mathbb{C})^{I_0} \text{ and } x^* = I_{s,t}{}^t\overline{x}I_{s,t} & \text{in Case A,} \\ M_{2n}(\mathbb{R})^{I_0} \text{ and } x^* = J_n{}^t\overline{x}J_n & \text{in Case C,} \\ M_n(\mathbb{H})^{I_0} \text{ and } x^* = -i{}^t\overline{x}i & \text{in Case D,} \end{cases} \tag{6.6}$$

where I_0 is the set of all field embeddings of F_0 into $\mathbb{R}$, $\mathbb{H} = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$ is the algebra of Hamilton quaternions, $J_n = \left(\begin{smallmatrix} 0 & -1_n \\ 1_n & 0 \end{smallmatrix}\right)$, $I_{s,t} = \left(\begin{smallmatrix} 1_s & 0 \\ 0 & -1_t \end{smallmatrix}\right)$ for the $t \times t$ identity matrix 1_t, and $x \mapsto \overline{x}$ is either complex conjugation or quaternion conjugation.

Suppose that $p > 2$ if we are in Case D. When $K^{(p)}$ is sufficiently small, Sh_K is smooth over S_0. This follows from the fact that the deformation ring of a quadruple $(X, \lambda, i, \overline{\eta}^{(p)})$ is always formally smooth (cf. [GIT] Proposition 6.15, and [K]). If $C = \mathrm{End}_B(V)$ is a division algebra, the big division algebra B sitting inside $\mathrm{End}_S(X) \otimes \mathbb{Q}$ for $S = Spec(\mathcal{K})$ with $\mathcal{K} = Frac(\mathcal{V})$ for a valuation ring $\mathcal{V}$ forces reduction of X modulo the maximal ideal $\mathfrak{m}_{\mathcal{V}}$ to be an abelian variety; so, by the valuative criterion of properness, Sh_{K/S_0} is proper. Since Sh_K is projective over the Mumford moduli $M_{\mathbb{Q}}$ which is quasi-projective over S_0, Sh_{K/S_0} has to be projective ([Ko] Section 5).

We now briefly describe the complex points of Sh_K. We can define the symmetric domain $\mathcal{X}$ as the collection of $h : \mathbb{C} \hookrightarrow C_\infty$ satisfying the positivity, etc., we described above. Since the stabilizer C_h of a fixed h in $G(\mathbb{R})$ is the product of the center and a maximal compact subgroup, the connected component of $\mathcal{X}$ is isomorphic to the symmetric domain $\mathcal{D} = G(\mathbb{R})/C_h$. An explicit form of $\mathcal{D}$ as a classical bounded matrix domain is given in [Sh1] (see also [ACM] Chapter VI for the domains in Case A and C), along with an explicit method of constructing all possible analytic families of abelian varieties over the domain. We have computed $\mathcal{D}$ for unitary groups (that is, groups of type A) already in Section 4. The complex analytic space $Sh_K(\mathbb{C})$ is given by $G(\mathbb{Q})\backslash G(\mathbb{A})/KC_h$, and its connected component is given by $\Gamma\backslash\mathcal{D}$ for the congruence subgroup $\Gamma = (gKg^{-1}G(\mathbb{R})_+) \cap G(\mathbb{Q})$ with a suitable $g \in G(\mathbb{A}^{(\infty)})$, where $G_+(\mathbb{R})$ is the identity component of the Lie group $G(\mathbb{R})$.

7. Formal Theory of Automorphic Forms

In this lecture, we describe the theory of false automorphic forms. The theory we describe is a generalization of the work of Deligne-Katz in the elliptic modular case (see [K1] Appendix III). The main purpose of this lecture is threefold:

1. Approximate p–adic automorphic forms by finite sums of classical forms;
2. Define the p–ordinary projector;
3. Find a set of (axiomatic) conditions which guarantees the VCT.

7.1. **True and False Automorphic Forms.** In our application, we remove supersingular locus from the moduli $M_{/W}$ of abelian schemes of PEL–type and write $S_{/W}$ for $M\left[\frac{1}{E}\right]$ for a lift E of the Hasse invariant. In this setting, sections in $H^0(S, \underline{\omega}^\kappa)$ are called *"false"* automorphic forms. On the other hand, sections in $H^0(M, \underline{\omega}^\kappa)$ are called *"true"* or *"classical"* automorphic forms.

7.1.1. *An analogue of the Igusa tower.* Let W be a mixed characteristic complete discrete valuation ring with residue characteristic p. Let ϖ be a uniformizing parameter. Write $W_m = W/p^m W$. Let S be a flat W–scheme. We put $S_m = S \times_W W_m$. Then S_m is a sequence of flat W_m–schemes, given with isomorphisms:

$$S_{m+1} \otimes_{W_{m+1}} W_m \cong S_m.$$

Let P be a rank g p–adic étale sheaf on the S_m's; thus, $P_{/S_{m+1}}$ induces $P_{/S_m}$, $P = \varprojlim_n P/p^n P$, and $P_n = P/p^n P$ is a twist of the constant sheaf $(\mathbb{Z}/p^n\mathbb{Z})^g$. We write S_∞ for the formal completion of S along S_1; so, $S_\infty = \varprojlim_m S_m$.

We can slightly generalize our setting and could suppose that there exists a finite extension $F/\mathbb{Q}$ with integer ring $O = O_F$ and a homomorphism: $O \hookrightarrow \mathrm{End}_{S_m}(P)$ such that $P_n \cong (O/p^n O)^g$ for all n locally under étale topology. Since we can transfer any of our results to this slightly general situation, just replacing $\mathbb{Z}_p$ by $O_p = O \otimes_{\mathbb{Z}} \mathbb{Z}_p$, we state our result only for P with $P_n \cong (\mathbb{Z}/p^n\mathbb{Z})^g$. This simplification also allows us to save some symbols.

Let $\underline{\omega}_m$ be the vector bundle $P \otimes_{\mathbb{Z}_p} \mathcal{O}_{S_m}$. We define

$$\pi_{m,n} : T_{m,n} = \mathrm{Isom}_{S_m}(P_n, (\mathbb{Z}/p^n\mathbb{Z})^g) \to S_m$$

to be a finite étale S_m–scheme which represents the following functor on $SCH_{/S_m}$:

$$(\pi : X \to S_m) \mapsto \left\{ \text{isomorphisms } \psi_n : P_{n/X} \cong (\mathbb{Z}/p^n\mathbb{Z})^g_{/X} \right\}.$$

The representability follows from the theory of Hilbert schemes as we have seen. By definition, $T_{m,n/S_m}$ is étale. Since each geometric fiber of $T_{m,n}$ over S_m is isomorphic to $GL_g(\mathbb{Z}/p^r\mathbb{Z})$ everywhere, it is faithfully flat and finite. Therefore $T_{m,n}$ is affine over S_m. We define $V_{m,n} = H^0(T_{m,n}, \mathcal{O}_{T_{m,n}})$.

The group $GL_g(\mathbb{Z}/p^n\mathbb{Z})$ acts on $T_{m,n}$ freely by $\psi \mapsto g\psi$ for $g \in GL_g(\mathbb{Z}/p^n\mathbb{Z})$, and we have $T_{m,n}/\Gamma_{n,n'} \cong T_{m,n'}$ for all $n' \le n$, where

$$\Gamma_{n,n'} = \left\{ x \in GL_g(\mathbb{Z}/p^n\mathbb{Z}) \middle| x \equiv 1 \mod p^{n'} \right\}.$$

Then we have a tower:

$$V_{m,0} \subset V_{m,1} \subset \cdots \subset V_{m,n}$$

with $V_{m,0} = H^0(S_m, \mathcal{O}_{S_m})$. We put $V_{m,\infty} = \bigcup_n V_{m,n}$ and $T_{m,\infty} = \varprojlim_n T_{m,n}$.

7.1.2. *Rational representations and vector bundles.* For a given ring A or a sheaf of rings A over a scheme, we look at the projective scheme $\mathcal{F}_{/A} = \mathrm{Flag}_{B/A}$ of all maximal flags in A^g (cf. [PAF] 6.1.3). We write $B \subset GL(g)$ for the upper triangular Borel subgroup. Let U be its unipotent radical, and put $T = B/U$ for the torus. Then $\mathcal{F} \cong GL(g)/B$. We define $\mathcal{H}_{/A} = \mathrm{Flag}_{U/A} = GL(g)/U$. Write $\mathbf{1}$ for the origin of $\mathcal{H}$ represented by the coset U. Then

$$R_A = H^0(\mathcal{H}, \mathcal{O}_{\mathcal{H}}) = \bigoplus_{\kappa \in X(T)_+} R_A[\kappa]$$

for the space $R_A[\kappa]$ of weight κ. Here

$$R_A[\kappa] = \left\{ f : GL(g)/U \to \mathbf{A}^1 \in \Gamma(GL(g)/U, \mathcal{O}_{GL(g)/U}) \middle| f(ht) = \kappa(t) f(h) \right\}$$

for $t \in T$ for the diagonal torus $T \cong B/U \cong {}^tB/{}^tU$. The pull-back action of $GL(g)$ on $R_A[\kappa]$: $f(x) \mapsto \rho(h)f(x) = f(h^{-1}x)$ gives a representation $\rho = \rho_\kappa$ such that $R_A[\kappa]^U \cong A$ on which T acts by $-w_0\kappa$, where w_0 is the longest element of the Weyl group of T. The dual $R_A[\kappa]^* = \mathrm{Hom}_A(R_A[\kappa], A)$ is the universal representation of highest weight κ (cf. [RAG] II.2.13). Thus the coinvariant space $R_A[\kappa]_U$ (on which T acts by $-\kappa$) is A–free of rank 1, and there is a unique U–invariant linear form $\ell_{can} : R_A[\kappa] \to A$ (up to A–unit multiple), which generates $(R_A[\kappa]^*)^U$. We can normalize ℓ_{can} so that it is the evaluation of $\phi \in R_A[\kappa]$ at the origin $\mathbf{1} \in GL(g)/U$. Then we have a tautological embedding $R_A[\kappa] \hookrightarrow \Gamma(GL(g)/U, \mathcal{O}_{GL(g)/U})$ given by

$$\phi \mapsto \{h \mapsto \ell_{can}(\rho(h^{-1})\phi)\}.$$

If $h^{-1} \in M_{g\times g}$ for the $g \times g$ matrix algebra $M_{g\times g}$ as a multiplicative semi-group scheme, the action of $\rho(h)$ is well defined on $R_A[\kappa]$ for any A.

In [RAG], a slightly different module is considered:

$$H^0(-\kappa) = \mathrm{ind}_{{}^tB}^{GL(g)} A(-\kappa) = \left\{ f : GL(g)/{}^tU \to \mathbb{A}^1 \middle| f(xt) = t^{-\kappa} f(x)\ \forall t \in T \right\}.$$

The action of $GL(g)$ is given by $hf(x) = f(h^{-1}x)$. In this context, $-\kappa$ is a positive weight with respect to ${}^tB$, and the $H^0(-\kappa)^U$ contains the highest weight vector. Using conjugation by w_0, we can remove the use of the lower triangular Borel subgroup ${}^tB$, but we need to modify the results of [RAG] accordingly, when we quote them (this will be done without further warning).

Let $f \in R_{A/p^mA}[\kappa]$. By definition, f induces a function on $GL_g(\mathbb{Z}_p)$ by $f(h) = \rho_\kappa(h)f(1)$. Therefore we see that $h \mapsto \ell_{can} \circ f(h)$ is an element in $R_{A/p^mA}[\kappa]$ by tautology. This shows the following fact:

(c) *We have a canonical map* $R_A[\kappa] \to \mathcal{C}(GL_g(\mathbb{Z}_p)/U(\mathbb{Z}_p), A)[\kappa]$*,*

which is injective if A is flat over $\mathbb{Z}_p$. Here $\mathcal{C}(GL_g(\mathbb{Z}_p)/U(\mathbb{Z}_p), A)$ is the space of (p–adic) continuous functions with values in A on $GL_g(\mathbb{Z}_p)/U(\mathbb{Z}_p)$, and "$[\kappa]$" indicates the κ–eigenspace under the right action of $T(\mathbb{Z}_p)$ on $GL_g(\mathbb{Z}_p)/U(\mathbb{Z}_p)$. The cokernel of the map (c) is large, because it is the continuous induction from $B(\mathbb{Z}_p)$ to $GL_g(\mathbb{Z}_p)$ for a p–adic ring A if P_n is constant. When A is a finite ring, the space of continuous functions $\mathcal{C}(GL_g(\mathbb{Z}_p)/U(\mathbb{Z}_p), A)$ is equal to the space of locally constant functions $\mathcal{LC}(GL_g(\mathbb{Z}_p)/U(\mathbb{Z}_p), A)$, and we use $\mathcal{LC}$ instead of $\mathcal{C}$ when A is finite.

7.1.3. *Weight of automorphic forms and representations.* We define a coherent sheaf $\underline{\omega}_{m,n}$ on $T_{m,n}$ by $(\pi^*_{m,n}P_m) \otimes_{\mathbb{Z}} \mathcal{O}_{T_{m,n}}$. On $T_{m,n}$ with $n \geq m$, we have the universal isomorphism

$$I_{can} : \pi^*_{m,n}P_n \cong (\mathbb{Z}/p^n\mathbb{Z})^g;$$

so we have an action of $\mathrm{Gal}(T_{m,\infty}/S_m)$ on $\pi^*_{m,n}P_m$, and

$$\omega_{can} = I_{can} \otimes \mathrm{id} : \underline{\omega}_{m,n} \cong \mathcal{O}^g_{T_{m,n}}$$

is an isomorphism. Then we can identify $\mathcal{H}_{/T_{m,n}}$ with

$$p_Y : Y = Y_{m,n} = GL(\underline{\omega}_{m,n})/U_{can} \to T_{m,n}$$

on $V_{m,n}$ $(n \geq m)$, where U_{can} is the pull back of U under ω_{can}. Thus ω_{can} induces an isomorphism:

$$\omega^\kappa_{can} : p_{Y,*}(\mathcal{O}_Y[\kappa]) \cong R_{V_{m,n}}[\kappa].$$

We write $\underline{\omega}^\kappa_{m,n}$ for the sheaf $p_{Y,*}(\mathcal{O}_Y[\kappa])$ on $T_{m,n}$. By definition, $GL_g(\mathbb{Z}/p^n\mathbb{Z})$ acts on Y on the left. The Galois group $\mathrm{Gal}(T_{m,n}/S_m) = GL_g(\mathbb{Z}/p^n\mathbb{Z})$ acts on $\underline{\omega}^\kappa_{m,n}$ via the rational structure given from $\underline{\omega}_m$, and we then descend the sheaf to $\underline{\omega}^\kappa_m$ on S_m. In other words, for an $\mathcal{O}_{T_{m,n}}$–algebra A, $f \in H^0(Spec_{T_{m,n}}(A), \underline{\omega}^\kappa_{m,n})$ is a functorial rule assigning $f(X, \psi) \in R_A[\kappa]$ to $X_{/A}$ and $\psi : P_{n/X} \cong (\mathbb{Z}/p^n\mathbb{Z})^g_{/X}$. We let $h \in$

$GL_g(\mathbb{Z}/p^n\mathbb{Z}) = \mathrm{Gal}(T_{m,n}/S_m)$ act on f by $f \mapsto \{(X,\psi) \mapsto \rho(h)^{-1}f(X,h\psi)\}$. Then for any $\mathcal{O}_{T_{m,0}}$–algebra A,

$$A \mapsto H^0(\mathrm{Gal}(T_{m,n}/S_m), H^0(Spec(A) \times_{S_m} T_{m,n}, \underline{\omega}^{\kappa}_{m,n}))$$

defines a coherent sheaf on S_m (by the Hochschild–Serre spectral sequence), which we write $\underline{\omega}^{\kappa}_m$. We have

$$\begin{aligned}&H^0(S_m, \underline{\omega}^{\kappa}_m) = \\ &\quad \left\{f \in H^0(T_{m,m}, R_{V_{m,m}}[\kappa]) \middle| f(X,h\psi) = \rho(h)f(X,\psi) \text{ for } \forall h \in GL_g(\mathbb{Z}/p^m\mathbb{Z})\right\}.\end{aligned}$$

There is another description of $\underline{\omega}^{\kappa}_{m/S_m}$. Since $P_m \cong (\mathbb{Z}/p^m\mathbb{Z})^g$ on $T_{m,m}$, the action of $\mathrm{Gal}(T_{m,n}/S_m)$ on P_m extends to an action of the Galois group on $R_{\mathbb{Z}/p^m\mathbb{Z}}[\kappa]$, which determines an étale torsion sheaf P^{κ}_m over S_m. Then we have

$$\underline{\omega}^{\kappa}_m = \mathcal{O}_{S_m} \otimes_{\mathbb{Z}} P^{\kappa}_m.$$

In this construction, we have $\det^k(\underline{\omega}_m) = (\bigwedge^g \underline{\omega})^{\otimes k}$ and $Sym^k(\underline{\omega}_m) = \underline{\omega}^{k\omega_1}_m$ for the first standard dominant weight ω_1.

By our definition, for each $f \in H^0(S_m, \underline{\omega}^{\kappa}_m)$,

$$\varphi(f) = \omega^{\kappa}_{can}(f) \in H^0(T_{m,m}, R_{T_{m,m}}[\kappa]),$$

which can be regarded as a functorial rule assigning each test object

$$(X_{/S_m}, \psi : P_{m/X} \cong (\mathbb{Z}/p^m\mathbb{Z})^g_{/X})$$

a value $\varphi(f)(X,\psi) \in H^0(X, R_{\mathcal{O}_X}[\kappa])$ such that $\varphi(f)(X,h\psi) = \rho(h)\varphi(f)(X,\psi)$ for all $h \in GL_g(\mathbb{Z}/p^m\mathbb{Z})$ and $\varphi(f)(Y,\phi^*\psi) = \phi^*(\varphi(f)(X,\psi))$ for any morphism $\phi : Y \to X$ of S_m–schemes. Similarly, $\varphi \in V_{m,n}$ is a functorial rule assigning (X,ψ) a value $\varphi(X,\psi_n : P_n \cong (\mathbb{Z}/p^n\mathbb{Z})^g) \in H^0(X,\mathcal{O}_X)$ such that $\varphi(Y,\phi^*\psi_n) = \phi^*(\varphi(X,\psi_n))$ for any morphism $\phi : Y \to X$ of S_m–schemes. Thus we have a natural map of $H^0(T_{m,m}, R_{T_{m,m}}[\kappa])$ into $V_{m,m}$ associated to each linear form $\ell \in R_{V_{m,m}}[\kappa]^*$. The map associates $f \in H^0(T_{m,m}, R_{T_{m,m}}[\kappa])$ with a rule: $(X,\psi) \mapsto \ell(\varphi(f)(X,\psi))$, which is a matrix coefficient of $\varphi(f)(X,\psi)$.

We let $h \in GL_g(\mathbb{Z}_p)$ act on test objects by $(X,\psi) \mapsto (X,h\psi)$. In this way, we identify $GL_g(\mathbb{Z}_p)$ with $\mathrm{Gal}(T_{m,\infty}/S_m)$. For the Borel subgroup $B \subset GL(g)$, we put $T^B_{m,n}$ for the quotient $T_{m,n}/B(\mathbb{Z}/p^n\mathbb{Z})$. Thus $V^B_{m,n} = H^0(T^B_{m,n}, \mathcal{O}_{T^B_{m,n}})$ is made of a functorial rule $(X,\psi_n) \mapsto \varphi(X,\psi_n) \in H^0(X,\mathcal{O}_X)$ such that $\varphi(X,b\psi) = \varphi(X,\psi)$ for all ψ and $b \in B(\mathbb{Z}_p)$. We define similarly $V^U_{m,n}$ and $T^U_{m,n}$ for the unipotent subgroup $U \subset B$.

Let $e_1,\dots,e_g$ be the standard base $e_j = {}^t(\overbrace{0,\dots,0,1}^{j},0,\dots,0)$ of $(\mathbb{Z}/p^n\mathbb{Z})^g$, and we consider the standard filtration $\mathbf{1}_n : (\mathbb{Z}/p^n\mathbb{Z})^g = L_g \supset L_{g-1} \supset \cdots \supset L_0 = \{0\}$ given by $L_j = \sum_{i=1}^j (\mathbb{Z}/p^n\mathbb{Z})e_i$. Then $\psi_n^*\mathbf{1}_n$ gives a (full) filtration $fil = fil_{\psi_n}$ of P_n, and all full filtrations $P_n = P_n^{(g)} \supset P_n^{(g-1)} \supset \cdots \supset P^{(0)} = \{0\}$ of P_n are given in this way. Since the stabilizer of $\mathbf{1}_n$ is $B(\mathbb{Z}/p^n\mathbb{Z})$, we may regard $\varphi \in V^B_{m,n}$ as a functorial rule assigning a value $\varphi(X, fil_n) \in H^0(X,\mathcal{O}_X)$ to a test object (X, fil_n). To describe $V^U_{m,n}$ in this way, we need to bring in an isomorphism of graded modules: $\phi_n : gr(fil_n) \cong \oplus_{j=1}^g (\mathbb{Z}/p^n\mathbb{Z})$ inducing $\phi_n^{(j)} : P^{(j)}/P^{(j-1)} \cong (\mathbb{Z}/p^n\mathbb{Z})$. In other words, $T^U_{m,n}$ classifies triples (X, fil_n, ϕ_n). Since we pulled back the filtration $\mathbf{1}_n$

by ψ_n, $h \in GL_g(\mathbb{Z}_p)$ acts on $(X, fil_{\psi_n}, \phi_n)$ by $fil_n \mapsto \psi^{-1}h^{-1}\psi fil_n = (h\psi_n)^*\mathbf{1}_n$ and $\phi_n \mapsto \psi^{-1}h^{-1}\psi\phi_n$.

We can think of the image of $R_{V_{m,m}}[\kappa]$ inside $V^U_{m,m}[\kappa]$, which is the homomorphic image of $H^0(S_m, \underline{\omega}^\kappa_m)$ under $f \mapsto \ell_{can} \circ \varphi(f)$. Thus we have a natural map

$$\beta : H^0(S_m, \underline{\omega}^\kappa_m) \to V^U_{m,\infty}[\kappa], \tag{7.1}$$

where $V^U_{m,m}[\kappa]$ is the κ–eigenspace of the right action of T. The above map is injective if $m = \infty$. Then we define

$$R'_m = \bigoplus_{\kappa \in X(T)_+, \kappa \gg 0} H^0(S_m, \underline{\omega}^\kappa_m). \tag{7.2}$$

Here "$\gg$" implies *sufficiently regular*. See [PAF] 5.1.3 for a definition of regularity.

We assume to have a locally free sheaf $\underline{\omega}_{/S}$ of finite rank such that $\underline{\omega} \otimes_W W_m = \underline{\omega}_m$ for all m. From $\underline{\omega}$, we can create $\underline{\omega}^\kappa_{/S}$ as $\pi_*\mathcal{O}_{\mathrm{Flag}_U(\underline{\omega})}[\kappa]$ for $\pi : \mathrm{Flag}_U \to S$. The global sections $H^0(S, \underline{\omega})$ inject into $H^0(S_\infty, \underline{\omega}_\infty) = \varprojlim_m H^0(S_m, \underline{\omega}_m)$. We define

$$R' = \bigoplus_\kappa H^0(S, \underline{\omega}^\kappa) \hookrightarrow R'_\infty = \varprojlim_m R'_m.$$

We call an element of $H^0(S, \underline{\omega}^\kappa_{/S})$ a *false* automorphic form of weight κ. A *true* automorphic form is a global section in $H^0(M, \underline{\omega}^\kappa_{/M})$ for a compactification $M \supset S$ of S we will specify later. In other words, false automorphic forms are meromorphic sections over M with a specified location of their poles.

7.1.4. *Density theorems.* We suppose now that for all $\kappa \gg 0$, the short exact sequence:

$$0 \longrightarrow \underline{\omega}^\kappa \xrightarrow{p^m} \underline{\omega}^\kappa \longrightarrow \underline{\omega}^\kappa_m \longrightarrow 0$$

gives rise to an exact sequence:

$$\text{(Hyp1)} \qquad 0 \longrightarrow H^0(S, \underline{\omega}^\kappa) \xrightarrow{p^m} H^0(S, \underline{\omega}^\kappa) \longrightarrow H^0(S_m, \underline{\omega}^\kappa_m) \longrightarrow 0;$$
$$\text{(Hyp2)} \qquad V^U_{m,\infty} = V^U/p^mV^U.$$

This condition is obviously satisfied when S_m is affine. From this, we have

$$R'/p^mR' \cong R'_m \quad \text{and} \quad H^0(S, \underline{\omega}^\kappa)/p^mH^0(S, \underline{\omega}^\kappa) \cong H^0(S_m, \underline{\omega}^\kappa_m). \tag{7.3}$$

We now define a homomorphism

$$\beta(m) : R'_m \to V^U_{m,m}$$

in the following way. Over $T_{m,m}$, we have a canonical isomorphism $\omega_{can} = I_{can} \otimes \mathrm{id} : \underline{\omega}_{m,m} \cong \mathcal{O}^g_{T_{m,m}}$. Then

$$\beta(m)(\sum_{\kappa \gg 0} f_\kappa) = \{(X_{/T_{m,m}}, \psi) \mapsto \sum_\kappa \ell_{can}(\omega^\kappa_{can}(f_\kappa(X, \psi))\}$$

for $f_\kappa \in H^0(S_m, \underline{\omega}^\kappa_m)$. Here, the image of $\beta(m)$ actually falls in $V^U_{m,n}$ because $\ell_{can} \circ \rho_\kappa(u) = \ell_{can}$ for all $u \in U(\mathbb{Z}_p)$, and $\omega^\kappa_{can}(f_\kappa) \in R'_{V_{m,m}}[\kappa]$. By construction, $\beta(n) \mod p^m = \beta(m)$ for all $n > m$. Thus taking the projective limit, we have

$$\beta(\infty) : R'_\infty \to V^U = \varprojlim_m V^U_{m,\infty}.$$

Since S_m is flat over $W_m = W/p^mW$ and $T^U_{m,n} = T_{m,n}/U(\mathbb{Z}/p^n\mathbb{Z})$ is étale over S_m, $V^U_{m,\infty}$ is flat over W_m. Therefore, V^U is a W–flat $GL_g(\mathbb{Z}_p)$–module. This is a subtle point. If $\underline{\omega}$ extends to the compactification M_m, assuming M_m to be W_m–flat, $H^0(M_m, \underline{\omega}^\kappa)$ is also W_m–flat. It is easy to create an example in the Hilbert modular case such that $H^0(M_1, \underline{\omega}^\kappa)/H^0(M_1, \underline{\omega}_\kappa) \neq 0$ for the interior M of the Satake compactification of S if κ is not parallel but $\kappa \mod |(O/pO)^\times|$ is parallel. By the Koecher principle, if $\underline{\omega}^\kappa$ extends to the Satake compactification M^* as a line bundle, we have $H^0(M^*_{/W}, \underline{\omega}^\kappa) = H^0(M^*_{/W}, \underline{\omega}_\kappa)$; so, we cannot expect the good base-change property.

Since B normalizes U, we can think of the action of $T = B/U$ on V^U and the κ–eigenspace $V^U[\kappa]$ of V^U. By definition, $\beta = \beta(\infty)$ induces

$$\beta = \beta_\kappa : H^0(S, \underline{\omega}^\kappa) \hookrightarrow \varprojlim{}_m H^0(S_m, \underline{\omega}^\kappa_m) \to V^U[\kappa].$$

Proposition 7.1. *Suppose* (Hyp1,2) *for S. The above map β_κ is an injection.*

Proof. Since $T_{m,n}$ is faithfully flat and étale over S_m, we may make a base-change: $T^U_{m,n/S_m}$ to $T^U_{m,n/S_m} \times_{S_m} T_{m,m}$, and hence we may suppose that P is constant. Then $V^U_{m,\infty}$ is made up of locally constant functions on $GL_g(\mathbb{Z}_p)/U(\mathbb{Z}_p)$ with values in $V_{m,0}$. By taking the limit, V^U is the space $\mathcal{C}(GL_g(\mathbb{Z}_p)/U(\mathbb{Z}_p), V_{\infty,0})$ of continuous function on $GL_g(\mathbb{Z}_p)/U(\mathbb{Z}_p)$ with values in $V_{\infty,0} = \varprojlim{}_m V_{m,0}$. Then $H^0(S, \underline{\omega}^\kappa)$ is inside the limit of global sections of $\varprojlim{}_m R_{V_{m,0}}[\kappa]$, which injects into $\mathcal{C}(GL_g(\mathbb{Z}_p)/U(\mathbb{Z}_p), V_{\infty,0})[\kappa]$. This shows the assertion. □

We now put, for $\beta = \beta(\infty)$

$$D' = \beta(R')\left[\frac{1}{p}\right] \bigcap V^U = \beta(\bigoplus_{\kappa \gg 0} H^0(S, \underline{\omega}^\kappa))\left[\frac{1}{p}\right] \bigcap V^U.$$

Theorem 7.2. *Suppose* (Hyp1-2) *for S. The inclusion $\beta = \beta(\infty) : D' \to V^U$ induces an isomorphism*

$$D'/p^m D' \cong V^U/p^m V^U \quad \text{for all } m.$$

In other words, D' is p–adically dense in V.

Proof. The injectivity of $D'/p^m D' \to V^U/p^m V^U$ follows from the definition (see [K1] Appendix III) (or as easily seen after faithfully flat extension to $T_{m,m}$).

We thus need to prove that $D'/pD' \to V^U/pV^U = V^U_{1,\infty}$ is surjective. Since $T_{m,n}/S_m$ is étale finite, replacing S_m by $T_{m,\infty}$, we may assume that P is constant (see [K1] Appendix III pages 364-5), because we can recover the global sections of $\underline{\omega}^\kappa$ over S_m as Galois invariants of that over $T_{m,n}$. Then

$$\mathcal{O}_{T^U_{1,n}} = \mathcal{O}_{S_1} \otimes_W W[GL_g(\mathbb{Z}/p^n\mathbb{Z})/U(\mathbb{Z}/p^n\mathbb{Z})] = \mathcal{O}_{T^U_{1,0}}[GL_g(\mathbb{Z}/p^n\mathbb{Z})/U(\mathbb{Z}/p^n\mathbb{Z})].$$

This shows $V^U_{1,\infty} = \mathcal{LC}(GL_g(\mathbb{Z}_p)/U(\mathbb{Z}_p), V_{1,0})$, where $\mathcal{LC}(GL_g(\mathbb{Z}_p)/U(\mathbb{Z}_p), V_{1,0})$ is the space of locally constant functions on the p–adic analytic space $GL_g(\mathbb{Z}_p)/U(\mathbb{Z}_p)$ with values in $H^0(S_1, \mathcal{O}_{S_1}) = V_{1,0}$. Writing $V^U_{\infty,0}$ as a union of W–free modules X of finite rank, we have $\mathcal{LC}(GL_g(\mathbb{Z}_p)/U(\mathbb{Z}_p), V^U_{\infty,0}) = \bigcup_X \mathcal{LC}(GL_g(\mathbb{Z}_p)/U(\mathbb{Z}_p), X)$. Thus we need to prove that

$$\mathcal{LC}(GL_g(\mathbb{Z}_p)/U(\mathbb{Z}_p), X/pX) = \mathcal{D}_X/p\mathcal{D}_X,$$

where $\mathcal{D}_X$ is the space of polynomial functions of homogeneous degree $\gg 0$ (with coefficients in $K = W\left[\frac{1}{p}\right]$ on the flag manifold Flag_U) which has values in X over $GL_g(\mathbb{Z}_p)$. This last fact follows from Mahler's theorem of the density of the linear span of the binomial polynomials in the space of continuous functions on $\mathbb{Z}_p$ with values in $\mathbb{Z}_p$ (see [PAF] Theorem 8.3 for more details of the use of Mahler's theorem). $\square$

We now assume that there exists a proper flat scheme $M_{/W}$ such that $S \subset M$, and $M - S$ is a proper closed subscheme of codimension ≥ 1. We further assume that $\underline{\omega}_{/S}$ extends to M. Then automatically $\underline{\omega}_m^\kappa$ extends to M_m by the theory of flag varieties. The sheaf $\underline{\omega}_{/M}$ is uniquely determined by $\{\underline{\omega}_{m/M_m}\}_{m=1,2,...}$ by the formal existence theorem of Grothendieck [EGA] III.5.1.4. By the properness of M, $H^1(M, \underline{\omega}^\kappa)$ is a W–module of finite type. Thus taking the projective limit with respect to m of the exact sequences:

$$0 \longrightarrow H^0(M, \underline{\omega}^\kappa) \otimes_W W_m \longrightarrow H^0(M_m, \underline{\omega}_m^\kappa) \longrightarrow H^1(M_m, \underline{\omega}^\kappa)[p^m] \longrightarrow 0,$$

we get $\varprojlim_m H^0(M_m, \underline{\omega}_m^\kappa) = H^0(M, \underline{\omega}_m^\kappa)$. Let $R_m = \bigoplus_{\kappa \gg 0} H^0(M_m, \underline{\omega}_m^\kappa)$ and $R = \bigoplus_{\kappa \gg 0} H^0(M, \underline{\omega}_{/M}^\kappa)$. Then we know that R is p–adically dense in $R_\infty = \varprojlim_m R_m$. By definition, $R \subset R'$. Note that $\det(\underline{\omega})^{p-1}$ is trivial on S_1. Let $a \in H^0(S_1, \det(\underline{\omega})^{p-1})$ be the section corresponding to $1 \in \det(\underline{\omega}_1)^{p-1} \cong \mathcal{O}_{S_1}$. We assume that a extends to M_1 so that it vanishes outside S_1. Suppose that we have a section $E \in H^0(M, \det(\underline{\omega})^{t(p-1)})$ such that $E \mod \varpi = a^t$. By further raising power, that is, replacing E by E^{p^m}, we may assume that $E \mod p = a^t$. Then by definition,

$$H^0(S_m, \underline{\omega}_m^\kappa) = \varinjlim_n \frac{H^0(M_m, \underline{\omega}_m^\kappa) \otimes \det^{nt(p-1)}(\underline{\omega}_m))}{E^n}.$$

We would like to show that $\beta(R[\frac{1}{p}]) \cap V^U$ is dense in V^U. Pick $\sum_\kappa f_\kappa \in p^m V^U$ for $f_\kappa \in H^0(S, \underline{\omega}^\kappa)$. We need to approximate $f = f_\kappa$ modulo $p^{m+1}V^U$ by an element in $H^0(M, \underline{\omega}^\kappa \otimes \det(\underline{\omega})^k)$. This section $f \in H^0(S, \underline{\omega}^\kappa)$ can be written as $f \equiv g_\ell/E^\ell \mod p^{m+1}$ for $g_\ell \in H^0(M, \underline{\omega}^\kappa \otimes \det^\ell(\underline{\omega}))$. Then for $k > \ell$, we have $f \equiv g_\ell E^{k-\ell}/E^k \mod p^{m+1}$. Thus we may assume that $k = p^m t(p-1)$. Then as a function of (X, ψ), $E^k \mod p^{m+1}$ is a constant. Thus $f \equiv g_\ell E^{k-\ell} \mod p^{m+1} R_\infty$. This shows the density of $\beta(R\left[\frac{1}{p}\right]) \bigcap V^U$ in V^U.

Corollary 7.3. *Suppose the following conditions in addition to* (Hyp1-2) *for S:*

1. *$S \subset M$ for a proper flat scheme $M_{/W}$ such that $S_m \subset M_m = M \otimes_W W_m$ is Zariski dense for all m;*
2. *$\underline{\omega}$ extends to a locally free vector bundle on M of rank g;*
3. *there exist an integer $t > 0$ and a section $E \in H^0(M, \det^{t(p-1)}(\underline{\omega}))$ such that $E \mod \varpi$ is the constant section 1 generating $\det^{t(p-1)}(\underline{\omega}_1) \cong \mathcal{O}_{S_1}$;*
4. *$M - S$ is the zero locus of the section E.*

Put

$$D = \beta\left(\bigoplus_{\kappa \gg 0} H^0(M, \underline{\omega}^\kappa)\right)\left[\frac{1}{p}\right] \bigcap V^U.$$

Then D is p–adically dense in V^U.

7.1.5. *p–Ordinary automorphic forms.* We now suppose to have a projector e (so $e^2 = e$) acting (continuously) on V^U, which projects down $V^U[\kappa]$ onto a W–free module of finite rank (for all $\kappa \gg 0$). We put $\mathcal{V}_U = \varinjlim_n V^U/p^n V^U = \varinjlim_n V^U_{n,\infty}$.

We have $\mathcal{V}_U[\kappa] = \varinjlim_m V^U_{m,m}[\kappa]$. Since V^U is W–flat, $\mathcal{V}_U$ is p–divisible, and its direct summand $e\mathcal{V}_U$ is p–divisible.

In practice, the projector e will be constructed so that it brings $\mathcal{V}_U[\kappa]$ down onto $eH^0(M_{/W}, \underline{\omega}^\kappa \otimes \mathbb{T}_p)$ for $\kappa \gg 0$, where $\mathbb{T}_p = \mathbb{Q}_p/\mathbb{Z}_p$. This implies

$$e\mathcal{V}_U[\kappa] = eH^0(S_{/W}, \underline{\omega}^\kappa \otimes_{\mathbb{Z}_p} \mathbb{T}_p) = eH^0(M_{/W}, \underline{\omega}^\kappa \otimes_{\mathbb{Z}_p} \mathbb{T}_p) \tag{7.4}$$

if $\kappa \gg 0$. By (Hyp1), $H^0(S_{/W}, \underline{\omega}^\kappa \otimes_{\mathbb{Z}_p} \mathbb{T}_p)$ is p–divisible. By assuming (7.4), $e\mathcal{V}_U[\kappa]$ is p–divisible. Since $H^0(M_{/W}, \underline{\omega}^\kappa)$ is a W–module of finite type, $e\mathcal{V}_U[\kappa]$ is a p–divisible module of finite corank. In any case, we just assume that $e\mathcal{V}[\kappa]$ is p–divisible and of finite corank for $\kappa \gg 0$.

Let $\mathcal{V}_U^*$ be the Pontryagin dual module of $\mathcal{V}_U$. Since $\mathcal{V}_U$ is a discrete $T(\mathbb{Z}_p)$–module, $\mathcal{V}_U^*$ is a compact $W[[T(\mathbb{Z}_p)]]$–module. Let $T(\mathbb{Z}_p)_p = \Gamma_T$ be the p–profinite part of $T(\mathbb{Z}_p)$. Then $T(\mathbb{Z}_p) = \Gamma_T \times \Delta$ for a finite group Δ of order prime to p. We fix a character $\overline{\chi} : \Delta \to \mathbb{F}^\times$ for $\mathbb{F} = W/\varpi W$. Then we write $\tilde{\chi} : \Delta \to W^\times$ for the Teichmüller lift of $\overline{\chi}$. We write $X_{\tilde{\chi}} \subset X_+(T)$ for the set of algebraic characters $\kappa : T \to \mathbb{G}_m$ such that $\kappa \equiv \tilde{\chi} \mod \varpi$ and κ is sufficiently regular so that the above equation (7.4) holds. Then $X_{\tilde{\chi}}$ is Zariski-dense in *Spec*$(W[[\Gamma_T]])(W)$. We write $\mathcal{V}^*_{ord}$ for $e\mathcal{V}_U^*$. Let us decompose

$$\mathcal{V}^*_{ord} = \bigoplus_{\tilde{\chi} \in \widehat{\Delta}} \mathcal{V}^*_{ord}[\tilde{\chi}]$$

into the direct sum of the $\tilde{\chi}$–eigenspaces under the action of Δ. The $\widetilde{\chi}$–eigenspace $\mathcal{V}^*_{ord}[\tilde{\chi}]$ is a compact module over $W[[\Gamma_T]]$. Then by (7.4), $\mathcal{V}^*_{ord}[\tilde{\chi}] \otimes_{W[[\Gamma_T]],\kappa} W$ is W–free of finite rank $s(\tilde{\chi})$ for $\kappa \in X_{\tilde{\chi}}$. Thus, by topological Nakayama's lemma, $\mathcal{V}^*_{ord}[\tilde{\chi}]$ is a $W[[\Gamma_T]]$–module of finite type with minimum number $s(\tilde{\chi})$ of generators. Since $X_{\tilde{\chi}}$ is Zariski-dense in *Spec*$(W[[\Gamma_T]])$, we see that $\mathcal{V}^*_{ord}[\tilde{\chi}]$ is $W[[\Gamma_T]]$–free of rank $s(\tilde{\chi})$. Thus we have, assuming (7.4) for the middle equality,

$$\begin{aligned} \operatorname{rank}_{W[[\Gamma_T]]} \mathcal{V}^*_{ord}[\tilde{\chi}] &= \operatorname{rank}_W \mathcal{V}^*_{ord}[\tilde{\chi}] \otimes_{W[[\Gamma_T]],\kappa} W \\ &= \operatorname{rank}_W (eH^0(M, \underline{\omega}^\kappa) \otimes_{\mathbb{Z}_p} \mathbb{T}_p)^* = \operatorname{rank}_W eH^0(M, \underline{\omega}^\kappa) \end{aligned} \tag{7.5}$$

for all $\kappa \in X_{\tilde{\chi}}$. Therefore we get

Theorem 7.4. *Suppose* (Hyp1-2)*, the existence of the idempotent* $e : V^U \to V^U$ *as above and the assumptions of Corollary 7.3. Then* $\mathcal{V}^*_{ord}$ *is a well controlled* $W[[T(\mathbb{Z}_p)]]$*–projective module of finite type. If we assume* (7.4)*, this means that*

$$\mathcal{V}^*_{ord} \otimes_{W[[T(\mathbb{Z}_p)]],\kappa} W \cong \operatorname{Hom}_W(eH^0(M, \underline{\omega}^\kappa), W)$$

canonically if κ *is sufficiently regular. For each* $\tilde{\chi}$*–component,* $\mathcal{V}^*_{ord}[\tilde{\chi}]$ *is free of finite rank over* $W[[\Gamma_T]]$ *for the maximal* p*–profinite subgroup* Γ_T *of* $T(\mathbb{Z}_p)$*.*

7.1.6. *Construction of the projector* e_{GL}. We are going to construct an approximation e_{GL} of the projector e. In the paper [H02] Section 2.6, we wrote: "e_{GL} is constructed using solely local data of the Galois group $\mathrm{Gal}(T_{m,\infty}/S_m) = GL_n(\mathbb{Z}_p)$, while the projector e will be constructed as $e = e_G e_{GL}$ for a global projector e_G depending on the group G." This statement is misleading. We actually need a global input. To explain this point, let us introduce the expanding semi-group of $GL_n(\mathbb{Q}_p)$. Writing $? = B$ and U, we introduce two subgroups $I_{B,n}$ and $I_{U,n}$ of $GL_g(\mathbb{Z}_p)$ by

$$I_{?,n} = \left\{ x \in GL_g(\mathbb{Z}_p) \middle| x \mod p^n \in ?(\mathbb{Z}/p^n\mathbb{Z}) \right\}. \tag{7.6}$$

Let diag$[X_1,\dots,X_j]$ denote the diagonal matrix whose diagonal blocks are given by X_1 to X_j from the top. We first look at the semi-group given by

$$D = D_{GL(g)} = \left\{\mathrm{diag}[p^{e_1},\dots,p^{e_g}]\middle| e_1 \le e_2 \le \cdots \le e_g\right\}.$$

Then $\Delta^?_n = I_{?,n} D I_{?,n}$ $\Delta^U_\infty = U(\mathbb{Z}_p)DU(\mathbb{Z}_p)$ and $\Delta^B_\infty = B(\mathbb{Z}_p)DB(\mathbb{Z}_p)$ are semi-groups, and we call them expanding semi-groups. If confusion is unlikely, we simply write Δ for one of these semi-groups.

The global input we need comes from the fact that $T_{m,n}$ in our application classifies not just trivializations of P_n but abelian varieties X with $X[p^n]^{et} \cong P_n$. In other words, each $g \in GL_n(\mathbb{Q}_p)$ acts on S_m by an appropriate isogeny of abelian varieties classified, and it acts not only the étale quotient of the p–divisible group of the abelian variety but also on the connected component of the p–divisible group. The action changes the isomorphism class of the abelian varieties, and hence it acts on S_m through endomorphisms (not necessarily through automorphisms).

Since at this point, we do not assume that S_m classifies abelian varieties, we instead assume to have such an action of the expanding semi-group (as defined below) on S_m which is at worst "radiciel" mod p; so, it does not affect the étale trivialization P_n. This action $\delta : S_m \to S_m$ sends an S_m–scheme $X \xrightarrow{f} S_m$ to $\delta \cdot X = X \times_{S_m,\delta} S_m$.

We consider the following triples:

$$\mathcal{X} = (X_{/S_m}, fil_n, \phi_n : \oplus_{j=1}^g \mathbb{Z}/p^n\mathbb{Z} \cong gr(fil_n)),$$

where $fil_n : P_{n/X} = P_n^{(g)} \supset P_n^{(g-1)} \supset \cdots \supset P_n^{(0)} = \{0\}$ with $\phi_j : \mathbb{Z}/p^n\mathbb{Z} \cong P_n^{(j)}/P_n^{(j-1)}$ for $j = 1,\dots,g$. If P is constant, the space classifying the above test objects over S_m is given by $T^U_{m,n} = T_{m,n}/U(\mathbb{Z}/p^n\mathbb{Z})$. Similarly, the classifying space of couples (X, fil_n) over S_m is given by $T^B_{m,n} = T_{m,n}/B(\mathbb{Z}/p^n\mathbb{Z})$. On test objects over $T_{m,n}$, we have a natural action of $h \in GL_n(\mathbb{Z}_p))$ given by $(X,\psi) \mapsto (X, h\psi)$. Writing $fil_n = \psi^{-1}\mathbf{1}_n$, we then see that $\psi^{-1}h^{-1}\mathbf{1}_n = \psi^{-1}\mathbf{1}_n = h \cdot fil_n$. Thus the Galois action on filtrations is given by $h \cdot fil = (\psi)^{-1}h^{-1}\psi)P_n^{(j)})$ and $h \cdot \phi = (\psi)^{-1}h^{-1}(\psi)\phi$, where $\psi : P_n \cong (\mathbb{Z}/p^n\mathbb{Z})^g$ such that $\psi^*(\mathbf{1}_n, \mathrm{id}) = (fil_n, \phi_n)$ for the standard identification $\mathrm{id} : gr(\mathbf{1}_n) \cong (\mathbb{Z}/p^n\mathbb{Z})^g$. Thus these test objects are always invariant under $U(\mathbb{Z}_p)$. The new test objects (X, fil_n, ϕ_n) are useful in defining an isogeny action of $\delta \in \Delta$ and in constructing the idempotent e_{GL}, although we may stick to the test objects $(X, \psi_n : P_n \cong (\mathbb{Z}/p^n\mathbb{Z})^g)$ if we want. We assume that

(d1) δ induces an isomorphism $\delta^* fil_{n/\delta\cdot X} \cong fil_{n/X}$ compatible with the action of the semi-group on fil_n (this holds if δ mod p is radiciel), where the action of $\delta \in \Delta$ on fil_n is the multiplication by δ up to scalars (as we specify later);

(d2) $h \cdot X = X$ if $h \in GL_n(\mathbb{Z}_p)$.

Here is how to create the idempotent e_{GL} using p–Hecke operators (modulo p^m). We study Hecke operators t_j $(j = 1,\dots,g)$ acting on V^U and $V^?_{m,n}$ for $? = B$ and U. We can thus think of the Hecke ring $R(I_{?,n},\Delta)$ $(n = 1,2,\dots,\infty)$ made of $\mathbb{Z}$–linear combinations of double cosets $I_{?,n}\delta I_{?,n}$ for $\delta \in \Delta^?_n$. These two algebras are commutative and all isomorphic to the polynomial ring $\mathbb{Z}[t_1,\dots,t_g]$ for $t_j = I_{?,n}\alpha_j I_{?,n}$ with $\alpha_j = \mathrm{diag}[1_{g-j}, p1_j]$. A key to getting this isomorphism is that once we choose a decomposition: $U(\mathbb{Z}_p)\alpha_j U(\mathbb{Z}_p) = \bigsqcup_j U(\mathbb{Z}_p)\delta_j$, then $I_{?,n}\alpha_j I_{?,n}$ for any n and ? is decomposed in the same way: $I_{?,n}\alpha_j I_{?,n} = \bigsqcup_j I_{?,n}\delta_j$ (see [PAF]

(5.3)). We have for $\alpha = \prod_{j=1}^{g-1} \alpha_j$

$$I_{?,n+1}\backslash I_{?,n+1}\alpha I_{?,n} \cong I_{?,n}\backslash I_{?,n}\alpha I_{?,n} \cong I_{?,n+1}\backslash I_{?,n+1}\alpha I_{?,n+1}. \tag{7.7}$$

For $\delta \in \Delta_1^B$, the action $\rho(\delta^{-1})$ is well defined on $R_A[\kappa]$ for any p–adic ring A, because $\rho(\delta^{-1})\phi(y) = \phi(\delta y)$ for $y \in GL(g)/U$. Decompose $I_{?,n}\delta I_{?,n} = \bigsqcup_j I_{?,n}\delta_j$ and regarding $f \in H^0(T^?_{m,n}, \underline{\omega}^\kappa)$ as a function of test objects $\mathcal{X}_{/T_{m,\infty}}$, we define

$$f|[I_{?,n}\delta I_{?,n}](\mathcal{X}) = \sum_j \rho_\kappa(\delta_j^{-1}) f(\delta_j \mathcal{X}), \tag{7.8}$$

where $\delta\mathcal{X} = (\delta \cdot X, \delta(\delta^* fil_n))$. The sum above is actually "heuristic", because if the action of δ on S is wildly ramified (that is, purely inseparable in characteristic p), we need to replace the sum by the trace as already described in 3.1.3. We will clarify this point in 8.2.1 more carefully; so, for the moment, we content ourselves with this heuristic action.

Although we have not yet specified the action of the element $\delta \in \Delta_n^?$ on $\delta^* fil_n$, if it exists, then the operator is well defined independent of the choice of δ_j because for $u \in I_{?,n}$,

$$\rho((u\delta_j)^{-1}) f(u\delta_j\mathcal{X}) = \rho((\delta_j)^{-1})\rho(u)^{-1} f(u\delta_j\mathcal{X}) = \rho(\delta_j^{-1}) f(\delta_j\mathcal{X}).$$

Further, by (7.7),

$$\textit{for } f \in H^0(T^B_{m,n}, \underline{\omega}^\kappa),\ f|t(p)^{n-1} \in H^0(T^B_{m,1}, \underline{\omega}^\kappa), \tag{7.9}$$

where $t(p) = \prod_{j=1}^{g-1} t_j$. When P_n comes from a universal abelian scheme, we have a natural isogeny action on test objects, and in this way, we can define Hecke operator on $H^0(S_m, \underline{\omega}^\kappa)$.

Since fil_n is an element of the flag variety of $(\mathbb{Z}/p^n\mathbb{Z})^g$, to study the action of Δ on filtrations, we study general flag varieties. For each commutative ring A, we consider the free module $L = L(A) = A^g$ and the flag space

$$y(A^g) = \left\{(L_i)\middle|L = L_g,\ L_i \supset L_{i-1},\ L_i/L_{i-1} \cong A \text{ for } i = 1, \cdots, g\right\}, \tag{7.10}$$

$$\mathcal{Y}(A^g) = \left\{(L_i, \phi_i)\middle|(L_i) \in y(A),\ \phi_i : A \cong L_i/L_{i-1} \text{ for } i = 1, \ldots, g\right\}. \tag{7.11}$$

We can extend the above definition to P_n over S_m: We define for each scheme T/S_m

$$y(P_n)_{/T} = \left\{(P^{(i)}_{n/T})\middle|P_n = P^{(g)}_n,\ P^{(i)}_n \supset P^{(i-1)}_n,\ \frac{P^{(i)}_n}{P^{(i-1)}_n} \cong \mathbb{Z}/p^n\mathbb{Z}\ (i = 1, \cdots, g)\right\},$$

$$\mathcal{Y}(P_n)_{/T} = \left\{(P^{(i)}_{n/T}, \phi_i)\middle|(P^{(i)}_n) \in y(P_n)_{/T},\ \phi_i : \mathbb{Z}/p^n\mathbb{Z} \cong \frac{P^{(i)}_n}{P^{(i-1)}_n}\ (i = 1, \ldots, g)\right\}.$$

After a finite étale extension to T/S_m, the spaces $y(P_n)$ and $\mathcal{Y}(P_n)$ get isomorphic to $y((\mathbb{Z}/p^n\mathbb{Z})^g) \times_{S_m} T$ and $\mathcal{Y}((\mathbb{Z}/p^n\mathbb{Z})^g) \times_{S_m} T$. Writing the standard base of L as $e_1, \ldots, e_g$, we define $\mathbf{1} = (\sum_{j=1}^i Ae_i, \phi_i = \mathrm{id}) \in \mathcal{Y}(A^g)$, which we call the origin. We may let $GL_g(A)$ act on $\mathcal{Y}(A^g)$ and $y(A^g)$ by $x((L_i), \phi_i) = (xL_i, x \circ \phi_i)$. Then $GL_g(A)/U(A) \cong \mathcal{Y}(A^g)$ by $xU(A) \mapsto x\mathbf{1}$. Now we assume that A to be a p–adic ring, that is, $A = \varprojlim_n A/p^nA$. We then define

$$\mathcal{Y}_n(A^g) = \{(L_i, \phi_i)|(L_i/p^nL_i) = \mathbf{1} \in y(A/p^nA)\}.$$

Similarly, we define $\mathcal{Y}_{n/T}(P_{n'})$ for $n \leq n' \leq \infty$. We note that $\mathcal{Y}_n(\mathbb{Z}_p^g) \cong I_{U,n}/U(\mathbb{Z}_p)$ via $x\mathbf{1} \leftrightarrow x$ and similarly $y_n(\mathbb{Z}_p^g) \cong I_{B,n}/U(\mathbb{Z}_p)$. So we have the conjugate action of Δ on these spaces introduced in Section 4.

We now write down explicitly the conjugate action of the semi-group Δ on $\mathcal{Y}_n(A^g)$. Since $y(\mathbb{Z}_p^g) = y(\mathbb{Q}_p^g)$ (because $y = \mathrm{Flag}_B$ is projective), the group $GL_g(\mathbb{Q}_p)$ acts naturally on $y(\mathbb{Q}_p^g)$. This action is described as follows: Take $x \in GL_g(\mathbb{Q}_p)$. Then $x(L_i) = (xL_i \otimes \mathbb{Q}_p \cap L(\mathbb{Z}_p)) \in y(\mathbb{Z}_p^g)$. We write $x(L_i) = (x \cdot L_i)$, that is, $x \cdot L_i = xL_i \otimes \mathbb{Q}_p \cap L(\mathbb{Z}_p)$. We now define an action of the semi-group $\Delta_n^B = I_{B,n} D_{GL(g)} I_{B,n}$ on $\mathcal{Y}_n(\mathbb{Z}_p^g)$. For each $u\delta u' \in \Delta_n^B$ with $u, u' \in I_{B,n}$ and $\delta \in D_{GL(g)}$. We write $\delta = \mathrm{diag}[p^{e_1(\delta)}, \ldots, p^{e_g(\delta)}]$. Then for $(L_i, \phi_i) \in \mathcal{Y}_n(\mathbb{Z}_p^g)$, $p^{-e_i(\delta)}\delta : L_i/L_{i-1} \to \delta \cdot L_i/\delta \cdot L_{i-1}$ is a surjective isomorphism as shown in [H95] page 438. Since $I_{B,n}$ acts naturally on flag varieties, the above action of $D_{GL(g)}$ extends an action of the semi-group Δ_n^B. For a given $\mathcal{X} = (X, fil, \phi) = (X, \psi^{-1}\mathbf{1})$, ψ brings "fil" to $\mathbf{1}$, and hence the action of Δ_n^B defined on the neighborhood of $\mathbf{1}$ (after conjugation by ψ) is enough to get an association: $\mathcal{X} \mapsto \{\delta_j \mathcal{X}\}$. By this, after a change of the base scheme S_m (for example to $T = T_{m,\infty}$) to trivialize $\mathcal{Y}(P)$, we have an action of Δ_n^B on $\mathcal{Y}(P)$. However this is sufficient to define the Hecke operators $[I_{?,n}\delta I_{?,n}]$ acting on $H^0(T_{m,n/S_m}, \underline{\omega}^\kappa)$ by the following reason: After extending scalar, define $f|[I_{?,n}\delta I_{?,n}]$ by (7.8). The formation of $f|[I_{?,n}\delta I_{?,n}]$ commutes with the base-change, in other words, it commutes with the Galois action of the base: $\mathrm{Gal}(T/S_m)$; so, $f|[I_{?,n}\delta I_{?,n}]$ is actually defined over the original base scheme $T^?_{m,n/S_m}$. This justifies the contraction property (7.9).

Let $t_j = U(\mathbb{Z}_p)\alpha_j U(\mathbb{Z}_p)$ in $R(U(\mathbb{Z}_p), \Delta_\infty)$ with $\alpha_j = \mathrm{diag}[1_j, p1_{g-j}]$, and define $t(p) = \prod_{j=1}^g t_j$. As shown in [H95] Lemma 3.1, α for $\alpha = \prod_{j=1}^{g-1} \alpha_j$ contracts $y_n(\mathbb{Z}/p^{n+1}\mathbb{Z})$ to the origin $\mathbf{1}_{n+1}$. Identifying $y_1(\mathbb{Z}/p^n\mathbb{Z})$ with $I_{B,1}/I_{U,n}$, if the filtration fil_n corresponds to $x \in I_{B,n}$, then the filtration is given by $\sum_{j=1}^i (\mathbb{Z}/p^n\mathbb{Z})x_j$ for the j–th column vector x_j of x. Choose a representative set $U(\mathbb{Z}_p)\alpha U(\mathbb{Z}_p) = \bigsqcup_{u \in R} U(\mathbb{Z}_p)\alpha_u$. Then we have $I_{B,n+1}\alpha I_{U,n} = \bigsqcup_{u \in R} I_{B,n+1}\alpha_u$, and $\alpha_u x^{-1} = x_u^{-1}\alpha_{u'}$ for some $u' \in R$. This $x^{-1} \mapsto x_u^{-1}$ coincides with the action of α_u on the flag variety $\mathcal{Y}_1((\mathbb{Z}/p^n\mathbb{Z})^g)$ if one identifies elements in $I_{B,1}$ with a flag. Here we need to use x^{-1} instead of x, because the action of $h \in GL_g(\mathbb{Z}_p)$ on filtrations is given by $fil \mapsto h \cdot fil = (\psi^{-1}h^{-1}\psi)fil$ as already explained. The element x_u gives rise to a couple $(\alpha_u \cdot X, fil_{n+1,u} = \alpha_u(\alpha_u^* fil_{n+1}))$, which is uniquely determined independently of the choice of α_u. We then define for $f \in V_{m,n+1}^B$ $f|\mathbf{t}(p)(X, fil_n) = \sum_{u \in R} f(\alpha_u \cdot X, fil_{n+1,u})$. Similarly, if we start from $f \in V_{m,n}^U$, by the same process, we get $x_u \in I_{U,n}/U(\mathbb{Z}_p)$ corresponding to $(X, fil_{u,n}, \phi_u)$, because we still have $I_{U,n}\alpha I_{U,n} = \bigsqcup_{u \in R} I_{U,n}\alpha_u$. We then define $f|\mathbf{t}(p)(X, fil_n, \phi_n) = \sum_{u \in R} f(\alpha_u \cdot X, fil_{u,n}, \phi_u)$ and define the idempotent e_{GL} by $e_{GL} = \lim_{n \to \infty} \mathbf{t}(p)^{n!}$ whenever it is well defined.

As we have seen in Section 4, $\kappa(\alpha)\mathbf{t}(p) = t(p)$ on $H^0(S_\infty, \underline{\omega}_\infty^\kappa)$, because on $\underline{\omega}^\kappa$, we used the action of Δ coming from schematic induction.

7.1.7. *Axiomatic control result.* In this subsection, we describe a simple prerequisite to have the control theorem relating false automorphic forms (sections over S) to true automorphic forms (sections over M). Later we will verify the requirement for automorphic vector bundles on Shimura varieties of PEL type.

Since in this general situation, S_m is not supposed to classify anything; so, we cannot define Hecke operators acting on $H^0(S_m,\underline{\omega}^\kappa)$ in this generality. Anyway, we suppose to have a Hecke operator $\tau(p)$ acting on $H^0(S_m,\underline{\omega}^\kappa)$ and $H^0(M_m,\underline{\omega}^\kappa)$ such that $\tau(p)\equiv \mathbf{t}(p)$ on $H^0(S_1,\underline{\omega}^\kappa)$ if $\kappa\gg 0$. We define $e_0=\lim_{n\to\infty}\tau(p)^{n!}$.

Now suppose that there exist further two projectors e_G acting on $V^U_{m,\infty}$ and e°_G on $H^0(M_m,\underline{\omega}^\kappa)$ for $\kappa\gg 0$ (depending on the reductive group G) such that $e_G e_{GL}=e_G$, $e^\circ_G e_0=e^\circ_G$ and $e_G\equiv e^\circ_G \mod p$, that is, they are equal each other on $e_G H^0(S_1,\underline{\omega}^\kappa)$. In addition to the above conditions, writing K for the field of fractions of W, we suppose the following two conditions:

(C) $e_G(Ef)=E(e_Gf)$ *for* $f\in H^0(S_1,\underline{\omega}^\kappa)$,

(F) $\dim_K e^\circ_G H^0(M_{/K},\underline{\omega}^\kappa\otimes\det^k(\underline{\omega})_{/K})$ *is bounded independent of* k.

Let $\overline{f}_1,\overline{f}_2,\dots$ be a sequence of linearly independent elements in $e_GH^0(S_1,\underline{\omega}^\kappa)$ over W_1. Since $H^0(S_{/W},\underline{\omega}^\kappa_{/W})\otimes_W W_1=H^0(S_1,\underline{\omega}^\kappa)$ (Hyp1), we can lift $\overline{f}_i$ to $f_i\in H^0(S_{/W},\underline{\omega}^\kappa)_{/W})$ so that $\overline{f}_i=(f_i \mod p)$. Then for any given integer $N>0$, we can find a sufficiently large integer m such that $E^mf_i\in H^0(M_{/W},\underline{\omega}^\kappa_{/W})$. Since multiplication by E is an isomorphism on S_1 (by definition of $S_1=M_1\left[\frac{1}{E}\right]$), by (C) and $e_G\equiv e^\circ_G \mod p$, $\{(e^\circ_G(E^mf_i) \mod p)\}_{i=1,\dots,N}$ are linearly independent over W_1; so, $\{e^\circ_G(E^mf_i)\}_{i=1,\dots,N}$ are linearly independent over W. This implies

$$\dim_K e^\circ_G H^0(M_{/K},\underline{\omega}^\kappa\otimes\det{}^{t(p-1)m}(\underline{\omega})_{/K})\ge N.$$

If $\mathrm{rank}_{W_1} e_GH^0(S_1,\underline{\omega}^\kappa)=\infty$, we can take N to be arbitrarily large, which contradicts the boundedness (F) of the dimension. Thus $\mathrm{rank}_{W_1} e_GH^0(S_1,\underline{\omega}^\kappa)$ has to be finite, and $\mathrm{rank}_{W_1} e_GH^0(S_1,\underline{\omega}^\kappa\otimes\det^{k(p-1)}(\underline{\omega}))$ is independent of k. Thus the existence of the desired projector follows from (F), (C) and (7.4).

The condition (F) can be proven in our application via group cohomology using the (generalized) Eichler-Shimura isomorphism combined with the p–adic density of D_{cusp} in D'_{cusp} (see [H95] for such boundedness for forms of $GL(n)$, [TiU] for inner forms of $GSp(2n)$ and [Mo] for more general groups).

The condition (C) can be proven either by q–expansion or the fact that Hasse invariant does not change after dividing an abelian variety by an étale subgroup.

8. Vertical Control for Projective Shimura Varieties

8.1. **Deformation Theory of Serre and Tate.** Let W be a complete discrete valuation ring of mixed characteristic with residue field $\mathbb{F}$ of characteristic p. We suppose that $\mathbb{F}$ is an algebraic closure over $\mathbb{F}_p=\mathbb{Z}/p\mathbb{Z}$. In this section, we describe deformation theory of abelian schemes over local W_m–algebras for $W_m=W/p^mW$. We follow principally Katz's exposition [K].

8.1.1. *A Theorem of Drinfeld.* Let R be a local W_m–algebra. Let $G:R\text{–}LR\to AB$ be a covariant functor into the category AB of abelian groups. When $m=\infty$, the category R–LR (resp. W_∞–LR) is made of p–adically complete local R–algebras $A=\varprojlim_n A/p^nA$ and morphisms are supposed to be p–adically continuous. For simplicity, we always assume that rings we consider are noetherian. Thus if we regard G as a functor from the category of affine R schemes (or formal schemes), it is contravariant. Suppose that, for any faithfully flat extension of finite type $A\hookrightarrow B$ of R–algebras,

1. The group $G(A)$ injects into $G(B)$, that is, $G(A) \hookrightarrow G(B)$;
2. Let $B' = B \otimes_A B$ and $B'' = B \otimes_A B \otimes_A B$. Write $\iota_i : B \hookrightarrow B'$ $(i = 1, 2)$ two inclusions (that is, $\iota_1(r) = r \otimes 1$) and $\iota_{ij} : B' \hookrightarrow B''$ be three inclusions (i.e. $\iota_{12}(r \otimes s) = r \otimes s \otimes 1$). If $x \in G(B)$ satisfies $y = G(\iota_1)(x) = G(\iota_2)(x)$ and $G(\iota_{12})(y) = G(\iota_{23})(y) = G(\iota_{13})(y)$, then x is in the image of $G(A)$.

Such a G is called an abelian sheaf on R–LR with the $fppf$–topology (or simply *abelian $fppf$–sheaf*). If $X_{/R}$ is an abelian scheme or a torus (a multiplicative group, like $\mathbb{G}_m$), then $G(A) = X(A) = \mathrm{Hom}_S(Spec(A), X)$ $(S = Spec(R)$ or $Spf(R))$ is an $fppf$–sheaf.

We call G *p–divisible* if for any $x \in G(A)$, there exists a finite faithfully flat extension B of A and a point $y \in G(B)$ such that $x = py$. If G comes from an abelian scheme X, it is p–divisible (e.g. [GME] Corollary 4.1.18). This also shows that $X[p^\infty] = \bigcup_n X[p^n]$ for $X[p^n] = \mathrm{Ker}(p^n : X \to X)$ is p–divisible.

Let R be a local W_m–algebra and I be an ideal of R such that $I^{\nu+1} = 0$ and $NI = 0$ for a power N of p. We define a new functor G_I and $\widehat{G}$ by

$$G_I(A) = \mathrm{Ker}(G(A) \to G(A/I)) \quad \text{and} \quad \widehat{G}(A) = \mathrm{Ker}(G(A) \to G(A/\mathfrak{m}_A)),$$

where $\mathfrak{m}_A$ is the maximal ideal of A. When $\widehat{G}(A) = \mathrm{Hom}_{R\text{-}LR}(\mathcal{R}, A)(= G(A))$ for $\mathcal{R} = R[[T_1, \ldots, T_n]]$ (that is $G_{/R} = Spf(\mathcal{R})_{/R}$) and the identity element $\mathbf{0}$ corresponding to the ideal $(T_1, \ldots, T_n)$, we call G a formal group. If G is formal, $G_I(A) = \{(t_1, \ldots, t_n) \in I\}$ by $\mathrm{Hom}_{R\text{-}LR}(\mathcal{R}, A) \ni \phi \mapsto (\phi(T_1), \ldots, \phi(T_n))$.

Suppose that $G_{/R}$ is formal. Then multiplication by $[N]$ induces a continuous algebra homomorphism $[N] : \mathcal{R} \to \mathcal{R}$. Then on the tangent space at the origin: $t_G = (T_1, \ldots, T_n)/(T_1, \ldots, T_n)^2$, the addition induced by the group law of G coincides with the addition of the tangent vectors (cf. [ABV] Section 11). Thus $[N](T_i) \equiv NT_i \mod (T_1, \ldots, T_n)^2$, and $[N](G_I(A)) = G_{I^2}(A)$ because $NI = 0$. Similarly, we have inductively, $[N](G_{I^a}(A)) = G_{I^{a+1}}(A)$, and $[N^\nu]G_I = G_0 = \{\mathbf{0}\}$. We get

$$G_I \subset \mathrm{Ker}([N^\nu] : G \to G) \quad \text{if } G \text{ is formal.} \tag{8.1}$$

Theorem 8.1 (Drinfeld)**.** *Let G and H be abelian $fppf$–sheaf over R–LR and I be as above. Let G_0 and H_0 be the restriction of G and H to R/I–LR. Suppose*

(i) *G is p–divisible;*
(ii) *$\widehat{H}$ is formal;*
(iii) *$H(A) \to H(A/J)$ is surjective for any nilpotent ideal (H is formally smooth).*

Then

1. $\mathrm{Hom}_{R\text{-}Gp}(G, H)$ *and* $\mathrm{Hom}_{R/I\text{-}Gp}(G_0, H_0)$ *is p–torsion-free, where "$\mathrm{Hom}_{X\text{-}Gp}$" stands for the homomorphisms of abelian $fppf$–sheaves over X–LR;*
2. *The natural map, so-called*
$$\text{“reduction} \mod I\text{”} : \mathrm{Hom}_{R\text{-}Gp}(G, H) \to \mathrm{Hom}_{R/I\text{-}Gp}(G_0, H_0)$$
is injective;
3. *For any $f_0 \in \mathrm{Hom}_{R/I\text{-}Gp}(G_0, H_0)$, there exists a unique $\Phi \in \mathrm{Hom}_{R\text{-}Gp}(G, H)$ such that $\Phi \mod I = N^\nu f_0$. We write $\widetilde{N^\nu} f$ for Φ even if f exists only in* $\mathrm{Hom}_{R\text{-}Gp}(G, H) \otimes_{\mathbb{Z}} \mathbb{Q}$*;*
4. *In order that $f \in \mathrm{Hom}_{R\text{-}Gp}(G, H)$, it is necessary and sufficient that $\widetilde{N^\nu} f$ kills $G[N^\nu]$.*

Proof. The first assertion follows from p–divisibility, because if $pf(x) = 0$ for all x, taking y with $py = x$, we find $f(x) = pf(y) = 0$ and hence $f = 0$.

We have an exact sequence: $0 \to H_I \to H \to H_0 \to 0$; so, we have another exact sequence:

$$0 \to \mathrm{Hom}(G, H_I) \to \mathrm{Hom}(G, H) \xrightarrow{\bmod I} \mathrm{Hom}(G, H_0) = \mathrm{Hom}(G_0, H_0),$$

which tells us the injectivity since H_I is killed by N^ν and $\mathrm{Hom}(G, H)$ is p–torsion-free.

To show (3), take $f_0 \in \mathrm{Hom}(G_0, H_0)$. By surjectivity of $H(A) \to H_0(A/I)$, we can lift $f_0(x \mod I)$ to $y \in H(A)$. The class $y \mod \mathrm{Ker}(H \to H_0)$ is uniquely determined. Since $\mathrm{Ker}(H \to H_0)$ is killed by N^ν, for any $x \in G(A)$, therefore $N^\nu y$ is uniquely determined; so, $x \mapsto N^\nu y$ induces functorial map: $\widetilde{N}^\nu f : G(A) \to H(A)$. This shows (3).

The assertion (4) is then obvious from p–divisibility of G. The uniqueness of f follows from the p–torsion-freeness of $\mathrm{Hom}(G, H)$. □

8.1.2. *A Theorem of Serre-Tate.* Let $\mathcal{A}_{/R}$ be the category of abelian schemes defined over R. We consider the category $Def(R, R/I)$ of triples (X_0, D, ϵ), where X_0 is an abelian scheme over R/I, $D = \bigcup_n D_n$ with finite flat group scheme $D^{(n)}$ over R with inclusion $D^{(n)} \hookrightarrow D^{(n+1)}$, which is p–divisible, and $\epsilon : D_0 \cong X_0[p^\infty]$. We have a natural functor $\mathcal{A}_{/R} \to Def(R, R/I)$ given by $X \mapsto (X_0 = X \mod I, X[p^\infty], \mathrm{id})$.

Theorem 8.2 (Serre-Tate)**.** *The above functor: $\mathcal{A}_{/R} \to Def(R, R/I)$ is a canonical equivalence of categories.*

Proof. By the Drinfeld theorem applied to $X[p^\infty]$ and X (both abelian $fppf$–sheaf), the functor is fully faithful (see [K] for details).

For a given triple, $(X_0, D, \varepsilon)_{/R}$, we need to create $X_{/R}$ which gives rise to $(X_0, D, \varepsilon)_{/R}$. It is known that we can lift X_0 to an abelian scheme Y over R. This follows from the deformation theory of Grothendieck ([GIT] Section 6.3). When R/I is a finite field, by a theorem of Tate, X_0 has complex multiplication. By the theory of abelian varieties with complex multiplication, X_0 can be lifted to a unique abelian scheme Y over R with complex multiplication (because the isomorphism classes of such abelian varieties of CM type corresponds bijectively to the lattice in a CM field). Thus we have an isomorphism $\alpha_0^{(p)} : Y_0[p^\infty] \to X_0[p^\infty]$. Then we have a unique lifting (by the Drinfeld theorem) $f = \widetilde{N}^\nu \alpha^{(p)} : Y[p^\infty] \to D$ of $N^\nu \alpha_0^{(p)}$. Since the special fiber is an isogeny having inverse $(\alpha_0^{(p)})^{-1}$, f is an isogeny, whose (quasi) inverse is the lift of $N^\nu(\alpha_0^{(p)})^{-1})$. Thus $\mathrm{Ker}(f)$ is a finite flat group subscheme of Y. The geometric quotient of Y by a finite flat group subscheme exists (see [ABV] Section 12) and is an abelian scheme over R. Then dividing Y by $\mathrm{Ker}(f)$, we get the desired $X_{/R} \in \mathcal{A}_{/R}$. □

8.1.3. *Deformation of an Ordinary Abelian Variety.* Let A be a ring of characteristic p and (X, ω) be a pair of an abelian variety over $S = Spec(A)$ of relative dimension g and a base ω of $H^0(X, \Omega_{X/A})$ over A. We have the absolute Frobenius endomorphism $F_{abs} : X_{/A} \to X_{/A}$. Let $\mathcal{T}_{X/S}$ be the relative tangent bundle; so, $H^0(X, \mathcal{T}_{X/S})$ is spanned by the dual base $\eta = \eta(\omega)$. For each derivation D of $\mathcal{O}_{X,\mathbf{0}}$, by the Leibnitz formula, we have

$$D^p(xy) = \sum_{j=0}^{p} \binom{p}{j} D^{p-j}xD^jy = xD^py + yD^px.$$

Thus D^p is again a derivation. The association: $D \mapsto D^p$ induces an F_{abs}–linear endomorphism F^* of $\mathcal{T}_{X/S}$. Then we define $H(X,\omega) \in A$ by $F^* \bigwedge^g \eta = H(X,\omega) \bigwedge^g \eta$. Since $\eta(\lambda\omega) = \lambda^{-1}\eta(\omega)$ for $\lambda \in GL_g(A)$, we see

$$\begin{aligned} H(X,\lambda\omega) \bigwedge^g \eta(\lambda\omega) &= F^* \bigwedge^g \eta(\lambda\omega) = F^*(\det(\lambda)^{-1} \bigwedge^g \eta(\omega)) \\ &= \det(\lambda)^{-p} F^* \bigwedge^g \eta(\omega) = \det(\lambda)^{-p} H(X,\omega) \bigwedge^g \eta(\omega) \\ &= \det(\lambda)^{-p} H(X,\omega) \det(\lambda) \bigwedge^g \eta(\lambda\omega) = \det(\lambda)^{1-p} H(X,\omega) \bigwedge^g \eta(\lambda\omega). \end{aligned}$$

Thus we get

$$H(X,\lambda\omega) = \det(\lambda)^{1-p} H(X,\omega).$$

We call X ordinary if $X[p] \cong (\mathbb{Z}/p\mathbb{Z})^g \times \mu_p^g$ étale locally. In the same manner as in the elliptic curve case, we know

$$H(X,\omega) = 0 \iff X \text{ is not ordinary}.$$

Let $\mathbb{F}$ be an algebraic closure of $\mathbb{F}_p$. Let R be a pro-artinian local ring with residue field $\mathbb{F}$. Write $CL_{/R}$ be the category of complete local R–algebras with residue field $\mathbb{F}$. We fix an *ordinary* abelian variety $X_{0/\mathbb{F}}$. Write $\widehat{X}_{/R}$ for the dual abelian scheme of an abelian scheme $X_{/R}$. We write $TX[p^\infty]^{et}$ for the Tate module of the maximal étale quotient of $X[p^\infty]$. We consider the following deformation functor: $\widehat{\mathcal{P}} : CL_{/R} \to SETS$ given by

$$\widehat{\mathcal{P}}_{X_0}(A) = \left[(X_{/A}, \iota_X) \middle| \; X \text{ is an abelian scheme over } A \text{ and } \iota_X : X \otimes_A \mathbb{F} \cong X_0 \right].$$

Here $f : (X, \iota_X)_{/A} \cong (X', \iota_{X'})_{/A}$ if $f : X \to X'$ is an isomorphism of abelian schemes with the following commutative diagram:

$$\begin{array}{ccc} X \otimes_A \mathbb{F} & \xrightarrow{f_0} & X' \otimes_A \mathbb{F} \\ {\scriptstyle \iota_X}\downarrow\wr & & {\scriptstyle \iota_{X'}}\downarrow\wr \\ X_0 & = & X_0. \end{array}$$

Theorem 8.3 (Serre-Tate)**.** *We have*

1. *A canonical isomorphism* $\widehat{\mathcal{P}}(A) \cong \mathrm{Hom}_{\mathbb{Z}_p}(TX_0[p^\infty]^{et} \times T\widehat{X}_0[p^\infty]^{et}, \widehat{\mathbb{G}}_m(A))$ *taking* $(X_{/A}, \iota_X)$ *to* $q_{X/A}(\ ,\ )$.
2. *The functor* $\widehat{\mathcal{P}}$ *is represented by the formal scheme*
$$\mathrm{Hom}_{\mathbb{Z}_p}(TX_0[p^\infty]^{et} \times T\widehat{X}_0[p^\infty]^{et}, \widehat{\mathbb{G}}_m) \cong \widehat{\mathbb{G}}_m^{g^2}.$$
3. $q_{X/A}(x,y) = q_{\widehat{X}/A}(y,x)$ *under the canonical identification:* $\widehat{\widehat{X}} = X$.
4. *Let* $f_0 : X_{0/\mathbb{F}} \to Y_{0/\mathbb{F}}$ *be a homomorphism of two ordinary abelian varieties with the dual map:* $\widehat{f}_0 : \widehat{Y}_0 \to \widehat{X}_0$. *Then* f_0 *is induced by a homomorphism* $f : X_{/A} \to Y_{/A}$ *for* $X \in \widehat{\mathcal{P}}_{X_0}(A)$ *and* $Y \in \widehat{\mathcal{P}}_{Y_0}(A)$ *if and only if* $q_{X/A}(x, \widehat{f}_0(y)) = q_{Y/A}(f_0(x), y)$.

Proof. We are going to give a sketch of the construction of $q_{X/A}$.

We prepare some facts. Let $f : X \to Y$ be an isogeny; so, $\mathrm{Ker}(f)$ is a finite flat group scheme over S. Pick $x \in \mathrm{Ker}(f)$, and let $\mathcal{L} \in \mathrm{Ker}(\widehat{f}) \subset \widehat{Y}$ be the line bundle on Y with $\mathbf{0}^*\mathcal{L} = \mathcal{O}_S$ ($S = Spec(A)$ for an artinian R–algebra A). Thus $f^*\mathcal{L} = \mathcal{O}_X$. Cover Y by affine subsets $U_i = Spec(A_i)$ so that $\mathcal{L}|_{U_i} = \phi_i^{-1}\mathcal{O}_{U_i}$. Since $\mathbf{0}_Y^*\mathcal{L} = \mathcal{O}_S$, we may assume that $(\phi_i/\phi_j) \circ \mathbf{0}_Y = 1$. Since $f : X \to Y$ is finite, it is affine. Write $V_i = f^{-1}(U_i) = Spec(B_i)$. Then $f^*\mathcal{L}|_{V_i} = \varphi_i^{-1}\mathcal{O}_{V_i}$ with $\varphi_i = \phi_i \circ f$, and we have, regarding $x : S \to \mathrm{Ker}(f)$,

$$\frac{\varphi_i \circ x}{\varphi_j \circ x} = \frac{\phi_i \circ f \circ x}{\phi_j \circ f \circ x} = \frac{\phi_i \circ \mathbf{0}_Y}{\phi_j \circ \mathbf{0}_Y} = 1.$$

Thus $\varphi_i \circ x$ glue into a morphism $[x, \mathcal{L}] : S \to \mathbb{G}_m$, and in this way, we get a pairing

$$e_f : \mathrm{Ker}(f) \times \mathrm{Ker}(\widehat{f}) \to \mathbb{G}_m.$$

Since X is a $\mathrm{Ker}(f)$–torsor over Y, we have $X \times_Y X \cong \mathrm{Ker}(f) \times_S Y$. Thus for any homomorphism $\zeta : \mathrm{Ker}(f) \to \mathbb{G}_m$, we can find a morphism $\phi : \mathrm{Ker}(f) \times_S Y \to \mathbf{P}^1$ such that $\phi(y+t) = \zeta(t)\phi(y)$ for $t \in \mathrm{Ker}(f)$. This function ϕ gives rise to a divisor D on $Y_X = Y \times_S X$. By definition $f_X^*\mathcal{L}(D) = \mathcal{O}_{X_X}$, and $e_f(x, \mathcal{L}(D)) = \zeta(x)$. Thus, over X, $e_{f/X} : \mathrm{Ker}(f)_{/X} \times \mathrm{Ker}(\widehat{f})_{/X} \to \mathbb{G}_m$ is a perfect pairing. Since $X \to S$ is faithfully flat, we find that the original e_f is perfect.

We apply the above argument to $f = [p^n] : X \to X$, write the pairing as e_n and verify the following points (e.g. [GME] 4.1.5):

(P1) $e_n(\alpha(x), y) = e_n(x, \widehat{\alpha}(y))$ for $\alpha \in \mathrm{End}(X_{/A})$;

(P2) Write $X_0[p^n]^\circ = \mu_{p^n}^g \subset X_0[p^n]$. Then e_n induces an isomorphism of group schemes: $X_0[p^n]^\circ \cong \mathrm{Hom}(\widehat{X}_0[p^n]^{et}, \mu_{p^n})$;

(P3) Taking limit of the above isomorphisms with respect to n, we find

$$X^\circ \cong \mathrm{Hom}(T\widehat{X}[p^\infty]^{et}, \widehat{\mathbb{G}}_m) \cong \mathrm{Hom}(T\widehat{X}_0[p^\infty]^{et}, \widehat{\mathbb{G}}_m)$$

as formal groups. We denote the induced pairing by

$$E_X : X^\circ \times T\widehat{X}_0[p^\infty]^{et} \to \widehat{\mathbb{G}}_m.$$

In particular $X^\circ = \widehat{\mathbb{G}}_m^g$.

The structure of the p–divisible group $X[p^\infty]$ is uniquely determined by the extension class of:

$$0 \to \mathrm{Hom}(T\widehat{X}_0[p^\infty]^{et}, \widehat{\mathbb{G}}_m)[p^n] \to X[p^n] \xrightarrow{\pi} X_0[p^n]^{et} \to 0 \tag{8.2}$$

for $n = 1, 2, \ldots, \infty$. Take $x = \varprojlim_n x_n \in TX_0[p^\infty]^{et}$ for $x_n \in X[p^n]^{et}$. Lift x_n to $v_n \in X[p^n]$ so that $\pi(v_n) = x_n$. Then $q_n(x) = \widetilde{p^n}v_n \in \mathrm{Hom}(T\widehat{X}_0[p^\infty]^{et}, \widehat{\mathbb{G}}_m)[p^n]$. Take the limit of q_n to get $q(x) \in \mathrm{Hom}(T\widehat{X}_0[p^\infty]^{et}, \widehat{\mathbb{G}}_m(A))$. This $q(x)$ completely determines the extension class of (8.2) so the deformation $X_{/A}$ because it is determined by $(X_0, X[p^\infty])$ by the Serre-Tate theorem in the previous subsection. Then we define $q_{X/A}(x, y) = q(x)(y)$.

It is known that for any given $q(x, y)$ as above an extension (8.2) exists by the theory of Barsotti-Tate groups studied by Messing (see [CBT] Appendix). This shows the assertions (1) and (2). All other assertions follows from (P1-3) easily. $\square$

8.1.4. *Symplectic Case.* We now fix a polarization $\lambda_0 : X_0 \to \widehat{X}_0$ of degree prime to p. We consider the functor

$$\widehat{\mathcal{P}}_{X_0,\lambda_0}(A) = \left[(X_{/A}, \iota_X, \lambda)\middle|(X, \iota_X) \in \widehat{\mathcal{P}}_{X_0}(A) \text{ and } \lambda \text{ induces } \lambda_0\right].$$

Here we call $f : (X, \lambda_X, \iota_X) \to (Y, \lambda_Y, \iota_Y)$ an isomorphism if $f : (X, \iota_X) \cong (Y, \iota_Y)$ and $\widehat{f} \circ \lambda_Y \circ f = \lambda_X$. Note that by Drinfeld theorem, $\mathrm{End}(X_{/A})$ is torsion-free, and hence, $\mathrm{End}(X_{/A}) \hookrightarrow \mathrm{End}^{\mathbb{Q}}(X_{/A}) = \mathrm{End}(X_{/A}) \otimes_{\mathbb{Z}} \mathbb{Q}$. We write $\alpha^* = \lambda_0^{-1} \circ \widehat{\alpha} \circ \lambda_0$ for $\alpha \in \mathrm{End}(X_0) \otimes_{\mathbb{Z}} \mathbb{Q}$. Since $\mathrm{End}(X_{/A}) \subset \mathrm{End}(X_0)$ again by Drinfeld's theorem, the involution keeps $\mathrm{End}^{\mathbb{Q}}(X_{/A})$ stable (because on $\mathrm{End}^{\mathbb{Q}}(X_{/A})$, it is given by $\alpha^* = \lambda^{-1} \circ \widehat{\alpha} \circ \lambda$). The involution $\alpha \mapsto \alpha^*$ is known to be positive (see [ABV] Section 21). The polarization λ_0 induces an isomorphism $\lambda_0 : X[p^\infty]^{et} \cong \widehat{X}[p^\infty]^{et}$. We identify $TX_0[p^\infty]^{et}$ and $T\widehat{X}_0[p^\infty]^{et}$ by λ_0. Then the involution $\alpha \mapsto \widehat{\alpha}$ is replaced by the positive involution "$*$". Then it is clear from the previous theorem that

$$\widehat{\mathcal{P}}_{X_0,\lambda_0}(A) \cong \mathrm{Hom}_{\mathbb{Z}_p}(Sym^2 TX_0[p^\infty]^{et}, \widehat{\mathbb{G}}_m(A)) \cong \widehat{\mathbb{G}}_m^{g(g+1)/2}(A).$$

8.2. **Proof of the VCT in the Co-compact Case.** We first describe the deformation space in the unitary case, and then we prove the VCT for such groups.

8.2.1. *Unitary Case.* We fix a division algebra B with positive involution "$*$". The center of B is either a CM field F ($*$ inducing complex conjugation on F) or a totally real field on which $*$ is trivial. We fix a B–module V with $*$–hermitian alternating form $\langle\ ,\ \rangle$ satisfying conditions (L1-2) in Section 6. Out of these data, we define the group

$$G_1(A) = \{x \in C \otimes_{\mathbb{Q}} A | xx^* = 1\},$$

where $C = \mathrm{End}_B(V)$ and $\langle xv, w\rangle = \langle v, x^*w\rangle$. For simplicity, we suppose that $F = \mathbb{Q}[\sqrt{-D}]$ for a positive integer D (we suppose that $-D$ is the discriminant of $F/\mathbb{Q}$). In particular, we have $Sh_{K/E} \cong Sh_K^{(p)} \times_{S_0} E$ for K maximal at p ([PAF] Theorem 7.5). The group G_1 is an inner form of a unitary group of signature (m, n). Let $\varepsilon = \mathrm{diag}[1, 0, \ldots, 0] \in O_{B,p}$. By the condition (det), the representation of F on $\varepsilon(Lie(X))$ for $(X, \lambda, i, \eta^{(p)}) \in \mathcal{P}(A)$ ($A \in W$–CL) is $m\,\mathrm{id} + nc$ for $\mathrm{id} : O_F \hookrightarrow W$ and non-trivial automorphism c of F. We fix an O_B–lattice L of V such that $\langle\ \ \rangle$ induces a self duality of $L_p = L \otimes_{\mathbb{Z}} \mathbb{Z}_p$. We suppose that $p = \mathfrak{p}\overline{\mathfrak{p}}$ in F; so, $O_{B,p} = O_B \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong M_r(O_{F,\mathfrak{p}}) \oplus M_r(O_{F,\overline{\mathfrak{p}}})$. Supposing that $X_{/\mathbb{F}}$ is ordinary, we have $Lie(X_{/\mathbb{F}}) \cong Lie(X[p]^\circ_{/\mathbb{F}})$, where $X[p]^\circ$ is the connected component of $X[p]$. Thus we may assume that $T_pX[\mathfrak{p}^\infty]^{et} \cong M_{r\times n}(O_{F,\mathfrak{p}})$ and $T_pX[\overline{\mathfrak{p}}^\infty]^{et} \cong M_{r\times m}(O_{F,\overline{\mathfrak{p}}})$.

For an artinian local W–algebra A with residue field $\mathbb{F} = W/\mathfrak{m}_W = \overline{\mathbb{F}}_p$ and $(X, \lambda, i, \eta^{(p)}) \in \mathcal{P}(A)$, we consider $D_X = \varepsilon(X[p^\infty])$. Since $X[p^\infty] \cong D_X^r$ as Barsotti-Tate p–divisible groups, the abelian scheme X as a deformation of $X_0 = X \otimes_A \mathbb{F}$ is completely determined by D_X.

Suppose that X_0 is ordinary. We write the $O_{F,\mathfrak{p}}$–component of $T_pD_X^{et} = T_pD_{X_0}^{et}$ as $T_{\mathfrak{p}}D_X^{et}$. Then the symmetric pairing

$$q_{X/A}(\ ,\ ) : T_pX_0[p^\infty]^{et} \times T_pX_0[p^\infty]^{et} \to \widehat{\mathbb{G}}_m(A)$$

induces a homomorphism:

$$q_{X/A} : T_{\mathfrak{p}}D_X^{et} \otimes_{\mathbb{Z}_p} T_{\overline{\mathfrak{p}}}D_X^{et} \to \widehat{\mathbb{G}}_m(A),$$

because the pairing is c–hermitian (that is, the involution $*$ induces complex conjugation c). Since the level N–structure outside p lifts uniquely to deformations, we can ignore the level structure while we study deformations of $(X_0, \lambda_0, i_0, \eta_0^{(p)})_{/\mathbb{F}}$. So we consider the functor

$$\widehat{\mathcal{P}}_{X_0,\lambda_0,i_0}(A) = \left[(X_{/A}, \iota_X, \lambda, i)\middle|(X, \iota_X, \lambda) \in \widehat{\mathcal{P}}_{X_0,\lambda_0}(A) \text{ and } i \text{ induces } i_0\right].$$

Then the above argument combined with the theorem of Serre-Tate (Theorem 8.3) shows

$$\widehat{\mathcal{P}}_{X_0,\lambda_0}(A) \cong \mathrm{Hom}_{\mathbb{Z}_p}(T_{\mathfrak{p}} D_X^{et} \otimes_{O_{F_0,\mathfrak{p}}} T_{\overline{\mathfrak{p}}} D_X^{et}, \widehat{\mathbb{G}}_m(A)) = \widehat{S}(A),$$

because the symmetric (c–hermitian) form on

$$(T_{\mathfrak{p}} D_X^{et} \times T_{\overline{\mathfrak{p}}} D_X^{et}) \otimes (T_{\mathfrak{p}} D_X^{et} \times T_{\overline{\mathfrak{p}}} D_X^{et})$$

is determined by its restriction on $(T_{\mathfrak{p}} D_X^{et} \times \{0\}) \times (\{0\} \times T_{\overline{\mathfrak{p}}} D_X^{et})$.

8.2.2. *Hecke Operators on Deformation Space.* Let $O_C = \{x \in C | xL \subset L\}$. We write $G_1(\widehat{\mathbb{Z}})$ for $\widehat{O}_C^{\times} \cap G_1(\mathbb{A}^{(\infty)})$, where $\widehat{O}_C = O_C \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$. We fix an open compact subgroup $K \subset G_1(\widehat{\mathbb{Z}})$ such that $K = K_p \times K^{(p)}$ with $K_p \cong GL_{m+n}(\mathbb{Z}_p)$ via the projection to $\mathfrak{p}$–factor.

We consider p–ordinary test objects $\underline{X} = (X, \lambda, i, \overline{\eta}^{(p)})_{/A}$ over a local artinian W–algebra A. Since the pairing $q_{X/A} \in \widehat{S}(A)$ is actually determined by its restriction to $q_{X/A} : T_{\mathfrak{p}} D_X^{et} \times T_{\overline{\mathfrak{p}}} D_X^{et}$, we only look into this restriction. We study the $O_{F,p}$–linear endomorphism algebra $\mathrm{End}_{BT}(\varepsilon X[\mathfrak{p}^\infty]_{/A})$ of the Barsotti-Tate group $\varepsilon X[\mathfrak{p}^\infty]_{/A}$. Write each endomorphism α as $\left(\begin{smallmatrix} a_\alpha & b_\alpha \\ 0 & d_\alpha \end{smallmatrix}\right)$ with

$$a_\alpha \in \mathrm{End}_{BT}(\varepsilon X[\mathfrak{p}^\infty]^\circ),\ b_\alpha \in \mathrm{Hom}_{BT}(\varepsilon X[\mathfrak{p}^\infty]^{et}, \varepsilon X[\mathfrak{p}^\infty]^\circ) \text{ and}$$
$$c_\alpha \in \mathrm{End}_{O_{F_0,\mathfrak{p}}}(\varepsilon T_{\mathfrak{p}}^{et}(X)).$$

If A is an algebraically closed field of characteristic p, the étale-connected exact sequence $\varepsilon X_0[\mathfrak{p}^\infty]^\circ \hookrightarrow \varepsilon X_0[\mathfrak{p}^\infty] \twoheadrightarrow \varepsilon X_0[\mathfrak{p}^\infty]^{et}$ is (non-canonically) split. In any case, α acts on $T_{X,\mathfrak{p}} = T_{\mathfrak{p}} D_X^{et} \oplus T_{\overline{\mathfrak{p}}} D_X^{et}$ diagonally via a_α and d_α. We regard $T = \mathbb{G}_m^n$ as a maximal split torus of $GL_m(O_{F,\overline{\mathfrak{p}}}) \times GL_n(O_{F,\mathfrak{p}})$, which is the automorphism group of $\varepsilon X[\mathfrak{p}^\infty]^{et} \times \varepsilon X[(\mathfrak{p}^c)^\infty]^{et}$.

Let $\mathbb{X}_{/Sh_K}$ be the universal abelian scheme. We write $P_t = \varepsilon \mathbb{X}_{/S}[\mathfrak{p}^\infty]^{et}[p^t]$ and $P'_s = \varepsilon \mathbb{X}_{/S}[\overline{\mathfrak{p}}^\infty]^{et}[p^s]$ and apply the theory developed in Section 7 to each piece P_t and P'_s; so, we obtain the theory of false automorphic forms for $GL_n(O_{F,\mathfrak{p}}) \times GL_m(O_{F,\overline{\mathfrak{p}}})$ ($O_{F,\mathfrak{p}} \cong O_{F,\overline{\mathfrak{p}}} \cong \mathbb{Z}_p$). Since p is unramified in F, $O_F \otimes_{\mathbb{Z}_p} W \cong W^I$ for the set of embedding $I = \{\sigma = \mathrm{id}, c\}$ of F into $\overline{\mathbb{Q}}$. Then we consider filtrations fil_σ and fil_c of $\varepsilon X[\mathfrak{p}^t]^{et}$ and $\varepsilon X[\overline{\mathfrak{p}}^s]^{et}$, and consider the following test objects: $\underline{X}_{(t,s)} = \{X, \lambda, i, \overline{\eta}^{(p)}, fil_\sigma, fil_c\}_{/A}$. Let $M = Sh_K$ and $S = M\left[\frac{1}{E}\right]$, where E is a lift of the Hasse invariant. We write $T^{\mathcal{B}}_{\ell,t,s/S_\ell}$ for the étale covering over $S_\ell = S \otimes_W W_\ell$ ($W_\ell = W/p^\ell W$) classifying the above test objects. Similarly, $T_{\ell,t,s}$ classifies

$$(X, \lambda, i, \overline{\eta}^{(p)}, (O_F/\mathfrak{p}^t)^n \times (O_F/\overline{\mathfrak{p}}^s)^m \cong \varepsilon X[\mathfrak{p}^t]^{et} \times \varepsilon X[\mathfrak{p}^s]^{et}).$$

The covering $T_{\ell,\infty,\infty}/S_\ell$ is an étale Galois covering with Galois group isomorphic to

$$GL_m(O_{F,\overline{\mathfrak{p}}}) \times GL_n(O_{F,\mathfrak{p}}).$$

We had an action of the expanding semi-group on filtrations of P_t ($0 < t \in \mathbb{Z}$) for an étale sheaf $P_t \cong (\mathbb{Z}/p^t\mathbb{Z})^n$. If we have a p–isogeny $\beta : P_\infty = \bigcup_t P_t \to P_\infty$ preserving a filtration of P_∞, we may assume that the matrix form of β is given by

$\beta_j = \begin{pmatrix} 1_{n-j} & * \\ 0 & p1_j \end{pmatrix}$ with respect to a base compatible with the filtration. Then the action of β_j is to give a new filtration on P_∞.

Since we cannot separate P_t and P'_t (which is sitting in the single universal abelian scheme $\mathbb{X}$), we define $\alpha_{m+j} = \mathrm{diag}[\beta_j, p1_m]$ and consider an isogeny of type α_{m+j}. We can thus interpret the operator action in terms of the quadruple: $(X, \lambda, i, \overline{\eta}^{(p)}, fil_\sigma, fil_c)_{/\mathbb{F}}$ as follows: Take an isogeny $\alpha : X \to X_\alpha$ of type α_{m+j} as above (inducing β on P_∞ and multiplication by p on P'_∞). Then we get a new filtration $\beta(fil_\sigma)$. The p–isogeny is insensitive to the level N–structure, and X_α has an induced polarization, still written as λ. Then we have

$$\text{(8.3)}\quad f|(t_j(\mathfrak{p}) \times t_m(\overline{\mathfrak{p}}))(X, \lambda, i, \overline{\eta}^{(p)}, fil_\sigma, fil_c) = \sum_\beta \widetilde{\rho}_\kappa(\beta^{-1}) f(X_\alpha, \lambda, i, \overline{\eta}^{(p)}, \beta(fil_\sigma), fil_c).$$

Thus we have a $GL(n)\times GL(m)$–Hecke operator $t_j(\mathfrak{p})\times t_m(\overline{\mathfrak{p}})$ acting on the coherent cohomology $H^0(S_\ell, \underline{\omega}^\kappa)$. This is actually an *over-simplified* version. The exact sequence:

$$0 \to \mathrm{Hom}(P'_t, \mathbb{G}_m) \to \mathbb{X}[\mathfrak{p}^t] \to P_t \to 0$$

may not split over S_ℓ; so, the isogeny α can be defined only over a finite flat extension $S_\ell^{\alpha_{m+j}}$ of S_ℓ (which is radiciel over an étale extension of S_ℓ). In other words, if we replace the term: $f(X_\alpha, \lambda, i, \overline{\eta}^{(p)}, \beta(fil_\sigma), fil_c)$ in (8.3) by the trace $\mathrm{Tr}_{S_\ell^{\alpha_{m+j}}/S_\ell}(f(X_\beta, \lambda, i, \overline{\eta}^{(p)}, \beta(fil_\sigma), fil_c))$, we can relate t_j to a global Hecke operator $U_i(\mathfrak{p})$ which is divisible by the degree of $S_\ell^{\alpha_{m+j}}$ over the maximal étale cover of S_ℓ under $S_\ell^{\alpha_{m+j}}$. The operator t_j is not well defined on coherent cohomology, although it is well defined on ℓ–adic étale cohomology (because ℓ–adic étale cohomology is insensitive to radiciel base-change).

For the moment, we pretend that the over-simplified version (8.3) is valid (and we later justify our argument). Thus for a while, our argument is just heuristic. Here we are having resort to the theory in Subsection 7.1.7 applied to $GL_m \times GL_n$, although the construction of the idempotent $e = e_G$ is global (so some modification necessary). So, under the notation used in Subsection 7.1.7, we would like to relate e_G with $e_{GL_n} \times e_{GL_m}$.

Since $G_1(\mathbb{Q}_p) \cong GL_{m+n}(\mathbb{Q}_p)$, we can embed $GL(n) \times GL(m)$ into $G_1(\mathbb{Q}_p)$ by $(x, y) \mapsto \mathrm{diag}[x, y]$. This implies the p–isogeny whose kernel sits in $X[\mathfrak{p}^\infty]^{et}$ (resp. $X[\overline{\mathfrak{p}}^\infty]^{et}$) corresponds to x (resp. y). Write $P = P_{m,n}$ for the upper triangular parabolic subgroup of G_1 whose Levi subgroup is given by the image of $GL_n\times GL_m$. Let $U_{m,n}$ be the unipotent radical of $P_{m,n}$.

Write $\alpha_j = \alpha_j(\mathfrak{p}) \in G(\mathbb{Q}_p)$ whose projection to $C_\mathfrak{p} = C \otimes_F F_\mathfrak{p}$ is given by $\mathrm{diag}[1_{m+n-j}, p \cdot 1_j]$ and $\nu(\alpha_j) = p$. We then have Hecke operators

$$U_j(\mathfrak{p}) = U(\alpha_j(\mathfrak{p})) = U_\mathcal{B} \alpha_j U_\mathcal{B},$$

where $U_\mathcal{B}$ is the upper unipotent subgroup of $G_1(\mathbb{Z}_p)$. Since we identify $G_1(\mathbb{Q}_p)$ with $GL_{m+n}(\mathbb{Q}_p)$ by projecting down $C \otimes_\mathbb{Q} \mathbb{Q}_p$ to the first component $C_\mathfrak{p}$, as a double coset, we see (symbolically)

$$U_j(\mathfrak{p}) = \begin{cases} \bigsqcup_{u\in U_{m,n}/\alpha_j^{-1}U_{m,n}\alpha_j} U_\mathcal{B}(t_{j-n}(\mathfrak{p}) \times t_m(\overline{\mathfrak{p}}))u & \text{if } j > n, \\ \bigsqcup_{u\in U_{m,n}/\alpha_j^{-1}U_{m,n}\alpha_j} U_\mathcal{B}(t_0(\mathfrak{p}) \times t_j(\overline{\mathfrak{p}}))u & \text{if } j \le n, \end{cases}$$

where we mean, for example, by $(t_{j-n}(\mathfrak{p}) \times t_m(\overline{\mathfrak{p}}))$, the double coset:

$$\widetilde{U}(\mathrm{diag}[1_{2n-j}, p \cdot 1_{j-n}] \times p \cdot 1_m)\widetilde{U}$$

in $GL_n(F_{\mathfrak{p}}) \times GL_m(F_{\overline{\mathfrak{p}}})$ for the upper triangular unipotent subgroup $\widetilde{U}$. This shows that the Hecke operator $U_j(\mathfrak{p})$ induces

$$[U_{m,n} : \alpha_j^{-1} U_{m,n} \alpha_j](t_{j-n}(\mathfrak{p}) \times t_m(\overline{\mathfrak{p}})) \quad \text{or} \quad [U_{m,n} : \alpha_j^{-1} U_{m,n} \alpha_j](t_0(\mathfrak{p}) \times t_m(\overline{\mathfrak{p}}))$$

according as $j > n$ or not. By computation, we get the following heuristic multiplicity formula:

$$(8.4) \qquad [U_{m,n} : \alpha_j(\mathfrak{p})^{-1} U_{m,n} \alpha_j(\mathfrak{p})] = \mu_{m,n}(\alpha_j) = \begin{cases} |p|_{\mathfrak{p}}^{-n(m+n-j)} & \text{if } j > n, \\ |p|_{\mathfrak{p}}^{-mj} & \text{if } j \leq n. \end{cases}$$

This formula suggests us that $U_j(\mathfrak{p})$ is divisible by $\mu_{m,n}(\alpha_j)$, which we will justify later.

Since the universal deformation space of $(X, \lambda, i, \overline{\eta}^{(p)}, fil_{\mathfrak{p}}, fil_{\overline{\mathfrak{p}}})_{/\overline{\mathbb{F}}_p}$ is isomorphic to

$$\widehat{S} = \mathrm{Hom}(T_{\mathfrak{p}} D_X^{et} \otimes_{O_{F_0,\mathfrak{p}}} T_{\overline{\mathfrak{p}}} D_X^{et}, \widehat{\mathbb{G}}_m),$$

as already seen, we can think of the effect of the isogeny $\beta : \widehat{X}_{/\widehat{S}} \to \widehat{X}'_{/\widehat{S}}$ of type α_j on the universal deformation space $\widehat{X}_{/\widehat{S}}$, which sends

$$\mathrm{Hom}(T_{\mathfrak{p}} D_X^{et} \times T_{\overline{\mathfrak{p}}} D_X^{et}, \widehat{\mathbb{G}}_m) \ni q(x,y) \mapsto q(\alpha(x), \alpha(y))/p.$$

We need to divide by p as above by the following reason: Since $q \in \widehat{S}$ measures the depth of non-splitting of the exact sequence $\mathrm{Hom}(P_t', \mathbb{G}_m) \hookrightarrow \mathbb{X}[\mathfrak{p}] \twoheadrightarrow P_t$, and the sequence for $t = 1$ is split if q is a p–power. Thus the isogeny α exists over $\widehat{S}^{1/p}$. Here we have written the group structure on $\widehat{\mathbb{G}}_m$ additively; so, "division by p" would become "taking p–th root" if we had formulated the group structure multiplicatively. The isogeny is defined over a smaller covering $\widehat{S}[(q \circ \alpha)^{1/p}] = Spf(\widehat{\mathcal{O}}_S[(q \circ \alpha)^{1/p}])$ by definition; so, $\widehat{S}^{\alpha}/\widehat{S}$ is given by $\widehat{S}[(q \circ \alpha)^{1/p}]$. At this point, we are taking p–th roots, and hence *pure inseparability* (we pretended not to have) comes in. Then the action of the isogeny α of type α_j on $\widehat{S}$ only depends on its effect on $T_{\mathfrak{p}} D_X^{et}$ and $T_{\overline{\mathfrak{p}}} D_X^{et}$ not on the individual choice α. This means that the covering $\widehat{S}^{\alpha}$ over $\widehat{S}$ carrying the isogeny α only depends on the image of α in the Levi-quotient of P. Indeed, taking a base $(x_i)_i$ of $T_{\mathfrak{p}} D_X^{et}$ and $(y_k)_k$ of $T_{\overline{\mathfrak{p}}} D_X^{et}$ so that the matrix of the isogeny is exactly α_j, the effect on $T = (T_{k,l}) = (e(x_k, y_l))$ is given by

$$(8.5) \qquad \left(\begin{smallmatrix} 1_m & T \\ 0 & 1_n \end{smallmatrix}\right) \mapsto \alpha_j(\mathfrak{p}) \left(\begin{smallmatrix} 1_m & T \\ 0 & 1_n \end{smallmatrix}\right) \alpha_j(\mathfrak{p})^{-1},$$

and S_1^{α} has degree of pure inseparability given by the value in (8.4). Hereafter we write $S_\ell^{\alpha_j}$ for S_ℓ^{α}.

Here is the justification of our argument. Write the multiplicative variable on $\widehat{S}$ as an $m \times n$ variable matrix $t = (t_{k,l})$. The conjugation: $\left(\begin{smallmatrix} 1_m & T \\ 0 & 1_n \end{smallmatrix}\right) \mapsto \alpha_j \left(\begin{smallmatrix} 1_m & T \\ 0 & 1_n \end{smallmatrix}\right) \alpha_j^{-1}$ induces $T_{k,l} \mapsto p^{-1} T_{k,l}$ for some indices (k,l). We split the set of indices (k,l) into a disjoint union $I \sqcup J$ of two subset so that the conjugation by α_j induces $T_{k,l} \mapsto p^{-1} T_{k,l}$ if and only if $(k,l) \in I$. The covering $\widehat{S}^{\alpha_j}$ is given by $Spf(\widehat{\mathcal{O}}_S[t_{k,\ell}^{1/p}]_{(k,l) \in I})$. Thus a formal function on $\widehat{S}$ has expansion $\sum_{\xi} a_{\xi} t_{\xi} \in W\widehat{[t_{k,l}, t_{k,l}^{-1}]}$ for $\xi \in \mathbb{Z}^{I \sqcup J}$.

Writing $\xi(I)$ for the I–part of the index ξ, a formal function f on $\widehat{S}_\ell^{\alpha_j}$ has expansion $f = \sum_\xi a_\xi t^{\xi(I)/p+\xi(J)}$, and we have

$$\mathrm{Tr}_{\widehat{S}^{\alpha_j}/\widehat{S}}(f) = \mu_{m,n}(\alpha_j) \sum_{\xi:\xi(I)\equiv 0 \mod p} a_\xi t^{\xi(I)/p+\xi(J)},$$

because $\mu_{m,n}(\alpha_j)$ is the degree of the (purely) wildly ramified covering $\widehat{S}^{\alpha_j} \twoheadrightarrow \widehat{S}$ and $\mathrm{Tr}(t_{k,l}^{i/p}) = \mu_{m,n}(\alpha_j)t_{k,l}^{i/p}$ or 0 according as $p|i$ or not. Thus by replacing the term: $f(X_\alpha, \lambda, i, \overline{\eta}^{(p)}, \alpha(fil_\sigma), \alpha(fil_c))$ in (8.3) by the trace

$$\mathrm{Tr}_{S^\alpha/S}(f(X_\alpha, \lambda, i, \overline{\eta}^{(p)}, \alpha(fil_\sigma), \alpha(fil_c))),$$

we get the p–divisibility of the operator $U_j(\mathfrak{p})$ as the (heuristic) multiplicity formula (8.4) suggests. This justifies the heuristic argument we gave (the heuristic argument is actually valid for ℓ–adic étale cohomology with $\ell \neq p$ as already explained).

Let $\widehat{S}_\ell = \widehat{S} \otimes_W W_\ell$. On the universal deformation $\widehat{X}_{\widehat{S}}$, the sheaf $\varepsilon(Lie(\widehat{X}))_{/\widehat{S}_\ell}$ is given by $\varepsilon(Lie(\widehat{X}[p^\ell]^\circ))$. By duality, $\underline{\omega}_{\widehat{S}_\ell} = \mathcal{O}_{\widehat{S}_\ell} \otimes_{\mathbb{Z}_p} \widehat{X}[p^\ell]^{et}$, which again only depends on $X_0[p^\ell]^{et}$; so, the Hecke operator $U_j(\mathfrak{p})$ is still divisible by $\mu_{m,n}(\alpha_j) = [U_\mathfrak{p} : \alpha_j^{-1}U_\mathfrak{p}\alpha_j]$ on $\underline{\omega}^\kappa_{\widehat{S}}$ for all $\kappa > 0$. Thus the action of the correspondence of characteristic 0 on $H^0(S_\infty, \underline{\omega}^\kappa)$ is exactly a multiple by the number in (8.4) of the operator induced by the mod p correspondence, which is an integral operator. From this our claim is clear.

In any case, we can divide the action of $U_j(\mathfrak{p})$ by the number in (8.4) keeping the integrality of the operator on $\underline{\omega}^\kappa$.

Lemma 8.4. *Let the notation be as above. We have a well defined integral operator* $[U_\mathfrak{p} : \alpha_j^{-1}U_\mathfrak{p}\alpha_j]^{-1}U_j(\mathfrak{p})$ *on* $H^0(T_{m,n}, \underline{\omega}^\kappa \otimes \Omega_{S/W})$.

We then define

$$e_G = \lim_{n\to\infty} (U(\mathfrak{p}))^{n!}, \tag{8.6}$$

where

$$U(\mathfrak{p}) = \prod_{j=1}^{m+n} [U_\mathfrak{p} : \alpha(\mathfrak{p})_j^{-1}U_\mathfrak{p}\alpha(\mathfrak{p})_j]^{-1}U_j(\mathfrak{p}).$$

As for $T_j(\mathfrak{p})$, if $\kappa \geq \mu_{m,n}$ (that is, $\kappa - \mu_{m,n}$ is in the Weyl chamber), $T_j(\mathfrak{p}) \equiv U_j(\mathfrak{p})$ mod p on $H^0(M_\ell, \underline{\omega}^\kappa) \otimes \omega^\circ_{M/W})$ for $M_\ell = Sh_{K/W} \otimes_W W_\ell$. The operator $T_j(\mathfrak{p})$ is well defined on $\underline{\omega}^\kappa$ over M as a linear operator, using moduli theoretic interpretation.

Let $\widetilde{U}$ be the upper unipotent subgroup of $\mathrm{Gal}(T_{\ell,\infty,\infty}/S_\ell)$. Following Lecture 7, we can define the space of p–adic automorphic forms $V^{\widetilde{U}}$ on S_∞ (which is the formal completion of S along S_1). Thus

$$V = \varprojlim_\ell \left(\varinjlim_{t,s} V_{\ell,t,s}^{\widetilde{U}}\right)$$

for $V_{\ell,t,s} = H^0(T_{\ell,t,s}, \mathcal{O}_{T_{\ell,t,s}})^{\widetilde{U}}$. We also define

$$\mathcal{V} = \varinjlim_\ell \left(\varinjlim_{t,s} V_{\ell,t,s}^{\widetilde{U}}\right)$$

The boundedness condition (F) in Section 7 is verified in [H95] in Case A because $G_1(\mathbb{Q}_p) \cong GL_{m+n}(\mathbb{Q}_p)$. The hypotheses (Hyp1-2) are clear because $S = Sh_K[\frac{1}{E}]$ is an affine scheme in the cocompact case. The value $f|U_j(\mathfrak{p})(X,\lambda,i,\overline{\eta}^{(p)},fil,\omega)$ is the sum (more precisely, the trace) of $f(X/C,\lambda',i',\overline{\eta'}^{(p)},fil',\omega')$ for étale subgroups C of $X[p]$. Since the Hasse invariant is insensitive to étale isogeny (by its definition), the commutativity condition (C) in Section 7 holds. Then we have

Theorem 8.5. *Let W be a $\mathfrak{p}$–adic completion of the integer ring of the Galois closure of $F/\mathbb{Q}$. Suppose that $M_{/W} = Sh_{K/W}$ is proper over W. Let $\widetilde{\mathcal{B}}$ be the upper triangular Borel subgroup of $GL_m(O_{F,\overline{\mathfrak{p}}}) \times GL_n(O_{F,\mathfrak{p}})$ and $\widetilde{U}$ is the unipotent radical of $\widetilde{\mathcal{B}}$. Let $T = \widetilde{\mathcal{B}}/\widetilde{U}$, and regard it as a diagonal torus of $G_1(\mathbb{Q}_p)$. We say $\kappa \in X(T)$ positive if κ is positive with respect to the opposite Borel subgroup of $\widetilde{\mathcal{B}}$. We write $X_+(T)$ for the set of positive weights κ.*

1. *There exists a canonical inclusion for $A = \mathbb{Z}_p$ and $\mathbb{Q}_p/\mathbb{Z}_p$*
$$\beta : \bigoplus_{\kappa \in X_+(T)} H^0(M_{/W}, \underline{\omega}^\kappa \otimes_{\mathbb{Z}_p} A) \hookrightarrow V \otimes_{\mathbb{Z}_p} A.$$
2. $\mathrm{Im}(\beta)[\frac{1}{p}] \cap V$ *is dense in V;*
3. *Write $U(p) = \prod_{1 \le j \le m+n} u_j(\mathfrak{p})$ for the standard Hecke operators at p of level p^∞, that is,*
$$u_j(\mathfrak{p}) = \frac{U_j(\mathfrak{p})}{[U_\mathfrak{p} : \alpha_j^{-1} U_\mathfrak{p} \alpha_j]}$$
for the unipotent radical $U_\mathfrak{p}$ of the upper triangular maximal parabolic subgroup of $GL_{m+n}(O_{F,\mathfrak{p}})$ with Levi-subgroup isomorphic to $GL(m) \times GL(n)$, and define the ordinary projector $e = \lim_{n\to\infty} U(p)^{n!}$ on V. Then

   $eH^0(S, \underline{\omega}^\kappa \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)$ is a p–divisible module with finite corank.
4. *If $\kappa > \mu_{m,n}$ is sufficiently regular, $eH^0(S, \underline{\omega}^\kappa \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)$ (resp. $eH^0(S_\infty, \underline{\omega}^\kappa)$) is isomorphic to $H^0_{ord}(M, \underline{\omega}^\kappa \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)$ (resp. $H^0_{ord}(M, \underline{\omega}^\kappa)$);*
5. *Let $\mathcal{V}^*_{ord}$ be the Pontryagin dual $W[[T(\mathbb{Z}_p)]]$–module of $e\mathcal{V}$ (which is isomorphic to the W–dual of eV). Then $\mathcal{V}^*_{ord}$ is $W[[\Gamma_T]]$–free of finite rank, where Γ_T is the maximal p–profinite subgroup of $T(\mathbb{Z}_p)$;*
6. *If $\kappa \in X_+(T)$,*
$$\mathcal{V}^*_{ord} \otimes_{W[[T(\mathbb{Z}_p)]],\kappa} W \cong \mathrm{Hom}_W(eH^0(S_\infty, \underline{\omega}^\kappa), W).$$

Although we restricted ourselves to cocompact unitary cases here, a similar result can be obtained in more general settings of cusp forms on a non-compact Shimura varieties of unitary groups and symplectic groups (see [H02]). In [H02], we have given the heuristic argument for the divisibility of $U(p)$, but it can be justified using the trace (in place of the sum of the values) from (wildly ramified) finite flat covering (carrying specified p–isogeny of the universal abelian scheme) over the Shimura variety as we did; so, the final result in [H02] is intact.

9. Hilbert Modular Forms

We shall give concrete examples in the non-co-compact case. These are Hilbert modular varieties. We give a sketch of the proof of the vertical control theorems. More details can be found in Chapter 4 of [PAF].

9.1. **Hilbert Modular Varieties.** We first recall the toroidal compactification of the Hilbert-Blumenthal moduli space. Main references are [C], [K2] and [Ra] (and [HT], [DT]).

Let $A = \varprojlim_n A/p^n A$ be a p–adic ring. Let F be a totally real field with integer ring O_F and N be an integer ≥ 3 prime to p. So our groups are given by $G = \mathrm{Res}_{O_F/\mathbb{Z}} GL(2)$ and $G_1 = \mathrm{Res}_{O_F/\mathbb{Z}} SL(2)$. We write T for the diagonal torus of G_1 defined over $\mathbb{Z}$; thus, we have $T(A) = (O_F \otimes_{\mathbb{Z}} A)^\times$. We consider a triple

$$(X, \lambda, \phi : (\mathfrak{d}^{-1} \otimes_{\mathbb{Z}} \mu_N) \oplus (O_F \otimes_{\mathbb{Z}} \mathbb{Z}/N\mathbb{Z}) \cong X[N])$$

over a scheme S made of an abelian variety with real multiplication by O_F (an AVRM). This means that $X_{/S}$ is an abelian scheme with $O_F \hookrightarrow \mathrm{End}(X_{/S})$ such that $Lie(X)$ is free of rank 1 over $\mathcal{O}_S \otimes_{\mathbb{Z}} O_F$. Here $\mathfrak{d}$ is the absolute different of F, and λ is a $\mathfrak{c}_+$–polarization for a fractional F–ideal $\mathfrak{c}$. This means that $\lambda : \widehat{X} \cong X \otimes_{O_F} \mathfrak{c}$. The word $\mathfrak{c}_+$–polarization means that the set of symmetric morphisms induced (fiber by fiber) by ample invertible sheaves: $P_+(X) \subset \mathrm{Hom}(X, \widehat{X})$ is isomorphic to $\mathfrak{c}_+$ by λ. This notion only depends on the strict ideal class of $\mathfrak{c}$. Thus hereafter we assume that $\mathfrak{c} \subset O_F$.

Tensoring X over O_F with the following exact sequence:

$$0 \to \mathfrak{c} \to O_F \to O_F/\mathfrak{c} \to 0,$$

we get another exact sequence:

$$0 \to \mathrm{Tor}_1(O_F/\mathfrak{c}, X) \to X \otimes \mathfrak{c} \to X \to 0.$$

Thus the above condition on polarization can be stated as

$$\widehat{X}/\widehat{X}[\mathfrak{c}] \cong X$$

for $\widehat{X}[\mathfrak{c}] = \{x \in (X \otimes \mathfrak{c}) | \mathfrak{c}x = \mathbf{0}\}$. We also note that

$$X = X'/X'[\mathfrak{a}] \iff X' = X \otimes \mathfrak{a} \iff X = X' \otimes \mathfrak{a}^{-1},$$

which will be useful.

To describe the toroidal compactification, let

$$C = \{\xi \in F_\infty | \xi^\sigma \geq 0 \text{ for all } \sigma : F \hookrightarrow \mathbb{R}\}$$

be the cone of totally positive numbers in $F_\infty = F \otimes_{\mathbb{Q}} \mathbb{R}$. Choose a cone decomposition $C = \bigsqcup_\sigma \sigma$ such that

1. σ is a non-degenerate open rational polyhedral cone without containing any entire line. Here the word "rational" implies that the cone is generated by a finitely many elements in $F_+ = F \cap C$ over $\mathbb{R}_+$;
2. the set of cones $\{\sigma\}$ is permuted under multiplication of $T(\mathbb{Z})(N)^2$, where
$$T(\mathbb{Z})(N) = \{\varepsilon \in O_F^\times | \varepsilon \equiv 1 \mod N\}.$$
There are only finitely many cones modulo $T(\mathbb{Z})(N)$, and $\varepsilon(\overline{\sigma}) \cap \overline{\sigma} \neq \{0\}$ implies that $\varepsilon = 1$ (see [LFE] Theorem 2.7.1 for an exposition of such decomposition);
3. σ is smooth (that is, generated by a part of a base of O_F);
4. $\{\sigma\}$ is sufficiently fine so that the toroidal compactification is projective (see [C] and [DAV] IV.2.4 for an exact condition for projectivity).

Let $\sigma^\vee$ be the dual cone:

$$\sigma^\vee = \{x \in F_\infty | \mathrm{Tr}_{F/\mathbb{Q}}(x\sigma) \geq 0\}.$$

Then $C = \bigcap_\sigma \sigma^\vee$.

Here is an oversimplified description of how to make the toroidal compactification over $W_{/\mathbb{Z}_p}$, where W is the discrete valuation ring we took as the base ring. Each Hilbert modular form f (defined over a ring W) has a q–expansion $f(q) = \sum_{\xi \in \mathfrak{a} \cap C} a(\xi; f) q^\xi$ for an ideal $\mathfrak{a}$. Thus we convince ourselves that the formal stalk of the minimal compactification at the cusp ∞ is the ring $R_\infty(\mathfrak{a}) = W[[q^\xi]]_{\mathfrak{a}\cap C} = W[[\mathfrak{a} \cap C]]$, which is the completion of the monoid ring of the semigroup $\mathfrak{a} \cap C$. Thus $R_\infty(\mathfrak{a}) = \bigcap_\sigma R_\sigma(\mathfrak{a})$ where $R_\sigma(\mathfrak{a}) = W[[q^\xi]]_{\mathfrak{a}\cap\sigma^\vee} = W[[\mathfrak{a} \cap \sigma^\vee]]$. For each σ as above, by enlarging $\mathfrak{a}$ if necessary, we may assume that $\mathfrak{a} \cap \sigma$ is generated over $\mathbb{Z}$ by $t_1, \ldots, t_r$ ($0 < r \leq [F : \mathbb{Q}] = g$). Then we have a base $\xi_1, \ldots, \xi_g$ of $\sigma^\vee$ so that $\mathrm{Tr}(t_i \xi_j) = \delta_{ij}$ for $1 \leq i \leq j \leq r$ and $\mathrm{Tr}(\sigma \xi_{r+j}) = 0$ ($j > 0$). Then each $\xi \in \mathfrak{a} \cap \sigma^\vee$ can be uniquely written as $\xi = \sum_i m_i \xi_i$ with $m_j \in \mathbb{Z}$ and $m_j \geq 0$ if $j \leq r$. Thus writing $T_j = q^{\xi_j}$, we find

$$R_\sigma(\mathfrak{a}) = W[[T_1, \ldots, T_g]][\frac{1}{T_{r+1}}, \ldots, \frac{1}{T_g}].$$

Thus $Spf(R_\sigma(\mathfrak{a})) = \widehat{\mathbf{A}}^r \times \widehat{\mathbb{G}}_m^{g-r}$ which is a compactification of $Spf(R_\tau(\mathfrak{a}))$ for each face τ of σ. Thus we can glue $\{Spf(R_\sigma(\mathfrak{a}))\}_\sigma$ on the ring in the common intersection of the $\sigma^\vee$'s, and getting a formal scheme $\mathcal{X}$ on which $T(\mathbb{Z})(N)$ acts by translation. Then make a quotient $\mathcal{X}/T(\mathbb{Z})(N)$. The algebraization of the quotient is the toroidal compactification at the infinity cusp.

We consider the moduli space $\mathcal{M}_{\mathfrak{c},N/W}$ of test objects $(X, \lambda, \phi)_{/A}$ for W–algebras A, where W is a discrete valuation ring containing all conjugates of O_F. We assume that W is unramified over $\mathbb{Z}_p$ and that $N\mathfrak{c}$ is prime to p. From the above data, we get a unique toroidal compactification $M = M_{\mathfrak{c},N}$ of $\mathcal{M}_{\mathfrak{c},N}$, which carries a (universal) semi-AVRM $\mathcal{G} = \mathcal{G}_{\mathfrak{c},N}$ with a level structure $\mathfrak{d} \otimes \mu_N \hookrightarrow \mathcal{G}[N]$. The semi-AVRM coincides with the universal abelian scheme $\mathcal{A} = \mathcal{A}_{\mathfrak{c},N}$ over $\mathcal{M}_{\mathfrak{c},N}$. Let M_∞ be the formal completion of M along $M_1 = M \otimes_W W_1$. Write $S_\infty \subset M_\infty$ for the ordinary locus, that is, S_∞ is the maximal formal subscheme of M_∞ on which the connected component $\mathcal{G}[p]^\circ$ of $\mathcal{G}[p]$ is isomorphic to μ_p^d locally under étale topology, and thus S_∞ is the formal completion of $S = M\left[\frac{1}{E}\right]$ along $S_1 = S \otimes_W W_1$, where E is a lift of Hasse invariant. Then we put $S_m = S \times_W W_m$. Let

$$T_{m,n/W_m} = \mathrm{Isom}_{O_F}(\mathfrak{d}^{-1} \otimes \mu_{p^n}, \mathcal{G}[p^n]^\circ) \cong \mathrm{Isom}_{O_F}(\widehat{\mathcal{G}[p^n]^\circ}, O_F/p^n O_F).$$

Then $T_{m,n}/S_m$ is an étale covering with Galois group $T(\mathbb{Z}/p^n\mathbb{Z}) = (O_F/p^n)^\times$ for $T = \mathrm{Res}_{O_F/\mathbb{Z}} \mathbb{G}_m$. By a result of K. Ribet [Ri] (see also [PAF] Theorem 4.21 and [DT] Section 12), $T_{m,n}$ is irreducible.

The sheaf $\underline{\omega}_{\infty/S_\infty} = \mathcal{O}_{S_\infty} \otimes_{\mathbb{Z}_p} \varprojlim_n \widehat{\mathcal{G}[p^n]^\circ}$ is isomorphic to the dual of $f_* Lie(\mathcal{G}_{/M})$ for $f : \mathcal{G} \to M$. In other words, $\underline{\omega}_{/M} = Hom(f_* Lie(\mathcal{G}_{/M}), \mathcal{O}_M)$ is the algebraization of the formal sheaf $\underline{\omega}_\infty$ on S_∞ (which is unique). Identifying $X(T) = \mathrm{Hom}_{alg-gp}(T, \mathbb{G}_{m/W})$ with $\mathbb{Z}[I]$ for the set I of embeddings of O_F into W, we write $\underline{\omega}^k$ for the sheaf associated to $k \in X(T)_+$. We then define

$$M^* = Proj(\bigoplus_{j \geq 0} H^0(M, \underline{\omega}^{jt})),$$

where $t = \sum_{\sigma\in I}\sigma$. Then $S^* \subset M^*$ is defined by

$$S^* = Spec(\bigoplus_{j\geq 0} H^0(M, \underline{\omega}^{jt})/(E-1))$$

for the lift of the Hasse invariant E. Write $\pi : M \to M^*$.

The only thing we need to verify is (Hyp1):

$$H^0(S, \underline{\omega}_k) \otimes_W W_m = H^0(S_m, \underline{\omega}_k \otimes_W W_m)$$

for the sheaf $\underline{\omega}_k \subset \underline{\omega}^k$ of cusp forms of weight k. Since $H^0(S, \underline{\omega}_k) = H^0(S^*, \pi_*(\underline{\omega}_k))$ and S^* is affine, we need to verify

$$\pi_*(\underline{\omega}_{k/S}) \otimes_W W_m = \pi_*(\underline{\omega}_{k/S} \otimes_W W_m).$$

We shall do this stalk by stalk. Outside the cusps, the two sheaves are the same; so, nothing to prove.

Now we have for each cusp x associated to the ideals $\mathfrak{a}\mathfrak{b}^{-1} = \mathfrak{c}$ and a p–adic W–algebra $A = \varprojlim_m A/p^m A$:

$$\begin{aligned}(9.1)\quad \widehat{\pi_*(\underline{\omega}^k_{/A})_x} &\cong H^0(T(\mathbb{Z})(N)^2, A[[\frac{1}{N}(\mathfrak{a}\mathfrak{b})_+]]) \\ &= \left\{ \sum_{\xi\in\frac{1}{N}(\mathfrak{a}\mathfrak{b})_+} a(\xi)q^\xi \in A[[\frac{1}{N}(\mathfrak{a}\mathfrak{b})_+]] \middle| a(\varepsilon^2\xi) = \varepsilon^k a(\xi)\ \forall \varepsilon \in T(\mathbb{Z})(N) \right\},\end{aligned}$$

where $(\mathfrak{a}\mathfrak{b})_+ = \{\xi \in \mathfrak{a}\mathfrak{b} | \xi \gg 0\} \bigcup \{0\} = C \cap \mathfrak{a}\mathfrak{b}$, and $T(\mathbb{Z})(N) = \{u \in O_F^\times | u \equiv 1 \mod N\}$ acts on $A[[\frac{1}{N}(\mathfrak{a}\mathfrak{b})_+]]$ by $\varepsilon \sum_{\xi\in(\mathfrak{a}\mathfrak{b})_+} a(\xi)q^\xi = \sum_{\xi\in(\mathfrak{a}\mathfrak{b})_+} \varepsilon^{-k} a(\varepsilon^2\xi)q^\xi$. When $N \geq 3$, for each $\varepsilon^2 \in T(\mathbb{Z})(N)^2$, there is a unique $\varepsilon \in T(\mathbb{Z})(N)$; so, there is no ambiguity of $(\pm\varepsilon)^{-k}$ in the above formula. We define $\underline{\omega}_k$ by requiring its stalk at every cusp is given by those q–expansions vanishing at the cusp. The group cohomology $H^0(T(\mathbb{Z})(N)^2, X)$ commutes with $\otimes_W W_m$ if X is $A[T(\mathbb{Z})(N)^2]$–free. Then from the above fact, we get

$$(9.2)\qquad \widehat{\pi_*(\underline{\omega}_{k/W})_x} \otimes_W W_m \cong \widehat{\pi_*(\underline{\omega}_{k/W_m})_x}.$$

We put

$$V^{cusp}_{m,n} = H^0(T_{m,n}, \mathcal{O}_{T_{m,n}}(-D)), \quad \mathcal{V}_{cusp} = \varinjlim_m V^{cusp}_{m,\infty}$$
$$V_{cusp} = \varprojlim_m V^{cusp}_{m,\infty}, \quad R^{cusp}_\ell = \bigoplus_{k>\ell} H^0(M, \underline{\omega}_k)$$
$$D^{cusp}_\ell = \beta(R^{cusp}_\ell)[\frac{1}{p}] \bigcap V_{cusp}.$$

Here $k > \ell$ means that $k_\sigma > \ell_\sigma$ for all $\sigma \in I$, and $D = \pi^{-1}(\sum_{x:cusp} x)$ is the cuspidal divisor on the toroidal compactification.

In this $GL(2)$–case, it is known that we have two Hecke operators $U(p)$ acting on cusp forms of level divisible by p and $T(p)$ acting on cusp forms of level prime to p, normalized as in Lemma 8.4 to keep integrality of $\underline{\omega}_k$. The operator $U(p)$ has its effect on q–expansion $a(\xi, f|U(p)) = a(\xi p, f)$ and decreases the level to the minimum as long as it is p^n for $n > 0$, and if $k > 2t$, then $T(p) \equiv U(p) \mod p$. Let e (resp. e°) be the idempotent attached to $U(p)$ (resp. $T(p)$). We attach a subscript or superscript "*ord*" to the object after applying the idempotent e or e° (depending on the setting). From this, we conclude

Theorem 9.1. *Let F be a totally real field of degree d and N be an integer $N \geq 3$. Suppose that p is prime to $NN_{F/\mathbb{Q}}(\mathfrak{d}\mathfrak{c})$. Then we have the following facts:*

1. *D_ℓ^{cusp} is dense in V_{cusp};*
2. *The Pontryagin dual $\mathcal{V}_{cusp}^{ord,*}$ (which is isomorphic to $\mathrm{Hom}_W(V_{cusp}^{ord}, W)$) of $\mathcal{V}_{cusp}^{ord}$ is a projective $W[[T(\mathbb{Z}_p)]]$–module of finite type;*
3. *$\mathcal{V}_{cusp}^{ord,*} \otimes_{W[[T(\mathbb{Z}_p)]],k} W \cong \mathrm{Hom}_W(H^0_{ord}(S_\infty, \underline{\omega}_k), W)$ if $k \geq 3t$;*
4. *If $k \geq 3t$ ($t = \sum_\sigma \sigma$), e induces an isomorphism*
$$H^0_{ord}(S_\infty, \underline{\omega}_k) \cong H^0_{ord}(M, \underline{\omega}_k),$$
where $H^0_{ord}(S_\infty, ?) = eH^0(S_\infty, ?)$ and $H^0_{ord}(M, ?) = e^\circ H^0(M, ?)$.

We shall give a very brief sketch of the proof (see [PAF] Theorem 4.8 for more details).

Proof. The assertions (1) and (2) follows from the general argument, using the theory of false modular forms (Section 7). Then the assertions (3) and (4) follow for sufficiently large k. It is known that $\dim H^0_{ord}(M, \underline{\omega}_k)$ depends only on $k|_{T(\mathbb{F}_p)}$ if $k \geq 3t$ (see [H88] Theorems 2.1 and 8.1 and [PAF] Theorem 4.37). From this, the assertion (3) and (4) for small k follows. □

9.1.1. *Moduli problem of* $\Gamma_1(N)$*–type.* Let $\widehat{\Gamma}_1(N)$ be an open compact subgroup in $GL_2(\widehat{O}_F)$ ($\widehat{O}_F = O_F \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$) consisting of elements congruent to upper triangular matrices of the form $\left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right)$ modulo N. Let $\widehat{\Gamma}(N)$ be the subgroup of $\widehat{\Gamma}_1(N)$ consisting of matrices congruent to 1 module N.

In place of the full level N–structure, we could have started with the moduli problem classifying test objects $(X, \lambda, \phi : \mathfrak{d}^{-1} \otimes_{\mathbb{Z}} \mu_N \hookrightarrow X[N])_{/A}$. As long as the moduli problem is representable (that is, N is sufficiently deep), we get the same assertions as in Theorem 9.1 replacing $M_{\mathfrak{c},N}$ by the moduli $M_{\mathfrak{c},\widehat{\Gamma}_1(N)}$ for weight k with $k_\sigma \equiv k_\tau \mod 2$ for all $\sigma, \tau : F \hookrightarrow \overline{\mathbb{Q}}$. This parity condition is necessary to define $\varepsilon \mapsto \varepsilon^{k/2}$ for totally positive units $\varepsilon \in T(\mathbb{Z})$ (since $\varepsilon^t = 1$ for such units). In this subsection, we only consider the moduli problem of $\widehat{\Gamma}_1(N)$–type, and accordingly, we define V, $\mathcal{V}$, $\mathcal{V}_{cusp}^{ord,*}$ and $\mathcal{V}_{cusp}^{ord}$ for $\widehat{\Gamma}_1(N)$. For simplicity, we hereafter assume that k is even (so we write $2k$ instead of k), since the general case is exposed already in [H96] Chapter 2. Since these spaces actually depend on the choice of the ideal $\mathfrak{c}$, we add subscript $\mathfrak{c}$ if we want to make explicit the dependence on $\mathfrak{c}$.

We consider a test object $(X, \lambda, \phi)_{/A}$ of level $\widehat{\Gamma}_1(N)$. For any ideal $\mathfrak{a}$ prime to Np, we make a quotient $X' = X/X[\mathfrak{a}] = X \otimes_{\mathbb{Z}} \mathfrak{a}$; thus, $X = X' \otimes \mathfrak{a}$. Then $X'[N] \cong X[N]$ canonically; so, ϕ induces a level $\widehat{\Gamma}_1(N)$–structure ϕ'. Let $P(X) = \{\lambda \in \mathrm{Hom}(X, \widehat{X}) | \widehat{\lambda} = \lambda\}$ and $P_+(X) \subset P(X)$ is the subset made of polarizations. Then we have an isomorphism: $\lambda : P \cong \mathfrak{c}$ of O_F–modules taking P_+ onto the subset $\mathfrak{c}_+$ of totally positive elements of $\mathfrak{c}$. Dualizing the exact sequence:
$$0 \to X[\mathfrak{a}] \to X \to X' \to 0,$$
we get another exact sequence:
$$0 \to \widehat{X}'[\mathfrak{a}] \to \widehat{X}' \to \widehat{X} \to 0,$$
because $\widehat{X}'[\mathfrak{a}]$ is the Cartier dual of $X[\mathfrak{a}]$. This shows $\widehat{X}' \cong \widehat{X} \otimes \mathfrak{a} \cong X \otimes \mathfrak{c}\mathfrak{a} \cong X' \otimes \mathfrak{c}\mathfrak{a}^2$, λ induces $\lambda' : P(X')_+ \cong (\mathfrak{c}\mathfrak{a}^2)_+$. Thus $(X, \lambda, \phi) \mapsto (X', \lambda', \phi')$ induces

$[\mathfrak{a}]: V_{\mathfrak{c}\mathfrak{a}^2} \cong V_{\mathfrak{c}}$. We identify $V_{\mathfrak{c}}$ and $V_{\mathfrak{c}\mathfrak{a}^2}$ by $[\mathfrak{a}]$. Thus $V_{\mathfrak{c}}$ only depends on the strict ideal class of $\mathfrak{c}$ (and also modulo square ideal classes).

We then define

$$\mathcal{V}^{ord}_{cusp}(\widehat{\Gamma}_1(N)) = \bigoplus_{\mathfrak{c}} \mathcal{V}^{ord}_{cusp,\mathfrak{c}} \text{ and } \mathcal{V}^{ord,*}_{cusp}(\widehat{\Gamma}_1(N)) = \bigoplus_{\mathfrak{c}} \mathcal{V}^{ord,*}_{cusp,\mathfrak{c}}, \tag{9.3}$$

where $\mathfrak{c}$ runs over strict equivalence classes of ideals modulo square classes; thus, it runs over the group $Cl^+_F/(Cl^+_F)^2$, where Cl^+_F is the strict ideal class group.

Note that

$$PGL_2(F_{\mathbb{A}}) = \bigsqcup_{a \in Cl^+_F/(Cl^+_F)^2} PGL_2(F) \left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) \overline{\Gamma}_1(N) PGL_2^+(F_\infty),$$

where a runs over a complete representative set for $Cl^+_F/(Cl^+_F)^2$ in $F^\times_{\mathbb{A}}$; $F_\infty = F \otimes_{\mathbb{Q}} \mathbb{R}$; $PGL_2^+(F_\infty)$ is the identity connected component of $PGL_2(F_\infty)$, and $\overline{\Gamma}_1(N)$ is the image of $\widehat{\Gamma}_1(N)$ in $PGL_2(F^{(\infty)}_{\mathbb{A}})$. Thus we may regard $\mathcal{V}^{ord,*}_{cusp}(\widehat{\Gamma}_1(N))$ as the W–dual of the space of p–adic cusp forms of level $\widehat{\Gamma}_1(N)$ on $PGL_2(F_{\mathbb{A}})$. For a given modular form $f = (f_{\mathfrak{c}})$ the above spaces, say in $\mathcal{V}_{cusp,\mathfrak{c}}$, it has q–expansion at the cusp $\infty = (\mathfrak{a} = O_F, \mathfrak{b} = \mathfrak{c}^{-1})$:

$$f_{\mathfrak{c}} = \sum_{\xi \in \mathfrak{c}^{-1}_+} a(\xi; f_{\mathfrak{c}}) q^{\xi}.$$

If $f \in \mathcal{V}_{cusp,\mathfrak{c}}[2k]$, as we have already seen for level N–modular forms, $a(\varepsilon^2\xi; f) = \varepsilon^{2k} a(\xi; f)$ for $\varepsilon \in T(\mathbb{Z}) \subset SL_2(O_F)$. Since we only have level $\widehat{\Gamma}_1(N)$–structure, f satisfies invariance under the matrix $\left(\begin{smallmatrix} \varepsilon & 0 \\ 0 & 1 \end{smallmatrix}\right)$ for totally positive units ε in addition to the invariance under $\left(\begin{smallmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{smallmatrix}\right)$ for arbitrary units ε. Thus we actually have

$$a(\varepsilon\xi; f) = \varepsilon^k a(\xi; f) \text{ for all totally positive units } \varepsilon.$$

Choose a finite idele c so that $cO_F = \mathfrak{c}$ and $c_p = 1$. For finite integral ideles y and $f \in \mathcal{V}_{cusp,\mathfrak{c}}$, we may define a continuous function $y \mapsto \mathbf{a}_p(y; f) \in K/W$ for the field of fractions K of W as follows: Choose ideal representatives (prime to Np) $\mathfrak{c}$ and $\mathfrak{a}$ so that $yO_F = \xi\mathfrak{c}\mathfrak{a}^2$ for $\xi \in (\mathfrak{c}\mathfrak{a}^2)^{-1}_+$, and write $y = u\xi c a^2$ for ideles u, c, a with $a_p = c_p = 1$, $cO_F = \mathfrak{c}$, $aO_F = \mathfrak{a}$ and $a_p = c_p = 1$. Then we define

$$\mathbf{a}_p(y; f) = u_p^k a(\xi; f_{\mathfrak{c}}|[\mathfrak{a}]).$$

We can verify that $\mathbf{a}_p(uy; f) = \mathbf{a}_p(y; f)$ for $u \in \widehat{O}^\times_F$ with $u_p = 1$ and if $f \in \mathcal{V}_{cusp,\mathfrak{c}}[2k]$, then $\mathbf{a}_p(uy; f) = u_p^k \mathbf{a}_p(y; f)$ for $u \in \widehat{O}^\times_F$. Thus $\mathbf{a}_p$ is well defined independently of the choice of c, and for an integral ideal $\mathfrak{n}$ prime to p, choosing a finite idele n so that $nO_F = \mathfrak{n}$ and $n_p = 1$, $\mathbf{a}_p(ny; f)$ is well defined independent of the choice of n. We write $\mathbf{a}_p(y\mathfrak{n}; f) = \mathbf{a}_p(ny; f)$.

We extend the function $\mathbf{a}_p$ outside integral ideles by defining it to be 0 and extend it to general $f \in \mathcal{V}_{cusp,\mathfrak{c}}$ using the fact that $\mathcal{V}_{cusp,\mathfrak{c}} = \sum_{2k} \mathcal{V}_{cusp,\mathfrak{c}}[2k]$. By the q–expansion principle due to Ribet (which we will prove in a more general setting in the last lecture: Section 10), the p–adic modular form is determined by the function $\mathbf{a}_p$ on integral ideles. An important fact (see [H96] 2.4) is the following formula for integral ideals $\mathfrak{n}$ prime to p and the Hecke operator $T(\mathfrak{n})$:

$$\mathbf{a}_p(y; f|T(\mathfrak{n})) = \sum_{\mathfrak{l} \supset \mathfrak{n} + yO_F} N(\mathfrak{l})^{-1} \mathbf{a}_p(y\mathfrak{n}/\mathfrak{l}^2; f). \tag{9.4}$$

For $w \in O_{F,p} \cap F_p^\times$, we write $T(w)$ for the normalized Hecke operator corresponding to the double coset $U(\mathbb{Z}_p)\left(\begin{smallmatrix}1 & 0\\ 0 & w\end{smallmatrix}\right)U(\mathbb{Z}_p)$. Then we have

$$\mathbf{a}_p(y; f|T(w)) = \mathbf{a}_p(yw; f). \tag{9.5}$$

Lemma 9.2. *Let $\mathbf{h}$ be the subalgebra of $End(\mathcal{V}^{ord}_{cusp}(\widehat{\Gamma}_1(N)))$ generated topologically by $T(w)$ for $w \in O_{F,p} \cap F_p^\times$ and $T(\mathfrak{n})$ for integral ideals $\mathfrak{n}$ prime to p. Then $\mathbf{h} \cong \mathcal{V}^{ord,*}_{cusp}(\widehat{\Gamma}_1(N))$ as $W[[T(\mathbb{Z}_p)]]$–modules.*

Proof. We shall give a sketch of a proof. We consider the following pairing: $(\ ,\ ) : \mathcal{V}^{ord}_{cusp}(\widehat{\Gamma}_1(N)) \times \mathbf{h} \to K/W$ given by $(f, h) = \mathbf{a}_p(1, f|h)$. Then $(T(w)T(\mathfrak{n}), f) = \mathbf{a}_p(w\mathfrak{n}; f)$ by (9.4) and (9.5), and hence, by the q–expansion principle, if $(f, \mathbf{h}) = 0$, then $f = 0$. By the perfectness of the Pontryagin duality, we thus have a surjective $\mathbf{h}$–linear morphism: $\mathbf{h} \twoheadrightarrow \mathcal{V}^{ord,*}_{cusp}(\widehat{\Gamma}_1(N))$ of Hecke modules. Since $\mathbf{h}$ acts faithfully on $\mathcal{V}^{ord,*}_{cusp}(\widehat{\Gamma}_1(N))$, we conclude the injectivity. $\square$

Since a similar duality holds between the weight $2k$ Hecke algebra $\mathbf{h}_{2k}(\widehat{\Gamma}_1(N); W)$ acting on $\bigoplus_{\mathfrak{c}} H^0_{ord}(M_{\widehat{\Gamma}_1(N),\mathfrak{c}}, \underline{\omega}_{2k})$ and $\mathcal{V}^{ord}_{cusp}(\widehat{\Gamma}_1(N))[2k]$, Theorem 9.1 implies the control result for the Hecke algebra:

$$\mathbf{h} \otimes_{W[[T(\mathbb{Z}_p)]],2k} W \cong \mathbf{h}_{2k}(\widehat{\Gamma}_1(N); W) \quad \text{for all } 2k \geq 3t. \tag{9.6}$$

We can extend this result to $GL(2)$ (from $PGL(2)$). Let $Z = Cl^+_F(p^\infty)$ be the ray class group modulo $p^\infty\infty$, that is, $\varprojlim_r Cl^+_F(p^r)$. We decompose $Z = \Gamma_Z \times \Delta_Z$ so that Γ_Z is p–profinite and Δ_Z has order prime to p.

Since the universal nearly p–ordinary Hecke algebra for $p > 2$ on $\mathrm{Res}_{F/\mathbb{Q}} GL(2)$ is the Pontryagin dual of $\mathcal{C}(Z, \mathcal{V}^{ord}_{cusp})$, the Hecke algebra is isomorphic to

$$\mathcal{V}^{ord,*}_{cusp}(\widehat{\Gamma}_1(N)) \widehat{\otimes}_W W[[Z]]$$

as $W[[Z \times T(\mathbb{Z}_p)]]$–modules (see [MFG] Theorem 5.6.1 for a proof when $p > 2$ and $N = 1$ and [PAF] 4.2.12 for more general results). Thus we have the following facts when N is sufficiently deep so that the $\widehat{\Gamma}_1(N)$–moduli problem is representable:

Corollary 9.3. *Let $p \nmid 2NN_{F/\mathbb{Q}}(\mathfrak{d})$ be a prime. Suppose either $p \geq 5$ or that N is sufficiently deep so that the $\widehat{\Gamma}_1(N)$–moduli problem is representable. Then we have*

1. *The universal p–nearly ordinary Hecke algebra of auxiliary level $\widehat{\Gamma}_1(N)$ is $W[[\Gamma_Z \times \Gamma_T]]$–free of finite rank;*
2. *The specialization of the universal Hecke algebra at each arithmetic point $P \in Spec(W[[Z \times T(\mathbb{Z}_p)]])(\overline{\mathbb{Q}}_p)$ inducing weight $k > 2t$ in $X_+(T)$ produces the nearly ordinary Hecke algebra of level $\widehat{\Gamma}_1(N)$ and weight P without any error terms. When $k \geq 2t$, the specialization produces the Hecke algebra of weight k with level $\widehat{\Gamma}_1(N) \cap \widehat{\Gamma}_0(p)$.*

See [PAF] 4.2.12 for the proof when $p \geq 5$.

9.2. **Elliptic Λ–adic Forms Again.** We describe how to view Λ–adic forms as a p–adic modular forms defined over Λ. Once this is done, we can evaluate Λ–adic forms at elliptic curves, which gives us a convenient method of constructing and analyzing p–adic L–functions. Then, we shall give a short account of the Λ–adic Eisenstein series and examples of Λ–adic L–functions.

All arguments presented here can be generalized to Hilbert modular case, Siegel-Hilbert modular case and quasi-split unitary cases, which will be treated in a forthcoming work.

9.2.1. *Generality of Λ–adic forms.* For simplicity, we assume that $p > 2$ and only consider the Λ–adic forms of level p^∞. Let $\Lambda = \mathbb{Z}_p[[T]]$. In the third lecture, we introduced the space $G(\chi;\Lambda)$ of p–ordinary Λ–adic forms, which is a free Λ–module of finite rank with

$$G(\chi;\Lambda)\otimes_{\Lambda,k}\mathbb{Z}_p \cong G_k^{ord}(\Gamma_0(p),\chi\omega^{-k};\mathbb{Z}_p)$$

for all $k \geq 2$. Here $k : \Lambda \to \mathbb{Z}_p$ is the evaluation at $u^k - 1$ of the power series. If we identify Λ with the Iwasawa algebra $\mathbb{Z}_p[[1+p\mathbb{Z}_p]]$ (via $1+T \leftrightarrow u \in 1+p\mathbb{Z}_p$), k is induced by the character $1+p\mathbb{Z}_p \ni z \mapsto z^k \in \mathbb{Z}_p^\times$.

We write $G(\Lambda)$ for the Λ–module made of formal q–expansions

$$\Phi = \sum_{n\geq 0} a(n;\Phi)(T)q^n \in \Lambda[[q]]$$

such that $\Phi(u^k-1) \in V[k]$ for infinitely many k. Thus we have $\bigoplus_\chi G(\chi;\Lambda) \subset G(\Lambda)$, where χ runs over (actually even) powers of Teichmüller characters.

We now consider the space of p–adic modular forms $V_{/\Lambda}$ over Λ of level p^∞. In other words, we shall make base-change $T_{m,n/\mathbb{Z}_p}$ to $T_{m,n/\Lambda} = T_{m,n/\mathbb{Z}_p} \times_{\mathbb{Z}_p} \Lambda$ and consider p–adic modular forms over Λ. The functions in $V_{/\Lambda} = V\widehat{\otimes}_{\mathbb{Z}_p}\Lambda$ classify couples: $(E, \phi : \mu_{p^\infty} \hookrightarrow E[p^\infty])_{/R}$ defined over p–adic Λ–algebras R, and $f \in V_{/\Lambda}$ is a functorial rule assigning the value $f(E,\phi) \in R$ for each couple $(E,\phi)_{/R}$ as above.

This space has two Λ–module structures: One coming from the base ring Λ and another coming from the action of $\mathrm{Gal}(T_{m,\infty}/S_m) = \mathbb{Z}_p^\times$ by diamond operators $\langle z\rangle$. Let $\nu : 1+p\mathbb{Z}_p \to \Lambda^\times$ be the universal character given by $\nu(z) = [z] \in 1+p\mathbb{Z}_p$. Then we can define

$$\mathbb{G}(\Lambda) = \left\{ f \in V_{/\Lambda} \middle| f|\langle z\rangle = \nu(z) f \ \forall z \in 1+p\mathbb{Z}_p \right\}. \tag{9.7}$$

Each $\Phi \in \mathbb{G}(\Lambda)$ has a q–expansion at ∞: $\Phi(T,q) = \sum_{n\geq 0} a(n;\Phi)(T)q^n$. By definition, we have a natural map:

$$V_{/\Lambda}\otimes_{\Lambda,s}\mathbb{Z}_p \to V_{/\mathbb{Z}_p}$$

for each $s : \Lambda \to \mathbb{Z}_p$ taking $\Phi(T)$ to $\Phi(u^s - 1)$ for $s \in \mathbb{Z}_p$. Here the tensor product is taken using Λ–module structure induced by the diamond operators. The map is injective by the q–expansion principle. Since on $\mathbb{G}(\Lambda)$, the two Λ–module structures coincide, this map brings $\Phi \in \mathbb{G}(\Lambda)$ to a p–adic modular from of weight s. Therefore, Φ is a Λ–adic form.

Conversely, starting from a Λ–adic form Φ, we regard Φ as a bounded measure on $1+p\mathbb{Z}_p$ having values in $V_{/\mathbb{Z}_p}$. Here we use the fact that $\mathbb{Z}_p[[1+p\mathbb{Z}_p]]$ is canonically isomorphic to the measure space on $1+p\mathbb{Z}_p$ by $a(T) \mapsto \int x^s da = a(u^s-1)$. Thus Φ is a bounded $\mathbb{Z}_p$–linear map of $\mathcal{C}(1+p\mathbb{Z}_p,\mathbb{Z}_p)$ into $V_{/\mathbb{Z}_p}$. Then for each test object $(E,\phi)_{/R}$ for a p–adic Λ–algebra R, regarding R as a p–adic $\mathbb{Z}_p$–algebra, we can evaluate $\int \phi d\Phi \in V_{/\mathbb{Z}_p}$ at $(E,\phi_p)_{/R}$, getting a bounded $\mathbb{Z}_p$–linear form from the space $\mathcal{C}(1+p\mathbb{Z}_p,\mathbb{Z}_p)$ into R, which we write $\Phi(E,\phi)(T) \in R\widehat{\otimes}_{\mathbb{Z}_p}\Lambda = R[[T]]$. Since R is already a Λ–algebra, the Λ–module structure $\Lambda\widehat{\otimes}R \to R$ given by $\lambda\otimes r = \lambda r$ induces a surjective algebra homomorphism $m : R\widehat{\otimes}_{\mathbb{Z}_p}\Lambda \twoheadrightarrow R$. We then define $\Phi(E,\phi)$ by $m(\Phi(E,\phi)(T))$. Then the assignment: $(E,\phi) \mapsto \Phi(E,\phi)$ satisfies the axiom of the p–adic modular forms defined over Λ. It is easy to check that this p–adic modular from is in $\mathbb{G}(\Lambda)$ having the same q–expansion at ∞ as Φ. Thus we have found:

Theorem 9.4. *The subspace $\mathbb{G}(\Lambda) \subset V_{/\Lambda}$ is isomorphic to the space $G(\Lambda)$ of all Λ–adic forms via q–expansion at the cusp ∞. In particular, we have*

$$\bigoplus_{\chi} G(\chi;\Lambda) \cong e(\mathbb{G}(\Lambda))$$

for the p–ordinary projector $e : V_{/\Lambda} \twoheadrightarrow V^{ord}_{/\Lambda}$.

Let $(E,\omega)_{/W}$ be an elliptic curve with complex multiplication by an imaginary quadratic field $F = \mathbb{Q}[\sqrt{-D}]$. We suppose that ω is defined over $\mathcal{W} = W \cap \overline{\mathbb{Q}}$ fixing an embedding $i_p : \overline{\mathbb{Q}} \hookrightarrow \widehat{\overline{\mathbb{Q}}}_p$. Suppose that p splits in F and write $p = \mathfrak{p}\overline{\mathfrak{p}}$. Under this assumption, E has ordinary good reduction modulo p; so, E is p–ordinary. We may assume that $W = W(\overline{\mathbb{F}}_p)$ and $E[\overline{\mathfrak{p}}^\infty]$ is the étale part of $E[p^\infty]$ over W. Thus we have $\phi : \mu_{p^\infty} \cong E[\mathfrak{p}^\infty]$. In this way, we can evaluate a given Λ–adic form Φ at (E,ϕ).

Corollary 9.5. *If $\Phi(E,\phi) = 0$ for infinitely many distinct E with complex multiplication, then $\Phi = 0$. There exists a finitely many elliptic curves $(E_i,\phi_i)_{/W}$ such that any given linear form $G(\chi;\Lambda) \to \Lambda$ is a Λ–linear combination of evaluation at (E_i,ϕ_i).*

If $\Phi \in G(\chi;\Lambda)$ and further if $f = \Phi(u^k - 1) \in G_k(\Gamma_0(p), \chi\omega^{-k}; \mathcal{W})$, then $f(E,\omega) \in \mathcal{W}$. The morphism $\mu_{p^\infty} \hookrightarrow E$ induces a canonical differential $\omega_{can} = \phi_* \frac{dt}{t}$. Then $\omega = \Omega_p \omega_{can}$, and we have a result of Katz [K2] Chapter II:

$$\text{(EQ1)} \qquad f(E,\omega) = \frac{f(E,\omega_{can})}{\Omega_p^k} = \frac{\Phi(E,\phi)(u^k-1)}{\Omega_p^k} \in \mathcal{W} \subset \overline{\mathbb{Q}}.$$

We may assume that $E(\mathbb{C}) = \mathbb{C}/O_F$. Let w be the variable of $\mathbb{C}$. Then dw induces a canonical differential ω_∞ on $E(\mathbb{C})$. Then $\omega = \Omega_\infty \omega_\infty$, and we get a result of Shimura [Sh4]:

$$\text{(EQ2)} \qquad f(E,\omega) = \frac{f(E,\omega_\infty)}{\Omega_\infty^k} = \frac{f(E,\omega_{can})}{\Omega_p^k} \in \mathcal{W} \subset \overline{\mathbb{Q}}.$$

The lattice $O_F = H_1(E,\mathbb{Z}) \subset \mathbb{C}$ is generated over O_F by a single element $\gamma = 1$ and

$$\Omega_\infty = \int_\gamma \omega,$$

because $\int_\gamma dw = 1$.

9.2.2. *Some p–adic L–functions.* For simplicity, we assume that $p > 2$ and only consider the Λ–adic Eisenstein series of level p^∞. Let us fix an even power $\chi = \omega^a$ of the Teichmüller character. For simplicity, we choose $a \not\equiv 0 \mod p-1$; so, χ is non-trivial. Then we consider the Kubota-Leopoldt p–adic L–function $-\frac{1}{2}L_p(1-s,\chi) = a_0(u^s - 1)$ $(u = 1+p)$ with $a_0 \in \mathbb{Z}_p[[T]]$ (cf. [LFE] 3.4-5). Then we have

$$L_p(1-k,\chi) = (1 - \chi\omega^{-k}(p)p^{k-1})L(1-k,\chi\omega^{-k})$$

for positive integers k. Then we define an element $\mathcal{E}_\chi \in G(\chi;\Lambda)$ by

$$a(n,\mathcal{E}_\chi) = \sum_{0<d|n, p\nmid d} \chi(d)d^{-1}(1+T)^{\log(d)/\log(u)} \quad \text{and} \quad a(0,\mathcal{E}_\chi) = a_0(T).$$

We want to relate $\mathcal{E}_\chi(u^k-1)$ to the following classical Eisenstein series:

$$E_k(E,\omega) = \frac{1}{2}\sum_{(m,n)\neq(0,0)} \frac{1}{(mw_1+nw_2)^k},$$

where $(E,\omega)_{/\mathbb{C}}$ corresponds to the lattice $L = \mathbb{Z}w_1 + \mathbb{Z}w_2$ by Weierstrass theory (that is, $E(\mathbb{C}) = \mathbb{C}/L$ and $\omega = dw$ for the variable $w \in \mathbb{C}$). As is well known (e.g. [LFE] 5.1), for even $k > 2$

$$E_k = c(k)\left(-\frac{1}{2}\zeta(1-k) + \sum_{n>0}(\sum_{0<d|n} d^{k-1})q^n\right)$$

for $c(k) = \frac{\Gamma(k)}{(2\pi\sqrt{-1})^k}$. Thus shows that if $\omega^k = \chi$, then

$$\mathcal{E}_\chi(u^k-1) = c(k)^{-1}\left(E_k - p^{k-1}E_k(pz)\right).$$

If we take the elliptic curve (E,ω) defined by $y^2 = 1-x^4$ with $\omega = \frac{dx}{y}$, then it has complex multiplication by $\mathbb{Q}[\sqrt{-1}]$ and for $k > 2$ with $\omega^k = \chi$, we have

$$\frac{1}{2}E_k(E,\omega) = \frac{L(k,\lambda_k)}{\Omega_\infty^k},$$

where λ_k is a Hecke character of conductor 1 such that $\lambda_k(\alpha) = \alpha^k$. Since in $SL_2(\mathbb{Z})\left(\begin{smallmatrix} p & 0 \\ 0 & 1\end{smallmatrix}\right)SL_2(\mathbb{Z})$, we can find α such that $(w_1,w_2)\alpha$ is a base of $\mathfrak{p}$; so, we rediscover Katz's p–adic interpolation of Hurwitz numbers:

$$\begin{aligned}\frac{\mathcal{E}_\chi(E,\omega)(u^k-1)}{\Omega_p^k} &= 2c(k)^{-1}\frac{(L(k,\lambda_k) - p^{k-1}\lambda_k(\overline{\mathfrak{p}})p^{-k}L(k,\lambda_k))}{\Omega_\infty^k} \\ &= 2c(k)^{-1}(1-p^{-1}\lambda_k(\overline{\mathfrak{p}}))\frac{L(k,\lambda_k)}{\Omega_\infty^k}.\end{aligned}$$

This is a Λ–adic version of Katz's way of constructing the p–adic Hecke L–function: $L_p(s) = \mathcal{E}_\chi(E,\omega)(u^s-1)$ ([K2] and [HT]).

A p–adic Rankin product can be constructed similarly. Let Φ be a normalized Hecke eigenform in $G(\chi;\Lambda)$. Writing $\mathbb{L}$ for the field of fractions of Λ. As seen in Section 4, the Hecke algebra acts semi-simply on $G(\chi;\Lambda)$; so, we can decompose uniquely $G(\chi;\Lambda)\otimes_\Lambda \mathbb{L} = \mathbb{L}\Phi \oplus \mathbb{X}$ as Hecke modules. Let $\ell : G(\chi;\Lambda) \to \mathbb{L}$ be the linear form defined by $\Psi = \ell(\Psi)\Phi + x$ for $x \in \mathbb{X}$. We consider two copies of Λ, say $\mathbb{Z}_p[[S]]$ and $\mathbb{Z}_p[[T]]$. Take two Hecke eigenforms $\Phi \in G(\chi;\mathbb{Z}_p[[T]])$ and $\Psi \in G(\psi;\mathbb{Z}_p[[S]])$. Extend linearly ℓ to $G(\chi;\mathbb{Z}_p[[T]])\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p[[S]] \to \mathbb{L}\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p[[S]]$. Then we define

$$L_p(S,T) = \ell(e(\Psi(S)\mathcal{E}_{\chi\psi^{-1}}((1+T)(1+S)^{-1}-1))),$$

where $e : V_{/\mathbb{Z}_p[[T]]} \to V^{ord}_{/\mathbb{Z}_p[[T]]}$ is the p–ordinary projector. Then we see

$$\Psi(S)\mathcal{E}_{\chi\psi^{-1}}((1+T)(1+S)^{-1})|_{S=u^m-1,T=u^k-1} = \Psi(u^m-1)\mathcal{E}_{\chi\psi^{-1}}(u^{k-m}-1).$$

Thus $\Psi(u^m-1)\mathcal{E}_{\chi\psi^{-1}}(u^m(1+T)-1) \in G(\chi;\mathbb{Z}_p[[T]])$ and hence $L_p(u^m-1,u^k-1)$ is the coefficient of $\Psi(u^m-1)E_{k-m}$ in $\Phi(u^k-1)$ for a suitable Eisenstein series

E_{k-m} of weight $k-m$. As is shown by Shimura, this coefficients can be computed by the Rankin product value

$$\frac{D(k-1,\Phi(u^k-1),\Psi(u^m-1))}{(\Phi(u^k-1),\Phi(u^k-1))} \quad (k>m)$$

for the Petersson inner product (,) up to an explicit constant; so, L_p gives p–adic interpolation of the Rankin product. For an explicit evaluation formula for $L_p(S,T)$, see [LFE] Chapter 7 and 10 and [H96] Chapter 6.

What I would like to emphasize is that the we have used almost everywhere are:

1. Vertical Control Theorem;
2. The q–expansion principle (irreducibility of the Igusa tower).

10. Igusa Towers

We sketch a proof of irreducibility of the generalized Igusa tower by using the determination of the automorphism group of the arithmetic automorphic function field by Shimura and his students. The method is classical and goes back to works of Deuring [Du] and Igusa [I]. By this result, the q–expansion principle holds for p–adic modular forms on symplectic groups, and for unitary groups, one need to modify it in an appropriate way. We can construct, as Panchishkin did for Siegel modular forms, the p–adic Eisenstein measure for quasi-split unitary groups. The difference of our result from Panchishkin's treatment is that our measure has values in the space of p–adic automorphic forms (not just in the formal q–expansion ring in Panchishkin's work), since we dispose the q–expansion principle. A detailed proof of the result presented here and a further generalization are in [PAF] Section 8.4.

10.1. **Automorphism Groups of Shimura Varieties.** Let the notation be as in Section 6. For a number field X, we write I_X for the set of all field embeddings of X into the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ in $\mathbb{C}$. Let W be the ring of Witt vectors $W(\mathbb{F})$ for an algebraic closure $\mathbb{F}$ of $\mathbb{F}_p$, and we identify W with a subring of the p–adic completion of an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. We fix an embedding $i_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ and write $\mathcal{W}$ for the pull back image W under i_p. We write $\mathfrak{m}_{\mathcal{W}}$ (resp. $\mathfrak{m}$) for the maximal ideal of $\mathcal{W}$ (resp. W). Recall the setting in Section 6 that F_0 is a totally real finite extension of $\mathbb{Q}$, that $F=F_0$ in Case C and D and that in Case A, F is a totally imaginary quadratic extension of a totally real field F_0. Let the algebraic group G be as in Section 6 and also assume that we are in Case A or C. Presumably Case D can be treated similarly, but the results of Shimura we need are often formulated only for groups of type A and C. Towards the end of this lecture, we assume for simplicity that $F_0=\mathbb{Q}$ and G in Case A is quasi-split over $\mathbb{Q}$.

10.1.1. *Automorphism Groups of Automorphic Function Fields.* For the moment, we do not assume that $F_0=\mathbb{Q}$. The group G is indefinite at ∞, that is, $G(\mathbb{R})$ is not compact modulo its center $Z(\mathbb{R})$. We use the formulation of Sh_K described in Section 6 which represents the functor $\mathcal{P}$ classifying quadruples $(X,i,\lambda,\overline{\eta})_{/S}$ for E–schemes S, where E is the reflex field. Thus E is the minimal field of definition of the complex representation of B on V_1 in Section 6. Take a finite Galois extension $F'/\mathbb{Q}$ containing F. When we are in Case A, writing formally the signature of G as $s=\sum_\sigma m(\sigma)\sum_{\tau\in\mathfrak{R}'_{F'}}\tau\sigma$ for embeddings $\sigma : F\hookrightarrow F'$ and for $\mathfrak{R}'_{F'}=\mathrm{Gal}(F'/F)$,

E is the fixed field of $\mathfrak{R}_{F'} = \{\sigma \in \mathrm{Gal}(F'/\mathbb{Q}) | s\sigma = s\}$. Then we can define $m'(\sigma)$ for $\sigma \in I_E$ by

$$\sum_{\sigma\in I_F} m(\sigma) \sum_{\tau\in\mathfrak{R}'_{F'}} (\tau\sigma)^{-1} = \sum_{\sigma\in I_E} m'(\sigma) \sum_{\tau\in\mathfrak{R}_{F'}} \tau\sigma.$$

Then $\theta = \sum_{\sigma\in I_E} m'(\sigma)\sigma$ can be regarded as a character of $\mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_{m/E}$ with values in $\mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_{m/F}$ (see [Sh3] Section 1). Then $\theta(E_{\mathbb{A}}^\times) = \theta(\mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_{m/E}(\mathbb{A}))$ is a closed subgroup of $F_{\mathbb{A}}^\times = \mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_{m/F}(\mathbb{A})$.

Kottwitz formulated the Shimura variety over $O_E \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$, but we only need Deligne's models over E to define the automorphic function field $\mathfrak{K}$. We then take a tower $\{V_K\}$ (allowing K not necessarily maximal at p) of the geometrically irreducible component of Sh_K so that $V_K(\mathbb{C}) = \Gamma_K\backslash\mathfrak{Z}$ for $\Gamma_K = KG(\mathbb{R})_+ \cap G(\mathbb{Q})$ and V_K is covered by V'_K if $K' \subset K$, where $\mathfrak{Z}$ is the symmetric hermitian domain of $G(\mathbb{R})_+$. Then the union $\mathfrak{K}$ of the function field $\mathbb{Q}(V_K)$ of V_K is independent of the choice of the tower (up to isomorphisms), since V_K is the canonical model in Shimura's sense ([ACM] and [AAF] Chapters I and II). Since the group $G(\mathbb{A}^{(\infty)})$ acts on the functor $\mathcal{P}$ by isogenies, we let $G(\mathbb{A})$ act on $\mathcal{P}$ through the projection $G(\mathbb{A}^{(\infty)})$. Let $\mathcal{G}_+ \subset G(\mathbb{A})$ be the stabilizer of the tower $\{V_K\}_K$. Since $V_K(\mathbb{C}) = \Gamma_K\backslash\mathfrak{Z}$, the closure of $\bigcup_K \Gamma_K G(\mathbb{R})_+$ is contained in $\mathcal{G}_+$.

We now suppose that G is an inner form of $GSp(2n)_{/\mathbb{Q}}$ in Case C and in Case A

$$G(\mathbb{Q}) = \left\{\alpha \in GL_{2n}(F) \middle| {}^t\overline{\alpha}J_n\alpha = \nu(\alpha)J_n \text{ for } \nu(\alpha) \in \mathbb{Q}\right\}$$

with J_n as in (6.6) for an imaginary quadratic field F. Therefore $E = F_0 = \mathbb{Q}$, $\theta = \mathrm{id} : \mathbb{Q} \hookrightarrow F$, B is either $\mathbb{Q}$ or a quaternion algebra over $\mathbb{Q}$ in Case C, and in Case A, $B = F$ and G in Case A is quasi-split over $\mathbb{Q}$ (any quasi-split unitary group acting on a hermitian space of dimension $2n$). In this case, we have an explicit description of $\mathcal{G}_+$ by a work of Shimura ([ACM] 26.8, [AAF] 8.10, [Mik], [Mit] and [MiS]):

(Sh1) $\mathcal{G}_+ = \psi^{-1}((F^\times)^{1-c}\mathbb{R}_+^\times)$ in $G(\mathbb{A})$ for $\psi = \det/\nu^n : G(\mathbb{A}) \to F_{\mathbb{A}}^\times$ in Case A; $\mathcal{G}_+ = G(\mathbb{A})_+ = \{x \in G(\mathbb{A}) | \nu(x_\infty) > 0\}$ in Case C; so, we have $\mathcal{G}_+ \supset SG(\mathbb{A})$ ($SG(A) = \{x \in G(A) | \det(x) = \nu(x) = 1\}$) and $\nu(\mathcal{G}_+) = \mathbb{A}_+^\times = (\mathbb{A}^{(\infty)})^\times \times \mathbb{R}_+^\times$;

(Sh2) (Shimura's reciprocity map) Let $Z \subset G$ be the center. Then we have a canonical exact sequence:

$$1 \to Z(\mathbb{Q})G(\mathbb{R})_+ \to \mathcal{G}_+ \xrightarrow{\tau} \mathrm{Aut}(\mathfrak{K}) \to 1,$$

and τ is continuous and open under the Krull topology on $\mathfrak{K}$ (see [IAT] 6.3 for the topology and [PAF] Theorem 7.7 for a description of τ).

(Sh3) The maximal abelian extension $\mathbb{Q}_{ab}$ of $\mathbb{Q}$ is the field of scalars of $\mathfrak{K}$, that is, $\mathfrak{K} \supset \mathbb{Q}_{ab}$ and $\mathfrak{K}$ and $\mathbb{C}$ are linearly disjoint over $\mathbb{Q}_{ab}$. In particular, $\tau(x)$ acts on $\mathbb{Q}_{ab}$ through the image of $\nu(x)$ under the projection: $\mathbb{A}^\times \to \mathbb{A}^\times/\mathbb{R}_+^\times\mathbb{Q}^\times \cong \mathrm{Gal}(\mathbb{Q}_{ab}/\mathbb{Q})$ (Artin reciprocity map).

(Sh4) The subfield E_K of $\mathbb{Q}_{ab}$ fixed by $\nu(K)$ is the field of definition of V_K, that is, E_K is isomorphic to the algebraic closure of $\mathbb{Q}$ in $\mathbb{Q}(V_K)$.

(Sh5) The extension $\mathfrak{K}/\mathbb{Q}(V_K)$ is a Galois extension with

$$\mathrm{Gal}(\mathfrak{K}/\mathbb{Q}(V_K)) = \tau(KG(\mathbb{R})_+).$$

The first three terms of the exact sequence of (Sh2) are proven in [Sh3] and [Mik] along with finiteness of the cokernel of τ. The surjectivity of τ can be shown, using the result in [Mit] (see [MiS]). When $F_0 \neq \mathbb{Q}$, we need to replace $\mathbb{Q}^\times G(\mathbb{R})_+$ by the adelic closure $\overline{F^\times G(\mathbb{R})_+}$ in (Sh2) and $\mathrm{Coker}(\tau)$ is non-trivial (basically given by $\mathrm{Aut}(F_0)$), and the notation $\overline{\mathcal{G}}_+$ is often used in place of $\mathcal{G}_+$ in the literature we quoted.

We suppose the following condition:

(ord) p split in F (in Case A).
(spt) G in Case C is split over $\mathbb{Q}_p$.

Thus, identifying $G(\mathbb{Q}_p)$ with the symplectic or unitary similitude group of J_n, we have the parabolic subgroup $P_n \subset G$ given by $\left\{\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \middle| * \text{ is of size } n \times n\right\}$.

We fix a place $\mathfrak{P}$ of $\mathbb{Q}_{ab}$ over p. For an open compact subgroup $K = K_p \times K^{(p)}$ with $K_p = GL_g(\mathbb{Z}_p) \times \mathbb{Z}_p^\times$ in Case A and $GSp_g(\mathbb{Z}_p)$ in Case C, we know that $Sh_K^{(p)}$ (and hence V_K) has good reduction at $\mathfrak{P}$ and hence $\widetilde{V}_K = (V_K \mod \mathfrak{P})$ is irreducible (as described in Section 6). Recall that such an open compact subgroup is called *maximal at p*. Let v be a valuation of $\mathfrak{K}$ over $\mathfrak{P}$ such that the residue field of v restricted to $\mathbb{Q}(V_K)$ is the function field of $V_K \mod \mathfrak{P}$ for any open compact subgroup maximal at p. In other words, the field $\bigcup_{K:\text{max at } p} \overline{\mathbb{F}}_p(\widetilde{V}_K)$ for K maximal at p is the residue field of v restricted to $\mathfrak{K}^{(p)} = \bigcup_{K:\text{max at } p} \mathbb{Q}(V_K) \subset \mathfrak{K}$. The valuation $v|_{\mathfrak{K}^{(p)}}$ is unique and is discrete, because $\mathfrak{K}^{(p)}$ is the function field of a smooth model $\varprojlim_{K:\text{max at } p} V_{K/\mathcal{W}}$ over the discrete valuation ring $\mathcal{W}$. Since $\mathfrak{K}/\mathfrak{K}^{(p)}$ is algebraic, $v|_{\mathfrak{K}^{(p)}}$ extends to a valuation v on $\mathfrak{K}$ (which is not discrete).

Let

$$\mathcal{D} = \left\{\sigma \in \mathrm{Aut}(\mathfrak{K}) \middle| v \circ \sigma = v\right\}.$$

Thus $\mathcal{D}$ is the decomposition (or monodromy) group of v inside $\mathrm{Aut}(\mathfrak{K})$. Since $\mathfrak{K}/\mathfrak{K}^{(p)}$ is algebraic, $\mathcal{D}$ is unique up to conjugations in $\mathrm{Aut}(\mathfrak{K})$.

We now state our main theorem:

Theorem 10.1. *Let the notation and assumptions be as above. Suppose that we are in Case A or C with $F_0 = E = \mathbb{Q}$. In addition to* (ord) *and* (spt)*, we suppose that G in Case A is quasi split isomorphic to $U(n,n)$. Then the group $\mathcal{D}$ is the image (under τ) of a conjugate in $\mathcal{G}_+$ of*

$$\mathbf{P} = \left\{x \in \Big(P_n(\mathbb{Q}_p) \times G(\mathbb{A}^{(p)})\Big) \cap \mathcal{G}_+ \middle| \nu(x) \in \overline{\mathbb{Q}_p^\times \mathbb{Q}^\times \mathbb{R}_+^\times}\right\},$$

where $\mathbb{R}_+^\times$ is the identity connected component of $\mathbb{R}^\times$.

We will prove the theorem in Case A in the following section. See [PAF] 6.4.3 and Section 8.4 for the proof valid for more general Shimura varieties in Cases A and C.

Suppose that K is maximal at p and $K^{(p)}$ is sufficiently small. Let $S = Sh_K\left[\frac{1}{E}\right]$ for a lift E of the Hasse invariant H. Let S° be a geometrically connected component of $S_{/W}$. Since S is smooth over W, by the existence of the projective compactification of $Sh^{(p)}_{K/\mathcal{W}}$ as described at the end of 6.1.2, $S_1^\circ = S^\circ \otimes_W \mathbb{F}$ is geometrically connected. Let $T_{1,\infty}/S_1$ be the Igusa tower as in Sections 7 and 8. Since we only care $T_{1,\infty}$, we simply write T_∞ for $T_{1,\infty}$. Let L_n be the Levi subgroup of $G_1 \cap P_n$. Thus $L_n(\mathbb{Z}_p)$ is isomorphic to $GL_n(\mathbb{Z}_p)$ in Case C and to $GL_n(O_{\mathfrak{p}}) \times GL_n(O_{\overline{\mathfrak{p}}})$ in Case A, writing $O = O_F$ for the integer ring of F. By construction, $L_n(\mathbb{Z}_p)$ acts transitively on the set of geometrically connected components of T_∞ over S_1°. Thus $\mathrm{Gal}(T_\infty^\circ/S_1^\circ)$ for a geometrically connected component T_∞° of T_∞ is a subgroup of

$L_n(\mathbb{Z}_p)$. In Case A, by (ord), we have $\Sigma = \{\mathfrak{p}, \overline{\mathfrak{p}}\}$. We define a subgroup $\mathfrak{G}$ of $L_n(\mathbb{Z}_p)$ by

$$(10.1)\quad \mathfrak{G} = \begin{cases} \{(g_{\mathfrak{p}}, g_{\overline{\mathfrak{p}}}) \in GL_n(O_{\mathfrak{p}}) \times GL_n(O_{\overline{\mathfrak{p}}}) \big| \det(g_{\mathfrak{p}}) = \det(g_{\overline{\mathfrak{p}}})\} & \text{in Case A,} \\ GL_n(\mathbb{Z}_p) & \text{in Case C.} \end{cases}$$

Let $\underline{\omega}_\sigma$ be the σ–eigenspace of the action of O on $\underline{\omega}$, where $\sigma : O \hookrightarrow W$ is an embedding. For the moment we suppose that we are in the unitary case. Extending scalar to $\mathbb{C}$ (from W), the automorphic factor $j_\sigma(g, z)$ defining $\underline{\omega}_\sigma$ satisfies

$$\det(j_{c\sigma}(g, z)) = \det(g)^{-1} \det(j_\sigma(g, z)).$$

In Subsection 4.2, $j_{c\sigma}(g, z)$ (resp. $j_\sigma(g, z)$) is written as $h(g, z)$ (resp. $j(g, z)$). These sheaves are actually defined over W, and the difference (which is $\det(g)$) factors through the map $\tau|_{\mathbb{Q}_{ab}}$ (because basically $\det = \nu^n$ on $\mathcal{G}_+$). Thus the two sheaves $\det(\underline{\omega}_\sigma)$ and $\det(\underline{\omega}_{c\sigma})$ are equivalent over $W = W(\overline{\mathbb{F}}_p)$.

We take a geometrically connected component T°_∞ of $T_\infty \times_S S^\circ_1$ containing the infinity cusp. Since $\underline{\omega}_{/S^\circ_1} = P_1 \otimes_{\mathbb{Z}} \mathcal{O}_{S^\circ_1}$, the Galois group $\mathrm{Gal}(T^\circ_\infty/S^\circ_1)$ in Case A has to be contained in $\mathfrak{G}$. By (Sh5), we now conclude from the theorem that the Galois group $\mathrm{Gal}(T^\circ_\infty/S^\circ_1)$ of the Igusa tower contains $\mathfrak{G}$; so, they are equal.

Corollary 10.2. *Let the assumption be as in the theorem. The Galois group* $\mathrm{Gal}(T^\circ_\infty/S^\circ_1)$ *is equal to the above group* $\mathfrak{G}$. *In the symplectic case,* $T_{1,m} \times_S S^\circ_1$ *is geometrically irreducible. In the unitary quasi-split case, each geometrically irreducible components of the Igusa tower* T°_∞ *has Galois group over* S°_1 *isomorphic to* $\mathfrak{G}$ *as in* (10.1), *which is a proper subgroup of* $GL_n(O_{\mathfrak{p}}) \times GL_n(O_{\overline{\mathfrak{p}}}) \cong L_n(\mathbb{Z}_p)$ *and hence* $T_{1,m} \times_S S^\circ_1$ *for* $m \geq 1$ *is not irreducible.*

The irreducibility was first implicitly proven by Deuring [Du] and explicitly by Igusa [I] in the elliptic modular case and was generalized to the Hilbert modular case by Ribet [Ri] and to the Siegel modular case $GSp(2n)_{/\mathbb{Q}}$ by Faltings-Chai [DAV] V.7. There is a further generalization in [PAF] Section 8.4.

Corollary 10.3. *Let the assumption be as in the theorem. We assume that* $G = GSp(2n)_{/\mathbb{Q}}$. *Then a p–adic automorphic form (in* V^U*) on* G *is determined by its q–expansion at the infinity (or any other cusps unramified over* Sh_K*). If f and g in* $V^U_{/W}$ *have congruences* $a(\xi; f) \equiv a(\xi; g) \mod p^k$, *then* $f = g$ *in* $V^U_{k,\infty} = V^U/p^kV^U$.

10.1.2. *q-Expansion Principle for Quasi-split Unitary Groups.* Hereafter we assume that F is an imaginary quadratic field with $(p) = \mathfrak{p}\overline{\mathfrak{p}}$ and that G is given by the quasi-split group $GU(n, n)$. As stated in Corollary 10.2, the original Igusa tower is not irreducible; so, to get the q–expansion principle, we need to take a smaller tower.

Let us explain how to define a smaller (irreducible) tower. Let $\mathbb{X}$ be the universal abelian scheme over S°_1. Then we write P_m (resp. $\overline{P}_m$) for the étale quotient $\mathbb{X}[\mathfrak{p}^m]^{et}$ of $\mathbb{X}[p^m]$ (resp. $\mathbb{X}[\overline{\mathfrak{p}}^m]^{et}$). The original tower T_m/S°_1 represents the functor $\mathrm{Isom}((O/\mathfrak{p}^m)^n \times (O/\overline{\mathfrak{p}}^m)^n, P_m \times \overline{P}_m)$ taking an S°_1–scheme T to the set of O–linear isomorphisms $\psi : (O/\mathfrak{p}^m)^n \times (O/\overline{\mathfrak{p}}^m)^n \cong P_m \times \overline{P}_m$. By the shape of $\mathfrak{G}$ in (10.1), we find that

$$Q_{m/S^\circ_1} = \bigwedge^{2n}(P_m \oplus \overline{P}_m) \cong (\wedge^n P_m) \otimes \left(\wedge^n \overline{P}_m\right)$$

is constant over S°_1 because $\mathfrak{G}$ acts trivially on $Q_{m/\mathbb{F}}$. Thus fixing an isomorphism

$$\iota_m : (\mathbb{Z}/p^m\mathbb{Z}) \times S^\circ_1 \cong Q_m \ (m = 1, 2, \dots)$$

over $\mathbb{F}$ so that ι_{m+1} induces ι_m, the irreducible component $T_m^\circ(\iota_m)/S_1^\circ$ (corresponding to ι_m) represents a subfunctor

$$T \mapsto \left\{\psi \in \mathrm{Isom}_T((O/\mathfrak{p}^m)^n \times (O/\overline{\mathfrak{p}}^m)^n, P_m \times \overline{P}_m) \middle| \wedge^{2n} \psi = \iota\right\}$$

of $\mathrm{Isom}((O/\mathfrak{p}^m)^n \times (O/\overline{\mathfrak{p}}^m)^n, P_m \times \overline{P}_m)$. Considering the tower $T^\circ_{\alpha,m}(\iota_m)$ over W_α, we can think of the ring of global sections $V_{\alpha,m}(\iota_m) = H^0(T^\circ_{\alpha,m}(\iota_m)_{/W_\alpha}, \mathcal{O}_{T^\circ_{\alpha,m}})$, and define

$$V^U(\iota_\infty) = \varprojlim{}_\alpha V_{\infty,\alpha}(\iota_\infty)^U.$$

This space of p–adic modular form is a subspace of V^U we considered before.

The formal scheme $T_{\infty,m} = \varprojlim{}_\alpha T_{\alpha,m}$ is étale over the formal completion S°_∞ and extends to a unique toroidal compactification $\overline{T}_{m,\infty}$ étale over the toroidal compactification $\overline{S}_\infty$ of S°_∞. In other words, taking the semi-abelian scheme $\mathbf{G}_{/\overline{S}_\infty}$ extending the universal abelian scheme $\mathbb{X}_{/S^\circ_\infty}$, we have $\overline{T}_{\infty,m} = \mathrm{Isom}(O/p^mO, \widehat{\mathbf{G}}[p^m]^\circ)$, where $\widehat{\mathbf{G}}[p^m]^\circ$ is the Cartier dual of the connected component $\mathbf{G}[p^m]^\circ$ (which naturally extends $\mathbb{X}[p^m]^{et} = P_m \oplus \overline{P}_m$ by the duality). Since $\overline{S}_\infty$ contains the infinity cusp, we have a well chosen infinity cusp of $\overline{T}_{\infty,m}$ regarding it as a formal subscheme of a suitable level p^m moduli scheme. Thus we can talk about the infinity cusp of the irreducible component of $\overline{T}_{\infty,\infty}$ containing $T^\circ_{\infty,\infty}$. Then we can state

Corollary 10.4. *Let the assumption be as in the theorem. Suppose that G is given by $GU(n,n)$ for an imaginary quadratic field F where p splits. Then a p–adic automorphic form (in $V^U(\iota_\infty)$) on G is determined by its q–expansion at the infinity (or any other cusps unramified over Sh_K). If f and g in $V^U(\iota_\infty)_{/W}$ have congruences $a(\xi; f) \equiv a(\xi; g) \mod p^k$, then $f = g$ in $V^U_{k,\infty}(\iota_\infty) = V^U(\iota_\infty)/p^kV^U(\iota_\infty)$.*

10.2. **Quasi-split Unitary Igusa Towers.** We shall give a sketch of a proof of the theorem in the quasi-split unitary case of even dimension at the end of this lecture. The proof in the split symplectic case is basically the same and actually easier (see [PAF] 6.4.3).

10.2.1. *Preliminaries.* First we describe necessary ingredients of the proof. Recall that $J_n = \left(\begin{smallmatrix} 0 & -1_n \\ 1_n & 0 \end{smallmatrix}\right)$. Then $G = GU(n,n)$ can be identified with the following group functor:

$$G(A) = \left\{\alpha \in GL_{2n}(A \otimes_{\mathbb{Q}} F) \middle| \alpha J_n {}^t\alpha^c = \nu(\alpha) J_n,\ \nu(\alpha) \in A^\times\right\}.$$

Here c is the non-trivial automorphism of $F/\mathbb{Q}$ extended to $A \otimes_{\mathbb{Q}} F$ for each $\mathbb{Q}$–algebra A. We consider the F–vector space V of dimension $2n$ and the alternating pairing $\langle x, y\rangle = xJ_n {}^t y^c$. Then $\langle bx, y\rangle = \langle x, b^c y\rangle$; so, the positive involution $*$ on $B = F$ is given by c. Then $C = \mathrm{End}_F V = M_{2n}(F)^{opp}$; in other words $M_{2n}(F)$ acts on V by the right multiplication. Let $L \subset V$ be a O–lattice with $L_p \cong \mathrm{Hom}_{\mathbb{Z}_p}(L_p, \mathbb{Z}_p)$ under $\langle\ ,\ \rangle$. We take $h : \mathbb{C} \hookrightarrow C_\infty = C \otimes_{\mathbb{Q}} \mathbb{R}$ to be $h(i) = -J_n \otimes 1$. In this case, the representation of O on V_1 is just a multiple of the regular representation of O; so, its (p–adic) isomorphism class is unique under (ord). We consider the following moduli problem for an integer $N > 0$ prime to p: To each W–scheme S, we associate the set of isomorphism classes: $\left[(X, i, \lambda, \eta^{(p)})_{/S} \middle| (\det)\right]$ such that

- $i : O \hookrightarrow \mathrm{End}(X_{/S})$ taking 1 to id_X;
- $\overline{\eta}^{(p)}$ is made of a pair of $\eta_1^{(p)} : T^{(p)}(X_s) \cong \widehat{L}^{(p)}$ modulo $\widehat{\Gamma}^{(p\infty)}$ as $\widehat{O}^{(p)}$–modules for any geometric point $s \in S$ and $\eta_N : L/NL \cong X[N]$, where for $\widehat{L} = L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$, $\widehat{\Gamma} = \widehat{\Gamma}_L = \{x \in G(\mathbb{A}^{(\infty)}) | x\widehat{L} = \widehat{L}\}$.

- $\lambda : X \to \widehat{X}$ is a polarization which induces $\langle\ ,\ \rangle$ on $\widehat{L}^{(p)}$ under η_1 and of degree prime to p.

The open compact subgroup of $G(\mathbb{A}^{(\infty)})$ corresponding to this moduli problem is:

$$\widehat{\Gamma}(N) = \widehat{\Gamma}_L(N) = \left\{x \in \widehat{\Gamma}_L \middle| (x-1)\widehat{L} = N\widehat{L}\right\}.$$

Suppose that N is sufficiently large so that the moduli problem has a solution, that is, we have a fine moduli scheme M.

We want to know the exact objects the generic fiber M_η classifies. For a given quadruple $(X, i, \lambda, \overline{\eta}^{(p)})$, if it has a generic fiber X_η, $T(X_\eta)$ can be embedded (as skew hermitian O–modules) into $V \otimes_{\mathbb{Q}} \mathbb{A}^{(\infty)}$ so that the embedding coincides with η at ℓ for each prime $\ell \nmid p$. Thus we know the isomorphism class of the localizations of $H_1(X_\eta, \mathbb{Z})$ as skew hermitian O–lattice in V outside p. Let L' be the image of $H_1(X_\eta, \mathbb{Z})$ in V.

For any given O–lattice $\Lambda \subset V$, we define

- $\mu(\Lambda)$ to be the ideal of $\mathbb{Z}$ generated by $\langle x, x\rangle$ for all $x \in \Lambda$;
- $\mu_0(\Lambda)$ to be the O_F–ideal generated by $\langle x, y\rangle$ for all $x, y \in \Lambda$.

If Λ is maximal among lattices having the same μ, we call it maximal. By the self duality at p of L', $\mu_0(L'_p) = O_{F,p}$. It is easy to see that $\mu(\Lambda)O \subset \mu_0(\Lambda) \subset \mathfrak{d}^{-1}\mu(\Lambda)$ for the relative different $\mathfrak{d}$ of $F/\mathbb{Q}$. If $L''_p \supset L'_p$ with $\mu(L''_p) = \mu(L'_p)$, then by (ord), we find

$$\mu(L'_p)O_{F,p} \subset \mu_0(L'_p) \subset \mu_0(L''_p) \subset \mu(L'_p)O_{F,p}.$$

Thus $L'_p = L''_p$ and hence L'_p is maximal with $\mu(L'_p) = \mathbb{Z}_p$. By the same argument, L_p is maximal with $\mu(L_p) = \mathbb{Z}_p$. Then by a lemma of Shimura proven in the 1960's ([EPE] 5.9 or [Ko] Corollary 7.3), there exists $x_p \in G_1(\mathbb{Q}_p)$ so that $L'_p = x_p L_p$. By self duality of L_p and L'_p, we see that $x_p^2 \in \widehat{\Gamma}_p$ and hence $x_p \in \widehat{\Gamma}_p$. Thus we find that $L'_p = L_p$ as skew hermitian $O_{F,p}$–modules.

This shows that there are only finitely many isomorphism classes of hermitian O–lattices in the genus class of L (approximation theorem). Thus the generic (geometrically) irreducible component of M classifies $(X, i, \lambda, \eta^{(p)})_{/S}$ satisfying the following conditions:

- $i : O \hookrightarrow \mathrm{End}(X_{/S})$ taking 1 to id_X;
- $\overline{\eta}^{(p)}$ is made of a pair of $\eta_1 : H_1(X_s, \mathbb{Z}) \cong L$ up to isomorphisms as skew hermitian O–modules for any geometric point $s \in S$ and $\eta_N : L/NL \cong X[N]$.
- $\lambda : X \to \widehat{X}$ is a polarization which induces $\langle\ ,\ \rangle$ on L under η_1.

This type of moduli problem has been studied over the reflex field E by Shimura (see, for example, [ACM] Section 26, [AAF] Chapters I and II and [Sh2]). In the formulation of [AAF] Section 4, the above conditions are summarized into a PEL type: $\Omega = (V, \Psi, L, J_n, t_1, \ldots, t_{2n})$, where t_j are generators of L/NL over O and Ψ is the isomorphism class over $\mathbb{Q}$ of the representation of F on $V_1 \subset V \otimes_{\mathbb{Q}} \mathbb{C}$ on which $h(\sqrt{-1})$ acts by the multiplication by $\sqrt{-1}$. A quadruple $(X, i, \lambda, \overline{\eta}^{(p)})$ over $\mathbb{C}$ is called of type Ω if we have a real analytic isomorphism $V_\infty \xrightarrow{\xi} X(\mathbb{C})$ with $\mathrm{Ker}(\xi) = L$ such that

- ξ induces an identification of $V_1 \cong Lie(X)$ as complex vector space on which F acts by Ψ,
- ξ induces the polarization $\langle\ ,\ \rangle$ on V (up to positive rational multiple). This means that $\langle x, y\rangle = \mathrm{Tr}_{F/\mathbb{Q}}(xJ_n{}^t y^c)$,

- $\xi(at) = i(a)\xi(t)$ for $t \in V$ and $a \in O$,
- $\overline{\eta} : (O/NO)^{2n} \to X[N]$ given by $\overline{\eta}(a_1, \dots, a_{2n}) = \sum_j a_j \xi(t_j)$.

The condition on Ψ is equivalent to (det) over $E = \mathbb{Q}$. We can think this moduli problem over E for an arbitrary N and get a tower of moduli space M_N. We now take L to be O^{2n}. Then each geometrically irreducible component of M_N is defined over $\mathbb{Q}[\zeta_N]$. The component $V_{N/\mathbb{Q}[\mu_N]} = V_{\widehat{\Gamma}_L(N)/\mathbb{Q}[\mu_N]}$ classifies quadruples (X, i, λ, η) over $\mathbb{Q}[\mu_N]$ under an extra condition that $e_N(t_i, \widehat{t}_j) = \zeta_N^{N\langle t_i, \widehat{t}_j\rangle}$ for the duality pairing $e_N : X[N] \times \widehat{X}[N] \to \mu_N$ and the dual base $\widehat{t}_j$ of t_j under $\langle\ ,\ \rangle$ localized at N. We then consider the union of the tower of fields $\mathbb{Q}[\mu_N](V_N) = \mathbb{Q}(V_N)$, and write the field as $\mathfrak{K}$. Naturally the group $x \in G(\mathbb{A}^{(\infty)})$ acts on $M = \varprojlim_N M_N$ by changing L to xL and t_j to xt_j, and if $x \in G_1(\mathbb{A}^{(\infty)})$, x keeps $\langle\ ,\ \rangle$. Let

$$H(A) = \left\{x \in G(A) \middle| \det(x) = \nu(x)^n, \nu(x) \in A^\times\right\}.$$

Then we have the following explicit description ([AAF] 8.8):

$$\mathcal{G}_+ = H(\mathbb{A}^{(\infty)})G(\mathbb{Q})_+G(\mathbb{R})_+ = (\widehat{\Gamma}_L(N) \cap H(\mathbb{A}^{(\infty)}))\iota(\widehat{\mathbb{Z}}^\times)G(\mathbb{Q})_+G(\mathbb{R})_+,$$

where $G(\mathbb{R})_+$ is the identity connected component of $G(\mathbb{R})$, $G(\mathbb{Q})_+ = G(\mathbb{Q}) \cap G(\mathbb{R})_+$, and $\iota(s) = \mathrm{diag}[1_n, s^{-1}1_n] \in H(\mathbb{A})$ for $s \in \mathbb{A}^\times$. To see this, we write $\psi = \det/\nu^n : G(\mathbb{A}) \to F_{\mathbb{A}}^\times$. Then $H(\mathbb{A}) = \mathrm{Ker}(\psi)$. By (Sh1), for a given $x \in \mathcal{G}_+$, we have $\psi(x) = \zeta u$ for $u \in \mathbb{R}_+^\times$ and $\zeta = \xi^{1-c}$ for $\xi \in M^\times$ ($\Leftrightarrow \zeta\zeta^c = 1$). Taking $\alpha \in G(\mathbb{Q})_+$ with $\psi(\alpha) = \zeta$, we find that $x\alpha^{-1} \in H(\mathbb{A})$, which shows the first equality of the above expression. For the second equality, we refer the reader to [AAF] 8.8. Since F is imaginary quadratic, it has only finitely many units; so, $(F^\times)^{1-c}\mathbb{R}_+^\times$ is a closed subgroup of $F_{\mathbb{A}}^\times$, and hence $\mathcal{G}_+$ is a closed subgroup of $G(\mathbb{A})$. This is the reason why we do not need to take closure of $(F^\times)^{1-c}\mathbb{R}_+^\times$ in $F_{\mathbb{A}}^\times$ in the definition of $\mathcal{G}_+$ in (Sh1).

For $p \nmid N$, we have

$$\mathrm{Gal}(\mathbb{Q}[\mu_{Np^m}, V_{Np^m}]/\mathbb{Q}[\mu_{Np^m}, V_N]) \cong \Gamma(Np^m)/\Gamma(N) \cong SG(\mathbb{Z}/p^m\mathbb{Z})$$

for N sufficiently large by (Sh3,5) (and the strong approximation theorem). Here $SG(A) = SL_{2n}(A \otimes_{\mathbb{Z}} O) \cap G_1(A)$ and

$$\Gamma(N) = \left\{\gamma \in SG(\mathbb{Q}) \middle| \gamma L = L \text{ and } (\gamma - 1)L_\ell \subset NL_\ell\ \forall \ell | N\right\}.$$

The moduli variety M_{Np^m} classifies quadruple $(X, \overline{\lambda}, \overline{\eta}, \phi : (O/p^mO)^{2n} \hookrightarrow X[p^m])$ for a level $\widehat{\Gamma}(N)$–structure $\overline{\eta}$. Thus taking the universal abelian variety $\mathbb{X}$ over M_N, we have, for each $Sh_{\widehat{\Gamma}_L(N)}$–scheme T

$$M_{Np^m}(T) = \left\{\phi : (O/p^mO)^{2n}_{/T} \cong \mathbb{X}[p^m]_{/T}\right\}.$$

The action of $g \in G(\mathbb{Z}_p)$ on M_{Np^m} is induced by the action on the level structure $\phi \mapsto \phi \circ g$. Thus taking $2n$–th exterior power $\bigwedge_O^{2n} \mathbb{X}[p^m]$, we find that $g \in G(\mathbb{Z}_p)$ acts by $\det(g) \in O_p^\times$. Then by the description of the stabilizer $\mathcal{G}_+$ of a geometrically irreducible component Sh, we find that the action is trivial on $\bigwedge_O^{2n} \mathbb{X}[p^m]$ if $g \in \mathcal{G}_+ \cap SG(\mathbb{A}^{(\infty)})$. Thus the group scheme

$$\bigwedge_O^{2n} \mathbb{X}[p^m] = \left(\wedge^{2n}\mathbb{X}[\mathfrak{p}^m]\right) \oplus \left(\wedge^{2n}\mathbb{X}[\overline{\mathfrak{p}}^m]\right)$$

is constant on each geometrically irreducible component V_N of M_N. In other words, $\wedge^{2n}\mathbb{X}[\mathfrak{p}^m]_{/V_N}$ is a base-change of $\mu_{p^m}^n \times (\mathbb{Z}/p^m\mathbb{Z})^n$ from $\overline{\mathbb{Q}}$ to $V_{N/\overline{\mathbb{Q}}}$.

Now we look into the Kottwitz model $Sh^{(p)}_{\widehat{\Gamma}(N)/\mathcal{W}}$ for N prime to p. In the rest of the paper, we always suppose that N is prime to p. Since each geometrically irreducible component of $Sh^{(p)}_{\widehat{\Gamma}(N)} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q} = M_{N/\mathbb{Q}}$ is defined over $\mathbb{Q}[\mu_N]$ in the sense of Weil, it remains irreducible after taking spacial fiber modulo $\mathfrak{m}_{\mathcal{W}}$ (Zariski's connectedness theorem combined with the existence of a smooth projective compactification). Thus we can talk about geometrically irreducible component $V^{(p)}_{N/\mathcal{W}}$ of $Sh^{(p)}_{\widehat{\Gamma}(N)/\mathcal{W}}$ whose generic fiber is $V_{N/\mathbb{Q}}$ and whose special fiber is the special fiber of the schematic closure of V_N in $Sh^{(p)}_{\widehat{\Gamma}(N)/\mathcal{W}}$.

Since the universal abelian scheme $\mathbb{X}_{/V_N}$ extends to the universal abelian scheme of the Kottwitz model $V^{(p)}_{N/\mathcal{W}}$,

$$Q_m = \bigwedge^{2n} \mathbb{X}[p^m]^{et} = \left(\wedge^n \mathbb{X}[\mathfrak{p}^m]^{et}\right) \otimes \left(\wedge^n \mathbb{X}[\overline{\mathfrak{p}}^m]^{et}\right)$$

is constant over $S^\circ_{1/\mathbb{F}}$. Since the Igusa tower T_m over $S^\circ_{1/\mathbb{F}}$ is given by

$$\mathrm{Isom}((O/p^mO)^n_{/V^{(p)}}, \mathbb{X}[p^m]^{et}),$$

T_m cannot be irreducible, and each irreducible component of $T_{m/\mathbb{F}}$ is contained in $T_m(\iota_m)$ for an isomorphism $\iota_m : \mathbb{Z}/p^m\mathbb{Z} \cong \bigwedge^{2n} \mathbb{X}[p^n]^{et} = Q_m$. Thus, for a geometrically irreducible component T°_∞ of T_∞, the Galois group $\mathrm{Gal}(T^\circ_\infty/S^\circ_1)$ is a subgroup of $\mathfrak{G}$ in (10.1). We reached the same conclusion before stating Corollary 10.2 by looking into vector bundles $\underline{\omega}_\sigma$. In any case, we need to show that

$$\mathrm{Gal}(T^\circ_\infty/S^\circ_1) = \mathfrak{G}$$

to prove Corollary 10.2.

Since p splits in F, we have $SG(\mathbb{Z}/p^m\mathbb{Z}) \cong SL_{2n}(\mathbb{Z}/p^m\mathbb{Z})$. Since we have a smooth model of M_N over $\mathcal{W}$, we take the valuation v of $\mathfrak{K}_N = \mathbb{Q}[\mu_N](V_N)$ corresponding to the generic point of $V_N \mod \mathfrak{P} = V_N \otimes_{\mathcal{W}} \mathbb{F}$ containing the infinity cusp. Since $M^*_N = Proj(\mathcal{G}_{\widehat{\Gamma}(N)})$ under the notation in Section 6, we can write the Satake compactification of $V_{N/\mathcal{W}}$ as $Proj(R)$ for $R = \bigoplus_{j \geq 0} R_j$ with $R_j = H^0(V_N, \det(\underline{\omega})^j_{/\mathcal{W}})$. By q–expansion at ∞, we can embed R into $\mathcal{W}[[q^\xi]]_{\xi \in M_n(F)_+}$, where

$$M_n(F)_+ = \{{}^tx^c = x \in M_n(F) |\ x \text{ is totally non-negative}\}.$$

and the symbol $A[[q^\xi]]_{\xi \in M_n(F)_+}$ indicates the completion by the augmentation ideal of the monoid algebra of the additive semi-group $M_n(F)_+$ with q^ξ indicating the element represented by ξ. Each $f \in R_j$ has q–expansion $\sum_\xi a(\xi; f)q^\xi \in \mathcal{W}[[q^\xi]]_{\xi \in M_n(F)_+}$. Replacing q^ξ by $\exp(2\pi i \mathrm{Tr}(\xi z))$ $z \in \mathfrak{Z}$, we get the Fourier expansion at ∞ of f (regarding $\mathcal{W} \hookrightarrow \mathbb{C}$).

We take a valuation v of $\mathfrak{K}$ which is induced by a valuation v on R given by

$$v(\sum_\xi a_\xi q^\xi) = \mathrm{Inf}_\xi\, ord_p(a_\xi),$$

where ord_p is the discrete valuation of $\mathcal{W}$ with $ord_p(p) = 1$. Here we used the existence of the smooth toroidal compactification of V_N $(p \nmid N)$ worked out by

Fujiwara ([F]) and the q–expansion principle for $f \in R_j$ on $\Gamma(N)$ with $p \nmid N$ to assure that the residue field of v restricted to $\mathbb{Q}(V_N)$ for $p \nmid N$ is the function field of $V_N \mod \mathfrak{P}$. Since the Satake compactification of $M_{Np^m/\mathbb{Q}}$ is again given by $Proj(\mathcal{R})_{/\mathbb{Q}[\mu_{Np^m}]}$ for $\mathcal{R} = \bigoplus_{j\geq 0} \mathcal{R}_j$ with $\mathcal{R}_j = H^0(M_{Np^m/\mathbb{Q}}, \det(\underline{\omega})^j_{/\mathbb{Q}})$, we can extend the valuation v to $\mathcal{R}$ by the same formula in terms of the unique extention of ord_p to $\mathcal{W}[\mu_{p^m}]$. This extension induces a valuation on $\mathbb{Q}(V_{Np^m}) = \mathfrak{K}_{Np^m}$ and on $\mathfrak{K} = \bigcup_m \bigcup_N \mathfrak{K}_{Np^m}$. We are going to show that the decomposition group D_v of v in $\mathrm{Aut}(\mathfrak{K})$ contains $P_n(\mathbb{A}) \cap \mathcal{G}_+$ and $G_1(\mathbb{A}^{(p\infty)})$.

10.2.2. *Proof of the irreducibility theorem.* Let $L \subset V$ be an O–lattice satisfying (L1-2) of Section 6 and recall

$$\widehat{\Gamma}_L = \left\{x \in G(\mathbb{A}^{(\infty)}) \middle| x\widehat{L} = \widehat{L}\right\}$$
$$\widehat{\Gamma}_L(N) = \left\{x \in \widehat{\Gamma}_L \middle| (x-1)\widehat{L} = N\widehat{L}\right\},$$

where $\widehat{L} = L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$. Let $\mathbb{X}_N$ be the universal abelian scheme over $V_N \subset Sh_{\widehat{\Gamma}(N)}$ for N sufficiently large. We have the following specification of the action of $\mathcal{G}_+$ (see [AAF] Theorem 8.10):

1. $x \in \mathcal{G}_+$ acts on the maximal abelian extension $\mathbb{Q}_{ab}$ of $\mathbb{Q}$ by the image of $\nu(x)$ under the reciprocity map of class field theory.
2. If $\gamma \in G(\mathbb{Q})_+ = G(\mathbb{Q}) \cap G(\mathbb{R})_+$, γ regarded as an element of $G(\mathbb{Q})_+ \subset \mathcal{G}_+$ satisfies $f^{\tau(\gamma)} = f \circ \gamma$.
3. If $x = \left(\begin{smallmatrix} 1 & 0 \\ 0 & s1_n \end{smallmatrix}\right)$ with $s \in \widehat{\mathbb{Z}}^\times$, then $a(\xi; f^{\tau(x)}) = \sigma(a(\xi; f))$ for all $\xi \in M_n(F)_+$, where σ is the automorphism of $\mathbb{Q}_{ab}$ corresponding to s^{-1} by class field theory. Strictly speaking, writing $f = \frac{h}{g}$ for $g, h \in \mathcal{R}_j$, we have $f^{\tau(x)} = \frac{h^\sigma}{g^\sigma}$ with $a(\xi; x^\sigma) = \sigma(a(\xi; x))$.
4. The natural action of $\widehat{\Gamma}_L(N)^{(p)}$ on $V_{tN} = \mathrm{Isom}(L/tL_{/V_N}, \mathbb{X}_N[t]_{/V_N})$ induces the action of $\widehat{\Gamma}_L(N)^{(p)} \cap \mathcal{G}_+$ on $E(V_{\widehat{\Gamma}_L(Nt)})$.

By (1), $\mathcal{D} = D_v$ is contained in the image (under τ) of

$$\{x \in \mathcal{G}_+ | \nu(x) \in \overline{\mathbb{Q}_p^\times \mathbb{Q}^\times \mathbb{R}_+^\times}\}.$$

Let U_n be the unipotent radical of P_n. By (2), we have $\tau(P_n(\mathbb{Q})) \subset \mathcal{D}$, since $\left(\begin{smallmatrix} a & c \\ 0 & d \end{smallmatrix}\right) \in P_n(\mathbb{Q})$ acts on q–expansion just by $q^\xi \mapsto \exp(2\pi i \mathrm{Tr}(\xi c d^{-1})) q^{d^{-1}\xi a}$. Then by density of $\mathbb{Q}$ in $\mathbb{A}$, we conclude that

$$\tau(U_n(\mathbb{A})) \subset \mathcal{D}. \tag{U}$$

By the strong approximation theorem, cusps of $\Gamma_K = K \cap SG(\mathbb{Q})$ are in bijection to $K\backslash SG(\mathbb{A}^{(\infty)})/P_n(\mathbb{Q})$. Choosing K to be maximal, by Iwasawa decomposition: $SG(\mathbb{A}^{(\infty)}) \subset KP_n(\mathbb{A}^{(\infty)})$, the above set of cusps is embedded into $K \cap P_n(\mathbb{A}^{(\infty)})\backslash P_n(\mathbb{A}^{(\infty)})/P_n(\mathbb{Q})$. We have $G(\mathbb{A}^{(\infty)}) = \bigsqcup_{b\in B} G(\mathbb{Q})bK$ for a finite set $B \subset P_n(\mathbb{A}^{(p\infty)})$. From this, $K\backslash \left(P_n(\mathbb{A}^{(\infty)}) \cap SG(\mathbb{A}^{(p\infty)})\right)/P_n(\mathbb{Q})$ is a finite set indexed by ideal classes. Thus the action of $b \in P_n(\mathbb{A}^{(p)})$ brings the q–expansion of $f \in \mathcal{R}_j$ to its q–expansion at other cusps. If K_p is maximal, V_K is smooth over the valuation ring of $\mathfrak{P}$, and hence the action preserves v restricted to $\beta(H^0(V_K, \underline{\omega}^\kappa))$ $(p \nmid N)$, where β is the embedding into the space V^U of p–adic modular functions (with respect to $\widehat{\Gamma}(N)$) we studied in Section 7. Note that the integral closure of the graded algebra R $(V_K^* = Proj(R))$ in $\mathfrak{K}^{U(L)}$ for $U(L) = U_n(\mathbb{Z}_p) \times \widehat{\Gamma}_L(N)^{(p)}$ is contained in V^U by definition. Let $\underline{\omega}_\kappa \subset \underline{\omega}^\kappa$ be the sheaf of cusp forms. Since $D_{cusp} = \left(\bigoplus_\kappa H^0(V_K, \underline{\omega}_{\kappa/\mathbb{Q}})\right) \cap V^U_{cusp}$ is p–adically dense in V^U_{cusp} (the density theorem in Section 7), we conclude that the action of $P_n(\mathbb{A}^{(p\infty)}) \cap \mathbf{P}$ preserves v restricted to $\mathfrak{K}^{(p)} = \bigcup_{p\nmid N} \mathbb{Q}_{ab}(V_N)$ and also $\mathfrak{K}^{U_n(\mathbb{Z}_p)} = \bigcup_L \mathfrak{K}^{U(L)} \supset \mathfrak{K}^{(p)}$ for the unipotent radical U_n of P_n, because $\mathfrak{K}^{U(L)}$ is generated by ratios $\frac{f}{g}$ of cusp forms f and g in D_{cusp}. Here L runs over all lattices satisfying (L1-2) in Lecture 6. Thus $\mathcal{D} \cdot \tau(U_n(\mathbb{Z}_p))/\tau(U_n(\mathbb{Z}_p)) \subset \mathrm{Aut}(\mathfrak{K}^{U_n(\mathbb{Z}_p)})$ contains the image of $\big(P_n(\mathbb{A}^{(p\infty)}) \cap \mathbf{P}\big) \times U_n(\mathbb{Z}_p)$ in $\mathrm{Aut}(\mathfrak{K}^{U_n(\mathbb{Z}_p)})$.

Then by (U), we conclude $\mathcal{D}$ contains the image under τ of

$$\mathbf{P} \cap \left(P_n(\mathbb{A}^{(p\infty)}) \times U_n(\mathbb{Z}_p)\right).$$

By the same argument applied to $K^{(p)}$, we find that $\tau(\mathbf{P} \cap K^{(p)}) \subset \mathcal{D}$. Note that

$$\mathbf{P} \cap \left(\bigcup_L P_n(\mathbb{A}^{(p\infty)}) P_n(\mathbb{Q}) \widehat{\Gamma}_L(N)^{(p)}\right)$$

is dense in $\mathbf{P}$ and hence $\mathcal{D} \supset \tau(\mathbf{P})$ (see the proof of Theorem 6.27 in [PAF] for a different argument giving this inclusion).

Since $g = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in \mathbf{P} \cap G(\mathbb{Z}_p)$ acts on T_m through its diagonal entries (a, d), we find that $\mathrm{Gal}(T^\circ_\infty/S^\circ_1)$ has to contain $\mathfrak{G}$ because the matrices (a, d) fills $\mathfrak{G}$. This proves Corollary 10.2.

The reverse inclusion: $\tau(\mathbf{P}) \supset \mathcal{D}$ follows from the existence of the exact sequence:

$$0 \to \mathbb{X}[p^\infty]^\circ_{/S^\circ_1} \to \mathbb{X}[p^\infty]_{/S^\circ_1} \to \mathbb{X}[p^\infty]^{et}_{/S^\circ_1} \to 0.$$

See [PAF] Theorem 6.28 and 8.4.3 for more details of how to prove the reverse inclusion from the above exact sequence. This finishes the proof of Theorem 10.1 in Case A.

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Department of Mathematics, UCLA, Los Angeles, CA 90095-1555, U.S.A.
E-mail address: hida@math.ucla.edu