TAYLOR-WILES PATCHING LEMMAS À LA KISIN

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Contents

We present an exposition of Kisin's patching argument which simplified and generalized earlier argument by Wiles, Taylor and Fujiwara. Main references are Kisin's two papers [K1] and [K2].

1. Patching lemmas

Let $K/\mathbb{Q}_p$ be a finite extension with p-adic integer ring W.

Lemma 1.1. *If $\varphi : R \to T$ is a surjective algebra homomorphism of noetherian integral domains and $\dim T = \dim R$, then φ is an isomorphism.*

Proof. Since $\mathrm{Spec}(T)$ is a closed subscheme of an integral irreducible scheme $\mathrm{Spec}(R)$; so, if it is proper closed subscheme, $\dim T < \dim R$, a contradiction; so, $\mathrm{Spec}(R) = \mathrm{Spec}(T)$, which implies $R \cong T$. □

There is a version of the above lemma:

Lemma 1.2. *Let B be a complete local noetherian domain and T be a complete local algebra. If $\varphi : B[[x_1, \ldots, x_m]] \to T$ is a surjective algebra homomorphism and $\dim T = \dim B + m$, then φ is an isomorphism.*

Proof. By assumption, T is noetherian; so, $\mathrm{Spec}(T)^{red}$ is a finite union of integral irreducible components. Let $\mathrm{Spec}(T_0) \subset \mathrm{Spec}(T)^{red}$ be an irreducible component of maximal dimension; so, $\dim T = \dim T_0$. Thus we have the surjective algebra homomorphism $T \twoheadrightarrow T_0$. Composing this with $B[[x_1, \ldots, x_m]] \twoheadrightarrow T$, we have a surjective algebra homomorphism: $B[[x_1, \ldots, x_m]] \twoheadrightarrow T_0$. Applying the above lemma to this morphism, we get $B[[x_1, \ldots, x_m]] \cong T_0$, and hence $B[[x_1, \ldots, x_m]] \cong T$. □

Here is a simpler version by Kisin ([K1] Proposition 3.4.1) of the Taylor-Wiles patching theorem (see [HMI] Theorem 3.23):

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Proposition 1.3 (Taylor-Wiles, Fujiwara, Kisin). *Let B be a complete local W-domain of dimension $d+1$, and let $\varphi : R \to T$ be a surjective homomorphism of B-algebras. Assume to have two non-negative integers h and j such that for each positive integer n, there is a commutative diagram of W-algebras:*

$$\begin{array}{lccccc} & \widetilde{B} := B[[x_1,\dots,x_{h+j-d}]] & \twoheadrightarrow & R_n & \xrightarrow{\varphi_n} & T_n \\ (1.1) & & \nearrow & \downarrow & & \downarrow \\ & \Lambda := W[[y_1,\dots,y_h,t_1,\dots,t_j]] & & R & \xrightarrow{\varphi} & T, \end{array}$$

which satisfies

(0) *The square consists of homomorphisms of B-algebras;*
(1) *The horizontal and vertical morphisms are surjective;*
(2) *$(y_1,\dots,y_h)R_n = \mathrm{Ker}(R_n \to R)$ and $(y_1,\dots,y_h)T_n = \mathrm{Ker}(T_n \to T)$;*
(3) *If $\mathfrak{b}_n = \mathrm{Ker}(W[[y_1,\dots,y_h,t_1,\dots,t_j]] \to T_n)$, then*
$$\mathfrak{b}_n \subset ((1+y_1)^{p^n}-1,\dots,(1+y_h)^{p^n}-1),$$
and T_n is finite free over $\Lambda/\mathfrak{b}_n$ (so, in particular, T is finite free over $\Lambda_f := W[[t_1,\dots,t_j]]$);
(4) *R_n is a quotient of a power series ring $\widetilde{B}$ (the bounded tangential dimension).*

Then the morphism φ is an isomorphism, as are the maps φ_n for all $n \geq 1$, and B is a Cohen-Macaulay ring.

A noetherian local ring A is Cohen-Macaulay if $\operatorname{depth} A = \dim A$ (see [CRT] Section 17). In an application, T is the minimal level Hecke algebra for Hilbert modular forms over a totally real field F with integer ring O, R is the minimal level universal deformation ring (for deformations ramified at the minimal finite set S including p and ∞). Then for a set $Q_n = \{\mathfrak{q}_1,\dots,\mathfrak{q}_h\}$ of h primes $\mathfrak{q}_j$ outside S with $N(\mathfrak{q}_j) \equiv 1 \bmod p^n$, T_n is the Hecke algebra of level $S \sqcup Q_n$ and R_n is the universal deformation ring unramified outside $S \sqcup Q_n$. Let Δ_n be the p-Sylow subgroup of $\prod_j (O/\mathfrak{q}_j)^\times$. By the diamond operator (the central action), T_n is an algebra over $W[\Delta_n]$. Write $\Delta_n = \prod C_j$ for the cyclic Sylow subgroup C_j of $(O/\mathfrak{q}_j)^\times$ with generator δ_j. Then the order p^{m_j} of C_j is equal to or greater than p^n; so, we can identify $W[\Delta_n]$ with $W[[y_1,\dots,y_h]]/((1+y_1)^{p^{m_1}}-1,\dots,(1+y_h)^{p^{m_h}}-1)$ sending the generator δ_j to $(1+y_j)$; so,

$$\mathfrak{b}_n = ((1+y_1)^{p^{m_1}}-1,\dots,(1+y_h)^{p^{m_h}}-1) \subset ((1+y_1)^{p^n}-1,\dots,(1+y_h)^{p^n}-1).$$

This proposition then proves the "$R = T$" theorem for minimal deformation ring. The ring B is the local deformation ring (versal and often an Iwasawa algebra), and $B[[t_1,\dots,t_j]]$ is the universal framed local deformation ring in Kisin's setting; so, t_i is framed variables; so, we write $\Lambda_f = W[[t_1,\dots,t_j]]$. In the original setting of Taylor-Wiles, $j = 0$ and $B = W$; so, they need to assume representability of the local deformation functor (the p-distinguishedness condition). Since the framed deformation functor is always representable, $B[[t_1,\dots,t_j]]$ is universal.

Proof. Write $\mathbb{F}$ for the residue field of W. A key point is that $\mathbb{F}$ is a finite field. Let $s = \mathrm{rank}_{\Lambda_f}(T) = \mathrm{rank}_{\Lambda/\mathfrak{b}_n}(T_n)$, and put $r_n = snp^n(h+j)$ with

$$\mathfrak{c}_n = (\mathfrak{m}_W^n) + ((1+y_1)^{p^n} - 1, \dots, (1+y_h)^{p^n} - 1, t_1^{p^n}, \dots, t_j^{p^n}) \subset \Lambda.$$

Note that $|T_n/\mathfrak{c}_n T_n| = q^{r_n}$ for $q = |\mathbb{F}|$, and $\mathrm{length}_R\, \mathfrak{m}_{T_n/\mathfrak{c}_n T_n} < r_n$; in particular, $\mathfrak{m}_R^{r_m}(T_m/\mathfrak{c}_m T_m) = 0$. Thus, $R \xrightarrow{\varphi} T$ induces a homomorphism $R/(\mathfrak{c}_m + \mathfrak{m}_R^{r_m}) \xrightarrow{\varphi} T/\mathfrak{c}_m T$. A patching datum (D, A) of level m is a commutative diagram of W-algebras

$$\begin{array}{ccccc} \widetilde{B} & \twoheadrightarrow & D & \xrightarrow{\varphi_m} & A \\ & \nearrow & \downarrow & & \downarrow \\ \Lambda/\mathfrak{c}_m & & R/(\mathfrak{c}_m + \mathfrak{m}_R^{r_m})R & \xrightarrow{\varphi} & T/\mathfrak{c}_m T, \end{array}$$

where we suppose $\mathfrak{m}_D^{r_m} = 0$ and to have a surjection of B-algebras $\widetilde{B} \twoheadrightarrow D$, and the square consists of surjective homomorphisms of B-algebras and $\mathfrak{m}_D^{r_m} = 0$. An isomorphism between patching data (D, A) and (D', A') are isomorphisms of each term of diagrams and the 3-dimensional diagram created by these two diagrams has to be commutative. For a given m, the order of D is bounded by the bounded tangential dimension condition (4) and $\mathfrak{m}_D^{r_m} = 0$; so, there are only finitely many isomorphism classes of patching data of level m. Reducing the diagram (1.1) modulo $\mathfrak{c}_m$, we get infinitely many patching data of level m for each $n \geq m$. Thus, by Dirichlet's drawer argument, we can find a subsequence indexed by an infinite set $I \subset \mathbb{N}$ such that the patching data for $n \in I$

$$\begin{array}{ccccc} \widetilde{B} & \twoheadrightarrow & R_n/(\mathfrak{c}_n + \mathfrak{m}_R^{r_n})R_n & \xrightarrow{\varphi_n} & T_n/\mathfrak{c}_n T_n \\ & \nearrow & \downarrow & & \downarrow \\ \Lambda/\mathfrak{c}_n & & R/(\mathfrak{c}_n + \mathfrak{m}_R^{r_n})R & \xrightarrow{\varphi} & T/\mathfrak{c}_n T, \end{array}$$

form a projective system (including the surjections: $\widetilde{B} \twoheadrightarrow R_n/\mathfrak{c}_n R_n$ induced by the original ones: $\widetilde{B} \twoheadrightarrow R_n$ in (1.1)) under reduction modulo $\mathfrak{c}_m$ for $m \in I$ with $m \leq n$. Passing to the projective limit, we get a new commutative diagram

$$\begin{array}{ccccc} \widetilde{B} & \twoheadrightarrow & R_\infty & \xrightarrow{\varphi_\infty} & T_\infty \\ & \nearrow & \downarrow & & \downarrow \\ \Lambda & & R & \xrightarrow{\varphi} & T \end{array}$$

with a surjective algebra homomorphism $\widetilde{B} \twoheadrightarrow R_\infty$. Since T_∞ is free of finite rank over $\Lambda = \varprojlim_{n\in I} \Lambda/\mathfrak{b}_n\Lambda = \varprojlim_{n\in I} \Lambda/((1+y_1)^{p^n} - 1, \dots, (1+y_h)^{p^n} - 1)\Lambda$, we conclude $\dim T_\infty = \dim \Lambda = h + j + 1 = \dim B + (h + j - d) = \dim \widetilde{B} \geq \dim T_\infty$; so, by the above lemma, we find that

$$\widetilde{B} \cong R_\infty \cong T_\infty,$$

in particular, φ_∞ is an isomorphism. Then T_∞ is Cohen-Macaulay because it is free of finite rank over the regular ring Λ; so, B is Cohen-Macaulay, since $\dim B[[x_1, \dots, x_m]] = m + \dim B$ and $\mathrm{depth}\, B[[x_1, \dots, x_m]] = m + \mathrm{depth}\, B$. $\square$

To state a slightly advanced patching lemma, we need the notion of the Fitting ideal. For a commutative ring A and a finitely presented A-module M, taking a presentation

$A^b \xrightarrow{\Phi} A^a \to M \to 0$, the fitting ideal $\mathrm{Fitt}_A(M)$ is defined by the ideal generated by $b \times b$-minors and $a \times a$-minors of the matrix expression of Φ. When $A = \mathbb{Z}$ and if $M = \mathbb{Z}/N\mathbb{Z}$, we find $\mathbb{Z} \xrightarrow{N} \mathbb{Z} \to M \to 0$ is a presentation, and hence $\mathrm{Fitt}(M) = |N|$. More generally if M is a finite $\mathbb{Z}$-module, $\mathrm{Fitt}(M) = (|M|)$. If A is a DVR and M is torsion, $\mathrm{Fitt}(M) = \mathfrak{m}_A^{\mathrm{length}_A M}$. If $A = W[[T]]$, then the reflexive closure (the double Λ-dual) of $\mathrm{Fitt}_A(M)$ is the characteristic ideal of M. The reflexive closure of an ideal $\mathfrak{a}$ of A is equal to the intersection of all principal ideal containing $\mathfrak{a}$. Since tensor product preserves presentation, if $\varphi : A \to B$ is an algebra homomorphism, we have $\mathrm{Fitt}_B(M \otimes_A B) = \mathrm{Fitt}_A(M)B$. Here is a lemma of Kisin:

Lemma 1.4. *Let $\{A_n, \nu_n : A_n \to A_{n-1}\}_{n\geq 1}$ and $\{M_n, \mu_n : M_n \to M_{n-1}\}_{n\geq 1}$ be a projective system of local artinian rings A_n and A_n-modules M_n of finite presentation. If ν_n and μ_n are all surjective and the number of generators of M_n over A_n is bounded independently of n, then $\mathrm{Fitt}_A M = \varprojlim_n \mathrm{Fitt}_{A_n} M_n$ for $M = \varprojlim_n M_n$ and $A = \varprojlim_n A_n$.*

This is an easy exercise (see [K1] 3.4.10).

The following patching theorem is an invention of Kisin (see [K1] Proposition 3.4.6) which allows to treat nonminimal deformation rings.

Proposition 1.5 (Kisin)**.** *Let B be a complete local neotherian flat W-domain with $\dim_W B = d > 0$. Suppose that for all localization of B at height 1 primes is a discrete valuation ring and that $\mathrm{Spec}(B/\mathfrak{m}_W B)$ has nonempty reduced open subscheme (the nilpotent locus of $\mathrm{Spec}(B/\mathfrak{m}_W B)$ is proper closed subscheme). Let $R \xrightarrow{\varphi} T \to T_\emptyset$ be surjective homomorphisms of B-algebras, with T reduced (that is, no nontrivial nilradical). Suppose that there exist non-negative integers h, j and r such that for each non-negative integer n, there is a commutative diagram*

$$(1.2)\qquad \begin{array}{ccccccc} B[[x_1,\dots,x_{h+j-d+r}]] & \xrightarrow{\phi_n} & R_n & \xrightarrow{\varphi_n} & T_n & \longrightarrow & T_{\emptyset,n} \\ \uparrow & & \downarrow & & \downarrow & & \downarrow \\ \Lambda := W[[y_1,\dots,y_h,t_1,\dots,t_j]] & & R & \xrightarrow{\varphi} & T & \longrightarrow & T_\emptyset, \end{array}$$

which satisfies the following conditions:

1. *The horizontal maps and downward pointing maps are surjective,*
2. *We have $(y_1,\dots,y_h)R_n = \mathrm{Ker}(R_n \to R)$, $(y_1,\dots,y_h)T_n = \mathrm{Ker}(T_n \to T)$ and $(y_1,\dots,y_h)T_{\emptyset,n} = \mathrm{Ker}(T_{\emptyset,n} \to T_\emptyset)$,*
3. *If $\mathfrak{b}_n = \mathrm{Ker}(\Lambda \to R_n)$, then*
$$\mathfrak{b}_n \subset ((y_1+1)^{p^n}-1,\dots,(y_h+1)^{p^n}-1)$$
and $T_{\emptyset,n}$ and T_n are finite free over $\Lambda/\mathfrak{b}_n$,
4. *The kernel, $\overline{J}_n$, of the induced map $B[[x_1,\dots,x_{h+j-d+r}]]/\mathfrak{b}_n \twoheadrightarrow R_n$ is generated by at most r elements, and $T_{\emptyset,n}$ is a quotient of $\widetilde{B}$ (but we do not require the compatibility of the $\widetilde{B}$-algebra structure with ϕ_n),*
5. *For each n there exists a faithful T_n module M_n, and a faithful $T_{\emptyset,n}$-module $M_{\emptyset,n}$ with the following properties:*

(a) *$M_{\emptyset,n}$ and M_n are finite free over $\Lambda/\mathfrak{b}_n$ of rank equal to* $\mathrm{rank}_{\Lambda/\mathfrak{b}_0} T_\emptyset$ *and* $\mathrm{rank}_{\Lambda/\mathfrak{b}_0} T$*, respectively,*

(b) *There exist maps of T_n-modules:*

$$M_{\emptyset,n} \xrightarrow{i} M_n \xrightarrow{i^\dagger} M_{\emptyset,n}$$

such that i is injective with free cokernel over $\Lambda/\mathfrak{b}_n$, $i^\dagger$ is surjective, and the composite $i^\dagger \circ i$ is multiplication by an element $\zeta_n \in T_{\emptyset,n}$, which has the following property: If $I \subset R_n$ is an ideal such that the natural map $R_n/I \to T_{\emptyset,n}/I$ has kernel $\mathfrak{p}_I$, and admits a section, then the Fitting ideal of the $T_{\emptyset,n}/I$-module $\mathfrak{p}_I/\mathfrak{p}_I^2$ satisfies

$$\zeta_n T_{\emptyset,n}/I \subset \mathrm{Fitt}_{T_{\emptyset,n}/I}(\mathfrak{p}_I/\mathfrak{p}_I^2).$$

Then φ is an isomorphism.

Lemma 1.6. *Let the notation and assumption be as in the proposition. Then $T_\emptyset$ is reduced, and ζ_0 is not a zero divisor of $T_\emptyset$.*

Proof. Let $\mathcal{K}$ be the quotient field of $\Lambda/\mathfrak{b}_0 = W[[t_1, \dots, t_j]]$. Then $T_\mathcal{K} = T \otimes_\Lambda \mathcal{K}$ is a finite dimensional semi-simple algebra over $\mathcal{K}$; so, $T_\mathcal{K} = \mathcal{K}_1 \oplus \cdots \oplus \mathcal{K}_s$ and $T_{\emptyset,\mathcal{K}} = T_\emptyset \otimes_\Lambda \mathcal{K} = \mathcal{K}_1 \oplus \cdots \oplus \mathcal{K}_t$ with finite extensions $\mathcal{K}_i/\mathcal{K}$ for $t \leq s$. Thus $T_{\emptyset,\mathcal{K}}$ is reduced, and hence $T_\emptyset \subset T_{\emptyset,\mathcal{K}}$ is reduced. Since $M_\mathcal{K} = M_0 \otimes_\Lambda \mathcal{K}$ is faithful over $T_\mathcal{K}$ with $\dim_\mathcal{K} T_\mathcal{K} = \dim_\mathcal{K} M_\mathcal{K}$, we conclude $M_\mathcal{K} \cong T_\mathcal{K}$. Similarly, we have $M_{\emptyset,0,\mathcal{K}} = M_{\emptyset,0} \otimes_\Lambda \mathcal{K} \cong T_{\emptyset,\mathcal{K}}$. Thus after tensoring $\mathcal{K}$, $i^\dagger \circ i$ gives an automorphism of $M_{\emptyset,0,\mathcal{K}}$; so, it is not a zero divisor of $T_\emptyset$. $\square$

In an application, $T_{\emptyset,n}$ is the level $Q_n \sqcup S$ Hecke algebra flat at p, and T_n is the level $Q_n \sqcup S$ Hecke algebra without imposing the flatness condition at p. The rings R_n is the universal deformation ring unramified outside $S \sqcup Q_n$ without imposing the flatness condition at p. The module $M_{\emptyset,n}$ (resp. M_n) is the space of automorphic forms (with integral coefficients in W) of level $S \sqcup Q_n$ on a definite quaternion algebra everywhere unramified at finite places. Here $\emptyset$ indicates that we impose flatness at p. The morphism $i : M_{\emptyset,n} \to M_n$ is the natural inclusion and $i^\dagger$ is its adjoint under the Petersson inner product.

For the proof, we need the following lemmas which we do not prove. The first one is a famous lemma by Lenstra generalizing an argument of Wiles in his Fermat's last theorem paper:

Lemma 1.7 (Lenstra)**.** *Let V be a discrete valuation ring, R be a complete noetherian local V-algebra, T a finite flat local V-algebra, and $\varphi : R \to T$ and $\pi : T \to V$ be surjective V-algebra homomorphisms. Then the following three conditions are equivalent:*

(1) $\mathrm{length}_V(\mathrm{Ker}(\pi \circ \varphi)/\mathrm{Ker}(\pi \circ \varphi)^2)$ *is finite, and*

$$\mathrm{length}_V(\mathrm{Ker}(\pi \circ \varphi)/\mathrm{Ker}(\pi \circ \varphi)^2) \leq \mathrm{length}_V(V/(\mathrm{Ann}_T(\mathrm{Ker}(\pi))V)$$

(which is equivalent to $(\mathrm{Ann}_T \mathrm{Ker}(\pi))V \subset \mathrm{Fitt}_V(\mathrm{Ker}(\pi \circ \varphi)/\mathrm{Ker}(\pi \circ \varphi)^2)$);

(2) $\mathrm{length}_V(\mathrm{Ker}(\pi\circ\varphi)/\mathrm{Ker}(\pi\circ\varphi)^2)$ *is finite, and*
$$\mathrm{length}_V(\mathrm{Ker}(\pi\circ\varphi)/\mathrm{Ker}(\pi\circ\varphi)^2) = \mathrm{length}_V(V/(\mathrm{Ann}_T(\mathrm{Ker}(\pi))V)$$
(which is equivalent to $(\mathrm{Ann}_T\mathrm{Ker}(\pi))V = \mathrm{Fitt}_V(\mathrm{Ker}(\pi\circ\varphi)/\mathrm{Ker}(\pi\circ\varphi)^2)$*);*

(3) T *is a complete intersection,* $(\mathrm{Ann}_T(\mathrm{Ker}(\pi))V \neq 0$ *and* φ *is an isomorphism.*

See [L] for Lenstra's proof.

We now prove Kisin's proposition Proposition 1.5.

Proof. Now a patching datum $(D, A, A_\emptyset, N, N_\emptyset)$ of level m consists of

(1) a commutative diagram of W-algebras:
$$\begin{array}{ccccccc} B[[x_1,\dots,x_{h+j-d+r}]] & \longrightarrow & D & \xrightarrow{\varphi_n} & A & \longrightarrow & A_\emptyset \\ \uparrow & & \downarrow & & \downarrow & & \downarrow \\ \Lambda/\mathfrak{c}_m & & R/(\mathfrak{c}_m+\mathfrak{m}_R^{r_m})R & \xrightarrow{\varphi} & T/\mathfrak{c}_mT & \longrightarrow & T_\emptyset/\mathfrak{c}_mT_\emptyset, \end{array}$$
where $\mathfrak{m}_D^{r_m}=0$,

(2) An A-module N and an $A_\emptyset$-module $N_\emptyset$ such that N and $N_\emptyset$ are free over $\Lambda/\mathfrak{c}_m\Lambda$ of rank equal to $\mathrm{rank}_{\Lambda_f} T$ and $\mathrm{rank}_{\Lambda_f} T_\emptyset$, respectively, and equipped with maps of A-modules $N_\emptyset \to N \to N_\emptyset$.

A morphism of pairing data
$$(D, A, A_\emptyset, N, N_\emptyset) \to (D', A', A'_\emptyset, N', N'_\emptyset)$$
consists of

(i) a morphism of corresponding elements of the data making the total diagram out of (1.2) commutative when we take the identity maps for the lower rows and $B[[x_1,\dots,x_{h+j-d+r}]]$,

(ii) morphisms $N \to N'$ of A-modules and $N_\emptyset \to N'_\emptyset$ of $A_\emptyset$-modules, which makes the following diagram commutes:
$$\begin{array}{ccccc} N_\emptyset & \longrightarrow & N & \longrightarrow & N_\emptyset \\ \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\ N'_\emptyset & \longrightarrow & N' & \longrightarrow & N'_\emptyset. \end{array}$$

Again we reduce the diagram (1.2) for each $n \geq m$ modulo $\mathfrak{c}_m + \mathfrak{m}_R^{r_m}$ and the finiteness of the isomorphism classes of patching data of level m, we find an infinite subset $I = \{n_m \geq m\}_{m\in\mathbb{N}} \subset \mathbb{N}$ and a coherent projective system of the data (1.2) modulo $\mathfrak{c}_m+\mathfrak{m}_R^{r_m}$:
$$\left\{\frac{R_{n_m}}{(\mathfrak{c}_m+\mathfrak{m}_R^{r_m})R_{n_m}}, \frac{T_{n_m}}{\mathfrak{c}_mT_{n_m}}, \frac{T_{\emptyset,n_m}}{\mathfrak{c}_mT_{\emptyset,n_m}}, \frac{M_{n_m}}{\mathfrak{c}_mM_{n_m}}, \frac{M_{\emptyset,n_m}}{\mathfrak{c}_mM_{\emptyset,n_m}}\right\}_{n_m\in I}.$$

Passing to the projective limit, we get, writing for simplicity $\mathbb{T} = \varprojlim_m T_{n_m}$ and $\mathbb{T}_\emptyset := \varprojlim_m T_{\emptyset,n_m}$,
$$\begin{array}{ccccccc} B[[x_1,\dots,x_{h+j-d+r}]] & \longrightarrow & R_\infty & \xrightarrow{\varphi_\infty} & \mathbb{T} & \longrightarrow & \mathbb{T}_\emptyset \\ \uparrow & & \downarrow & & \downarrow & & \downarrow \\ \Lambda := W[[y_1,\dots,y_h,t_1,\dots,t_j]] & & R & \xrightarrow{\varphi} & T & \longrightarrow & T_\emptyset, \end{array} \tag{1.3}$$

$\mathbb{M} = \varprojlim_m M_{n_m}/\mathfrak{c}_m M_{n_m}$ and $\mathbb{M}_\emptyset = \varprojlim_m M_{n_m}/\mathfrak{c}_m M_{\emptyset,n_m}$ with $\mathbb{M}_\emptyset \xrightarrow{i} \mathbb{M} \xrightarrow{i^\dagger} \mathbb{M}_\emptyset$ giving $i^\dagger \circ i = \zeta_\infty \in T_{\emptyset,\infty}$. They satisfy

(a) $\mathbb{T} := T_\infty$ and $\mathbb{T}_0 := T_{\emptyset,\infty}$ are free of finite rank over $\Lambda = W[[y_1, \dots, y_h, t_1, \dots, t_j]]$ with rank $r = \mathrm{rank}_{W[[t_1,\dots,t_j]]} T$ and $r_\emptyset = \mathrm{rank}_{W[[t_1,\dots,t_j]]} T_\emptyset$, respectively;
(b) $J_\infty = \mathrm{Ker}(B[[x_1, \dots, x_{h+j-r+d}]] \to R_\infty)$ is generated by (at most) r elements;
(c) $\mathbb{M}$ and $\mathbb{M}_\emptyset$ are faithful modules over T_∞ and $T_{\emptyset,\infty}$, respectively. They are free of finite rank over Λ of rank equal to r and $r_\emptyset$ respectively,
(d) We have surjective W-algebra homomorphism $\widetilde{B} \twoheadrightarrow \mathbb{T}_\emptyset$.

By (a), $\dim \mathbb{T}_\emptyset = \dim \widetilde{B}$. By (d) and Lemma 1.2, we have $\mathbb{T}_\emptyset \cong B[[z_1, \dots, z_{h+j-d}]]$ for variables z_i. By (a), $\mathbb{T}_\emptyset = \bigcap_P \mathbb{T}_{\emptyset,P}$ for all height 1 primes P of Λ, and hence $\mathbb{T}_\emptyset = \bigcap_P \mathbb{T}_{\emptyset,P}$ over all height 1 primes P of $\mathbb{T}_\emptyset$. In particular,

$$\mathbb{T}_\emptyset \text{ is a normal noetherian integral domain,} \tag{1.4}$$

since B is regular in codimension 1. Choosing $\widetilde{z}_i \in R_\infty$ whose image in $\mathbb{T}_\emptyset$ is equal to z_i, sending z_i to $\widetilde{z}_i$ for each i produces a section $\iota : \mathbb{T}_\emptyset \to R_\infty$ of the projection map $R_\infty \twoheadrightarrow \mathbb{T}_\emptyset$. For any ideal I with artinian quotient R_∞/I, we find some n such that $R_\infty \twoheadrightarrow R_\infty/I$ factors through R_n. Then by (5b), for $\mathfrak{p}_I = \mathrm{Ker}(R_n/I \to T_{\emptyset,n}/I) = \mathrm{Ker}(R_\infty/I \to \mathbb{T}_\emptyset/I)$, we have $\zeta_n T_{\emptyset,n}/I \subset \mathrm{Fitt}_{T_{\emptyset,n}/I}(\mathfrak{p}_I/\mathfrak{p}_I^2)$, which is identical to $\zeta_\infty \mathbb{T}_\emptyset/I \subset \mathrm{Fitt}_{\mathbb{T}_\emptyset/I}(\mathfrak{p}_I/\mathfrak{p}_I^2)$. Since any closed ideal I of R_∞ is an intersection of ideals I' with Artinian quotient R_∞/I' and the section ι induces a section of $R_\infty/I \twoheadrightarrow \mathbb{T}_\emptyset/I$, we actually have $\zeta_\infty \mathbb{T}_\emptyset/I \subset \mathrm{Fitt}_{\mathbb{T}_\emptyset/I}(\mathfrak{p}_I/\mathfrak{p}_I^2)$ for any closed ideal I (by Lemma 1.4); in particular, $I = 0$. Since $\zeta_\infty \bmod (y_1, \dots, y_h) = \zeta_0 \neq 0$ (Lemma 1.6), $\zeta \neq 0 \in \mathbb{T}_\emptyset \cong \widetilde{B}$, which is an integral domain. Thus for $\mathfrak{p}_0 = \mathfrak{p}_I$ for $I = (0)$, $\mathrm{Fitt}_{\mathbb{T}_\emptyset}(\mathfrak{p}_0/\mathfrak{p}_0^2) \supset \zeta_\infty \mathbb{T}_\emptyset$ implies for any hight 1 prime P of $\mathbb{T}_\emptyset$, $\mathrm{length}_{\mathbb{T}_{\emptyset,P}}(\mathfrak{p}_{0,P}/\mathfrak{p}_{0,P}^2) < \infty$.

Since $T \cong \mathbb{T}/(y_1, \dots, y_h)\mathbb{T}$ and $T_\emptyset \cong \mathbb{T}_\emptyset/(y_1, \dots, y_h)\mathbb{T}_\emptyset$ are reduced (Lemma 1.6), for an nonempty open subscheme U of $\mathrm{Spec}(\mathbb{T})$, U is reduced. Since each irreducible component of $\mathrm{Spec}(\mathbb{T})$ has intersection with $\mathrm{Spec}(\mathbb{T}/(y_1, \dots, y_h)\mathbb{T})$ (because $\mathbb{T}$ is finite flat over Λ), U intersects with each irreducible component nontrivially. Thus for minimal primes P of $\mathbb{T}$, $\mathbb{T}_P$ is reduced; in other words, writing the nilradical $\mathfrak{n}_\mathbb{T}$ of $\mathbb{T}$ as $\mathfrak{n}_\mathbb{T} = \bigcap_{P \in \mathrm{Ass}(\mathbb{T})} P$, $\mathfrak{n}_\mathbb{T} \neq 0$ implies $\mathrm{Ass}(\mathbb{T})$ has embedded (nonminimal) primes. Since $\mathbb{T}$ is Cohen-Macaulay (that is, $\mathrm{depth}\, \mathbb{T} = \dim \mathbb{T}$, because $\mathbb{T}$ is flat over Λ), all associated primes of $\mathbb{T}$ is minimal ([CRT] Theorem 17.3). Thus $(0) = \mathfrak{n}_\mathbb{T} = \cap_P P$ for minimal primes P in $\mathbb{T}$; in particular, $\mathbb{T} \hookrightarrow \prod_P \mathbb{T}/P$ by the Chinese remainder theorem; so, $\mathbb{T}$ is reduced.

Now pick any height one prime $P \subset \mathbb{T}_\emptyset$ such that $P \cap W = \mathfrak{m}_W = (\pi)$. Then $V = \mathbb{T}_{\emptyset,P}$ is a discrete valuation ring (because $\mathbb{T}_\emptyset$ is a normal noetherian ring; see [CRT] Theorems 10.2 and 10.4). Since $R_\infty \twoheadrightarrow \mathbb{T}_\emptyset$ has a section (of B-algebras), regarding R_∞ as a $\mathbb{T}_\emptyset$-algebra, we can localize R_∞ at P. We now check Lenstra's criterion to show $R_{\infty,P} \cong \mathbb{T}_P$. Write $\mathfrak{p} = \mathrm{Ker}(\mathbb{T} \to \mathbb{T}_\emptyset)$ (then $\mathfrak{p}_P = \mathrm{Ker}(\mathbb{T}_P \to \mathbb{T}_{\emptyset,P} = V)$. The criterion is:

$$\mathrm{Ann}_{\mathbb{T}_P}(\mathfrak{p}_P)V \subset \mathrm{Fitt}_V((\mathfrak{p}_0/\mathfrak{p}_0^2)_P) = \mathrm{Fitt}_V(\mathfrak{p}_{0,P}/\mathfrak{p}_{0,P}^2)$$

for $\mathfrak{p}_0 = \mathrm{Ker}(R_\infty \to \mathbb{T}_\emptyset)$. We look at $\mathbb{M}_\emptyset \xrightarrow{i} \mathbb{M} \xrightarrow{i^\dagger} \mathbb{M}_\emptyset$ which are made up of homomorphism of $\mathbb{T}_\emptyset$-modules. Let $\mathbb{K}$ be the quotient field of Λ. Since $\mathbb{T}$ (resp. $\mathbb{T}_\emptyset$) is reduced, $\mathbb{T}_\mathbb{K} = \mathbb{T}\otimes_\Lambda \mathbb{K}$ (resp. $\mathbb{T}_{\emptyset,\mathbb{K}} = \mathbb{T}_\emptyset \otimes_\Lambda \mathbb{K}$) is a finite dimensional semi-simple algebra over $\mathbb{K}$, and by the argument proving Lemma 1.6, we find that $\mathbb{M}_\mathbb{K} = \mathbb{M}\otimes_\Lambda \mathbb{K}$ (resp. $\mathbb{M}_{\emptyset,\mathbb{K}} = \mathbb{M}_\emptyset \otimes_\Lambda \mathbb{K}$) is free of rank 1 over $\mathbb{T}_\mathbb{K}$ (resp. $\mathbb{T}_{\emptyset,\mathbb{K}}$), and $\mathbb{T}_\mathbb{K} = X \oplus \mathbb{T}_{\emptyset,\mathbb{K}}$ as an algebra direct sum. Thus $\mathbb{M}_\mathbb{K}[\mathfrak{p}] = M[\mathfrak{p}]\otimes_\Lambda \mathbb{K}$ for $\mathbb{M}[\mathfrak{p}] = \{x \in \mathbb{M} | \mathfrak{p}x = 0\}$ is free of rank 1 over $\mathbb{T}_{\emptyset,\mathbb{K}}$. Since $\mathbb{M}[\mathfrak{p}] = \{x \in \mathbb{M} | \mathfrak{p}x = 0\}$ contains $\mathrm{Im}(i)$, $\mathrm{Im}(i)\backslash\mathbb{M}[\mathfrak{p}] \hookrightarrow \mathrm{Coker}(i)$ is a torsion Λ-module; however, $\mathrm{Coker}(i)$ is free over Λ; so, $\mathrm{Im}(i) = \mathbb{M}[\mathfrak{p}]$. Thus $i^\dagger$ induces a surjection

$$\mathbb{M}/\mathbb{M}[\mathfrak{p}] \twoheadrightarrow M_\emptyset/i^\dagger \circ i(\mathbb{M}_\emptyset) = \mathbb{M}_\emptyset/\zeta_\infty \mathbb{M}_\emptyset.$$

Note that $\mathfrak{p} = (X \oplus 0) \cap \mathbb{T}$, $\mathrm{Ann}_\mathbb{T}(\mathfrak{p}) = \mathbb{T} \cap (0 \oplus \mathbb{T}_{\emptyset,\mathbb{K}})$ and $\mathbb{M}[\mathfrak{p}] = (0 \oplus \mathbb{M}_{\emptyset,\mathbb{K}}) \cap \mathbb{M}$, and hence $\mathbb{M}/\mathbb{M}[\mathfrak{p}]$ is a module over the image $\mathbb{T}/\mathrm{Ann}_\mathbb{T}(\mathfrak{p})$ of $\mathbb{T}$ in X. Thus $\mathbb{M}_\emptyset/i^\dagger \circ i(\mathbb{M}_\emptyset)$ is a module over $\mathbb{T}_\emptyset/\mathrm{Ann}_\mathbb{T}(\mathfrak{p})\mathbb{T}_\emptyset$, which implies $(\mathrm{Ann}_\mathbb{T}\mathfrak{p})\mathbb{T}_\emptyset \subset \zeta_\infty \mathbb{T}_\emptyset$. In particular,

$$(\mathrm{Ann}_{\mathbb{T}_P}\mathfrak{p}_P)V \subset \zeta_\infty V \subset \mathrm{Fitt}_V(\mathfrak{p}_{0,P}/\mathfrak{p}_{0,P}^2)$$

as desired. Lenstra's lemma shows that $R_{\infty,P} \cong \mathbb{T}_P$ for all height 1 prime $P \subset \mathbb{T}_\emptyset$ over $\mathfrak{m}_W$. Such height 1 primes (of $\mathbb{T}_\emptyset$ over $\mathfrak{m}_W$) are in bijection with the irreducible components of $\mathbb{T}_\emptyset/\mathfrak{m}\mathbb{T}_\emptyset$. Thus the set of irreducible components of $\mathrm{Spec}(R_\infty/\mathfrak{m}R_\infty)$ of maximal dimension $h+j$ is in bijection with the set of irreducible components of $\mathrm{Spec}(\mathbb{T}/\mathfrak{m}_W\mathbb{T})$.

To prove $R_\infty \cong \mathbb{T}$, it is enough to prove $R_\infty/\mathfrak{m}_W R_\infty \cong \mathbb{T}/\mathfrak{m}_W\mathbb{T}$, because $R_\infty \cong \varprojlim_n R_\infty/\mathfrak{m}_W^n R_\infty$, $\mathbb{T} = \varprojlim_n \mathbb{T}/\mathfrak{m}_W^n\mathbb{T}$ and

$$R_\infty/\mathfrak{m}_W R_\infty \twoheadrightarrow \mathfrak{m}^{n-1}R_\infty/\mathfrak{m}_W^n R_\infty \twoheadrightarrow \mathfrak{m}^{n-1}\mathbb{T}/\mathfrak{m}_W^n\mathbb{T} \cong \mathbb{T}/\mathfrak{m}_W\mathbb{T}$$

(because $\mathfrak{m}_W = (\pi)$ and $\mathbb{T}$ is W-flat). Let $A = B[[x_1, \dots, x_{h+j-d+r}]]$, which is Cohen-Macaulay (because B is by Proposition 1.3). Since $J_m = \mathrm{Ker}(A/(\mathfrak{c}_m + \mathfrak{m}_A^{r_m}) \twoheadrightarrow R_\infty/\mathfrak{c}_m + \mathfrak{m}_{R_\infty}^{r_m} = R_m/\mathfrak{c}_m + \mathfrak{m}_{R_m}^{r_m})$ is generated by r elements, $J_\infty = \varprojlim_m J_m = \mathrm{Ker}(A \twoheadrightarrow R_\infty)$ is generated by r elements. By a result of Raynaud (the lemma following this proof), for any given irreducible component $C_1 \subset \mathrm{Spec}(R_\infty/\mathfrak{m}_W R_\infty)$, there exists an irreducible component $C_2 \subset \mathrm{Spec}(R_\infty/\mathfrak{m}_W R_\infty)$ containing an irreducible component of $C_2' \subset \mathrm{Spec}(\mathbb{T}_\emptyset/\mathfrak{m}_W\mathbb{T}_\emptyset)$ such that $\dim(C_1 \cap C_2) \geq h+j+r+1-r-1 = h+j$ which is equal to the dimension of C_2' and C_2. Thus $C_2' = C_2$ which has maximal dimension, we conclude $C_1 = C_2$. Thus any irreducible component of $\mathrm{Spec}(R_\infty/\mathfrak{m}_W R_\infty)$ is of maximal dimension; hence, J_∞ must be generated by a regular sequence of length r ([CRT] Theorem 17.4). Thus $R_\infty/\mathfrak{m}_W R_\infty$ is Cohen-Macaulay, and there is no embedded primes of $R_\infty/\mathfrak{m}_W R_\infty$. In particular, $\mathrm{Spec}(R_\infty/\mathfrak{m}_W R_\infty)$ is equi-dimensional, and the full set I of minimal primes of $R_\infty/\mathfrak{m}_W R_\infty$ is in bijection with the set of irreducible components of $\mathrm{Spec}(R_\infty/\mathfrak{m}_W R_\infty)$. By the isomorphism of the localizationa at height 1 primes of $\mathbb{T}_\emptyset$ over $\mathfrak{m}_W$, $(R_\infty/\mathfrak{m}_W R_\infty)_P \cong (\mathbb{T}/\mathfrak{m}_W\mathbb{T})_P$ for all $P \in I$, and the kernel of the surjection $R_\infty/\mathfrak{m}_W R_\infty \twoheadrightarrow \mathbb{T}/\mathfrak{m}_W\mathbb{T}$ is contained in $\bigcap_{P\in I} P = (0)$ (no embedded primes). Thus we conclude $R_\infty/\mathfrak{m}_W R_\infty \cong \mathbb{T}/\mathfrak{m}_W\mathbb{T}$. □

We now state Raynaud's lemma as formulated in [SW] Proposition A.1 and Corollary A.2:

Lemma 1.8. *Let A be a local Cohen-Macaulay ring of dimension d, and suppose that $\mathfrak{a} = (f_1, \dots, f_r)$ is an ideal of A with $r \leq d-2$. Let I be the set of irreducible components of* $\mathrm{Spec}(A/\mathfrak{a})$*. If $I = I_1 \bigsqcup I_2$ is a partition of I with $I_1 \neq \emptyset$ and $I_2 \neq \emptyset$, then there exist irreducible components $C_1 \in I_1$ and $C_2 \in I_2$ such that $C_1 \cap C_2$ contains a prime of dimension $d - r - 1$.*

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