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Hecke fields of Hilbert modular analytic families

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To the memory of Ilya Pyatetski-Shapiro

Abstract. Take a non CM $\mathfrak{p}$-slope 0 analytic family of Hilbert modular forms of level $\mathfrak{N}\mathfrak{p}^\infty$ for a prime $\mathfrak{p}$ of the base totally real field F. We prove that the Hecke field over $\mathbb{Q}[\mu_{p^\infty}]$ of members of the family grows indefinitely large over any infinite set of arithmetic points with fixed weight. The condition: $p > 2$ made in [**H11**] for $F = \mathbb{Q}$ is also eliminated in this paper for the assertion.

Fix a prime p and field embeddings $\mathbb{C} \overset{\iota_\infty}{\hookleftarrow} \overline{\mathbb{Q}} \overset{\iota_p}{\hookrightarrow} \overline{\mathbb{Q}}_p \subset \mathbb{C}_p$, where $\overline{\mathbb{Q}}$ is an algebraic closure of $\mathbb{Q}$. Fix a finite totally real extension $F_{/\mathbb{Q}}$ inside $\overline{\mathbb{Q}}$ with integer ring O. We define a prime ideal of O by $\mathfrak{p} = \{\alpha \in O : |\iota_p(\alpha)|_p < 1\}$ and fix an O-ideal $\mathfrak{N}$ prime to $\mathfrak{p}$.

Let $S_\kappa(\mathfrak{N}, \epsilon; \mathbb{C})$ denote the space of weight κ adelic Hilbert cusp forms $\mathbf{f} : GL_2(F)\backslash GL_2(F_\mathbb{A}) \to \mathbb{C}$ of level $\mathfrak{N}$ with Neben character ϵ modulo $\mathfrak{N}$, where $\mathfrak{N}$ is a non-zero ideal of O. Here the weight $\kappa = (\kappa_1, \kappa_2)$ is the Hodge weight of the rank 2 pure motive $M(\mathbf{f})$ over F with coefficient in the Hecke field $\mathbb{Q}(\mathbf{f})$ associated to the Hecke eigen new form $\mathbf{f} \in S_\kappa(\mathfrak{N}, \epsilon; \mathbb{C})$ (see [**BR**]). Since each (non-zero) Hecke eigen cusp form $\mathbf{f}$ generates a unique automorphic representation which contain a unique new form $\mathbf{f}^{\mathrm{new}}$, we sometime abuse our language saying that $M(\mathbf{f}^{\mathrm{new}})$ is associated to $\mathbf{f}$ and write $M(\mathbf{f})$ for $M(\mathbf{f}^{\mathrm{new}})$. Strictly speaking, each classical member of a primitive p-adic analytic family is a Hecke eigenform but may not be a new form (even not a p-stabilized form of a new form but a minimal form as we will describe below). For each field embedding $\sigma : F \hookrightarrow \overline{\mathbb{Q}}$, $M(\mathbf{f}) \otimes_{F,\iota_\infty \circ \sigma} \mathbb{C}$ has Hodge weight $(\kappa_{1,\sigma}, \kappa_{2,\sigma})$ and $(\kappa_{2,\sigma}, \kappa_{1,\sigma})$, and the motivic weight $\kappa_{1,\sigma} + \kappa_{2,\sigma}$ is independent of σ. Thus this constancy of $\kappa_{1,\sigma} + \kappa_{2,\sigma}$ as a function of σ will be imposed always for our weight κ, and the constant is written as $[\kappa] = \kappa_{1,\sigma} + \kappa_{2,\sigma}$. In addition, we normalize the weight imposing $\kappa_{1,\sigma} \leq \kappa_{2,\sigma}$. This normalization is the one in [**HMI**, (SA1–3)].

Denote by I the set of all field embeddings of F in $\overline{\mathbb{Q}}$. Let $I_\mathfrak{p}$ be the subset of I consisting of those $\sigma \in I$ for which $\iota_p \circ \sigma$ is continuous with respect to the $\mathfrak{p}$-adic topology on F, hence factors through $F \subset F_\mathfrak{p}$. Thus $I_\mathfrak{p}$ can be identified with the set of $\mathbb{Q}_p$-linear embeddings of $F_\mathfrak{p}$ into $\mathbb{C}_p$. We split $I = I_\mathfrak{p} \sqcup I^\mathfrak{p}$. The projection of

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$\kappa \in \mathbb{Z}[I] \times \mathbb{Z}[I]$ to $\mathbb{Z}[I_\mathfrak{p}] \times \mathbb{Z}[I_\mathfrak{p}]$ (resp. $\mathbb{Z}[I^\mathfrak{p}] \times \mathbb{Z}[I^\mathfrak{p}]$) is denoted by $\kappa_\mathfrak{p}$ (resp. $\kappa^\mathfrak{p}$). Often we use I to denote $\sum_\sigma \sigma \in \mathbb{Z}[I]$. If the Hodge weight is given by $\kappa = (0, kI)$ for an integer $k \geq 1$, traditionally, the integer $k+1$ is called the weight (of the cusp forms in $S_\kappa(\mathfrak{N}, \epsilon; \mathbb{C})$) at all σ, but we use here the Hodge weight κ.

The "Neben character" we use is again not a traditional one (but the one introduced in [**HMI**]). It is a set of three characters $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_+)$, where $\epsilon_+ : F_\mathbb{A}^\times / F^\times \to \mathbb{C}^\times$ is the central character of the automorphic representation $\pi_\mathbf{f}$ of $GL_2(F_\mathbb{A})$ generated by any Hecke eigenform $0 \neq \mathbf{f} \in S_\kappa(\mathfrak{N}, \epsilon; \mathbb{C})$. The character ϵ_+ has infinity type $I - \kappa_1 - \kappa_2$, and therefore its finite part has values in $\overline{\mathbb{Q}}^\times$. The characters ϵ_j is a $\overline{\mathbb{Q}}$-valued continuous character of $\widehat{O}^\times = \varprojlim_{0<N\in\mathbb{Z}} (O/NO)^\times$ with $\epsilon_1\epsilon_2 = \epsilon_+|_{\widehat{O}^\times}$. The two characters ϵ_j are purely local and may not extend to Hecke characters of the idele class group $F_\mathbb{A}^\times/F^\times$. Put $\epsilon^- := \epsilon_1\epsilon_2^{-1}$, and we assume that ϵ^- factors through $(O/\mathfrak{N})^\times$; so, the conductor of ϵ^- is a factor of $\mathfrak{N}$ (which could be a proper factor of $\mathfrak{N}$). Then for the level group

$$U = U_0(\mathfrak{N}) = \{u = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(\widehat{O}) \ \text{ with } c \in \widehat{\mathfrak{N}} = \mathfrak{N}\widehat{O}\},$$

we have $\mathbf{f}(gu) = \epsilon(u)\mathbf{f}(g)$ for all $g \in GL_2(F_\mathbb{A})$ and $u \in U$, where

$$\epsilon(u) = \epsilon_2(\det(u))\epsilon^-(a_\mathfrak{N}) = \epsilon_1(\det(u))(\epsilon^-)^{-1}(d_\mathfrak{N})$$

for the projection $d_\mathfrak{N}$ of d to $\prod_{\mathfrak{l}|\mathfrak{N}} F_\mathfrak{l}$. The characters ϵ_j for $j = 1, 2$ factor through $(O/\mathfrak{N}_j)^\times$ for some multiple $\mathfrak{N}_j$ of $\mathfrak{N}$ but we do not insist on $\mathfrak{N} = \mathfrak{N}_j$. If the local component $\pi_\mathfrak{l}$ of $\pi_\mathbf{f}$ is a principal series $\pi(\alpha_\mathfrak{l}, \beta_\mathfrak{l})$ or Steinberg $\sigma(\alpha_\mathfrak{l}, \beta_\mathfrak{l})$ and $(\alpha_\mathfrak{l}|_{O_\mathfrak{l}^\times}, \beta_\mathfrak{l}|_{O_\mathfrak{l}^\times}) = (\epsilon_1|_{O_\mathfrak{l}^\times}, \epsilon_2|_{O_\mathfrak{l}^\times})$ in this order for the primes $\mathfrak{l}$ in the level, the minimal vector in $\pi_\mathfrak{l}$ is unique up to scalar multiple (see [**H89b**, Section 2]). If $\pi_\mathfrak{l}$ is super-cuspidal, we suppose that $\mathbf{f}$ is new at $\mathfrak{l}$. Such a form we call a minimal form $\mathbf{f}^\circ \in S_\kappa(\mathfrak{N}^\circ, \epsilon; \mathbb{C})$ in $\pi_\mathbf{f}$ with minimal level $\mathfrak{N}^\circ|\mathfrak{N}$. Though this minimal level $\mathfrak{N}^\circ$ of $\pi_\mathbf{f}$ is a factor of the conductor of $\pi_\mathbf{f}$ but could be a **proper** factor of it. These minimal forms are $\mathfrak{p}$-adically interpolated (not the new forms). Indeed, if $\epsilon_{1,\mathfrak{p}}$ and $\epsilon_{2,\mathfrak{p}}$ are both non-trivial, the new form $\mathbf{f}^{\mathrm{new}}$ has infinite $\mathfrak{p}$-slope. Though our "Neben Typus" appears complicated, to formulate the Hilbert modular "$R = T$" theorems in Galois deformation theory, without using such a level structure, the appropriate Hecke algebra giving the universal deformation ring cannot be produced (cf. [**HMI**] §3.2.4), and also the primitive p-adic L-functions can be only constructed via such level structure (see, for example, [**H09**] §3.1–3.2). A detailed description of cusp forms in $S_\kappa(\mathfrak{N}, \epsilon; \mathbb{C})$ will be recalled in Section 1.9.

Hereafter, throughout the paper, $\mathfrak{N}$ denotes an O-ideal prime to $\mathfrak{p}$, and we work with cusp forms of (minimal) level $\mathfrak{N}\mathfrak{p}^{r+1}$ (for $r \geq 0$). Extend ϵ_j to $(F_\mathbb{A}^{(\infty)})^\times$ (trivial outside the level $\mathfrak{N}_j$ and trivial at a choice of uniformizer $\varpi_\mathfrak{l}$ at each prime $\mathfrak{l}$), and extend the character ϵ of U to the semi group

$$\Delta_0(\mathfrak{N}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(F_\mathbb{A}^{(\infty)}) \middle| d\widehat{O} + \widehat{\mathfrak{N}} = \widehat{O},\ c \in \widehat{\mathfrak{N}} \right\}$$

by $\epsilon\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \epsilon_1(ad - bc)(\epsilon^-)^{-1}(d_\mathfrak{N})$. The Hecke operator $T(y)$ of the double coset $U\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right)U = \bigsqcup_\delta \delta U$ is defined by $\mathbf{f}|T(y)(g) = \sum_\delta \epsilon(\delta)^{-1}\mathbf{f}(g\delta)$ (see (1.14)). As shown in [**PAF**, Section 4.2] or [**HMI**, Section 4.3], $\mathbb{T}(y) = y_p^{-\kappa_1}T(y)$ is optimally p-integral. For a Hecke eigenform $\mathbf{f}$, the eigenvalue $a(y, \mathbf{f})$ of $T(y)$ depends only on the ideal $\mathfrak{y} = y\widehat{O} \cap F$ (see (1.19)); so, for each prime $\mathfrak{l}$ of F, we write $a(\mathfrak{l}, \mathbf{f})$ for $a(\varpi_\mathfrak{l}, \mathbf{f})$ and put $T(\mathfrak{l}) := T(\varpi_\mathfrak{l})$. Therefore the y-th Fourier coefficient $c(y, \mathbf{f})$

of $\mathbf{f}$ is $\epsilon_1(y)a(y,\mathbf{f})$ for each Hecke eigenform $\mathbf{f}$ normalized so that $c(1,\mathbf{f})=1$, and the Fourier coefficient depends on y (if $\epsilon_1\neq 1$) not just on the ideal $\mathfrak{y}$. A $T(\mathfrak{p})$-eigenform $\mathbf{f}$ has $\mathfrak{p}$-*slope* 0 if $|y_p^{-\kappa_1}\iota_p(a(\mathfrak{p},\mathbf{f}))|_p=1$. A $\mathfrak{p}$-slope 0 form can have positive slope at primes $\mathfrak{p}'|p$ different from $\mathfrak{p}$. For a Hecke eigenform $\mathbf{f}\in S_\kappa(\mathfrak{N}\mathfrak{p}^{r+1},\epsilon;\mathbb{C})$ ($\mathfrak{p}\nmid\mathfrak{N}, r\geq 0$) and a subfield K of $\overline{\mathbb{Q}}$, the Hecke field $K(\mathbf{f})$ inside $\mathbb{C}$ is generated over K by the eigenvalues $a(\mathfrak{l},\mathbf{f})$ of $\mathbf{f}$ for the Hecke operators $T(\mathfrak{l})$ for all prime ideals $\mathfrak{l}$ and the values of ϵ over finite ideles.

Let $\mathbf{\Gamma}\cong\mathbb{Z}_p^m$ be the maximal torsion-free quotient of $O_\mathfrak{p}^\times$. We fix once and for all a splitting of the projection: $O_\mathfrak{p}^\times\twoheadrightarrow\mathbf{\Gamma}$ and decompose $O_\mathfrak{p}^\times=\mathbf{\Gamma}\times\Delta$ for a finite group Δ. We fix a $\mathbb{Z}_p$-basis $\{\gamma_j\}_{j=1,\dots,m}\subset\mathbf{\Gamma}$ so that $\mathbf{\Gamma}=\prod_j\gamma_j^{\mathbb{Z}_p}$ and identify the Iwasawa algebra $\mathbf{\Lambda}=W[[\mathbf{\Gamma}]]$ with the power series ring $W[[T]]$ ($T=\{T_j\}_{j=1,\dots,m}$) by $\mathbf{\Gamma}\ni\gamma_j\mapsto(1+T_j)\in\mathbf{\Lambda}$ (for a sufficiently large discrete valuation ring W flat over $\mathbb{Z}_p$). Putting $t_j=1+T_j$, we have $W[[T]]=\varprojlim_n W[t,t^{-1}]/(t^{p^n}-1)$, where $t=(t_j)_j$, $t^{-1}=(t_j^{-1})_j$ and $(t^{p^n}-1)$ is the ideal $(t_1^{p^n}-1,\dots,t_m^{p^n}-1)$ in $W[[T]]$. In this way, we identify the formal spectrum $\mathrm{Spf}(\mathbf{\Lambda})$ with $\widehat{\mathbb{G}}_m\otimes_{\mathbb{Z}_p}\mathbf{\Gamma}^*$ for $\mathbf{\Gamma}^*=\mathrm{Hom}_{\mathbb{Z}_p}(\mathbf{\Gamma},\mathbb{Z}_p)$, as t_j^m giving the character of $\mathbf{\Gamma}^*$ corresponding $t_j^m(\gamma_i^*)=\delta_{ij}$ for the dual basis $\{\gamma_j^*\}_j$ of $\{\gamma_j\}_j$. The group $\mathbb{Z}[I_\mathfrak{p}]$ can be considered to be the character group of $\mathbb{T}_\mathfrak{p}=\mathrm{Res}_{O_\mathfrak{p}/\mathbb{Z}_p}\mathbb{G}_m$. The formal completion $\widehat{\mathbb{T}}_\mathfrak{p}$ of the torus $\mathbb{T}_\mathfrak{p}$ along the origin splits over W, and hence we identify $\widehat{\mathbb{G}}_m\otimes_{\mathbb{Z}_p}\mathbf{\Gamma}^*$ with $\widehat{\mathbb{T}}_\mathfrak{p}$ regarding $\mathbb{Z}[I_\mathfrak{p}]\subset\mathrm{Hom}_{\mathbb{Z}_p}(\mathbf{\Gamma},W)$ by sending $\kappa_\mathfrak{p}$ to the homomorphism $\gamma_j\mapsto\sum_{\sigma\in I_\mathfrak{p}}\log_p(\sigma(\gamma_j)^{\kappa_{\mathfrak{p},\sigma}})/\log_p(\sigma(\gamma_j))$. This gives a rational structure on the formal torus $\widehat{\mathbb{G}}_m\otimes_{\mathbb{Z}_p}\mathbf{\Gamma}^*$ as a formal completion of the algebraic torus $\mathbb{T}_\mathfrak{p}$.

Fix a weight $\kappa\in\mathbb{Z}[I]^2$ satisfying $\kappa_2-\kappa_1\geq I$. A p-adic $\mathfrak{p}$-slope 0 analytic family of eigenforms $\mathcal{F}=\{\mathbf{f}_P|P\in\mathrm{Spec}(\mathbb{I})(\mathbb{C}_p)\}$ is indexed by points of $\mathrm{Spec}(\mathbb{I})(\mathbb{C}_p)$, where $\mathbb{I}$ is a torsion-free domain of finite rank over $\mathbf{\Lambda}$ (in this sense, we call $\mathrm{Spec}(\mathbb{I})$ a finite torsion-free covering of $\mathrm{Spec}(\mathbf{\Lambda})$). For each $P\in\mathrm{Spec}(\mathbb{I})(\mathbb{C}_p)$, $\mathbf{f}_P$ is a p-adic Hecke eigenform of $\mathfrak{p}$-slope 0 of level $\mathfrak{N}\mathfrak{p}^\infty$ for a fixed prime to $\mathfrak{p}$-level $\mathfrak{N}$. The family is called analytic because $P\mapsto a(y,\mathbf{f}_P)$ is a p-adic analytic function on the rigid analytic space associated to the formal spectrum $\mathrm{Spf}(\mathbb{I})$ in the sense of Berthelot (cf. [**dJ**, §7]). We call $P\in\mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$ *arithmetic* of weight $\kappa(P)=\kappa_\mathfrak{p}(P)+\kappa^\mathbf{p}\in\mathbb{Z}[I]$ ($\kappa_\mathfrak{p}(P)\in\mathbb{Z}[I_\mathfrak{p}]$) with character $\epsilon=(\epsilon_1,\epsilon_2,\epsilon_+)$ if $\kappa_2(P)-\kappa_1(P)\geq I$, $\epsilon_1|_\mathbf{\Gamma}$ has values in $\mu_{p^\infty}(\overline{\mathbb{Q}}_p)$ and $P(t_j-\epsilon_1^{-1}(\gamma_j)\gamma_j^{\kappa_{1,\mathfrak{p}}})=0$ for all j (regarding P as a W-algebra homomorphism $P:\mathbb{I}\to\overline{\mathbb{Q}}_p$). Here $\gamma^k=\prod_{\sigma\in I_\mathfrak{p}}\sigma(\gamma)^{k_\sigma}$ for $\gamma\in O_\mathfrak{p}$ and $k=\sum_{\sigma\in I_\mathfrak{p}}k_\sigma\sigma$, and $k\geq I$ means $k_\sigma\geq 1$ for all $\sigma\in I$. Classicity of a member of an analytic family of modular forms depends on weight $\kappa^\mathbf{p}$ outside $\mathfrak{p}$. The weight $(0,1)$ at an infinite place σ corresponds classical weight 2 at σ; so, weight $\kappa=(-I^\mathbf{p},I+I^\mathbf{p})$ is perfectly a $\mathfrak{p}$-arithmetic weight with $[\kappa]=1$.

Recall that we have fixed a weight $\kappa\in\mathbb{Z}[I]^2$ satisfying $\kappa_2-\kappa_1\geq I$. We remark that each maximal (non-constant) slope 0 family comes from an irreducible subscheme $\mathrm{Spec}(\mathbb{I})$ of the spectrum $\mathrm{Spec}(\mathbf{h}^{\mathfrak{p}.ord})$ of the big $\mathfrak{p}$-ordinary Hecke algebra $\mathbf{h}^{\mathfrak{p}.ord}$ of level $\mathfrak{N}\mathfrak{p}^\infty$, though the full proof of this fact in [**W**] is not published. Here $\mathbf{h}^{\mathfrak{p}.ord}$ is the Hecke algebra cut out by $\mathfrak{p}$-ordinary idempotent $e_\mathfrak{p}=\lim_{n\to\infty}\mathbb{T}(\mathfrak{p})^{n!}$ from the Hecke algebra of level $\mathfrak{N}\mathfrak{p}^\infty$ and fixed weight $\kappa^\mathbf{p}$ outside $I_\mathfrak{p}$). Since [**W**] is not widely available, we suppose the following facts (minimally necessary in this work) as axioms:

(A1) We have a $\mathbf{\Lambda}$-algebra $\mathbf{h}$ with specific element $\mathbb{T}(y)$ for each $y \in \widehat{O} \cap F_{\mathbb{A}}^{\times}$ which is finite torsion-free over $\mathbf{\Lambda}$;

(A2) For each arithmetic point $P \in \mathrm{Spec}(\mathbf{\Lambda})(\overline{\mathbb{Q}}_p) = \mathrm{Hom}_{W\text{-alg}}(\mathbb{I}, \overline{\mathbb{Q}}_p)$ with $\kappa(P) = \kappa$, we have an algebra homomorphism $\mathbf{h}$ to the $\mathfrak{P}$-ordinary part $h_\kappa^{\mathfrak{p}.ord}(\mathfrak{N}\mathfrak{p}^{r(P)+1}, \epsilon_P; \overline{\mathbb{Q}}_p)$ of the Hecke algebra of $S_\kappa^{\mathfrak{p}.ord}(\mathfrak{N}\mathfrak{p}^{r(P)+1}, \epsilon_P; \overline{\mathbb{Q}}_p)$ sending $\mathbb{T}(y)$ to $\mathbb{T}(y)$ inducing an embedding

$$\mathbf{h} \otimes_{\mathbf{\Lambda},P} \overline{\mathbb{Q}}_p \hookrightarrow h_\kappa^{\mathfrak{p}.ord}(\mathfrak{N}\mathfrak{p}^{r(P)+1}, \epsilon_P; \overline{\mathbb{Q}}_p).$$

By (A2), if P is arithmetic of weight κ, $\mathbf{f}_P$ is known to be a $\mathfrak{p}$-stabilized classical Hecke eigenform and has a set of "Neben characters" $\epsilon_P = (\epsilon_{P,1}, \epsilon_{P,2}, \epsilon_{P+})$ whose restriction to $\mathbf{\Gamma}$ is still written by ϵ_P. Since we can move essentially from one central character to another by Hecke character twist ($\pi_{\mathbf{f}} \mapsto \pi_{\mathbf{f}} \otimes \chi$), without losing generality, we may (and do) fix $\epsilon_{P+} = \epsilon_+$ throughout the paper. Thus the characters $\epsilon_{P,j}$ of $(O/\mathfrak{N}_j)^\times \times O_{\mathfrak{p}}^\times$ satisfy $\epsilon_{P,1}\epsilon_{P,2} = \epsilon_+|_{\widehat{O}^\times}$. We write $\mathfrak{p}^{r(P)+1}$ for the level of ϵ_P at $\mathfrak{p}$ in the sense that the classical $\mathfrak{p}$-stabilized form $\mathbf{f}_P$ is a minimal form of minimal level $\mathfrak{N}\mathfrak{p}^{r(P)+1}$; so, we have $r(P) \geq 0$. We assume now that $\mathrm{Spec}(\mathbb{I})$ is one of the irreducible components of $\mathrm{Spec}(\mathbf{h})$. See Section 1.9 for more details of Hecke algebras.

Though we stated the minimum facts necessary as the axioms (A1–2), actually we can vary $\kappa_{\mathfrak{p}}(P)$ in $\mathbb{Z}[I_{\mathfrak{p}}]$ (while fixing $\kappa^{\mathbf{p}}$), and if we take $\mathbf{h} = \mathbf{h}^{\mathfrak{p}.ord}$, the embedding in (A2) can be proven to be an isomorphism as long as $\kappa_{2,\mathfrak{p}} - \kappa_{1,\mathfrak{p}} \geq I_{\mathfrak{p}}$ (the proof of these facts will be given in [**HHA**]). Just for logical completeness, once a $\mathfrak{p}$-slope 0 family is given, rather than the precise control theorem asserting $\mathbf{h}^{\mathfrak{p}.ord} \otimes_{\mathbf{\Lambda},P} \overline{\mathbb{Q}}_p \cong h_\kappa^{\mathfrak{p}.ord}(\mathfrak{N}\mathfrak{p}^{r(P)+1}, \epsilon_P; \overline{\mathbb{Q}}_p)$, we only need the boundedness of the dimension of $S_\kappa^{\mathfrak{p}.ord}(\mathfrak{N}\mathfrak{p}^{r(P)+1}, \epsilon_P; \overline{\mathbb{Q}}_p)$ independent of arithmetic P with $\kappa(P) = \kappa$, which we will prove as Theorem 1.1. We can take $\mathbf{h}$ to be the image in $\mathbf{h}^{\mathfrak{p}.ord}$ of the full nearly p-ordinary Hecke algebra parameterizing p-slope 0 forms (in [**PAF**, Section 4.2]) assuming $p|\mathfrak{N}$, and then the assertions (A1–2) have been known for long time (see [**H89a**]).

Pick an infinite set $\mathcal{A}$ of arithmetic points in $\mathrm{Spec}(\mathbb{I})$ of weight $\kappa_{\mathfrak{p}}(P) = \kappa_{\mathfrak{p}}$; so, $\mathbf{f}_P$ for $P \in \mathcal{A}$ has fixed weight κ. We define the following Hecke field $\mathbb{Q}_{\mathcal{A}}(\mathcal{F})$ out of $\mathcal{F}$:

(H) $\mathbb{Q}_{\mathcal{A}}(\mathcal{F})$ is the composite of $\mathbb{Q}(\mathbf{f}_P)$ inside $\overline{\mathbb{Q}}$ for all arithmetic $P \in \mathcal{A}$.

Since $\mathbb{Q}_{\mathcal{A}}(\mathcal{F})$ contains the values of ϵ_j for all $\mathbf{f}_P$, it contains the cyclotomic $\mathbb{Z}_p^\times$-extension $\mathbb{Q}[\mu_{p^\infty}]$. If the family contains a theta series of weight κ of the norm form of a quadratic extension $M_{/F}$, M is a CM field, and all forms indexed by $\mathrm{Spec}(\mathbb{I})$ have CM by the same CM field M, and hence $\mathbb{Q}_{\mathcal{A}}(\mathcal{F})$ is contained in a finite extension of $\mathbb{Q}(\mu_{p^\infty})$ (Proposition 3.3).

We prove the following so-called horizontal theorem.

THEOREM. *Let the notation and the assumptions be as above. The field $\mathbb{Q}_{\mathcal{A}}(\mathcal{F})$ for a fixed weight κ (with $\kappa_2 - \kappa_1 \geq I$) is a finite extension of $K := \mathbb{Q}(\mu_{p^\infty})$ if, and only if $\mathcal{F}$ contains some theta series of weight κ of a CM quadratic extension of F. Moreover, for a non-CM family $\mathcal{F}$, we have*

$$\lim_{P \in \mathcal{A}} [K(\mathbf{f}_P) : K] = \infty,$$

where the limit is taken with respect to the filter of $\mathcal{A}$ made of (all) complements of finite subsets of $\mathcal{A}$.

We will prove a stronger version of the horizontal theorem as Theorem 3.1 in the text. Note that the Zariski closure of $\mathcal{A}$ in $\mathrm{Spf}(\mathbb{I})$ may have as its image in $\widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \mathbf{\Gamma}^*_{/W}$ a totally transcendental formal subscheme of $\widehat{\mathbb{T}}_{\mathfrak{p}/W}$ (non-algebraizable to any closed subscheme of $\mathbb{T}_{\mathfrak{p}}$) as it may not have arithmetic points of weight different from κ. There is a vertical version varying the weight $\kappa_{\mathfrak{p}}$ (while fixing $\kappa^{\mathbf{p}}$) as discussed in [**H11**] for elliptic cusp forms, but we do not touch it in this paper.

As was in [**H11**], the proof of the above theorem is based on the finiteness of Weil l-numbers of given weight in $\mathbb{Q}[\mu_{p^\infty}]$ up to multiplication by roots of unity and a rigidity lemma on a formal subscheme of the formal torus $\widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \mathbf{\Gamma}^*$ for $\mathbf{\Gamma}^* = \mathrm{Hom}_{\mathbb{Z}_p}(\mathbf{\Gamma}, \mathbb{Z}_p)$ stable under the action $t \mapsto t^z$ for $t \in \widehat{\mathbb{G}}_m$ (see Section 4). A main difficulty we encountered in trying to generalize the method of [**H11**] in the Hilbert modular case is that in the elliptic modular case, the Zariski closure $\overline{\mathcal{A}}$ of $\mathcal{A}$ projects down to the full torus $\widehat{\mathbb{G}}_m$; so, $\overline{\mathcal{A}}$ is algebraic, and we could change weight of arithmetic points. In the many variable situation, as already mentioned, the image of the closure of $\mathcal{A}$ can have a transcendental image in $\widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \mathbf{\Gamma}^*$, and hence all the arithmetic points in the closure of $\mathcal{A}$ may have a constant weight (so, we lose freedom of varying weight). Another difficulty is that our rigidity lemmas proven in [**H11**] are given for power series of one variable (i.e., given for one dimensional multiplicative formal groups). We prove a rigidity lemma (Lemma 4.1) in several variables barely sufficient to prove our claim (and therefore, the lemma is not an optimal form we hoped to prove; see [**C1**, Remark 6.6.1 (iv)]).

For a finite Galois extension E/F, we can consider a $\mathrm{Gal}(E/F)$-invariant analytic family for the Hecke algebra $\mathbf{h}_{/E}$ of $GL(2)_{/E}$. If the base-change from F to E is proven, this family comes from a family for $GL(2)_{/F}$; so, the above theorem tells us the behavior of Hecke fields for such Galois invariant families. Since we do not know the existence of base-change in general, we briefly describe the relative case at the end of this paper (see Theorem 7.2).

Here are general notation used in this paper. We denote by a Gothic letter an ideal of a number field. The corresponding Roman letter denotes the residual characteristic if a Gothic letter is used for a prime ideal. For each prime $\mathfrak{l}$ of F, we write $F_{\mathfrak{l}}$ for the $\mathfrak{l}$-adic completion of F, and for an integral ideal $\mathfrak{a}$, we put $F_{\mathfrak{a}} = \prod_{\mathfrak{l}|\mathfrak{a}} F_{\mathfrak{l}}$. For an idele $x \in F_{\mathbb{A}}^\times$, let $x_{\mathfrak{a}}$ (resp. x_∞) denote the projection of x to $F_{\mathfrak{a}}^\times$ (resp. to $F_\infty^\times = (F \otimes_{\mathbb{Q}} \mathbb{R})^\times$). Then we write $x = x^{(\mathfrak{a})} x_{\mathfrak{a}}$ and $x = x^{(\mathfrak{a}\infty)} x_{\mathfrak{a}} x_\infty$ and put $F_{\mathbb{A}}^{(\mathfrak{a}\infty)} = \{x^{(\mathfrak{a}\infty)} | x \in F_{\mathbb{A}}^\times\}$. Put $G = \mathrm{Res}_{O/\mathbb{Z}} GL(2)$ with center Z and the maximal diagonal torus T_0. For a local ring A, its maximal ideal is denoted by $\mathfrak{m}_A$.

Contents

1. Hilbert Modular Forms of $\mathfrak{p}$-power level

We give a brief summary of arithmetic theory of Hilbert modular forms and p-adic modular forms of level $\mathfrak{N}\mathfrak{p}^\infty$. The theory is similar to the one of level $\mathfrak{N}p^\infty$, but the description of the theory seems not to be found in the existing literature. Hence giving an account would be useful, though logically speaking, the results necessary for this work are in Subsections 1.8–1.14 (and the earlier sections are to prepare the readers for the results in these subsections). Perhaps, new results appear starting with §1.11, and if the reader is familiar with the subjects, he or she can go directly to Section 2.

1.1. Abelian varieties with real multiplication. Put $O^* = \{x \in F | \mathrm{Tr}(xO) \subset \mathbb{Z}\}$ (which is the different inverse $\mathfrak{d}^{-1}$). Recall the level ideal $\mathfrak{N}$, and fix a fractional ideal $\mathfrak{c}$ of F prime to $p\mathfrak{N}$. We write A for a fixed base algebra, in which $N(\mathfrak{c})$ and the prime-to-p part of $N(\mathfrak{N})$ is invertible. To include the case where p ramifies in the base field F, we use the moduli problem of Deligne–Pappas in [**DeP**] to define the Hilbert modular variety. As explained in [**Z**, Sections 2 and 3], if p is unramified in F, the resulting p-integral model of the Hilbert modular Shimura variety is canonically isomorphic to the one defined by Kottwitz (described also in [**PAF**, Chapter 4]). Let $\mathfrak{c}$ be an integral ideal of F prime to p. Writing $\mathfrak{c}_+$ for the monoid of totally positive elements in $\mathfrak{c}$, giving data $(\mathfrak{c}, \mathfrak{c}_+)$ is equivalent to fix a strict ideal class of $\mathfrak{c}$. The Hilbert modular variety $\mathfrak{M} = \mathfrak{M}(\mathfrak{c}; \mathfrak{N})$ of level $\mathfrak{N}$ classifies triples $(X, \Lambda, i)_{/S}$ formed by

- An abelian scheme $\pi : X \to S$ of relative dimension $d = [F : \mathbb{Q}]$ over an A–scheme S with an embedding: $O \hookrightarrow \mathrm{End}(X_{/S})$;
- An O–linear polarization $X \otimes \mathfrak{c} \underset{\sim}{\xrightarrow{\Lambda}} X^t := \mathrm{Pic}^0_{X/S}$ inducing an isomorphism $(\mathfrak{c}, \mathfrak{c}_+) \cong (\mathrm{Hom}^{Sym}_S(X_{/S}, X^t_{/S}), \mathcal{P}(X, X^t_{/S}))$, where $\mathrm{Hom}^{Sym}_S(X_{/S}, X^t_{/S})$ is the O-module of symmetric O-linear homomorphisms and $\mathcal{P}(X, X^t_{/S}) \subset \mathrm{Hom}^{Sym}_S(X_{/S}, X^t_{/S})$ is the positive cone made up of O-linear polarizations;
- A closed O–linear immersion $i = i_{\mathfrak{N}} : (\mathbb{G}_m \otimes_{\mathbb{Z}} O^*)[\mathfrak{N}] \hookrightarrow X$ for the group $(\mathbb{G}_m \otimes_{\mathbb{Z}} O^*)[\mathfrak{N}]$ of $\mathfrak{N}$-torsion points of the multiplicative O-module scheme $\mathbb{G}_m \otimes_{\mathbb{Z}} O^*$.

By Λ, we identify the O–module $\mathrm{Hom}^{Sym}_S(X_{/S}, X^t_{/S})$ of symmetric O–linear homomorphisms inside $\mathrm{Hom}_S(X_{/S}, X^t_{/S})$ with $\mathfrak{c}$. We require that the (multiplicative) monoid of symmetric O–linear isogenies induced locally by ample invertible sheaves be identified with the set of totally positive elements $\mathfrak{c}_+ \subset \mathfrak{c}$. The quasi projective scheme $\mathfrak{M} = \mathfrak{M}(\mathfrak{c}; \mathfrak{N})_{/A}$ is the coarse moduli scheme of the following functor $\wp$ from the category of A–schemes into the category $SETS$: $\wp(S) = \left[(X, \Lambda, i)_{/S}\right]$, where $[\ \ ] = \{\ \ \}/\cong$ is the set of isomorphism classes of the objects inside the brackets, and $(X, \Lambda, i) \cong (X', \Lambda', i')$ if we have an O–linear isomorphism $\phi : X_{/S} \to X'_{/S}$ such that $\Lambda' = \phi^t \circ \Lambda \circ (\phi \otimes 1)$ and $i'^* \circ \phi = i^* (\Leftrightarrow \phi \circ i = i')$. The scheme $\mathfrak{M}$ is a fine moduli if $\mathfrak{N}$ is sufficiently deep (see [**DeP**]).

1.2. Geometric Hilbert modular forms. In the definition of the functor $\wp$ in §1.1, we could impose local $\mathcal{O}_S \otimes_{\mathbb{Z}} O$-freeness of the $\mathcal{O}_S \otimes_{\mathbb{Z}} O$-module $\pi_*(\Omega_{X/S})$ as was done by Rapoport in [**R**]. We consider an open subfunctor $\wp^R$ of $\wp$ which is defined by imposing locally freeness of $\pi_*(\Omega_{X/S})$ over $\mathcal{O}_S \otimes_{\mathbb{Z}} O$. Over $\mathbb{Z}[\frac{1}{D_F}]$ for the discriminant D_F of F, the two functors $\wp^R$ and $\wp$ coincide (see [**DeP**]). We write $\mathfrak{M}^R(\mathfrak{c};\mathfrak{N})$ for the open subscheme of $\mathfrak{M}(\mathfrak{c};\mathfrak{N})$ representing $\wp^R$. For ω with $\pi_*(\Omega_{X/S}) = (\mathcal{O}_S \otimes_{\mathbb{Z}} O)\omega$, we consider the functor classifying quadruples (X, Λ, i, ω):

$$\mathcal{Q}(S) = \left[(X, \Lambda, i, \omega)_{/S}\right].$$

Let $T = \mathrm{Res}_{O/\mathbb{Z}}\mathbb{G}_m$. We let $a \in T(S) = H^0(S, (\mathcal{O}_S \otimes_{\mathbb{Z}} O)^\times)$ act on $\mathcal{Q}(S)$ by $(X, \Lambda, i, \omega) \mapsto (X, \Lambda, i, a\omega)$. By this action, $\mathcal{Q}$ is a T–torsor over the open subfunctor $\wp^R$ of $\wp$; so, $\mathcal{Q}$ is representable by an A–scheme $\mathcal{M} = \mathcal{M}(\mathfrak{c};\mathfrak{N})$ affine over $\mathfrak{M}^R = \mathfrak{M}^R(\mathfrak{c};\mathfrak{N})_{/A}$. For each weight $k \in X^*(T) = \mathrm{Hom}_{\text{gp-sch}}(T, \mathbb{G}_m)$, if $F \neq \mathbb{Q}$, the k^{-1}–eigenspace of $H^0(\mathcal{M}_{/A}, \mathcal{O}_{\mathcal{M}/A})$ is the space of modular forms of weight k integral over a ring A. We write $G_k(\mathfrak{c}, \mathfrak{N}; A)$ for this space of A–integral modular forms, which is an A–module of finite type. When $F = \mathbb{Q}$, we need to take the subsheaf of sections with logarithmic growth towards cusps (the condition (G0) below). Thus $f \in G_k(\mathfrak{c}, \mathfrak{N}; A)$ is a functorial rule (i.e., a natural transformation $f : \mathcal{Q} \to \mathbb{G}_a$) assigning a value in B to each isomorphism class of $(X, \Lambda, i, \omega)_{/B}$ (defined over an A–algebra B) satisfying the following three conditions:

(G1) $f(X, \Lambda, i, \omega) \in B$ if (X, Λ, i, ω) is defined over B;
(G2) $f((X, \Lambda, i, \omega) \otimes_B B') = \rho(f(X, \Lambda, i, \omega))$ for each morphism $\rho : B_{/A} \to B'_{/A}$;
(G3) $f(X, \Lambda, i, a\omega) = k(a)^{-1} f(X, \Lambda, i, \omega)$ for $a \in T(B)$.

By abusing the language, we pretend f to be a function of isomorphism classes of test objects $(X, \Lambda, i, \omega)_{/B}$ hereafter. The sheaf of k^{-1}–eigenspace $\mathcal{O}_{\mathcal{M}}[k^{-1}]$ under the action of T is an invertible sheaf on $\mathfrak{M}^R_{/A}$. We write this sheaf as $\underline{\omega}^k$ (imposing (G0) when $F = \mathbb{Q}$). Then we have

$$G_k(\mathfrak{c}, \mathfrak{N}; A) \cong H^0(\mathfrak{M}^R(\mathfrak{c};\mathfrak{N})_{/A}, \underline{\omega}^k_{/A}) \text{ canonically,}$$

as long as $\mathfrak{M}^R(\mathfrak{c};\mathfrak{N})$ is a fine moduli space. Writing $\underline{\mathbb{X}} := (\mathbb{X}, \boldsymbol{\lambda}, \mathbf{i}, \boldsymbol{\omega})$ for the universal abelian scheme over $\mathfrak{M}^R$, $s = f(\underline{\mathbb{X}})\boldsymbol{\omega}^k$ ($\boldsymbol{\omega}^k = \overbrace{\boldsymbol{\omega} \otimes \cdots \otimes \boldsymbol{\omega}}^{k}$) gives rise to the section of $\underline{\omega}^k$. Conversely, for any section $s \in H^0(\mathfrak{M}^R(\mathfrak{c};\mathfrak{N}), \underline{\omega}^k)$, taking a unique morphism $\phi : \mathrm{Spec}(B) \to \mathfrak{M}^R$ such that $\phi^*\underline{\mathbb{X}} = \underline{X}$ for $\underline{X} := (X, \Lambda, i, \omega)_{/B}$, we can define $f \in G_k$ by $\phi^* s = f(\underline{X})\omega^k$.

We suppose that the fractional ideal $\mathfrak{c}$ is prime to $\mathfrak{N}p$, and take two ideals $\mathfrak{a}$ and $\mathfrak{b}$ prime to $\mathfrak{N}p$ such that $\mathfrak{a}\mathfrak{b}^{-1} = \mathfrak{c}$. To $(\mathfrak{a}, \mathfrak{b})$, we attach the Tate AVRM $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)$ (symbolically given as $\mathbb{G}_m \otimes \mathfrak{a}^*/q^{\mathfrak{b}}$ in [**K**]) defined over the completed group ring $\mathbb{Z}((\mathfrak{a}\mathfrak{b}))$ made of formal series $f(q) = \sum_{\xi \gg -\infty} a(\xi) q^\xi$ $(a(\xi) \in \mathbb{Z})$. Here ξ runs over all elements in $\mathfrak{a}\mathfrak{b}$, and there exists a positive integer n (dependent on f) such that $a(\xi) = 0$ if $\sigma(\xi) < -n$ for some $\sigma \in I$. We write $A[[(\mathfrak{a}\mathfrak{b})_{\geq 0}]]$ for the subring of $A[[\mathfrak{a}\mathfrak{b}]]$ made of formal series f with $a(\xi) = 0$ for all ξ with $\sigma(\xi) < 0$ for at least one embedding $\sigma : F \hookrightarrow \mathbb{R}$.

Strictly speaking, $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)$ is defined over $\mathbb{Z}[[C^*]] := \mathbb{Z}[[q^\xi]]_{\xi \in C^*}$ for the dual cone C^* under the trace pairing of a cone C inside $\mathfrak{a}\mathfrak{b}_{\geq 0} = \{\xi \gg 0 | \xi \in \mathfrak{a}\mathfrak{b}\}$. When we evaluate a modular form at $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)$, it has values in $\mathbb{Z}[[\mathfrak{a}\mathfrak{b}_{\geq 0}]] = \bigcap_C \mathbb{Z}[[C^*]]$; so, by abusing the language, we proceed as if $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)$ were defined over the non-noetherian ring $\mathbb{Z}[[\mathfrak{a}\mathfrak{b}_{\geq 0}]]$. In addition, we skipped a step of introducing the toroidal compactification of $\mathfrak{M}^R$ and $\mathfrak{M}$ (done in [**R**] and [**DeP**]) whose (completed) stalk at the cusp corresponding to $(\mathfrak{a}, \mathfrak{b})$ actually carries $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)$. The scheme $\mathfrak{M}_{/A}$ is proper normal by [**DeP**] and hence by Zariski's connected theorem, it is geometrically connected. Since $\mathfrak{M}^R$ is open dense in each fiber of $\mathfrak{M}$ (as shown by [**DeP**]), it is geometrically connected. Therefore the q-expansion principle holds for $H^0(\mathfrak{M}^R(\mathfrak{c};\mathfrak{N}),\underline{\omega}^k)$. We refer details of these facts to [**K**, Chapter I], [**C**], [**DiT**], [**Di**], [**DeP**], [**HiT**, Section 1] and [**PAF**, 4.1.4]. The scheme $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)$ can be extended to a semi-abelian scheme over the toroidal compactification adding the fiber $\mathbb{G}_m \otimes \mathfrak{a}^*$ over the cusp "$(\mathfrak{a},\mathfrak{b})$". Since $\mathfrak{a}$ is prime to p, $\mathfrak{a}_p = O_p$. Thus if A is a $\mathbb{Z}_p$–algebra, we have a canonical isomorphism:

$$Lie(\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q) \mod \mathfrak{A}) = Lie(\mathbb{G}_m \otimes \mathfrak{a}^*) \cong A \otimes_{\mathbb{Z}} \mathfrak{a}^* \cong A \otimes_{\mathbb{Z}} O^*.$$

By duality, we have $\Omega_{\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)/A[[(\mathfrak{ab})_{\geq 0}]]} \cong A[[(\mathfrak{ab})_{\geq 0}]]$. Indeed we have a canonical generator ω_{can} of $\Omega_{\mathrm{Tate}(q)}$ induced by $\frac{dt}{t} \otimes 1$ on $\widehat{\mathbb{G}}_m \otimes \mathfrak{a}^* = \widehat{\mathbb{G}}_m \otimes O_p$ for the formal completion $\widehat{\mathbb{G}}_m \otimes \mathfrak{a}^*$ sat the identity $1 \in \widehat{\mathbb{G}}_m \otimes \mathfrak{a}^*(\overline{\mathbb{F}}_p)$ of $\widehat{\mathbb{G}}_m \otimes \mathfrak{a}^*$. We have a canonical inclusion $(\mathbb{G}_m \otimes O^*)[\mathfrak{N}] = (\mathbb{G}_m \otimes \mathfrak{a}^*)[\mathfrak{N}]$ into $\mathbb{G}_m \otimes \mathfrak{a}^*$, which induces a canonical closed immersion $i_{can} : (\mathbb{G}_m \otimes O^*)[\mathfrak{N}] \hookrightarrow \mathrm{Tate}(q)$. As described in [**K**, (1.1.14)] and [**HiT**, page 204], $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)$ has a canonical $\mathfrak{c}$–polarization Λ_{can}. Thus we can evaluate $f \in G_k(\mathfrak{c},\mathfrak{N};A)$ at $(\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q),\Lambda_{can},i_{can},\omega_{can})$. The value $f(q) = f_{\mathfrak{a},\mathfrak{b}}(q)$ actually falls in $A[[(\mathfrak{ab})_{\geq 0}]]$ (if $F \neq \mathbb{Q}$: Koecher principle) and is called the q–expansion at the cusp $(\mathfrak{a},\mathfrak{b})$. When $F = \mathbb{Q}$, we impose f to have values in the power series ring $A[[(\mathfrak{ab})_{\geq 0}]]$:

(G0) $f_{\mathfrak{a},\mathfrak{b}}(q) \in A[[(\mathfrak{ab})_{\geq 0}]]$ for all $(\mathfrak{a},\mathfrak{b})$.

1.3. p-Adic Hilbert modular forms of level $\mathfrak{N}\mathfrak{p}^\infty$. Let $\mathfrak{p}$ be the p-adic place of F induced by ι_p, and regard $\mathfrak{p}$ as a prime ideal of O. Suppose that $A = \varprojlim_n A/p^nA$ (such a ring is called a p-adic ring) and that $\mathfrak{N}$ is prime to $\mathfrak{p}$. Put $\mathbf{p} = \prod_{\mathfrak{p}'|p,\mathfrak{p}'\neq\mathfrak{p}} \mathfrak{p}'$ for the product taken over all prime factors $\mathfrak{p}'|p$ different from $\mathfrak{p}$. Since $O \otimes_{\mathbb{Z}} A = O \otimes_{\mathbb{Z}} \mathbb{Z}_p \otimes_{\mathbb{Z}_p} A = O_p \otimes_{\mathbb{Z}_p} A = A_{\mathfrak{p}} \times A_{\mathbf{p}}$ for $O_p = O_{\mathfrak{p}} \times O_{\mathbf{p}}$, $A_{\mathfrak{p}} = A \otimes_{\mathbb{Z}_p} O_{\mathfrak{p}}$ and $A_{\mathbf{p}} = A \otimes_{\mathbb{Z}_p} O_{\mathbf{p}}$, we can think of the $O_{\mathbf{p}}$-part $\Omega_{X/A,\mathbf{p}} = A_{\mathbf{p}}\Omega_{X/A} = O_{\mathbf{p}}\Omega_{X/A}$ for p-adic rings A. We consider a functor into sets

$$\widehat{\wp}(A) = \left[(X,\Lambda,i_{\mathfrak{p}},\omega^{\mathbf{p}},i_{\mathfrak{N}})_{/S}\right]$$

defined over the category of p–adic A–algebras $B = \varprojlim_n B/p^nB$. Here $\omega^{\mathbf{p}}$ is a generator over $B_{\mathbf{p}}$ of $\pi_*(\Omega_{X/B,\mathbf{p}})$. An important point is that we consider an embedding of ind-group schemes $i_{\mathfrak{p}} : \mu_{p^\infty} \otimes_{\mathbb{Z}_p} O_{\mathfrak{p}}^* \hookrightarrow X[\mathfrak{p}^\infty]$ (in place of the $\mathfrak{p}$-part $\omega_{\mathfrak{p}}$ of a differential ω), which induces $\widehat{\mathbb{G}}_m \otimes O_{\mathfrak{p}}^* \cong \widehat{X}_{\mathfrak{p}} := \widehat{X} \otimes_O O_{\mathfrak{p}}$ for the formal completion $\widehat{X}$ along the identity section of the characteristic p–fiber of the abelian scheme X over A.

We call an AVRM X over a characteristic p ring A $\mathfrak{p}$-ordinary if the Barsotti–Tate group $X[\mathfrak{p}^\infty]$ is ordinary; in other words, its (Frobenius) Newton polygon has only two slopes 0 and 1. In the moduli $\mathfrak{M}(\mathfrak{c};\mathfrak{N})_{/\mathbb{F}_p}$, locally under Zariski topology, $\mathfrak{p}$-ordinary locus is an open dense subscheme of $\mathfrak{M}(\mathfrak{c};\mathfrak{N})$ (see [**G**] and [**Z**, Section 3]). Indeed, the locus is obtained by inverting the partial Hasse invariant $\prod_i h_{\mathfrak{p},i}$ given in [**AG**, §7.12] (over $\mathfrak{M}(\mathfrak{c};\mathfrak{N})_{/\mathbb{F}_p}$). So, the $\mathfrak{p}$-ordinary locus $\mathfrak{M}^{\mathfrak{p}.ord}(\mathfrak{c};\mathfrak{N})$ inside $\mathfrak{M}^R(\mathfrak{c};\mathfrak{N})$ is open dense in $\mathfrak{M}^R(\mathfrak{c};\mathfrak{N})$. Let $\widehat{\mathfrak{M}}^R(\mathfrak{c};\mathfrak{N})$ be the formal completion of $\mathfrak{M}^R(\mathfrak{c};\mathfrak{N})$ along $\mathfrak{M}^{\mathfrak{p}.ord}(\mathfrak{c};\mathfrak{N})_{/\mathbb{F}_p}$ with universal abelian scheme $\mathbf{A}$.

By a standard argument, the functor

$$\mathcal{I}_{\mathfrak{p}}(A) = \left[(X,\Lambda,i_{\mathfrak{p}},i_{\mathfrak{N}})_{/S}\right]$$

is represented by the Igusa $\mathfrak{p}$-tower

$$Ig_{\mathfrak{p}} := \mathrm{Isom}_{\widehat{\mathfrak{M}}^R(\mathfrak{c};\mathfrak{N})}(\mu_{p^\infty} \otimes_{\mathbb{Z}_p} O_{\mathfrak{p}}^*, \mathbf{A}[\mathfrak{p}^\infty]^\circ)$$

for the connected component $\mathbf{A}[\mathfrak{p}^\infty]^\circ$. Then $\widehat{\wp}$ is represented by a $T_{\mathbf{p}}$-torsor $\widetilde{Ig}_{\mathfrak{p}}$ over $Ig_{\mathfrak{p}}$ for $T_{\mathbf{p}} := \mathrm{Res}_{O_{\mathbf{p}}/\mathbb{Z}_p}\mathbb{G}_m$ (via the multiplication action $\omega^{\mathbf{p}} \mapsto a\omega^{\mathbf{p}}$ for $a \in T(A)$); so, $\widehat{\wp}$ is represented by the scheme $\widetilde{Ig}_{\mathfrak{p}}$ (e.g., [**PAF**, 4.1.9]).

Taking a character $k \in \mathbb{Z}[I]$, write $k^{\mathbf{p}} \in \mathbb{Z}[I^{\mathbf{p}}]$ for its projection to the $I^{\mathbf{p}}$-part. Then we regard it as a character of $T_{\mathbf{p}}$. A $\mathfrak{p}$–adic modular form $f_{/A}$ of weight $k^{\mathbf{p}}$ over a p–adic ring A is a function (strictly speaking, a functorial rule) of isomorphism classes of $(X, \Lambda, i_{\mathfrak{p}}, \omega^{\mathbf{p}}, i_{\mathfrak{N}})_{/B}$ ($i_{\mathfrak{N}} : \mathbb{G}_m \otimes_{\mathbb{Z}} O^*[\mathfrak{N}] \hookrightarrow X$) satisfying the following three conditions:

- (P1) $f(X, \Lambda, i_{\mathfrak{p}}, \omega^{\mathbf{p}}, i_{\mathfrak{N}}) \in B$ if $(X, \Lambda, i_{\mathfrak{p}}, \omega^{\mathbf{p}}, i_{\mathfrak{N}})$ is defined over B;
- (P2) $f((X, \Lambda, i_{\mathfrak{p}}, \omega^{\mathbf{p}}, i_{\mathfrak{N}}) \otimes_B B') = \rho(f(X, \Lambda, i_{\mathfrak{p}}, \omega^{\mathbf{p}}, i_{\mathfrak{N}}))$ for each continuous A–algebra homomorphism $\rho : B \to B'$;
- (P3) $f_{\mathfrak{a},\mathfrak{b}}(q) \in A[[(\mathfrak{ab})_{\geq 0}]]$ for all $(\mathfrak{a}, \mathfrak{b})$ prime to $\mathfrak{N}p$ (this condition is automatic if $F \neq \mathbb{Q}$ by Koecher principle);
- (P4) $f(X, \Lambda, i_{\mathfrak{p}}, a\omega^{\mathbf{p}}, i_{\mathfrak{N}}) = a^{-k^{\mathbf{p}}} f(X, \Lambda, i_{\mathfrak{p}}, \omega^{\mathbf{p}}, i_{\mathfrak{N}})$ for all $a \in T(A)$.

We write $V_{k^{\mathbf{p}}}(\mathfrak{c}, \mathfrak{N}\mathfrak{p}^\infty; A)$ for the space of p–adic modular forms satisfying (P1-4). This space $V_{k^{\mathbf{p}}}(\mathfrak{c}, \mathfrak{N}\mathfrak{p}^\infty; A)$ is a p–adically complete A–module and is the $k^{\mathbf{p}}$ eigenspace of $H^0(\widetilde{Ig}_{\mathfrak{p}}, \mathcal{O}_{\widetilde{Ig}_{\mathfrak{p}}})$ under the action of $T_{\mathbf{p}}$.

The q-expansion principle is valid both for classical modular forms and p-adic modular forms f:

(q-exp) *The q–expansion: $f \mapsto f_{\mathfrak{a},\mathfrak{b}}(q) \in A[[(\mathfrak{ab})_{\geq 0}]]$ determines f uniquely.*

This follows from the irreducibility of the level $\mathfrak{p}^\infty$ Igusa tower, which can be proven as in [**DeR**] and [**PAF**, 4.2.4]; the argument in [**PAF**] proving irreducibility also works well for the partial tower.

Since $\widehat{\mathbb{G}}_m \otimes O^*$ has a canonical invariant differential $\frac{dt}{t}$, we have $\omega_{\mathfrak{p}} = i_{\mathfrak{p},*}(\frac{dt}{t})$ on $X_{/B}$ (under the notation of (P1–4)). Since over the p-adic ring B, ω_p can be written as a sum of the $O_{\mathfrak{p}}$-eigen part $\omega_{\mathfrak{p}}$ and the $O_{\mathbf{p}}$-eigen part $\omega^{\mathbf{p}}$ uniquely, the sum $\omega_{\mathfrak{p}} + \omega^{\mathbf{p}}$ is a generator over $O \otimes_{\mathbb{Z}} B$ of $H^0(X, \Omega_{X/B})$. This allows us to regard $f \in G_k(\mathfrak{c}, \mathfrak{N}; A)$ a p–adic modular form by

$$f(X, \Lambda, i_{\mathfrak{p}}, \omega^{\mathbf{p}}, i_{\mathfrak{N}}) := f(X, \Lambda, i_{\mathfrak{N}}, \omega_{\mathfrak{p}} + \omega^{\mathbf{p}}).$$

By (q-exp), this gives an injection of $G_k(\mathfrak{c}, \mathfrak{N}; A)$ into $V_{k^{\mathbf{p}}}(\mathfrak{c}, \mathfrak{N}\mathfrak{p}^\infty; A)$ preserving q–expansions.

1.4. Complex analytic Hilbert modular forms. Over $\mathbb{C}$, the category of test objects (X, Λ, i, ω) is equivalent to the category of triples $(\mathcal{L}, \Lambda, i)$ made of the following data (by the theory of theta functions): $\mathcal{L}$ is an O–lattice in $O \otimes_{\mathbb{Z}} \mathbb{C} = \mathbb{C}^I$, an alternating pairing $\Lambda : \mathcal{L} \wedge_O \mathcal{L} \cong \mathfrak{c}^*$ and $i : \mathfrak{N}^*/O^* \hookrightarrow F\mathcal{L}/\mathcal{L}$. The alternating form Λ is supposed to be positive in the sense that $\Lambda(u, v)/\mathrm{Im}(uv^c)$ is totally positive definite. The differential ω can be recovered by $\iota : X(\mathbb{C}) = \mathbb{C}^I/\mathcal{L}$ so that $\omega = \iota^* du$ where $u = (u_\sigma)_{\sigma \in I}$ is the variable on $\mathbb{C}^I$. Conversely

$$\mathcal{L}_X = \left\{ \int_\gamma \omega \in O \otimes_{\mathbb{Z}} \mathbb{C} \middle| \gamma \in H_1(X(\mathbb{C}), \mathbb{Z}) \right\}$$

is a lattice in $\mathbb{C}^I$, and the polarization $\Lambda : X^t \cong X \otimes \mathfrak{c}$ induces $\mathcal{L} \wedge \mathcal{L} \cong \mathfrak{c}^*$.

Using this equivalence, we can relate our geometric definition of Hilbert modular forms with the classical analytic definition. Define $\mathfrak{Z}$ by the product of I copies of the upper half complex plane $\mathfrak{H}$. We regard $\mathfrak{Z} \subset F \otimes_{\mathbb{Q}} \mathbb{C} = \mathbb{C}^I$. For each $z \in \mathfrak{Z}$, we define

$$\mathcal{L}_z = 2\pi\sqrt{-1}(\mathfrak{b}z + \mathfrak{a}^*),\ \Lambda_z(2\pi\sqrt{-1}(az+b), 2\pi\sqrt{-1}(cz+d)) = -(ad-bc) \in \mathfrak{c}^*$$

with $i_z : \mathfrak{N}^*/O^* \to \mathbb{C}^I/\mathcal{L}_z$ given by $i_z(a \mod O^*) = (2\pi\sqrt{-1}a \mod \mathcal{L}_z)$.

Consider the following congruence subgroup $\Gamma_1^1(\mathfrak{N};\mathfrak{a},\mathfrak{b})$ given by

$$\left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(F) \middle| a, d \in O,\ b \in (\mathfrak{a}\mathfrak{b})^*,\ c \in \mathfrak{N}\mathfrak{a}\mathfrak{b}\mathfrak{d} \text{ and } d-1 \in \mathfrak{N} \right\}.$$

Write $\Gamma_1^1(\mathfrak{c};\mathfrak{N}) = \Gamma_1^1(\mathfrak{N}; O, \mathfrak{c}^{-1})$. We let $g = (g_\sigma) \in SL_2(F \otimes_{\mathbb{Q}} \mathbb{R}) = SL_2(\mathbb{R})^I$ act on $\mathfrak{Z}$ by linear fractional transformation of g_σ on each component z_σ. It is easy to verify

$$(\mathcal{L}_z, \Lambda_z, i_z) \cong (\mathcal{L}_w, \Lambda_w, i_w) \iff w = \gamma(z) \text{ for } \gamma \in \Gamma_1^1(\mathfrak{N};\mathfrak{a},\mathfrak{b}).$$

The set of pairs $(\mathfrak{a},\mathfrak{b})$ with $\mathfrak{a}\mathfrak{b}^{-1} = \mathfrak{c}$ is in bijection with the set of cusps (unramified over ∞) of $\Gamma_1^1(\mathfrak{N};\mathfrak{a},\mathfrak{b})$. Two cusps are equivalent if they transform each other by an element in $\Gamma_1^1(\mathfrak{N};\mathfrak{a},\mathfrak{b})$. The standard choice of the cusp is $(O, \mathfrak{c}^{-1})$, which we call the infinity cusp of $\mathfrak{M}(\mathfrak{c};\mathfrak{N})$. For each ideal $\mathfrak{t}$, $(\mathfrak{t}, \mathfrak{t}\mathfrak{c}^{-1})$ gives another cusp. The two cusps $(\mathfrak{t}, \mathfrak{t}\mathfrak{c}^{-1})$ and $(\mathfrak{s}, \mathfrak{s}\mathfrak{c}^{-1})$ are equivalent under $\Gamma_1^1(\mathfrak{c};\mathfrak{N})$ if $\mathfrak{t} = \alpha\mathfrak{s}$ for an element $\alpha \in F^\times$ with $\alpha \equiv 1 \mod \mathfrak{N}$ in $F_{\mathfrak{N}}^\times$. We have

$$\mathfrak{M}(\mathfrak{c};\mathfrak{N})(\mathbb{C}) \cong \Gamma_1^1(\mathfrak{c};\mathfrak{N})\backslash\mathfrak{Z}, \text{ canonically.}$$

Take the following open compact subgroup of $G(\mathbb{A}^{(\infty)})$:

$$U_1^1(\mathfrak{N}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(\widehat{\mathbb{Z}}) \middle| c \in \mathfrak{N}\widehat{O} \text{ and } a \equiv d \equiv 1 \mod \mathfrak{N}\widehat{O} \right\},$$

and put $K = K_1^1(\mathfrak{N}) = \left(\begin{smallmatrix} d & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1} U_1^1(\mathfrak{N}) \left(\begin{smallmatrix} d & 0 \\ 0 & 1 \end{smallmatrix}\right)$ for an idele d with $d\widehat{O} = \widehat{\mathfrak{d}}$ and $d_{\mathfrak{d}} = 1$. Here for an idele and an O-ideal $\mathfrak{a} \neq 0$, we write $x_{\mathfrak{a}}$ for the projection of x to $\prod_{\mathfrak{l}|\mathfrak{a}} F_{\mathfrak{l}}^\times$ and $x^{(\mathfrak{a})} = x x_{\mathfrak{a}}^{-1}$. Then taking an idele c with $c\widehat{O} = \widehat{\mathfrak{c}}$ and $c_{\mathfrak{c}} = 1$, we see that

$$\Gamma_1^1(\mathfrak{c};\mathfrak{N}) \subset \left(\left(\begin{smallmatrix} c & 0 \\ 0 & 1 \end{smallmatrix}\right) K \left(\begin{smallmatrix} c & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1} \cap G(\mathbb{Q})_+ \right) \subset O^\times \Gamma_1^1(\mathfrak{c};\mathfrak{N})$$

for $G(\mathbb{Q})_+$ made up of all elements in $G(\mathbb{Q})$ with totally positive determinant. Choosing a complete representative set $\{c\} \subset F_{\mathbb{A}}^\times$ for the strict ray class group $Cl_F^+(\mathfrak{N})$ modulo $\mathfrak{N}$, we find by the approximation theorem that

$$G(\mathbb{A}) = \bigsqcup_{c \in Cl_F^+(\mathfrak{N})} G(\mathbb{Q}) \left(\begin{smallmatrix} c & 0 \\ 0 & 1 \end{smallmatrix}\right) K \cdot G(\mathbb{R})^+$$

for the identity connected component $G(\mathbb{R})^+$ of the Lie group $G(\mathbb{R})$. This shows

$$G(\mathbb{Q})\backslash G(\mathbb{A})/KC_{\mathbf{i}} \cong G(\mathbb{Q})_+\backslash G(\mathbb{A})_+/KC_{\mathbf{i}} \cong \bigsqcup_{\mathfrak{c} \in Cl_F^+(\mathfrak{N})} \mathfrak{M}(\mathfrak{c};\mathfrak{N})(\mathbb{C}), \tag{1.1}$$

where $G(\mathbb{A})_+ = G(\mathbb{A}^{(\infty)})G(\mathbb{R})^+$ and $C_{\mathbf{i}}$ is the stabilizer of $\mathbf{i} = (\sqrt{-1}\ldots,\sqrt{-1}) \in \mathfrak{Z}$ in $G(\mathbb{R})^+$. By (1.1), a $Cl_F^+(\mathfrak{N})$–tuple $(f_{\mathfrak{c}})_{\mathfrak{c}}$ with $f_{\mathfrak{c}} \in G_k(\mathfrak{c},\mathfrak{N};\mathbb{C})$ can be viewed as a single automorphic form defined on $G(\mathbb{A})$.

Recall the identification $X^*(T)$ with $\mathbb{Z}[I]$ so that $k(x) = \prod_\sigma \sigma(x)^{k_\sigma}$. Regarding $f \in G_k(\mathfrak{c}, \mathfrak{N}; \mathbb{C})$ as a function of $z \in \mathfrak{Z}$ by $f(z) = f(\mathcal{L}_z, \Lambda_z, i_z)$, it satisfies the following automorphic property:

$$f(\gamma(z)) = f(z)\prod_\sigma (c^\sigma z_\sigma + d^\sigma)^{k_\sigma} \quad \text{for all } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1^1(\mathfrak{c}; \mathfrak{N}). \tag{1.2}$$

The holomorphy of f follows from the functoriality (G2). The function f has the Fourier expansion

$$f(z) = \sum_{\xi \in (\mathfrak{ab})_{\geq 0}} a(\xi)\mathbf{e}_F(\xi z)$$

at the cusp corresponding to $(\mathfrak{a}, \mathfrak{b})$. Here $\mathbf{e}_F(\xi z) = \exp(2\pi\sqrt{-1}\sum_\sigma \xi^\sigma z_\sigma)$. This Fourier expansion gives the q–expansion $f_{\mathfrak{a},\mathfrak{b}}(q)$ substituting q^ξ for $\mathbf{e}_F(\xi z)$.

1.5. Γ_0-level structure and Hecke operators. We now assume that the base algebra A is a $\mathcal{W}$–algebra. Choose a prime $\mathfrak{q}$ of F. We are going to define Hecke operators $U(\mathfrak{q}^n)$ and $T(1, \mathfrak{q}^n)$ assuming for simplicity that $\mathfrak{q} \nmid p\mathfrak{N}$, though we may extend the definition for arbitrary $\mathfrak{q}$ (see [**PAF**, 4.1.10]). Then $X[\mathfrak{q}^r]$ is an étale group over B if X is an abelian scheme over an A–algebra B. We call a subgroup $C \subset X$ cyclic of order $\mathfrak{q}^r$ if $C \cong O/\mathfrak{q}^r$ over an étale faithfully flat extension of B.

We can think of quintuples $(X, \Lambda, i, C, \omega)_{/S}$ adding an additional information C of a cyclic subgroup scheme $C \subset X$ cyclic of order $\mathfrak{q}^r$. We define the space of classical modular forms $G_k(\mathfrak{c}, \mathfrak{N}, \Gamma_0(\mathfrak{q}^r); A)$ (resp. the space $V_{k\mathbf{P}}(\mathfrak{c}, \mathfrak{N}\mathfrak{p}^\infty, \Gamma_0(\mathfrak{q}^r); A)$ of p–adic modular forms) of prime-to-$\mathfrak{p}$ level $(\mathfrak{N}, \Gamma_0(\mathfrak{q}^r))$ by (G1-4) (resp. (P1-4)) replacing test objects (X, Λ, i, ω) (resp. $(X, \Lambda, i_{\mathfrak{N}}, i_{\mathfrak{p}}, \omega^{\mathbf{P}})$) by $(X, \Lambda, i, C, \omega)$ (resp. $(X, \Lambda, i_{\mathfrak{N}}, C, i_{\mathfrak{p}}, \omega^{\mathbf{P}})$).

Our Hecke operators are defined on the space of prime-to-$\mathfrak{p}$ level $(\mathfrak{N}, \Gamma_0(\mathfrak{q}^r))$. The operator $U(\mathfrak{q}^n)$ is defined only when $r > 0$ and $T(1, \mathfrak{q}^n)$ is defined only when $r = 0$. For a cyclic subgroup C' of $X_{/B}$ of order $\mathfrak{q}^n$, we can define the quotient abelian scheme X/C' with projection $\pi : X \to X/C'$. The polarization Λ and the differential ω induce a polarization $\pi_*\Lambda$ and a differential $(\pi^*)^{-1}\omega$ on X/C'. If $C' \cap C = \{0\}$ (in this case, we call that C' and C are *disjoint*), $\pi(C)$ gives rise to the level $\Gamma_0(\mathfrak{q}^r)$–structure on X/C'. Then we define for $f \in G_k(\mathfrak{c}\mathfrak{q}^n; \mathfrak{N}, \Gamma_0(\mathfrak{q}^r); A)$,

$$f|U(\mathfrak{q}^n)(X, \Lambda, C, i, \omega) = \frac{1}{N(\mathfrak{q}^n)}\sum_{C'} f(X/C', \pi_*\Lambda, \pi \circ i, \pi(C), (\pi^*)^{-1}\omega), \tag{1.3}$$

where C' runs over all cyclic subgroups of order $\mathfrak{q}^n$ *disjoint* from C. Since $\pi_*\Lambda = \pi \circ \Lambda \circ \pi^t$ is a $\mathfrak{c}\mathfrak{q}^n$–polarization, the modular form f has to be defined for abelian varieties with $\mathfrak{c}\mathfrak{q}^n$–polarization. Since $\mathfrak{q} \nmid \mathfrak{N}$, forgetting the $\Gamma_0(\mathfrak{q}^n)$–structure, we define for $f \in G_k(\mathfrak{c}\mathfrak{q}^n; \mathfrak{N}; A)$

$$f|T(1, \mathfrak{q}^n)(X, \Lambda, i, \omega) = \frac{1}{N(\mathfrak{q}^n)}\sum_{C'} f(X/C', \pi_*\Lambda, \pi \circ i, (\pi^*)^{-1}\omega), \tag{1.4}$$

where C' runs over all cyclic subgroups of order $\mathfrak{q}^n$. We check that $f|U(\mathfrak{q}^n)$ and $f|T(1, \mathfrak{q}^n)$ belong to $V_{k\mathbf{P}}(\mathfrak{c}, \mathfrak{N}\mathfrak{p}^\infty, \Gamma_0(\mathfrak{q}^r); A)$ and stays in $G_k(\mathfrak{c}, \mathfrak{N}, \Gamma_0(\mathfrak{q}^r); A)$ if $f \in G_k(\mathfrak{c}\mathfrak{q}^n, \mathfrak{N}, \Gamma_0(\mathfrak{q}^r); A)$. We have

$$U(\mathfrak{q}^n) = U(\mathfrak{q})^n.$$

1.6. Hilbert modular Shimura varieties. We extend the level structure i limited to $\mathfrak{N}$–torsion points to a far bigger structure $\eta^{(p)}$ including all prime-to–p torsion points. Let $\mathbb{Z}_{(p)} = \mathbb{Q} \cap \mathbb{Z}_p$ (the localization of $\mathbb{Z}$ at (p)). Triples $(X, \overline{\Lambda}, \eta^{(p)})_{/S}$ for $\mathbb{Z}_{(p)}$–schemes S are classified by an integral model $Sh^{(p)}_{/\mathbb{Z}_{(p)}}$ (cf. [**Ko**]) of the Shimura variety $Sh_{/\mathbb{Q}}$ associated to the algebraic $\mathbb{Z}_{(p)}$–group G (in the sense of Deligne [**De**, 4.22] interpreting Shimura's original definition in [**Sh**] as a moduli of abelian schemes up to isogenies). Here the classification is up to prime-to–p isogenies, and $\overline{\Lambda}$ is an equivalence class of polarizations up to multiplication by totally positive elements in F prime to p.

To give a description of the functor represented by $Sh^{(p)}$, we introduce some more notations. We consider the fiber category $\mathcal{A}_F^{(p)}$ over schemes defined by

(Object) abelian schemes X with real multiplication by O;
(Morphism) $\mathrm{Hom}_{\mathcal{A}_F^{(p)}}(X, Y) = \mathrm{Hom}(X, Y) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$.

Isomorphisms in this category are isogenies with degree prime to p (called "prime-to–p isogenies"), and hence the degree of polarization Λ is supposed to be also prime to p. Two polarizations are equivalent if $\Lambda = c\Lambda' = \Lambda' \circ i(c)$ for a totally positive c prime to p. We fix an O–lattice $L \subset V = F^2$ with O–hermitian alternating pairing $\langle \cdot, \cdot \rangle$ inducing a self duality on $L_p = L \otimes_{\mathbb{Z}} \mathbb{Z}_p$.

For an open-compact subgroup K of $G(\mathbb{A}^{(\infty)})$ *maximal at* p (i.e. $K = GL_2(O_p) \times K^{(p)}$), we consider the following functor from $\mathbb{Z}_{(p)}$–schemes into $SETS$:

$$\wp_K^{(p)}(S) = \left[(X, \overline{\Lambda}, \overline{\eta}^{(p)})_{/S} \text{ with (det)} \right]. \tag{1.5}$$

Here $\overline{\eta}^{(p)} : L \otimes_{\mathbb{Z}} \mathbb{A}^{(p\infty)} \cong V^{(p)}(X) = T(X) \otimes_{\mathbb{Z}} \mathbb{A}^{(p\infty)}$ is an equivalence class of $\eta^{(p)}$ modulo multiplication $\eta^{(p)} \mapsto \eta^{(p)} \circ k$ by $k \in K^{(p)}$ for the Tate module $T(X) = \varprojlim_{\mathfrak{N}} X[\mathfrak{N}]$ (in the sheafified sense that $\eta^{(p)} \equiv (\eta')^{(p)} \mod K$ étale-locally), and a $\Lambda \in \overline{\Lambda}$ induces the self-duality on L_p. As long as $K^{(p)}$ is sufficiently small, $\wp_K^{(p)}$ is representable over any $\mathbb{Z}_{(p)}$–algebra A (cf. [**Ko**], [**DeP**] and [**Z**, Section 3]) by a scheme $Sh_{K/A} = Sh/K$, which is smooth over $\mathrm{Spec}(\mathbb{Z}_{(p)})$ if p is unramified in $F_{/\mathbb{Q}}$ and singular if $p|D_F$ but is smooth outside a closed subscheme of codimension 2 in the p-fiber $Sh^{(p)} \times_{\mathbb{Z}_{(p)}} \mathbb{F}_p$ by the result of [**DeP**]. We let $g \in G(\mathbb{A}^{(p\infty)})$ act $Sh^{(p)}_{/\mathbb{Z}_{(p)}}$ by

$$x = (X, \overline{\Lambda}, \eta) \mapsto g(x) = (X, \overline{\Lambda}, \eta \circ g),$$

which gives a right action of $G(\mathbb{A})$ on $Sh^{(p)}$ through the projection $G(\mathbb{A}) \twoheadrightarrow G(\mathbb{A}^{(p\infty)})$.

By the universality, we have a morphism $\mathfrak{M}^R(\mathfrak{c}; \mathfrak{N}) \to Sh^{(p)}/\widehat{\Gamma}_1^1(\mathfrak{c}; \mathfrak{N})$ for the open compact subgroup: $\widehat{\Gamma}_1^1(\mathfrak{c}; \mathfrak{N}) = \left(\begin{smallmatrix} c & 0 \\ 0 & 1 \end{smallmatrix}\right) K_1^1(\mathfrak{N}) \left(\begin{smallmatrix} c & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} cd^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right) U_1^1(\mathfrak{N}) \left(\begin{smallmatrix} cd^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1}$ maximal at p. The image of $\mathfrak{M}^R(\mathfrak{c}; \mathfrak{N})$ gives a geometrically irreducible component of $Sh^{(p)}/\widehat{\Gamma}_1^1(\mathfrak{c}; \mathfrak{N})$. If $\mathfrak{N}$ is sufficiently deep, we can identify $\mathfrak{M}^R(\mathfrak{c}; \mathfrak{N})$ with its image in $Sh^{(p)}/\widehat{\Gamma}_1^1(\mathfrak{c}; \mathfrak{N})$. By the action on the polarization $\Lambda \mapsto \alpha\Lambda$ for a suitable totally positive $\alpha \in F$, we can bring $\mathfrak{M}^R(\mathfrak{c}; \mathfrak{N})$ into $\mathfrak{M}^R(\alpha\mathfrak{c}; \mathfrak{N})$; so, the image of $\varprojlim_{\mathfrak{N}} \mathfrak{M}^R(\mathfrak{c}; \mathfrak{N})$ in $Sh^{(p)}$ only depends on the strict ideal class of $\mathfrak{c}$ in $\varprojlim_{\mathfrak{N}:\mathfrak{N}+(p)=O} Cl_F^+(\mathfrak{N})$.

1.7. Level structure with "Neben" character. In order to make a good link between classical modular forms and adelic automorphic forms (which we will describe in the following subsection), we would like to introduce "Neben" characters. We fix an integral ideal $\mathfrak{N}' \subset O$. We think of the following level structure on an AVRM X:

$$i : (\mathbb{G}_m \otimes O^*)[\mathfrak{N}'] \hookrightarrow X[\mathfrak{N}'] \text{ and } i' : X[\mathfrak{N}'] \twoheadrightarrow O/\mathfrak{N}', \tag{1.6}$$

where the sequence

$$1 \to (\mathbb{G}_m \otimes O^*)[\mathfrak{N}'] \xrightarrow{i} X[\mathfrak{N}'] \xrightarrow{i'} O/\mathfrak{N}' \to 0 \tag{1.7}$$

is exact and is required to induce a canonical duality between $(\mathbb{G}_m \otimes O^*)[\mathfrak{N}']$ and $O/\mathfrak{N}'$ under the polarization Λ. Here, if $\mathfrak{N}' = (N)$ for an integer $N > 0$, a canonical duality pairing

$$\langle \cdot, \cdot \rangle : (\mathbb{G}_m \otimes O^*)[N] \times O/N \to \mu_N$$

is given by $\langle \zeta \otimes \alpha, m \otimes \beta \rangle = \zeta^{m\mathrm{Tr}(\alpha\beta)}$ for $(\alpha, \beta) \in O^* \times O$ and $(\zeta, m) \in \mu_N \times O/N$ identifying $(\mathbb{G}_m \otimes O^*)[N] = \mu_N \otimes O^*$ and $O/N = (\mathbb{Z}/N\mathbb{Z}) \otimes_{\mathbb{Z}} O$. In general, taking an integer $0 < N \in \mathfrak{N}'$, the canonical pairing between $(\mathbb{G}_m \otimes O^*)[\mathfrak{N}']$ and $O/\mathfrak{N}'$ is induced by the one for (N) via the canonical inclusion $(\mathbb{G}_m \otimes O^*)[\mathfrak{N}'] \hookrightarrow (\mathbb{G}_m \otimes O^*)[N]$ and the quotient map $O/(N) \twoheadrightarrow O/\mathfrak{N}'$.

We fix two characters $\epsilon_1 : (O/\mathfrak{N}')^\times \to A^\times$ and $\epsilon_2 : (O/\mathfrak{N}')^\times \to A^\times$, and we insist for $f \in G_k(\mathfrak{c}, \mathfrak{N}; A)$ on the version of (G0-3) for quintuples $(X, \Lambda, i \cdot a, d \cdot i', \omega)$ and the equivariancy:

$$f(X, \overline{\Lambda}, i \cdot d, a \cdot i', \omega) = \epsilon_1(d)\epsilon_2(a) f(X, \overline{\Lambda}, i, i', \omega) \text{ for } a, d \in (O/\mathfrak{N})^\times. \tag{Neben}$$

Here the order $\epsilon_1(d)\epsilon_2(a)$ is correct as the diagonal matrix $\left(\begin{smallmatrix} d & 0 \\ 0 & a \end{smallmatrix}\right)$ in $T_0(O/\mathfrak{N}')$ acts on the quotient $O/\mathfrak{N}'$ by a and the submodule $(\mathbb{G}_m \otimes O^*)[\mathfrak{N}']$ by d. Here $\overline{\Lambda}$ is the polarization class modulo multiple of totally positive numbers in F prime to p. We write $G_k(\mathfrak{c}, \Gamma_0(\mathfrak{N}), \epsilon; A)$ $(\epsilon = (\epsilon_1, \epsilon_2))$ for the A–module of geometric modular forms satisfying these conditions.

1.8. Adelic Hilbert modular forms. Let us interpret what we have said so far in automorphic language and give a definition of the adelic Hilbert modular forms and their Hecke algebra of level $\mathfrak{N}$ (cf. [**H96**, Sections 2.2-4] and [**PAF**, Sections 4.2.8–4.2.12]).

We consider the following open compact subgroup of $G(\mathbb{A}^{(\infty)})$:

$$\begin{aligned} U_0(\mathfrak{N}) &= \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(\widehat{O}) \middle| c \equiv 0 \mod \mathfrak{N}\widehat{O} \right\}, \\ U_1^1(\mathfrak{N}) &= \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in U_0(\mathfrak{N}) \middle| a \equiv d \equiv 1 \mod \mathfrak{N}\widehat{O} \right\}, \end{aligned} \tag{1.8}$$

where $\widehat{O} = O \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ and $\widehat{\mathbb{Z}} = \prod_\ell \mathbb{Z}_\ell$. Then we introduce the following semi-group

$$\Delta_0(\mathfrak{N}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(\mathbb{A}^{(\infty)}) \cap M_2(\widehat{O}) \middle| c \equiv 0 \mod \mathfrak{N}\widehat{O}, d_{\mathfrak{N}} \in O_{\mathfrak{N}}^\times \right\}, \tag{1.9}$$

where $d_{\mathfrak{N}}$ is the projection of $d \in \widehat{O}$ to $O_{\mathfrak{N}} := \prod_{\mathfrak{q}|\mathfrak{N}} O_{\mathfrak{q}}$ for prime ideals $\mathfrak{q}$. Writing T_0 for the maximal diagonal torus of $GL(2)_{/O}$ and putting

$$D_0 = \left\{ \mathrm{diag}[a, d] = \left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) \in T_0(F_{\mathbb{A}}^{(\infty)}) \cap M_2(\widehat{O}) \middle| d_{\mathfrak{N}} = 1 \right\}, \tag{1.10}$$

we have (e.g. [**MFG**, 3.1.6] and [**PAF**, Section 5.1])

$$\Delta_0(\mathfrak{N}) = U_0(\mathfrak{N}) D_0 U_0(\mathfrak{N}). \tag{1.11}$$

In this section, the group U is assumed to be a subgroup of $U_0(\mathfrak{N}\mathfrak{p}^\alpha)$ with $U \supset U_1^1(\mathfrak{N}\mathfrak{p}^\alpha)$ for some $0 < \alpha \leq \infty$ (though we do not assume that $\mathfrak{N}$ is prime to $\mathbf{p}$). Formal finite linear combinations $\sum_\delta c_\delta U\delta U$ of double cosets of U in $\Delta_0(\mathfrak{N}\mathfrak{p}^\alpha)$ form a ring $R(U, \Delta_0(\mathfrak{N}\mathfrak{p}^\alpha))$ under convolution product (see [**IAT**, Chapter 3] or [**MFG**, 3.1.6]). Recall the prime element $\varpi_\mathfrak{q}$ of $O_\mathfrak{q}$ for each prime $\mathfrak{q}$ fixed in the introduction. The algebra is commutative and is isomorphic to the polynomial ring over the group algebra $\mathbb{Z}[U_0(\mathfrak{N}\mathfrak{p}^\alpha)/U]$ with variables $\{T(\mathfrak{q}), T(\mathfrak{q},\mathfrak{q})\}_\mathfrak{q}$. Here $T(\mathfrak{q})$ (resp. $T(\mathfrak{q},\mathfrak{q})$ for primes $\mathfrak{q} \nmid \mathfrak{N}\mathfrak{p}^\alpha$) corresponds to the double coset $U\left(\begin{smallmatrix}\varpi_\mathfrak{q} & 0\\ 0 & 1\end{smallmatrix}\right)U$ (resp. $U_0\varpi_\mathfrak{q}U$). The group element $u \in U_0(\mathfrak{N}\mathfrak{p}^\alpha)/U$ in the group algebra $\mathbb{Z}[U_0(\mathfrak{N}\mathfrak{p}^\alpha)/U]$ corresponds to the double coset UuU (cf. [**H95**, Section 2]).

As in the introduction, we extend ϵ_j to a character of $(F_\mathbb{A}^{(\infty)})^\times \subset \widehat{O}^\times \times \prod_\mathfrak{q} \varpi_\mathfrak{q}^\mathbb{Z}$ trivial on the factor $\prod_\mathfrak{q} \varpi_\mathfrak{q}^\mathbb{Z}$, and denote the extended character by the same symbol ϵ_j. In [**HMI**, (ex0–3)], ϵ_2 is extended as above, but the extension of ϵ_1 taken there is to keep the identity $\epsilon_+ = \epsilon_1\epsilon_2$ over $(F_\mathbb{A}^{(\infty)})^\times$. The present extension is more convenient in this paper.

The double coset ring $R(U, \Delta_0(\mathfrak{N}\mathfrak{p}^\alpha))$ naturally acts on the space of modular forms on U. We now recall the action (which is a slight simplification of the action of $[UxU]$ given in [**HMI**, (2.3.14)]). Recall the diagonal torus T_0 of $GL(2)_{/O}$; so, $T_0 = \mathbb{G}_{m/O}^2$. Since $T_0(O/\mathfrak{N}')$ is canonically a quotient of $U_0(\mathfrak{N}')$ for an ideal $\mathfrak{N}'$, a character $\epsilon : T_0(O/\mathfrak{N}') \to \mathbb{C}^\times$ can be considered as a character of $U_0(\mathfrak{N}')$. If ϵ_j is defined modulo $\mathfrak{N}_j$, we can take $\mathfrak{N}'$ to be any multiple of $\mathfrak{N}_1 \cap \mathfrak{N}_2$. Writing $\epsilon(\left(\begin{smallmatrix}a & 0\\ 0 & d\end{smallmatrix}\right)) = \epsilon_1(a)\epsilon_2(d)$, if $\epsilon^- = \epsilon_1\epsilon_2^{-1}$ factors through $(O/\mathfrak{N})^\times$ for for an ideal $\mathfrak{N}|\mathfrak{N}'$, then we can extend the character ϵ of $U_0(\mathfrak{N}')$ to $\Delta_0(\mathfrak{N})$ by putting $\epsilon(\delta) = \epsilon_1(\det(\delta))(\epsilon^-)^{-1}(d_\mathfrak{N})$ for $\delta = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) \in \Delta_0(\mathfrak{N})$ (as before). In this sense, we hereafter assume that ϵ is defined modulo $\mathfrak{N}$ and regard ϵ as a character of the group $U_0(\mathfrak{N})$ and the semi-group $\Delta_0(\mathfrak{N})$. We fix a Hecke character $\epsilon_+ : F_\mathbb{A}^\times/F^\times \to \mathbb{C}^\times$ with infinity type $(1-[\kappa])I$ (for the integer $[\kappa] = \kappa_{1,\sigma} + \kappa_{2,\sigma}$) such that $\epsilon_+(z) = \epsilon_1(z)\epsilon_2(z)$ for $z \in \widehat{O}^\times$.

Writing I for the set of all embeddings of F into $\overline{\mathbb{Q}}$ and T^2 for $\mathrm{Res}_{O/\mathbb{Z}}T_0$ (the diagonal torus of G), the group of geometric characters $X^*(T^2)$ is isomorphic to $\mathbb{Z}[I]^2$ so that $(m,n) \in \mathbb{Z}[I]^2$ send $\mathrm{diag}[x,y] \in T^2$ to $x^m y^n = \prod_{\sigma\in I}(\sigma(x)^{m_\sigma}\sigma(y)^{n_\sigma})$. Taking $\kappa = (\kappa_1,\kappa_2) \in \mathbb{Z}[I]^2$, we assume $[\kappa]I = \kappa_1 + \kappa_2$, and we associate with κ a factor of automorphy:

$$J_\kappa(g,\tau) = \det(g_\infty)^{\kappa_1 - I} j(g_\infty,\tau)^{\kappa_2-\kappa_1+I} \quad \text{for } g \in G(\mathbb{A}) \text{ and } \tau \in \mathfrak{Z}. \tag{1.12}$$

We define $S_\kappa(U,\epsilon;\mathbb{C})$ for an open subgroup $U \subset U_0(\mathfrak{N})$ by the space of functions $\mathbf{f} : G(\mathbb{A}) \to \mathbb{C}$ satisfying the following three conditions (e.g. [**HMI**, (SA1–3)] and [**PAF**, 4.3.1]):

- (S1) $\mathbf{f}(\alpha xuz) = \epsilon(u)\epsilon_+(z)\mathbf{f}(x)J_\kappa(u,\mathbf{i})^{-1}$ for $\alpha \in G(\mathbb{Q})$, $u \in U\cdot C_\mathbf{i}$ and $z \in Z(\mathbb{A})$.
- (S2) Choose $u \in G(\mathbb{R})$ with $u(\mathbf{i}) = \tau$ for $\tau \in \mathfrak{Z}$, and put $\mathbf{f}_x(\tau) = \mathbf{f}(xu)J_\kappa(u,\mathbf{i})$ for each $x \in G(\mathbb{A}^{(\infty)})$ (which only depends on τ). Then $\mathbf{f}_x$ is a holomorphic function on $\mathfrak{Z}$ for all x.
- (S3) $\mathbf{f}_x(\tau)$ for each x is rapidly decreasing as $\eta_\sigma \to \infty$ ($\tau = \xi + \mathbf{i}\eta$) for all $\sigma \in I$ uniformly.

If we replace the word "rapidly decreasing" in (S3) by "slowly increasing", we get the definition of the space $G_\kappa(U,\epsilon;\mathbb{C})$. It is easy to check (e.g. [**HMI**, (2.3.5)] that the function $\mathbf{f}_x$ in (S2) satisfies

$$\mathbf{f}_x(\gamma(\tau)) = \epsilon^{-1}(x^{-1}\gamma x)\mathbf{f}_x(\tau)J_\kappa(\gamma,\tau) \text{ for all } \gamma \in \Gamma_x(U), \tag{1.13}$$

where $\Gamma_x(U) = xUx^{-1}G(\mathbb{R})^+ \cap G(\mathbb{Q})$. Also by (S3), $\mathbf{f}_x$ is rapidly decreasing towards all cusps of Γ_x; so, it is a cusp form. If we restrict $\mathbf{f}$ as above to $SL_2(F_{\mathbb{A}})$, the determinant factor $\det(g)^{\kappa_1 - I}$ in the factor $J_\kappa(g,\tau)$ disappears, and the automorphy factor becomes only dependent on $k = \kappa_2 - \kappa_1 + I \in \mathbb{Z}[I]$; so, the classical modular form in G_k has single digit weight $k \in \mathbb{Z}[I]$. Via (1.1), we have an embedding of $S_\kappa(\mathfrak{N}', \epsilon; \mathbb{C})$ into $G_k(\Gamma_0(\mathfrak{N}'), \epsilon; \mathbb{C}) = \bigoplus_{[\mathfrak{c}] \in Cl_F^+} G_k(\mathfrak{c}, \Gamma_0(\mathfrak{N}'), \epsilon; \mathbb{C})$ ($\mathfrak{c}$ running over a complete representative set prime to $\mathfrak{N}'$ for the strict ideal class group Cl_F^+) bringing $\mathbf{f}$ into $(\mathbf{f}_\mathfrak{c})_{[\mathfrak{c}]}$ for $\mathbf{f}_\mathfrak{c} = \mathbf{f}_x$ (as in (S3)) with $x = \left(\begin{smallmatrix} cd^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)$ (for $d \in F_{\mathbb{A}}^\times$ with $d\widehat{O} = \widehat{\mathfrak{d}}$). The cusp form $\mathbf{f}_\mathfrak{c}$ is determined by the restriction of $\mathbf{f}$ to $x \cdot SL_2(F_{\mathbb{A}})$. Thus $\mathbf{f}_x$ is not exactly the restriction of $\mathbf{f}$ to $SL_2(F_{\mathbb{A}})$, and the embedding $\mathbf{f} \mapsto (f_\mathfrak{c})_\mathfrak{c}$ into $\bigoplus_{[\mathfrak{c}] \in Cl_F^+} G_k(\mathfrak{c}, \Gamma_0(\mathfrak{N}'), \epsilon; \mathbb{C})$ depends on κ and ϵ_+, and hence the uniform definition of Hecke operators in (1.4) (appeared dependent only on k) produces different eigenvalues dependent on κ and ϵ_+. In other words, if we vary the weight κ keeping $k = \kappa_2 - \kappa_1 + I$, the image of S_κ in $G_k(\Gamma_0(\mathfrak{N}'), \epsilon; \mathbb{C})$ transforms accordingly. By this identification, the Hecke operator $T(\mathfrak{q})$ for non-principal $\mathfrak{q}$ makes sense as an operator acting on a single space $G_\kappa(U, \epsilon; \mathbb{C})$, and its action depends on the choice of κ. The $SL(2)$–weight of an automorphic representation π of $SL_2(F_{\mathbb{A}})$ generated by $\mathbf{f}|_{SL_2(F_{\mathbb{A}})}$ for $\mathbf{f} \in G_\kappa(U, \epsilon; \mathbb{C})$ is given by k (which specifies the infinity type of π_∞ as a discrete series representation of $SL_2(F_{\mathbb{R}})$). Though in (1.13), ϵ^{-1} shows up, the Neben character of the direct factor $G_k(\mathfrak{c}, \Gamma_0(\mathfrak{N}'), \epsilon; \mathbb{C})$ is given by ϵ, since in (Neben), the order of (a, d) is reversed to have $\epsilon_1(d)\epsilon_2(a)$. It is easy to see that $G_\kappa = 0$ unless $\kappa_1 + \kappa_2 = [\kappa]I$ for the integer $[\kappa] \in \mathbb{Z}$.

In the introduction, we have extended ϵ_j to $(F_{\mathbb{A}}^{(\infty)})^\times$ and ϵ to $\Delta_0(\mathfrak{N})$ (as long as ϵ^- is defined modulo $\mathfrak{N}$), and we have $\epsilon(\delta) = \epsilon_1(\det(\delta))(\epsilon^-)^{-1}(d_\mathfrak{N})$ for $\delta = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Delta_0(\mathfrak{N})$. Let $\mathcal{U}$ be the unipotent algebraic subgroup of $GL(2)_{/O}$ defined by $\mathcal{U}(A) = \left\{\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right) \middle| a \in A\right\}$. Note here that $\mathcal{U}(\widehat{O}) \subset \mathrm{Ker}(\epsilon)$; so, $\epsilon(tu) = \epsilon(t)$ if $t \in D_0$ and $u \in \mathcal{U}(\widehat{O})$. For each $UyU \in R(U, \Delta_0(\mathfrak{N}\mathfrak{p}^\alpha))$, we decompose $UyU = \bigsqcup_{t \in D_0, u \in \mathcal{U}(\widehat{O})} utU$ for finitely many u and t (see [**IAT**, Chapter 3] or [**MFG**, 3.1.6]) and define

$$[UyU](\mathbf{f})(x) = \sum_{t,u} \epsilon(t)^{-1}\mathbf{f}(xut). \tag{1.14}$$

We check that this operator preserves the spaces of automorphic forms: $G_\kappa(\mathfrak{N}, \epsilon; \mathbb{C})$ and $S_\kappa(\mathfrak{N}, \epsilon; \mathbb{C})$, and depends only on UyU not the choice of y as long as $y \in D_0$. However it depends on the choice of $\varpi_\mathfrak{q}$ as the character ϵ (extended to $\Delta_0(\mathfrak{N})$) depends on $\varpi_\mathfrak{q}$. This action for y with $y_\mathfrak{N} = 1$ is independent of the choice of the extension of ϵ to $T_0(F_{\mathbb{A}})$. When $y_\mathfrak{N} \neq 1$, we may assume that $y_\mathfrak{N} \in D_0 \subset T_0(F_{\mathbb{A}})$, and in this case, t can be chosen so that $t_\mathfrak{N} = y_\mathfrak{N}$ (so $t_\mathfrak{N}$ is independent of single right cosets in the double coset). If we extend ϵ to $T_0(F_{\mathbb{A}}^{(\infty)})$ by choosing another prime element $\varpi'_\mathfrak{q}$ and write the extension as ϵ', then we have

$$\epsilon(t_\mathfrak{N})[UyU] = \epsilon'(t_\mathfrak{N})[UyU]',$$

where the operator on the right-hand-side is defined with respect to ϵ'. Thus the sole difference is the root of unity $\epsilon(t_\mathfrak{N})/\epsilon'(t_\mathfrak{N}) \in \mathrm{Im}(\epsilon/\epsilon'|_{T_0(O/\mathfrak{N}')})$. Since it depends on the choice of $\varpi_\mathfrak{q}$, we make the choice once and for all, and write $T(\mathfrak{q})$

for $\left[U\left(\begin{smallmatrix}\varpi_{\mathfrak{q}} & 0\\ 0 & 1\end{smallmatrix}\right)U\right]$ (if $\mathfrak{q}\nmid\mathfrak{N}$), which coincides with $T(1,\mathfrak{q})$ in (1.4) if $\mathfrak{q}\nmid\mathfrak{N}'$. By linearity, these action of double cosets extends to the ring action of the double coset ring $R(U,\Delta_0(\mathfrak{N}\mathfrak{p}^\alpha))$.

To introduce rationality of modular forms, we recall Fourier expansion of adelic modular forms (cf. [**HMI**, Proposition 2.26]). Recall the embedding $\iota_\infty:\overline{\mathbb{Q}}\hookrightarrow\mathbb{C}$, and identify $\overline{\mathbb{Q}}$ with the image of ι_∞. Recall also the differental idele $d\in F_{\mathbb{A}}^\times$ with $d^{(\mathfrak{d})}=1$ and $d\widehat{O}=\mathfrak{d}\widehat{O}$. Each member $\mathbf{f}$ of $S_\kappa(U,\epsilon;\mathbb{C})$ has its Fourier expansion:

$$\mathbf{f}\left(\begin{smallmatrix}y & x\\ 0 & 1\end{smallmatrix}\right)=|y|_{\mathbb{A}}\sum_{0\ll\xi\in F}c(\xi yd,\mathbf{f})(\xi y_\infty)^{-\kappa_1}\mathbf{e}_F(i\xi y_\infty)\mathbf{e}_F(\xi x),\tag{1.15}$$

where $\mathbf{e}_F:F_{\mathbb{A}}/F\to\mathbb{C}^\times$ is the additive character with $\mathbf{e}_F(x_\infty)=\exp(2\pi i\sum_{\sigma\in I}x_\sigma)$ for $x_\infty=(x_\sigma)_\sigma\in\mathbb{R}^I=F\otimes_{\mathbb{Q}}\mathbb{R}$. Here $y\mapsto c(y,\mathbf{f})$ is a function defined on $y\in F_{\mathbb{A}}^\times$ only depending on its finite part $y^{(\infty)}$. The function $c(y,\mathbf{f})$ is supported by the set $(\widehat{O}\times F_\infty)\cap F_{\mathbb{A}}^\times$ of *integral* ideles.

Let $F[\kappa]$ be the field fixed by $\{\sigma\in\mathrm{Gal}(\overline{\mathbb{Q}}/F)|\kappa\sigma=\kappa\}$, over which the character $\kappa\in X^*(T^2)$ is rational. Write $O[\kappa]$ for the integer ring of $F[\kappa]$. We also define $O[\kappa,\epsilon]$ for the integer ring of the field $F[\kappa,\epsilon]$ generated by the values of ϵ over $F[\kappa]$. For any $F[\kappa,\epsilon]$–algebra A inside $\mathbb{C}$, we define

$$S_\kappa(U,\epsilon;A)=\left\{\mathbf{f}\in S_\kappa(U,\epsilon;\mathbb{C})\middle|c(y,\mathbf{f})\in A\ \text{ as long as } y \text{ is integral}\right\}.\tag{1.16}$$

As we have seen, we can interpret $S_\kappa(U,\epsilon;A)$ as the space of A–rational global sections of a line bundle of a variety defined over A; so, by the flat base-change theorem (e.g. [**GME**, Lemma 1.10.2]),

$$S_\kappa(\mathfrak{N},\epsilon;A)\otimes_A\mathbb{C}=S_\kappa(\mathfrak{N},\epsilon;\mathbb{C}).\tag{1.17}$$

Thus for any $\overline{\mathbb{Q}}_p$–algebras A, we may consistently define

$$S_\kappa(U,\epsilon;A)=S_\kappa(U,\epsilon;\overline{\mathbb{Q}})\otimes_{\overline{\mathbb{Q}},\iota_p}A.\tag{1.18}$$

By linearity, $y\mapsto c(y,\mathbf{f})$ extends to a function on $F_{\mathbb{A}}^\times\times S_\kappa(U,\epsilon;A)$ with values in A. For $u\in\widehat{O}^\times$, we know from [**HMI**, (2.3.20)]

$$c(yu,\mathbf{f})=\epsilon_1(u)c(y,\mathbf{f}).\tag{1.19}$$

If $\mathbf{f}$ is a normalized Hecke eigenform, its eigenvalue $a(y,\mathbf{f})$ of $T(y)$ is given by $\epsilon_1(y)^{-1}c(y,\mathbf{f})$ which depends only on the ideal $\mathfrak{y}:=y\widehat{O}\cap F$ by the above formula as claimed in the introduction. We define the q–expansion coefficients (at p) of $\mathbf{f}\in S_\kappa(U,\epsilon;A)$ by

$$\mathbf{c}_p(y,\mathbf{f})=y_p^{-\kappa_1}c(y,\mathbf{f}).\tag{1.20}$$

The formal q–expansion of an A–rational $\mathbf{f}$ has values in the space of functions on $(F_{\mathbb{A}}^{(\infty)})^\times$ with values in the formal monoid algebra $A[[q^\xi]]_{\xi\in F_+}$ of the multiplicative semi-group F_+ made up of totally positive elements, which is defined by

$$\mathbf{f}(y)=\mathcal{N}(y)^{-1}\sum_{\xi\gg0}\mathbf{c}_p(\xi yd,\mathbf{f})q^\xi,\tag{1.21}$$

where $\mathcal{N}:F_{\mathbb{A}}^\times/F^\times\to\overline{\mathbb{Q}}_p^\times$ is the character given by $\mathcal{N}(y)=y_p^{-I}|y^{(\infty)}|_{\mathbb{A}}^{-1}$.

We now define for any p–adically complete $O[\kappa,\epsilon]$–algebra A in $\mathbb{C}_p$

$$S_\kappa(U,\epsilon;A)=\left\{\mathbf{f}\in S_\kappa(U,\epsilon;\mathbb{C}_p)\middle|\mathbf{c}_p(y,\mathbf{f})\in A \text{ for integral } y\right\}.\tag{1.22}$$

As we have already seen, these spaces have geometric meaning as the space of A–integral global sections of a line bundle defined over A of the Hilbert modular variety of level U, and the q–expansion above for a fixed $y = y^{(\infty)}$ gives rise to the geometric q–expansion at the infinity cusp of the classical modular form $\mathbf{f}_x$ for $x = \left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right)$ (see [**H91**, (1.5)] and [**PAF**, (4.63)]).

We have chosen a complete representative set $\{c_i\}_{i=1,\dots,h}$ in finite ideles for the strict idele class group $F^\times \backslash F_{\mathbb{A}}^\times / \widehat{O}^\times F_{\infty+}^\times$, where h is the strict class number of F. Let $\mathfrak{c}_i = c_i O$. Write $t_i = \left(\begin{smallmatrix} c_i d^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and consider $\mathbf{f}_i = \mathbf{f}_{t_i}$ as defined in (S2). The collection $(\mathbf{f}_i)_{i=1,\dots,h}$ determines f, because of the approximation theorem. Then $\mathbf{f}(c_i d^{-1})$ gives the q–expansion of $\mathbf{f}_i$ at the Tate abelian variety with $\mathfrak{c}_i$–polarization $\mathrm{Tate}_{\mathfrak{c}_i^{-1},O}(q)$ ($\mathfrak{c}_i = c_i O$). By (q–exp), the q–expansion $\mathbf{f}(y)$ determines $\mathbf{f}$ uniquely.

We write $T(y)$ for the Hecke operator acting on $S_\kappa(U, \epsilon; A)$ corresponding to the double coset $U \left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right) U$ for an integral idele y. We renormalize $T(y)$ to have a p–integral operator $\mathbb{T}(y)$: $\mathbb{T}(y) = y_p^{-\kappa_1} T(y)$. Since this only affects $T(y)$ with $y_p \neq 1$, $\mathbb{T}(\mathfrak{q}) = T(\varpi_{\mathfrak{q}}) = T(\mathfrak{q})$ if $\mathfrak{q} \nmid p$. However depending on weight, we can have $\mathbb{T}(\mathfrak{p}) \neq T(\mathfrak{p})$ for primes $\mathfrak{p}|p$. The renormalization is optimal to have the stability of the A–integral spaces under Hecke operators. We define $\langle \mathfrak{q} \rangle = N(\mathfrak{q}) T(\mathfrak{q}, \mathfrak{q})$ with $T(\mathfrak{q}, \mathfrak{q}) = [U \varpi_{\mathfrak{q}} U]$ for $\mathfrak{q} \nmid \mathfrak{N}' \mathfrak{p}^\alpha$ ($\mathfrak{N}' = \mathfrak{N}_1 \cap \mathfrak{N}_2$), which is equal to the central action of a prime element $\varpi_{\mathfrak{q}}$ of $O_{\mathfrak{q}}$ times $N(\mathfrak{q}) = |\varpi_{\mathfrak{q}}|_{\mathbb{A}}^{-1}$. We have the following formula of the action of $\mathbb{T}(\mathfrak{q})$ (e.g., [**HMI**, (2.3.21)] or [**PAF**, 4.2.10]):

$$\mathbf{c}_p(y, \mathbf{f}|\mathbb{T}(\mathfrak{q})) = \begin{cases} \mathbf{c}_p(y\varpi_{\mathfrak{q}}, \mathbf{f}) + \mathbf{c}_p(y\varpi_{\mathfrak{q}}^{-1}, \mathbf{f}|\langle \mathfrak{q} \rangle) & \text{if } \mathfrak{q} \text{ is outside } \mathfrak{N}\mathfrak{p} \\ \mathbf{c}_p(y\varpi_{\mathfrak{q}}, \mathbf{f}) & \text{otherwise,} \end{cases} \tag{1.23}$$

where the level $\mathfrak{N}$ of U is the ideal maximal under the condition: $U_1^1(\mathfrak{N}) \subset U \subset U_0(\mathfrak{N})$. Thus we have $\mathbb{T}(\varpi_{\mathfrak{q}}) = (\varpi_{\mathfrak{q}})_p^{-\kappa_1} U(\mathfrak{q})$ when $\mathfrak{q}$ is a factor of the level of U (even when $\mathfrak{q}|p$; see [**PAF**, (4.65–66)]). Writing the level of U as $\mathfrak{N}\mathfrak{p}^\alpha$, we assume

$$\text{either } p|\mathfrak{N}\mathfrak{p}^\alpha \text{ or } [\kappa] \geq 0, \tag{1.24}$$

since $\mathbb{T}(\mathfrak{q})$ and $\langle \mathfrak{q} \rangle$ preserve the space $S_\kappa(U, \epsilon; A)$ under this condition (see [**PAF**, Theorem 4.28]). We define the Hecke algebra $h_\kappa(U, \epsilon; A)$ (resp. $h_\kappa(\mathfrak{N}, \epsilon; A)$) with coefficients in A by the A–subalgebra of the A–linear endomorphism algebra $\mathrm{End}_A(S_\kappa(U, \epsilon; A))$ (resp. $\mathrm{End}_A(S_\kappa(\mathfrak{N}, \epsilon; A))$) generated by the action of the finite group $U_0(\mathfrak{N}\mathfrak{p}^\alpha)/U$, $\mathbb{T}(\mathfrak{q})$ and $\langle \mathfrak{q} \rangle$ for all $\mathfrak{q}$.

1.9. Hecke algebras. We have canonical projections for $U_\alpha = U_0(\mathfrak{N}) \cap U_1^1(\mathfrak{p}^\alpha)$:

$$R(U_\alpha, \Delta_0(\mathfrak{N}\mathfrak{p}^\alpha)) \twoheadrightarrow R(U, \Delta_0(\mathfrak{N}\mathfrak{p}^\alpha)) \twoheadrightarrow R(U_0(\mathfrak{N}p^\beta), \Delta_0(\mathfrak{N}p^\beta))$$

for all $\alpha \geq \beta$ ($\Leftrightarrow \alpha(\mathfrak{p}) \geq \beta(\mathfrak{p})$ for all $\mathfrak{p}|p$) taking canonical generators to the corresponding ones, which are compatible with inclusions

$$S_\kappa(\mathfrak{N}\mathfrak{p}^\beta, \epsilon; A) \hookrightarrow S_\kappa(U, \epsilon; A) \hookrightarrow S_\kappa(U_\alpha, \epsilon; A).$$

We decompose $O_{\mathfrak{p}}^\times = \mathbf{\Gamma} \times \Delta$ as in the introduction and hence $\mathbf{G} = \mathbf{\Gamma} \times \Delta \times (O/\mathfrak{N}')^\times$. We fix κ and ϵ_+ and the initial $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_+)$. We suppose that ϵ_j ($j = 1, 2$) factors through $\mathbf{G}/\mathbf{\Gamma} = \Delta \times (O/\mathfrak{N}')^\times$ for $\mathfrak{N}'$ prime to $\mathfrak{p}$. We write $\mathfrak{N}$ for a factor of $\mathfrak{N}'$ such that ϵ^- is defined modulo $\mathfrak{N}\mathfrak{p}^{r_0+1}$ for some $\mathfrak{p}^{r_0+1}|\mathcal{P}$. Then we get a projective system of Hecke algebras $\{h_\kappa(U, \epsilon; A)\}_U$ (U running through open subgroups of $U_0(\mathfrak{N}\mathfrak{p}^{r_0+1})$ containing U_∞), whose projective limit (when $\kappa_2 - \kappa_1 \geq I$) gives rise to the universal Hecke algebra $\mathbf{h}_\kappa(\mathfrak{N}\mathfrak{p}^\infty,\epsilon;A) = \varprojlim_U h_\kappa(U,\epsilon;A)$ for a complete p–adic algebra A.

We have a continuous character $T : \widehat{O}^\times \to \mathbf{h}_\kappa(\mathfrak{N}\mathfrak{p}^\infty,\epsilon;A)$ given by $u \mapsto T(u)$ where $\mathbf{f}|T(u)(x) = \epsilon_1(u)^{-1}\mathbf{f}\left(x\left(\begin{smallmatrix} u & 0\\ 0 & 1\end{smallmatrix}\right)\right)$ for $u \in \widehat{O}^\times$. This character T factors through $\mathbf{\Gamma} = \mathbf{G}/(\Delta \times (O/\mathfrak{N}')^\times)$ and induces a canonical algebra structure of $\mathbf{h}_\kappa(\mathfrak{N}\mathfrak{p}^\infty,\epsilon;A)$ over $A[[\mathbf{\Gamma}]]$.

Let W be a sufficiently large complete discrete valuation ring inside $\overline{\mathbb{Q}}_p$ (as before). Define $W[\epsilon] \subset \overline{\mathbb{Q}}_p$ by the W-subalgebra generated by the values of ϵ (over the finite adeles). It has canonical generators $\mathbb{T}(y)$ over $\mathbf{\Lambda} = W[[\mathbf{\Gamma}]]$. Here note that the operator $\langle\mathfrak{q}\rangle$ acts via multiplication by $N(\mathfrak{q})\epsilon_+(\mathfrak{q})$ for the fixed central character ϵ_+, where $N(\mathfrak{q}) = |O/\mathfrak{q}|$.

We write $h_\kappa^{\mathfrak{p}.ord}(U,\epsilon;W)$, $h_\kappa^{\mathfrak{p}.ord}(\mathfrak{N}p^\alpha,\epsilon;W)$ and $\mathbf{h}^{\mathfrak{p}.ord} = \mathbf{h}_\kappa^{\mathfrak{p}.ord} = \mathbf{h}_\kappa^{\mathfrak{p}.ord}(\mathfrak{N}\mathfrak{p}^\infty,\epsilon;W)$ for the image of the (nearly) $\mathfrak{p}$-ordinary projector $e = \lim_n \mathbb{T}(\varpi_\mathfrak{p})^{n!}$ ($\mathbb{T}(\varpi_\mathfrak{p})$ may depend on the choice of $\varpi_\mathfrak{p}$ but e is independent of the choice). By Brad Wilson's thesis [**W**] at UCLA, this algebra $\mathbf{h}^{\mathfrak{p}.ord}$ is shown to be independent of $\kappa_\mathfrak{p}$ (as long as $\kappa_2 - \kappa_1 \geq I$) but dependent on $\kappa^\mathbf{p}$. We plan to give a full details of a proof of this fact in [**HHA**] as well as the control theorem we referred in the introduction, though we only need the the axioms (A1–2) in this paper. We write $\mathbf{h}_\kappa^{\mathfrak{p}.ord}$ if the relevance of the weight is important. The algebra $\mathbf{h}_\kappa^{\mathfrak{p}.ord}$ is by definition the universal nearly $\mathfrak{p}$-ordinary Hecke algebra over $\mathbf{\Lambda}$ of level $\mathfrak{N}\mathfrak{p}^\infty$ with "Neben character" ϵ. We also note here that, if $\mathfrak{p}$ is the unique prime in F above p, this algebra $\mathbf{h}_\kappa^{\mathfrak{p}.ord}(\mathfrak{N}\mathfrak{p}^\infty,\epsilon;W)$ is exactly the one $\mathbf{h}_\kappa(\psi^+,\psi')$ employed in [**HiT**, page 240] (note that in [**HiT**] we assumed $\kappa_1 \geq \kappa_2$ reversing our normalization here).

Note that $\mathbf{\Gamma}$ is isomorphic to the additive group $\mathbb{Z}_p^m$ for $m = [F_\mathfrak{p} : \mathbb{Q}_p]$. Take a point $P \in \mathrm{Spf}(\mathbf{\Lambda})(\overline{\mathbb{Q}}_p)$. If P is arithmetic, $\epsilon_P = P\kappa_\mathfrak{p}(P)^{-1}$ is a character of $\mathbf{\Gamma}$. By abusing a symbol, we write ϵ_P for the character $(\epsilon_{P,1,},\epsilon_{P,2},\epsilon_+)$ given by $\epsilon_{P,j}$ on $\mathbf{\Gamma}$ and ϵ_j on $\Delta \times (O/\mathfrak{N}')^\times$. Writing the conductor of $\epsilon_P^-|_{O_\mathfrak{p}^\times}$ as $\mathfrak{p}^{f(P)}$, we define $r(P) \geq 0$ by $\mathfrak{p}^{r(P)+1} = \mathfrak{p}^{f(P)} \cap \mathfrak{p}$. Let $\kappa(P) = \kappa_\mathfrak{p}(P) + \kappa^\mathbf{p}$ for the fixed κ. As long as P is arithmetic with $\kappa(P) = \kappa$, we have a canonical specialization morphism:

$$\mathbf{h}_\kappa^{\mathfrak{p}.ord}(\mathfrak{N}\mathfrak{p}^\infty,\epsilon;W) \otimes_{\mathbf{\Lambda},P} W[\epsilon_P] \twoheadrightarrow h_\kappa^{\mathfrak{p}.ord}(\mathfrak{N}\mathfrak{p}^{r(P)+1},\epsilon;W[\epsilon_P]).$$

The specialization morphism takes the generators $\mathbb{T}(y)$ to $\mathbb{T}(y)$. We show in [**HHA**] that this morphism is an isogeny (surjective and of finite kernel). Instead in this paper, we prove a weaker bounded dimensionality of $h_\kappa^{\mathfrak{p}.ord}(\mathfrak{N}\mathfrak{p}^{r(P)+1},\epsilon;K[\epsilon_P])$ independent of $r(P)$ for the field of fractions K of W, which is logically sufficient to prove the theorem in the introduction under the axiom (A1–2). We prove the bounded dimensionality in §1.11 after defining $\mathfrak{p}$-slope 0 analytic families.

1.10. Analytic families of Hecke eigenforms. For a fixed κ and ϵ_+, we have the algebra $\mathbf{h}$ as in (A1–2). We may take $\mathbf{h}$ to be the image of the nearly p-ordinary Hecke algebra of level $\mathfrak{N}p^\infty$ in [**PAF**, §4.2.12] in the Hecke algebra generated in $\mathbf{h}_\kappa^{\mathfrak{p}.ord} := \mathbf{h}_\kappa(\mathfrak{N}\mathfrak{p}^\infty,\epsilon;W)$, or any $\mathbf{h}$ giving a closed subscheme $\mathrm{Spec}(\mathbf{h})$ of $\mathrm{Spec}(\mathbf{h}_\kappa^{\mathfrak{p}.ord})$ satisfying (A1–2).

By fixing an isomorphism $\mathbf{\Gamma} \cong \mathbb{Z}_p^m$ with $m = [F_\mathfrak{p} : \mathbb{Q}_p]$, we have identified $\mathbf{\Lambda}$ with $W[[T_1,\dots,T_m]]$ for $\{t_i = 1 + T_i\}_{i=1,\dots,m}$ corresponding to a $\mathbb{Z}_p$-basis $\{\gamma_i\}_{i=1,\dots,m}$ of $\mathbf{\Gamma}$. Regard $\kappa_{1,\mathfrak{p}}$ as a character of $O_\mathfrak{p}^\times$ whose value at γ_i is

$$\gamma_i^{\kappa_{1,\mathfrak{p}}} = \prod_{\sigma \in I_\mathfrak{p}} \sigma(\gamma_i)^{\kappa_{1,\mathfrak{p},\sigma}}.$$

We may write an arithmetic prime P as a prime $\mathbf{\Lambda}$-ideal

$$P = (t_i - \epsilon_1(\gamma_i)^{-1}\gamma_i^{\kappa_{1,\mathfrak{p}}})W[\epsilon][[\mathbf{\Gamma}]] \cap \mathbf{\Lambda}.$$

When $\kappa_{1,\mathfrak{p}} = k_1 I_{\mathfrak{p}}$, our choice of the extension $\gamma^{\kappa_{1,\mathfrak{p}}}$ is given by $\gamma \mapsto N(\gamma)^{k_1}$ for the norm map $N = N_{F_{\mathfrak{p}}/\mathbb{Q}_p}$ on $O_{\mathfrak{p}}^{\times}$. For a point $P \in \mathrm{Spec}(\mathbf{\Lambda})(\overline{\mathbb{Q}}_p)$ killing $(t_i - \zeta_i^{-1}\gamma_i^{\kappa_{1,\mathfrak{p}}})$ for $\zeta_i \in \mu_{p^\infty}(W)$, we make explicit the character ϵ_P. First we define a character $\epsilon_{P,1,\mathbf{\Gamma}} : O_{\mathfrak{p}}^{\times} \to \mu_{p^\infty}(W)$ factoring through $\mathbf{\Gamma} = O_{\mathfrak{p}}^{\times}/\Delta$ by $\epsilon_{P,1,\mathbf{\Gamma}}(\gamma_i) = \zeta_i$ for all i. Then for the fixed ϵ_+, we put $\epsilon_{P,2,\mathbf{\Gamma}} = (\epsilon_+|_{\mathbf{\Gamma}})\epsilon_{P,1,\mathbf{\Gamma}}^{-1}$. With the fixed data $\epsilon_1^{(\mathbf{\Gamma})} := \epsilon_1|_{(O/\mathfrak{N}')^\times \times \Delta}$ and $\epsilon_2^{(\mathbf{\Gamma})} := \epsilon_2|_{(O/\mathfrak{N}')^\times \times \Delta}$, we put $\epsilon_{P,j} = \epsilon_{j,P,\mathbf{\Gamma}}\epsilon_j^{(\mathbf{\Gamma})}$. In this way, we form $\epsilon_P = (\epsilon_{P,1}, \epsilon_{P,2}, \epsilon_+)$.

Let $\mathrm{Spec}(\mathbb{I})$ be a reduced irreducible component $\mathrm{Spec}(\mathbb{I}) \subset \mathrm{Spec}(\mathbf{h})$. Since $\mathbf{h}$ is torsion-free of finite rank over $\mathbf{\Lambda}$, $\mathrm{Spec}(\mathbb{I})$ is a finite torsion-free covering of $\mathrm{Spec}(\mathbf{\Lambda})$. Write $a(y)$ and $a(\mathfrak{l})$ for the image of $T(y)$ and $T(\mathfrak{l})$ in $\mathbb{I}$ (so, $a(\varpi_{\mathfrak{p}})$ is the image of $T(\varpi_{\mathfrak{p}})$). We also write $\mathbf{a}(y)$ for the image of $\mathbb{T}(y)$; so, $\mathbf{a}(y) = y_p^{-\kappa_1}a(y)$. If $P \in \mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$ induces an arithmetic point P_0 of $\mathrm{Spec}(\mathbf{\Lambda})$, we call it again an *arithmetic* point of $\mathrm{Spec}(\mathbb{I})$, and put $\kappa(P) = \kappa(P_0)$. If P is arithmetic, by (A2), we have a Hecke eigenform $\mathbf{f}_P \in S_\kappa(\mathfrak{N}\mathfrak{p}^{r(P)+1}, \epsilon_P; \overline{\mathbb{Q}}_p)$ such that its eigenvalue for $\mathbb{T}(\mathfrak{l})$ and $\mathbb{T}(y)$ is given by $a_P(\mathfrak{l}) := P(a(\mathfrak{l})), a_P(y) := P(a(y)) \in \overline{\mathbb{Q}}_p$ for all $\mathfrak{l}$ and $y \in F_p^{\times}$. Thus $\mathbb{I}$ gives rise to a family $\mathcal{F} = \mathcal{F}_{\mathbb{I}} = \{\mathbf{f}_P | \text{arithemtic } P \in \mathrm{Spec}(\mathbb{I}) \text{ with } \kappa(P) = \kappa\}$ of classical Hecke eigenforms. We call this family a *p-adic analytic family of $\mathfrak{p}$-slope* 0 (of weight κ with coefficients in $\mathbb{I}$) associated to an irreducible component $\mathrm{Spec}(\mathbb{I}) \subset \mathrm{Spec}(\mathbf{h})$. There are sub-family corresponding to any closed integral subscheme $\mathrm{Spec}(\mathbb{J}) \subset \mathrm{Spec}(\mathbb{I})$ as long as $\mathrm{Spec}(\mathbb{J})$ has densely populated arithmetic points. Abusing our language slightly, for any covering $\pi : \mathrm{Spec}(\widetilde{\mathbb{I}}) \twoheadrightarrow \mathrm{Spec}(\mathbb{I})$, we consider the pulled back family $\mathcal{F}_{\widetilde{\mathbb{I}}} = \{\mathbf{f}_P = \mathbf{f}_{\pi(P)} | \text{arithemtic } P \in \mathrm{Spec}(\widetilde{\mathbb{I}}) \text{ with } \kappa(P) = \kappa\}$. The choice of $\widetilde{\mathbb{I}}$ is often the normalization of $\mathbb{I}$ or the integral closure of $\mathbb{I}$ in a finite extension of the quotient field of $\mathbb{I}$.

Identify $\mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$ with $\mathrm{Hom}_{W\text{-alg}}(\mathbb{I}, \overline{\mathbb{Q}}_p)$ so that each element $a \in \mathbb{I}$ gives rise to a "function" $a : \mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p) \to \overline{\mathbb{Q}}_p$ whose value at $(P : \mathbb{I} \to \overline{\mathbb{Q}}_p) \in \mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$ is $a_P := P(a) \in \overline{\mathbb{Q}}_p$. Then a is an analytic function of the rigid analytic space associated to $\mathrm{Spf}(\mathbb{I})$. We call such a family $\mathfrak{p}$-slope 0 because $|\mathbf{a}_P(\varpi_{\mathfrak{p}})|_p = 1$ for the p-adic absolute value $|\cdot|_p$ of $\overline{\mathbb{Q}}_p$ (it is also called a $\mathfrak{p}$-ordinary family).

1.11. Bounded Dimensionality. We define a character $\widetilde{\epsilon} = (\widetilde{\epsilon}_1, \widetilde{\epsilon}_2, \widetilde{\epsilon}_+)$ with values in $W^\times$ by the Teichmüller lift of the reduction modulo $\mathfrak{m}_W$ of the characters ϵ. By the following theorem, $\mathbf{h}_\kappa(\mathfrak{N}\mathfrak{p}^\infty, \epsilon; W)$ is finite over $\mathbf{\Lambda}$; so, the essence of [**W**] is torsion-freeness over $\mathbf{\Lambda}$ of this algebra and the control theorem.

THEOREM 1.1. *Fix a weight κ with $\kappa_2 - \kappa_1 \geq I$ and a level $\mathfrak{N}$. Then the dimension*

$$\dim_{\mathbb{C}_p} S_\kappa^{\mathfrak{p}.ord}(\mathfrak{N}\mathfrak{p}^{r(P)+1}, \epsilon_P; \mathbb{C}_p)$$

is bounded independently of arithmetic points $P \in \mathrm{Spec}(\mathbf{\Lambda})(\overline{\mathbb{Q}}_p)$ with $\kappa(P) = \kappa$.

Let $Y_0(\mathfrak{N}) = G(\mathbb{Q})\backslash G(\mathbb{A})/U_0(\mathfrak{N})Z \cdot SO_2(F_{\mathbb{R}})$ for the center Z nof $G(\mathbb{A})$ and $F_{\mathbb{R}} = F \otimes_{\mathbb{Q}} \mathbb{R}$. If the assertion of the theorem holds for $\mathfrak{N}$ contained in the principal ideal (N) for a sufficiently large integer N, the assertion holds for all $\mathfrak{N}$, because by the theory of primitive forms [**MFM**] 4.6 and [**H88**] Section 3 (or the strong multiplicity one theorem [**AAG**] Sections 4 and 10), one can recover the dimension of each space with lower level. Therefore, replacing $\mathfrak{N}$ by deeper level, we may assume that $Y_0(\mathfrak{N})$ is smooth. Define $n = \kappa_2 - \kappa_1 - I$. Taking a sufficiently large algebra R in $\mathbb{C}_p$ containing all conjugates of O, consider the representation $\rho_\sigma : GL_2(O) \to GL_2(R)$ induced by $\sigma \in I$, and take its symmetric n-th tensor representation $\bigotimes_\sigma \rho_\sigma^{sym\otimes n_\sigma}$. As in [**PAF**, §4.3.2], we realize this representation on the polynomials in $\{(X_\sigma, Y_\sigma)\}_\sigma$ with coefficients in R having homogeneous degree n_σ, where R is a $W[\epsilon_P]$-algebra. Then we twist the action by ϵ suitably (as in [**PAF**, §4.3.2]) and write the resulting module by $L(\kappa^*\epsilon; R)$. Further deepening $\mathfrak{N}$ if necessary, this module $L(\kappa^*\epsilon_P; R)$ gives rise to a locally constant sheaf over $Y_0(\mathfrak{N})$. Identifying $\mathbb{C}_p$ and $\mathbb{C}$ compatibly with our chosen embeddings $\mathbb{C} \hookleftarrow \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$. Then the Eichler-Shimura isomorphism (in [**PAF**, §4.3.4]) gives rise to a Hecke-equivariant embedding

$$S_\kappa(\mathfrak{N}, \epsilon; \mathbb{C}) \hookrightarrow H^d(Y_0(\mathfrak{N}), L(\kappa^*\epsilon; \mathbb{C})) \quad (d = [F : \mathbb{Q}]).$$

Thus to bound the dimension, we need to bound the rank of $H^d_{\mathfrak{p}.ord}(Y_0(\mathfrak{N}), L(\kappa^*\epsilon; W[\epsilon_P]))$, where $H^d_{\mathfrak{p}.ord}$ indicates the image of the $\mathfrak{p}$-ordinary projector e.

Proof. Take W to be the ring of Witt vectors with coefficients in an algebraic closure $\mathbb{F}$ of $\mathbb{F}_p$. We simply write L_P for $L(\kappa^*\epsilon; W[\epsilon_P])$. By the long exact sequence attached to $0 \to L_P \xrightarrow{x \mapsto \varpi \cdot x} L_P \to L_P \otimes_W \mathbb{F} \to 0$ for a prime element ϖ of $W[\epsilon_P]$, we have the following inclusion:

$$H^d(Y_0(\mathfrak{N}\mathfrak{p}^{r(P)+1}), L_P) \otimes_{W[\epsilon_P]} \mathbb{F} \hookrightarrow H^d(Y_0(\mathfrak{N}\mathfrak{p}^{r(P)+1}), L_P \otimes_{W[\epsilon_P]} \mathbb{F}).$$

Thus we need to bound

$$\dim_{\mathbb{F}} H^d(Y_0(\mathfrak{N}\mathfrak{p}^{r(P)+1}), L_P \otimes_{W[\epsilon_P]} \mathbb{F}),$$

since

$$\dim_{\mathbb{C}} H^d_{\mathfrak{p}.ord}(Y_0(\mathfrak{N}\mathfrak{p}^{r(P)+1}), L_P \otimes_{W[\epsilon_P]} \mathbb{C}) \leq \dim_{\mathbb{F}} H^d_{\mathfrak{p}.ord}(Y_0(\mathfrak{N}\mathfrak{p}^{r(P)+1}), L_P) \otimes_{W[\epsilon_P]} \mathbb{F}.$$

By the description of ϵ_P in the previous subsection, we have $\epsilon_P \equiv \widetilde{\epsilon} \mod \mathfrak{m}_{W[\epsilon_P]}$. Thus we have

$$H^d(Y_0(\mathfrak{N}\mathfrak{p}^{r(P)+1}), L_P \otimes_{W[\epsilon_P]} \mathbb{F}) \cong H^d(Y_0(\mathfrak{N}\mathfrak{p}^{r(P)+1}), L(\kappa^*\widetilde{\epsilon}, \mathbb{F})).$$

Note that $L(\widetilde{\kappa\epsilon}; \mathbb{F})$ is well defined over $Y_0(\mathfrak{N}\mathfrak{p})$. Since we have a natural bijection

$$U_0(\mathfrak{N}\mathfrak{p})\alpha_r U_0(\mathfrak{N}\mathfrak{p}^{r+1})/U_0(\mathfrak{N}\mathfrak{p}^{r+1}) \cong U_0(\mathfrak{N}\mathfrak{p})\alpha_r U_0(\mathfrak{N}\mathfrak{p})/U_0(\mathfrak{N}\mathfrak{p})$$

for $\alpha_r = \left(\begin{smallmatrix} \varpi_{\mathfrak{p}}^r & 0 \\ 0 & 1 \end{smallmatrix}\right)$, writing u_r for the normalized operator

$$u_r := \varpi_{\mathfrak{p}}^{-\kappa_1}[U_0(\mathfrak{N}\mathfrak{p})\alpha_r U_0(\mathfrak{N}\mathfrak{p}^{r+1})],$$

we get the following commutative diagram for $r = r(P)$:

$$\begin{array}{ccc} H^d(Y_0(\mathfrak{N}\mathfrak{p}^{r+1}), L_P \otimes_{W[\epsilon_P]} \mathbb{F}) & \xrightarrow{\text{Res}} & H^d(Y_0(\mathfrak{N}\mathfrak{p}), L(\kappa^*\widetilde{\epsilon}, \mathbb{F})) \\ \downarrow \mathbb{T}(\varpi_{\mathfrak{p}}^r) & \searrow & \downarrow \mathbb{T}(\varpi_{\mathfrak{p}}^r) \\ H^d(Y_0(\mathfrak{N}\mathfrak{p}^{r+1}), L_P \otimes_{W[\epsilon_P]} \mathbb{F}) & \xrightarrow[\text{Res}]{} & H^d(Y_0(\mathfrak{N}\mathfrak{p}), L(\kappa^*\widetilde{\epsilon}, \mathbb{F})), \end{array}$$

where the middle south-east arrow is given by u_r. Here, the original action of α_r on $\bigotimes_\sigma \rho_\sigma^{sym\otimes n_\sigma}$ is $\det(\alpha_r)^{-\kappa_1}$ times the action of α_r of the twisted module L_P (up to p-adic unit multiple); so, the operator u_r is intrinsically integral defined on the cohomology group $H^d(Y_0(\mathfrak{N}\mathfrak{p}^{r+1}), L_P \otimes_{W[\epsilon_P]} \mathbb{F})$ of characteristic p (and hence the above diagram commutes). This diagram shows the identity of dimension:

$$\dim_{\mathbb{F}} H^d_{\mathfrak{p}.ord}(Y_0(\mathfrak{N}\mathfrak{p}^{r(P)+1}), L_P \otimes_{W[\epsilon_P]} \mathbb{F}) = \dim_{\mathbb{F}} H^d_{\mathfrak{p}.ord}(Y_0(\mathfrak{N}\mathfrak{p}), L(\kappa^*\widetilde{\epsilon}, \mathbb{F})),$$

and therefore $\dim_{\mathbb{F}} H^d_{\mathfrak{p}.ord}(Y_0(\mathfrak{N}\mathfrak{p}^{r(P)+1}), L_P \otimes_{W[\epsilon_P]} \mathbb{F})$ is bounded. This finishes the proof. $\square$

1.12. Modular Galois representations. Each (reduced) irreducible component $\mathrm{Spec}(\mathbb{I})$ of the Hecke spectrum $\mathrm{Spec}(\mathbf{h})$ has a 2-dimensional semi-simple (actually absolutely irreducible) continuous representation $\rho_{\mathbb{I}}$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ with coefficients in the quotient field of $\mathbb{I}$ (see [**H86a**] and [**H89b**]). The representation $\rho_{\mathbb{I}}$ restricted to the p-decomposition group D_p is reducible (see [**HMI**, §2.3.8]). Define the p-adic avatar $\widehat{\epsilon}_+ : (F_{\mathbb{A}}^{(\infty)})^\times/F^\times \to \overline{\mathbb{Q}}_p^\times$ by $\widehat{\epsilon}_+(y) = \epsilon_+(y)y_p^{I-\kappa_1-\kappa_2}$. We write $\rho_{\mathbb{I}}^{ss}$ for its semi-simplification over D_p. As is well known now (e.g., [**HMI**, §2.3.8]), $\rho_{\mathbb{I}}$ is unramified outside Np and satisfies

$$\text{(Gal)} \qquad \mathrm{Tr}(\rho_{\mathbb{I}}(\mathrm{Frob}_{\mathfrak{l}})) = a(\mathfrak{l}) \quad \text{for all prime } \mathfrak{l} \nmid \mathfrak{p}\mathfrak{N},$$

and
(Loc)
$$\rho_{\mathbb{I}}^{ss}([\gamma_i^s, F_{\mathfrak{p}}]) \sim \begin{pmatrix} t_i^{-s} & 0 \\ 0 & t_i^s\widehat{\varepsilon}_+([\gamma_i^s, F_{\mathfrak{p}}])N_{F_{\mathfrak{p}}/\mathbb{Q}_p}(\gamma_i)^{-s} \end{pmatrix} \text{ and } \rho_{\mathbb{I}}^{ss}([\varpi_{\mathfrak{p}}, F_{\mathfrak{p}}]) \sim \left(\begin{smallmatrix} * & 0 \\ 0 & \mathbf{a}(\varpi_{\mathfrak{p}}) \end{smallmatrix}\right),$$

where we have written $\gamma_i^s \in \mathbf{\Gamma}$ for $s \in \mathbb{Z}_p$ via the multiplicative $\mathbb{Z}_p$-module structure of $\mathbf{\Gamma}$ and $[x, F_{\mathfrak{p}}]$ is the local Artin symbol.

By (Gal) and Chebotarev density, $\mathrm{Tr}(\rho_{\mathbb{I}})$ has values in $\mathbb{I}$; so, for any integral closed subscheme $\mathrm{Spec}(\mathbb{J}) \subset \mathrm{Spec}(\mathbb{I})$ with projection $\pi : \mathbb{I} \to \mathbb{J}$, $\pi \circ \mathrm{Tr}(\rho_{\mathbb{I}}) : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathbb{J}$ gives rise to a pseudo-representation of Wiles (e.g., [**MFG**, §2.2]). Then by a theorem of Wiles, we can make a unique 2-dimensional semi-simple continuous representation $\rho_{\mathbb{J}} : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to GL_2(Q(\mathbb{J}))$ unramified outside $\mathfrak{N}p$ with $\mathrm{Tr}(\rho_{\mathbb{J}}(\mathrm{Frob}_{\mathfrak{l}})) = \pi(a(\mathfrak{l}))$ for all primes $\mathfrak{l}$ outside $\mathfrak{N}p$, where $Q(\mathbb{J})$ is the quotient field of $\mathbb{J}$. If $\mathrm{Spec}(\mathbb{J})$ is one point $P \in \mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$, we write ρ_P for $\rho_{\mathbb{J}}$. This is the Galois representation associated to the Hecke eigenform $\mathbf{f}_P$ (given in [**H89b**]). Then the above condition (Loc) implies
(Ram)
$$\rho_P^{ss}([u, F_{\mathfrak{p}}]) \sim \begin{pmatrix} \epsilon_{P,1}(u)u^{-\kappa_1} & 0 \\ 0 & \epsilon_{P,2}(u)u^{-\kappa_2} \end{pmatrix} \text{ for } u \in O_{\mathfrak{p}}^\times \text{ and } \rho_P^{ss}([y, F_{\mathfrak{p}}]) \sim \left(\begin{smallmatrix} * & 0 \\ 0 & \mathbf{a}_P(y) \end{smallmatrix}\right)$$

for each arithmetic point P.

1.13. CM theta series. Following the description in [**H04**, §6.2], we construct CM theta series with $\mathfrak{p}$-slope 0 and describe the CM component which gives rise to such theta series. We recall a cusp form $\mathbf{f}$ on $GL_2(F_{\mathbb{A}})$ with complex multiplication by a CM field M. Let M/F be a CM field with integer ring $\mathfrak{O}$ and choose a CM type Σ:

$$I_M = \mathrm{Hom}_{\text{field}}(M, \overline{\mathbb{Q}}) = \Sigma \sqcup \Sigma c$$

for complex conjugation c. To assure the $\mathfrak{p}$-slope 0 condition, we need to assume that the CM type Σ is $\mathfrak{p}$–ordinary, that is, the set $\Sigma_{\mathfrak{p}}$ of $\mathfrak{p}$–adic places induced by $\iota_p \circ \sigma$ for $\sigma \in \Sigma$ is disjoint from $\Sigma_{\mathfrak{p}^c}$ (its conjugate by the generator c of $\mathrm{Gal}(M/F)$). The existence of such a $\mathfrak{p}$-ordinary CM type implies that the prime $\mathfrak{p}$ of F split in M/F. Thus $\Sigma = \Sigma_{\mathfrak{p}} \sqcup \Sigma'$ and $I_M = \Sigma_{\mathfrak{p}} \sqcup \Sigma' \sqcup \Sigma' \circ c \sqcup \Sigma_{\mathfrak{p}^c}$. Write $\mathfrak{p} = \mathfrak{P}\mathfrak{P}^c$ in $\mathfrak{O}$

for two primes $\mathfrak{P} \neq \mathfrak{P}^c$ such that $\mathfrak{P}$ is induced by ι_p on M. For each $k \in \mathbb{Z}[I]$ and $X = \Sigma, \Sigma_{\mathfrak{p}}, \Sigma'$, we write $kX = \sum_{\sigma \in X} k_{\sigma|F}\sigma$.

We choose $\kappa_2 - \kappa_1 \geq I$ with $\kappa_1 + \kappa_2 = [\kappa]I$ for an integer $[\kappa]$. We then choose a Hecke character λ of conductor $\mathfrak{C}\mathfrak{P}^e$ ($\mathfrak{C}$ prime to $\mathfrak{p}$) such that

$$\lambda((\alpha)) = \alpha^{c\kappa_1\Sigma + \kappa_2\Sigma} \text{ for } \alpha \in M^\times \text{ with } \alpha \equiv 1 \mod \mathfrak{C}\mathfrak{P}^e,$$

where $\mathfrak{P}^e = \mathfrak{P}^{e(\mathfrak{P})}\mathfrak{P}^{ce(\mathfrak{P}^c)}$ for $e = e(\mathfrak{P})\mathfrak{P} + e(\mathfrak{P}^c)\mathfrak{P}^c$. If we need to indicate that $\mathfrak{C}$ is the prime-to-$\mathfrak{p}$ conductor of λ, we write $\mathfrak{C}(\lambda)$ for $\mathfrak{C}$. We also decompose $\mathfrak{C} = \prod_{\mathfrak{L}} \mathfrak{L}^{e(\mathfrak{L})}$ for prime ideals $\mathfrak{L}$ of M. We extend λ to a p–adic idele character $\widehat{\lambda} : M_{\mathbb{A}}^\times / M^\times M_\infty^\times \to \overline{\mathbb{Q}}_p^\times$ so that $\widehat{\lambda}(a) = \lambda(aO)a_p^{-\kappa_2\Sigma - c\kappa_1\Sigma}$. By class field theory, we may regard $\widehat{\lambda}$ as a character of $\mathrm{Gal}(\overline{F}/M)$. Any character φ of $\mathrm{Gal}(\overline{F}/M)$ of the form $\widehat{\lambda}$ as above is called "of weight κ". For a prime ideal $\mathfrak{L}$ of M outside p, we write $\lambda_{\mathfrak{L}}$ for the restriction of $\widehat{\lambda}$ to $M_{\mathfrak{L}}^\times$; so, $\lambda_{\mathfrak{L}}(x) = \widehat{\lambda}(x) = \lambda(x)$ for $x \in M_{\mathfrak{L}}^\times$. For a any prime ideal $\mathfrak{P}'|p$ of M, we put $\lambda_{\mathfrak{P}'}(x) = \widehat{\lambda}(x)x^{\kappa_2\Sigma + c\kappa_1\Sigma} = \lambda(x)$ for $x \in M_{\mathfrak{P}'}^\times$. In particular, for the prime $\mathfrak{P}|\mathfrak{p}$, we have $\lambda_{\mathfrak{P}}(x) = \widehat{\lambda}(x)x^{\kappa_2\Sigma_{\mathfrak{p}}}$ for $x \in M_{\mathfrak{P}}^\times$, and $\lambda_{\mathfrak{P}^c}(x) = \widehat{\lambda}(x)x^{c\kappa_1\Sigma_{\mathfrak{p}}}$ for $x \in M_{\mathfrak{P}^c}^\times$. Then $\lambda_{\mathfrak{L}}$ for all prime ideals $\mathfrak{L}$ (including those above p) is a continuous character of $M_{\mathfrak{L}}^\times$ with values in $\overline{\mathbb{Q}}$ whose restriction to $\mathfrak{O}_{\mathfrak{L}}^\times$ is of finite order. By the condition $\kappa_1 \neq \kappa_2$, $\widehat{\lambda}$ cannot be of the form $\widehat{\lambda} = \phi \circ N_{M/F}$ for an idele character $\phi : F_{\mathbb{A}}^\times / F^\times F_{\infty+}^\times \to \overline{\mathbb{Q}}_p^\times$.

We define a function $(F_{\mathbb{A}}^{(\infty)})^\times \ni y \mapsto c(y, \theta(\lambda))$ supported by integral ideles by

$$c(y, \theta(\lambda)) = \sum_{x \in (M_{\mathbb{A}}^{(\infty)})^\times, xx^c = y} \lambda(x) \quad \text{if } y \text{ is integral,} \tag{1.25}$$

where x runs over elements in $M_{\mathbb{A}}^\times / (\widehat{\mathfrak{O}}^{(\mathfrak{C}\mathfrak{P}^e)})^\times$ satisfying the following four conditions: (0) $x_\infty = 1$, (i) $x\mathfrak{O}$ is an integral ideal of M, (ii) $N_{M/F}(x) = y$ and (iii) $x_{\mathfrak{Q}} = 1$ for prime factors $\mathfrak{Q}$ of the conductor $\mathfrak{C}\mathfrak{P}^e$. The q–expansion determined by the coefficients $c(y, \theta(\lambda))$ gives a unique element $\theta(\lambda) \in S_\kappa(\mathfrak{N}_\theta, \epsilon'_\lambda; \overline{\mathbb{Q}})$ ([**HiT**, Theorem 6.1] and [**HMI**, Theorem 2.72]), where $\mathfrak{N}_\theta = N_{M/F}(\mathfrak{C}\mathfrak{P}^e)d(M/F)$ for the discriminant $d(M/F)$ of M/F and ϵ'_λ is a suitable "Neben" character. We have

(C) The central character $\epsilon_{\lambda+}$ of the automorphic representation $\pi(\lambda)$ generated by $\theta(\lambda)$ is given by the product: $x \mapsto \lambda(x)|x|_{\mathbb{A}} \left(\frac{M/F}{x}\right)$ for $x \in F_{\mathbb{A}}^\times$ and the quadratic character $\left(\frac{M/F}{}\right)$ of the CM quadratic extension M/F.

We describe the Neben character $\epsilon_\lambda = (\epsilon_{\lambda,1}, \epsilon_{\lambda,2}, \epsilon_{\lambda+})$ of the minimal form $\mathbf{f}(\lambda)$ in the automorphic representation $\pi(\lambda)$. For that, we choose a decomposition $\mathfrak{C} = \mathfrak{F}\mathfrak{F}_c\mathfrak{I}$ so that $\mathfrak{F}\mathfrak{F}_c$ is a product of split primes and $\mathfrak{I}$ for the product of inert or ramified primes, $\mathfrak{F} + \mathfrak{F}_c = \mathfrak{O}$ and $\mathfrak{F} \subset \mathfrak{F}_c^c$. If we need to make the dependence on λ of these symbols explicit, we write $\mathfrak{F}(\lambda) = \mathfrak{F}$, $\mathfrak{F}_c(\lambda) = \mathfrak{F}_c$ and $\mathfrak{I}(\lambda) = \mathfrak{I}$. We put $\mathfrak{f} = \mathfrak{F} \cap F$ and $\mathfrak{i} = \mathfrak{I} \cap F$. Define $\lambda^-(\mathfrak{a}) = \lambda(\mathfrak{a}^{c-1})$ (with $\mathfrak{a}^{c-1} = \mathfrak{a}^c\mathfrak{a}^{-1}$), and write its conductor as $\mathfrak{C}(\lambda^-)$. Decompose as above $\mathfrak{C}(\lambda^-) = \mathfrak{F}(\lambda^-)\mathfrak{F}^c(\lambda^-)\mathfrak{I}(\lambda^-)$ so that we have the following divisibility of radicals $\sqrt{\mathfrak{F}(\lambda^-)}|\sqrt{\mathfrak{F}(\lambda)}$ and $\sqrt{\mathfrak{F}_c(\lambda^-)}|\sqrt{\mathfrak{F}_c(\lambda)}$. Let $T_0 = \mathrm{Res}_{\mathfrak{O}/O}\mathbb{G}_m$. The $\mathfrak{l}$-component $\epsilon_{\lambda,j,\mathfrak{l}}$ ($j = 1, 2$) of the character $\epsilon_{\lambda,j}$ is given as follows:

(hk1) For $\mathfrak{l}|\mathfrak{f}$, we identify $T_0(O_{\mathfrak{l}}) = \mathfrak{O}_{\mathfrak{L}}^\times \times \mathfrak{O}_{\mathfrak{L}^c}^\times$ with this order for the prime ideal $\mathfrak{L}|(\mathfrak{l}\mathfrak{O} \cap \mathfrak{F})$ and define $\epsilon_{\lambda,1,\mathfrak{l}} \times \epsilon_{\lambda,2,\mathfrak{l}}$ by the restriction of $\lambda_{\mathfrak{L}} \times \lambda_{\mathfrak{L}^c}$ to $T_0(O_{\mathfrak{l}})$.

(hk2) For $\mathfrak{P}|\mathfrak{p}$, we identify $T_0(O_\mathfrak{p}) = \mathfrak{O}_\mathfrak{P}^\times \times \mathfrak{O}_{\mathfrak{P}^c}^\times$ and define $\epsilon_{\lambda,1,\mathfrak{p}} \times \epsilon_{\lambda,2,\mathfrak{p}}$ by the restriction of $\lambda_\mathfrak{P} \times \lambda_{\mathfrak{P}^c}$ to $T_0(O_\mathfrak{p})$.

(hk3) For $\mathfrak{l}|(\mathfrak{I}(\lambda) \cap O)d(M/F)$ but $\mathfrak{l} \nmid (\mathfrak{I}(\lambda^-) \cap O)$, we can choose a character $\phi_\mathfrak{l} : F_\mathfrak{l}^\times \to \mathbb{C}^\times$ such that $\lambda_\mathfrak{L} = \phi_\mathfrak{l} \circ N_{M_\mathfrak{L}/F_\mathfrak{l}}$. Then we define $\epsilon_{\lambda,1,\mathfrak{l}}(a) = \left(\frac{M_\mathfrak{L}/F_\mathfrak{l}}{a}\right)\phi_\mathfrak{l}(a)$ and $\epsilon_{\lambda,2,\mathfrak{l}}(d) = \phi_\mathfrak{l}(d)$, where $\mathfrak{L}$ is the prime factor of $\mathfrak{l}$ in M and $\left(\frac{M_\mathfrak{L}/F_\mathfrak{l}}{d}\right)$ is the character of $M_\mathfrak{L}/F_\mathfrak{l}$.

(hk4) For $\mathfrak{l}|(\mathfrak{I}(\lambda^-) \cap O)$, $\epsilon_{\lambda,1,\mathfrak{l}} = \epsilon_{\lambda+,\mathfrak{l}}|_{O_\mathfrak{l}^\times}$ and $\epsilon_{\lambda,2,\mathfrak{l}} = 1$ for the central character $\epsilon_{\lambda+}$ given in (C).

We now give an explicit description of the automorphic representation $\pi(\lambda)$. In Cases (hk1–3), taking a prime $\mathfrak{L}|\mathfrak{l}$ in M, we have

$$\pi_\mathfrak{p}(\lambda) \cong \begin{cases} \pi(\lambda_\mathfrak{P}, \lambda_{\mathfrak{P}^c}) & \text{in Case (hk1)}, \\ \pi(\lambda_\mathfrak{L}, \lambda_{\mathfrak{L}^c}) & \text{in Case (hk2)}, \\ \pi(\left(\frac{M_\mathfrak{L}/F_\mathfrak{l}}{}\right)\phi_\mathfrak{l}, \phi_\mathfrak{l}) & \text{in Case (hk3)}. \end{cases} \tag{1.26}$$

In Case (hk4), $\pi_\mathfrak{l}(\lambda)$ is the super-cuspidal representation giving rise to $\mathrm{Ind}_{M_\mathfrak{l}}^{F_\mathfrak{l}} \widehat{\lambda}|_{\mathrm{Gal}(\overline{F}_\mathfrak{l}/M_\mathfrak{l})}$.

To describe of $\mathbf{f}(\lambda)$, we split $\mathfrak{N}_\theta$ into a product of co-prime ideals $\mathfrak{N}_{nc}$ and $\mathfrak{N}_{cusp}$ so that $\mathfrak{N}_{nc}$ is made up of primes in Cases (hk1–3). For $\mathfrak{l}|\mathfrak{N}_{nc}$, writing $\pi_\mathfrak{l}(\lambda) = \pi(\eta_\mathfrak{l}, \eta'_\mathfrak{l})$ for characters $\eta_\mathfrak{l}, \eta'_\mathfrak{l} : F_\mathfrak{l}^\times \to \mathbb{C}^\times$, we write $C_\mathfrak{l}$ for the conductor of $\eta_\mathfrak{l}^{-1}\eta'_\mathfrak{l}$. Define the minimal level of $\pi(\lambda)$ by

$$\mathfrak{N}(\lambda) = \mathfrak{N}_{cusp} \prod_{\mathfrak{l}|\mathfrak{N}_{nc}} C_\mathfrak{l},$$

where $\mathfrak{l}$ runs over primes satisfying one of the three conditions (hk1–3). Put

$$\Xi = \{\mathfrak{L}|\mathfrak{L} \supset \mathfrak{F}\mathfrak{P}, \mathfrak{L} \supset \mathfrak{N}(\lambda)\}$$

for primes $\mathfrak{L}$ of M. Then the minimal form $\mathbf{f}(\lambda)$ has the following q-expansion coefficient:

$$\mathbf{c}_p(y, \mathbf{f}(\lambda)) = \begin{cases} \sum_{xx^c=y, x_\Xi=1} \widehat{\lambda}(x) & \text{if } y \text{ is integral}, \\ 0 & \text{otherwise}, \end{cases} \tag{1.27}$$

where x runs over $(\widehat{\mathfrak{O}} \cap M_{\mathbb{A}^{(\infty)}}^\times/(\mathfrak{O}^{(\Xi)})^\times$ with $x_\mathfrak{L} = 1$ for $\mathfrak{L} \in \Xi$. See [**H04**, §6.2] for more details of this construction (though in [**H04**], the order of (κ_1, κ_2) is interchanged so that $\kappa_1 > \kappa_2$).

1.14. CM components. We fix $\kappa^\mathbf{p}$ and vary $\kappa_\mathfrak{p}$. We fix a Hecke character λ of type κ as in the previous subsection, and we continue to use the symbols defined above. We may regard the Galois character $\widehat{\lambda}$ as a character of $Cl_M(\mathfrak{C}p^\infty)$.

We consider the ray class group $Cl_M(\mathfrak{C}(\lambda^-)\mathfrak{p}^\infty)$ modulo $\mathfrak{C}(\lambda^-)\mathfrak{p}^\infty$. Since $\lambda^-(\mathfrak{a}^c) = (\lambda^-)^{-1}(\mathfrak{a})$, we have $\mathfrak{C}(\lambda^-) = \mathfrak{C}(\lambda^-)^c$. Thus $\mathrm{Gal}(M/F) = \langle c \rangle$ acts naturally on $Cl_M(\mathfrak{C}(\lambda^-)\mathfrak{p}^\infty)$. We define the anticyclotomic quotient of $Cl_M(\mathfrak{C}(\lambda^-)\mathfrak{p}^\infty)$ by

$$Cl_M^-(\mathfrak{C}(\lambda^-)\mathfrak{p}^\infty) := Cl_M(\mathfrak{C}(\lambda^-)\mathfrak{p}^\infty)/Cl_M(\mathfrak{C}(\lambda^-)\mathfrak{p}^\infty)^{1+c}.$$

We have canonical identities:

$$\mathfrak{O}_\mathfrak{p}^\times = \mathfrak{O}_\mathfrak{P}^\times \times \mathfrak{O}_{\mathfrak{P}^c}^\times = O_\mathfrak{p}^\times \times O_\mathfrak{p}^\times$$

on which c acts by interchanging the components. The natural inclusion $\mathfrak{O}_{\mathfrak{p}}^{\times}/\overline{\mathfrak{O}}^{\times} \hookrightarrow Cl(\mathfrak{C}(\lambda^-)\mathfrak{p}^\infty)$ therefore induces an inclusion $\mathbf{\Gamma} \hookrightarrow Cl_M^-(\mathfrak{C}(\lambda^-)\mathfrak{p}^\infty)$. Decompose $Cl_M^-(\mathfrak{C}(\lambda^-)\mathfrak{p}^\infty) = \mathbf{\Gamma}_M \times \Delta_M$ with the maximal finite subgroup Δ_M so that $\mathbf{\Gamma}_M \supset \mathbf{\Gamma}$. Then $\mathbf{\Gamma}$ is an open subgroup in $\mathbf{\Gamma}_M$. In particular, $W[[\mathbf{\Gamma}_M]]$ is a regular domain finite flat over $\mathbf{\Lambda}_W$. Thus we call $P \in \mathrm{Spec}(W[[\mathbf{\Gamma}_M]])(\overline{\mathbb{Q}}_p)$ arithmetic if P is above an arithmetic point of $\mathrm{Spec}(\mathbf{\Lambda}_W)(\overline{\mathbb{Q}}_p)$. Regard the tautological character

$$\upsilon : Cl_M(\mathfrak{C}p^\infty) \xrightarrow{\text{projection}} \mathbf{\Gamma}_M \hookrightarrow W[\mathbf{\Gamma}_M]]^\times$$

as a Galois character $\upsilon : \mathrm{Gal}(\overline{M}/M) \to W[\mathbf{\Gamma}_M]]^\times$.

The composite $\upsilon_P = P \circ \upsilon$ for an arithmetic point $P \in \mathrm{Spec}(W[[\mathbf{\Gamma}_M]])$ is of the form $\widehat{\varphi}_P$ for a Hecke character φ_P with p-type $\kappa'_{P,2}\Sigma_{\mathfrak{p}} + \kappa'_{P,1}\Sigma_{\mathfrak{p}}^c$ for $\kappa'_P = (\kappa'_{P,1}, \kappa'_{P,2}) \in \mathbb{Z}[I_{\mathfrak{p}}]^2$ satisfying $\kappa_{\mathfrak{p},2} + \kappa'_{P,2} - (\kappa_{1,\mathfrak{p}} + \kappa'_{P,1}) \geq I_{\mathfrak{p}}$. Assume that $\widehat{\lambda}$ has values in $W^\times$ (enlarging W if necessary). We then consider the product $\widehat{\lambda}\upsilon : \mathrm{Gal}(\overline{M}/M) \to W[[\mathbf{\Gamma}_M]]^\times$ and $\rho_{W[[\mathbf{\Gamma}_M]]} := \mathrm{Ind}_M^F \widehat{\lambda}\upsilon : \mathrm{Gal}(\overline{M}/M) \to GL_2(W[[\mathbf{\Gamma}_M]])$. Define $\mathbb{I}_M \subset W[[\mathbf{\Gamma}_M]]$ by the $\mathbf{\Lambda}_W$-subalgebra generated by $\mathrm{Tr}(\rho_{W[[\Gamma_M]]})$. Then we have the localization identity $\mathbb{I}_{M,P} = W[[\mathbf{\Gamma}_M]]_P$ for any arithmetic point P (this follows from the irreducibility of $\rho_P = P \circ \rho_{W[[\mathbf{\Gamma}_M]]} = \mathrm{Ind}_M^F \widehat{\lambda}\upsilon_P$).

We have a surjective projection $\pi_\lambda : \mathbf{h}_\kappa^{\mathfrak{p}.ord} \to \mathbb{I}_M$ sending $T(\mathfrak{l})$ to $\mathrm{Tr}(\rho_{W[[\mathbf{\Gamma}]]}(\mathrm{Frob}_{\mathfrak{l}}))$ for primes $\mathfrak{l}$ outside $\mathfrak{N}(\lambda)$. If π_λ factors through $\mathbf{h}$, $\mathrm{Spec}(\mathbb{I}_M)$ is an irreducible component of $\mathrm{Spec}(\mathbf{h})$ by (A1). In particular, $\rho_{\mathbb{I}_M} = \rho_{W[[\mathbf{\Gamma}_M]]}$. Replacing $\mathrm{Spec}(\mathbf{h})$ by $\mathrm{Spec}(\mathbf{h}) \cup (\bigcup_{\mathbb{I}_M} \mathrm{Spec}(\mathbb{I}_M)$ for all possible CM components $\mathbb{I}_M$, we assume that any CM component is contained in $\mathrm{Spec}(\mathbf{h})$. Since $\mathbb{I}_M$ is torsion-free and finite over $\mathbf{\Lambda}$, the axioms (A1–2) are intact under this change. In the same manner as in [**HMI**, Proposition 3.78], we can prove the following facts assuming (A1–2):

Proposition 1.2. *Let the notation and the assumptions be as above. Then for the reduced part $\mathbf{h}^{red}$ of $\mathbf{h}$ and each arithmetic point $P \in \mathrm{Spec}(\mathbf{\Lambda})(\overline{\mathbb{Q}}_p)$ with $\kappa(P) = \kappa$, $\mathrm{Spec}(\mathbf{h}_P^{red})$ is étale finite over $\mathrm{Spec}(\mathbf{\Lambda}_P)$. In particular, no irreducible components cross each other at a point above arithmetic point of $\mathrm{Spec}(\mathbf{\Lambda})$ of weight κ.*

A component $\mathbb{I}$ is called a *CM component* if there exists a nontrivial character $\chi : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathbb{I}^\times$ such that $\rho_{\mathbb{I}} \cong \rho_{\mathbb{I}} \otimes \chi$. We also say that $\mathbb{I}$ has *complex multiplication* if $\mathbb{I}$ is a CM component. In this case, we call the corresponding family $\mathcal{F}$ a CM family (or we say $\mathcal{F}$ has complex multiplication). It is known essentially by deformaton theory of Galois characters (cf. [**H11**, §4]) that any CM component is given by $\mathrm{Spec}(\mathbb{I}_M)$ as above for a specific choice of λ.

If $\mathcal{F}$ is a CM family associated to $\mathbb{I}$ with $\rho_{\mathbb{I}} \cong \rho_{\mathbb{I}} \otimes \chi$, then χ is a quadratic character of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ which cuts out an imaginary quadratic field M, i.e., $\chi = \left(\frac{M/F}{}\right)$. Write $\widetilde{\mathbb{I}}$ for the integral closure of Λ inside the quotient field of $\mathbb{I}$. The following three conditions are known to be equivalent:

- (CM1) *$\mathcal{F}$ has CM and $\rho_{\mathbb{I}} \cong \rho_{\mathbb{I}} \otimes \left(\frac{M/F}{}\right)$ ($\Leftrightarrow$ $\rho_{\mathbb{I}} \cong \mathrm{Ind}_M^F \widehat{\lambda}$ for a character $\widehat{\lambda} : \mathrm{Gal}(\overline{\mathbb{Q}}/M) \to \widetilde{\mathbb{I}}^\times$);*
- (CM2) *For all arithmetic P of $\mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$, $\mathbf{f}_P$ is a binary theta series of the norm form of M/F;*

(CM3) *For some arithmetic P of* $\mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$, $\mathbf{f}_P$ *is a binary theta series of the norm form of M/F.*

Indeed, (CM1) is equivalent to $\rho_{\mathbb{I}} \cong \mathrm{Ind}_M^F \widehat{\lambda}$ for a character $\widehat{\lambda} : \mathrm{Gal}(\overline{\mathbb{Q}}/M) \to \widetilde{\mathbb{I}}^\times$ unramified outside Np (e.g., [**MFG**, Lemma 2.15]). Since the characteristic polynomial of $\rho_{\mathbb{I}}(\sigma)$ has coefficients in $\mathbb{I}$, its eigenvalues fall in $\widetilde{\mathbb{I}}$; so, the character $\widehat{\lambda}$ has values in $\widetilde{\mathbb{I}}^\times$ (see, [**H86b**, Corollary 4.2]). Then by (Gal) and (Loc), $\widehat{\lambda}_P = P \circ \widehat{\lambda} : \mathrm{Gal}(\overline{\mathbb{Q}}/M) \to \overline{\mathbb{Q}}_p^\times$ for an arithmetic $P \in \mathrm{Spec}(\widetilde{\mathbb{I}})(\overline{\mathbb{Q}}_p)$ is a locally algebraic p-adic character, which is the p-adic avatar of a Hecke character $\lambda_P : M_{\mathbb{A}}^\times/M^\times \to \mathbb{C}^\times$ of type A_0 of the quadratic extension $M_{/F}$. Then by the characterization (Gal) of $\rho_{\mathbb{I}}$, $\mathbf{f}_P$ is the theta series with q-expansion $\sum_{\mathfrak{a}} \lambda_P(\mathfrak{a}) q^{N(\mathfrak{a})}$, where $\mathfrak{a}$ runs over all integral ideals of M. By $\kappa_2(P) - \kappa_1(P) \geq I$ (and (Gal)), M has to be a CM field in which $\mathfrak{p}$ is split (as the existence of Hecke characters of infinity type corresponding to such $\kappa(P)$ forces that M/F is a CM quadratic extension). This shows (CM1)⇒(CM2)⇒(CM3). If (CM2) is satisfied, we have an identity $\mathrm{Tr}(\rho_{\mathbb{I}}(\mathrm{Frob}_{\mathfrak{l}})) = a(\mathfrak{l}) = \chi(\mathfrak{l})a(\mathfrak{l}) = \mathrm{Tr}(\rho_{\mathbb{I}} \otimes \chi(\mathrm{Frob}_{\mathfrak{l}}))$ with $\chi = \left(\frac{M/F}{}\right)$ for all primes $\mathfrak{l}$ outside a finite set of primes (including prime factors of $\mathfrak{N}(\lambda)p$). By Chebotarev density, we have $\mathrm{Tr}(\rho_{\mathbb{I}}) = \mathrm{Tr}(\rho_{\mathbb{I}} \otimes \chi)$, and we get (CM1) from (CM2) as $\rho_{\mathbb{I}}$ is semi-simple. If a component $\mathrm{Spec}(\mathbb{I})$ contains an arithmetic point P with theta series $\mathbf{f}_P$ of M/F as above, either $\mathbb{I}$ is a CM component or otherwise P is in the intersection in $\mathrm{Spec}(\mathbf{h})$ of a component $\mathrm{Spec}(\mathbb{I})$ not having CM by M and another component having CM by M (as all families with CM by M are made up of theta series of M by the construction of CM components as above). The latter case cannot happen as two distinct components never cross at an arithmetic point in $\mathrm{Spec}(\mathbf{h})$ (i.e., the reduced part of the localization $\mathbf{h}_P$ is étale over $\mathbf{\Lambda}_P$ for any arithmetic point $P \in \mathrm{Spec}(\mathbf{\Lambda})(\overline{\mathbb{Q}}_p)$; see Proposition 1.2). Thus (CM3) implies (CM2). We call a binary theta series of the norm form of a CM quadratic extension of F a *CM theta series.*

Remark 1.1. If $\mathrm{Spec}(\mathbb{J})$ is an integral closed subscheme of $\mathrm{Spec}(\mathbb{I})$, we write the associated Galois representation as $\rho_{\mathbb{J}}$. By abuse of language, we say $\mathbb{J}$ has CM by M if $\rho_{\mathbb{J}} \cong \rho_{\mathbb{J}} \otimes \left(\frac{M/F}{}\right)$. Thus (CM3) is equivalent to having ρ_P with CM for some arithmetic point P. More generally, if we find some arithmetic point P in $\mathrm{Spec}(\mathbb{J})$ and ρ_P has CM, $\mathbb{J}$ and $\mathbb{I}$ have CM.

2. Weil numbers

Since $\overline{\mathbb{Q}}$ sits inside $\mathbb{C}$, it has "the" complex conjugation c. For a prime l, a Weil l-number $\alpha \in \overline{\mathbb{Q}}$ of integer weight $k \geq 0$ is defined by the following two properties:

(1) α is an algebraic integer;
(2) $|\alpha^\sigma| = l^{k/2}$ for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$ for the complex archimedean absolute value $|\cdot|$.

Note that $\mathbb{Q}(\alpha)$ is in a CM field finite over $\mathbb{Q}$ (e.g., [**Ho**, Proposition 4]), and the Weil number is realized by the Frobenius eigenvalue of a CM abelian variety over a finite field of characteristic l. If we ease the condition (1) above to

(1′) $l^m\alpha$ is an algebraic integer for $0 \leq m \in \mathbb{Z}$,

we call the number satisfying (1′) and (2) a *generalized Weil number* of weight k (generalized Weil numbers includes Frobenius eigenvalues of Tate twists of a CM abelian variety over a finite field of characteristic l). We call two nonzero numbers $a, b \in \overline{\mathbb{Q}}$ equivalent (written as $a \sim b$) if a/b is a root of unity. The following fact is proven in [**H11**, Corollary 2.5]:

PROPOSITION 2.1. *Let d be a positive integer. Let $\mathcal{K}_d$ be the set of all finite extensions of $\mathbb{Q}[\mu_{p^\infty}]$ of degree d inside $\overline{\mathbb{Q}}$ whose ramification at l is tame. Then there are only finitely many Weil l-numbers of a given weight in the set-theoretic union $\bigcup_{L \in \mathcal{K}_d} L$ (in $\overline{\mathbb{Q}}$) up to equivalence.*

Here is another lemma proven in [**H11**, Lemma 2.6]:

LEMMA 2.2. *Let $\mathcal{K}_?$ be one of $\mathcal{K}_{\mathcal{L}}$ and $\mathcal{K}_d$. Suppose $\mathcal{K}_? \neq \emptyset$. Then the group of roots of unity in the composite $\mathbf{L}$ of L for $L \in \mathcal{K}_?$ in $\overline{\mathbb{Q}}$ contains $\mu_{p^\infty}(K)$ as a subgroup of finite index.*

3. Theorems and conjectures

Hereafter, we fix a weight $\kappa \in \mathbb{Z}[I]^2$ satisfying

(W) $\kappa_2 - \kappa_1 \geq I$ *and* $\kappa_1 + \kappa_2 = [\kappa]I$ *for an integer* $[\kappa] \geq 1$.

As for the level, we assume

(L) $\mathfrak{N}\mathfrak{p}^{r_0}$ ($r_0 \geq 0, \mathfrak{p} \nmid \mathfrak{N}$) *is the conductor of ϵ^- prime to $\mathfrak{p}$ and $\mathfrak{N}'\mathfrak{p}^{r'_0}$ ($r'_0 \geq 0, \mathfrak{p} \nmid \mathfrak{N}'$) is the maximal ideal such that ϵ_j ($j = 1, 2$) are defined on $(O/\mathfrak{N}'\mathfrak{p}^{r'_0})^\times$.*

In addition to the weight, we fix the central character ϵ_+ and the starting Neben character ϵ, but the Neben character ϵ_P varies in such a way that $\epsilon^{-1}\epsilon_P$ factors through $\mathbf{\Gamma}$ having values in $\mu_{p^\infty}(\overline{\mathbb{Q}}_p)$. Let $\mathbf{f} \in S_\kappa(\mathfrak{N}\mathfrak{p}^{r+1}, \epsilon; W)$ be a Hecke eigenform normalized so that $\mathbf{f}|\mathbb{T}(y) = \mathbf{a}_p(y, \mathbf{f})\mathbf{f}$ for all y. Here $\mathbf{a}_p(y, \mathbf{f}) = y_p^{-\kappa_1} a(y, \mathbf{f})$. For primes $\mathfrak{l} \nmid \mathfrak{N}\mathfrak{p}$, write $\mathbf{f}|T(\mathfrak{l}) = (\alpha_\mathfrak{l} + \beta_\mathfrak{l})\mathbf{f}$ and $\alpha_\mathfrak{l}\beta_\mathfrak{l} = \epsilon_+(\mathfrak{l})l^{f_\mathfrak{l}}$ if $\mathfrak{l} \nmid \mathfrak{N}\mathfrak{p}^{r+1}$ ($\alpha_\mathfrak{l}, \beta_\mathfrak{l} \in \overline{\mathbb{Q}}$), where $|O/\mathfrak{l}| = l^{f_\mathfrak{l}}$. If $\mathfrak{l}|\mathfrak{N}\mathfrak{p}$, we put $\beta_\mathfrak{l} = 0$ and define $\alpha_\mathfrak{l} \in \overline{\mathbb{Q}}$ by $\mathbf{f}|U(\mathfrak{l}) = \alpha_\mathfrak{l}\mathbf{f}$. The inverse of the Hecke polynomial $H_\mathfrak{l}(X) = (1 - \alpha_\mathfrak{l}X)(1 - \beta_\mathfrak{l}X)$ for $\mathfrak{l} \nmid \mathfrak{N}'\mathfrak{p}$ gives the Euler $\mathfrak{l}$-factor of $L^{(\mathfrak{N}\mathfrak{p}')}(s, \pi_\mathbf{f}) = \sum_{\mathfrak{n}, \mathfrak{n}+\mathfrak{N}'=O} a(\mathfrak{n}, \mathbf{f})N(\mathfrak{n})^{-s}$ after replacing X by $|O/\mathfrak{l}|^{-s} = N(\mathfrak{l})^{-s}$.

Let $\mathcal{F} = \{\mathbf{f}_P\}_{P \in \mathrm{Spec}(\mathbb{I})(\mathbb{C}_p)}$ be a p-adic analytic family of Hecke eigen cusp forms of $\mathfrak{p}$-slope 0. Without assuming (W), the function $P \mapsto \mathbf{a}_{p,P}(y) = \mathbf{a}_p(y, \mathbf{f}_P)$ is a function on $\mathrm{Spec}(\mathbb{I})$ in the structure sheaf $\mathbb{I}$. Since $a(\varpi_\mathfrak{p}, \mathbf{f}_P) = \varpi_\mathfrak{p}^{\kappa_1}\mathbf{a}_p(\varpi_\mathfrak{p}, \mathbf{f}_P)$ for a fixed κ_1 in (W), $P \mapsto a(\varpi_\mathfrak{p}, \mathbf{f}_P) = \alpha_{\mathfrak{p},P}$ is also an element in $\mathbb{I}$. We write $\alpha_{\mathfrak{l},P}, \beta_{\mathfrak{l},P}$ for $\alpha_\mathfrak{l}, \beta_\mathfrak{l}$ for $\mathbf{f}_P$, which does not depend on the choice of $\varpi_\mathfrak{p}$ by (1.19). By [**B**] (and [**H88**, Lemma 12.2]), $\alpha_{\mathfrak{l},P}$ is a generalized Weil l-number of weight $[\kappa]f_\mathfrak{l}$ for $f_\mathfrak{l}$ given by $|O/\mathfrak{l}| = l^{f_\mathfrak{l}}$. Writing $|\kappa_1| = \max_{\sigma \in I \text{ with } \kappa_{1,\sigma} < 0}(|\kappa_{1,\sigma}|)$, the Hodge weight $\kappa + (|\kappa_1|I, |\kappa_1|I)$ has all non-negative coefficients, and hence $l^{f_\mathfrak{l}|\kappa_1|}\alpha_{\mathfrak{l},P}$ is a Weil l-number.

We state the horizontal theorem in a form slightly stronger than the theorem in the introduction:

THEOREM 3.1. *Pick an infinite set $\mathcal{A} \subset \mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$ of arithmetic points P with fixed weight κ satisfying $\kappa_2 - \kappa_1 \geq I$. Write $M_\mathcal{A}(\mathcal{F}) \subset \overline{\mathbb{Q}}$ for the field generated over $K := \mathbb{Q}(\mu_{p^\infty})$ by $\{\alpha_{\mathfrak{p},P}\}_{P \in \mathcal{A}}$, where P runs over all arithmetic points in $\mathcal{A}$. Then the field $M_\mathcal{A}(\mathcal{F})$ is a finite extension of $\mathbb{Q}(\mu_{p^\infty})$ if, and only if $\mathbf{f}_P$ is a CM theta series for some arithmetic P with $\kappa(P) = \kappa$. Moreover we have $\lim_{P \in \mathcal{A}}[K(\alpha_{\mathfrak{p},P}) : K] = \infty$ unless $\mathcal{F}$ has complex multiplication.*

We prove this theorem in Section 6. For a prime $\mathfrak{l} \nmid p$, we may conjecture the $\mathfrak{l}$-version of the stronger form in the horizontal case also:

CONJECTURE 3.2 (Horizontal $\mathfrak{l}$-version). *Write $M^{(\mathfrak{l})}_{\mathcal{A}}(\mathcal{F})$ for the field generated over $K := \mathbb{Q}(\mu_{p^\infty})$ by $\{\alpha_{\mathfrak{l},P}\}_{P\in\mathcal{A}}$. Then $M^{(\mathfrak{l})}_{\mathcal{A}}(\mathcal{F})$ is a finite extension of K if, and only if for some arithmetic P with $k(P) \geq 1$, either $\mathbf{f}_P$ is a CM theta series or the automorphic representation generated by $\mathbf{f}_P$ is square-integrable at l. If $[M^{(\mathfrak{l})}_{\mathcal{A}}(\mathcal{F}) : K] = \infty$, we have*

$$\lim_{P\in\mathcal{A}}[K(a(\mathfrak{l},\mathbf{f}_P)) : K] = \infty \quad (\Leftrightarrow \lim_{P\in\mathcal{A}}[K(\alpha_{\mathfrak{l},P}) : K] = \infty).$$

In the same manner as was done after Conjecture 3.4 in [**H11**], one can show $[M^{(l)}_{\mathcal{A}}(\mathcal{F}) : K] < \infty$ if $\mathbf{f}_{P_0}$ is square-integrable at a prime $\mathfrak{l} \nmid p$ for one arithmetic P_0. We can prove the following statements in exactly the same manner as in the elliptic modular case treated in [**H11**] as Proposition 3.5 (WH):

PROPOSITION 3.3. *Let $\widetilde{M}_{H,\kappa}(\mathcal{F})$ be the field generated over $\mathbb{Q}$ by $\{\alpha^2_{\mathfrak{l},P}, \beta^2_{\mathfrak{l},P}\}_{\mathfrak{l},P}$, where P runs over all arithmetic point with $\kappa(P) = \kappa$ for a fixed κ with $\kappa_2 - \kappa_1 \geq I$ and l runs over all primes. The field $\widetilde{M}_{H,\kappa}(\mathcal{F})$ is a finite extension of $\mathbb{Q}(\mu_{p^\infty})$ for a fixed κ if, and only if $\mathbf{f}_P$ is a CM theta series for some arithmetic P with $\kappa(P) = \kappa$.*

We add one more lemma:

LEMMA 3.4. *Let $\mathcal{F}$ be a $\mathfrak{p}$-slope 0 p-adic analytic family of Hecke eigenforms with coefficients in $\mathbb{I}$. Let $K = \mathbb{Q}(\mu_{p^\infty})$ and fix κ with $\kappa_2 - \kappa_1 \geq I$. Then the degree $[K(\mathbf{f}_P) : K(\alpha_{\mathfrak{p},P})]$ for arithmetic P with $\kappa(P) = \kappa$ is bounded independently of P.*

PROOF. As we have seen, $\epsilon_P\epsilon^{-1}$ has values in μ_{p^∞} for all arithmetic points P. We prove that the degree $[K'(\mathbf{f}_P) : K'(\alpha_{\mathfrak{p},P})]$ is bounded independently of P for $K' = \mathbb{Q}(\mu_{p^\infty}, \epsilon)$. Here K' is the field generated by the values of ϵ over finite ideles over $\mathbb{Q}(\mu_{p^\infty})$. Note here that K' is a finite extension of K, and hence the lemma follows from the boundedness of $[K'(\mathbf{f}_P) : K'(\alpha_{\mathfrak{p},P})]$. Then by Theorem 1.1, $\dim_{\mathbb{C}_p} S^{\mathfrak{p}.ord}_{\kappa}(\mathfrak{N}\mathfrak{p}^{r(P)+1}, \epsilon_P; \mathbb{C}_p)$ is bounded by a constant d independent of P with $\kappa(P) = \kappa$. Since ϵ_P has values in K', if $\sigma \in \mathrm{Aut}(\mathbb{C}_p/K(\alpha_{\mathfrak{p},P}))$, $\mathbf{f}^\sigma_P$ is another Hecke eigenform within the same space $S^{\mathfrak{p}.ord}_{\kappa}(\mathfrak{N}\mathfrak{p}^{r(P)+1}, \epsilon_P; \mathbb{C}_p)$. Thus

$$[K'(\mathbf{f}_P) : K'(a(p, \mathbf{f}_P))] = \#\{\mathbf{f}^\sigma_P | \sigma \in \mathrm{Aut}(\mathbb{C}_p/K'(\alpha_{\mathfrak{p},P}))\} \leq d$$

as desired. □

4. Rigidity lemmas

We study formal subschemes of $\widehat{G} := \widehat{\mathbb{G}}^n_m$ stable under the action of $t \mapsto t^z$ for all z in an open subgroup U of $\mathbb{Z}^\times_p$.

LEMMA 4.1. *Let $X = \mathrm{Spf}(\mathcal{X})$ be a closed formal subscheme of $\widehat{G} = \widehat{\mathbb{G}}^n_{m/W}$ flat geometrically irreducible over W (i.e., $\mathcal{X} \cap \overline{\mathbb{Q}}_p = W$). Suppose there exists an open subgroup U of $\mathbb{Z}^\times_p$ such that X is stable under the action $\widehat{G} \ni t \mapsto t^u \in \widehat{G}$ for all $u \in U$. If there exists a subset $\Omega \subset X(\mathbb{C}_p) \cap \mu^n_{p^\infty}(\mathbb{C}_p)$ Zariski dense in X, then $\zeta^{-1}X$ is a formal subtorus for some $\zeta \in \Omega$.*

A similar assertion is not valid for a formal group $\widehat{\mathbb{G}}^2_{m/K} = \mathrm{Spec}(K[[T, T']])$ over a characteristic 0 field K. Writing $t = 1 + T$ and $t' = 1 + T'$ for multiplicative variables, the formal subscheme Z defined by $t^{\log(t')} = 1$ is not a formal torus, but it is stable under $(t, t') \mapsto (t^m, t'^m)$ for any $m \in \mathbb{Z}$. See [**C1**, Remark 6.6.1 (iv)] for an optimal expected form of the assertion similar to the above lemma.

Proof. Let X_s be the singular locus of the associated scheme $X^{sh} = \mathrm{Spec}(\mathcal{X})$ over W, and put $X^\circ = X^{sh} \setminus X_s$. The scheme X_s is a closed formal subscheme of X. To see this, we note, by the structure theorem of complete neotherian ring, that $\mathcal{X}$ is finite over a power series ring $W[[X_1, \dots, X_d]] \subset \mathcal{X}$ for $d = \dim_W X$ (cf. [**CRT**, §29]). The sheaf of continuous differentials $\Omega_{\mathcal{X}/\mathrm{Spf}(W[[X_1,\dots,X_d]]}$ is a torsion $\mathcal{X}$-module, and X_s is the support of the formal sheaf of $\Omega_{\mathcal{X}/\mathrm{Spf}(W[[X_1,\dots,X_d]]}$ (which is a closed formal subscheme of X). The regular locus of X° is open dense in the generic fiber $X^{sh}_{/K} := X^{sh} \times_W K$ of X^{sh} (for the field K of fractions of W). Then $\Omega^\circ := X^\circ \cap \Omega$ is Zariski dense in $X^{sh}_{/K}$.

In this proof, by extending scalars, we always assume that W is sufficiently large so that for $\zeta \in \Omega$ we focus on, we have $\zeta \in \widehat{G}(W)$ and that we have a plenty of elements of infinite order in $X(W)$ and in $X^\circ(K) \cap X(W)$, which we simply write as $X^\circ(W) := X^\circ(K) \cap X(W)$.

Note that the stabilizer U_ζ of $\zeta \in \Omega$ in U is an open subgroup of U. Indeed, if the order of ζ is equal to p^a, then $U_\zeta = U \cap (1 + p^a\mathbb{Z}_p)$. Thus making a variable change $t \mapsto t\zeta^{-1}$ (which commutes with the action of U_ζ), we may assume that the identity $\mathbf{1}$ of $\widehat{G}$ is in Ω°.

Let $\widehat{G}^{an}$, X_{an} and X^s_{an} be the rigid analytic spaces associated to X and X° (in Berthelot's sense in [**dJ**, §7]). We put $X^\circ_{an} = X_{an} \setminus X^s_{an}$, which is an open rigid analytic subspace of X_{an}. Then we apply the logarithm $\log : \widehat{G}^{an}(\mathbb{C}_p) \to \mathbb{C}_p^n = Lie(\widehat{G}^{an}_{/\mathbb{C}_p})$ sending $(t_j)_j \in \widehat{G}^{an}(\mathbb{C}_p)$ (the p-adic open unit ball centered at $\mathbf{1} = (1, 1, \dots, 1)$) to $(\log_p(t_j))_j \in \mathbb{C}_p^n$ for the p-adic Iwasawa logarithm map $\log_p : \mathbb{C}_p^\times \to \mathbb{C}_p$. Then for each smooth point $x \in X^\circ(W)$, taking a small analytic open neighborhood V_x of x (isomorphic to an open ball in W^d for $d = \dim_W X$) in $X^\circ(W)$, we may assume that $V_x = G_x \cap X^\circ(W)$ for an n-dimensional open ball G_x in $\widehat{G}(W)$ centered at $x \in \widehat{G}(W)$. Since $\Omega^\circ \neq \emptyset$, $\log(X^\circ(W))$ contains the origin $0 \in \mathbb{C}_p^n$. Take $\zeta \in \Omega^\circ$. Write T_ζ for the Tangent space at ζ of X. Then $T_\zeta \cong W^d$ for $d = \dim_W X$. The space $T_\zeta \otimes_W \mathbb{C}_p$ is canonically isomorphic to the tangent space T_0 of $\log(V_\zeta)$ at 0.

If $\dim_W X = 1$, there exists an infinite order element $t_1 \in X(W)$. We may (and will) assume that $U = (1 + p^m\mathbb{Z}_p)$ for $0 < m \in \mathbb{Z}$. Then X is the (formal) Zariski closure $\overline{t_1^U}$ of

$$t_1^U = \{t_1^{1+p^m z} | z \in \mathbb{Z}_p\} = t_1\{t_1^{p^m z} | z \in \mathbb{Z}_p\},$$

which is a coset of a formal subgroup Z. The group Z is the Zariski closure of $\{t_1^{p^m z} | z \in \mathbb{Z}_p\}$; in other words, regarding t_1^u as a W-algebra homomorphism $t_1^u : \mathcal{X} \to \mathbb{C}_p$, we have $t_1 Z = \mathrm{Spf}(\mathcal{Z})$ for $\mathcal{Z} = \mathcal{X}/\bigcap_{u \in U} \mathrm{Ker}(t_1^u)$. Since t_1^U is an infinite set, we have $\dim_W Z > 0$. From geometric irreducibility and $\dim_W X = 1$, we conclude $X = t_1 Z$ and $Z \cong \widehat{\mathbb{G}}_m$. Since X contains roots of unity $\zeta \in \Omega \subset \mu^n_{p^\infty}(W)$, we confirm that $X = \zeta Z$ for $\zeta \in \Omega \cap \mu^n_{p^{m'}}$ for $m' \gg 0$. Replacing t_1 by $t_1^{p^m}$ for m as above if necessary, we have the translation $\mathbb{Z}_p \ni s \mapsto \zeta t_1^s \in Z$ of one parameter subgroup $\mathbb{Z}_p \ni s \mapsto t_1^s$. Thus we have $\log(t_1) = \frac{dt_1^s}{ds}|_{s=0} \in T_\zeta$, which is sent by "$\log : \widehat{G} \to \mathbb{C}_p^n$" to $\log(t_1) \in T_0$. This implies that $\log(t_1) \in T_0$ and hence

$\log(t_1) \in T_\zeta$ for any $\zeta \in \Omega^\circ$ (under the identification of the tangent space at any $x \in \widehat{G}$ with $Lie(\widehat{G})$). Therefore T_ζ's over $\zeta \in \Omega^\circ$ can be identified canonically. This is natural as Z is a formal torus, and the tangent bundle on Z is constant, giving $Lie(Z)$.

Suppose that $d = \dim_W X > 1$. Consider the Zariski closure Y of t^U for an infinite order element $t \in V_\zeta$ (for $\zeta \in \Omega^\circ$). Since U permutes finitely many geometrically irreducible components, each component of Y is stable under an open subgroup of U. Therefore $Y = \bigcup \zeta' \mathcal{T}_{\zeta'}$ is a union of formal subtori $\mathcal{T}_{\zeta'}$ of dimension ≤ 1, where ζ' runs over a finite set inside $\mu^n_{p^\infty}(\mathbb{C}_p) \cap X(\mathbb{C}_p)$. Since $\dim_W Y = 1$, we can pick $\mathcal{T}_{\zeta'}$ of dimension 1 which we denote simply by $\mathcal{T}$. Then $\mathcal{T}$ contains t^u for some $u \in U$. Applying the argument in the case of $\dim_W X = 1$ to $\mathcal{T}$, we find $u \log(t) = \log(t^u) \in T_\zeta$; so, $\log(t) \in T_\zeta$ for any $\zeta \in \Omega^\circ$ and $t \in V_\zeta$. Summarizing our argument, we have found

(T) The Zariski closure of t^U in X for an element $t \in V_\zeta$ of infinite order contains a coset $\xi\mathcal{T}$ of one dimensional subtorus $\mathcal{T}$, $\xi^{p^{m'}} = 1$ and $t^{p^{m'}} \in \mathcal{T}$ for some $m' > 0$;

(D) Under the notation as above, we have $\log(t) \in T_\zeta$.

Moreover, the image $\overline{V}_\zeta$ of V_ζ in $\widehat{G}/\mathcal{T}$ is isomorphic to $(d-1)$-dimensional open ball. If $d > 1$, therefore, we can find $\overline{t}' \in \overline{V}_\zeta$ of infinite order. Pulling back $\overline{t}'$ to $t' \in V_\zeta$, we find $\log(t), \log(t') \in T_\zeta$, and $\log(t)$ and $\log(t')$ are linearly independent in T_ζ. Inductively arguing this way, we find infinite order elements $t_1, \ldots, t_d$ in V_ζ such that $\log(t_i)$ span over the quotient field K of W the tangent space $T_{\zeta/K} = T_\zeta \otimes_W K \hookrightarrow T_0$ (for any $\zeta \in \Omega^\circ$). We identify $T_{\mathbf{1}/K} \subset T_0$ with $T_{\zeta/K} \subset T_0$. Thus the tangent bundle over $X^\circ_{/K}$ is constant as it is constant over the Zariski dense subset Ω°. Therefore X° is close to an open dense subscheme of a coset of a formal subgroup. We pin-down this fact.

Take $t_j \in V_\zeta$ as above ($j = 1, 2, \ldots, d$) which give rise to a basis $\{\partial_j = \log(t_j)\}_j$ of the tangent space of $T_{\zeta/K} = T_{\mathbf{1}/K}$. Note that $t_j^u \in X$ and $u\partial_j = \log(t_j^u) = u\log(t_j) \in T_{\mathbf{1}/K}$ for $u \in U$. The embedding $\log : V_\zeta \hookrightarrow T_{\mathbf{1}} \subset Lie(\widehat{G}_{/W})$ is surjective onto a open neighborhood of $0 \in T_{\mathbf{1}}$ (by extending scalars if necessary). For $t \in V_\zeta$, if we choose t closer to ζ, $\log(t)$ getting closer to 0. Thus replacing $t_1, \ldots, t_d$ inside V_ζ by elements in V_ζ closer to ζ, we may assume that $\log(t_i) \pm \log(t_j)$ for all $i \neq j$ is in $\log(V_\zeta)$.

So, for each pair $i \neq j$, we can find $t_{i\pm j} \in V_\zeta$ such that $\log(t_i t_j^{\pm 1}) = \log(t_i) \pm \log(t_j) = \log(t_{i\pm j})$. The element $\log(t_{i\pm j})$ is uniquely determined in $\log(\widehat{G}_{an}(\mathbb{C}_p)) \cong \widehat{G}_{an}(\mathbb{C}_p)/\mu^n_{p^\infty}(\mathbb{C}_p)$. Thus we conclude $\zeta'_{i\pm j} t_i t_j^{\pm 1} = t_{i\pm j}$ for some $\zeta'_{i\pm j} \in \mu^n_{p^N}$ for sufficiently large N. Replacing X by its image under the p-power isogeny $\widehat{G} \ni t \mapsto t^{p^N} \in \widehat{G}$ and t_i by $t_i^{p^N}$, we may assume that $t_i t_j^{\pm 1} = t_{i\pm j}$ all in X. Since $t_i^U \subset X$, by (T), for a sufficiently large $m' \in \mathbb{Z}$, we find a one dimensional subtorus $\widehat{H}_i$ containing $t_i^{p^{m'}}$ such that $\zeta_i \widehat{H}_i \subset X$ with some $\zeta_i \in \mu^n_{p^{m'}}$ for all i. Thus again replacing X by the image of the p-power isogeny $\widehat{G} \ni t \mapsto t^{p^{m'}} \in \widehat{G}$, we may assume that the subgroup $\widehat{H}$ (Zariski) topologically generated by $t_1, \ldots, t_d$ is contained in X. Since $\{\log(t_i)\}_i$ is linearly independent, we conclude $\dim_W \widehat{H} \geq d = \dim_W X$, and hence X must be the formal subgroup $\widehat{H}$ of $\widehat{G}$. Since X is geometrically irreducible, $\widehat{H} = X$ is a formal subtorus. Pulling it back by the p-power isogenies we have used, we conclude $X = \zeta\widehat{H}$ for the original X and $\zeta \in \mu^n_{p^{m'N}}(W)$. Since Ω is Zariski dense in X, we may assume that $\zeta \in \Omega$. This finishes the proof. $\square$

Corollary 4.2. *Let W be a complete discrete valuation ring in $\mathbb{C}_p$. Write $W[[T]] = W[[T_1, \dots, T_n]]$ for the tuple of variable $T = (T_1, \dots, T_n)$. Let*

$$\widehat{G} := \widehat{\mathbb{G}}^n_m = \mathrm{Spf}(W[t_1, \widehat{t_1^{-1} \dots, t_n}, t_n^{-1}]),$$

and identify $W[t_1, \widehat{t_1^{-1} \dots, t_n}, t_n^{-1}]$ with $W[[T]]$ for $t_j = 1 + T_j$. Let $\Phi(T_1, \dots, T_n) \in W[[T]]$, Suppose that there is a Zariski dense subset $\Omega \subset \mu^n_{p^\infty}(\mathbb{C}_p)$ in $\widehat{G}(\mathbb{C}_p)$ such that $\Phi(\zeta - 1) \in \mu_{p^\infty}(\mathbb{C}_p)$ for all $\zeta \in \Omega$. Then there exists $\zeta_0 \in \mu_{p^\infty}(W)$ and $z = (z_j)_j \in \mathbb{Z}_p^n$ with $z_j \in \mathbb{Z}_p$ such that $\zeta_0^{-1}\Phi(t) = \prod_j (t_j)^{z_j}$, where $(1+T)^x = \sum_{n=0}^\infty \binom{x}{n} T^n$ with $x \in \mathbb{Z}_p$.

Proof. Pick $\eta = (\eta_j) \in \Omega$. Making variable change $T \mapsto \eta^{-1}(T+1) - 1$ (i.e., $T_j \mapsto \eta_j^{-1}(T_j + 1) - 1$ for each j) replacing W by its finite extension if necessary, we may replace Ω by $\eta^{-1}\Omega \ni 1$; so, rewriting $\eta^{-1}\Omega$ as Ω, we may assume that $\mathbf{1} \in \Omega$. Then $\Phi(0) = \zeta_0 \in \mu_{p^\infty}$. Thus again replacing Φ by $\zeta_0^{-1}\Phi$, we may assume that $\Phi(0) = 1$.

For $\sigma \in \mathrm{Gal}(K(\mu_{p^\infty})/K)$ with the quotient field K of W,

$$\Phi(\zeta^\sigma - 1) = \Phi(\zeta - 1)^\sigma.$$

Writing $\phi(\zeta) = \Phi(\zeta - 1)$, the above identity means $\phi(\zeta^\sigma) = \phi(\zeta)^\sigma$. Identify $\mathrm{Gal}(K(\mu_{p^\infty})/K)$ with an open subgroup U of $\mathbb{Z}_p^\times$. This is possible as W is a discrete valuation ring, while $W[\mu_{p^\infty}]$ is not. Writing $\sigma_u \in \mathrm{Gal}(F(\mu_{p^\infty})/F)$ for the element corresponding to $u \in U$, we find that

$$\Phi \circ u(\zeta - 1) = \Phi(\zeta^u - 1) = \Phi(\zeta^{\sigma_u} - 1) = \Phi(\zeta - 1)^{\sigma_u} = u \circ \Phi(\zeta - 1).$$

We find that $u \circ \phi = \phi \circ u$ is valid on the Zariski dense subset Ω of $\mathrm{Spec}(W[[T]])$; so, ϕ as a formal scheme morphism of $\widehat{\mathbb{G}}^n_m$ into $\widehat{\mathbb{G}}_m$ commutes with the action of $u \in U$.

Regard $W[[T]]$ as the affine ring of the formal torus $\widehat{\mathbb{G}}^n_{m/W}$ (so that T is given by $t-1$). Note that $u \in \mathbb{Z}_p^\times$ acts on $\widehat{\mathbb{G}}^n_m$ as a group automorphism induced by a W-bialgebra automorphism of $W[[T]]$ sending $t = (1+T) \mapsto t^u = (1+T)^u = \prod_j (t_j)^u$. More generally, take a morphism of formal schemes $\phi \in \mathrm{Hom}_{SCH_{/W}}(\widehat{\mathbb{G}}_m)$, which sends 1 to 1. Put $\widehat{\mathbf{G}} := \widehat{\mathbb{G}}^n_m \times \widehat{\mathbb{G}}_{m/W}$. We consider the graph Γ_ϕ of ϕ which is an irreducible formal subscheme $\Gamma_\phi \subset \widehat{\mathbb{G}}^n_m \times \widehat{\mathbb{G}}_m$ smooth over W. Writing the variable on $\widehat{\mathbf{G}}$ as (T, T'), Γ_ϕ is the geometrically irreducible closed formal subscheme containing the identity $\mathbf{1} \in \widehat{\mathbf{G}}$ defined by the principal ideal $(t' - \phi(t))$. If $\phi \circ u = u \circ \phi$ for all u in an open subgroup U of $\mathbb{Z}_p^\times$ (where U acts on the source $\widehat{\mathbb{G}}^n_m$ and on the target $\widehat{\mathbb{G}}_m$ by $t \mapsto t^u$), Γ_ϕ is stable under the diagonal action of U on $\widehat{\mathbf{G}}$ and is finite flat over $\widehat{\mathbb{G}}^r_m$ (the left factor of $\widehat{\mathbf{G}}$). Then, applying Lemma 4.1 to Γ_ϕ, we find that Γ_ϕ is a subtorus of rank $n-1$ surjecting down to the last factor $\widehat{\mathbb{G}}_m$. Since any subtorus of rank $n-1$ in $\widehat{\mathbf{G}}$ whose projection to the last factor is étale surjective is defined by the equation $t' = (1+T)^z$. Thus $t' - \Phi(T) = (t' - (1+T)^z)u(T, T')$ for a unit power series $u(T, T') \in W[[T, T']]$. Thus $t' = t'u(T, T')$, and hence $u(T, T') = 1$. $\square$

5. A Frobenius eigenvalue formula

Recall the fixed weight κ with $\kappa_2 - \kappa_1 \geq I$. In this section, we assume the following conditions and notations:

(J1) Let $\mathrm{Spec}(\mathbb{J})$ be a closed reduced geometrically irreducible subscheme of $\mathrm{Spec}(\mathbb{I})$ flat over $\mathrm{Spec}(W)$ of relative dimension r with Zariski dense set $\mathcal{A}$ of arithmetic points of the fixed weight κ.

(J2) We identify $\mathrm{Spf}(\mathbf{\Lambda})$ for $\mathbf{\Lambda} = W[[\mathbf{\Gamma}]]$ with $\widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \mathbf{\Gamma}^*$ for $\mathbf{\Gamma}^* := \mathrm{Hom}_{\mathbb{Z}_p}(\mathbf{\Gamma}, \mathbb{Z}_p)$ naturally.

Then for any direct $\mathbb{Z}_p$-summand $\Gamma \subset \mathbf{\Gamma}$, $\widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \Gamma^*$ is a closed formal torus of $\widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \mathbf{\Gamma}^*$.

LEMMA 5.1. *Let the notation and the assumption be as in* (J1–2). *Then, after making extension of scalars to a sufficiently large complete discrete valuation ring $W \subset \mathbb{C}_p$, we can find a $\mathbb{Z}_p$-direct summand Γ of $\mathbf{\Gamma}$ with rank $\dim_W \mathrm{Spf}(\mathbb{J})$ and an arithmetic point $P_0 \in \mathcal{A} \cap \mathrm{Spec}(\mathbb{J})(W)$ such that we have the following cartesian diagram:*

$$\begin{array}{ccc} \mathrm{Spf}(\mathbb{I}) & \longrightarrow & \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \mathbf{\Gamma}^* = \mathrm{Spf}(\mathbf{\Lambda}_W) \\ \cup\uparrow & & \uparrow\cup \\ \mathrm{Spf}(\mathbb{J}) & \longrightarrow & P_0 \cdot (\widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \Gamma^*), \end{array}$$

where $P_0 \cdot (\widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \Gamma^*)$ is the image of the multiplication by the point $P_0 \in \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \mathbf{\Gamma}^*$ inside $\widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \mathbf{\Gamma}^*$.

PROOF. Let $\pi : \mathrm{Spec}(\mathbb{J}) \to \mathrm{Spec}(\mathbf{\Lambda})$ be the projection. Then the smallest reduced closed subscheme $Z \subset \mathrm{Spec}(\mathbf{\Lambda})$ containing the topological image of π contains an infinitely many arithmetic points of weight κ. Take a basis $\{\gamma_1, \ldots, \gamma_m\}$ of $\mathbf{\Gamma}$, and write $\widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \mathbf{\Gamma}^*$ as $\mathrm{Spf}(\widehat{W[t_j, t_j^{-1}]}_{j=1,\ldots,m})$ for the variable t_j corresponding to the dual basis $\{\gamma_j^*\}_j$ of $\mathbf{\Gamma}^*$. Let $P_1 \in Z$ be an arithmetic point of weight κ under $P \in \mathrm{Spec}(\mathbb{J})(W)$ (after replacing W by its finite extension, we can find P). Then by the variable change $t \mapsto P_1^{-1} \cdot t$ (which can be written as $t_j \mapsto \zeta_j \gamma_j^{-\kappa_1} t_j$ for suitable $\zeta_j \in \mu_{p^\infty}(W)$), the image of arithmetic points of $\mathrm{Spec}(\mathbb{J})$ of weight κ in Z is contained in $\mu_{p^\infty}^m(\overline{\mathbb{Q}}_p)$. Since Z is defined over W, $\Omega := Z(\mathbb{C}_p) \cap \mu_{p^\infty}^n(\mathbb{C}_p)$ is stable under $\mathrm{Gal}(K[\mu_{p^\infty}]/K)$ for the quotient field K of W. Then by Lemma 4.1, we may assume, after making further variable change $t \mapsto \eta^{-1} t$ for $\eta \in \mu_{p^\infty}^m(W)$ (again replacing W by its finite extension if necessary), that Z contains $\widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \Gamma^*$ for a rank $\dim_W \mathrm{Spf}(\mathbb{J})$ direct summand Γ of $\mathbf{\Gamma}$. Then putting $P_0 = P_1 \cdot \eta$, we get the desired result. □

If a prime $\mathfrak{l}$ is a factor of $\mathfrak{N}$ (so $\mathfrak{l} \neq \mathfrak{p}$) and $\mathbf{f}_P$ (or more precisely the automorphic representation generated by $\mathbf{f}_P$) is Steinberg (resp. super-cuspidal) at $\mathfrak{l}$ for an arithmetic point P, then all members of $\mathcal{F}$ are Steinberg (resp. super-cuspidal) at $\mathfrak{l}$ (see the remark after Conjecture 3.4 in [**H11**]).

Take a prime $\mathfrak{l} \nmid \mathfrak{N}\mathbf{p}$ of O with $\alpha_{\mathfrak{l},P} \neq 0$ for some P (so, $\mathfrak{l}$ can be equal to $\mathfrak{p}$). If $\mathfrak{l} \nmid \mathfrak{N}\mathfrak{p}$, replacing $\mathbb{I}$ by its finite extension, we assume that $\det(T - \rho_{\mathbb{I}}(\mathrm{Frob}_l)) = 0$ has roots in $\mathbb{I}$. Since $\alpha_{\mathfrak{l},P} \neq 0$ for some P (and hence $\alpha_{\mathfrak{l},P}$ is a p-adic unit), $\mathbf{f}_P$ is not super-cuspidal at $\mathfrak{l}$ for any arithmetic P.

Take Γ as in Lemma 5.1 given for $\mathbb{J}$, and write $\Lambda = W[[\Gamma]]$. Fix a basis $\gamma_1, \dots, \gamma_r \in \Gamma$ and identify Λ with $W[[T]](T = (T_i)_{i=1,\dots,r})$ by $\gamma_i \leftrightarrow t_i = 1 + T_i$. Let Q be the quotient field of Λ and fix its algebraic closure $\overline{Q}$. We embed $\mathbb{J}$ into $\overline{Q}$. We introduce one more notation:

(J3) If $\mathfrak{l} = \mathfrak{p}$, let A be the image $a(\varpi_{\mathfrak{p}})$ in $\mathbb{J}$, and if $\mathfrak{l} \nmid \mathfrak{N}\mathfrak{p}$, fix a root A in $\overline{Q}$ of $\det(T - \rho_{\mathbb{J}}(\mathrm{Frob}_{\mathfrak{l}})) = 0$. Replacing $\mathbb{J}$ by its finite extension, we assume that $A \in \mathbb{J}$.

Recall $A_P = P(A)$. Take and fix p^n-th root t_i^{1/p^n} of t_i in $\overline{Q}$ $(i = 1, 2, \dots, r)$ and consider

$$W[\mu_{p^n}][[T]][t^{1/p^n}] := W[\mu_{p^n}][[T_1, \dots, T_r]][t_1^{1/p^n}, \dots, t_r^{1/p^n}] \subset \overline{Q}$$

which is independent of the choice of t^{1/p^n}. Take a basis $\{\gamma = \gamma_1, \dots, \gamma_m\}$ of $\mathbf{\Gamma}$ over $\mathbb{Z}_p$ (containing $\{\gamma_1, \dots, \gamma_r\}$) and write $N : Cl(\mathfrak{N}\mathfrak{p}^\infty) \to \mathbb{Z}_p^\times$ for the norm map $N_{F_{\mathfrak{p}}/\mathbb{Q}_p}$. We write t_j $(t := t_1)$ for the variable of $\widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \mathbf{\Gamma}^*$ corresponding to the dual basis of $\{\gamma_j\}_j$ of $\mathbf{\Gamma}^*$.

Proposition 5.2 (Frobenius eigenvalue formula). *Let the notation and the assumption be as in* (J1–3). *Pick a prime ideal $\mathfrak{l}$ and define A as in* (J3). *Write $K := \mathbb{Q}[\mu_{p^\infty}]$ and $L_P = K(A_P)$ for each arithmetic point P with $\kappa(P) = \kappa$. Suppose*

($\mathrm{BT}_{\mathfrak{l}}$) *$L_P/K$ is a finite extension of bounded degree independently of $P \in \mathcal{A}$ and in L_P/K, the prime $\mathfrak{l}$ is at worst tamely ramified for all $P \in \mathcal{A}$.*

Then, after making extension of scalars to a sufficiently large W, we have

$$A \in W[\mu_{p^n}][[T_1, \dots, T_r]][t_1^{1/p^n}, \dots, t_r^{1/p^n}] \cap \mathbb{J}$$

in $\overline{Q}$ for $0 \le n \in \mathbb{Z}$, and there exists $s = (s_i) \in \mathbb{Q}_p^r$ and a constant $c \in W^\times$ such that $A(T) = c(1+T)^s = c\prod_i t_i^{s_i}$.

Let $E_P = K[\alpha_{\mathfrak{p},P}]$. We will see in the next section that c is actually a generalized Weil l-number for any $\mathfrak{l}$ under the following condition:

($\mathrm{B}_{\mathfrak{p}}$) E_P/K is a finite extension of bounded degree independently of $P \in \mathcal{A}$.

To simplify the notation, for $k = r$ and m, we often write $(\zeta\gamma^{-\kappa_{1,\mathfrak{p}}}t - 1)$ for the ideal in $W[[T_1, \dots, T_k]]$ generated by a tuple $(\zeta_j\gamma_j^{-\kappa_{1,\mathfrak{p}}}t_j - 1)$ for $j = 1, 2, \dots, k$ (where $\zeta = (\zeta_j)$ is also a tuple in $\mu_{p^\infty}^k(\overline{\mathbb{Q}}_p)$). The value of k should be clear in the context.

Proof. Since $\mathcal{A}$ is Zariski dense in $\mathrm{Spec}(\mathbb{J})$, for any $\mathrm{Gal}(K[\mu_{p^\infty}]/K)$ for the field K of fractions of W, $\mathcal{A}_{st} := \bigcup_{\sigma \in \mathrm{Gal}(K[\mu_{p^\infty}]/K)} \mathcal{A}^\sigma$ is Zariski dense in $\mathrm{Spec}(\mathbb{J})$. We replace $\mathcal{A}$ by $\mathcal{A}_{st}$. Let $Z = \mathrm{Spec}(\mathbf{\Lambda}/\mathfrak{a})$ for $\mathfrak{a} := \mathrm{Ker}(\mathbf{\Lambda} \to \mathbb{J})$ be the image of $\mathrm{Spec}(\mathbb{J})$ in $\mathrm{Spec}(\mathbf{\Lambda})$, and identify $\mathcal{A}$ with its image in Z. By Proposition 2.1, we have only a finite number of generalized Weil l-numbers α of weight $[\kappa]f_{\mathfrak{l}}$ with bounded l-power denominator (i.e., $l^B\alpha$ is a Weil number of weight $2B[\kappa]f_{\mathfrak{l}}$ for some $B > 0$) in $\bigcup_{P \in \mathcal{A}} L_P$ up to multiplication by roots of unity. Here we can take $B = |\kappa_1|$. Hence, replacing $\mathcal{A}$ by its subset of infinite cardinality, we may assume that A_P for all $P \in \mathcal{A}$ hits one α of such generalized Weil l-numbers of weight $[\kappa]f_{\mathfrak{l}}$, up to roots of unity, since the automorphic representation generated by $\mathbf{f}_P$ is not Steinberg because $\mathfrak{l} \nmid \mathfrak{N}$.

Let P_0 be as in Lemma 5.1 for this $\mathcal{A}$. Adding one point to $\mathcal{A}$ does not change the setting; so, taking $\mathcal{A} = \mathcal{A} \cup \{P_1 = (\gamma_j^{-\kappa_{1,\mathfrak{p}}}t_j - 1)_{j=1,\dots,m}\}$, we may assume that $P_0 = P_1$. By making a variable change $t \mapsto P_0 \cdot t$, we may assume that

$P_0 = (t_j - 1)_{j=1,\dots,m}$, and $\mathcal{A} \subset \mu^r_{p^\infty}(K)$, where we regard $\mu^r_{p^\infty}$ as a subgroup of $\widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \Gamma^*$ (for $\Gamma \cong \mathbb{Z}_p^r$ as in Lemma 5.1) isomorphic to $\mathrm{Spf}(W[[\Gamma]]) = \mathrm{Spf}(W\widehat{[t_1, t_1^{-1}, \dots, t_r, t_r^{-1}]}) = \mathrm{Spf}(W[[T_1 \dots, T_r]])$ with $t_j = 1 + T_j$. Write simply $\widehat{H} := \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \Gamma^*$ and $\widehat{G} = \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \mathbf{\Gamma}^*$. We have $X^*(\widehat{G}) = \mathrm{Hom}_{\text{formal gp}}(\widehat{G}_{/W}, \widehat{\mathbb{G}}_{m/W}) = \Gamma$ and $X_*(\widehat{G}) = \Gamma^*$.

Suppose for the moment $\mathbb{J} = P_0 \cdot \widehat{H}$ for the base discrete valuation ring W finite flat over $\mathbb{Z}_p$. Choosing $\gamma_1, \dots, \gamma_r$ to be a generator of Γ for $r = \mathrm{rank}_{\mathbb{Z}_p} \Gamma$, we may assume that the projection $\mathbf{\Lambda} \to W[[T]] = \mathbb{J}$ has kernel $(t_{r+1} - 1, \dots, t_m - 1)$. In down to earth terms, for $A = A(T)$ in (J3), the variable change $t \mapsto P_0 \cdot t$ is the variable change $T_j \mapsto Y_j = \gamma_j^{-\kappa_{1,\mathfrak{p}}}(1 + T_j) - 1$ with $Y = (Y_1, \dots, Y_m)$, and we have $A(Y)|_{Y=0} = A(T)|_{T_j = \gamma_1^{\kappa_{1,\mathfrak{p}}} - 1}$. Let $\Omega_1 = \{\epsilon_{P,1}(\gamma) | P \in \mathcal{A}\}$ which is an infinite set in $\mu_{p^\infty}(K)$. Let

$$\Phi_1(Y) := \alpha^{-1} A(Y) = A(\gamma^{-\kappa_{1,\mathfrak{p}}}(1 + T) - 1) \in W[[Y]]$$

and $\mathbf{L}$ be the composite of L_P for P running through $\mathcal{A}$. By this variable change, $\mathcal{A}$ is brought into an infinite subset Ω_1 of $\mu^r_{p^\infty}(\overline{\mathbb{Q}}_p) \subset \widehat{\mathbb{G}}^r_m = \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \Gamma^*$ made up of $\zeta \in \Omega_1$ such that $\Phi_1(\zeta - 1)$ is a root of unity in $\mathbf{L}$. By Lemma 2.2, the group of roots of unity of $\mathbf{L}$ contains $\mu_{p^\infty}(K)$ as a subgroup of finite index, and we find an infinite subset $\Omega \subset \Omega_1$ and a root of unity ζ_1 such that $\{\Phi_1(\zeta - 1) | \zeta \in \Omega\} \subset \zeta_1 \mu_{p^\infty}(K)$. Then $\Phi = \zeta_1^{-1} \Phi_1$ satisfies the assumption of Corollary 4.2, and for a root of unity ζ, we have $A(Y) = \zeta\alpha(1 + Y)^s$ for $s \in \mathbb{Z}_p^r$, and $A(T) = \zeta\alpha(\gamma^{-\kappa_{1,\mathfrak{p}}}(1 + T))^s$. Thus $A(T) = c(1 + T)^s$ for a non-zero p-adic unit $c = \zeta\alpha\gamma^{-\kappa_{1,\mathfrak{p}} s} \in W^\times$ as desired.

We now assume that $A \in W[[T]][(1 + T)^{1/p^n}]$. Since

$$\mathrm{Spf}(W[[T]][t^{1/p^n}]]) \cong \widehat{\mathbb{G}}_m \xrightarrow{t \mapsto t^{p^n}} \widehat{\mathbb{G}}_m = \mathrm{Spf}(W[[T]]),$$

by applying the same argument as above to $W[[T]][t^{1/p^n}]]$, we get $A(T) = \zeta_0(1 + T)^{s/p^n}$ for $s \in \mathbb{Z}_p^r$.

We thus need to show $A \in W[\mu_{p^n}][[T]][t^{1/p^n}]$ for sufficient large n, and then the result follows from the above argument. Again we make the variable change $T \mapsto Y$ we have already done. Replacing A by $\alpha^{-1} A$ for a suitable Weil l-number α of weight k (up to $\mu_{p^\infty}(\overline{\mathbb{Q}}_p)$), we may assume that there exists an infinite set $\mathcal{A}_0 \subset \mathrm{Spec}(\mathbb{J})(\overline{\mathbb{Q}}_p)$ such that $P \cap \Lambda = (1 + Y - \zeta_P)$ for $\zeta_P \in \mu^r_{p^\infty}(\overline{\mathbb{Q}}_p)$ and $A_P \in \mu_{p^\infty}(\overline{\mathbb{Q}}_p)$ for all $P \in \mathcal{A}_0$. By another variable change $(1 + Y) \mapsto \zeta(1 + Y)$ for a suitable $\zeta \in \mu^r_{p^\infty}(\overline{\mathbb{Q}}_p)$, we may further assume that we have $P_0 \in \mathcal{A}_0$ with $\zeta_{P_0} = 1$ and $A_{P_0} = 1$ (i.e., choosing α well in $\alpha \cdot \mu_{p^\infty}(\overline{\mathbb{Q}}_p)$). We now write $\mathbb{K}$ for the subalgebra of $\mathbb{I}$ topologically generated by A over $\Lambda = W[[Y]]$. Then we have $\mathbb{K} = \Lambda[A] \subset \mathbb{J}$. Since $\mathbb{J}$ is geometrically irreducible, the base ring W is integrally closed in $\mathbb{K}$. Since A is a unit in $\mathbb{J}$, we may embed the irreducible formal scheme $\mathrm{Spf}(\mathbb{K})$ into $\widehat{\mathbb{G}}^r_m \times \widehat{\mathbb{G}}_m = \mathrm{Spf}(W\widehat{[y, y^{-1}, t', t'^{-1}]})$ by the surjective W-algebra homomorphism $\pi : W\widehat{[y, y^{-1}, t', t'^{-1}]} \twoheadrightarrow \mathbb{K}$ sending (y, t') to $(1 + Y, A)$. Write $Z \subset \widehat{\mathbb{G}}^r_m \times \widehat{\mathbb{G}}_m$ for the reduced image of $\mathrm{Spf}(\mathbb{K})$. Thus we are identifying Λ with $W\widehat{[y, y^{-1}]}$ by $y \leftrightarrow 1 + Y$. Then $P_0 \in Z$ is the identity element of $(\widehat{\mathbb{G}}^r_m \times \widehat{\mathbb{G}}_m)(\overline{\mathbb{Q}}_p)$. Since A is integral over Λ, it is a root of a monic polynomial $\Phi(t') = \Phi(y, t') = t'^d + a_1(y) t'^{d-1} + \dots + a_d(y) \in \Lambda[t']$ irreducible over the quotient field Q of Λ, and we have $\mathbb{K} \cong \Lambda[t']/(\Phi(y, t'))$. Thus $\mathbb{J}$ is free of rank d over Λ; so, $\pi : Z \to \widehat{\mathbb{G}}^r_m = \mathrm{Spf}(\Lambda)$ is a finite flat morphism of degree d. We let $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ act on Λ by $\sum_{n=0}^{\infty} a_n Y^n \mapsto \sum_{n=0}^{\infty} a_n^{\sigma} Y^n$ and on $\Lambda[t']$ by $\sum_j A_j(Y)t'^j \mapsto \sum_j A_j^{\sigma}(Y)t'^j$ for $A_j(Y) \in \Lambda$. Note that $\Phi(\zeta_P, A_P) = 0$ for $P \in \mathcal{A}_0$. Since $A_P \in \mu_{p^\infty}(\overline{\mathbb{Q}}_p)$, $A_P^{\sigma} = A_P^{\nu(\sigma)}$ for the p-adic cyclotomic character $\nu : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathbb{Z}_p^{\times}$. Since W is a discrete valuation ring, for its quotient field F, the image of ν on $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$ is an open subgroup U of $\mathbb{Z}_p^{\times}$. Thus we have $\Phi^{\sigma}(\zeta_P^{\nu(\sigma)}, A_P^{\nu(\sigma)}) = \Phi(\zeta_P, A_P)^{\sigma} = 0$ for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and if $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$, $\Phi^{\sigma} = \Phi$. Thus we get

$$\Phi(\zeta_P^{\nu(\sigma)}, A_P^{\nu(\sigma)}) = \Phi(\zeta_P, A_P)^{\sigma} = 0 \quad \text{for all } P \in \mathcal{A}_0.$$

For $s \in \mathbb{Z}_p^{\times}$, consider the integral closed formal subscheme $Z_s \subset \widehat{\mathbb{G}}_m^r \times \widehat{\mathbb{G}}_m$ defined by $\Phi(y^s, t'^s) = 0$. If $s \in U$, we have $\mathcal{A}_0 \subset Z \cap Z_s$. Since Z and Z_s are finite flat over Λ and $\mathcal{A}_0$ is an infinite set, we conclude $Z = Z_s$. Thus $Z \subset \widehat{\mathbb{G}}_m^r \times \widehat{\mathbb{G}}_m$ is stable under the diagonal action $(y, t') \mapsto (y^s, t'^s)$ for $s \in U$. By Lemma 4.1, Z is a formal multiplicative group and is a formal subtorus of $\widehat{\mathbb{G}}_m^r \times \widehat{\mathbb{G}}_m$, because $\mathbf{1} = P_0 \in Z$. The projection $\pi : Z \to \mathrm{Spf}(\Lambda) = \widehat{\mathbb{G}}_m^r$ is finite flat of degree d. So $\pi : Z \to \widehat{\mathbb{G}}_m^r$ is an isogeny. Thus we conclude $\mathrm{Ker}(\pi) \cong \prod_{j=1}^r \mu_{p^{m_j}}$ and hence $d = p^m$ for $m = \sum_j m_j \geq 0$. This implies $\mathbb{K} = \Lambda[A] \subset W[\mu_{p^n}][[Y]][(1+Y)^{p^{-n}}] = W[\mu_{p^n}][[T]][t^{p^{-n}}]$ for $n = \max(m_j|j)$, as desired. □

6. Proof of the horizontal theorem: Theorem 3.1

The way of the proof of Theorem 3.1 in this paper is far simpler than the earlier one given in [**H11**], and we do not need the existence of infinitely many arithmetic points on $\mathrm{Spec}(\mathbb{J})$ of different weight which was used in the proof of [**H11**]. Indeed, Theorem 3.1 follows from the following result:

Theorem 6.1. *Let the notation be as in* Proposition 5.2, *and write* $K := \mathbb{Q}[\mu_{p^\infty}]$ *and* $E_P = K(\alpha_{\mathfrak{p},P})$ *for each arithmetic point* P *with* $\kappa(P) = \kappa$. *Suppose that there exists an infinite set* $\mathcal{A}$ *of arithmetic points with* $\kappa(P) = \kappa$ *satisfying the following condition:*

($\mathrm{B}_{\mathfrak{p}}$) E_P/K *is a finite extension of bounded degree independent of* $P \in \mathcal{A}$.

Then we have a CM quadratic extension $M_{/F}$ *(in which* $\mathfrak{p}$ *splits) such that the component* $\mathbb{I}$ *has complex multiplication by* M. *In particular, the constant* c *in* Proposition 5.2 *is a generalized Weil number.*

Proof. By ($\mathrm{B}_{\mathfrak{p}}$) and Lemma 3.4, the condition ($\mathrm{BT}_{\mathfrak{l}}$) of Proposition 5.2 is satisfied if l is sufficiently large. We now assume that l is sufficiently large so that tameness in ($\mathrm{BT}_{\mathfrak{l}}$) is satisfied. As proved in the proof of Proposition 5.2, we have $A \in W[\mu_{p^n}][[T_1, \ldots, T_r]][t_1^{p^{-n}}, \ldots, t_r^{p^{-n}}]$. Since $\mathrm{rank}_{W[[T_1,\ldots,T_r]]} \mathbb{J} \geq p^{n(r+1)}$, the exponent n is bounded independent of $\mathfrak{l}$. Taking the maximum n_0 of n and replacing W by $W[\mu_{p^{n_0}}]$, by the variable change $t_j \mapsto t_j^{p^{n_0}}$, we may assume that $\mathbb{J} = W[[T_1, \ldots, T_r]]$ and $A \in W[[T_1, \ldots, T_r]]$. We use the symbols introduced in the proof of Proposition 5.2. Since we now move $\mathfrak{l}$, we write $A_{\mathfrak{l}}$ for A defined for a prime $\mathfrak{l} \nmid Np$ and regard $A_{\mathfrak{l}}$ as a function of $t = (t_j)$ with $t_j = (1 + T_j)$. By Proposition 5.2, we have $A_{\mathfrak{l}}(\zeta\gamma^{\kappa_{1,\mathfrak{p}}}) = \zeta^s \alpha_{\mathfrak{l}}$ with $s \in \mathbb{Z}_p^r$ (dependent on $\mathfrak{l}$) for all $\zeta \in \mu_{p^\infty}^r(\overline{\mathbb{Q}}_p)$, where $\alpha_{\mathfrak{l}}$ is a generalized Weil l-number.

Pick two distinct points $P, P' \in \mathcal{A}$ and write $f = \mathbf{f}_P$ and $g = \mathbf{f}_{P'}$. Thus $\alpha_{\mathfrak{l},P'} = \zeta_{\mathfrak{l}} \alpha_{\mathfrak{l},P}$ for a $\zeta_{\mathfrak{l}} \in \mu_m^r(\overline{\mathbb{Q}}_p)$ for a p-power $m = p^e$ (i.e., $\epsilon_{P,1}/\epsilon_{P',1}(\gamma_i) \in$

$\mu_m(\overline{\mathbb{Q}}_p)$). Consider the compatible system of Galois representations associated to f and g (cf. [**B**] for compatibility). Fix one more prime q and an embedding $\iota_q : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_q$. Write $\mathfrak{q}$ for the place of $\mathbb{Q}(f,g)$ associated to the induced embedding $\iota_q : \mathbb{Q}(f,g) = \mathbb{Q}(f)(g) \hookrightarrow \overline{\mathbb{Q}}_q$. Write $\rho_f = \rho_{f,\mathfrak{q}}$ (resp. $\rho_g = \rho_{g,\mathfrak{q}}$) for $\mathfrak{q}$-adic member of the compatible system associated to f and g.

Since $\zeta_{\mathfrak{l}} = \zeta^s$, $\zeta_{\mathfrak{l}}$ has the order m bounded independently of $\mathfrak{l}$. Thus we have $\alpha^m_{\mathfrak{l},P'} = \alpha^m_{\mathfrak{l},P}$ for all eigenvalues (suitably ordered) of $\rho_{?,\mathfrak{q}}(\mathrm{Frob}_{\mathfrak{l}})$ for all $\mathfrak{l} \nmid Np$ with sufficiently large l. We consider the function $\rho_?^m$ on $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ sending σ to the m-th power $\rho_?(\sigma)^m$ of the matrix $\rho_?(\sigma)$. In particular, $\mathrm{Tr}(\rho_f^m(\mathrm{Frob}_{\mathfrak{l}})) = \mathrm{Tr}(\rho_g^m(\mathrm{Frob}_{\mathfrak{l}}))$ for all primes $\mathfrak{l} \nmid Npq$ with $l \gg 0$. Since $\mathrm{Tr}(\rho_?^m) : g \mapsto \mathrm{Tr}(\rho_?^m(g)) \in \overline{\mathbb{Q}}_q$ is a continuous function, by Chebotarev density theorem, we get $\mathrm{Tr}(\rho_f^m) = \mathrm{Tr}(\rho_g^m)$ all over $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$. Writing $\rho^{sym\otimes j}$ for the symmetric n-th tensor representation of any 2-dimensional matrix representation ρ, we have $\mathrm{Tr}(\rho^m) = \mathrm{Tr}(\rho^{sym\otimes m}) - \mathrm{Tr}(\rho^{sym\otimes(m-2)} \otimes \det\rho)$. Thus we get

$$\mathrm{Tr}(\rho_f^{sym\otimes m} \oplus (\rho_g^{sym\otimes(m-2)} \otimes \det\rho_g)) = \mathrm{Tr}(\rho_g^{sym\otimes m} \oplus (\rho_f^{sym\otimes(m-2)} \otimes \det\rho_f)) \tag{6.1}$$

Assume on the contrary to the desired assertion that the family does not have complex multiplication (hence by (CM1–3), f and g do not have complex multiplication), and we aim to get a contradiction. By [**Di1**, §0.1], if q is sufficiently large and f and g do not have complex multiplication, the image of its residual representation contains $SL_2(\mathbb{F}_q)$ up to conjugation. If $q > m$, as is well known, the j-th symmetric tensor representations $\rho_?^{sym\otimes j}$ of $\rho_?$ ($? = f, g$) (even reduced modulo $\mathfrak{q}$) are all absolutely irreducible and distinct for $0 \le j \le m$. Thus the above identity implies $\rho_f^{sym\otimes m} \cong \rho_g^{sym\otimes m}$. Thus the m-th symmetric power of the compatible system of f is isomorphic to that of g. This is contradictory. To see this, write $\rho_{?,\mathfrak{p}}$ for the $\mathfrak{p}$-adic component of the compatible system of $? = f, g$ for the place $\mathfrak{p}$ induced by $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. Then we have an isomorphism $\rho_{?,\mathfrak{p}}|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)} \cong \left(\begin{smallmatrix} \epsilon_? & * \\ 0 & \delta_? \end{smallmatrix}\right)$. Then we have

$$\{\epsilon_f^{m-j}\delta_f^j | j = 0, \dots, m\} = \{\epsilon_g^{m-j}\delta_g^j | j = 0, \dots, m\}.$$

By (Ram), we have $\epsilon_?^{m-j}\delta_?^j([u, F_{\mathfrak{p}}]) = u^{-\kappa_1 m + (\kappa_1 - \kappa_2)j}$ (for each j) up to finite order characters. Therefore, by $\kappa_2 - \kappa_1 \ge I$, we conclude

$$\epsilon_f^{m-j}\delta_f^j = \epsilon_g^{m-j}\delta_g^j \quad \text{for each } j.$$

By (Ram) and $\det\rho_f = \det\rho_g = \nu\epsilon_+$, for $\varepsilon = \epsilon_{P,1}/\epsilon_{P',1}$, we find $\epsilon_f/\epsilon_g = \varepsilon$ and $\delta_f/\delta_g = \varepsilon^{-1}$ on the inertia group at $\mathfrak{p}$. By our choice of m, we have $\varepsilon^m = 1$, and these identities combined tells us

$$\epsilon_f^{m-j}\delta_f^j = \epsilon_g^{m-j}\delta_g^j = \varepsilon^{m-2j}\epsilon_f^{m-j}\delta_f^j = \varepsilon^{-2j}\epsilon_f^{m-j}\delta_f^j$$

for all $j = 1, \dots, m$. Thus $\varepsilon^{2j} = 1$ ($j = 1, \dots, m$) and hence $\varepsilon^2 = 1$. This is impossible if we choose ε having order > 2. Thus f and hence g must have complex multiplication. The multiplication field M/F has to be the same for f and g by (CM1–3). Then c is a generalized Weil number by the explicit form in (CM1) of the Galois representations attached to CM forms in a p-adic family. □

7. Relative version

If $F = \mathbb{Q}$, by the solution of Serre's mod p modularity conjecture by Khare–Wintenberger, all two dimensional odd compatible systems of Galois representation come from modular forms, and this fact is heavily used to show a finiteness property of some class of rational abelian varieties in [**H12**]. Over general F, we do not know yet the generalized Serre's conjecture. Only available results are Taylor's potential modularity, and therefore, it would be useful to study relative version of our main results if one expects this type of applications of our horizontal theorem. Since the proof is the same for the relative version, we just sketch here the outcome briefly.

Let E/F be a totally real finite Galois extension with Galois group $\mathfrak{G}$. We assume

(UR) the fixed prime $\mathfrak{p}$ is unramified in E/F.

If we have well established theory of base-change, we can explicitly relate the maximal $\mathfrak{G}$-invariant quotient of a Hecke algebra for $GL(2)_{/E}$ with level group U to the Hecke algebra for $GL(2)_{/F}$ for the corresponding level group. Since the theory of base-change is not known in general (except for soluble $\mathfrak{G}$), we can think of the $\mathfrak{G}$-invariant quotient directly and would like to state a version of the horizontal theorem for such Hecke algebras. Since the construction and the proof are the same, we only describe necessary notations and state the result without going into details. For the space of Hilbert modular cusp forms and Hecke algebras, we add subscript "$/E$" to indicate their dependence on E, for example, $h_\kappa(\mathfrak{N},\epsilon;A)$ for E is written as $h_{\kappa/E}(\mathfrak{N},\epsilon;A)$.

Let $S := S_\mathfrak{p}$ be the set of all primes of E over the fixed prime $\mathfrak{p}$ of F. Write R for the integer ring of E, and denote by $\mathbf{\Gamma}_E$ for the maximal torsion-free quotient of $R_S^\times$ for the S-completion $R_S = \prod_{\mathfrak{P}\in S} R_\mathfrak{P}$. We put $I_S = \bigsqcup_{\mathfrak{P}\in S} I_\mathfrak{P}$ and split $I_E := \mathrm{Hom}_{\mathrm{field}}(E,\overline{\mathbb{Q}}) = I_S \sqcup I^S$. The projection of $\kappa \in \mathbb{Z}[I]\times\mathbb{Z}[I]$ to $\mathbb{Z}[I_S]\times\mathbb{Z}[I_S]$ (resp. $\mathbb{Z}[I^S]\times\mathbb{Z}[I^S]$) is denoted by κ_S (resp. κ^S). Often we use I_E to denote $\sum_\sigma \sigma \in \mathbb{Z}[I_E]$. The Neben typus in this setting is again a set of three characters $\epsilon = (\epsilon_1,\epsilon_2,\epsilon_+)$ as before.

Let W be a sufficiently large complete valuation ring inside $\mathbb{C}_p$ and fix an R-ideal $\mathfrak{N} \neq 0$ prime to S. We write $h^{S.ord}_{\kappa/E}(U,\epsilon;W)$, $h^{S.ord}_{\kappa/E}(\mathfrak{N}p^\alpha,\epsilon;W)$ and $\mathbf{h}^{S.ord}_{\kappa^S/E} = \mathbf{h}^{S.ord}_{\kappa^S/E}(\mathfrak{N}\mathfrak{p}^\infty,\epsilon;W)$ for the image of the (nearly) S-ordinary projector $e_S = \prod_{\mathfrak{P}\in S} e_\mathfrak{P}$ for $e_\mathfrak{P} = \lim_n \mathbb{T}(\varpi_\mathfrak{p})^{n!}$, where $\varpi_\mathfrak{p}$ is a prime element in $R_\mathfrak{P}$. The algebra $\mathbf{h}^{S.ord}_{\kappa^S/E}$ is by definition the universal nearly ordinary Hecke algebra over $W[[\mathbf{G}_E]]$ of level $\mathfrak{N}\mathfrak{p}^\infty$ with "Neben character" ϵ. Here $\mathbf{G}_E = R_S^\times \times (R/\mathfrak{N})^\times$ (the E-version of $\mathbf{G}$). We write $\mathbf{\Gamma}_E$ for the maximal torsion-free quotient of $R_S^\times$. We fix a section of the projection $R_S^\times \twoheadrightarrow \mathbf{\Gamma}_E$ and regard $\mathbf{\Gamma}_E$ as a subgroup of $\mathbf{G}_E$. Choosing the section well, we may assume that $\mathbf{\Gamma}_E = \prod_{\mathfrak{P}\in S}\mathbf{\Gamma}_\mathfrak{P}$ with $\mathbf{\Gamma}_\mathfrak{P} \subset R_\mathfrak{P}^\times$. As before, we write $\mathbf{\Lambda} = \mathbf{\Lambda}_{/E}$ for the group algebra $W[[\mathbf{\Gamma}_E]]$. Choosing a basis $\{\gamma_{i,\mathfrak{P}}\}_{i=1,\dots,m'}$ of $\mathbf{\Gamma}_\mathfrak{P}$, we identify $\mathbf{\Lambda}_{/E}$ with $W[[T_{i,\mathfrak{P}}]]_{i,\mathfrak{P}\in S}$ so that $\gamma_{i,\mathfrak{P}}$ corresponds $t_{i,\mathfrak{P}} = 1+T_{i,\mathfrak{P}}$. Since E/F is a Galois extension, $m' = \mathrm{rank}_{O_\mathfrak{p}} R_\mathfrak{P}$ is independent of $\mathfrak{P}\in S$. For a tuple $(\zeta_{i,\mathfrak{P}})_{i,\mathfrak{P}}$ of p-power root of unity and $\kappa_S \in \mathbb{Z}[I_S]$ with $\kappa_{1,S} - \kappa_{2,S} \geq I_S$, we call $P \in \mathrm{Spec}(\mathbf{\Lambda}_{/E})(\overline{\mathbb{Q}}_p)$ arithmetic if $P(t_{i,\mathfrak{P}} - \zeta_{i,\mathfrak{P}}^{-1}\gamma_{i,\mathfrak{P}}^{\kappa_{1,\mathfrak{P}}}) = 0$ for all tuples $(i,\mathfrak{P}) \in \{1,2,\dots,m'\}\times S$, where $\kappa_{1,\mathfrak{P}}$ is the projection of κ to $\mathbb{Z}[I_\mathfrak{P}]$. Write simply $\mathbf{h}_{/E} = \mathbf{h}^{S.ord}_{\kappa^S/E}(\mathfrak{N}\mathfrak{p}^\infty,\epsilon;W)$. Then we call a point $P \in \mathrm{Spec}(\mathbf{h}_{/E})(\overline{\mathbb{Q}}_p)$ arithmetic if P is over an arithmetic point of $\mathrm{Spec}(\mathbf{\Lambda}_E)(\overline{\mathbb{Q}}_p)$. For each arithmetic $P \in \mathrm{Spec}(\mathbf{h}_{/E})(\overline{\mathbb{Q}}_p)$, we define the $\mathfrak{P}$-level exponent $r(P)_\mathfrak{P}$ of $\mathbf{f}_P$ in the same manner as $r(P)$ (replacing $(\mathfrak{p},F)$ by $(\mathfrak{P},E)$). Then we put formally $r(P) = \sum_{\mathfrak{P}\in S} r(P)_\mathfrak{P}\mathfrak{P}$ and define $\mathfrak{p}^{r(P)+1} = \prod_{\mathfrak{P}\in S}\mathfrak{P}^{r(P)_\mathfrak{P}+1}$.

For a fixed κ^S and ϵ_+, we assume that the algebra $\mathbf{h}_{/E}$ is characterized by the following two properties (stronger than (A1–2)):

(a1) $\mathbf{h}_{/E}$ is torsion-free of finite rank over $\mathbf{\Lambda}_W$ equipped with $\mathbb{T}(\mathfrak{l}), \mathbb{T}(y) \in \mathbf{h}_{/E}$ for all primes $\mathfrak{l}$ prime to p and $y \in R_p^\times$,

(a2) if $\kappa_2 - \kappa_1 \geq I$ and P is an arithmetic point of $\mathrm{Spec}(\mathbf{\Lambda}_W)$, we have a surjective W-algebra homomorphism:

$$\mathbf{h}_{/E} \otimes_{\mathbf{\Lambda}_E} \mathbf{\Lambda}_E/P \to h^{S.ord}_{\kappa(P)/E}(\mathfrak{N}\mathfrak{p}^{r(P)+1}, \epsilon_P; W[\epsilon_P]) \text{ with finite kernel,}$$

sending $\mathbb{T}(\mathfrak{l})$ to $\mathbb{T}(\mathfrak{l})$ (and $\mathbb{T}(y)$ to $\mathbb{T}(y)$).

Until we give detailed proof of the assertion (a1–2) in [**HHA**], we just admit them in this paper.

The Galois group $\mathfrak{G}$ acts naturally on $\mathbf{G}_E$, $R_S^\times$ and its quotient $\mathbf{\Gamma}_E$. We assume that the section $\mathbf{\Gamma}_E \hookrightarrow R_S^\times$ (of the projection $R_S^\times \twoheadrightarrow \mathbf{\Gamma}_E$) is $\mathfrak{G}$-equivariant. This is possible as the section can be chosen to be $\exp_{\mathfrak{P}} \circ \log_{\mathfrak{P}}$ for the exponential map/logarithm map for each component $R_{\mathfrak{P}}^\times$ of $R_S^\times$. Thus $\mathbf{\Lambda}_{/E}$ has natural $\mathfrak{G}$-action through W-algebra automorphisms. Then we consider the maximal $\mathfrak{G}$-invaiant quotient $\mathbf{\Lambda}_{/E}/\sum_{\sigma\in\mathfrak{G}} \mathbf{\Lambda}_{/E}(\sigma-1)\mathbf{\Lambda}_{/E}$.

Lemma 7.1. *Under* (UR), *by the canonical surjection induced by the norm map relative to $R_S/O_{\mathfrak{p}}$, we have $\mathbf{\Lambda}_{/E}/\sum_{\sigma\in\mathfrak{G}} \mathbf{\Lambda}_{/E}(\sigma-1)\mathbf{\Lambda}_{/E} \cong \mathbf{\Lambda}_{/F}$. Similarly, we have*

$$W[[\mathbf{G}_E]]/\sum_{\sigma\in\mathfrak{G}} W[[\mathbf{G}_E]](\sigma-1)W[[\mathbf{G}_E]] \cong W[[\mathbf{G}]].$$

Proof. Let K (resp. k) be the residue field of $R_{\mathfrak{P}}$ (resp. $O_{\mathfrak{p}}$). By (UR), all norm 1 elements of $R_{\mathfrak{P}}^\times$ is of the form $x^{\phi-1}$ for $x \in R_{\mathfrak{P}}^\times$. Thus

$$\mathrm{Ker}(N_{E_{\mathfrak{P}}/F_{\mathfrak{p}}} : \mathbf{\Gamma}_{\mathfrak{P}} \to \mathbf{\Gamma}) = \mathrm{Im}(\phi - 1 : \mathbf{\Gamma}_{\mathfrak{P}} \to \mathbf{\Gamma}_{\mathfrak{P}}),$$

where $\mathbf{\Gamma}_{\mathfrak{P}}$ is the maximal torsion-free quotient of $R_{\mathfrak{P}}^\times$. This is equivalent to

$$W[[\mathbf{\Gamma}_{\mathfrak{P}}]]/\sum_{\sigma\in\mathrm{Gal}(E_{\mathfrak{P}}/F_{\mathfrak{p}})} W[[\mathbf{\Gamma}_{\mathfrak{P}}]](\sigma-1)W[[\mathbf{\Gamma}_{\mathfrak{P}}]] \cong \mathbf{\Lambda}_{/F}$$

by the homomorphism induced by the local norm map at $\mathfrak{P}/\mathfrak{p}$. Since $\mathbf{\Lambda}_{/E} = \widehat{\bigotimes}_{\mathfrak{P}\in S} W[[\mathbf{\Gamma}_{\mathfrak{P}}]]$ (the tensor product taken over W), from the above identity, we obtain

$$\mathbf{\Lambda}_{/E}/\sum_{\sigma\in\mathfrak{G}} \mathbf{\Lambda}_{/E}(\sigma-1)\mathbf{\Lambda}_{/E} \cong \mathbf{\Lambda}_{/F}.$$

The proof for

$$W[[\mathbf{G}_E]]/\sum_{\sigma\in\mathfrak{G}} W[[\mathbf{G}_E]](\sigma-1)W[[\mathbf{G}_E]] \cong W[[\mathbf{G}]]$$

is similar and simpler. □

Suppose $\mathfrak{N}^\sigma = \mathfrak{N}$ and $(\kappa^S)^\sigma = \kappa^S$ for all $\sigma \in \mathfrak{G}$, where $\sigma \in \mathfrak{G}$ acts on $\mathbb{Z}[I_E]$ by right multiplication. Then $\mathfrak{G}$ acts on $\mathbf{h}_{/E}$ by $\mathbb{T}(y) \mapsto \mathbb{T}(y^\sigma)$. The structure homomorphism $W[[\mathbf{G}_E]] \to \mathbf{h}_{/E}$ is $\mathfrak{G}$-equivariant. Thus we may consider the quotient

$$\mathbf{h}_{\mathfrak{G}} := \mathbf{h}_{/E}/\sum_{\sigma\in\mathfrak{G}} \mathbf{h}_{/E}(\sigma-1)\mathbf{h}_{/E}. \tag{7.1}$$

Thus the quotient is a $\mathbf{\Lambda}_{/F}$-algebra by Lemma 7.1. Write $\mathbf{H} = \mathbf{H}_{/F}$ for the maximal $\mathbf{\Lambda}_{/F}$-torsion free quotient of the algebra $\mathbf{h}_{\mathfrak{G}}$ in (7.1) which is a pseudo-isomorphic to $\mathbf{h}_{\mathfrak{G}}$ (as $\mathbf{\Lambda}_{/F}$-modules). By definition, $\mathrm{Spec}(\mathbf{h}_{\mathfrak{G}})$ is the maximal subscheme of $\mathrm{Spec}(\mathbf{h})$ fixed by $\mathfrak{G}$. Identify $\mathrm{Spec}(W[[\mathbf{G}]])$ with the maximal subscheme of $\mathrm{Spec}(W[[\mathbf{G}_E]])$ fixed by $\mathfrak{G}$ by Lemma 7.1. Pick an arithmetic point P of $\mathrm{Spec}(W[[\mathbf{G}]])$ and consider the localization $\mathbf{h}_{/E,P}$ and its reduced part $\mathbf{h}^{red}_{/E,P}$. Similarly to [**HMI**], we can prove that $\mathbf{h}^{red}_{/E,P}$ is étale finite over $W[[\mathbf{G}]]_P = \mathbf{\Lambda}_{W,P}$. Thus after extending scalars to a étale finite extension A over $\mathbf{\Lambda}_{W,P}$, we get a trivialization $\mathbf{h}^{red}_{/E,P} \otimes_{\mathbf{\Lambda}_{W,P}} A \cong A^d$ for $d = \mathrm{rank}_{\mathbf{\Lambda}_{W,P}} \mathbf{h}_{/E,P}$ (i.e., $\mathrm{Spec}(\mathbf{h}^{red}_{/E,P}) \times_{\mathrm{Spec}(\mathbf{\Lambda}_{W,P})} \mathrm{Spec}(A) = \bigsqcup \mathrm{Spec}(A)$ for d copies of $\mathrm{Spec}(A)$). Then the action of $\mathfrak{G}$ on $\mathbf{h}^{red}_{/E,P} \otimes_{\mathbf{\Lambda}_{W,P}} A$ factors through the permutation action on A^d, and hence by descent,

$$\mathbf{h}^{red}_{\mathfrak{G},P} \cong \mathbf{h}^{red}_{/E,P} / \sum_{\sigma \in \mathfrak{G}} \mathbf{h}^{red}_{/E,P}(\sigma - 1)\mathbf{h}^{red}_{/E,P} \cong \mathbf{H}^{red}_P$$

for the reduced part $\mathbf{h}^{red}_{\mathfrak{G}}$ and its localization $\mathbf{h}^{red}_{\mathfrak{G},P}$. Thus from (a2), we conclude

(a3) $\mathbf{H}^{red}_P / \mathbf{H}^{red}_P \cong \mathbf{h}^{red}_{\mathfrak{G},P} / P\mathbf{h}^{red}_{\mathfrak{G},P}$ is canonically isomorphic to the reduced part of the maximal $\mathfrak{G}$-invariant quotient of $h^{S.ord}_{\kappa(P)}(\mathfrak{N}\mathfrak{p}^\alpha, \epsilon_P; K)$ for the quotient field K of W.

Let $\mathrm{Spec}(\mathbb{I})$ be a reduced irreducible component $\mathrm{Spec}(\mathbb{I}) \subset \mathrm{Spec}(\mathbf{H})$. By the above description, $\mathrm{Spec}(\mathbb{I})$ is a finite torsion-free covering of $\mathrm{Spec}(\mathbf{\Lambda})$. Since $\mathfrak{G}$ acts on $\mathbf{h}_{\mathfrak{G}}$ trivially, defining the inner conjugate $\rho^\sigma_{\mathbb{I}}(\tau) = \rho_{\mathbb{I}}(\widetilde{\sigma}\tau\widetilde{\sigma}^{-1})$ for an extension $\widetilde{\sigma}$ of $\sigma \in \mathfrak{G}$ to $\overline{\mathbb{Q}}$, we have $\mathrm{Tr}(\rho^\sigma_{\mathbb{I}}(\mathrm{Frob}_{\mathfrak{L}})) = \mathbb{T}(\mathfrak{L}^\sigma) = \mathbb{T}(\mathfrak{L}) = \mathrm{Tr}(\rho_{\mathbb{I}}(\mathrm{Frob}_{\mathfrak{L}}))$ for all prime $\mathfrak{L}$ of R outside $\mathfrak{N}\mathfrak{p}$. By Chebotarev density, we conclude $\rho^\sigma_{\mathbb{I}} \cong \rho_{\mathbb{I}}$. Since $H^2(\mathfrak{G}, \overline{Q}^\times) = 0$ by the divisibility of $\overline{Q}^\times$, $\rho_{\mathbb{I}}$ extends to $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ (after replacing $\mathbb{I}$ by a suitable finite flat extension). The extension is unique up to twists by characters of $\mathfrak{G}$ (see [**MFG**, §4.3.5]). We choose an extension and write it as $\rho_{\mathbb{I}}$ still. We call $\mathbb{I}$ a CM component if (CM1) is satisfied for the extended $\rho_{\mathbb{I}}$. This notion does not depends on the choice of the extension. Then for a finite idele $y \in F^\times_{\mathbb{A}}$, we write $a_F(\mathfrak{l})$ for $\mathrm{Tr}(\rho_{\mathbb{I}}(\mathrm{Frob}_{\mathfrak{L}}))$ in $\mathbb{I}$ for primes $\mathfrak{l}$ of O outside $\mathfrak{N}p$. Writing $\rho_{\mathbb{I}}|_{D_{\mathfrak{p}}} \cong \left(\begin{smallmatrix} \varepsilon & * \\ 0 & \delta \end{smallmatrix}\right)$ so that δ restricted to $I_{\mathfrak{P}}$ coincides with $\epsilon_{2,\mathfrak{P}}$ for all $\mathfrak{P}|\mathfrak{p}$ by local class field theory, we put $\mathbf{a}_{F,p}(y) = \delta([y, F_{\mathfrak{p}}])$ and $a_F(y) = y_p^{\kappa_{2,\mathfrak{p}}} \delta([y, F_{\mathfrak{p}}])$ for $y \in F^\times_{\mathfrak{p}}$. So, $a_F(\varpi_{\mathfrak{p}})^f$ is equal to $a(\varpi_{\mathfrak{P}})$ for a suitable exponent f, and $a(\mathfrak{L})$ can be written as a polynomial of $a_F(\mathfrak{l})$ and $\epsilon_+(\mathfrak{l})$ for any prime $\mathfrak{L}$ of E above $\mathfrak{l}$. If P is arithmetic, by (a3), we have a $\mathfrak{G}$-invariant Hecke eigenform $\mathbf{f}_P \in S_{\kappa(P)}(\mathfrak{N}\mathfrak{p}^{r(P)+1}, \epsilon_P; \overline{\mathbb{Q}}_p)$ such that its eigenvalue for $\mathbb{T}(\mathfrak{L})$ and $\mathbb{T}(y)$ is given by $a_P(\mathfrak{l}) := P(a(\mathfrak{L})), a_P(y) := P(a(y)) \in \overline{\mathbb{Q}}_p$ for all $\mathfrak{L}$ and $y \in E^\times_p$. Thus $\mathbb{I}$ gives rise to a family $\mathcal{F} = \{\mathbf{f}_P | \text{arithemtic } P \in \mathrm{Spec}(\mathbb{I})\}$ of Hecke eigenforms. Under this circumstance, we put $\alpha_{\mathfrak{p},P} = a_P(\varpi_{\mathfrak{p}})$ for arithmetic points P. As before, $\mathbb{Q}[\mu_{Np^\infty}](\alpha_{\mathfrak{p},P})$ is well defined independent of the choice of $\varpi_{\mathfrak{p}}$, where N is the prime-to-p part of order of $\epsilon_{P,1}|_{O^\times_{\mathfrak{p}}}$ (which is independent of P). Then exactly in the same manner as we have done in the proof of Theorem 3.1, we get

THEOREM 7.2. *Let the notation and the assumptions be in the setting relative to E/F as above. Pick an infinite set $\mathcal{A} \subset \mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$ of arithmetic points P with fixed weight κ satisfying $\kappa_2 - \kappa_1 \geq I$. Write $M_{\mathcal{A}}(\mathcal{F}) \subset \overline{\mathbb{Q}}$ for the field generated over $K := \mathbb{Q}(\mu_{p^\infty})$ by $\{\alpha_{\mathfrak{p},P}\}_{P \in \mathcal{A}}$, where P runs over all arithmetic points in $\mathcal{A}$. Then the field $M_{\mathcal{A}}(\mathcal{F})$ is a finite extension of $\mathbb{Q}(\mu_{p^\infty})$ if, and only if $\mathbf{f}_P$ is a CM* *theta series for an arithmetic P with $k(P) \geq 1$. Moreover we have*

$$\lim_{P \in \mathcal{A}} [K(\alpha_{\mathfrak{p},P}) : K] = \infty$$

unless $\mathcal{F}$ has complex multiplication.

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