Consider the universal minimal $p$-ordinary deformation $\rho_T : \text{Gal}(\overline{Q}/Q) \rightarrow \text{GL}_2(T)$ (for a prime $p \geq 5$) of an induced representation $\text{Ind}_F^Q \varphi$ from a real quadratic field $F$. For almost all primes $p$ split in $F$, we describe an isomorphism $T \cong \Lambda[\sqrt{\langle \epsilon \rangle - 1}]$ for the Iwasawa algebra $\Lambda$, where $\langle \epsilon \rangle := (1 + T)^{\log_p(\epsilon)/\log_p(1+p)} \in \mathbb{Z}_p[[T]] \subset \Lambda$ for a fundamental unit $\epsilon$ of $F$. This implies that the dual adjoint Selmer group of $\rho_T$ is isomorphic to $\Lambda/(\langle \epsilon \rangle - 1)$ as $\Lambda$-modules, and in particular, it is a semi-simple $\Lambda$-module after extension of scalars to $Q_p$ from $\mathbb{Z}_p$. 

*The author is partially supported by the NSF grant: DMS 1464106.
§0. Setting.

- \( p \geq 5 \) a fixed prime. \( \infty : F \subset \mathbb{R} \): a real quadratic field.
- \( O \): the integer ring of \( F; (p) = pp^\sigma \) (fix \( p \) and \( \sigma|_F \neq 1 \)).
- \( \bar{\rho} = \text{Ind}_F^Q \varphi \) (\( \varphi : \text{Gal}(\overline{Q}/F) \to \mathbb{F}^\times) \); \([F : F_p] < \infty \); the Teichmüller lift \( \varphi \) of \( \bar{\varphi} \). For simplicity, assume \( F = F_p \);
- \( \varphi^-(\tau) = \varphi(\tau) \varphi(\sigma\tau^{-1}\sigma^{-1}) \).
- \( \varphi^-|_{\text{Gal}(\overline{Q}_p/Q_p)} \neq 1 \) (locally) and \( \varphi^- \) has order \( \geq 3 \) (globally).
- \( c\infty \): the conductor of \( \varphi \), assume \( c + c^\sigma = O \) and \( c + p^\sigma = O \).
- \( F(\rho) \): the splitting field of a Galois representation \( \rho \),
- \( F(p)(\bar{\rho}) \): the maximal \( p \)-profinite extension of \( F(\bar{\rho}) \) unramified outside \( p \). \( G = \text{Gal}(F(p)(\bar{\rho})/\mathbb{Q}) \triangleright H = \text{Gal}(F(p)(\bar{\rho})/F) \)
- \( p \nmid h_F h_{F(\varphi^-)} \) for the class number \( h_X \) of a number field \( X \).
- \( (T, \rho_T : G \to \text{GL}_2(\mathbb{T})) \): the universal pair among \( p \)-ordinary deformations with coefficients in local \( p \)-profinite \( \mathbb{Z}_p \)-algebras with residue field \( \mathbb{F}_p \). (CNL: category of such algebras)
- \( \mathbb{T} \) is an algebra over the Iwasawa algebra \( \Lambda = \mathbb{Z}_p[[T]] \) via \( \det(\rho_T)([u, \mathbb{Q}_p]) = t^{\log_p(u)/\log_p(1+p)} (t = 1 + T) \) for \( u \in 1 + p\mathbb{Z}_p \).
§1. Main Theorem. Suppose $p \nmid h_F h_F(\overline{\varphi})$. We have

$$T = \Lambda[\sqrt{\langle \varepsilon \rangle - 1}]$$

for $\langle \varepsilon \rangle = t^{\log_p(\varepsilon)/\log_p(1+p)}$, where $\varepsilon$ is a fundamental unit of $F$.

- $Ad(\rho_T)$: the adjoint action of $\rho_T$ on $\mathfrak{sl}_2(\mathbb{T})$ and $\Lambda_\varepsilon := \Lambda/(\langle \varepsilon \rangle - 1)$.

**Corollary.** For the adjoint Selmer group $\text{Sel}(Ad(\rho_T))$ over $\mathbb{Q}$,

$$\text{Sel}(Ad(\rho_T)) \cong \Lambda_\varepsilon^\vee \quad \text{(the Pontryagin dual of $\Lambda_\varepsilon$)}.$$

So $\text{Sel}(Ad(\rho_T))^\vee \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a **semi-simple** $\Lambda$-module.

This follows from $\text{Sel}(Ad(\rho_T))^\vee \cong \Omega_{T/\Lambda}$ (a theorem of Mazur).
§2. Presentation Theorem. Since $\bar{\rho} \otimes \chi \cong \bar{\rho}$ for $\chi = \left( \frac{F/Q}{\cdot} \right)$, we have an involution $\iota \in \text{Aut}(\mathbb{T}/\Lambda)$ such that $\iota \circ \rho_{\mathbb{T}} \cong \rho_{\mathbb{T}} \otimes \chi$. Let $I := \mathbb{T}(\iota-1)\mathbb{T}$. Since $\iota$ acts trivially on $\mathbb{T}/I$, we have $(\rho_{\mathbb{T}} \mod I) \cong (\rho_{\mathbb{T}} \otimes \chi \mod I)$, and hence
\[(\rho_{\mathbb{T}} \mod I) \cong \text{Ind}_F^Q \Phi\]
for a character $\Phi : H \to (\mathbb{T}/I)\times$. By the universality of $(\mathbb{T}, \rho_{\mathbb{T}})$,

**Ind Lemma.** $\mathbb{T}/I \cong \Lambda_\epsilon$ and $\Phi : H \to \Lambda_\epsilon^\times$ is universal among characters of $H$ deforming $\overline{\varphi}$, where $\Lambda_\epsilon = \mathbb{Z}_p[(O_p^\times/\overline{O_p}^\times) \otimes_{\mathbb{Z}} \mathbb{Z}_p]$.

**Presentation Theorem.** Let $\mathbb{T}_+: = \{x \in \mathbb{T} | \iota(x) = x\}$.
\[\mathbb{T} \cong \Lambda[[X]]/(S) \quad X \mapsto \Theta \in \mathbb{T}, \quad \mathbb{T}_+ = \Lambda[[X^2]]/(S) \quad X^2 \mapsto \Theta^2 = \theta\]
with involution $\iota_\infty$ of $\Lambda[[X]]$ lifting $\iota$: $\iota_\infty(X) = -X, \iota_\infty(S) = S$.

This is done in “Cyclicity of adjoint Selmer groups and fundamental units” posted on the web by a $\iota$-compatible Taylor–Wiles patching argument.
§3. Weierstrass preparation. Note

\[ I = \mathbb{T}(\iota - 1)\mathbb{T} = (\Theta) \text{ and } \mathbb{T}/(\Theta) \cong \Lambda_\varepsilon. \]

**Theorem:** \( S(X) = D(X)U(X) \) with \( D(0) = \langle \varepsilon \rangle - 1 \) with a unique monic distinguished polynomial \( D(X) \) of degree \( e \equiv \text{rank}_\Lambda \mathbb{T} \).

For \( S = S(X) \in \Lambda[[X^2]] \),

\[ \mathbb{T}/(\Theta) = \Lambda[[X]]/(X, S(X)) = \Lambda/(S(0)) = \Lambda_\varepsilon \Rightarrow S(0) = \langle \varepsilon \rangle - 1. \]

- By Weierstrass preparation theorem, \( S(X) = D(X)U(X) \) for a distinguished polynomial \( D(X) = X^e + \cdots + D(0) \in \Lambda[X] \) for \( e = \text{rank}_\Lambda \mathbb{T} \) with \( D(X) \equiv X^e \mod m_\Lambda \) and \( U(X) \in \Lambda^X \).
- \( U(0)D(0) = S(0) = \langle \varepsilon \rangle - 1 \), replacing \( S(X) \) by \( U(0)^{-1}S(X) \), we may assume \( D(0) = \langle \varepsilon \rangle - 1 \).

Pick a prime divisor \( P \) of \( \langle \varepsilon \rangle - 1 \). Do the same argument in \( \hat{\Lambda}_P[[X]] \) over the localized-completed DVR \( \hat{\Lambda}_P \).
§4. **D(X) is Eisenstein.** We have

\[ S(X) = D_P(X)U_P(X) \] with \( D_P(X) \in \hat{\Lambda}_P[x] \) and \( U_P(X) \in \hat{\Lambda}_P[[X]]^\times \).

By the uniqueness of the decomposition, \( D = D_P \), so,

\[ D(X) \equiv X^e \mod P \] and \( P \parallel (\langle \epsilon \rangle - 1) = D(0) \).

Since \((\langle \epsilon \rangle - 1) = (t^{m-1}p - 1)\) \((m = \text{ord}_p(\epsilon^{p-1} - 1))\) is square-free,

\[ D(X) \text{ is an } \text{Eisenstein polynomial} \text{ with respect to } P. \]

**Irreducibility Theorem.** \( T = \Lambda[X]/(D(X)) \) is an integral domain fully ramified at each prime factor \( P \) of \( \langle \epsilon \rangle - 1 \).

**If** \( e = 2, \text{ we have } T \cong \Lambda[\sqrt{\langle \epsilon \rangle - 1}]. \)

\( T \) is regular \(\iff\) \( \text{ord}_p(\epsilon^{p-1} - 1) = m = 1 \) as \((\langle \epsilon \rangle - 1) = (t^{m-1}p - 1)\);

- **If** \( F = \mathbb{Q}[\sqrt{5}], m > 1 \iff p \) is **Wall-Sun-Sun primes** (1992).

No Wall-Sun-Sun primes \( \leq 2.6 \times 10^{17} \).

- \( p = 191, 693, \ldots \) are for such for \( \mathbb{Q}[\sqrt{10}] \).
§5. Pseudo character (start of the proof of $e = 2$).

- A pseudo-character $T : X \to A$ ($A \in CNL$) from an $A$-algebra $X$ is a function satisfying
  
  (T1) $T(1) = 2$;
  (T2) $T(rs) = T(sr)$ for all $r, s \in X$;
  (T3) $T(r)T(s)T(t) + T(rst) + T(tsr) - T(rs)T(t) - \cdots = 0$;
  (T4) $r \mapsto \det(T) = \frac{1}{2}(T(r)^2 - T(r^2))$ is multiplicative.

**Cayley–Hamilton** (CH) representation lifting $T$:

- An $A$-GMA $E$ is an $A$-algebra of the form $E = \begin{pmatrix} A & B \\ C & A \end{pmatrix}$ with $A$-modules $B$ and $C$ equipped with product law $B \otimes C \to A$. The product of $E$ is then given by usual matrix product.
- A homomorphism $\rho : X \to E$ is a CH representation of $T$ if
  \[ T = \text{Tr}(\rho), \quad \det(T) = \det(\rho), \quad \rho(r)^2 - T(r)\rho(r) + \det(T) = 0. \]
§6. Universal ordinary CH representation.

- A CH representation $\rho : \mathbb{Z}_p[H] \to E$ (continuous with respect to the topology induced by $\mathbb{Z}_p[[H]]$) is called ordinary if

$$b(D_p\sigma) = 0, \ a(I_p\sigma) = 1, \ c(D_p) = 0 \text{ and } d(I_p) = 1$$

writing $\rho = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. This definition is similar to the one by Wang-Erickson and Wake over Gal($\overline{\mathbb{Q}}/\mathbb{Q}$).

- There is a universal triple

$$(R^{ord}, E^{ord}, \rho^{ord} : H \to E^{ord})$$

deforming $\text{Tr}(\rho)|_H$ made of $R^{ord} \in \text{CNL}$, a $R^{ord}$-GMA $E^{ord} = \begin{pmatrix} R^{ord} & B^{ord} \\ \text{Cord} & R^{ord} \end{pmatrix}$ and a CH representation $\rho^{ord}$ such that we have a unique $\mathbb{Z}_p$-algebra morphism $\phi_R : R^{ord} \to A$ and an $R^{ord}$-GMA homomorphism $\phi_E : E^{ord} \to E$ such that $\text{Tr}(\rho) = \phi_R \circ \text{Tr}(\rho^{ord})$ and $\phi_E \circ \rho^{ord} \cong \rho$ for any ordinary CH representation $\rho : H \to E$ deforming $\varphi \oplus \overline{\varphi}\zeta$. 
§7. Universal ordinary reducible CH representation.

- Universal “reducibility” locus: $\text{Spec}(R^{\text{red}}) \subset \text{Spec}(R^{\text{ord}})$ such that $\rho^{\text{red}} = \rho^{\text{ord}} \otimes 1 : H \rightarrow E^{\text{ord}} \otimes_{R^{\text{ord}}} R^{\text{red}} =: E^{\text{red}} = \left( \frac{R^{\text{red}} B^{\text{red}}}{C^{\text{red}} R^{\text{red}}} \right)$ satisfies the universality among reducible ordinary CH deformation. Again by universality, $\rho^{\text{red}}_{11} = \Phi$ and $R^{\text{red}} \cong \Lambda_\varepsilon$.

- By Bellaïshe–Chenevier, the reducibility ideal $J$ of $R^{\text{ord}}$ giving $R^{\text{red}} = R^{\text{ord}}/J$ is the image in $R^{\text{ord}}$ of $B^{\text{ord}} \otimes_{R^{\text{ord}}} C^{\text{ord}}$.

Let $L_p/F(\Phi^-)$ be the maximal (only) $p$-abelian $p$-ramified extension with Galois group $\mathcal{Y}$. Write $\mathcal{Y}(\varphi^-)$ the $\varphi^-$-branch of $\mathcal{Y}$.

**Reducibility Theorem.** We have $E^{\text{red}} = \left( \frac{\Lambda_\varepsilon}{\sigma \mathcal{Y}(\varphi^-) \sigma^{-1} \Lambda_\varepsilon} \right)$; so,

$$B^{\text{red}} \cong \mathcal{Y}(\varphi^-) \cong \Lambda_\varepsilon \quad \text{and} \quad C^{\text{red}} \cong \sigma \mathcal{Y}(\varphi^-) \sigma^{-1} \cong \Lambda_\varepsilon$$

as $R^{\text{red}}$-modules (the cyclicity follows from cyclicity paper). The product map $B^{\text{red}} \otimes C^{\text{red}} \xrightarrow{0} R^{\text{red}}$ is the zero map.
§8. Univ Theorem: $R^{ord} \cong \mathbb{T}_+$, $E^{ord} = \begin{pmatrix} T_+ & T_- \\ T_- & T_+ \end{pmatrix}$, $\rho^{ord} = \rho_{\mathbb{T}}|_H$.

Proof. By universality, we have $\pi: R^{ord} \overset{onto}{\rightarrow} \mathbb{T}_+$ ($\pi$ is the dual of $\rho \mapsto \text{Tr}(\rho|_H)$); so, $J^n/J^{n+1} \rightarrow I^n/I^{n+1}$ for $I_+ := \mathbb{T}_+ \cap I = (\Theta^2)$.

- By cyclicity over $\Lambda_\epsilon$ of $\mathcal{Y}(\varphi^-)$, we find $B^{ord}/\mathcal{J}B^{ord}$ is cyclic over $R^{ord}$. By NAK, $B^{ord}$ is cyclic over $R^{ord}$; $C^{ord}$ is cyclic over $R^{ord}$.
- Since $\mathcal{J}$ is the image of $B^{ord} \otimes_{R^{ord}} C^{ord}$, $\mathcal{J}$ is cyclic; so, $\mathcal{J}$ is a principal ideal $(\eta)$ ($\eta \in R^{ord}$). For all $n > 0$ we have

$$R^{ord}/(\eta) = R^{red} = \Lambda_\epsilon \mapsto \Lambda_\epsilon = \mathbb{T}_+/\langle \theta \rangle$$

$$\begin{array}{c}
x \mapsto \eta^nx \\
\downarrow \\
J^n/J^{n+1}
\end{array} \quad \mapsto 
\begin{array}{c}
x \mapsto \theta^nx \\
\downarrow \\
I^n/I^{n+1}
\end{array}$$

- We find that $J^n/J^{n+1} \cong I^n/I^{n+1} \Rightarrow R^{ord} = \mathbb{T}_+$.
- Since $B^{ord}$ is generated by one element by NAK and $\mathbb{T}_-$ is free of rank 1 over $\mathbb{T}_+$, $E^{ord} = \begin{pmatrix} T_+ & T_- \\ T_- & T_+ \end{pmatrix}$ and $\rho^{ord} = \rho_{\mathbb{T}}|_H$. □
§9. \( p \)-Inertia Theorem.

\[ \rho_T(I) = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \bigg| a \in \mathbb{Z}_p, b \in \Theta^\Lambda \right\} \]

for the \( p \)-inertia subgroup \( I \subset \text{Gal}(F(\rho_T)/F(\bar{\rho})) \).

Corollary. For a Hecke eigen form \( f \) of weight \( \geq 2 \) whose Galois representation \( \rho_f \) is a deformation of \( \text{Ind}_F^Q \varphi \), the restriction of \( \rho_f \) to the inertia subgroup at \( p \) is indecomposable.

Write \( \rho^{ord} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \). Then the corresponding Wiles’ pseudo representation \( \pi^{ord} = (a, d, x) : H \to R^{ord} \) for \( x(h, h') = b(h)c(h') \) is universal among deformations of the Wiles’ pseudo representation associated to \( \rho(\bar{\rho}) \). Therefore \( (T_+, \pi^{ord}) \) is the universal couple also for Wiles’ pseudo deformation. This point is important as I work in Wiles’ theory. Though we go a different path, Betina found a criterion for the eigencurve \( C \) has ramification index 2 at \( \text{Ind}_F^Q \varphi \) over the weight space.
§10. Betina’s criterion for $e = 2$.

- Local ramification index to be 2 at $\varphi \sim \text{Ind}_F^Q \varphi \in C \Rightarrow T = \Lambda[\sqrt{\langle \varepsilon \rangle} - 1]$ (Irreducibility Theorem).
- Betina’s criterion: Let $M_\ell$ be the maximal $p$-abelian extension of $F(\varphi^-)$ unramified outside $\ell$ and $M_\infty := M_p M_p \sigma$. Let $M_\infty$ be the maximal multiple $\mathbb{Z}_p$-extensions of $F(Ad(\bar{\rho}))$ inside $M_\infty$ and $L_\infty / M_\infty$ be the maximal unramified $p$-abelian extension of $M_\infty$. $X := \text{Gal}(L_\infty / M_\infty)$ is a module over $\text{Gal}(M_\infty / Q)$ and $X_\chi$ be the maximal $\chi$-quotient of $X$. Thus $X_\chi = \text{Gal}(L_\chi / M_\infty)$. Let $F''$ be the maximal unramified extension of $F(\varphi^-)$ inside $M_\infty$. Then

$$e = 2 \text{ if for example } L_\chi / F'' \text{ is abelian.}$$

- Since $\text{Gal}(M_\infty / F(\varphi^-)) \cong \mathbb{Z}_p^s$ for some large $s$, it is difficult to verify his criterion.
- If this criterion is met, Betina shows the triviality of the pseudo representation $(a', d', x')$ associated to the tangent vector at the point $\varphi \in C$; so, $e = 2$. We try to prove triviality differently.

R. Pink gave a classification of $p$-profinite subgroup $G$ of $\text{SL}_2(\mathbb{T})$:

- Let $L : G \rightarrow \text{sl}_2(\mathbb{T})$ by $L(x) = x - \frac{1}{2} \text{Tr}(x) 1_2$, and define the topological $\mathbb{Z}_p$-span $\mathfrak{G}$ of $L(G)$. Then $\mathfrak{G}$ is a Lie algebra.
- Define the central descending sequence: $G_1 = G$, $G_2 = (G, G)$ (the commutator subgroup) and $G_n := (G, G_{n-1})$ for $n > 2$. Similarly $G_1 = G$, $G_3 = [G_1, G_2]$ (Lie bracket), and $G_n = [G, G_{n-1}]$ for $n > 2$. Then $L$ induces conjugate equivariant group isomorphism $G_n/G_{n+1} \cong \mathfrak{G}_n/\mathfrak{G}_{n+1}$ for $n \geq 2$ and $G/G_2 \hookrightarrow \mathfrak{G}/\mathfrak{G}_2$ becomes isomorphism after twisting suitable the abelian group structure of $\mathfrak{G}/\mathfrak{G}_2$.
- Split $\text{Gal}(F(\bar{\rho})/F) = \Delta \rtimes H_p$ with $\Delta \cong \text{Gal}(F(\bar{\rho})/F)$ and $H_p$ $p$-profinite. Let $G := \text{Ker}(\text{det} \circ \rho|_{H_p}) \cap \text{Ker}(\Phi^-)$. Note $H_p = Z \times G$ with $Z \cong t\mathbb{Z}_p$. For simplicity we ignore the center $Z$ so small.
- Pick a prime $P|T$. Then $\varphi = \Phi \mod P$, and $(A := \hat{T}_{+,P}, \pi^{ord})$ is a universal pair over $\hat{\Lambda}_P$ ($P$-adic localization completion) of Wiles’ pseudo representation deforming $\text{Tr}(\text{Ind}_F^G \varphi)|_H$. Writing $B = \hat{\Lambda}_P$. We show $t^* = t^*_{A/B} = m_A/(m_A^2 + m_B) = 0$; so, $A = B$ and hence $\mathbb{T}_+ = \Lambda$. 
§12. How to prove $T_+ = B$.

- By presentation $\dim_\kappa t^* \leq 1$ for $\kappa = A/m_A$ (residue field of $P$). We have $A/(m_A^2 + m_B) = \kappa[t^*] \hookrightarrow \kappa[\epsilon]$ for dual number $\epsilon$. Regard $\pi := \pi^{ord}$ mod $(m_A^2 + m_B)$ as having values in $\kappa[\epsilon]$. Need to show $\pi$ is the deformation of $(\Phi, \Phi_c, 0)$.

- By the action of $\Delta$, $\mathcal{G} = \mathcal{G}^+ \oplus \mathcal{G}^\Delta \oplus \mathcal{G}^-$, $\mathcal{G}^\pm$ being $(\varphi^-)^{\pm 1}$-eigenspace. Here $\mathcal{G}^+$ is upper nilpotent, $\mathcal{G}^\Delta$ is diagonal and $\mathcal{G}^-$ is lower nilpotent. $\exp(\mathcal{G}^\pm) = 1 + \mathcal{G}^\pm \subset \mathcal{G}$ is the inertia group at $p_\pm$.

- Let $\mathcal{D} \subset \mathcal{G}$ be made of diagonal matrices; so, $\mathcal{G} = \exp(\mathcal{G}^-)\mathcal{D}\exp(\mathcal{G}^+)$. On $\mathcal{D}\exp(\mathcal{G}^+) \cup \exp(\mathcal{G}^-)\mathcal{D}$, $x(g, h) = b(g)c(h) = 0$.

- One can show that $\pi$ factors through $\mathcal{G}/\mathcal{G}_3$ (Ballaïche-Chenevier theory). By vanishing of $x$, $x = 0$ on $\exp(\mathcal{G}^-)\mathcal{D}\mathcal{G}_3 \cup \mathcal{D}\exp(\mathcal{G}^+)\mathcal{G}_3$; so, $a(lg) = a(l)a(g)$ and $a(gr) = a(g)a(r)$ for $r \in \exp(\mathcal{G}^+)\mathcal{D}\mathcal{G}_3$, $l \in \mathcal{D}\exp(\mathcal{G}^-)\mathcal{G}_3$ and any $g \in \mathcal{G}$. For the unipotent part, $a = d = 1$; so, $a$ is determined by the restriction to $\mathcal{D}$ modulo $\mathcal{G}_3$ which is $\mathcal{G}$-conjugate invariant; so, $a$ is $\mathcal{G}$-conjugate invariant and hence is a character. $a$ being a deformation of $\varphi$ is easy.