# * Galois Deformation Ring and Base Change to a Real Quadratic Field

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Consider the universal minimal $p$-ordinary deformation $\rho_{\mathbb{T}} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{T})$ (for a prime $p \geq 5$) of an induced representation $\mathrm{Ind}_F^{\mathbb{Q}}\varphi$ from a real quadratic field $F$. For almost all primes $p$ split in $F$, we describe an isomorphism $\mathbb{T} \cong \Lambda[\sqrt{\langle\varepsilon\rangle - 1}]$ for the Iwasawa algebra $\Lambda$, where $\langle\varepsilon\rangle := (1+T)^{\log_p(\varepsilon)/\log_p(1+p)} \in \mathbb{Z}_p[[T]] \subset \Lambda$ for a fundamental unit $\varepsilon$ of $F$. This implies that the dual adjoint Selmer group of $\rho_{\mathbb{T}}$ is isomorphic to $\Lambda/(\langle\varepsilon\rangle - 1)$ as $\Lambda$-modules, and in particular, it is a semi-simple $\Lambda$-module after extension of scalars to $\mathbb{Q}_p$ from $\mathbb{Z}_p$.

## §0. Setting.

- $p \geq 5$ a fixed prime. $\infty : F \subset \mathbb{R}$: a real quadratic field.
- $O$: the integer ring of $F$; $(p) = \mathfrak{p}\mathfrak{p}^\sigma$ (fix $\mathfrak{p}$ and $\sigma|_F \neq 1$).
- $\overline{\rho} = \mathrm{Ind}_F^{\mathbb{Q}} \overline{\varphi}$ ($\overline{\varphi} : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathbb{F}^\times$); $[\mathbb{F} : \mathbb{F}_p] < \infty$; the Teichmüller lift $\varphi$ of $\overline{\varphi}$. For simplicity, assume $\underline{\mathbb{F} = \mathbb{F}_p}$;
- $\varphi^-(\tau) = \varphi(\tau)\varphi(\sigma\tau^{-1}\sigma^{-1})$.
- $\varphi^-|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)} \neq 1$ (locally) and $\varphi^-$ has order $\geq 3$ (globally).
- $\mathfrak{c}\infty$: the conductor of $\varphi$, assume $\mathfrak{c} + \mathfrak{c}^\sigma = O$ and $\mathfrak{c} + \mathfrak{p}^\sigma = O$.
- $F(\rho)$:the splitting field of a Galois representation $\rho$, $F^{(p)}(\overline{\rho})$: the maximal $p$-profinite extension of $F(\overline{\rho})$ **unramified outside** $p$. $G = \mathrm{Gal}(F^{(p)}(\overline{\rho})/\mathbb{Q}) \triangleright H = \mathrm{Gal}(F^{(p)}(\overline{\rho})/F)$
- $p \nmid h_F h_{F(\varphi^-)}$ for the class number $h_X$ of a number field $X$.
- $(\mathbb{T}, \rho_{\mathbb{T}} : G \to \mathrm{GL}_2(\mathbb{T}))$: the universal pair among $p$-ordinary deformations with coefficients in local $p$-profinite $\mathbb{Z}_p$-algebras with residue field $\mathbb{F}_p$. ($CNL$: category of such algebras)
- $\mathbb{T}$ is an algebra over the Iwasawa algebra $\Lambda = \mathbb{Z}_p[[T]]$ via $\det(\rho_{\mathbb{T}})([u, \mathbb{Q}_p]) = t^{\log_p(u)/\log_p(1+p)}$ ($t = 1 + T$) for $u \in 1 + p\mathbb{Z}_p$.

§**1. Main Theorem.** *Suppose* $p \nmid h_F h_{F(\overline{\varphi}^-)}$. *We have*

$$\mathbb{T} = \Lambda[\sqrt{\langle\varepsilon\rangle - 1}]$$

*for* $\langle\varepsilon\rangle = t^{\log_p(\varepsilon)/\log_p(1+p)}$, *where* $\varepsilon$ *is a fundamental unit of* $F$.

- $Ad(\rho_{\mathbb{T}})$: the adjoint action of $\rho_{\mathbb{T}}$ on $\mathfrak{sl}_2(\mathbb{T})$ and $\Lambda_\varepsilon := \Lambda/(\langle\varepsilon\rangle - 1)$.

**Corollary.** *For the adjoint Selmer group* $\mathrm{Sel}(Ad(\rho_{\mathbb{T}}))$ *over* $\mathbb{Q}$,

$$\mathrm{Sel}(Ad(\rho_{\mathbb{T}})) \cong \Lambda_\varepsilon^\vee \quad \textit{(the Pontryagin dual of } \Lambda_\varepsilon\textit{)}.$$

*So* $\mathrm{Sel}(Ad(\rho_{\mathbb{T}}))^\vee \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ *is a* **semi-simple** $\Lambda$*-module.*

This follows from $\mathrm{Sel}(Ad(\rho_{\mathbb{T}}))^\vee \cong \Omega_{\mathbb{T}/\Lambda}$ (a theorem of Mazur).

§2. **Presentation Theorem.** Since $\overline{\rho} \otimes \chi \cong \overline{\rho}$ for $\chi = \left(\frac{F/\mathbb{Q}}{}\right)$, we have an involution $\iota \in \mathrm{Aut}(\mathbb{T}/\Lambda)$ such that $\iota \circ \rho_{\mathbb{T}} \cong \rho_{\mathbb{T}} \otimes \chi$. Let $I := \mathbb{T}(\iota - 1)\mathbb{T}$. Since $\iota$ acts trivially on $\mathbb{T}/I$, we have $(\rho_{\mathbb{T}} \bmod I) = (\rho_{\mathbb{T}}^{\iota} \bmod I) \cong (\rho_{\mathbb{T}} \otimes \chi \bmod I)$, and hence

$$(\rho_{\mathbb{T}} \bmod I) \cong \mathrm{Ind}_F^{\mathbb{Q}} \Phi$$

for a character $\Phi : H \to (\mathbb{T}/I)^{\times}$. By the universality of $(\mathbb{T}, \rho_{\mathbb{T}})$,

**Ind Lemma.** *$\mathbb{T}/I \cong \Lambda_{\varepsilon}$ and $\Phi : H \to \Lambda_{\varepsilon}^{\times}$ is universal among characters of $H$ deforming $\overline{\varphi}$, where $\Lambda_{\varepsilon} = \mathbb{Z}_p[(O_p^{\times}/\overline{O^{\times}}) \otimes_{\mathbb{Z}} \mathbb{Z}_p]$.*

**Presentation Theorem.** *Let $\mathbb{T}_+ := \{x \in \mathbb{T} | \iota(x) = x\}$.*

$$\mathbb{T} \cong \Lambda[[X]]/(S) \;\; X \mapsto \Theta \in \mathbb{T}, \;\; \mathbb{T}_+ = \Lambda[[X^2]]/(S) \;\; X^2 \mapsto \Theta^2 = \theta$$

*with involution $\iota_\infty$ of $\Lambda[[X]]$ lifting $\iota$: $\iota_\infty(X) = -X, \iota_\infty(S) = S$.*

This is done in **“Cyclicity of adjoint Selmer groups and fundamental units”** posted on the web by a $\iota$-compatible Taylor–Wiles patching argument.

## §3. Weierstrass preparation. Note

$$I = \mathbb{T}(\iota - 1)\mathbb{T} = (\Theta) \quad \text{and} \quad \mathbb{T}/(\Theta) \cong \Lambda_\varepsilon.$$

**Theorem:** *$S(X) = D(X)U(X)$ with $D(0) = \langle\varepsilon\rangle - 1$ with a unique monic distinguished polynomial $D(X)$ of degree $e := \mathrm{rank}_\Lambda \mathbb{T}$.*

For $S = S(X) \in \Lambda[[X^2]]$,

$$\mathbb{T}/(\Theta) = \Lambda[[X]]/(X, S(X)) = \Lambda/(S(0)) = \Lambda_\varepsilon \Rightarrow S(0) = \langle\varepsilon\rangle - 1.$$

- By Weierstrass preparation theorem, $S(X) = D(X)U(X)$ for a distinguished polynomial $D(X) = X^e + \cdots + D(0) \in \Lambda[X]$ for $e = \mathrm{rank}_\Lambda \mathbb{T}$ with $D(X) \equiv X^e \mod \mathfrak{m}_\Lambda$ and $U(X) \in \Lambda^\times$.
- $U(0)D(0) = S(0) = \langle\varepsilon\rangle - 1$, replacing $S(X)$ by $U(0)^{-1}S(X)$, we may assume $D(0) = \langle\varepsilon\rangle - 1$.

Pick a prime divisor $P$ of $\langle\varepsilon\rangle - 1$. Do the same argument in $\widehat{\Lambda}_P[[X]]$ over the localized-completed DVR $\widehat{\Lambda}_P$.

§**4.** $D(X)$ **is Eisenstein.** We have

$S(X) = D_P(X)U_P(X)$ with $D_P(X) \in \widehat{\Lambda}_P[x]$ and $U_P(X) \in \widehat{\Lambda}_P[[X]]^\times$.

By the uniqueness of the decomposition, $D = D_P$, so,

$$D(X) \equiv X^e \mod P \text{ and } P \parallel (\langle\varepsilon\rangle - 1) = D(0).$$

Since $(\langle\varepsilon\rangle - 1) = (t^{p^{m-1}} - 1)$ $(m = \mathrm{ord}_{\mathfrak{p}}(\varepsilon^{p-1} - 1))$ is **square-free**,

$D(X)$ is an **Eisenstein polynomial** with respect to $P$.

**Irreducibility Theorem.** $\mathbb{T} = \Lambda[X]/(D(X))$ *is an integral domain fully ramified at each prime factor* $P$ *of* $\langle\varepsilon\rangle - 1$.
**If** $e = 2$**, we have** $\mathbb{T} \cong \Lambda[\sqrt{\langle\varepsilon\rangle - 1}]$.

$\mathbb{T}$ is regular $\Leftrightarrow$ $\mathrm{ord}_{\mathfrak{p}}(\varepsilon^{p-1} - 1) = m = 1$ as $(\langle\varepsilon\rangle - 1) = (t^{p^{m-1}} - 1)$;
- If $F = \mathbb{Q}[\sqrt{5}]$, $m > 1 \Leftrightarrow p$ is **Wall-Sun-Sun primes** (1992).

No Wall-Sun-Sun primes $\leq 2.6 \times 10^{17}$.
- $p = 191, 693, \cdots$ are for such for $\mathbb{Q}[\sqrt{10}]$.

## §5. Pseudo character (start of the proof of $e = 2$).

• A pseudo-character $T : X \to A$ ($A \in CNL$) from an $A$-algebra $X$ is a function satisfying

(T1) $T(1) = 2$;

(T2) $T(rs) = T(sr)$ for all $r, s \in X$;

(T3) $T(r)T(s)T(t) + T(rst) + T(tsr) - T(rs)T(t) - \cdots = 0$;

(T4) $r \mapsto \det(T) = \frac{1}{2}(T(r)^2 - T(r^2))$ is multiplicative.

**Cayley–Hamilton** (CH) representation lifting $T$:

• An $A$-GMA $E$ is an $A$-algebra of the form $E = \begin{pmatrix} A & B \\ C & A \end{pmatrix}$ with $A$-modules $B$ and $C$ equipped with product law $B \otimes C \to A$. The product of $E$ is then given by usual matrix product.

• A homomorphism $\rho : X \to E$ is a **CH representation** of $T$ if

$$T = \mathrm{Tr}(\rho),\ \det(T) = \det(\rho),\ \rho(r)^2 - T(r)\rho(r) + \det(T) = 0.$$

## §6. Universal ordinary CH representation.

• A CH representation $\rho : \mathbb{Z}_p[H] \to E$ (continuous with respect to the topology induced by $\mathbb{Z}_p[[H]]$) is called **ordinary** if

$$b(D_{\mathfrak{p}^\sigma}) = 0,\ a(I_{\mathfrak{p}^\sigma}) = 1,\ c(D_{\mathfrak{p}}) = 0 \text{ and } d(I_{\mathfrak{p}}) = 1$$

writing $\rho = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. This definition is similar to the one by Wang-Erickson and Wake over $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

• There is a universal triple

$$(R^{ord}, E^{ord}, \rho^{ord} : H \to E^{ord}) \text{ deforming } \mathrm{Tr}(\overline{\rho})|_H$$

made of $R^{ord} \in CNL$, a $R^{ord}$-GMA $E^{ord} = \left(\begin{smallmatrix} R^{ord} & B^{ord} \\ C^{ord} & R^{ord} \end{smallmatrix}\right)$ and a CH representation $\rho^{ord}$ such that we have a unique $\mathbb{Z}_p$-algebra morphism $\phi_R : R^{ord} \to A$ and an $R^{ord}$-GMA homomorphism $\phi_E : E^{ord} \to E$ such that $\mathrm{Tr}(\rho) = \phi_R \circ \mathrm{Tr}(\rho^{ord})$ and $\phi_E \circ \rho^{ord} \cong \rho$ for any ordinary CH representation $\rho : H \to E$ deforming $\overline{\varphi} \oplus \overline{\varphi}_\varsigma$.

## §7. Universal ordinary reducible CH representation.

• Universal "reducibility" locus: Spec$(R^{red}) \subset$ Spec$(R^{ord})$ such that $\rho^{red} = \rho^{ord} \otimes 1 : H \to E^{ord} \otimes_{R^{ord}} R^{red} =: E^{red} = \left(\begin{smallmatrix} R^{red} & B^{red} \\ C^{red} & R^{red} \end{smallmatrix}\right)$ satisfies the universality among reducible ordinary CH deformation. Again by universality, $\rho^{red}_{11} = \Phi$ and $\underline{R^{red} \cong \Lambda_\varepsilon}$.

• By Bellaïshe–Chenevier, the **reducibility** ideal $\mathcal{J}$ of $R^{ord}$ giving $R^{red} = R^{ord}/\mathcal{J}$ is the **image** in $R^{ord}$ of $B^{ord} \otimes_{R^{ord}} C^{ord}$.

Let $L_{\mathfrak{p}}/F(\Phi^-)$ be the maximal (only) $p$-abelian $\mathfrak{p}$-ramified extension with Galois group $\mathcal{Y}$. Write $\mathcal{Y}(\varphi^-)$ the $\varphi^-$-branch of $\mathcal{Y}$.

**Reducibility Theorem.** *We have* $E^{red} = \begin{pmatrix} \Lambda_\varepsilon & \mathcal{Y}(\varphi^-) \\ \sigma\mathcal{Y}(\varphi^-)\sigma^{-1} & \Lambda_\varepsilon \end{pmatrix}$*; so,*

$$B^{red} \cong \mathcal{Y}(\varphi^-) \cong \Lambda_\varepsilon \quad \textit{and} \quad C^{red} \cong \sigma\mathcal{Y}(\varphi^-)\sigma^{-1} \cong \Lambda_\varepsilon$$

*as* $R^{red}$*-modules (the cyclicity follows from* **cyclicity** *paper). The product map* $B^{red} \otimes C^{red} \xrightarrow{0} R^{red}$ *is the* **zero** *map.*

§**8. Univ Theorem:** $R^{ord} \cong \mathbb{T}_+$, $E^{ord} = \begin{pmatrix} \mathbb{T}_+ & \mathbb{T}_- \\ \mathbb{T}_- & \mathbb{T}_+ \end{pmatrix}$, $\rho^{ord} = \rho_{\mathbb{T}}|_H$.

*Proof.* By universality, we have $\pi : R^{ord} \xrightarrow{onto} \mathbb{T}_+$ ($\pi$ is the dual of $\rho \mapsto \mathrm{Tr}(\rho|_H)$); so, $\mathcal{J}^n/\mathcal{J}^{n+1} \twoheadrightarrow I_+^n/I_+^{n+1}$ for $I_+ := \mathbb{T}_+ \cap I = (\Theta^2)$.

- By cyclicity over $\Lambda_\varepsilon$ of $\mathcal{Y}(\varphi^-)$, we find $B^{ord}/\mathcal{J}B^{ord}$ is cyclic over $R^{ord}$. By NAK, $B^{ord}$ is cyclic over $R^{ord}$; $C^{ord}$ is cyclic over $R^{ord}$.
- Since $\mathcal{J}$ is the image of $B^{ord} \otimes_{R^{ord}} C^{ord}$, $\mathcal{J}$ is cyclic; so, $\mathcal{J}$ is a principal ideal $(\eta)$ ($\eta \in R^{ord}$). For all $n > 0$ we have

$$\begin{array}{ccc} R^{ord}/(\eta) = R^{red} = \Lambda_\varepsilon & \xrightarrow{=} & \Lambda_\varepsilon = \mathbb{T}_+/(\theta) \\ {\scriptstyle x\mapsto\eta^n x}\big\downarrow & & \wr\big\downarrow{\scriptstyle x\mapsto\theta^n x} \\ \mathcal{J}^n/\mathcal{J}^{n+1} & \underset{\twoheadrightarrow}{\longrightarrow} & I_+^n/I_+^{n+1} \end{array}$$

- We find that $\mathcal{J}^n/\mathcal{J}^{n+1} \cong I_+^n/I_+^{n+1} \Rightarrow R^{ord} = \mathbb{T}_+$.
- Since $B^{ord}$ is generated by one element by NAK and $\mathbb{T}_-$ is free of rank 1 over $\mathbb{T}_+$, $E^{ord} = \begin{pmatrix} \mathbb{T}_+ & \mathbb{T}_- \\ \mathbb{T}_- & \mathbb{T}_+ \end{pmatrix}$ and $\rho^{ord} = \rho_{\mathbb{T}}|_H$. $\square$

## §9. $p$-Inertia Theorem.

$$\rho_{\mathbb{T}}(I) = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \middle| a \in t^{\mathbb{Z}_p}, b \in \Theta\Lambda \right\}$$

*for the $\mathfrak{p}$-inertia subgroup $I \subset \mathrm{Gal}(F(\rho_{\mathbb{T}})/F(\overline{\rho}))$.*

**Corollary.** *For a Hecke eigen form $f$ of weight $\geq 2$ whose Galois representation $\rho_f$ is a deformation of $\mathrm{Ind}_F^{\mathbb{Q}} \overline{\varphi}$, the restriction of $\rho_f$ to the inertia subgroup at $p$ is* **indecomposable**.

Write $\rho^{ord} = \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix}$. Then the corresponding Wiles' pseudo representation $\pi^{ord} = (\mathbf{a}, \mathbf{d}, \mathbf{x}) : H \to R^{ord}$ for $\mathbf{x}(h, h') = \mathbf{b}(h)\mathbf{c}(h')$ is universal among deformations of the Wiles' pseudo representation associated to $\rho(\overline{\rho})$. Therefore $(\mathbb{T}_+, \pi^{ord})$ is **the universal couple also for Wiles' pseudo deformation**. This point is important as I work in Wiles' theory. Though we go a different path, Betina found a criterion for the **eigencurve** $C$ has ramification index 2 at $\mathrm{Ind}_F^{\mathbb{Q}} \varphi$ over the weight space.

## §10. Betina's criterion for $e = 2$.

• Local ramification index to be 2 at $\wp \sim \mathrm{Ind}_F^{\mathbb{Q}}\varphi \in C \Rightarrow \mathbb{T} = \Lambda[\sqrt{\langle\varepsilon\rangle - 1}]$ (**Irreducibility Theorem**).

• Betina's criterion: Let $\mathbf{M}_{\mathfrak{l}}$ be the maximal $p$-abelian extension of $F(\varphi^-)$ unramified outside $\mathfrak{l}$ and $\mathbf{M}_\infty := \mathbf{M}_{\mathfrak{p}}\mathbf{M}_{\mathfrak{p}^\sigma}$. Let $M_\infty$ be the maximal multiple $\mathbb{Z}_p$-extensions of $F(Ad(\overline{\rho}))$ inside $\mathbf{M}_\infty$ and $L_\infty/M_\infty$ be the maximal unramified $p$-abelian extension of $M_\infty$. $X :=$ Gal$(L_\infty/M_\infty)$ is a module over Gal$(M_\infty/\mathbb{Q})$ and $X_\chi$ be the maximal $\chi$-quotient of $X$. Thus $X_\chi =$ Gal$(L_\chi/M_\infty)$. Let $F''$ be the maximal unramified extension of $F(\varphi^-)$ inside $M_\infty$. Then

$$e = 2 \textbf{ if for example } L_\chi/F'' \textbf{ is abelian.}$$

• Since Gal$(M_\infty/F(\varphi^-)) \cong \mathbb{Z}_p^s$ for some large $s$, it is difficult to verify his criterion.

• If this criterion is met, Betina shows the triviality of the pseudo representation $(a', d', x')$ associated to the tangent vector at the point $\wp \in C$; so, $e = 2$. We try to prove triviality differently.

§**11. Pink Theory:** Compositio **88** (1993).
R. Pink gave a classification of $p$-profinite subgroup $\mathcal{G}$ of $\mathrm{SL}_2(\mathbb{T})$:
• Let $L : \mathcal{G} \to \mathfrak{sl}_2(\mathbb{T})$ by $L(x) = x - \frac{1}{2}\mathrm{Tr}(x)1_2$, and define the topological $\mathbb{Z}_p$-span $\mathfrak{G}$ of $L(\mathcal{G})$. Then $\mathfrak{G}$ is a Lie algebra.
• Define the central descending sequence: $\mathcal{G}_1 = \mathcal{G}$, $\mathcal{G}_2 = (\mathcal{G}, \mathcal{G})$ (the commutator subgroup) and $\mathcal{G}_n := (\mathcal{G}, \mathcal{G}_{n-1})$ for $n > 2$. Similarly $\mathfrak{G}_1 = \mathfrak{G}$, $\mathfrak{G}_3 = [\mathfrak{G}_1, \mathfrak{G}_2]$ (Lie bracket), and $\mathfrak{G}_n = [\mathfrak{G}, \mathfrak{G}_{n-1}]$ for $n > 2$. Then $L$ induces conjugate equivariant group isomorphism $\mathcal{G}_n/\mathcal{G}_{n+1} \cong \mathfrak{G}_n/\mathfrak{G}_{n+1}$ for $n \geq 2$ and $\mathcal{G}/\mathcal{G}_2 \hookrightarrow \mathfrak{G}/\mathfrak{G}_2$ becomes isomorphism after twisting suitable the abelian group structure of $\mathfrak{G}/\mathfrak{G}_2$. • Split $\mathrm{Gal}(F(\boldsymbol{\rho})/F) = \Delta \ltimes H_p$ with $\Delta \cong \mathrm{Gal}(F(\overline{\rho})/F)$ and $H_p$ $p$-profinite. Let $\mathcal{G} := \mathrm{Ker}(\det \circ \boldsymbol{\rho}|_{H_p}) \cap \mathrm{Ker}(\Phi^-)$. Note $H_p = Z \times \mathcal{G}$ with $Z \cong t^{\mathbb{Z}_p}$. For simplicity we ignore the center $Z$ so small.
• Pick a prime $P|T$. Then $\varphi = \Phi \mod P$, and $(A := \widehat{\mathbb{T}}_{+,P}, \pi^{ord})$ is a universal pair over $\widehat{\Lambda}_P$ ($P$-adic localization completion) of Wiles' pseudo representation deforming $\mathrm{Tr}(\mathrm{Ind}_F^{\mathbb{Q}} \varphi)|_H$. Writing $B = \widehat{\Lambda}_P$. We show $t^* = t^*_{A/B} = \mathfrak{m}_A/(\mathfrak{m}_A^2 + \mathfrak{m}_B) = 0$; so, $A = B$ and hence $\mathbb{T}_+ = \Lambda$.

## §12. How to prove $\mathbb{T}_+ = B$.

• By presentation $\dim_\kappa t^* \leq 1$ for $\kappa = A/\mathfrak{m}_A$ (residue field of $P$). We have $A/(\mathfrak{m}_A^2 + \mathfrak{m}_B) = \kappa[t^*] \hookrightarrow \kappa[\epsilon]$ for dual number $\epsilon$. Regard $\pi := \pi^{ord} \mod (\mathfrak{m}_A^2 + \mathfrak{m}_B)$ as having values in $\kappa[\epsilon]$. Need to show $\pi$ is the deformation of $(\Phi, \Phi_c, 0)$.

• By the action of $\Delta$, $\mathfrak{G} = \mathfrak{G}^+ \oplus \mathfrak{G}^\Delta \oplus \mathfrak{G}^-$, $\mathfrak{G}^\pm$ being $(\varphi^-)^{\pm 1}$-eigenspace. Here $\mathfrak{G}^+$ is upper nilpotent, $\mathfrak{G}^\Delta$ is diagonal and $\mathfrak{G}^-$ is lower nilpotent. $\exp(\mathfrak{G}^\pm) = 1 + \mathfrak{G}^\pm \subset \mathcal{G}$ is the inertia group at $\mathfrak{p}_\pm$.

• Let $\mathcal{D} \subset \mathcal{G}$ be made of diagonal matrices; so, $\mathcal{G} = \exp(\mathfrak{G}^-)\mathcal{D}\exp(\mathfrak{G}^+)$. On $\mathcal{D}\exp(\mathfrak{G}^+) \cup \exp(\mathfrak{G}^-)\mathcal{D}$, $\mathbf{x}(g,h) = \mathbf{b}(g)\mathbf{c}(h) = 0$.

• One can show that $\pi$ factors through $\mathcal{G}/\mathcal{G}_3$ (Ballaïche-Chenevier theory). By vanishing of $\mathbf{x}$, $x = 0$ on $\exp(\mathfrak{G}^-)\mathcal{D}\mathcal{G}_3 \cup \mathcal{D}\exp(\mathfrak{G}^+)\mathcal{G}_3$; so, $a(lg) = a(l)a(g)$ and $a(gr) = a(g)a(r)$ for $r \in \exp(\mathfrak{G}^+)\mathcal{D}\mathcal{G}_3$, $l \in \mathcal{D}\exp(\mathfrak{G}^-)\mathcal{G}_3$ and any $g \in \mathcal{G}$. For the unipotent part, $a = d = 1$; so, $a$ is determined by the restriction to $\mathcal{D}$ modulo $\mathcal{G}_3$ which is $\mathcal{G}$-conjugate invariant; so, $a$ is $\mathcal{G}$-conjugate invariant and hence is a character. $a$ being a deformation of $\varphi$ is easy.