IRREDUCIBILITY OF THE SIEGEL–IGUSA TOWER

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We prove the irreducibility of the Igusa tower over a Shimura variety by studying the decomposition group of the prime p inside the automorphism group of the arithmetic automorphic function field. This is the method employed in [PAF] Sections 6.4.3 and 8.4, which uses characteristic 0 results to prove the characteristic p assertion. As explained below (and in more details in [H06]), one can also give a purely characteristic p proof following the same line. There are some other arguments (of purely in characteristic p) to prove the same result (covering different families of reductive groups giving the Shimura variety) as sketched in [C] for the Siegel modular variety.

Here is a general axiomatic approach to prove the relative irreducibility of an étale covering $\pi : I \to S$ of an irreducible variety S over an algebraically closed field $\mathbb{F}$. Suppose the following two axioms:

(A1) A group $G = M \times G_1$ acts on I and S compatibly so that $M \subset \mathrm{Aut}(I/S)$, $G_1 \subset \mathrm{Aut}(S)$ and G_1 acts trivially on $\pi_0(I)$.

(A2) M acts on each fiber of I/S **transitively**; so, M acts transitively on $\pi_0(I)$.

Let $I^\circ_{/\mathbb{F}}$ be an irreducible component of $I_{/\mathbb{F}}$. We want to prove $I^\circ = \pi^{-1}(S) = I$ (relative irreducibility). Then $\mathrm{Gal}(I^\circ/S) \subset M$, and if $M = \mathrm{Gal}(I^\circ/S)$, we get $I^\circ = \pi^{-1}(S)$. Let D be the stabilizer of $I^\circ \in \pi_0(I)$ in G. Pick a point $x \in I$ (which can be a generic point), and look at the stabilizer $D_x \subset G$ of x. Since $g_x(x) \in I^\circ$ ($g_x \in M$) by the transitivity of the action, we have $g_x D_x g_x^{-1} \subset D$. Then we show that $M = G/G_1$ is generated by $\{g_x D_x g_x^{-1} | x \in I\}$, which implies $M = \mathrm{Gal}(I^\circ/S)$.

In the study of the Igusa tower $I_{/S}$, $\mathbb{F}$ is an algebraic closure of the finite field $\mathbb{F}_p$, and S is the ordinary locus of the reduction modulo p the Shimura variety (of level away from p) which is defined over a valuation ring of mixed characteristic with residue field $\mathbb{F}$. We can take G to be a closed subgroup of the automorphism group of the p-integral Shimura variety (studied by Shimura, Deligne and Milne-Shih in depth). In the case of the Siegel–Shimura variety, we have

$$G = M(\mathbb{Z}_p) \times G_1 \quad \text{with} \quad G_1 = Sp_{2n}(\mathbb{A}^{(p\infty)})$$

for the standard Levi-subgroup M of the Siegel parabolic subgroup P of $Sp(2n)$. Then we can take x to be the generic point of $I^\circ_{/\mathbb{F}}$ containing the infinity cusp (so, $x = I^\circ$). The group $M(\mathbb{Z}_p)$ is isomorphic to $GL_n(\mathbb{Z}_p)$ via $GL_n(\mathbb{Z}_p) \ni a \mapsto \left(\begin{smallmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{smallmatrix}\right) \in M(\mathbb{Z}_p)$. Irreducible components of I are in bijection with p-adic valuations v of the function field of the characteristic 0 Shimura variety (of level $\Gamma_1(p^\infty)$) unramified over v_0 extending

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the valuation v_0 given by the q-expansion coefficients at the infinity cusp of automorphic functions of level away from p (Lemma 2.2). This observation of the one-to-one correspondence between such valuations and irreducible components was first made by Deuring when he factored the modular equation of $X_0(p^m)$ over $\mathbf{P}^1(J)$ into irreducibles (and he could have proven the irreducibility of the elliptic Igusa tower by his argument before Igusa if he had the tower; of course, the special case for $X_0(p)$ of his result is the congruence relation of Kronecker). We show that the stabilizer D of a valuation v_∞ unramified over v_0 contains $M(\mathbb{Z}_{(p)})$ for $\mathbb{Z}_{(p)} = \mathbb{Z}_p \cap \mathbb{Q}$ (Lemma 1.1) and that $G_1 = Sp_{2n}(\mathbb{A}^{(p\infty)})$ fixes $\pi_0(I)$ (Lemma 2.3), and therefore, $D_x \supset M(\mathbb{Z}_{(p)})Sp_{2n}(\mathbb{A}^{(p\infty)})$, which implies $D_x = G$ and hence $M(\mathbb{Z}_p) = \mathrm{Gal}(I^\circ/S)$.

This choice of the generic point works well for Shimura varieties of $\mathrm{Res}_{F/\mathbb{Q}}Sp(2n)$ and $\mathrm{Res}_{F/\mathbb{Q}}SU(n,n)$ for totally real fields F (as was done in [H05] Section 10). If the Shimura variety of PEL type comes from a simply-connected inner form of a symplectic or unitary G_1 over a totally real field, we can have at least two choices of the points $x \in I$:

- (cm) all CM points $x \in I(\mathbb{F})$, whose stabilizer covers all maximal tori T_x of G anisotropic at ∞ (this choice is taken in [H06]);
- (gn) Taking the Serre–Tate coordinates $T = (T_1, \dots, T_d)$ around x and take the valuation $v_x(\sum_\alpha c(\alpha, f)T^\alpha) = \mathrm{Inf}_\alpha\, \mathrm{ord}_p(c(\alpha, f))$. Then the decomposition group D of v_x contains T_x (for all $x \in I^\circ$), and D is the stabilizer of the generic point of I° containing x (this choice is taken in [PAF] 8.4.4).

Again we can prove $D \supset G_1(\mathbb{A}^{(p\infty)})$. If we make the choice (cm), by p-adic approximation, $\{T_x(\mathbb{Z}_{(p)})\}_{x \in I^\circ(\mathbb{F})}$ and $G_1(\mathbb{A}^{(p\infty)})$ topologically generate $M \times G_1(\mathbb{A}^{(p\infty)})$ and we get $M = \mathrm{Aut}(I^\circ/S)$. If we make choice (gn), again D contains $\{T_x(\mathbb{Z}_{(p)})\}_{x \in I^\circ(\mathbb{F})}$, and the desired result follows by the same argument. Our axiomatic argument covers all type A and type C classical groups (as detailed in [PAF] 8.4, [H05] Section 10 and [H06]).

1. Siegel modular function fields

Fix a prime p and an algebraic closure $\mathbb{F}$ of $\mathbb{F}_p$. We fix a strict henselization $\mathcal{W} \subset \overline{\mathbb{Q}}$ of $\mathbb{Z}_{(p)}$ with quotient field $\mathcal{K}$ and residue field $\mathbb{F}$. For any integer N prime to p, we regard $\mathbb{Z}[\mu_N]$ as sitting inside $\mathcal{W}$. We have a continuous embedding $i_p : \mathcal{W} \hookrightarrow \overline{\mathbb{Q}}_p$. We fix a complex embedding $i_\infty : \mathcal{W} \hookrightarrow \mathbb{C}$. Put $G = GL_n(\mathbb{Z}_p) \times Sp_{2n}(\mathbb{A}^{(p\infty)})$ (with GL_n identified with the standard Levi subgroup M of the Siegel parabolic subgroup of $Sp(2n)$).

We consider the Mumford moduli $\mathfrak{M}(N)_{/\mathbb{Z}[\frac{1}{N}]}$ for an integer N prime to p which classifies triples $(X, \lambda, \phi_N)_{/A}$, where X is an abelian scheme over A of relative dimension n, $\lambda : X \to {}^tX$ is a principal polarization, $\phi_N : (\mathbb{Z}/N\mathbb{Z})^{2n} \cong X[N]$ with $\langle \phi_N(x), \phi_N(y)\rangle_\lambda = \zeta_N^{{}^txJy}$ for $J = \left(\begin{smallmatrix} 0 & -1_n \\ 1_n & 0 \end{smallmatrix}\right)$ and a fixed primitive root of unity $\zeta_N \in \mu_N$ (by the pairing $\langle \cdot, \cdot \rangle_\lambda : X[N] \times X[N] \to \mu_N$ induced by the polarization). If we consider level p^m-structure ϕ_p of type $\Gamma = \Gamma_?(p^m)$ $(? = 0, 1)$ given as follows: ϕ_p is a subgroup isomorphic to $\mu_{p^m}^n$ étale locally if $\Gamma = \Gamma_0(p^m)$ and $\phi_p : \mu_{p^m}^n \hookrightarrow X[p^m]$ if $\Gamma = \Gamma_1(p^m)$, we can think of the moduli space $\mathfrak{M}(N, \Gamma)_{/B}$ for the base ring $B = \mathbb{F}$ and $B = \mathbb{Q}$ which classifies quadruples $(X, \lambda, \phi_N, \phi_p)_{/A}$ over B-algebras A. The quasi-projective schemes $\mathfrak{M}(N)$ (resp. $\mathfrak{M}(N, \Gamma)$) can be regarded as a scheme over $\mathrm{Spec}(\mathbb{Z}[\frac{1}{N}, \mu_N])$ (resp. over $\mathrm{Spec}(\mathbb{Q}[\mu_N])$) by the pairing $\langle \cdot, \cdot \rangle_\lambda$. Note that $\mathbb{Z}[\frac{1}{N}, \mu_N] \subset \mathcal{W}$ if $p \nmid N$.

We can think of the p-integral Siegel–Shimura variety

$$Sh^{(p)}_{/\mathbb{Z}_{(p)}} = \varprojlim_{p\nmid N} \mathfrak{M}(N)_{/\mathbb{Z}_{(p)}},$$

and more generally over $\mathbb{Q}$,

$$Sh_{\Gamma/\mathbb{Q}} = \varprojlim_{p\nmid N} \mathfrak{M}(N,\Gamma)_{/\mathbb{Q}}$$

(regarding these schemes as $\mathbb{Z}_{(p)}$-schemes or $\mathbb{Q}$-schemes). The scheme $Sh^{(p)}$ is the p-integral model (of level away from p) Kottwitz considered. Let

$$\mathfrak{M}(N,\Gamma)_{/\mathcal{K}} = \mathfrak{M}(N,\Gamma)_{/\mathbb{Q}[\mu_N]} \times_{\mathbb{Q}[\mu_N]} \mathcal{K} \text{ and } \mathfrak{M}(N)_{/\mathcal{W}} = \mathfrak{M}(N)_{/\mathbb{Z}[\mu_N,\frac{1}{N}]} \times_{\mathbb{Z}[\mu_N,\frac{1}{N}]} \mathcal{W}.$$

The pro-schemes

$$\mathfrak{M}_{\Gamma/\mathcal{K}} = \varprojlim_{N} \mathfrak{M}(N,\Gamma)_{/\mathcal{K}} \text{ and } \mathfrak{M}^{(p)}_{\Gamma/\mathcal{W}} = \varprojlim_{p\nmid N} \mathfrak{M}(N)_{/\mathcal{W}}$$

give geometrically irreducible components of $Sh_{\Gamma/\mathbb{Q}}$ and $Sh^{(p)}_{/\mathbb{Z}_{(p)}}$ (the neutral components). If convenient, we write $Sh_{\Gamma_1(p^0)/\mathbb{Z}_{(p)}}$ for $Sh^{(p)}_{/\mathbb{Z}_{(p)}}$ (abusing the notation). By the interpretation of Deligne–Kottwitz, we have

$$Sh_\Gamma(A) \cong \frac{\{(X,\overline{\lambda},\eta : (\mathbb{A}^{(p\infty)})^{2n} \cong V(X), \phi_p)_{/A}\}}{\text{prime-to-}p\text{ isogenies}}, \tag{1.1}$$

where A runs over $\mathbb{Z}_{(p)}$-algebras if $\Gamma = \Gamma_1(p^0)$ and B-algebras ($B = \mathbb{F}$ or $\mathbb{Q}$) if $\Gamma = \Gamma_?(p^m)$ with $m > 0$ ($? = 0, 1$), $V(X) = \mathbb{A}^{(p\infty)} \otimes \varprojlim_{p\nmid N} X[N]$, and $\overline{\lambda}$ is a polarization class principal modulo multiplication by $\mathbb{Z}^\times_{(p)+}$. Thus $(a, g) \in G$ ($a \in M(\mathbb{Z}_p)$ and $g \in Sp_{2n}(\mathbb{A}^{(p\infty)})$) acts on Sh_Γ by

$$(X,\overline{\lambda},\eta,\phi_p) \mapsto (X,\overline{\lambda},\eta\circ g,\phi_p\circ a),$$

where $a \in GL_n(\mathbb{Z}_p) = M(\mathbb{Z}_p)$. Write $\mathfrak{F}_\Gamma$ for the function field $\mathcal{K}(\mathfrak{M}_\Gamma)$ and $\mathfrak{F}^{(p)}$ for $\mathcal{K}(\mathfrak{M}^{(p)})$ (the arithmetic automorphic function fields). By the above action, we have an isomorphism

$$\tau : G/\{\pm 1\} \hookrightarrow \mathrm{Aut}(\mathfrak{F}_{\Gamma_1(p^\infty)}/\mathcal{K}) = \mathrm{Aut}(\mathfrak{M}_{\Gamma_1(p^\infty)/\mathcal{K}}).$$

By Shimura (e.g., [PAF] Theorem 6.26), the action of $\tau(a, g)$ on the function field $\mathfrak{F}_\Gamma$ has the following property: For $a \in GL_n(\mathbb{Z}_{(p)})$ (corresponding to $\left(\begin{smallmatrix} a & 0 \\ 0 & {}^ta^{-1}\end{smallmatrix}\right)$ in $M(\mathbb{Z}_{(p)})$ diagonally embedded in $Sp_{2n}(\mathbb{A}^{(\infty)})$), we have for $f \in \mathcal{F}_\Gamma$

$$\tau(a)(f)(z) = f(a^{-1}z\,{}^ta^{-1}); \tag{1.2}$$

so, we have $\tau(\alpha)(f) = f(\alpha^{-1}(z))$ for $\alpha = \left(\begin{smallmatrix} a & 0 \\ 0 & {}^ta^{-1}\end{smallmatrix}\right)$. This formula is valid for general $\alpha \in GSp_{2n}(\mathbb{Z}_{(p)})$ if $f \in \mathfrak{F}^{(p)}$.

We define a valuation $v_\Gamma(f) = \inf_\xi \mathrm{ord}_p(c(\xi, f))$ defined on Siegel modular forms $f = \sum_\xi c(\xi, f)q^\xi \in H^0(\mathfrak{M}(N,\Gamma)_{/\mathcal{K}}, \det(\underline{\omega})^{\otimes k})$, where $\omega = \pi_*\Omega_{\mathbb{X}/\mathfrak{M}(N,\Gamma)}$ for the universal abelian scheme $\mathbb{X}_{/\mathfrak{M}(N,\Gamma)}$. Thus $\underline{\omega}$ is a vector bundle of rank n, and $\det(\underline{\omega}) = \bigwedge^n \underline{\omega}$, which is a line bundle. We write v_m for v_Γ if $\Gamma = \Gamma_1(p^m)$. Since all $\phi \in \mathfrak{F}_\Gamma$ can be written as $\phi = f/g$ for homogeneous polynomials f and g (of equal degree) of theta constants (existence of the minimal compactification over $\mathbb{Q}$), even if $\Gamma = \Gamma_?(p^m)$ with $m > 0$, the valuation

$v_\Gamma(\phi) = v_\Gamma(f) - v_\Gamma(g)$ is well defined. Thus the valuation $v_0 : \mathfrak{F}^{(p)} \to \mathbb{Z} \cup \{\infty\}$ has a standard unramified extension $v_\Gamma : \mathfrak{F}_\Gamma \twoheadrightarrow \mathbb{Z} \cup \{\infty\}$. Here are some easy facts:

Lemma 1.1. (1) *If $a \in GL_n(\mathbb{Z}_{(p)}) \cong M(\mathbb{Z}_{(p)})$ (with $M(\mathbb{Z}_{(p)})$ embedded diagonally in G), then $c(\xi, \tau(a)(f)) = c({}^t a\xi a, f)$. In particular, $M(\mathbb{Z}_{(p)}) \subset G$ preserves the valuation v_Γ;*

(2) *The vertical divisor $\mathfrak{M}^{(p)}_{/\mathbb{F}} := \mathfrak{M}^{(p)}_{/\mathcal{W}} \otimes_{\mathcal{W}} \mathbb{F}$ of $\mathfrak{M}^{(p)}_{/\mathcal{W}}$ is a prime divisor (geometrically irreducible) and gives rise to a* **unique** *valuation of $\mathfrak{F}^{(p)}$, whose explicit form is given by the valuation v_0.*

Proof. The first assertion follows directly from (1.2). By the existence of a smooth toroidal compactification (and the associated minimal compactification) of $\mathfrak{M}^{(p)}$ over $\mathcal{W}$ (due to Chai-Faltings [DAV]), Zariski's connectedness theorem tells us that $\mathfrak{M}^{(p)}_{/\mathbb{F}} = \mathfrak{M}^{(p)} \times_{\mathcal{W}} \mathbb{F}$ is irreducible. Thus the vertical Weil prime divisor $\mathfrak{M}^{(p)}_{/\mathbb{F}}$ gives rise to a unique valuation. By the irreducibility of $\mathfrak{M}^{(p)}_{/\mathbb{F}}$, a $\mathcal{W}$-integral modular form of level away from p vanishes on the divisor $\mathfrak{M}^{(p)}_{/\mathbb{F}}$ if and only if its q-expansion vanishes modulo p. Thus the valuation v_0 is the one corresponding to the vertical prime divisor $\mathfrak{M}^{(p)}_{/\mathbb{F}} \subset \mathfrak{M}^{(p)}_{/\mathcal{W}}$. □

2. Mod p connected components and the valuation v_m

Let S be the ordinary locus $\mathfrak{M}^{(p)}[\frac{1}{H}]_{/\mathbb{F}}$ for the Hasse invariant H. Then S is an irreducible variety over $\mathbb{F}$, because H is a global section of the ample sheaf $\det(\underline{\omega})^{\otimes(p-1)}$ of the minimal compactification of $\mathfrak{M}^{(p)}_{/\mathbb{F}}$. Consider the valuation ring V of $\mathfrak{F}^{(p)}$ of the valuation v_0. Thus the residue field $V/\mathfrak{m}_V$ is the function field $\mathbb{F}(S)$ of S. Let $\widehat{V} = \varprojlim_n V/p^n V$.

Let $\mathbb{X}_{/\mathfrak{M}^{(p)}}$ be the universal abelian scheme. Then we consider the Cartesian diagram for $\mathbb{X}_V = \mathbb{X} \times_{\mathfrak{M}^{(p)}} \mathrm{Spec}(V)$:

$$\begin{array}{ccc} \mathbb{X}_V & \xrightarrow{\hookrightarrow} & \mathbb{X} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(V) & \xrightarrow[\hookrightarrow]{} & \mathfrak{M}^{(p)}. \end{array}$$

Since any lift of H is inverted in V, $\mathbb{X}_V$ is an ordinary abelian scheme. Thus we can think of $I_{V,m} = \mathrm{Isom}_V(\mu^n_{p^m/V}, \mathbb{X}_V[p^m]_{/V})$, which is the scheme representing the functor sending a V-acheme R to the set of closed R-immersions: $\mu^n_{p^m/R} \hookrightarrow \mathbb{X}_V[p^m]_{/R}$. Let $\mathbb{X}_{\widehat{V}} = \mathbb{X}_V \times_V \widehat{V}$. We can restrict the above functor to p-adically complete formal $\widehat{V}$-schemes $\widehat{R}$. Then it is represented by the p-adic formal completion $\widehat{I}_{V,m}$ along the special fiber of $I_{V,m}$. Since $\mathbb{X}_{\widehat{V}}[p^m]$ has a well defined connected component $\mathbb{X}_{\widehat{V}}[p^m]^\circ$, $\widehat{I}_{V,m}$ is canonically isomorphic to $\mathrm{Isom}_V((\mathbb{Z}/p^m\mathbb{Z})^n, \mathbb{X}_{\widehat{V}}[p^m]^{et})$ by sending $\phi : \mu^n_{p^m/R} \cong \mathbb{X}_{\widehat{V}}[p^m]^\circ_{/R}$ to its cartier dual inverse ${}^t\phi^{-1}$. Then choosing a standard basis $e_1, \ldots, e_n$ of $(\mathbb{Z}/p^m\mathbb{Z})^n$ and sending ${}^t\phi^{-1}$ to ${}^t\phi^{-1}(e_1) \times {}^t\phi^{-1}(e_2) \times \cdots \times {}^t\phi^{-1}(e_n)$ we have an isomorphism from $\widehat{I}_{V,m}$ onto an étale closed formal subscheme of $\overbrace{\mathbb{X}_{\widehat{V}}[p^m]^{et} \times_{\widehat{V}} \cdots \times_{\widehat{V}} \mathbb{X}_{\widehat{V}}[p^m]^{et}}^{n}$ over $\mathrm{Spf}(\widehat{V})$.

Indeed, writing $\det : \overbrace{\mathbb{X}_{\widehat{V}}[p^m]^{et} \times_{\widehat{V}} \cdots \times_{\widehat{V}} \mathbb{X}_{\widehat{V}}[p^m]^{et}}^{n} \to \bigwedge^n_{\mathbb{Z}/p^m\mathbb{Z}} \mathbb{X}_{\widehat{V}}[p^m]^{et}$ for the determinant map, $\widehat{I}_{V,m}$ is identified with $\det^{-1}(\bigwedge^n_{\mathbb{Z}/p^m\mathbb{Z}} \mathbb{X}_{\widehat{V}}[p^m]^{et} - \bigwedge^n_{\mathbb{Z}/p^{m-1}\mathbb{Z}} \mathbb{X}_{\widehat{V}}[p^{m-1}]^{et})$.

Lemma 2.1. *The formal scheme $\widehat{I}_{V,m}/\mathrm{Spf}(\widehat{V})$ is an étale finite covering with a natural $GL_n(\mathbb{Z}_p)$-action, and $\widehat{I}_{V,m}$ is a $GL_n(\mathbb{Z}/p^m\mathbb{Z})$-torsor over* $\mathrm{Spf}(\widehat{V})$.

We may regard the moduli scheme $\mathfrak{M}(N,\Gamma)_{/\mathbb{F}}$ as a scheme over $\mathfrak{M}(N)[\frac{1}{H}]$ (forgetting the level p-structure). Since $\widehat{I}_{V,m}$ is étale faithfully flat over $\mathrm{Spf}(\widehat{V})$, it is affine, and we may write $\widehat{I}_{V,m} = \mathrm{Spec}(\widehat{V}_m)$. Then $\widehat{V}_m$ is a semi-local normal $\widehat{V}$-algebra étale faithfully flat over $\widehat{V}$ and hence is a product of finitely many valuation rings unramified over $\widehat{V}$.

We put $\mathfrak{M}_{\Gamma/\mathbb{F}} = \varprojlim_{p\nmid N} \mathfrak{M}(N,\Gamma)_{/\mathbb{F}}$. Then

$$\mathfrak{M}_{\Gamma_1(p^m)/\mathbb{F}} = \mathrm{Isom}_S(\mu^n_{p^m}, \mathbb{X}[p^m]^\circ) =: I_m$$

gives rise to the Igusa tower $I_\infty \twoheadrightarrow \cdots \twoheadrightarrow I_m \twoheadrightarrow \cdots \twoheadrightarrow I_1 \twoheadrightarrow S$ over S.

By the definition of the action of $(a,g) \in G$:

$$(X, \overline{\lambda}, \eta, \phi_p) \mapsto (X, \overline{\lambda}, \eta \circ g, \phi_p \circ a),$$

$G := M(\mathbb{Z}_p) \times Sp_{2n}(\mathbb{A}^{(p\infty)})$ acts on $I_{V,m}$ ($m = 1,2,\ldots,\infty$), $\mathrm{Spec}(V)$ (by Lemma 1.1 (2)), $\mathrm{Spf}(\widehat{V})$, $\mathfrak{F}_\Gamma$, I_m, $\mathfrak{M}_{\Gamma/\mathbb{F}}$ and $\mathfrak{M}_{\Gamma/\mathcal{K}}$. Thus we can make the étale quotient $\widehat{I}_{\Gamma_0(p^m)} := \widehat{I}_{V,m}/GL_n(\mathbb{Z}/p^m\mathbb{Z})$. Again we have $\widehat{I}_{\Gamma_0(p^m)} = \mathrm{Spf}(\widehat{V}_{\Gamma_0(p^m)})$, and by Lemma 2.1, $\widehat{V}_{\Gamma_0(p^m)}$ is a valuation ring unramified over $\widehat{V}$ sharing the same residue field. Thus $\widehat{V}_{\Gamma_0(p^m)} = \widehat{V}$. Indeed, there is a unique connected subgroup of X (isomorphic to $\mu^n_{p^m}$ étale locally) if $(X,\lambda,\phi_N)_{/A}$ gives rise to an A-point of $\mathfrak{M}(N,\Gamma_0(p^m))_{/\mathbb{F}}$. Thus $\mathfrak{M}(N,\Gamma_0(p^m))_{/\mathbb{F}} = S_{/\mathbb{F}}$. This shows that the residue field of $\widehat{V}_{\Gamma_0(p^m)}$ is the function field of S and that the quotient field of $V_{\Gamma_0(p^m)} = \widehat{V}_{\Gamma_0(p^m)} \cap \mathfrak{F}_{\Gamma_0(p^m)}$ is $\mathfrak{F}_{\Gamma_0(p^m)}$. This shows

Lemma 2.2. *We have the following one-to-one onto correspondences:*

$$\left\{v : \mathfrak{F}_{\Gamma_1(p^m)} \to \mathbb{Z} \middle| v|_{\mathfrak{F}_{\Gamma_0(p^m)}} = v_{\Gamma_0(p^m)}\right\} \leftrightarrow \mathrm{Max}(\widehat{V}_m) \leftrightarrow \pi_0(I_m),$$

where v is a p-adic valuation of $\mathfrak{F}_{\Gamma_1(p^m)}$ unramified (of degree 1) over v_0 and $\mathrm{Max}(\widehat{V}_m)$ is the set of maximal ideals of $\widehat{V}_m$.

Proof. By the above argument, $\widehat{V}_m \otimes_W \mathbb{F} = \prod_{I^\circ_m \in \pi_0(I_m)} \mathbb{F}(I^\circ_m)$; so, $\pi_0(I_m)$ injects into the set of valuations in the lemma. Since $GL_n(\mathbb{Z}/p^m\mathbb{Z}) = \mathrm{Gal}(\mathfrak{F}_{\Gamma_1(p^m)}/\mathfrak{F}_{\Gamma_0(p^m)})$ acts transitively on $\pi_0(I_m)$ and the set of valuations (by Hilbert's theory of extension of valuations), the injection is onto. □

Lemma 2.3. *The action of $G_1 := Sp_{2n}(\mathbb{A}^{(p\infty)})$ fixes $v_m = v_{\Gamma_1(p^m)}$ and each element of $\pi_0(I_m)$.*

Proof. Since $\mathfrak{F}_{\Gamma_1(p^m)}/\mathfrak{F}_{\Gamma_0(p^m)}$ is a finite Galois extension, the set of extensions of $v_{\Gamma_0(p^m)}$ to $\mathfrak{F}_{\Gamma_1(p^m)}$ is a finite set, and by the above lemma, it is in bijection with $\pi_0(I_m)$. Thus the action of $Sp_{2n}(\mathbb{A}^{(p\infty)})$ on $\pi_0(I_m)$ gives a finite permutation representation of $Sp_{2n}(\mathbb{A}^{(p\infty)})$. Since $Sp_{2n}(k)$ of any field k of characteristic 0 does not have nontrivial finite quotient group (because it is generated by divisible unipotent subgroups), the action of $Sp_{2n}(\mathbb{A}^{(p\infty)})$ fixes every irreducible component of $\pi_0(I_m)$. □

3. Proof of irreducibility of I_m

Let $v_\infty = v_{\Gamma_1(p^\infty)}$, and define

$$D_v = \left\{x \in (M(\mathbb{Z}_p) \times Sp_{2n}(\mathbb{A}^{(p\infty)}))/\{\pm 1\} \middle| v_\infty \circ \tau(x) = v_\infty\right\}.$$

Since $M(\mathbb{Z}_{(p)})$ and $Sp_{2n}(\mathbb{A}^{(p\infty)})$ fixes v_∞ (Lemmas 1.1 and 2.3) and $M(\mathbb{Z}_{(p)})Sp_{2n}(\mathbb{A}^{(p\infty)})$ is dense in $(M(\mathbb{Z}_p) \times Sp_{2n}(\mathbb{A}^{(p\infty)}))$, we have

Theorem 3.1. *We have $D_v = M(\mathbb{Z}_p) \times Sp_{2n}(\mathbb{A}^{(p\infty)})/\{\pm 1\}$.*

Let $K^{(p)}$ be an open compact subgroup of $Sp_{2n}(\mathbb{A}^{(p\infty)})$ and $K = K^{(p)} \times GSp_{2n}(\mathbb{Z}_p)$. Put $\mathfrak{M}_K = \mathfrak{M}^{(p)}/K^{(p)}$ (which is the level K Siegel modular variety). Let $I_K = I_\infty/K^{(p)}$, which is the Igusa tower over $\mathfrak{M}_K$. Since I_∞ is irreducible by $\mathrm{Gal}(I^\circ/S) = GL_n(\mathbb{Z}_p) = M(\mathbb{Z}_p)$ (the above theorem), I_K is irreducible. Thus we have reproved

Corollary 3.2 (Chai-Faltings)**.** *The Siegel-Igusa tower I_K over $\mathfrak{M}_{K/\mathbb{F}}$ is irreducible for any open compact subgroup K of $Sp_{2n}(\mathbb{A}^{(\infty)})$ maximal at p.*

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