CONTROL OF NEARLY ORDINARY HECKE ALGEBRAS

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Abstract. Let p be a prime and F be a totally real field. We describe the structure theory of the nearly-ordinary Hilbert modular p-adic Hecke algebra for F. In particular, if we fix a central character (without allowing the character to deform), we prove that the dimension (over $\mathbb{Z}_p$) of the cuspidal part is $[F:\mathbb{Q}]$ and the Eisenstein part is $1+\delta$ for the p-adic defect δ of the Leopoldt conjecture. We may be able to touch the control theorem and the Galois representation into $GL(2)$ with coefficients in the algebra.

We prepare some notation to define Hilbert modular Hecke algebras (for F) in a classical way. Write $I = \mathrm{Hom}_{\text{field}}(F, \overline{\mathbb{Q}})$. Let $G = \mathrm{Res}_{O/\mathbb{Z}} GL(2)$ for the integer ring O of F. Thus G is a group scheme with $G(A) = GL_2(A \otimes_{\mathbb{Z}} O)$; so, $G(\mathbb{Z}) = GL_2(O)$, $G(\mathbb{Q}) = GL_2(F)$, $G(\widehat{\mathbb{Z}}) = GL_2(\widehat{O})$ for $\widehat{\mathbb{Z}} = \prod_{\ell:\text{primes}} \mathbb{Z}_\ell$ and $\widehat{O} = \prod_{\mathfrak{l}:\text{ prime ideals}} O_{\mathfrak{l}} \cong O \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$, $G(\mathbb{A}) = GL_2(F_{\mathbb{A}})$ for the adele ring $\mathbb{A}$ of $\mathbb{Q}$ ($F_{\mathbb{A}} = F \otimes_{\mathbb{Q}} \mathbb{A}$) and $G(\mathbb{R}) = GL_2(\mathbb{R})^I$ since $F \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{R}^I$ by $\xi \otimes r \mapsto (\sigma(\xi) r)_{\sigma \in I}$. Then $G(\mathbb{R}) = \{(g_\sigma)_{\sigma \in I} \in G(\mathbb{R}) | g_\sigma \in GL_2(\mathbb{R})\}$ acts naturally on $\mathfrak{Z} = (\mathbb{C} - \mathbb{R})^I$ by component-wise linear fractional transformation. Let $Z \subset G$ be the center; so, $Z(A)$ consists of scalar matrices in $G(A)$. Let $T_0 = \mathbb{G}^2_{m/O}$ be the diagonal torus of $GL(2)_{/O}$, and put $T = \mathrm{Res}_{O/\mathbb{Z}} T_0$. Then T contains the center Z of G, and identify $\overline{T} = T/Z$ with $\mathrm{Res}_{O/\mathbb{Z}} \mathbb{G}_m$ via $y \mapsto \left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \overline{T}$. Take an open subgroup $U \subset G(\widehat{\mathbb{Z}})$ of the form $U = \prod_{\mathfrak{l}} U_{\mathfrak{l}}$ with $U_{\mathfrak{l}} \subset GL_2(O_{\mathfrak{l}})$ (such a group is called a level subgroup), and consider the (abstract) Hecke ring R_U with convolution product of compactly supported bi-U-invariant $\mathbb{Z}$-values functions on $G(\mathbb{A}^{(\infty)})$ (where $\mathbb{A} = \mathbb{A}^{(\infty)} \times \mathbb{R}$). Here the convolution product $f * g(x) = \int f(xy^{-1}) g(y) dy$ is defined under the Haar measure on $G(\mathbb{A}^{(\infty)})$ with $\int_U dy = 1$. For any open-compact bi-U-invariant set $X \subset G(\mathbb{A}^{(\infty)})$, we write $[X]$ for the characteristic function of X. Then

$$R_U = \bigoplus_{x \in G(\mathbb{A}^{(\infty)})} \mathbb{Z}[UxU]$$

and is an algebra with identity $[U]$. Thus shows $R_U = \bigotimes_{\mathfrak{l}} R_{U_{\mathfrak{l}}}$ for the convolution algebra $R_{U_{\mathfrak{l}}}$ with respect to $U_{\mathfrak{l}} \subset GL_2(F_{\mathfrak{l}})$. The standard level subgroup in $G(\widehat{\mathbb{Z}})$ of $\Gamma_0(\mathfrak{N})$-type for an integral ideal $\mathfrak{N}$ (called a level) is given by

$$\widehat{\Gamma}_0(\mathfrak{N}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(\widehat{\mathbb{Z}}) \middle| c \in \mathfrak{N}\widehat{O} \right\}. \tag{0.1}$$

We also define the Γ_1-*type level subgroup*

$$\widehat{\Gamma}_1(\mathfrak{N}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \widehat{\Gamma}_0(\mathfrak{N}) \middle| a-1, d-1 \in \mathfrak{N}\widehat{O} \right\}.$$

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We fix a level $\mathfrak{N}$ prime to p and a level subgroup U with $\widehat{\Gamma}_0(\mathfrak{N}) \supset U \supset \widehat{\Gamma}_1(\mathfrak{N})$. Since $U = \prod_{\mathfrak{l}} U_{\mathfrak{l}}$ with $U_{\mathfrak{l}} \subset GL_2(O_{\mathfrak{l}})$, $U_{\mathfrak{l}} = GL_2(O_{\mathfrak{l}})$ for $\mathfrak{l} \nmid \mathfrak{N}$. Each $f \in R_U$ is a tensor product of local functions $f_{\mathfrak{l}} : GL_2(F_{\mathfrak{l}}) \to \mathbb{Z}$ and $f_{\mathfrak{l}}$ is the characteristic function of $GL_2(O_{\mathfrak{l}})$ for almost all $\mathfrak{l} \nmid \mathfrak{N}$; in other words, $f(x) = \prod_{\mathfrak{l}} f_{\mathfrak{l}}(x_{\mathfrak{l}})$ and $[UxU] = \bigotimes_{\mathfrak{l}} [U_{\mathfrak{l}} x_{\mathfrak{l}} U_{\mathfrak{l}}]$, where $[U_{\mathfrak{l}} x_{\mathfrak{l}} U_{\mathfrak{l}}]$ is the characteristic function of the double coset $U_{\mathfrak{l}} x_{\mathfrak{l}} U_{\mathfrak{l}}$.

Write $U_a = U \cap \widehat{\Gamma}_0(p^a)$ and $U_a^1 = U \cap \widehat{\Gamma}_1(p^a)$. Hereafter we only consider a general level subgroup S with $U_a \supset S \supset U_a^1$ for some a. If we need to indicate the exponent a of the p-power level, we write S_a for S. At p, we consider, fixing a generator $\varpi_{\mathfrak{p}}$ of $\mathfrak{p}O_{\mathfrak{p}}$ for $\mathfrak{p}|p$,

$$\Delta = \left\{ \left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix} \right) \in M_2(O_p \times O_{\mathfrak{N}}) | a \in \prod_{\mathfrak{p}|p} \varpi_{\mathfrak{p}}^{\mathbb{Z}} \times F_{\mathfrak{N}}^{\times} \right\}.$$

It is easy to check that $\Delta_S = S_{p\mathfrak{N}} \Delta S_{p\mathfrak{N}}$ is a multiplicative semi-group. We consider

$$H_S = \{ f \in R_S | \operatorname{Supp}(f_{p\mathfrak{N}}) \subset \Delta_S \} \quad \text{where} \quad f_{p\mathfrak{N}} = \bigotimes_{\mathfrak{l}|p\mathfrak{N}} f_{\mathfrak{l}}.$$

Since Δ_S is a multiplicative semi-group, H_S is an algebra, and we can again factor $H_S = \bigotimes_{\mathfrak{l}} H_{S,\mathfrak{l}}$, and $H_{S,\mathfrak{l}} = R_{U_{\mathfrak{l}}}$ for all $\mathfrak{l} \nmid p$. For such groups $S = S_a \subset S'_b = S'$ with $a \geq b > 0$, the association $[SxS] \mapsto [S'xS']$ ($x \in G(\mathbb{A}^{(\infty)})$ with $x_{p\mathfrak{N}} \in \Delta$) induces a linear map $H_S \twoheadrightarrow H_{S'}$ ($a \geq b > 0$) and hence a linear map: $H_S \twoheadrightarrow H_{S'}$ (by the tensor product expression). By group theory, as rings, $H_{S',\mathfrak{p}} \cong H_{S,\mathfrak{p}} \cong \mathbb{Z}[U_S(\varpi_{\mathfrak{p}})]$ ($U_S(y) = [S \left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix} \right) S]$) by this map:

Lemma 0.1. *The algebra H_S is commutative, and the above linear map is an isomorphism of rings: $H_S \cong H_{S'}$ for all $a > b > 0$.*

We identify all H_S by the above isomorphism and write it as H. For any commutative ring R, we write $H(R)$ for the scalar extension $H \otimes_{\mathbb{Z}} R$.

If $U_a \supset S \supset S' \supset U_a^1$ (so, $S \triangleright S'$ with commutative quotient S/S'), we can think of the Hecke ring $H_{S'}^S = \bigoplus_{x \in \Delta_S} \mathbb{Z}[S'xS']$. Then $H_{S'}^S \cong H_S[S/S'] \cong H[S/S']$ (the group ring of S/S') by a natural map. Here is a general lemma.

Lemma 0.2. *Let M be a $\mathbb{Z}_p$-module of finite or cofinite type and $u : M \to M$ be a linear operator. Then the p-adic limit $e = \lim_{n\to\infty} u^{n!}$ exists in $\operatorname{End}_{\mathbb{Z}_p}(M)$ and satisfies $e^2 = e$.*

Proof. Taking the Pontryagin dual if necessary, we may assume that M is a $\mathbb{Z}_p$-module of finite type. Since $\mathbb{Z}_p[u] \subset \operatorname{End}_{\mathbb{Z}_p}(M)$ is a commutative $\mathbb{Z}_p$-algebra finite over $\mathbb{Z}_p$. Thus it is a product of finitely many local rings; so, we may assume that $\mathbb{Z}_p[u]$ is a local ring. Then the residue field $\mathbb{Z}_p[u]$ has order q, and hence u^{q-1} is either congruent to 0 or 1 modulo the maximal ideal $\mathfrak{m}$. Let $q_m = |\mathbb{Z}_p[u]/\mathfrak{m}^m|$; so, $q_m = (q-1)q^{a_m}$ for an increasing sequence $a_1 = 0 < a_2 < \cdots < a_m < \cdots$ of integers, and u^{q_m} is either congruent to 0 or 1 modulo the maximal ideal $\mathfrak{m}^m$ according as u^{q-1} is congruent to 0 or 1 $\mathfrak{m}$. Then we have $\lim_{n\to\infty} u^{n!} = \lim_{m\to\infty} u^{q_m} = \begin{cases} 1 & \text{if } u \notin \mathfrak{m}, \\ 0 & \text{if } u \in \mathfrak{m}. \end{cases}$ □

1. Axiomatic method

Let $K/\mathbb{Q}_p$ be a finite extension and $W \subset K$ be the p-adic integer ring of K. Fix U as above. We give ourselves $H_{S_a}^{U_a}(W)$-modules $\{M_S^*\}_S$. Since the action of $Z(\mathbb{A}^{(p\mathfrak{N}\infty)})$ is contained in the $H(W)$-module structure, the group $Z(\mathbb{A}^{(p\mathfrak{N}\infty)})$ acts on M_S^*, and suppose the action extends to $Z(\mathbb{A}^{(\infty)})$. We require $\{M_S^*\}_S$ to satisfy the following axiom (A0–A4) for $a \gg 0$:

- (A0) The center $Z(\mathbb{A}^{(\infty)})$ acts on M_S^* by a continuous character $\varepsilon : Z(\mathbb{A}^{(\infty)}) \to W^\times$;
- (A1) M_S^* is p-divisible of finite corank (its Pontryagin dual M_S is W-free of finite rank);
- (A2) If $U_\infty^1 \subset S'_b \subset S_a \subset U_1$ $(b \geq a)$, we have a $H_{S'_b}^{U_b}(W)$-linear maps $[SxS'] : M_{S'}^* \to M_S^*$ for $x \in \Delta_{S'}$ (such that $[SxS'] \circ [S'yS'']$ is induced by $[SxS] * [S'yS'']$ if $S \supset S' \supset S''$) and $\iota_{S/S'} : M_S^* \to M_{S'}^*$ forming an injective system $\{M_S^*, \iota_{S/S'}\}_S$, and if $T(\mathbb{Z}/p^b\mathbb{Z}) \supset (S'_b/U_b^1)$ surjects down to $(S_a/U_a^1) \subset T(\mathbb{Z}/p^a\mathbb{Z})$, we have $[S \left(\begin{smallmatrix} p^{b-a} & 0 \\ 0 & 1 \end{smallmatrix}\right) S'] \circ \iota_{S/S'} = U_S(p^{b-a})$ and $\iota_{S/S'} \circ [S \left(\begin{smallmatrix} p^{b-a} & 0 \\ 0 & 1 \end{smallmatrix}\right) S'] = U_{S'}(p^{b-a})$;
- (A3) If $U_a \supset S \supset S' \supset U_a^1$, $M_S \cong H_0(S/S', M_{S'})$ by $\iota_{S/S'}^*$ $(\Leftrightarrow M_S^* \cong H^0(S/S', M_{S'}^*))$.

We assume that the action of S/S' on $M_{S'}$ factors through $\overline{S/S'} = S \cdot Z(\mathbb{Q}_p)/S' \cdot Z(\mathbb{Q}_p)$ (this we can achieve by twisting by a character), for simplicity.

- (A4) If $U_a \supset S \supset S' \supset U_a^1$, for each minimal prime ideal $P \subset W[\overline{S/S'}]$, M_S/PM_S is free of finite rank over W $(\Leftrightarrow M_S^*[P] := \{x \in M_S^* | Px = 0\}$ is p-divisible).

Let M be a projective limit $\varprojlim_S M_S = \varprojlim_a M_{U_a^1}$. Since the finite group $\overline{U_a/U_a^1} \cong \overline{T}(\mathbb{Z}/p^a\mathbb{Z})$ acts on $M_{U_a^1}$, M is a $W[[\overline{T}(\mathbb{Z}_p)]]$-module as well as an $H(W)$-module. Giving M the topology of projective limit of the p-adic topology of M_S, the ring $\mathrm{End}_W(M)$ is a compact ring. The subring $\mathbf{h} = \mathbf{h}(M) \subset \mathrm{End}_W(M)$ generated by the operator in $H(W)$ and the action of $\overline{T}(\mathbb{Z}_p)$ is the compact Hecke algebra of M. For each S, we have an S-version $\mathbf{h}_S = \mathbf{h}(M_S) \subset \mathrm{End}(M_S) = \mathrm{End}(M_S^*)$ which is the algebra generated by the action of $H(W)$ and $\overline{T}(\mathbb{Z}_p)$ (factoring through $\overline{T}(\mathbb{Z}/p^a\mathbb{Z})$ if $S = S_a$). Then $\mathbf{h}(M_S)$ is W-free of finite rank (by (A1)) and $\mathbf{h} = \varprojlim_S \mathbf{h}_S$. Consider the $U(p^m) = U_S(p^m)$ operator $S \left(\begin{smallmatrix} p^m & 0 \\ 0 & 1 \end{smallmatrix}\right) S$. Since for any $S \supset S'$ as above, we check by computation the following fact:

Lemma 1.1. *Let $S = S_a \supset S' = S'_b$ with $b \geq a > 0$. If $m \geq b - a$ and the image of S' in $\overline{T}(\mathbb{Z}/p^b\mathbb{Z}) = \overline{U_b/U_b^1}$ modulo p^a is the image of S in $\overline{T}(\mathbb{Z}/p^a\mathbb{Z})$, we have*

$$S \begin{pmatrix} p^m & 0 \\ 0 & 1 \end{pmatrix} S' = \bigsqcup_{u \in O_p/p^m O_p} \begin{pmatrix} p^m & u \\ 0 & 1 \end{pmatrix} S'.$$

From this, it is easy to conclude $U(p^m) = U(p)^m$. Consider $e_S = \lim_{n \to \infty} U(p)^{n!} \in \mathbf{h}_S$ and $e = \varprojlim_S e_S \in \mathbf{h}$. Let $M_S^\circ = e_S \cdot M_S$, $M^\circ = e \cdot M = \varprojlim_S M_S^\circ$, $\mathbf{h}^\circ = e \cdot \mathbf{h}$ and $\mathbf{h}_S^\circ = e_S \mathbf{h}_S$. Split $\overline{T}(\mathbb{Z}_p) = \Gamma \times \mu$ for a torsion-free p-profinite subgroup Γ and a torsion subgroup μ.

Theorem 1.2. *We write $\mathbf{\Lambda}$ for $W[[\overline{T}(\mathbb{Z}_p)]]$ and Λ for $W[[\Gamma]]$. Then we have*

1. *For $S = S_a$, M_S° is a projective ($\Leftrightarrow$ flat) module of finite type over $W[\overline{U_a/S}]$;*
2. *M° is a projective $\mathbf{\Lambda}$-module of finite type (so, M° is free of finite rank over Λ);*
3. *For a level group $S = S_a$ with $a > 0$, regarding $\overline{T}(\mathbb{Z}/p^a\mathbb{Z}) \twoheadrightarrow \overline{U_a/S}$ as a quotient of $\overline{T}(\mathbb{Z}_p)$, put $\mathfrak{a}_S = \mathrm{Ker}(\mathbf{\Lambda} \twoheadrightarrow W[\overline{U_a/S}])$. Then $M_{S'}^\circ/\mathfrak{a}_S M_{S'}^\circ \cong M_S^\circ$ for all $S' \subset S$ and $M^\circ/\mathfrak{a}_S M^\circ \cong M_S^\circ$ canonically by the projection $M^\circ \twoheadrightarrow M_S^\circ$;*
4. *$\mathbf{h}^\circ$ is a Λ-torsion-free module of finite type.*

Proof. For simplicity, we assume that $X(U)$ is smooth; other wise, replace $\overline{T}(\mathbb{Z}_p)$ by $T' = 1 + p^{a_0}O_p \subset \overline{T}(\mathbb{Z}_p)$ (for a_0 such that $X(U^1_{a_0})$ is smooth) and use the fact that $\mathbf{\Lambda}$ is free (and faithfully flat) over $W[[T']]$ to recover the result for $\mathbf{\Lambda}$.

Let $U_a \supset S \supset S' \supset U^1_a$. Regard $M_{S'}$ as A-module for $A = W[\overline{S/S'}]$. To see $M^\circ_{S'}$ is A-projective, for each local ring $A_\mathfrak{m}$ of A, we need to prove that the localization $M_\mathfrak{m} := M^\circ_{S',\mathfrak{m}}$ is $A_\mathfrak{m}$-free. Since for any finite extension W'/W, $W'[\overline{S/S'}]$ is $W[\overline{S/S'}]$-free and hence is $W[\overline{S/S'}]$-faithfully flat. Thus freeness over $A_\mathfrak{m}$ and over $A_\mathfrak{m} \otimes_W W'$ is equivalent. Replacing W by W', we may assume that $A_\mathfrak{m}/P \cong W$ for all minimal ideals P of $A_\mathfrak{m}$. Let $n = \dim M_\mathfrak{m}/\mathfrak{m}M_\mathfrak{m}$. Choose a set of generators $x_1, \ldots, x_n$ of $M_\mathfrak{m}$ over $A_\mathfrak{m}$ by Nakayama's lemma, we have a surjection $\pi : A^n_\mathfrak{m} \twoheadrightarrow M_\mathfrak{m}$ sending $(a_1, \ldots, a_n)$ to $\sum_j a_j x_j$. By (A4), $M_\mathfrak{m}/PM_\mathfrak{m}$ is W-free for each minimal prime $P \subset A_\mathfrak{m}$. We have a surjection $\pi_P : W^n = (A_\mathfrak{m}/P)^n \twoheadrightarrow M_\mathfrak{m}/PM_\mathfrak{m}$. The minimal number of generators of $M_\mathfrak{m}/PM_\mathfrak{m}$ over $A_\mathfrak{m}/P = W$ is $\dim(M_\mathfrak{m}/PM_\mathfrak{m} \otimes_A A/\mathfrak{m}) = \dim M_\mathfrak{m} \otimes_A A/\mathfrak{m} = n$, then the LHS and RHS of π_P are free of rank n over W; so, π_P is an isomorphism; hence $\mathrm{Ker}(\pi) \subset \bigcap_P P^n - \{0\}$. Thus $M^\circ_{S'}$ is A-projective, proving (1). Passing to the limit, we get (2) by (A3).

We now prove (3). If $U_a \supset S_a \supset S'_a \supset U^1_a$, (3) follows from (A3). Thus we assume that $S' = S'_b \subset S_a = S$ with $b > a$, and we first assume first that the image of S' in $\overline{T}(\mathbb{Z}/p^b\mathbb{Z})$ modulo p^a is the image of S in $\overline{T}(\mathbb{Z}/p^a\mathbb{Z})$. By (A2) (and Lemma 1.1), we have the following commutative diagram:

$$\begin{array}{ccccc} & M_{S'} & \overset{\iota^*_{S/S'}}{\twoheadrightarrow} & M_S & \\ U_{S'}(p)^{b-a} & \downarrow & \nwarrow u^* & \downarrow & U_S(p)^{b-a} \\ & M_{S'} & \twoheadrightarrow & M_S & \end{array}$$

for $u = [S \left(\begin{smallmatrix} p^{b-a} & 0 \\ 0 & 1 \end{smallmatrix}\right) S']$. Since $U(p)$ is invertible on M°_S and $M^\circ_{S'}$, this shows that $M^\circ_{S'} \cong M^\circ_S$ as $H(W)$-modules and also as $\overline{T}(\mathbb{Z}_p)$-modules as long as the image of S' in $\overline{T}(\mathbb{Z}/p^b\mathbb{Z})$ modulo p^a is the image of S in $\overline{T}(\mathbb{Z}/p^a\mathbb{Z})$. In general, taking $S_a \supset S''_a \supset S'_b$ such that $U_a \supset S_a \supset S''_a \supset U^1_a$ and that the image of S' in $\overline{T}(\mathbb{Z}/p^b\mathbb{Z})$ modulo p^a is the image of S'' in $\overline{T}(\mathbb{Z}/p^a\mathbb{Z})$, we have $M^\circ_{S'} \cong M^\circ_{S''}$ by the above argument, and $M^\circ_{S''}/\mathfrak{a}_S M^\circ_{S''} \cong M^\circ_S$ by (A3). Passing to the limit, we get $M^\circ/\mathfrak{a}_S M^\circ \cong M_S$. Since we get

- $\mathbf{\Lambda}/\mathfrak{a}_{U^1_a} = W[\overline{T}(\mathbb{Z}/p^a\mathbb{Z})]$;
- $M^\circ_{U^1_b}/\mathfrak{a}_{U^1_a} M^\circ_{U^1_b} \cong M^\circ_{U^1_a}$ for all $b > a > 0$;
- $M^\circ_{U^1_a}$ is $W[\overline{T}(\mathbb{Z}/p^a\mathbb{Z})]$-projective of finite type for all $a > 0$,

passing to the limit, we conclude that M° is $\mathbf{\Lambda}$-projective of finite type. This proves (1), (2) and (3). Since $\mathbf{h}^\circ \subset \mathrm{End}_\Lambda(M^\circ) \cong M_r(\Lambda)$ for $r = \mathrm{rank}_\Lambda M^\circ$, $\mathbf{h}^\circ$ is Λ-torsion-free of finite rank, and (4) follows. $\square$

Remark 1.1. We can remove (A0) and (A4) deforming the central character. In that case, the limit group $\varprojlim_a U_a/U^1_a$ is isomorphic to $T(\mathbb{Z}_p)$. However, in application, automorphic forms of level S is also invariant under $Z(\mathbb{Q})$; so, we need to think of the quotient $\mathbf{G}_U := \varprojlim_a Z(\mathbb{Q})U_a/Z(\mathbb{Q})U^1_a \cong T(\mathbb{Z}_p)/\overline{Z_U}$, where $Z_U = Z(\mathbb{Q}) \cap U \subset Z(\mathbb{Z})$ is the group of U-units. The assertion of the above theorem is valid replacing $\overline{T}(\mathbb{Z}_p)$ by $\mathbf{G}_U$ which is isogenous to $\mathbb{Z}_p^{[F:\mathbb{Q}]+1+\delta}$ for the p-adic defect δ of the Leopoldt conjecture.

2. Choice of M_S

To describe the examples of M_S, we assume (by twists) that the central character ε is trivial for simplicity. We consider quaternionic Shimura varieties. Let D be a 4-dimensional central simple algebra over F (in short, a quaternion algebra over F). Fix a maximal order R of D, and define an algebraic group $G^D(A) = (R \otimes_{\mathbb{Z}} A)^\times$ for a ring A. For simplicity, we assume $G^D(\mathbb{Z}_\ell) \cong G(\mathbb{Z}_\ell)$ for all primes ℓ. Such a G^D always exists. Identifying $G^D(\widehat{\mathbb{Z}})$ with $G(\widehat{\mathbb{Z}})$, we can use the same level subgroups U and $\widehat{\Gamma}_*(\mathfrak{N})$ for G^D. Since G^D for any D shares the same center, we identify them with Z. Then we consider automorphic varieties $X(S) = G^D(\mathbb{Q})\backslash G^D(\mathbb{A})/S \cdot Z(\mathbb{A})C_D$, where C_D is the maximal compact subgroup of $G(\mathbb{R})$. If U is sufficiently small, $X(S)$ is a smooth complex manifold. If $G(\mathbb{R})/Z(\mathbb{R})$ is compact (that is, $D\otimes_{\mathbb{Q}}\mathbb{R}$ is a product of copies of the Hamilton quaternion algebra $\mathbb{H}_{/\mathbb{R}}$), $X(S)$ is just a set of finitely many points. If $G(\mathbb{R})/Z(\mathbb{R})$ is noncompact, we have a partition $I = I_D \bigsqcup I^D$ for non-empty I_D such that we have $D\otimes_{\mathbb{Q}}\mathbb{R} \cong M_2(\mathbb{R})^{I_D} \times \mathbb{H}^{I^D}$. Through the $M_2(\mathbb{R})^{I_D}$, $G^D(\mathbb{R})$ acts on $\mathfrak{Z}_D = (\mathbb{C}-\mathbb{R})^{I_D}$ by linear fractional transformation, and $X(S) \cong \bigsqcup_g \Gamma_{S^g}\backslash\mathfrak{Z}_D$ ($\Gamma_{S^g} = g^{-1}S \cdot Z(\mathbb{A})g \cap G^D(\mathbb{Q})$) for a complete representative set $\{g\}$ (with $g_p = 1$) of $G^D(\mathbb{Q})\backslash G^D(\mathbb{A}^{(\infty)})/S{\cdot}Z(\mathbb{A}^{(\infty)})$ (the set $\{g\}$ is finite: the approximation theorem). Thus $\dim_{\mathbb{C}} X(S) = d = |I_D|$. Under he condition that $G^D(\widehat{\mathbb{Z}}) \cong G(\widehat{\mathbb{Z}})$, $|I^D| \equiv [F : \mathbb{Q}] \mod 2$ (Hasse). In the case of $I_D \neq \emptyset$, by the theory of canonical models of Shimura, it is the manifold of $\mathbb{C}$-points of a quasi projective variety $X(S)_{/E}$ for the reflex field E. Here E is generated over $\mathbb{Q}$ by $\sum_{\sigma\in I_D}\sigma(\xi)$ for $\xi \in F$. In particular, $E = F$ if I_D is made of the identity embedding $F \hookrightarrow \overline{F}$, and we have $F = \mathbb{Q}$ if $I_D = I$ (the Hilbert modular case). An obvious choice of M_S is $H_d(X(S), W)$ (so, $M_S^* = H^d(X(S), K/W)$). We think of the Hecke algebra $\mathbf{h}(M_S) \subset \mathrm{End}_K(H_d(X(S), K))$ generated over W by the operators of $H(W)$ and $\overline{T}(\mathbb{Z}_p)$. Then we can define $\mathbf{h}^\circ(M_S)$ and $\mathbf{h}^\circ = \varprojlim_S \mathbf{h}^\circ(M_S)$. By the Jacquet-Langlands correspondence (e.g. [HMI] 2.3.6), we have

Lemma 2.1. *The algebras $\mathbf{h}^\circ$ and $\mathbf{h}^\circ_S$ are independent of the choice of D as long as $D \neq M_2(F)$. For two choices of such D and D', we have an isomorphism between them sending $[SxS']$ of the D-side to $[SxS']$ of the D'-side. If $D = M_2(F)$, the cuspidal part of $\mathbf{h}^\circ_S$ coincides with the corresponding Hecke algebra of a division D in the same way.*

Hereafter, we use the symbol $\mathbf{h}$ to indicate the cuspidal Hecke algebra common to all D and use $\mathbf{H}$ for the full Hecke algebra for $D = M_2(F)$; so, $\mathbf{H}^\circ$ has Eisenstein component. By the above lemma, to study $\mathbf{h}^\circ$, we can choose a quaternion algebra we like. If $|I_D| \leq 1$, M_S is W-torsion-free (because $\dim X(S) \leq 1$). This shows (A1), and (A2) follows from the fact that we have the restriction map $\iota_{S/S'} = \mathrm{Res}_{S/S'}$ for cohomology and the correspondence action of $[SxS']$ on homology and cohomology group. It is a left action on cohomology groups and by the Poincaré duality we get the homological right action.

Lemma 2.2. *Suppose $|I_D| \leq 1$ and put $M_S = H_d(X(S), W)$. Then the conditions* (A3–4) *are satisfied by $\{M_S\}_S$.*

The conditions (A3–4) follows trivially when $|I_D| = 0$; so, we may assume $|I_D| = 1$. We give a sketch of the proof.

Proof. A key is the inflation-restriction exact sequence for H^1:

$$0 \to H^1(S/S', K/W) \xrightarrow{\text{Inf}} H^1(X(S), K/W) \xrightarrow{\text{Res}} H^1(X(S'), K/W)^S \to H^2(S/S', K/W),$$

and it is easy by computation to show that Im(Inf) and Coker(Res) is killed by e; so, $H^1(X(S), K/W) \cong H^1(X(S'), K/W)^S$ and the Pontryagin dual version is (A3). Extending scalar if necessary, each minimal prime $P \subset W[S/S']$ is generated by $(s - \chi(s))$ for $s \in S/S'$ for a character $\chi : S/S' \to W^\times$. Since S/S' is the Galois group of the covering $X(S)/X(S')$, we write $K/W(\chi)_{/X(S)}$ and $W(\chi)_{/X(S)}$ for the twist of the constant sheaves by χ. Then by the same argument as above for $W(\chi^{-1})$, we get $H^1(X(S), K/W(\chi^{-1})) \cong H^1(X(S'), K/W(\chi^{-1}))^S$. The Pontryagin dual of the right-hand-side is $M_{S'}/PM_{S'}$. By the exact sequence $H^1(X(S), K(\chi^{-1})) \to H^1(X(S), K/W(\chi^{-1})) \to H^2(X(S), W(\chi^{-1}))$, since e kills $H^2(X(S), W(\chi))$, $H^1(X(S), K/W(\chi^{-1}))$ is p-divisible, and hence its dual $M_{S'}/PM_{S'}$ is W-free by (A4). □

Proposition 2.3. *Suppose $|I_D| \leq 1$. For each height 1 prime $P \supset \mathfrak{a}_S$ of $\mathbf{\Lambda}$, the localization M°_P is $\mathbf{h}^\circ_P$-free of rank $r = 2^{|I_D|}$. Moreover we have a surjective isogeny $\iota_S : \mathbf{h}^\circ/\mathfrak{a}_S\mathbf{h}^\circ \to \mathbf{h}^\circ_S$ sending $[U_\infty x U_\infty]$ to SxS for $x \in G(\mathbb{A}^{(\infty)})$ with $x_{p\mathfrak{N}} \in \Delta$.*

Proof. For finite level, $M_S \otimes_W K$ is known to be free of rank $r = 2^{|I_D|}$ over $\mathbf{h}_S \otimes_W K$. Thus for any height 1 prime $P \supset \mathfrak{a}_S$ of $\mathbf{\Lambda}$ (so, $\mathbf{\Lambda}/P$ is W-free),

$$M^\circ_P/PM^\circ_P \cong (M^\circ_S/PM^\circ_S) \otimes_W K \cong (\mathbf{h}_S/P\mathbf{h}_S)^r \otimes_W K \cong (\mathbf{h}^\circ_P/P\mathbf{h}^\circ_P)^r.$$

By Nakayama's lemma, we have a surjection $(\mathbf{h}^\circ_P)^r \to M^\circ_P$. If it has a kernel, M°_P has $\mathbf{\Lambda}$-torsion, since $\mathbf{h}^\circ_P$ is $\mathbf{\Lambda}$-torsion free of finite rank. This is a contradiction, and we get the freeness over $\mathbf{h}^\circ_P$. We also have, $(\mathbf{h}^\circ_P/P\mathbf{h}^\circ_P)^r \cong M^\circ_P/PM^\circ_P \cong M_S \otimes_W K/PM_S \otimes_W K \cong (\mathbf{h}_S \otimes_W K/P\mathbf{h}_S \otimes_W K)^r$; so, the projection $\mathbf{h}^\circ$ to $\mathbf{h}^\circ_S$ sending $[U_\infty x U_\infty]$ to $[SxS]$ induces $\mathbf{h}^\circ_P/P\mathbf{h}^\circ_P \cong (\mathbf{h}_S \otimes_W K/P\mathbf{h}_S \otimes_W K)$. Since $\mathfrak{a}_S = \bigcap_P P$ for such P, we conclude that the map $\mathbf{h}^\circ/\mathfrak{a}_S\mathbf{h}^\circ \twoheadrightarrow \mathbf{h}^\circ_S$ has finite kernel and is an isogeny. □

Remark 2.1. As we said, we can remove (A0) varying also the central character. We can take $M_S = H_d(Y(S), W)$ for $Y(S) = G^D(\mathbb{Q})\backslash G(\mathbb{A})/SC_D$. In that case, M° is $W[[\mathbf{\Gamma}]]$-torsionfree of finite rank for the torsion-free part $\mathbf{\Gamma} \subset T(\mathbb{Z}_p)/\overline{O^\times_+}$ of $\mathbf{G}_U$. All analogous results as above hold for $\mathbf{G}_U$ in place of $\overline{T}(\mathbb{Z}_p)$. The resulting Hecke algebras, we write $\mathbf{h}^{n.ord}$ for the cuspidal one and $\mathbf{H}^{n.ord}$ that with Eisenstein component.

We can consider higher weight case also. The set of algebraic characters $X(T) = \mathrm{Hom}_{\text{alg gp}}(T_{/\overline{\mathbb{Q}}}, \mathbb{G}_{m/\overline{\mathbb{Q}}})$ can be identified with $\mathbb{Z}[I]^2$ so that $\kappa = (\kappa_1, \kappa_2) \in \mathbb{Z}[I]^2$ induces the following character on $T(\mathbb{Q}) = F^\times \times F^\times$

$$T(\mathbb{Q}) \ni (\xi_1, \xi_2) \mapsto \kappa(\xi_1, \xi_2) = \xi_1^{\kappa_1}\xi_2^{\kappa_2} \in \overline{\mathbb{Q}}^\times,$$

where $\xi^{\kappa_j} = \prod_{\sigma \in I} \sigma(\xi_j)^{\kappa_{j,\sigma}} \in \overline{\mathbb{Q}}^\times$. Take K sufficiently large so that $G^D(K) = GL_2(K)^I$, and we write the projection to the $\sigma \in I$ component $GL_2(K)$ as σ again. Then the rational representation $L(\kappa; K)$ of G^D of highest weight $\kappa = (\kappa_1, \kappa_2)$, taking the ordering so that $\kappa > 0$ if $\kappa_{1,\sigma} < \kappa_{2,\sigma}$ for all $\sigma \in I$, is given by

$$L(\kappa, K) = \bigotimes_{\sigma \in I} \left((\det \circ \sigma)^{\kappa_{1,\sigma}} \sigma^{\otimes(\kappa_{2,\sigma} - \kappa_{1,\sigma})} \right),$$

where $\sigma^{\otimes n}$ is the n-th symmetric tensor representation. Letting $\sigma^{\otimes n}$ act on polynomials $WX^n + WX^{n-1}Y + \cdots + WY^n$ by $\gamma P((X,Y)) = P((X,Y)^t\sigma(\gamma)^\iota)$ for the involution ι with $\gamma^\iota + \gamma = \mathrm{Tr}(\gamma)$, this representation has W-integral structure. We write $L(\kappa;W)$ for the W-integral subspace on which $G^D(W)$ acts. We put $L(\kappa;A) = L(\kappa;W)\otimes_W A$ for any W-module A. Since $X(S) \cong \bigsqcup_{g\in\Omega}\Gamma_{S^g}\backslash\mathfrak{Z}_D$, we have a covering $\mathcal{L}(\kappa;A)$ on $X(S)$ which is given over $\Gamma_{S^g}\backslash\mathfrak{Z}_D$ as a quotient $\Gamma_{S^g}\backslash(\mathfrak{Z}_D \times L(\kappa;A))$ by the diagonal action.

There is an adelic version of the definition of $\mathcal{L}(\kappa;A)$. We let $u\in U$ act on $L(\kappa;A)$ from the right by $u_p^{-1}\in G_D(\mathbb{Z}_p)\subset G^D(W)$ and let $Z(\mathbb{A})$ act on $L(\kappa)$ by scalar multiplication of $\widehat{\varepsilon}(z_p)^{-1}$ for the p-adic avatar $\widehat{\varepsilon}$ of ε, $\mathcal{L}(\kappa;A)\cong G^D(\mathbb{Q})\backslash(G^D(\mathbb{A})\times L(\kappa;A)/S\cdot Z(\mathbb{A})C_D$.

Since $\Gamma_{S^g}\cap Z(\mathbb{Q})$ acts trivially on $\mathfrak{Z}_D$, this quotient to be well defined étale space, $\Gamma_{S^g}\cap Z(\mathbb{Q})\subset O_+^\times$ has to act trivially on $L(\kappa;A)$; so, at least, $\kappa_1+\kappa_2 = [\kappa]\sum_{\sigma\in I}\sigma$ (if $A = W$). By this condition, if one of κ_j is non-parallel (that is, not of the form $k\sum_{\sigma\in I}\sigma$), the other is also non-parallel. In particular $\kappa_1 \neq 0$. Thus the original $U(p)$ given by the action of $\alpha = \left(\begin{smallmatrix} p & 0\\ 0 & 1\end{smallmatrix}\right)\in G^D(\mathbb{Q}_p)$ on $L(\kappa;A)$ is divisible by $p^{\kappa_1} = \det(\alpha)^{\kappa_1} = p^{\sum_\sigma \kappa_{1,\sigma}}$. Thus to have nontrivial space, we need to divide the action of $[SxS]$ by $\det(x_p)^{\kappa_1}$ to make it optimally integral. Using this new operator $[SxS]^\circ = \det(x_p)^{-\kappa_1}[SxS]$, if $|I_D|\le 1$, we can verify (A0-4) for $M_S^* = M_{\kappa,S}^* = H^d(X(S), L(\kappa;K/W))$. We write $U^\circ(p^m)$ for $[S\alpha^m S]^\circ$. This modification only affect Hecke operators supported at p. We write $T^\circ(y)$ for $[S\left(\begin{smallmatrix} y & 0\\ 0 & 1\end{smallmatrix}\right)S]^\circ$ for $y\in\widehat{O}\cap F_{\mathbb{A}}^\times$. If $y\in O_p$, we write $U(y)$ for $[S\left(\begin{smallmatrix} y & 0\\ 0 & 1\end{smallmatrix}\right)S]$ and $U^\circ(y)$ for $[S\left(\begin{smallmatrix} y & 0\\ 0 & 1\end{smallmatrix}\right)S]^\circ$. If $U_{\mathfrak{l}} = GL_2(O_{\mathfrak{l}})$ and $\mathfrak{l}\nmid p$, $T(\varpi_{\mathfrak{l}})$ for a uniformizer $\varpi_{\mathfrak{l}}$ of $O_{\mathfrak{l}}$ is independent of the choice of $\varpi_{\mathfrak{l}}$; so, we write it as $T(\mathfrak{l})$. The resulting Hecke algebra of weight κ and the module M° of weight κ, we write $\mathbf{h}_\kappa^\circ$ and M_κ°. An important fact is

Theorem 2.4 (Independence of weight)**.** *We have canonical compatible isomorphisms* $\mathbf{h}^\circ\cong\mathbf{h}_\kappa^\circ$ *and* $M_\circ\cong M_\kappa^\circ$. *The isomorphism:* $\mathbf{h}^\circ\cong\mathbf{h}_\kappa^\circ$ *sends* $[SxS]$ *to* $[SxS]^\circ$ *for all* $x\in G(\mathbb{A}^{(\infty)})$ *with* $x_{p\mathfrak{N}}\in\Delta$.

Note that $\overline{T}(\mathbb{Z}_p)\cong O_p^\times$ by $\left(\begin{smallmatrix} y & 0\\ 0 & 1\end{smallmatrix}\right)\leftrightarrow y$, and y acted on M_κ° originally by $U(y)$; so, the above isomorphism becomes $\mathbf{\Lambda}$-linear if we use $U^\circ(y) = y_p^{-\kappa_1}U(y)$ for the action of y.

Proof. Here is a sketch of a proof. For simplicity, we assume that p is unramified in $F/\mathbb{Q}$; so, we can take W unramified over $\mathbb{Z}_p$. The evaluation of polynomials in $L(\kappa, W/p^mW)$ at $(X,Y) = (1,0)$ gives a U_m^1 isomorphism $L(\kappa,W/p^mW)\cong W/p^mW$; so, we get a $T^\circ(y)$-equivariant morphism of $i_m : M_{\kappa,U_m^1}\otimes_W W/p^mW \cong M_{U_m^1}\otimes_W W/p^mW$. After taking the limit, we get a morphism $i_\infty : M_\kappa\to M$. The inclusion $W/pW\ni a\mapsto aY^{\kappa_2-\kappa_1}\in L(\kappa;W/pW)$ is a morphism of U_1-module; so, we have $j : M_{U_1^1}\otimes_W W/pW\to M_{\kappa,U_1^1}\otimes_W W/pW$. Take $\tau = \left(\begin{smallmatrix} 0 & 1\\ p & 0\end{smallmatrix}\right)\in G^D(\mathbb{Q}_p)$ and $\delta = \left(\begin{smallmatrix} 0 & 1\\ 1 & 0\end{smallmatrix}\right)\in G^D(\mathbb{Z}_p)$. Then define $J = [U_1\delta U_1]\circ j\circ[U_1\tau U_1] : M_{U_1^1}\otimes_W W/pW\to M_{\kappa,U_1^1}\otimes_W W/pW$. By computation, we find, on $M_{U_1^1}\otimes_W W/pW$ and $M_{\kappa,U_1^1}\otimes_W W/pW$, $i\circ J = U^\circ(p)$ and $J\circ i = U(p)$. Thus $M_{U_1^1}^\circ\otimes_W W/pW\cong M_{\kappa,U_1^1}^\circ\otimes_W W/pW$. By Nakayama's lemma applied to the Jacobson radical of $\mathbf{\Lambda}$ implies that $i_\infty : M_\kappa^\circ\to M^\circ$ is surjective. Then by comparing the rank over Λ, i_∞ is an isomorphism satisfying $T(y)\circ i_\infty = i_\infty\circ T^\circ(y)$. □

Corollary 2.5. *Let* $\pi_\kappa : \mathbf{\Lambda}\to W$ *be the* W*-algebra homomorphism induced by* $\overline{T}(\mathbb{Z}_p)\ni z\mapsto z^{-\kappa_1}\in W^\times$. *If* $\kappa_1\le\kappa_2$, *we have a* W*-algebra isogeny* $\iota_\kappa : \mathbf{h}^\circ\otimes_{\mathbf{\Lambda},\pi_\kappa} W\twoheadrightarrow\mathbf{h}^\circ(M_{\kappa,U_1})$ *which sends* $T(y)$ *to* $T^\circ(y)$ *for all integral* F*-idele* y.

There is one more choice of M_S. If we have a good p-integral model $X(S)_{/W}$, we can think of $H^d(X(S),\mathcal{O}_{X(S)})$ which is the dual of $H^0(X(S),\Omega_{X(S)/W})$. This can be done taking "Igusa style model" and the cuspidal part $\Omega^{cusp}_{X(S)/W}$ of $\Omega_{X(S)/W}$ if $D = M_2(F)$, though it is quite involved to prove (A1–4). A good point is that $\mathrm{Hom}(H^0(X(S),\Omega^{cusp}_{X(S)/W}),W)$ is isomorphic to the corresponding Hecke algebra; so, we get (see [PAF] and [HMI] 4.3.9)

Theorem 2.6. *Suppose that $X(U)$ is smooth over W. Then $\mathbf{h}^\circ$ (resp. $\mathbf{h}^{n.ord}$) is free of finite rank over Λ (resp. $W[[\mathbf{\Gamma}]]$). The isogenies ι_S and ι_κ are isomorphisms.*

3. Galois representations

When $|I_D| = 1$ (so $[F:\mathbb{Q}]$ odd), taking M_S to be the étale homology of the Shimura curve $X(S)_{/F}$. Pick a prime $P \supset \mathfrak{a}_{U^1_a} \cap \Lambda$. Since $M^\circ_P \cong (\mathbf{h}^\circ_P)^2$ (if $X(U)$ is smooth) by Proposition 2.3, the Galois action produces a Galois representation $\rho_{\mathbf{h}} : \mathrm{Gal}(\overline{F}/F) \to GL_2(\mathbf{h}^\circ_P)$ (constructed in [68c] in [CPS], [H81] and [H86]). If $[F:\mathbb{Q}]$ is even, a similar construction is possible if we allow ramification of D at finite places. By the theory of pseudo-representation of Wiles, as he showed, we can actually construct $\rho_{\mathbf{h}} : \mathrm{Gal}(\overline{F}/F) \to GL_2((\mathbf{h}^\circ_P)^{red})$ even in the case of even degree ([W] and [H89b]).

Theorem 3.1 (A. Wiles, H. Hida)**.** *Let $\mathbb{T}$ be a reduced local ring of $\mathbf{h}^\circ$ or $\mathbf{h}^{n.ord}$. Then there exists a continuous Galois representation $\rho_{\mathbb{T}} : \mathrm{Gal}(\overline{F}/F) \to GL_2(\mathbb{T}_P)$ such that*

(1) *$\rho_{\mathbb{T}}$ is unramified outside p, ∞ and $\mathfrak{N}$;*
(2) *$\mathrm{Tr}(\rho_{\mathbb{T}}(Frob_{\mathfrak{l}})) = T(\mathfrak{l})|_{\mathbb{T}}$ with the arithmetic Frobenius element $Frob_{\mathfrak{l}}$ at primes $\mathfrak{l}$ outside p, ∞ and $\mathfrak{N}$;*
(3) *If $\mathbb{T} \subset \mathbf{h}^\circ$, $\det \rho_{\mathbb{T}} = \widehat{\varepsilon}\mathcal{N}$ for the p-adic cyclotomic character $\mathcal{N}$;*
(4) *For each decomposition group $D_{\mathfrak{p}}$ ($\mathfrak{p}|p$), $\rho_{\mathbb{T}}|_{D_{\mathfrak{p}}} \cong \left(\begin{smallmatrix}\beta & * \\ 0 & \alpha\end{smallmatrix}\right)$, $\alpha([y,F_{\mathfrak{p}}]) = U^\circ(y)|_{\mathbb{T}}$ and $\pi_\kappa \circ \alpha([u,F_{\mathfrak{p}}]) = u^{-\kappa_1}$ and $\pi_\kappa \circ \beta([u,F_{\mathfrak{p}}]) = u^{-\kappa_2} N_{F_{\mathfrak{p}}/\mathbb{Q}_p}(u)^{-1}$ for $u \in O^\times_{\mathfrak{p}}$, where $[x,F_{\mathfrak{p}}]$ is a local Artin symbol (with $\mathcal{N}([u,F_{\mathfrak{p}}]) = N_{F_{\mathfrak{p}}/\mathbb{Q}_p}(u)^{-1}$ and $\mathcal{N}(Frob_\ell) = \ell$).*

Remark 3.1. If the residual Galois representation $\rho_{\mathbb{T}} \mod \mathfrak{m}_{\mathbb{T}}$ for the maximal ideal $\mathfrak{m}_{\mathbb{T}}$ of $\mathbb{T}$ is absolutely irreducible, we have $\rho_{\mathbb{T}}$ with values in $GL_2(\mathbb{T})$.

Let $U = \widehat{\Gamma}_1(\mathfrak{N})$. Let $F^{p\mathfrak{N}}/F$ be the maximal abelian extension unramified outside $p\mathfrak{N}$. Then we split $C = \mathrm{Gal}(F^{p\mathfrak{N}}/F) = C_p \times C^{(p)}$ for the p-profinite part C_p and prime-to-p part $C^{(p)}$. Thus C_p is isogenous to $\mathbb{Z}_p^{1+\delta}$. We consider the continuous group algebra $W[[C_p]]$ and consider the inclusion $\iota : C_p \to W[[C_p]]^\times$. If p is odd, then $\rho \otimes \eta \not\cong \rho$ for any 2-dim Galois representation ρ and any character $\eta \neq 1$ of C_p. Since p is odd, we have a unique $\sqrt{\iota} : C_p \to C_p \subset W[[C_p]]^\times$. For $\mathbb{T} \subset \mathbf{h}^\circ$, $\rho_{\mathbb{T}} \otimes \sqrt{\iota}$ has determinant $\iota \det \rho_{\mathbb{T}} = \iota\widehat{\varepsilon}\mathcal{N}$ and is a promodular representation into $GL_2(\mathbb{T}[[C_p]])$. Since we have the Galois character $\boldsymbol{\delta} = \det(\rho_{\mathbf{h}^{n.ord}})$, $\mathbf{h}^\circ = \mathbf{h}^{n.ord}/(\boldsymbol{\delta}(\sigma) - \widehat{\varepsilon}\mathcal{N}(\sigma))_\sigma$, and we have a unique local ring $\mathbb{T}^{n.ord}$ of $\mathbf{h}^{n.ord}$ surjecting down to $\mathbb{T}$. Since $\rho_{\mathbb{T}} \otimes \sqrt{\iota}$ is realized by a quotient of $\mathbb{T}^{n.ord}$ (because order p character has square-free prime-to-p conductor), there exists an algebra homomorphism $\pi : \mathbb{T}^{n.ord} \to \mathbb{T}[[C_p]]$ such that $\pi \circ \rho_{\mathbb{T}^{n.ord}} \cong \rho_{\mathbb{T}}$.

Proposition 3.2. *If p is odd and $\mathbb{T}^{n.ord}$ and $\mathbb{T}$ are reduced, then $\mathbb{T}^{n.ord} \cong \mathbb{T}[[C_p]]$ by π.*

Anyway, $\dim \mathbb{T} = [F:\mathbb{Q}]$ for $\mathbb{T} \subset \mathbf{h}^\circ$ and $\dim \mathbb{T}^{ord} = [F:\mathbb{Q}] + 1 + \delta$. Write $\mathbf{H}_E^\circ$ and $\mathbf{H}_E^{n.ord}$ be the Eisenstein component. The Galois representation $\rho_\mathbb{T}$ for local ring $\mathbb{T}$ of $\mathbf{H}_E^\circ$ has trace $\widehat{\varepsilon}\mathcal{N}(\sigma)\iota + \iota^{-1}$. Since $\mathbb{T}^{n.ord} \cong \mathbb{T}^\circ[[C_p]]$, we have

Proposition 3.3. *The algebras $\mathbf{H}_E^\circ$ and $\mathbf{H}_E^{n.ord}$ are equidimensional and has dimension $1+\delta$ and $2+2\delta$, respectively.*

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