Local indecomposability of Tate modules of abelian varieties of $GL(2)$-type

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Abstract: We prove indecomposability of p-adic Tate modules over the p-inertia group for non CM (partially p-ordinary) abelian varieties with real multiplication. I will also discuss its application (given by Bin Zhao) to local indecomposability of Hilbert modular Galois representations.

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§1. Greenberg's Question.

Pick a totally real field $F \subset \overline{\mathbb{Q}}$ with integer ring O.

Take a non CM AVRM A over a number field $k \subset \overline{\mathbb{Q}}$ with integer ring $\mathfrak{O}$; so, $O \hookrightarrow \mathrm{End}(A_{/k})$, $\dim A = [F : \mathbb{Q}]$ and the centralizer of O in $\mathrm{End}(A_{/\overline{\mathbb{Q}}})$ is O.

Pick a prime $\mathfrak{p}|p$ of O and consider $\mathfrak{p}$-adic Tate module $T_{\mathfrak{p}}A$. Suppose that A has good reduction $\widetilde{A}$ modulo a prime $\mathfrak{P}|p$ of k, and assume that $\widetilde{A}[\mathfrak{p}](\overline{\mathbb{F}}_p) \cong O/\mathfrak{p}$ ($\mathfrak{p}$-ordinary at $\mathfrak{P}$).

Greenberg's Question: Is $T_{\mathfrak{p}}A$ indecomposable over the decomposition group $D_{\mathfrak{P}}$?

§2. Solution.

Theorem 1. *Yes it is indecomposable.*

I try to explain my far-fetched proof and its consequences, assuming that p is unramified in $F \cdot k$ (this assumption has been removed by Bin Zhao; so, the theorem is unconditional).

Fix a prime p and

- a finite set of rational primes $p \in \Xi$ unramified in $F \cdot k$;
- field embeddings $\mathbb{C} \xleftarrow{i_\infty} \overline{\mathbb{Q}} \xrightarrow{i_l} \mathbb{C}_l$ for all primes l.

Write $\mathfrak{l}$ (resp. $\mathfrak{L}$) be prime of O (resp. $\mathfrak{O}$) induced by $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_l$.

§3. Hilbert modular Shimura variety.

Let $\mathbb{Z}_{(\Xi)} = \mathbb{Q} \cap \prod_{l \in \Xi} \mathbb{Z}_l$, and $\mathbb{A}^{(\Xi)}$ be the adele ring away from $\Xi \cup \{\infty\}$. Put $V = O^2$ and $V(R) = V \otimes_{\mathbb{Z}} R$ for $\mathbb{Z}_{(\Xi)}$-algebras R.

Hilbert modular Shimura variety $Sh^{(\Xi)}_{/\mathbb{Z}_{(\Xi)}}$ classifies

$$(A, \eta^{(\Xi)}, \overline{\lambda})$$

made of an AVRM A, level structure for $TA = \varprojlim_N A[N]$

$$\eta^{(\Xi)} : V(\mathbb{A}^{(\Xi)}) \cong TA \otimes_{\widehat{\mathbb{Z}}} \mathbb{A}^{(\Xi)}$$

and prime-to-Ξ polarization class. Thus

$$Sh^{(\Xi)}(R) \cong \{(A, \eta^{(\Xi)}, \overline{\lambda})_{/R}\} / \approx,$$

where $\approx$ is by prime-to-Ξ F-linear isogenies. We remove "(Ξ)" from our notation if no confusion is likely.

§4. Aut(Sh).

Let $G = \mathrm{Res}_{F/\mathbb{Q}} GL(2)$. We let $G(\mathbb{A}^{(\Xi)})$ act on Sh by

$$\eta \mapsto \eta \circ g.$$

If $x \in Sh$ corresponds $(A_x, \eta_x, \overline{\lambda}_x)$, let

$$M_x = \mathrm{End}^0(A_x) = \mathrm{End}(A_x) \otimes \mathbb{Q}.$$

Then we can embed

$$M_x^\times \xrightarrow{\rho_x} G(\mathbb{A}^{(\Xi)}) \quad \text{by} \quad \alpha \circ \eta_x = \eta_x \circ \rho(\alpha).$$

Then $\rho(M_x^\times)$ gives rise to the stabilizer of x.

If $M_x = F$, the action of $M_x^\times$ is trivial; but, if M_x/F is a CM extension, the action factors through $M_x^\times/F^\times$.

§5. Serre–Tate deformation theory

Assume that $\underline{A} = (A, \eta, \overline{\lambda})$ has ordinary good reduction at $\mathfrak{L}$. Consider its reduction

$$\underline{A}_{\mathfrak{L}} = (A_{\mathfrak{L}}, \eta_{\mathfrak{L}}, \overline{\lambda}_{\mathfrak{L}}) = (A, \eta, \overline{\lambda}) \otimes \overline{\mathbb{F}}_{\mathfrak{L}}$$

for an algebraic closure $\overline{\mathbb{F}}_{\mathfrak{L}}$ of $\mathbb{F}_{\mathfrak{L}} := \mathfrak{O}/\mathfrak{L}$, which gives rise to a point $x_{\mathfrak{L}} \in Sh(\overline{\mathbb{F}}_{\mathfrak{L}})$.

Let $W_l = W(\overline{\mathbb{F}}_{\mathfrak{L}})$. Then the formal completion $\widehat{S}_l$ of Sh along $x_{\mathfrak{L}}$ is isomorphic to the Serre–Tate deformation space.

For any complete local ring R with residue field $\overline{\mathbb{F}}_{\mathfrak{L}}$,

$\widehat{S}_l(R) \cong \{\underline{\mathcal{A}} := (\mathcal{A}, \eta_{\mathcal{A}}, \overline{\lambda}_{\mathcal{A}})_{/R} | \underline{\mathcal{A}}_{/R} \otimes_R R/\mathfrak{m}_R = \underline{A}_{\mathfrak{L}}\}/\cong .$

As is well known, $\widehat{S}_l \cong \widehat{\mathbb{G}}_m \otimes O$.

§6. Serre–Tate coordinates

Identify $\widehat{\mathbb{G}}_m = \mathrm{Spf}(\widehat{W_l[t, t^{-1}]})$. Then consider the rigid analytic space $\widehat{S}_l^{an}$ associated to $\widehat{S}_l$ in the sense of Berthelot.

Taking the σ-component of "log(t)" given by

$$\tau_{l,\sigma} : \widehat{\mathbb{G}}_m \otimes O(W_l) = (1 + \mathfrak{m}_{W_l}) \otimes_{\mathbb{Z}} O \xrightarrow{\log_p} \prod_{\sigma} W_l \xrightarrow{\sigma} W_l,$$

we may identify $\widehat{S}_l^{an} = \mathrm{Sp}(\mathbb{C}_l\{\{\tau_{l,\sigma}\}\}_\sigma)$.

We have a decomposition

$$\Omega_{\widehat{S}_l^{an}/\mathbb{C}_l} = \bigoplus_{\sigma} \Omega^{an}_{\sigma/\mathbb{C}_l}$$

such that Ω^{an}_σ is generated by $d\tau_{l,\sigma}$.

§7. CM action on $\widehat{S}_l$

Let $M_{\mathfrak{L}} = \mathrm{End}^0_F(A_{\mathfrak{L}/\overline{\mathbb{F}}_{\mathfrak{L}}})$ which is a CM quadratic extension of F generated by the $N(\mathfrak{L})$-power Frobenius map $\phi_{\mathfrak{L}}$.

We can embed $\alpha \in M_{\mathfrak{L}}^{\times}$ into $G(\mathbb{A}^{(\Xi)})$ by $\alpha \circ \eta_{\mathfrak{L}} = \eta_{\mathfrak{L}} \circ \rho(\alpha)$.

Then we have, writing t for $t \otimes 1$ and $t^a = t \otimes a$,

$$\tau_{l,\sigma} \circ \rho(\alpha) = \alpha^{\sigma(1-c)} \tau_{l,\sigma} \ (\Leftrightarrow t \circ \rho(\alpha) = t^{\alpha^{1-c}}).$$

So we call $\tau_{l,\sigma}$ σ-eigen-coordinate. Any σ-eigen-coordinate on $\widehat{S}_l^{an}$ is **proportional** to $\tau_{l,\sigma}$.

The origin $\tau = 0$ gives rise to the canonical CM lift A^{cm} of $A_{\mathfrak{L}}$.

Hereafter we take $\mathcal{W} = \bigcap_{l \in \Xi} i_l^{-1}(W_l)$ inside $\overline{\mathbb{Q}}$.

§8. Point $x \in \widehat{S}_p \subset Sh$ of $(A, \eta, \overline{\lambda})$.

Suppose $T_{\mathfrak{p}}A$ is decomposable over $D_{\mathfrak{P}}$. If $\mathfrak{p}$ remains prime over p, by non-CM property, we have $t(A) \neq 1$ and hence $T_{\mathfrak{p}}A$ cannot be decomposable.

We may assume that there are more than two prime factors of (p) in F. For simplicity, we assume that $[F : \mathbb{Q}] = 2$; so, $(p) = \mathfrak{p}\mathfrak{p}'$. Write $\sigma : F \hookrightarrow \overline{\mathbb{Q}}_p$ corresponding to $\mathfrak{p}$ and $\sigma' : F \hookrightarrow \overline{\mathbb{Q}}_p$ corresponding to $\mathfrak{p}'$.

Then we have

$$\tau_{p,\sigma}(x) = 0 \quad \text{and} \quad \tau_{p,\sigma'}(x) \neq 0.$$

§9. Kodaira-Spencer map.

Let $\pi : \mathbf{A} \to Sh$ (resp. $\widehat{\pi} : \mathbf{A} \to \widehat{S}_l$) be the universal abelian varieties. By the O-action on $\mathbf{A}$, O acts on $\Omega_{\mathbf{A}/Sh}$ and $\Omega_{\mathbf{A}/\widehat{S}_l}$.

Writing $\omega = \pi_* \Omega_{\mathbf{A}/Sh}$ and $\omega_l = \widehat{\pi}_* \Omega_{\widehat{\mathbf{A}}/\widehat{S}_l}$, we have the following decomposition into σ-eigenspaces:

$$\omega = \bigoplus_\sigma \omega^{\otimes\sigma} \quad \text{and} \quad \omega_l = \bigoplus_\sigma \omega_l^{\otimes\sigma}.$$

The Kodaira-Spencer map induces a canonical isomorphism

$$\Omega_{\sigma,Sh/\mathcal{W}} \cong \omega^{\otimes 2\sigma}, \Omega_{\sigma,\widehat{S}_l/W_l} \cong \omega_l^{\otimes 2\sigma}.$$

§10. Stability under CM action.

Since $\tau_{p,\sigma} \circ \rho(\alpha) = \alpha^{\sigma(1-c)}\tau_{p,\sigma}$, the fiber $\omega_p^{\otimes 2\sigma}(x)$ at x of the invertible sheaf $\omega_{p/\widehat{S}_p}^{\otimes 2\sigma}$ is stable under the action of $\rho(M_{\mathfrak{P}}^{\times})$.

Since

$$\omega_p^{\otimes 2\sigma}(x) = \omega^{\otimes 2\sigma}(x) \otimes_{\mathcal{W}} W_p,$$

the fiber at x of the global sheaf $\omega^{\otimes 2\sigma}(x)$ is stable under $\rho(M_{\mathfrak{P}}^{\times})$.

Pick another prime $l \in \Xi$ so that A has ordinary good reduction at $\mathfrak{L}$. Then we have

$$\omega_l^{\otimes 2\sigma}(x) = \omega^{\otimes 2\sigma}(x) \otimes_{\mathcal{W}} W_l.$$

Thus $\omega_l^{\otimes 2\sigma}(x)$ is stable under $\rho(M_{\mathfrak{P}}^{\times})$; so,

$$\tau_{l,\sigma}(x) = 0 \quad \text{and} \quad \tau_{l,\sigma} \circ \rho(\alpha) = \alpha^{\sigma(1-c)}\tau_{l,\sigma}.$$

§11. CM contradiction.

By

$$\tau_{l,\sigma}(x) = 0 \quad \text{and} \quad \tau_{l,\sigma} \circ \rho(\alpha) = \alpha^{\sigma(1-c)} \tau_{l,\sigma},$$

$A_{\mathfrak{L}}$ has CM by the same $M_{\mathfrak{P}} = M_{\mathfrak{L}}$.

The choice of $\mathfrak{L}$ is arbitrary, by Chebotarev density applied to the Galois representation on $T_{\mathfrak{p}}A$, we can find l with $M_{\mathfrak{L}} \neq M_{\mathfrak{P}}$, a contradiction.

Thus $T_{\mathfrak{p}}A$ must be indecomposable over $D_{\mathfrak{P}}$.

The argument works well for any $\mathfrak{p}$-ordinary A and general F.

§12. Kodaira-Spencer map again.

We restart with a CM abelian variety A^{cm} with CM by $\mathfrak{O}$.

Recall $\omega = \pi_*\Omega_{\mathbf{A}/Sh}$ and $\omega_l = \widehat{\pi}_*\Omega_{\widehat{\mathbf{A}}/\widehat{S}_l}$, we have the following decomposition into σ-eigenspaces:

$$\omega = \bigoplus_\sigma \omega^{\otimes\sigma} \quad \text{and} \quad \omega_l = \bigoplus_\sigma \omega_l^{\otimes\sigma}.$$

The Kodaira-Spencer map induces a canonical isomorphism

$$\Omega_{\sigma,Sh/\mathcal{W}} \cong \omega^{\otimes 2\sigma} \quad \text{and} \quad \Omega_{\sigma,\widehat{S}_l/W_l} \cong \omega_l^{\otimes 2\sigma}.$$

Writing the formal group $\widehat{\mathbf{A}}$ of $\mathbf{A}_{/\widehat{S}_l}$ as $\widehat{\mathbf{A}} = \widehat{\mathbb{G}}_m \otimes O$ with $\widehat{\mathbb{G}}_m = \mathrm{Spf}(W\widehat{[s_l, s_l^{-1}]})$, the Kodaira-Spencer map is given by

$$d\tau_{l,\sigma} \leftrightarrow \left(\frac{ds_\sigma}{s_\sigma}\right)^{\otimes 2}.$$

§13. Katz period and proportionality.

Choose an algebraic differential ω^{cm} with

$$H^0(A^{cm}, \Omega_{A^{cm}/\mathcal{W}}) = (\mathcal{W} \otimes O)\omega^{cm}.$$

Assuming $A_{\mathfrak{P}}$ is ordinary, identifying $\widehat{A}^{cm} = \widehat{\mathbb{G}}_m \otimes O$ with $\widehat{\mathbb{G}}_m = \mathrm{Spf}(W\widehat{[s_l, s_l^{-1}]})$, Katz defined his p-adic period $\Omega_{p,\sigma} \in W_l^{\times}$ by

$$\omega^{cm}_{\sigma} = \Omega_{p,\sigma}\left(\frac{ds_l}{s_l}\right)_{\sigma}$$

comparing its σ-eigen components.

Comparing the fibers of the Kodair-Spencer map at $\tau = 0$, we get $d\tau_{p,\sigma}/d\tau_{l,\sigma} = \Omega^2_{p,\sigma}/\Omega^2_{l,\sigma}$. Since $\tau_{p,\sigma}$ and $\tau_{l,\sigma}$ are proportional,

Theorem 2. $\tau_{p,\sigma}/\tau_{l,\sigma} = \Omega^2_{p,\sigma}/\Omega^2_{l,\sigma}$

§14. Hilbert modular Galois representation.

Let $\mathbf{f}$ be a nearly $\mathfrak{p}$-**ordinary** weight 2 non CM Hilbert modular Hecke eigenform for a totally real field K.

Assume that its p-adic Galois representation $\rho_{\mathbf{f}}$ comes from an abelian variety $A_{\mathbf{f}}$ of $GL(2)$-type (e.g., $\mathbf{f}$ is the image of Jacquet-Langlands correspondence from a Shimura curve).

Bin Zhao removed the unramifiedness assumption by the result of Deligne-Pappas and showed that $A_{\mathbf{f}} \sim A^e$ for an absolute simple AVRM over a number field $k_{/K}$; so,

Theorem 3 (B. Zhao). $\rho_{\mathbf{f}}|_{D_{\mathfrak{P}}}$ *is indecomposable.*

Balasubramanyam, Ghate and Vatsal have got a similar result under a different set of assumptions.

§15. Application to Coleman's problem.

Assume $p > 3$ and let N be a positive number prime to p. Write $M_k^\dagger(\Gamma_1(N))$ for the space of elliptic overconvergent p-adic modular forms.

By the theta operator $\theta = q\frac{d}{dq}$, we have

$$\theta^{k-1} : M_{2-k}^\dagger(\Gamma_1(N)) \to M_k^\dagger(\Gamma_1(N)).$$

Coleman proved that for $k \geq 2$, every classical CM cuspidal eigenform of weight k and slope $k-1$ is in the image of θ^{k-1}.

Coleman's question is
Is there non-CM classical cusp forms in the image of θ^{k-1}?

§16. Answer is no for $k = 2$.

By p-adic Hodge theory (a result of Kisin), if f of $\mathfrak{p}$-slope $k-1$ is in the image of θ^{k-1}, ρ_f has to be decomposable at p (a remark made by Emerton).

Thus by the result of Zhao, we get

Theorem 4 (B. Zhao)**.** *Suppose $k = 2$. Then a $\mathfrak{p}$-slope 1 classical Hecke eigen cusp form is in the image of θ if and only if f has complex multiplication.*