Analytic variation of Tate–Shafarevich groups

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Analyzing elementary relations between $U(p)$ operators and Picard functoriality of the Jacobians of each tower of modular curves of $p$-power level, we get fairly exact control of the ordinary part of the limit Barsotti-Tate groups and the ($p$-adically completed) ind-limit Mordell-Weil groups with respect to the weight Iwasawa algebra. Computing Galois cohomology of these controlled Galois modules, we obtain good control of the (ordinary part of) limit Selmer groups and limit Tate-Shafarevich groups.
§0. Exotic $\Gamma_1$-type congruence subgroups:
Let $\Gamma := \mathbb{Z}_p^\times / \mu_{p-1} \cong 1 + p\mathbb{Z}_p$, for a prime $p \geq 5$. Fix an exact sequence of profinite groups $1 \to H_p \to \Gamma \times \Gamma \xrightarrow{\pi \Gamma} \Gamma \to 1$, and regard $H_p$ as a subgroup of $\Gamma \times \Gamma$. This implies $\pi \Gamma(a, d) = a^\alpha d^{-\delta}$ for a pair $(\alpha, \delta) \in \mathbb{Z}_p^2$ with $\alpha \mathbb{Z}_p + \delta \mathbb{Z}_p = \mathbb{Z}_p$. Let $H$ be the pull-back of $H_p$ to $\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times$. Define, for $\hat{\mathbb{Z}} = \prod_{l: \text{primes}} \mathbb{Z}_l$ and $0 < N \in \mathbb{Z}$,

\[
\hat{\Gamma}_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\hat{\mathbb{Z}}) \bigg| c \in N\hat{\mathbb{Z}} \right\},
\]

\[
\hat{\Gamma}_s = \hat{\Gamma}_{H,s} := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \hat{\Gamma}_0(Np^s) \bigg| (a_p, d_p) \in H/H_p^{p^s-1} \right\} \quad (p \nmid N),
\]

\[
\hat{\Gamma}_r^s = \hat{\Gamma}_{H,s}^r := \hat{\Gamma}_0(p^s) \cap \hat{\Gamma}_r \quad (s \geq r).
\]

The group $\Gamma_r := \hat{\Gamma}_r \cap \text{SL}_2(\mathbb{Q})$ is almost $\Gamma_0(N) \cap \Gamma_1(p^r)$ independent of $H$ and $(\alpha, \delta)$. 
§1. Exotic modular tower.

Let $X_r/Q$ and $X_s^r/Q$ be Shimura’s canonical models associated with $\hat{\Gamma}_r$ and $\hat{\Gamma}_s^r$. They are geometrically connected curves canonically defined over $\mathbb{Q}$ and the moduli of elliptic curves with certain level structure (which can be defined over $\mathbb{Z}(p)$).

We have an adelic expression of their complex points.

$$X_s^r(\mathbb{C}) - \{\text{cusps}\} = \text{GL}_2(\mathbb{Q})\backslash\text{GL}_2(\mathbb{A})/\hat{\Gamma}_s^r \cdot \mathbb{R}_+^\times \cdot \text{SO}_2(\mathbb{R}) \cong \Gamma_s^r \backslash \mathcal{H},$$

where $\Gamma_s^r = \hat{\Gamma}_s^r \cap \text{SL}_2(\mathbb{Q})$ and $\Gamma_r = \hat{\Gamma}_r \cap \text{SL}_2(\mathbb{Q})$. Note that $\Gamma_r$ and $\Gamma_s^r$ is independent of the choice of $(\alpha, \delta)$.

Write $J_{s}^r/Q = \text{Pic}^0_{X_s^r/Q}$ and $J_r/Q = \text{Pic}^0_{X_r/Q}$ for the corresponding Jacobian varieties.
§2. Galois representation.
Let \( f \in S_2(\Gamma_r) \) be a Hecke eigenform and \( \rho_f \) be its \( p \)-adic Galois representation, taking the choice \((\alpha, \delta) = (0, 1)\). Note that \( \det \rho_f = \nu \psi_f \) for a \( p \)-power order character \( \psi_f \) which has a unique square root \( \sqrt{\psi_f} \) of \( p \)-power order. The same \( f \) regarded as \( f dz \in H^0(X_r, \Omega_{X_r/\mathbb{C}}) \) gives rise to \( \rho_f \otimes \sqrt{\psi_f^{-1}} \) if \((\alpha, \delta) = (1, 1)\).

If we write the Mazur-Kitagawa \( p \)-adic \( L \)-function (interpolating \( L(s, f) \)) for \( f \) in a two variable nearly ordinary family as \( L(k, s) \) for the weight variable \( k \leftarrow f \) and the cyclotomic variable \( s \), the tower \( \{X_r\}_r \) for \((\alpha, \delta) \) gives the one variable variation the one variable \( p \)-adic \( L \)-function \( k \mapsto L(2\delta k + 2, \alpha k + 1) \). In particular, if \((\alpha, \delta) = (0, 1)\) gives the ordinary variation: the one variable \( p \)-adic \( L \)-function \( k \mapsto L(2k + 2, 1) \), and \((\alpha, \delta) = (1, 1)\) gives the central critical variation: the one variable \( p \)-adic \( L \)-function \( k \mapsto L(2k + 2, k + 1) \), which can be identically 0.
§3. Ordinary $\Lambda$-BT group.

Define $G = G_{\alpha,\delta} := \lim_{\to} J_s[p^\infty]^\text{ord}$ as an ind-group-scheme over $\mathbb{Q}$. Here “ord” indicates the image of the idempotent $e := \lim_{\to} U(p)^n!$. Since $\Gamma = (\Gamma \times \Gamma)/H = \lim_{\to} \widehat{\Gamma}_s/\widehat{\Gamma}_s^1$ naturally acts on $G$, $G$ has natural action of the weight Iwasawa algebra $\Lambda = \mathbb{Z}_p[[\Gamma]] = \mathbb{Z}_p[[T]]$ with $t = 1 + T$ generating $\Gamma$.

The $\Lambda$-BT group $G$ satisfies

(CT) For $G_s := J_s[p^\infty]^\text{ord}$, we have

$$G_s = G[t^{ps-1} - 1] := \ker(t^{ps-1} - 1 : G \to G)$$

(in particular, $G_s/R \hookrightarrow G/R$ is a closed immersion for $R = \mathbb{Z}_p(\mu_{p^\infty})$);

(DV) The geometric generic fiber $G(K)$ is isomorphic to $(\Lambda^\vee)^n$ for the Pontryagin dual $\Lambda^\vee := \text{Hom}_{\mathbb{Z}_p}(\Lambda, \mathbb{Q}_p/\mathbb{Z}_p)$; so, $T^\vee G = \text{Hom}_{\Lambda}(\Lambda^\vee, G(K))$ is $\Lambda$-free of finite rank.
§4. A diagram of cohomology groups.

Since $\Gamma^r_s \triangleright \Gamma_s$, consider the cyclic quotient group $C := \Gamma^r_s/\Gamma_s$ of order $p^{s-r}$. By the inflation restriction sequence, we have the following commutative diagram with exact rows for $T = S^1 = \mathbb{R}/\mathbb{Z}$:

$$
\begin{array}{cccccc}
H^1(C, T) & \hookrightarrow & H^1(\Gamma^r_s, T) & \rightarrow & H^1(\Gamma_s, T)^{\gamma^{p^r}=1} & \rightarrow & H^2(C, T) = 0 \\
\uparrow & & \downarrow \cup & & \uparrow \cup & & \uparrow \\
? & \rightarrow & J^r_s(C) & \rightarrow & J_s(\mathbb{C})[\gamma^{p^r-1} - 1] & \rightarrow & ?.
\end{array}
$$

Since $C$ is a finite cyclic group of order $p^{s-r}$ (with generator $g$) acting trivially on $T$, we have $H^1(C, T) = \text{Hom}(C, T) \cong C$ and

$$
H^2(C, T) = T/(1 + g + \cdots + g^{p^{s-r}-1}) = T/p^{s-r}T = 0.
$$
§5. The $U(p)$-isomorphism. By a cocycle computation, we confirm that $U(p)$ acts on $H^1(C, T)$ via multiplication by its degree $p$, and hence $U(p)^{s-r}$ kill $H^1(C, T)$. Hence $J^r_s \to J_s$ is an $U(p)$-isomorphism over $\mathbb{C}$ (meaning its kernel and cokernel are killed by a power of $U(p)$) and hence over $\mathbb{Q}$. We record what we have proven:

$$U(p)^{s-r}(H^1(C, T)) = H^2(C, T) = 0.$$ 

This fact for $\mathbb{T}_p = \mathbb{Q}_p/\mathbb{Z}_p \subset T$ has been exploited by the speaker to show (CT) and (DV).

By (DV), for any factor $\varpi|(t^p - 1)$, we have an exact sequence

$$0 \to G[\varpi] \to G \xrightarrow{\varpi} G \to 0$$

of fppf abelian sheaves.
§6. The $U(p)$-identity.

Note a simple identity:

$$U_r^{s}(p^{s-r}) := \Gamma_r \backslash \Gamma_s (\begin{pmatrix} 1 & 0 \\ 0 & p^{s-r} \end{pmatrix}) \Gamma_1(Np^r) = \left\{ \begin{pmatrix} 1 & u \\ 0 & p^{s-r} \end{pmatrix} \bigg| u \mod p^{s-r} \right\}$$

$$= \Gamma_1(Np^r) \backslash \Gamma_1(Np^r) \left( \begin{pmatrix} 1 & 0 \\ 0 & p^{s-r} \end{pmatrix} \right) \Gamma_1(Np^r) =: U(p^{s-r})$$

which implies the relation of $U(p^{s-r})$-operators:

$$J_{r/Q} \xrightarrow{\pi^*} J_{s/Q}^{r} \xrightarrow{\pi^*} J_{s/Q}^{r}$$

where the middle $u'$ is given by $U_r^{s}(p^{s-r})$ and $u$ and $u''$ are $U(p^{s-r})$.

Then the above diagram implies

$$J_{r/Q}[p^{\infty}]^{\text{ord}} \cong J_{s/Q}[p^{\infty}]^{\text{ord}}, \quad J_{r/Q}^{\text{ord}} \cong J_{s/Q}^{r,\text{ord}}.$$
§7. Replace \( H^1(X_s, T) \) by \( H^1(X_s, O_{X_s}^\times) \).

Note \( H^1_{\text{fppf}}(X, O_{X/Q}^\times) = \text{Pic}_{X/Q} \) for a smooth geometrically irreducible curve \( X \). Out of \( \check{H}^p(X_T/Y_T, \underline{H}^q(\mathbb{G}_m/Y)) \Rightarrow H^n_{\text{fppf}}(Y_T, O_{Y_T}^\times) \) (for \( T/Q \)), we have the following commutative diagram with exact rows and columns for \( X = X_s/Q \) and \( Y = X^r_s/Q \):

\[
\begin{array}{cccccccc}
0 & \rightarrow & \mathbb{Z} & \rightarrow & \mathbb{Z} & \rightarrow & 0 \\
\uparrow & & \downarrow\text{deg} & \text{onto} & \downarrow\text{deg} & \text{onto} & \\
\check{H}^1(H_Y^0) & \rightarrow & \text{Pic}_{Y/Q}(T) & \rightarrow & \check{H}^0(\underline{H}_Y^0, \text{Pic}_{Y/Q}(T)) & \rightarrow & \check{H}^2(H_Y^0) \\
\uparrow & & \uparrow & & \uparrow & & \\
?_1 & \rightarrow & J_{s}^r(T) & \rightarrow & \check{H}^0(\underline{H}_Y^0, J_s(T)) & \rightarrow & ?_2,
\end{array}
\]

Here \( H_Y^\bullet := H^\bullet(\underline{\mathbb{G}_m/Y})(U) = H^\bullet_{\text{fppf}}(U, O_U^\times) \) for a \( Y \)-scheme \( U \) as a presheaf. By Čech cohomology computation, one can easily show \( e(\check{H}^\bullet(H_Y^0)) = 0 \).
§8. Arithmetic points.
Define $h = h_{\alpha, \delta} := \Lambda[ T(n) | n = 1, 2, \ldots ] \subset \text{End}_\Lambda( TG )$. Take a connected component $\text{Spec}( T )$ and assume that $T$ is a unique factorization domain (this is usually the case).

Define $\mathcal{A}_T := \{ P \in \text{Spec}( T ) (\overline{\mathbb{Q}}_p) | P(t^{p^r} - 1) = 0 \text{ for some } r > 0 \}$. Then we have an abelian varieties $A_P \subset J_r$ and $J_r \rightarrow B_P$ associated to $P$ and a Hecke eigenform $f_P$ associated to $P$. Write $H_P = \mathbb{Q}( f_P ) \subset \text{End}( A_P / \mathbb{Q} ) \otimes \mathbb{Q}$ for the Hecke field of $f_P$. Put

$$\Omega_T = \{ P \in \mathcal{A}_T | A_P \text{ has potentially good reduction modulo } p \}.$$ 

Let $K$ be a number field and $K^S$ be the maximal extension of $K$ unramified outside $S := \{ v : \text{ places of } K|v \text{ over } Np\infty \}$. Recall the first fundamental sequence for $K' = K^S, \overline{K_v}$

$$0 \rightarrow \mathcal{G}_T( K' )[\varpi] \rightarrow \mathcal{G}_T( K' ) \xrightarrow{\varpi} \mathcal{G}_T( K' ) \rightarrow 0 \text{ for } P = (\varpi) \in \Omega_T,$$

where $\mathcal{G}_T = \mathcal{G} \otimes_h T$. 
§9. Second fundamental exact sequence

Define, for number fields $F/\mathbb{Q}$, $X^\text{ord}(F) := e(\lim_{\leftarrow n} X(F) \otimes \mathbb{Z}/p^n\mathbb{Z})$ $X^\text{ord}(K') = \bigcup_{K'/F/K} X^\text{ord}(F)$ for a factor $X$ of $J_s$ or $J'_s$ stable under $U(p)$, and put $J^\text{ord}_\infty := \lim_{\leftarrow s} J^\text{ord}_s$. Since $\mathbb{T}$ is a UFD, each prime $P \in A_{\mathbb{T}}$ is generated by $\varpi \in \mathbb{T}$ associated to a Hecke eigenform $f_P \in S_2(\hat{\Gamma}_r)$ and an abelian subvariety $A_P \subset J_r$ and an abelian quotient $J_r \twoheadrightarrow B_P$ isogenous to $A_P$. We get the following exact sequence of Galois modules:

$$0 \to A^\text{ord}_P(K') \to J^\text{ord}_\infty,_{\mathbb{T}}(K') \xrightarrow{\varpi} J^\text{ord}_\infty,_{\mathbb{T}}(K') \to B^\text{ord}_P(K') \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to 0,$$

where $J^\text{ord}_\infty,_{\mathbb{T}} = J^\text{ord}_\infty \otimes_{h} \mathbb{T}$. In other words, $A^\text{ord}_P \cong J^\text{ord}_{s,\mathbb{T}}[\varpi] = \text{Ker}(\varpi : J^\text{ord}_{s,\mathbb{T}} \to J^\text{ord}_{s,\mathbb{T}})$ for all $s \geq r$ and $B^\text{ord}_P \cong J^\text{ord}_{r,\mathbb{T}}/\varpi(J^\text{ord}_{r,\mathbb{T}})$, but the limit $\lim_{\rightarrow s \geq r} J^\text{ord}_{s,\mathbb{T}}/\varpi(J^\text{ord}_{s,\mathbb{T}})$ is isomorphic to $B^\text{ord}_P \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. 
§10. Arithmetic cohomology groups.

For a finite set of places $S$ of a number field $K$ containing all places above $N_p$ and $\infty$, write $K^S/K$ for the maximal extension unramified outside $S$. For a topological $\text{Gal}(K^S/K)$-module $M$ and $v \in S$, we write $H^\bullet(K^S/K, M)$ (resp. $H^\bullet(K_v, M)$ for the $v$-completion $K_v$ of $K$) for the continuous cohomology for the profinite group $\text{Gal}(K^S/K)$ (resp. $\text{Gal}(\overline{K}_v/K_v)$ for an algebraic closure $\overline{K}_v$ of $K_v$). Define

$$\text{III}(K^S/K, M) = \text{Ker}(H^1(K^S/K, M) \to \prod_{v \in S} H^1(K_v, M)) \otimes_{\mathbb{Z}} \mathbb{Z}_p.$$ 

In addition to the Mordell–Weil group $J_r(K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p / \mathbb{Z}_p$, we study the Tate–Shafarevich group $\text{III}_K(J_r^{\text{ord}})$, $\text{III}_K(K^S/K, J_r[p^\infty]^{\text{ord}})$ and the Selmer group

$$\text{Sel}_K(J_r^{\text{ord}}) = \text{Ker}(H^1(K^S/K, J_r[p^\infty]^{\text{ord}}) \to \prod_{v \in S} H^1(K_v, J_r^{\text{ord}})).$$
§11. Theorem for Tate–Shafarevich groups.

**Theorem III.** Suppose that $\mathcal{T}$ is a unique factorization domain.

1. If $\Sha_K(K^S/K, A_{P_0}[p^{\infty}]^{\text{ord}})$ is finite for a single point $P_0 \in \Omega_T$, then $\Sha_K(K^S/K, A_P[p^{\infty}]^{\text{ord}})$ is finite for almost all $P \in \Omega_T$.

2. If $\Sha_K(A_{P_0}^{\text{ord}})$ is finite and $\dim_{H_{P_0}} A_{P_0}(K) \otimes_{\mathbb{Z}} \mathbb{Q} \leq 1$ for a single point $P_0 \in \Omega_T$, then $\Sha_K(A_P^{\text{ord}})$ is finite for almost all $P \in \Omega_T$.

3. If $|\Sha_K(A_{P_0}^{\text{ord}})| < \infty$ and $\dim_{H_{P_0}} A_{P_0}(K) \otimes_{\mathbb{Z}} \mathbb{Q} \leq 1$ for a single point $P_0 \in \Omega_T$, then $\dim_{H_P} A_P(K) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ or 1 independent of $P$ for almost all $P \in \Omega_T$. 
§12. Theorem for Selmer groups.

**Theorem S.** Suppose that $\mathbb{T}$ is a unique factorization domain.

1. If $\text{Sel}_K(A_{P_0}^{\text{ord}})$ is finite for a single point $P_0 \in \Omega_{\mathbb{T}}$, then $\text{Sel}_K(A_P^{\text{ord}})$ is finite for almost all $P \in \Omega_{\mathbb{T}}$.

2. Suppose that all prime factors of $p$ in $K$ has residual degree 1. If $\text{Sel}_K(A_{P_0}^{\text{ord}}) = 0$ for a single point $P_0 \in \Omega_{\mathbb{T}}$ such that $A_{P_0}/\mathbb{Q}$ has good reduction modulo $p$ with $A_{P_0}(\mathbb{F}_p) = 0$, $\text{Sel}_K(A_P^{\text{ord}})$ is finite for all $P \in \Omega_{\mathbb{T}}$ without exception.

This type of control has been studied by other people, notably, J. Nekovář.
§13. Abelian variety of GL(2)-type.

A \(\mathbb{Q}\)-simple abelian variety (with a polarization) is “of GL(2)-type” if we have a subfield \(H_{A} \subset \text{End}^{0}(A/\mathbb{Q}) = \text{End}(A/\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Q}\) with \(\dim A = [H_{A} : \mathbb{Q}]\) (stable under Rosati-involution).

Then, for the two-dimensional compatible system \(\rho_{A}\) of Galois representation of \(A\) with coefficients in \(H_{A}\), \(H_{A}\) is generated by traces \(\text{Tr}(\rho_{A}(\text{Frob}_{l}))\) of Frobenius elements \(\text{Frob}_{l}\) for primes \(l\) of good reduction (i.e., the field \(H_{A}\) is uniquely determined by \(A\)). We always regard \(\mathbb{Q}\) as a subfield of the algebraic closure \(\overline{\mathbb{Q}}\). Thus \(O'_{A} := \text{End}(A/\mathbb{Q}) \cap H_{A}\) is an order of \(H_{A}\). Write \(O_{A}\) for the integer ring of \(H_{A}\). Replacing \(A\) by the abelian variety representing the group functor \(R \mapsto A(R) \otimes_{O'_{A}} O_{A}\), we may choose \(A\) so that \(O'_{A} = O_{A}\) in the \(\mathbb{Q}\)-isogeny class of \(A\).

To reformulate the result, we introduce congruence among abelian varieties.

For two abelian varieties $A$ and $B$ of GL(2)-type over $\mathbb{Q}$, we say that $A$ is congruent to $B$ modulo a prime $p$ over $\mathbb{Q}$ if we have a prime factor $p_A$ (resp. $p_B$) of $p$ in $O_A$ (res. $O_B$) and field embeddings $\sigma_A : O_A/p_A \hookrightarrow \overline{\mathbb{F}}_p$ and $\sigma_B : O_B/p_B \hookrightarrow \overline{\mathbb{F}}_p$ such that $(A[p_A] \otimes O_A/p_A, \sigma_A \overline{\mathbb{F}}_p)^{ss} \cong (B[p_B] \otimes O_B/p_B, \sigma_B \overline{\mathbb{F}}_p)^{ss}$ as semi-simplified $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-modules.

We call that $A$ is of $p_A$-type $(\alpha, \delta)$ if the $p_A$-adic Tate module produces a local representation $\rho_{p_A}$ of $\text{Gal} (\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ such that $\rho_{p_A}|_{I_p} \cong \begin{pmatrix} \nu_p & \epsilon^{-\delta} \\ 0 & \epsilon^\alpha \end{pmatrix}$ for a character $\epsilon : I_p \rightarrow \mu_{p\infty}$ of the inertia group $I_p$ at $p$. 
§15. Rational elliptic curves.

Let $E/\mathbb{Q}$ be an elliptic curve. Writing the Hasse–Weil $L$-function $L(s, E)$ as a Dirichlet series $\sum_{n=1} a_n n^{-s}$ ($a_n \in \mathbb{Z}$) (i.e., $1+p-a_p = |E(\mathbb{F}_p)|$ for each prime $p$ of good reduction for $E$), we call $p$ admissible for $E$ if $E$ has good reduction at $p$ and $(a_p \mod p)$ is not in $\Omega_E := \{\pm 1, 0\}$ (so, 2 and 3 are not admissible). Therefore, the maximal étale quotient of $E[p]$ over $\mathbb{Z}_p$ is not isomorphic to $\mathbb{Z}/p\mathbb{Z}$ up to unramified quadratic twists.

By the Hasse bound $|a_p| \leq 2\sqrt{p}$, $p \geq 7$ is not admissible if and only if $a_p \in \Omega_E$. Thus if $E$ does not have complex multiplication, the Dirichlet density of non-admissible primes is zero by a theorem of Serre as $L(s, E) = L(s, f)$ for a rational Hecke eigenform $f$. 
§16. Vanishing of $\mathbb{III}$ proliferates.
Take $E/\mathbb{Q}$ an elliptic curve with $|\mathbb{III}_K(E)| < \infty$ and rank $E(K) \leq 1$. Let $N$ be the conductor of $E$, and pick an admissible prime $p$ for $E$. Consider the set $\mathcal{A}_{E,p}$ made up of all $\mathbb{Q}$-isogeny classes of $\mathbb{Q}$-simple abelian varieties $A/\mathbb{Q}$ of $p_A$-type $(\alpha, \delta)$ with prime-to-$p$ conductor $N$ congruent to $E$ modulo $p$ over $\mathbb{Q}$.

**Theorem B.** There exists an explicit (computable) finite set $S_E$ of primes depending on $N$ but independent of $K$ such that if $p \notin S_E$ is admissible for $E$, almost all members $A \in \mathcal{A}_{E,p}$ have finite $\mathbb{III}_K(A)[p^\infty_A]$ and constant dimension $\dim_{H_A} A(K) \leq 1$. If further $E(K)_p = \mathbb{III}_K(E) = 0$ (i.e., Sel$_K(E) = 0$ in short) and $E$ can be embedded into $J_r$ for some $r > 0$, then as long as $p$ totally splits in $K/\mathbb{Q}$, every $A \in \mathcal{A}_{E,p}$ has finite $\mathbb{III}_K(A)[p^\infty_A]$ and Sel$_K(A)[p^\infty_A]$ as long as $p \notin S_E$. 
§17. More concrete statement.

**Corollary C.** Let \( N \in \{11, 14, 15, 17, 19, 20, 21, 24, 27, 32, 36, 49\} \) (all the cases when \( X_0(N) \) is an elliptic curve with finite \( X_0(N)(\mathbb{Q}) \)). Pick an admissible prime \( p \) for \( X_0(N) \). Then \(|\text{III}_\mathbb{Q}(A)[p^\infty]| < \infty\) and \(|\text{Sel}_\mathbb{Q}(A)[p^\infty]| < \infty\) for almost all \( A \) in \( \mathcal{A}_{X_0(N),p} \). If further \( X_0(N)(\mathbb{Q})_p = \text{III}_\mathbb{Q}(X_0(N))_p = 0 \), \( \text{Sel}_\mathbb{Q}(A)[p^\infty] \) and \( \text{III}_\mathbb{Q}(A)[p^\infty] \) are both finite for all \( A \) in \( \mathcal{A}_{X_0(N),p} \) without exception.

If \( E \subset J_0(37) \) with root number \(-1\) (so, rank \( E(\mathbb{Q}) = 1 \)), for an admissible prime \( p \) for \( E \), we have \(|\text{III}_\mathbb{Q}(A)[p^\infty]| < \infty\) for almost all \( A \) in \( \mathcal{A}_{E,p} \).
§18. Conjecture. Here is a conjecture:

**Conjecture 1.** Let $\text{Spec}(\mathbb{I})$ be a new irreducible component of $\text{Spec}(\mathfrak{h}_{\alpha,\delta})$, and pick a totally real field $K$.

1. Suppose $(\alpha, \delta) = (1, 1)$ and that the root number of $\mathbb{I}$ is $\epsilon := \pm 1$ over the totally real number field $K$. Then for almost all $P \in \Omega_{\mathbb{I}}$, we have $\dim_{H_P} A_P(K) \otimes_{\mathbb{Z}} \mathbb{Q} = \frac{1-\epsilon}{2}$.

2. Suppose $(\alpha, \delta) \neq (1, 1)$. Then for almost all $P \in \Omega_{\mathbb{I}}$, we have $\dim_{H_P} A_P(K) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$.

Combined with the solution of the parity conjecture by Nekovář and Dokchitser/Dokchitser with our result, the above conjecture holds in many cases.

For simplicity, assuming $K = \mathbb{Q}$, we now give a sketch of the proof of the theorems.
§19. Start of the proof for $\mathbf{III}(G) := \mathbf{III}(\mathbb{Q}^S/\mathbb{Q}, G)$.

Recall the exact sequence: $A_P[p^\infty](K') \hookrightarrow \mathcal{G}_T(K') \twoheadrightarrow \mathcal{G}_T(K')$. Then we get a commutative diagram with exact bottom two rows and exact columns:

$\begin{align*}
\text{Ker}(\iota_{III, *}) &\longrightarrow \mathbf{III}(A_P^{\text{ord}}[p^\infty]) \xrightarrow{\iota_{III, *}} \mathbf{III}(\mathcal{G}_T) \xrightarrow{\varpi_{III, *}} \mathbf{III}(\mathcal{G}_T) \\
\cap &\downarrow \cap &\downarrow \cap &\downarrow \\
E_{BT}^\infty(\mathbb{Q}) &\hookrightarrow H^1(A_P^{\text{ord}}[p^\infty]) \xrightarrow{\iota_*} H^1(\mathcal{G}_T) \xrightarrow{\varpi_*} H^1(\mathcal{G}_T) \\
\downarrow &\downarrow &\downarrow &\downarrow \\
E_{BT}^\infty(\mathbb{Q}_p) &\hookrightarrow H^1_S(A_P^{\text{ord}}[p^\infty]) \xrightarrow{\iota_{S,*}} H^1_S(\mathcal{G}_T) \xrightarrow{\varpi_{S,*}} H^1_S(\mathcal{G}_T),
\end{align*}$

where $E_{BT}^\infty(k) = \text{Coker}(\varpi : \mathcal{G}(k) \to \mathcal{G}(k))$ and $H^1_S(?) = \prod_{v \in S} H^1(\mathbb{Q}_v, ?)$. 
§20. Conclusion of the proof for $\III(G)$.

If $a_p \not\equiv 1 \mod p$, we have $G(\mathbb{Q}) = G(\mathbb{Q}_p) = 0$. If the residual representation of $\rho_{f_P}$ is irreducible, again $G(\mathbb{Q}) = 0$. It is easy to show $E^\infty_{BT}(\mathbb{Q})$ and $\prod_{v \in S} E^\infty_{BT}(\mathbb{Q}_v)$ are finite. Thus the sequence

$$0 \to \III(A_{P}^{\text{ord}}[p^\infty]) \to \III(G_T) \xrightarrow{\varpi} \III(G_T)$$

is exact up to finite error.

$|\III(A_{P_0}^{\text{ord}}[p^\infty])| < \infty \Rightarrow \III(G)^{\vee}$ is a torsion $\mathbb{T}$-module of finite type. So,

$|\III(A_{P_0}^{\text{ord}}[p^\infty])| < \infty \Rightarrow |\III(A_{P}^{\text{ord}}[p^\infty])| < \infty$ for most $P \in \Omega_{\mathbb{T}}$. 
§21. Start of the proof for $\text{Sel}(A_P^{\text{ord}}) := \text{Sel}_Q(A_P)^{\text{ord}}$.

Recall the second fundamental exact sequence:

$$0 \to A_P^{\text{ord}}(K') \to J_\infty^{\text{ord}}(K') \xrightarrow{\varpi} J_\infty^{\text{ord}}(K') \to B_P^{\text{ord}}(K') \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to 0,$$

where $J_\infty^{\text{ord}} = \varinjlim_s J_s^{\text{ord}}$ and $K' = \mathbb{Q}^S$ and $\overline{\mathbb{Q}}_l$.

We separate it into two short exact sequences:

$$0 \to A_P^{\text{ord}}(K') \to J_\infty^{\text{ord}}(K') \xrightarrow{\varpi} \varpi(J_\infty^{\text{ord}})(K') \to 0,$$

$$0 \to \varpi(J_\infty^{\text{ord}})(K') \to J_\infty^{\text{ord}}(K') \to B_P^{\text{ord}}(K') \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to 0.$$
§22. The proof for Sel($A_P^{\text{ord}}$) continues.

Look into the following commutative diagram of sheaves with exact rows:

\[
\begin{array}{cccccc}
A_P[p^\infty]^{\text{ord}} & \hookrightarrow & J_\infty^{\text{ord}}[p^\infty] & \xrightarrow{\varpi[p^\infty]} & J_\infty^{\text{ord}}[p^\infty] & \rightarrow & 0 \\
\downarrow & & \downarrow & & i \downarrow & & \downarrow \\
A^{\text{ord}}_P & \hookrightarrow & J_\infty^{\text{ord}} & \xrightarrow{\varpi} & J_\infty^{\text{ord}} & \rightarrow & B^{\text{ord}}_P \otimes \mathbb{Q}_p.
\end{array}
\]

Since $B^{\text{ord}}_P \otimes \mathbb{Q}_p$ is a sheaf of $\mathbb{Q}_p$-vector spaces and $J_\infty^{\text{ord}}[p^\infty]$ is $p$-torsion, the inclusion map $i$ factors through the image $\text{Im}(\varpi) = \varpi(J_\infty^{\text{ord}})$; so,

$$\varpi(J_\infty^{\text{ord}})[p^\infty] = J_\infty^{\text{ord}}[p^\infty].$$
§23. **Injectivity of** \(\text{Sel}(\varpi(J_\infty^{\text{ord}})) \to \text{Sel}(J_\infty^{\text{ord}})\).

From the exact sequence, \(\varpi(J^{\text{ord}}) \hookrightarrow J^{\text{ord}} \twoheadrightarrow B_P^{\text{ord}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p\), taking its cohomology sequence, we get the bottom sequence of the following commutative diagram with exact rows:

\[
\begin{array}{ccc}
0 & \rightarrow & \text{Sel}(\varpi(J_\infty^{\text{ord}})) \rightarrow \text{Sel}_\mathbb{Q}(J_\infty^{\text{ord}}) \\
\downarrow & & \downarrow \cap \\
0 & \rightarrow & H^1(\varpi(J_\infty^{\text{ord}})[p^\infty]) \rightarrow H^1(J_\infty^{\text{ord}}[p^\infty]) \\
\downarrow & & \downarrow i \\
\prod_{v \in S} E^*_\text{Sel}(\mathbb{Q}_v) & \hookrightarrow & H^1_S(\varpi(J_\infty^{\text{ord}})) \rightarrow H^1_S(J_\infty^{\text{ord}}),
\end{array}
\]

where we have written \(H^1_S(X) := \prod_{v \in S} H^1(\mathbb{Q}_v, X)\) and

\[E^*_\text{Sel}(\mathbb{Q}_v) := \text{Coker}(J_\infty^{\text{ord}}(\mathbb{Q}_v) \to B_P^{\text{ord}}(\mathbb{Q}_v) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p).\]
§24. Conclusion for injectivity.

By the snake lemma, we get an exact sequence

$$0 \to \text{Sel}_K(\varpi(J^\text{ord}_\infty)) \to \text{Sel}_K(J^\text{ord}_\infty) \to E^*_\text{Sel}(\mathbb{Q}_p),$$

since it is easy to see $B^\text{ord}_P(\mathbb{Q}_l) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = E^*_\text{Sel}(\mathbb{Q}_l) = 0$ if $l \neq p$. 

We look into:

\begin{equation}
\begin{array}{ccccccc}
\text{Ker}(\iota_*) & \overset{i_0}{\longrightarrow} & \text{Sel}(A^\text{ord}_P) & \overset{\iota_*}{\longrightarrow} & \text{Sel}(J^\text{ord}_{\infty,\mathbb{T}}) & \overset{\varpi_*}{\longrightarrow} & \text{Sel}(\varpi(J^\text{ord}_{\infty,\mathbb{T}})) \\
\downarrow i & & \cap \downarrow a & & \cap & & \\
E^\infty_{BT}(\mathbb{Q}) & \overset{\varepsilon}{\hookrightarrow} & H^1(A^\text{ord}_P[p^\infty]) & \overset{\iota_*}{\longrightarrow} & H^1(G) & \overset{\varpi_*}{\longrightarrow} & H^1(G) \\
\downarrow e & & \downarrow & & \downarrow & & \\
E^\infty(\mathbb{Q}_p) & \overset{\varepsilon_0}{\hookrightarrow} & H^1_S(A^\text{ord}_P) & \overset{\iota_{S,*}}{\longrightarrow} & H^1_S(J^\text{ord}_{\infty,\mathbb{T}}) & \overset{\varpi_*}{\longrightarrow} & H^1_S(\varpi(J^\text{ord}_{\infty,\mathbb{T}})). \\
\end{array}
\end{equation}

Here $E^\infty(k) = \lim_{s} \frac{\varpi(J^\text{ord}_{\infty,\mathbb{T}})(k)}{\varpi(J^\text{ord}_{\infty,\mathbb{T}})(k)}$. The error term $E^\infty(\mathbb{Q}_p)$ injects into

$$H^1(\mathbb{Q}_p[\mu_{p^\infty}]/\mathbb{Q}_p, A^\text{ord}_P(\mathbb{Q}_p[\mu_{p^\infty}])).$$
§26. Conclusion.
If $A_P$ has good reduction modulo $p$, by a result of P. Schneider on universal norm, $|E^\infty(\mathbb{Q}_p)| \leq |H^1(\mathbb{Q}_p[\mu_p]/\mathbb{Q}_\mathbb{Q}_p, A_P^{\text{ord}}(\mathbb{Q}_p[\mu_p])| = |A_P(\mathbb{F}_p)|^2$. If $A_P$ has good reduction over $\mathbb{Z}_p[\mu_p]$, we have an exact sequence for $K_r = \mathbb{Q}_p[\mu_p]$:

$$H^1(K_r/K, A_P^{\text{ord}}(K_r)) \rightarrow H^1(K_{\infty}/K, A_P^{\text{ord}}(K_{\infty}))$$
$$\rightarrow H^0(K_r/K, H^1(K_{\infty}/K_r, A_P^{\text{ord}}(K_{\infty}))) \rightarrow H^2(K_r/K, A_P^{\text{ord}}(K_r)),$$

we get the finiteness of $E^\infty(\mathbb{Q}_p)$.

As already seen, $E_{BT}^\infty(\mathbb{Q}) = \text{Coker}(\varpi : \mathcal{G}(\mathbb{Q}) \rightarrow \mathcal{G}(\mathbb{Q}))$ is finite, we get an exact sequence up to finite error

$$0 \rightarrow \text{Sel}(A_P^{\text{ord}}) \rightarrow \text{Sel}(J_{\infty, T}^{\text{ord}}) \xrightarrow{\varpi} \text{Sel}(J_{\infty, T}^{\text{ord}})$$

and hence if Sel$(A_P^{\text{ord}}_0)$ is finite, Sel$(J_{\infty, T}^{\text{ord}})^{\vee}$ is $\mathbb{T}$-torsion, so, Sel$(A_P^{\text{ord}})$ is finite for most of $P$. 
§27. **Case where** $\text{Sel}(A_{P_0}^{\text{ord}}) = 0$.

If $\text{Sel}(A_{P_0}^{\text{ord}}) = 0$ and $A_{P_0}$ has good reduction modulo $p$ with $A_{P_0}^{\text{ord}}(\mathbb{F}_p) = 0$, again by Schneider, the sequence

$$0 \rightarrow \text{Sel}(A_{P_0}^{\text{ord}}) \rightarrow \text{Sel}(J_{\infty,T}^{\text{ord}}) \rightarrow \text{Sel}(J_{\infty,T}^{\text{ord}})$$

is exact, and hence $\text{Sel}(J_{\infty,T}^{\text{ord}}) = 0$. This shows that finiteness of $\text{Sel}(A_{P}^{\text{ord}})$ for all $P \in \Omega_T$. 
§28. Diagram for \( \text{III}(A^{\text{ord}}) \).
The sequence \( A_P^{\text{ord}}[p^\infty](k) \hookrightarrow A_P^{\text{ord}}(k) \oplus \varpi(J^{\text{ord}})(k) \xrightarrow{\pi_k} J^{\text{ord}}(k) \) is exact. Writing \( \text{III}_S(?) := \text{III}(Q^S/Q,?) \) and \( \text{III}_S(A_P^{\text{ord}}[p^\infty]) \xrightarrow{t_{\text{III},*}} \text{III}_Q(A_P^{\text{ord}} \oplus \varpi(J^{\text{ord}})) \) for the natural map, we get the following commutative diagram:

\[
\begin{array}{c}
\frac{\text{III}_S(\hat{A}_P^{\text{ord}}[p^\infty])}{\text{Ker}(t_{\text{III},*})} \twoheadrightarrow \frac{\text{III}_Q(\hat{A}_P^{\text{ord}} \oplus \varpi(J^{\text{ord}}))}{\text{Ker}(t_{\text{III},*})} \twoheadrightarrow \frac{\text{III}_Q(J^{\text{ord}})}{\text{Ker}(t_{\text{III},*})} \\
\downarrow \quad \downarrow \quad \downarrow \\
\frac{H^1(\hat{A}_P^{\text{ord}}[p^\infty])}{\text{Coker}(\pi_Q)} \twoheadrightarrow \frac{H^1(\hat{A}_P^{\text{ord}} \oplus \varpi(J^{\text{ord}}))}{\text{Coker}(\pi_Q)} \twoheadrightarrow \frac{H^1(J^{\text{ord}})}{\text{Coker}(\pi_Q)} \\
\downarrow \quad \downarrow \quad \downarrow \\
\frac{H^1_S(\hat{A}_P^{\text{ord}}[p^\infty])}{\text{Coker}(\pi_{Q_p})} \twoheadrightarrow \frac{H^1_S(\hat{A}_P^{\text{ord}} \oplus \varpi(J^{\text{ord}}))}{\text{Coker}(\pi_{Q_p})} \twoheadrightarrow \frac{H^1_S(J^{\text{ord}})}{\text{Coker}(\pi_{Q_p})}.
\end{array}
\]
§29. Almost injectivity of $\text{III}_Q(\varpi(J^\text{ord}_\infty)) \to \text{III}_Q(J^\text{ord}_\infty)$.

Thus we have an exact sequence

$$0 \to \frac{\text{III}_S(\widehat{A}_P^{\text{ord}}[p^\infty])}{\text{Ker}(\iota_{\text{III},*})} \to \text{III}_Q(\widehat{A}_P^{\text{ord}} \oplus \varpi(J^\text{ord}_\infty)) \to \text{III}_Q(J^\text{ord}_\infty).$$

If $\text{III}_S(\widehat{A}_P^{\text{ord}}[p^\infty])$ is finite, for the natural map:

$$\text{III}_Q(\varpi(J^\text{ord}_\infty)) \to \text{III}_Q(J^\text{ord}_\infty)$$

induced from the inclusion: $\varpi(J^\text{ord}_\infty) \hookrightarrow J^\text{ord}_\infty$, we get

**finiteness** of $\text{Ker}(\text{III}_Q(\varpi(J^\text{ord}_\infty)) \to \text{III}_Q(J^\text{ord}_\infty))$. 

§30. Diagram involving $\mathbb{III}(A^{\text{ord}})$ and $\mathbb{III}(A^{\text{ord}}[p^\infty])$.
Consider Kummer sequences:

\[
\begin{array}{cccccc}
\text{Ker}_{MW} & \hookrightarrow & \mathbb{III}_S(\hat{A}_P^{\text{ord}}[p^\infty]) & \xrightarrow{\iota_{\mathbb{III},*}} & \mathbb{III}_Q(\hat{A}_P^{\text{ord}}) \\
\downarrow & & \cap & & \cap \\
\hat{A}_P^{\text{ord}}(\mathbb{Q}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p & \hookrightarrow & H^1(\hat{A}_P^{\text{ord}}[p^\infty]) & \xrightarrow{\iota_*} & H^1(\hat{A}_P^{\text{ord}}) \\
\delta \downarrow & & \text{Res}[p^\infty] & & \text{Res} \\
\hat{A}_P^{\text{ord}}(\mathbb{Q}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p & \xrightarrow{I_S} & H^1_S(\hat{A}_P^{\text{ord}}[p^\infty]) & \xrightarrow{\iota_{S,*}} & H^1_S(\hat{A}_P^{\text{ord}}). \\
\end{array}
\]

Thus $\mathbb{III}_S(\hat{A}_P^{\text{ord}}[p^\infty])$ is finite if $\dim_{H_P} \hat{A}_P^{\text{ord}}(\mathbb{Q}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \leq 1$ and $\mathbb{III}(A^{\text{ord}}_P)$ is finite.
§31. Relation between $\mathbf{III}(A^{\text{ord}})$ and $\mathbf{III}(A^{\text{ord}}[p^\infty])$.

The previous diagram produces a new one with exact rows and columns (up to finite error) if $|\text{III}_S(\hat{A}_P^{\text{ord}}[p^\infty])| < \infty$,

$$
\begin{array}{cccccccc}
\text{Ker}_{MW} & \hookrightarrow & \text{III}_S(\hat{A}_P^{\text{ord}}[p^\infty]) & & & \longmapsto & \text{III}_Q(\hat{A}_P^{\text{ord}}) \\
\downarrow & & \downarrow & & & \cap & \downarrow \\
0 & \rightarrow & \text{Coker}(I) & \rightarrow & \text{coker}(I) & \rightarrow & 0 \\
\delta & \downarrow & \downarrow & \rightarrow & \text{Res} & \downarrow \\
0 & \rightarrow & \text{Coker}(I_S) & \rightarrow & H^1(\hat{A}_P^{\text{ord}}) & \rightarrow & 0 \\
\end{array}
$$

By snake lemma, $|\text{III}_S(\hat{A}_P^{\text{ord}}[p^\infty])| < \infty \Rightarrow |\text{III}_Q(\hat{A}_P^{\text{ord}})| < \infty$. So

$|\text{III}_S(\hat{A}_P^{\text{ord}}[p^\infty])| < \infty \Rightarrow |\text{III}_S(\hat{A}_P^{\text{ord}}[p^\infty])| < \infty$ for most $P$

$\Rightarrow |\text{III}_S(\hat{A}_P)^{\text{ord}}| < \infty$ for most $P$. 
§32. \( \text{III}_\mathbb{Q}(\hat{A}_P)^{\text{ord}} \hookleftarrow \text{III}_\mathbb{Q}(J_{\infty,T})^{\text{ord}} \xrightarrow{\varpi} \text{III}_\mathbb{Q}(J_{\infty,T})^{\text{ord}} \) is exact.

We have the following diagram:

\[
\begin{array}{ccccccc}
\text{Ker}(\iota_{\text{III},*,T}) & \rightarrow & \text{III}_\mathbb{Q}(\hat{A}_P^{\text{ord}}) & \xrightarrow{\iota_{\text{III},*,T}} & \text{III}_\mathbb{Q}(J_{\text{ord}}^{\infty,T}) & \xrightarrow{\varpi_{\text{III},*,T}} & \text{III}_\mathbb{Q}(\varpi(J_{\text{ord}}^{\infty,T})) \\
\cap & \downarrow & \cap & \downarrow & \cap & \downarrow & \cap \\
E^{\infty}(\mathbb{Q}_T) & \xhookrightarrow{} & H^1(\hat{A}_P^{\text{ord}}) & \xrightarrow{\iota^*} & H^1(J_{\text{ord}}^{\infty,T}) & \xrightarrow{\varpi^*} & H^1(\varpi(J_{\text{ord}}^{\infty,T})) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
E^{\infty}(\mathbb{Q}_p)_T & \xhookrightarrow{} & H^1_S(\hat{A}_P^{\text{ord}}) & \xrightarrow{\iota_S^*} & H^1_S(J_{\text{ord}}^{\infty,T}) & \xrightarrow{\varpi_S^*} & H^1_S(\varpi(J_{\text{ord}}^{\infty,T})).
\end{array}
\]

Thus \( |\text{III}_S(A_{P_0}[p^{\infty}]^{\text{ord}})| < \infty \Rightarrow |\text{III}_\mathbb{Q}(\hat{A}_P^{\text{ord}})| < \infty \) for most \( P \Rightarrow \)

\[
0 \rightarrow \text{III}_\mathbb{Q}(\hat{A}_P)^{\text{ord}} \hookleftarrow \text{III}_\mathbb{Q}(J_{\infty,T})^{\text{ord}} \xrightarrow{\varpi} \text{III}_\mathbb{Q}(J_{\infty,T})^{\text{ord}}
\]

is exact up to finite error.
§33. Conclusion for $\mathbf{III}(A^{\text{ord}})$.

Assume that $\dim_{H_{P_0}} A_{P_0}(\mathbb{Q}) \otimes \mathbb{Q} \leq 1$ and $|\mathbf{III}_{\mathbb{Q}}(A^{\text{ord}}_{P_0})| < \infty$.

Then by Kummer theory,

$$|\mathbf{III}_{S}(A_{P_0}[p^{\infty}]^{\text{ord}})| < \infty \Rightarrow |\mathbf{III}_{S}(A_{P}[p^{\infty}]^{\text{ord}})| < \infty \text{ and } |\mathbf{III}_{S}(\hat{A}_{P}^{\text{ord}})| < \infty \text{ for most } P \text{ and}$$

$$0 \to \mathbf{III}_{\mathbb{Q}}(\hat{A}_{P})^{\text{ord}} \to \mathbf{III}_{\mathbb{Q}}(J_{\infty,T})^{\text{ord}} \xrightarrow{\varpi} \mathbf{III}_{\mathbb{Q}}(J_{\infty,T})^{\text{ord}}$$

is exact up to finite error for most $P$ including $P_0$.

Since limit Selmer groups and Tate–Shafarevich groups are well controlled, the limit Mordell–Weil groups are also well controled as in Theorem $\mathbf{III}$ (3). \qed