Abstract. The (pro) Λ-MW group is a projective limit of Mordell–Weil groups over a number field $k$ (made out of modular Jacobians) with an action of the Iwasawa algebra and the “big” Hecke algebra. We prove a control theorem of the ordinary part of the Λ-MW groups under mild assumptions. We have proven a similar control theorem for the dual inductive limit in [H15a].

1. Introduction

Fix a prime $p$. This article concerns weight 2 cusp forms of level $Np^r$ for $r > 0$ and $p \nmid N$, and for small primes $p = 2, 3$, they exist only when $N > 1$; thus, we may assume $Np^r \geq 4$. Then the open curve $Y_1(Np^r)$ (obtained from $X_1(Np^r)$ removing all cusps) gives the fine smooth moduli scheme classifying elliptic curves $E$ with an embedding $\mu_{Np^r} \hookrightarrow E$. We applied in [H86b] and [H14] the techniques of $U(p)$-isomorphisms to Barsotti–Tate groups of modular Jacobian varieties of high $p$-power level (with the fixed prime-to-$p$ level $N$). In this article, we apply the same techniques of $U(p)$-isomorphisms to the projective limit of Mordell–Weil groups of the Jacobians and see what we can say (see Section 3 for $U(p)$-isomorphisms). We study the (limit of) Tate–Shafarevich groups of the Jacobians in another article [H15c].

Let $X_r = X_1(Np^r)/Q$ be the compactified moduli of the classification problem of pairs $(E, \phi)$ of an elliptic curve $E$ and an embedding $\phi : \mu_{Np^r} \hookrightarrow E[Np^r]$. Write $J_r/Q$ for the Jacobian whose origin is given by the infinity cusp $\infty \in X_r(Q)$ of $X_r$. For a number field $k$, we consider the group of $k$-rational points $J_r(k)$. Put $\tilde{J}_r(k) = \lim_{r \to \infty} J_r(k)$ (as a compact $p$-profinite module). The Albanese functoriality of Jacobians (twisted by the Weil involutions) gives rise to a projective system $\{\tilde{J}_r(k)\}_r$ compatible with Hecke operators (see Section 6 for details of twisting), and we have

$$\tilde{J}_\infty(k) = \lim_{r \to \infty} \tilde{J}_r(k)$$

equipped with the projective limit compact topology. By Picard functoriality, we have an injective limits $J_\infty(k) = \lim_{r \to \infty} \tilde{J}_r(k)$ (with the injective limit of the compact topology of $\tilde{J}_r(k)$) and $J_\infty[p^\infty]/Q = \lim_{r \to \infty} J_r[p^\infty]/Q$ (the injective limit of the $p$-divisible Barsotti–Tate group). We define

$$J_\infty(k) = \lim_{n \to \infty} J_\infty(k)/p^n J_\infty(k).$$

An fppf sheaf $F$ (over Spec($k$)) is a presheaf functor from the fppf site over Spec($k$) to the category of abelian groups satisfying the sheaf condition for a covering $\{U_i\}$ of $T/k$, that is, the exactness of

$$(L) \quad 0 \to F(T) \xrightarrow{\text{Res}_{U_i/T}} \prod_i F(U_i) \xrightarrow{\text{Res}_{U_{ij}/U_i} - \text{Res}_{U_{ij}/U_j}} \prod_{i,j} F(U_{ij}),$$

where $\text{Res}_{U/V}$ indicates the restriction map relative to $U \to V$ and $U_{ij} := U_i \times_T U_j$. Since the category of fppf sheaves over $Q$ (e.g., [EAI, §4.3.7]) is an abelian category (cf. [ECH, II.2.15]), if we apply a left exact functor (of the category of abelian groups into itself) to the value of a sheaf, it preserves the sheaf condition given by the left exactness (L). Thus projective limit and injective limit

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exist inside the category of fppf sheaves. We may thus regard $R \mapsto \tilde{J}_\infty(R) := \varinjlim_r (J_r(R) \otimes_{\mathbb{Z}} \mathbb{Z}_p)$ and $R \mapsto J_\infty(R)$ as fppf sheaves over the fppf site over $\mathbb{Q}$ for an fppf extension $R/\mathbb{Q}$, though we do not use this fact much (as we compute $\tilde{J}_\infty(k)$ as a limit of $\tilde{J}_s(k)$ not using sheaf properties of $\tilde{J}_\infty$). If one extends $\tilde{J}_s$ to the ind-category of fppf extensions, we no longer have projective limit expression. We have given detailed description of the value $\tilde{J}_s(R)$ in [H15a, §2] and we will give a brief outline of this in Section 2 in the text. We can think of the sheaf endomorphism algebra $\text{End}(J_\infty/\mathbb{Q})$ (in which we have Hecke operators $T(r)$ of $\text{End}(J_s)$. We may thus regard $h$ as well known (cf. [H86b] and [O99]) and we will give an exposition on this in Section 6. $T(n)$ and $U(l)/l|N_p)$.

The Hecke operator $U(p)$ acts on $J_s(k)$, and the $p$-adic limit $e = \lim_{n \to \infty} U(p)^{n!}$ is well defined on $\tilde{J}_s(k)$. As is well known (cf. [H86b] and [O99]) and we will give an exposition on this in Section 6, $T(n)$ and $U(l)/l|N_p)$ are endomorphisms of the injective (resp. projective) systems $\{J_s(k)\}_s$ (resp. $\{\tilde{J}_s(k)\}_s$). The projective system comes from $w$-twisted Albanese functoriality for the Weil involution $w$ (as we need to twist in order to make the system compatible with $U(p)$; see Section 6 for the twisting). The image of $e$ is called the *ordinary* part. We add the superscript or the subscript “ord” to indicate the ordinary part. We call the groups $\tilde{J}_\infty(k)^\text{ord}$ pro $\Lambda$-MW groups. We studied the $\Lambda$-BT group $\mathcal{G}_\mathbb{Q} = J_\infty[p^\infty]^\text{ord}$ in [H14, §4]. Though in [H14], we made an assumption that $p \geq 5$, as for the results over $\mathbb{Q}$ in [H14, §4], they are valid without any change for $p = 2, 3$ as verified in [GK13] for $p = 2$ (and the prime $p = 3$ can be treated in the same manner as in [H86a] or [H14, §4]). Thus we use control result over $\mathbb{Q}$ of $\mathcal{G}$ in this paper without assuming $p \geq 5$. Its Tate module $T_G := \text{Hom}_{\mathbb{Z}_p}(\Lambda^\vee, \mathcal{G})$ is a continuous $\text{Gal}^{\text{ab}}(\overline{\mathbb{Q}}/\mathbb{Q})$-module under the profinite topology, where $\Lambda^\vee = \text{Hom}_{\mathbb{Z}_p}(\Lambda, \mathbb{Q}_p/\mathbb{Z}_p)$.

We define the big Hecke algebra $h = h(N)$ to be the $\Lambda$-subalgebra of $\text{End}_{\mathbb{S}}(T_G)$ generated by Hecke operators $T(n)(n = 1, 2, \ldots)$. Then $\tilde{J}_\infty(k)^{\text{ord}}$ and $\tilde{J}_\infty(k)^{\text{ord}}$ are naturally continuous $h$-modules. Take a connected component $\text{Spec}(T)$ of $\text{Spec}(h)$ and define the direct factors

$$\tilde{J}_s(k)^{\text{ord}} := \tilde{J}_s(k)^{\text{ord}} \otimes_{\mathbb{T}} T (s = 1, 2, \ldots, \infty)$$

of $\tilde{J}_\infty(k)^{\text{ord}}$ and $T_G$, respectively. In this introduction, for simplicity, we assume that the component $\mathbb{T}$ cuts out $\tilde{J}_\infty(k)^{\text{ord}}$ from $\tilde{J}_s(k)^{\text{ord}}$ a part with potentially good reduction modulo $p$ (meaning that $\mathcal{G}_T[p^r]-1$ extends to $\Lambda$-BT group over $\mathbb{Z}_p[\mu_{p^r}]$). This is to avoid technicality coming from potentially multiplicative reduction of factors of $J_s$ outside $\tilde{J}_s(k)^{\text{ord}}$.

The maximal torsion-free part $\Gamma$ of $\mathbb{Z}_p^\times$ (which is a $p$-profinite cyclic group) acts on these modules by the diamond operators. In other words, for modular curves $X_r$ and $X_0(Np^r)$, we identify $\text{Gal}(X_r/\mathbb{Q}_p)$ with $(\mathbb{Z}/Np^r\mathbb{Z})^\times$, and $\Gamma$ acts on $J_r$ through its image in $\text{Gal}(X_r/\mathbb{Q}_p)$.

Therefore the Iwasawa algebra $\Lambda = \mathbb{Z}_p[\Gamma]$ is the $\Lambda$-MW group, and $\mathcal{G}$ and its Tate module. Then $T_G$ is known to be free of finite rank over $\Lambda$ [H86b], [GK13] and [H14, §4]. A prime $p \in \text{Spec}(\lambda)(\overline{\mathbb{Q}}_p)$ is called arithmetic of weight $2$ if $p$ factors through $\text{Spec}(\mathbb{Z}_p[\Gamma/T^{p^r}])$ for some $r > 0$. In this article, we prove in Theorem 9.2 and Theorem 10.4 control results for the pro $\Lambda$-MW group $\tilde{J}_\infty(k)^{\text{ord}}$ and study the control of the ind $\Lambda$-MW-groups $\tilde{J}_\infty(k)^{\text{ord}}$ in the twin paper [H15a, Theorem 6.5]. Take a topological generator $\gamma = 1 + p^r$, and regard $\gamma$ as a group element of $\Lambda = \mathbb{Z}_p[\Gamma]$, where $\epsilon = 1$ if $p > 2$ and $\epsilon = 2$ if $p = 2$. We use this definition of $\epsilon$ throughout the paper (and we assume that $r \geq \epsilon$ if the exponent $r - \epsilon$ shows up in a formula). We fix a finite set $S$ of places of $\mathbb{Q}$ containing all places $v|Np$ and the archimedean place. Here is a simplified statement of our final result (see Theorem 10.5 for a slightly stronger result):

**Theorem.** Choose a finite set of places $S$ of $k$ containing all archimedean places and all places over $Np$. We have the following canonical exact sequence of Hecke modules:

$$0 \to \tilde{J}_\infty^{\text{ord}}(k)^{\text{ord}} \xrightarrow{\alpha} \tilde{J}_\infty^{\text{ord}}(k)^{\text{ord}} \xrightarrow{\rho_{\infty}} \tilde{J}_s^{\text{ord}}(k)^{\text{ord}} \xrightarrow{\pi_{\text{un}}} \Pi_{\mathbb{S}}^2(T_G)$$

for each $0 \leq r \in \mathbb{Z}$, where $\alpha$ is the multiplication by $\gamma^{p^r-\epsilon} - 1 \in \Lambda$ on the $\Lambda$-module $\tilde{J}_\infty^{\text{ord}}(k)$, and $\Pi_{\mathbb{S}}^2(T_G) = \text{Ker}(H^2(k^{\mathbb{S}}/k, T_G)) \to \prod_{\mathbb{S}} H^2(k_t, T_G) (j = 1, 2)$ for the maximal extension $k^{\mathbb{S}}$ of $k$ unramified outside $S$. In the sequence, the second Tate–Shafarevich group is in the quotation mark as the image of $\pi_{\text{un}}$ in $H^2(k^{\mathbb{S}}/k, T_G)$ can be slightly bigger than $\Pi_{\mathbb{S}}^2(T_G)$, up to finite error.
The exact sequence in the theorem is a Mordell–Weil analogue of a result of Nekovář in [N06, 12.7.13.4] for Selmer groups and implies that \( \hat{\mathcal{J}}_\infty(k) \) is a \( \Lambda \)-module of finite type.

Put \( \hat{\mathcal{J}}_\infty(k)_{\text{ord}} := \text{Hom}_R(\hat{\mathcal{J}}_\infty(k), \mathbb{Z}_p) \). In [H15a, Theorem 1.1], we proved the following exact sequence:

\[
\hat{\mathcal{J}}_\infty(k)_{\text{ord}, P} \overset{\alpha}{\to} \hat{\mathcal{J}}_\infty(k)_{\text{ord}, P} \to \hat{\mathcal{J}}_\varepsilon(k)_{\text{ord}, P} \to 0
\]

for arithmetic \( P \) of weight 2, in addition to the finiteness of \( \hat{\mathcal{J}}_\infty(k)_{\text{ord}} \) as a \( \Lambda \)-module. This sequence is a localization at \( P \) of the natural one. The two sequences should be dual each other if we have an \( \Lambda \)-adic version of the Néron–Tate height pairing. The theorem will be proven for more general \( \alpha \in \text{End}(\hat{\mathcal{J}}_\infty) \) in this paper (see Theorem 10.5 for the final result). The choice of \( \alpha \) will be specified in Section 7. We expect that the following fact should be true:

**Conjecture.** The second Tate–Shafarevich group \( \text{III}^2_{\hat{\mathcal{J}}}(T\mathcal{G}) \) is a torsion \( \Lambda \)-module of finite type.

The control of \( \hat{\mathcal{J}}_\infty(k)_{\text{ord}} \) quoted above has a good application described in [H15b] for the Mordell–Weil rank of abelian factors of \( \mathcal{J}_p \) \( (s = 1, 2, \ldots) \). If we are able to prove the above conjecture, we might get a more precise estimate of the Mordell–Weil rank of the abelian factors.

Here is some notation for Hecke algebras used throughout the paper. Let

\[
h_r(\mathbb{Z}) = \mathbb{Z}[T(n), U(l) : l \mid N_p, (n, N_p) = 1] \subset \text{End}(\mathcal{J}_r),
\]

and put \( h_r(R) = h_r(\mathbb{Z}) \otimes_{\mathbb{Z}} R \) for any commutative ring \( R \). Then we define \( \mathbf{h}_r = e(h_r(\mathbb{Z}_p)) \). The restriction morphism \( h_r(\mathbb{Z}) \ni h \mapsto h|_{\mathcal{J}_r} \in h_r(\mathbb{Z}) \) for \( s > r \) induces a projective system \( \{\mathbf{h}_r\} \) whose limit gives rise to a big ordinary Hecke algebra

\[
\mathbf{h} = \mathbf{h}(N) := \lim_{\Rightarrow} \mathbf{h}_r.
\]

Writing \( \langle l \rangle \) (the diamond operator) for the action of \( l \in (\mathbb{Z}/Np^s\mathbb{Z})^\times = \text{Gal}(X_r/X_0(Np^s)) \), we have an identity \( l\langle l \rangle = T(l)^2 - T(l^2) \in h_r(\mathbb{Z}_p) \) for all primes \( l \nmid N_p \). Thus we have a canonical \( \Lambda \)-algebra structure \( \Lambda = \mathbb{Z}_p[\Gamma] \hookrightarrow \mathbf{h} \). It is now well known that \( \mathbf{h} \) is a free of finite rank over \( \Lambda \) and \( \mathbf{h}_r = \mathbf{h} \otimes_{\Lambda} \Lambda/((\gamma^{p^{r-s}} - 1) \text{ (cf. [H86a])}) \). Though the construction of the big Hecke algebra is intrinsic, to relate an algebra homomorphism \( \lambda : \mathbf{h} \to \overline{\mathbb{Q}}_p \) killing \( \gamma^{p^r} - 1 \) for \( r > 0 \) to a classical Hecke eigenform, we need to fix (once and for all) an embedding \( \overline{\mathbb{Q}} \overset{i_p}{\to} \overline{\mathbb{Q}}_p \) of the algebraic closure \( \overline{\mathbb{Q}} \) in \( \mathbb{C} \) into a fixed algebraic closure \( \overline{\mathbb{Q}}_p \) of \( \mathbb{Q}_p \). We write \( i_\infty \) for the inclusion \( \overline{\mathbb{Q}} \subset \mathbb{C} \).

The following two sections Sections 3 and 4 (after a description of sheaves associated to abelian varieties) about \( U(p) \)-isomorphisms are an expanded version of a conference talk at CRM (see http://www.crm.umontreal.ca/Representations05/indexen.html) in September of 2005 which was not posted in the author’s web page, though the lecture notes of the two lectures [H05] at CRM earlier than the conference has been posted. While converting [H05] into a research article [H14], the author found an application to Mordell–Weil groups of modular Jacobians. The author is grateful for CRM’s invitation to speak.

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2. Sheaves associated to abelian varieties

Here is a general fact proven in [H15a, §2] about sheaves associated to abelian varieties. Let $0 \to A \to B \to C \to 0$ be an exact sequence of algebraic groups proper over a field $k$. The field $k$ is either a number field or a finite extension of the $l$-adic field $\mathbb{Q}_l$ for a prime $l$. We assume that $B$ and $C$ are abelian varieties. However $A$ can be an extension of an abelian variety by a finite (étale) group.

If $k$ is a number field, let $S$ be a set of places including all archimedean places of $k$ such that all members of the above exact sequence has good reduction outside $S$. We use the symbol $K$ for $k^S$ (the maximal extension unramified outside members of the above exact sequence has good reduction outside $S$). We use the symbol $\kappa$ for $k^S$ (an algebraic closure of $k$) if $k$ is a finite extension of $\mathbb{Q}_l$. A general field extension of $k$ is denoted by $\kappa$. We consider the étale topology, the smooth topology and the fppf topology on the small site over $\text{Spec}(k)$. Here under the smooth topology, covering families are made of faithfully flat smooth morphisms.

For the moment, assume that $k$ is a number field. In this case, for an extension $X$ of abelian variety defined over $k$ by a finite étale group scheme, we define $\tilde{X}(\kappa) := X(\kappa) \otimes_\mathbb{Z} \mathbb{Z}_p$ for an fppf extension $\kappa$ over $k$. By Mordell–Weil theorem (and its extension to fields of finite type over $\mathbb{Q}_l$; e.g., [RTP, IV]), we have $\tilde{X}(\kappa) = \tilde{X}(\kappa)/p^nX(\kappa)$ if $\kappa$ is a field extension of $k$ of finite type. We may regard the sequence $0 \to \tilde{A} \to \tilde{B} \to \tilde{C} \to 0$ as an exact sequence of fppf abelian sheaves over $k$ (or over any subring of $k$ over which $B$ and $C$ extends to abelian schemes). Since we find a complementary abelian subvariety $C'$ of $B$ such that $C'$ is isogenous to $C$ and $B = A + C'$ with finite $A \cap C'$, adding the primes dividing the order $|A \cap C'|$ to $S$, the intersection $A \cap C' \cong \text{Ker}(C' \to C)$ extends to an étale finite group scheme outside $S$; so, $\mathcal{C}(K) \to C(K)$ is surjective. Thus we have an exact sequence of $\text{Gal}(\kappa/k)$-modules

$$0 \to A(K) \to B(K) \to C(K) \to 0.$$ 

Note that $\tilde{A}(K) = A(K) \otimes_\mathbb{Z} \mathbb{Z}_p := \bigcup_{F} \tilde{A}(F)$ for $F$ running over all finite extensions of $k$ inside $K$. Then we have an exact sequence

$$0 \to \tilde{A}(K) \to \tilde{B}(K) \to \tilde{C}(K) \to 0.$$  

(2.1)

Now assume that $k$ is a finite extension of $\mathbb{Q}_l$. Again we use $F$ to denote a finite field extension of $k$. Then $A(F) \cong O_F^{\dim A} \oplus \Delta_F$ for a finite group $\Delta_F$ for the $l$-adic integer ring $O_F$ of $F$ (by [M55] or [T66]). Thus if $l \neq p$, $\tilde{A}(F) := \lim_{\leftarrow n} A(F)/p^nA(F) = \Delta_F \otimes_\mathbb{Z} \mathbb{Z}_p$. Recall $K = \overline{k}$. Then $\tilde{A}(K) = A[p^\infty](K)$ (for $A[p^\infty] := \lim_{\leftarrow n} A[p^n]$ with $A[p^n] = \text{Ker}(p^n : A \to A)$); so, defining $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ by $A[p^\infty]$, $B[p^\infty]$ and $C[p^\infty]$ as fppf abelian sheaves, we again have the exact sequence (2.1) of $\text{Gal}(\overline{k}/k)$-modules:

$$0 \to \tilde{A}(K) \to \tilde{B}(K) \to \tilde{C}(K) \to 0$$ 

and an exact sequence of fppf abelian sheaves

$$0 \to \tilde{A} \to \tilde{B} \to \tilde{C} \to 0$$

whose value at a finite field extension $\kappa/\mathbb{Q}_l$ coincides with $\tilde{X}(\kappa) = \lim_{\leftarrow n} X(\kappa)/p^nX(\kappa)$ for $X = A, B, C$.

Suppose $l = p$. For any finite $M$, we define $M^{(p)}$ by the maximal prime-to-$p$ torsion submodule of $M$. For $X = A, B, C$ and an fppf extension $R/k$, the sheaf $R \mapsto X^{(p)}(R) = \lim_{\leftarrow^{\text{proj}}} \tilde{X}[N](R)$ for any finite field extension $F/k$, over the étale site on $k$, $\tilde{X}$ is the sheaf associated to a presheaf $R \mapsto O_F^{\dim X} \oplus X[p^\infty](R)$. If $X$ has semi-stable reduction over $O_F$, we have $\tilde{X}(F) = X^0(O_F) + X[p^\infty](F) < X(F)$ for the formal group $X^0$ of the identity connected component of the Néron model of $X$ over $O_F$ [T66]. Since $X$ becomes semi-stable over a finite Galois extension $F_0/k$, in general $\tilde{X}(F) = H^0(\text{Gal}(F_0/F), X(F_0F))$ for any finite extension $F/K$ (or more generally for each finite étale extension $F/k$); so, $F \mapsto \tilde{X}(F)$ is a sheaf over the étale site on $k$. Thus by [ECH, II.1.5], the sheafification coincides over the étale site with the presheaf $F \mapsto \tilde{X}(F) = \lim_{\leftarrow n} X(F)/p^nX(F)$. Thus we conclude $\tilde{X}(F) = \lim_{\leftarrow n} X(F)/p^nX(F)$ for any
étagé finite extensions $F/k$. Moreover $\tilde{X}(K) = \bigcup_{K/F/k} \tilde{X}(F)$. Applying the snake lemma to the commutative diagram with exact rows (in the category of fppf abelian sheaves):

$$
\begin{array}{ccc}
A^{(p)} & \to & B^{(p)} \\
\cap & \downarrow & \cap \\
A & \to & B
\end{array}
$$

the cokernel sequence gives rise to an exact sequence of fppf abelian sheaves over $k$:

$$
0 \to \tilde{A} \to \tilde{B} \to \tilde{C} \to 0
$$

and an exact sequence of $\text{Gal}(\overline{k}/k)$-modules

$$
0 \to \tilde{A}(K) \to \tilde{B}(K) \to \tilde{C}(K) \to 0.
$$

In this way, we extended the sheaves $\tilde{A}, \tilde{B}, \tilde{C}$ to an fppf abelian sheaves keeping the exact sequence $\tilde{A} \to \tilde{B} \to \tilde{C}$ intact. However note that our way of defining $\tilde{X}$ for $X = A, B, C$ depends on the base field $k = \mathbb{Q}, \mathbb{Q}_p, \mathbb{Q}_l$. Here is a summary for fppf algebras $R/k$:

$$
\tilde{X}(R) = \begin{cases} 
X(R) \otimes_{\mathbb{Z}} \mathbb{Z}_p & \text{if } [k : \mathbb{Q}] < \infty, \\
X[p^\infty](R) & \text{if } [k : \mathbb{Q}_p] < \infty, \\
(X/X^{(p)})(R) & \text{as a sheaf quotient, if } [k : \mathbb{Q}_p] < \infty. 
\end{cases}
$$

Here is a sufficient condition when $\tilde{X}(\kappa)$ is given by the projective limit: $\lim \limits_n X(\kappa)/p^n X(\kappa)$ for $X = A, B$ or $C$:

$$
(\mathrm{S})
\tilde{X}(\kappa) = \lim \limits_n \tilde{X}(\kappa)/p^n \tilde{X}(\kappa)
\text{ if } \begin{cases} 
[k : \mathbb{Q}] < \infty \text{ and } \kappa \text{ is a field of finite type over } k \\
[k : \mathbb{Q}_p] < \infty \text{ with } l \neq p \text{ and } \kappa \text{ is a field of finite type over } k \\
[k : \mathbb{Q}_p] < \infty \text{ and } \kappa \text{ is a finite algebraic extension over } k.
\end{cases}
$$

A slightly weaker sufficient condition for $\tilde{X}(\kappa) = \lim \limits_n \tilde{X}(\kappa)/p^n \tilde{X}(\kappa)$ is proven in [H15a, Lemma 2.1].

For a sheaf $X$ under the topology $\mathbb{?}$, we write $H^*(X)$ for the cohomology group $H^*(\text{Spec}(\kappa), X)$ under the topology $\mathbb{?}$. If we have no subscript, $H^1(X)$ means the Galois cohomology $H^*(\text{Gal}(K/k), X)$ for the $\text{Gal}(K/k)$-module $X$.

The following fact is proven in [H15a, Lemma 2.2] (where it was proven for finite $S$ but the same proof works for infinite $S$ as is obvious from the fact that it works under fppf topology):

**Lemma 2.1.** Let $X$ be an extension of an abelian variety over $k$ by a finite étale group scheme of order prime to $p$. For any intermediate extension $K/k$, we have a canonical injection

$$
\lim \limits_n \tilde{X}(\kappa)/p^n \tilde{X}(\kappa) \hookrightarrow \lim \limits_n H^1(\text{Gal}(K/k), X[p^n]).
$$

Similarly, for any fppf or smooth extension $\kappa/k$ of finite type which is an integral domain, we have an injection

$$
\lim \limits_n \tilde{X}(\kappa)/p^n \tilde{X}(\kappa) \hookrightarrow \lim \limits_n H^1(X[p^n]).
$$

for $\mathbb{?} = \text{fppf or sm}$ according as $\kappa/k$ is an fppf extension or a smooth extension. For Galois cohomology, we have an exact sequence for $j = 0, 1$:

$$
0 \to \lim \limits_n H^j(\tilde{X}(\kappa))/p^n H^j(\tilde{X}(\kappa)) \to \lim \limits_n H^{j+1}(X[p^n]) \to \lim \limits_n H^{j+1}(X[p^n]).
$$

The natural map: $\lim \limits_n H^{j+1}(X[p^n]) \to \lim \limits_n H^{j+1}(X)[p^n]$ is surjective if either $k$ is local or $S$ is finite. In particular, $H^1(T_p X)$ for $T_p X = \lim \limits_n X[p^n]$ is equal to $\lim \limits_n H^1(X[p^n])$, and

$$
0 \to \tilde{X}(\kappa) \to H^1(T_p X) \to \lim \limits_n H^1(X)[p^n] \to 0
$$

is exact.
The above lemma is valid without finiteness of $S$ except for the surjectivity of $\pi$. As for the surjectivity of $\pi$, see the proof of the following lemma. Similarly we have

**Lemma 2.2.** Let $R = k$ if $k$ is local, and let $R$ be the $S$-integer ring of $k$ (i.e., primes in $S$ is inverted in $R$) if $k$ is a number field. Let $X$ be an étale $p$-divisible sheaf over Spec($R$) (i.e., $X \rightarrow \lim_{n} X[p^n]$ is surjective in the abelian category of étale sheaves). Suppose that $X[p^n](K)$ is a finite module for all $n$. Then for any intermediate extension $K/k$, $H^j(\mathbb{T}_p X)$ for the Tate module $\mathbb{T}_p X = \lim_n X[p^n]$ is equal to $\lim_n H^1(X[p^n])$, and

$$0 \rightarrow \lim_n H^j(X(\kappa))/p^n H^j(X(\kappa)) \rightarrow H^{j+1}(\mathbb{T}_p X) \rightarrow \lim_n H^{j+1}(X)[p^n]$$

is exact for $j = 0, 1$. In particular,

$$0 \rightarrow \lim_n X(\kappa)/p^n X(\kappa) \rightarrow H^1(\mathbb{T}_p X) \rightarrow \lim_n H^1(\mathbb{X})[p^n]$$

is exact. The natural map $H^1(\mathbb{T}_p X) \rightarrow \lim_n H^1(\mathbb{X})[p^n]$ is onto if either $k$ is local or $S$ is finite.

**Proof.** By $p$-divisibility, we have the sheaf exact sequence under the étale topology over Spec($k$)

$$0 \rightarrow X[p^n] \rightarrow X \stackrel{p^n}{\rightarrow} X \rightarrow 0.$$ 

This implies, we have an exact sequence

$$0 \rightarrow X[p^n](K) \rightarrow X(K) \stackrel{p^n}{\rightarrow} X(K) \rightarrow 0.$$ 

By the long exact sequence associated to this sequence, we have exactness of

$$(*) \rightarrow H^j(X(\kappa))/p^n H^j(X(\kappa)) \rightarrow H^{j+1}(X[p^n]) \rightarrow H^{j+1}(X)[p^n] \rightarrow 0.$$ 

Passing to the limit (with respect to $n$), we have the exactness of

$$0 \rightarrow \lim_n H^j(X(\kappa))/p^n H^j(X(\kappa)) \rightarrow H^{j+1}(\mathbb{T}_p X) \rightarrow \lim_n H^{j+1}(\mathbb{X})[p^n].$$

as $\lim_n H^j(X[p^n]) = H^j(\lim_n X[p^n]) = H^j(\mathbb{T}_p X)$ for $j = 1, 2$ (because of finiteness of $X[p^n](K)$ and $p$-divisibility of $X$; e.g., [CNF, Corollary 2.7.6] and [H15c, Lemma 7.1 (2)]). If $k$ is local or $S$ is finite, by Tate duality, all the terms of $(*)$ is finite; so, the surjectivity of $(*)$ is kept after passing to the limit. \hfill $\square$

For finite $S$, the following module structure of $H^1(\overline{A})$ is well known (see [ADT, Corollary I.4.15] or [H15a, Lemma 2.3]):

**Lemma 2.3.** Let $k$ be a finite extension of $\mathbb{Q}$ or $\mathbb{Q}_l$ for a prime $l$. Suppose that $S$ is finite if $k$ is a finite extension of $\mathbb{Q}$. Let $A$ be an abelian variety. Then $H^1(A) / \mathbb{Z}_p = H^1(\overline{A})$ is isomorphic to the discrete module $(\mathbb{Q}_p/\mathbb{Z}_p)^r \oplus \Delta$ for a finite $r \geq 0$ and a finite $p$-torsion group $\Delta$.

Hereafter we assume that $S$ is a finite set unless otherwise indicated.

3. $U(p)$-ISOMORPHISMS FOR GROUP COHOMOLOGY

For $\mathbb{Z}[U]$-modules $X$ and $Y$, we call a $\mathbb{Z}[U]$-linear map $f : X \rightarrow Y$ a $U$-injection (resp. a $U$-surjection) if $\text{Ker}(f)$ is killed by a power of $U$ (resp. $\text{Coker}(f)$ is killed by a power of $U$). If $f$ is an $U$-injection and $U$-surjection, we call $f$ a $U$-isomorphism. If $X \rightarrow Y$ is a $U$-isomorphism, we write $X \cong_U Y$. In terms of $U$-isomorphisms (for $U = U(p), U^+(p)$, we describe briefly the facts we need in this article (and in later sections, we fill in more details in terms of the ordinary projector $e$ and the co-ordinary projector $e^+ := \lim_{n \rightarrow \infty} U^+(p)^{n+1}$).

Let $N$ be a positive integer prime to $p$. We consider the (open) modular curve $Y_1(Np^r)/\mathbb{Q}$ which classifies elliptic curves $E$ with an embedding $\phi : \mu_{p^r} \hookrightarrow E[p^r] = \text{Ker}(p^r : E \rightarrow E)$ of finite flat groups. Let $R = \mathbb{Z}_p[\mu_{p^r}]$ and $K_1 = \mathbb{Q}[[\mu_{p^r}]]$. For a valuation subring or a subfield $R$ of $\mathbb{K}_t$ over $\mathbb{Z}_p$ with quotient field $K$, we write $X_{p^r/R}$ for the normalization of the $j$-line $\mathbb{P}(j)/R$ in the function field of $Y_1(Np^r)/K$. The group $e \in (\mathbb{Z}/p^n\mathbb{Z})^t$ acts on $X_\phi$ by $\phi \mapsto \phi \circ z$, as $\text{Aut}(\mu_{Np^r}) \cong (\mathbb{Z}/Np^r\mathbb{Z})^t$. 


Thus $\Gamma = 1 + p^\mathbb{Z}p = \gamma_{2p}^r$ acts on $X_r$ (and its Jacobian) through its image in $(\mathbb{Z}/Np^r\mathbb{Z})^\times$. Hereafter we take $U = U(p)$ for the Hecke–Atkin operator $U(p)$.

Let $J_{r/R} = \text{Pic}^0_{X_{r/R}}$ be the connected component of the Picard scheme. We state a result comparing $J_{r/R}$ and the Néron model of $J_{r/K}$ over $R$. Thus we assume that $R$ is a valuation ring. By [AME, 13.5.6, 13.11.4], $X_{r/R}$ is regular; the reduction $X_r \otimes_R \mathbb{F}_p$ is a union of irreducible components, and the component containing the $\infty$-cusps has geometric multiplicity 1. Then by [NMD, Theorem 9.5.4], $J_{r/R}$ gives the identity connected component of the Néron model of the Jacobian of $X_{r/R}$.

We write $X_s^r / R$ for the normalized of the $r$-line of the canonical $\mathbb{Q}$-curve associated to the modular curve for the congruence subgroup $\Gamma_s^r = \Gamma_1(Np^r) \cap \Gamma_0(p^r)$ for $0 < r \leq s$. We denote $\text{Pic}^0_{X_{s/R}}$ by $J^r_{s/R}$. Similarly, as above, $J^r_{s/K}$ is the connected component of the Néron model of $X_{s/K}$.

Note that, for $\alpha = \left( \begin{array}{c} 1 \\ 0 \end{array} \right)$, we take the component containing the Picard scheme. Then we have the following commutative diagram from the above identity, first over $\mathbb{C}$, then over $K$ and by Picard functoriality over $R$:

$$
\begin{array}{ccc}
J_{r/R} & \xrightarrow{\pi^r} & J^r_{s/R} \\
\downarrow u & \nearrow u' & \downarrow u'' \\
J_{r/R} & \xrightarrow{\pi^s} & J^s_{s/R}
\end{array}
$$

where the middle $u'$ is given by $U_r^s(p^{s-r})$ and $u$ and $u''$ are $U_r^s(p^{s-r})$. Thus

$$(u1) \; \pi^r : J_{r/R} \to J^r_{s/R} \text{ is a } U(p) \text{-isomorphism (for the projection } \pi : X_s^r \to X_r).$$

Taking the dual $U^*(p)$ of $U(p)$ with respect to the Rosati involution associated to the canonical polarization of the Jacobians, we have a dual version of the above diagram for $s > r > 0$:

$$
\begin{array}{ccc}
J_{r/R} & \xrightarrow{\pi^s_r} & J^s_{s/R} \\
\downarrow u^* & \nearrow u'^* & \downarrow u''^* \\
J_{r/R} & \xrightarrow{\pi^r_s} & J^r_{s/R}
\end{array}
$$

Here the superscript “*” indicates the Rosati involution of the canonical divisor of the Jacobians, and $u^* = U^*(p)^{s-r}$ for the level $\Gamma_1(Np^s)$ and $u''^* = U^*(p)^{s-r}$ for $\Gamma_s^r$. Note that these morphisms come from the following coset decomposition, for $\beta_m := \left( \begin{array}{c} p^m \\ 0 \\ 1 \end{array} \right) \Gamma_1(Np^r)$,

$$
\Gamma_s^r \backslash \Gamma_1(Np^s) \beta_{s-r} \Gamma_1(Np^r) = \left\{ \left( \begin{array}{c} p^{s-r} \\ a \\ 0 \\ 1 \end{array} \right) \bigg| a \mod p^{s-r} \right\} = \Gamma_1(Np^s) \backslash \Gamma_1(Np^r) \beta_{s-r} \Gamma_1(Np^r).
$$

From this, we get

$(u^*1) \; \pi^s_r : J_{r/R} \to J^r_{s/R}$ is a $U^*(p)$-isomorphism, where $\pi^s_r$ is the dual of $\pi^r$.

In particular, if we take the ordinary and the co-ordinary projector $e = \lim_{n \to \infty} U(p)^n$ and $e^* = \lim_{n \to \infty} U^*(p)^n$ on $J[p^n]$ for $J = J_{r/R}, J^r_{s/R}$, noting $U(p)^m = U(p)^m$, we have

$$
\pi^* : J_{r/R}[p^n] \cong J_{s/R}[p^n] \text{ and } \pi^* : J^r_{s/R}[p^n] \cong J^s_{s/R}[p^n]
$$

where “ord” indicates the image of the projector $e$. For simplicity, we write $G_{r/R} := J_{r/R}[p^n]/R$.

Pick a congruence subgroup $\Gamma$ defining the modular curve $X(\mathbb{C}) = \Gamma \backslash (\mathbb{H} \sqcup \tilde{\mathbb{P}}^1(\mathbb{Q}))$, and write its Jacobian as $J$. We now identify $J(\mathbb{C})$ with a subgroup of $H^1(\Gamma, T)$ (for the trivial $\Gamma$-module $T := R/\mathbb{Z} \cong \{ z \in \mathbb{C}^\times : |z| = 1 \}$ with trivial $\Gamma$-action). Since $\Gamma_s^r >> \Gamma_1(Np^r)$, consider the finite cyclic quotient group $C := \frac{\Gamma_s^r}{\Gamma_1(Np^r)}$. By the inflation restriction sequence, we have the following
commutative diagram with exact rows:

\[
\begin{array}{cccccc}
H^1(C, \mathbf{T}) & \longrightarrow & H^1(\Gamma_1, \mathbf{T}) & \longrightarrow & H^1(\Gamma_m, \mathbf{T}) & \twoheadrightarrow H^2(C, \mathbf{T}) \\
\uparrow & & \cup & & \uparrow & \uparrow \\
? & \longrightarrow & J'_s(\mathbb{C}) & \longrightarrow & J_s(\mathbb{C})[\gamma^{p^{-}} - 1] & \longrightarrow ?. \\
\end{array}
\]

Since \( C \) is a finite cyclic group of order \( p^{*} \) (with generator \( g \)) acting trivially on \( \mathbf{T} \), we have \( H^1(C, \mathbf{T}) = \text{Hom}(\mathbf{T}, \mathbb{C}) \cong \mathbb{C} \) and

\[
H^2(C, \mathbf{T}) = \mathbf{T}/(1 + g + \cdots + g^{p^{*} - 1}) = \mathbf{T}/p^{*} = 0.
\]

By the same token, replacing \( \mathbf{T} \) by \( \mathbb{G}_m \), we get \( H^2(\mathbb{G}_m, \mathbb{Z}/p) = 0 \). By a sheer computation (cf. [H86b, Lemma 6.1]), we confirm that \( U(p) \) acts on \( H^1(C, \mathbf{T}) \) and \( H^1(C, \mathbb{G}_m) \) via multiplication by its degree \( p \), and hence \( U(p)^{\cdot} \) kill \( H^1(C, \mathbf{T}) \) and \( H^1(C, \mathbb{G}_m) \). Hence \( J'_s \to J_s \) is a \( U \)-isomorphism over \( \mathbb{Z} \) and hence over \( K \). We record what we have proven:

\[
U(p)^{\cdot} H^1(C, \mathbb{G}_m) = H^2(C, \mathbf{T}) = H^2(C, \mathbb{G}_m) = 0.
\]

This fact has been exploited by the author (for example, [H86b] and [H14]) to study the modular Barsotti–Tate groups \( J_s[p^{*}] \).

4. \( U(p) \)-ISOMORPHISMS FOR ARITHMETIC COHOMOLOGY

To good extent, we reproduce the results and proofs in [H15a, §3] as it is important in the sequel. Let \( X \to Y \to S \) be proper morphisms of noetherian schemes. We now replace \( H^1(\Gamma, \mathbf{T}) \) in the above diagram (3.5) by

\[
H^p_{\text{fppf}}(T, R^1f_*\mathbb{G}_m) = R^1f_*O^\chi_X(T) = \text{Pic}_{X/S}(T)
\]

for \( S \)-scheme \( T \) and the structure morphism \( f : X \to S \), and do the same analysis as in Section 3 for arithmetic cohomology in place of group cohomology (via the moduli theory of Katz-Mazur and Drinfeld; cf., [AME]). Write the morphisms as \( X \to Y \to S \) with \( f = g \circ s \). Assume that \( \pi \) is finite flat.

Suppose that \( f \) and \( g \) have compatible sections \( s_g \to Y \to S \) and \( s_f \to X \) so that \( s \circ s_f = s_g \). Then we get (e.g., [NMD, Section 8.1])

\[
\text{Pic}_{X/S}(T) = \text{Ker}(s^1_f : H^1_{\text{fppf}}(X_T, O^\chi_X) \to H^1_{\text{fppf}}(T, O^\chi_T))
\]

\[
\text{Pic}_{Y/S}(T) = \text{Ker}(s^1_g : H^1_{\text{fppf}}(Y_T, O^\chi_Y) \to H^1_{\text{fppf}}(T, O^\chi_T))
\]

for any \( S \)-scheme \( T \), where \( s^1_f : H^q(X_T, O^\chi_X) \to H^q(T, O^\chi_T) \) and \( s^1_g : H^n(Y_T, O^\chi_Y) \to H^n(T, O^\chi_T) \) are morphisms induced by \( s_f \) and \( s_g \), respectively. Here \( X_T = X \times_ST \) and \( Y_T = Y \times_ST \). We suppose that the functors \( \text{Pic}_{X/S} \) and \( \text{Pic}_{Y/S} \) are representable by smooth group schemes (for example, if \( X, Y \) are curves and \( S = \text{Spec}(k) \) for a field \( k \); see [NMD, Theorem 8.2.3 and Proposition 8.4.2]). We then put \( J_f = \text{Pic}_{Y/S}(T) = \{X = Y, Y \} \). Anyway we suppose hereafter also that \( X, Y, S \) are varieties (in the sense of [ALG, II.4]).

For an fppf covering \( U \to Y \) and a presheaf \( P = P_Y \) on the fppf site over \( Y \), we define via Čech cohomology theory an fppf presheaf \( U \to H^*(U, P) \) denoted by \( H^*(F_Y) \) (see [ECH, III.2.2 (b)]). The inclusion functor from the category of fppf sheaves over \( Y \) into the category of fppf presheaves over \( Y \) is left exact. The derived functor of this inclusion of an fppf sheaf \( F = F_Y \) is denoted by \( H^*(F_Y) \) (see [ECH, III.1.5 (c)]). Thus \( H^n_{\text{fppf}}(U, O^\chi_U) = H^n_{\text{fppf}}(U, O^\chi_U) \) for a \( Y \)-scheme \( U \) as a presheaf (here \( U \) varies in the small fppf site over \( Y \).

Instead of the Hochschild-Serre spectral sequence producing the top row of the diagram (3.5), assuming that \( f \), \( g \), and \( \pi \) are all faithfully flat of finite presentation, we use the spectral sequence of Čech cohomology of the flat covering \( \pi : X \to Y \) in the fppf site over \( Y \) [ECH, III.2.7]:

\[
H^p(X_T/Y_T, \underline{H}^q(\mathbb{G}_m/Y)) \Rightarrow H^p_{\text{fppf}}(Y_T, O^\chi_{Y_T}) \Rightarrow H^p(Y_T, O^\chi_{Y_T})
\]

for each \( S \)-scheme \( T \). Here \( F \to H^p_{\text{fppf}}(Y_T, F) \) (resp. \( F \to H^p(Y_T, F) \)) is the right derived functor of the global section functor: \( F \to F(Y_T) \) from the category of fppf sheaves (resp. Zariski sheaves)
over $Y_T$ to the category of abelian groups. The canonical isomorphism $i$ is the one given in [ECH, III.4.9].

By the sections $s_?$, we have a splitting $H^q(X_T, O^\times_{X_T}) = \text{Ker}(s_?^q) \oplus H^q(T, O^\times_T)$ and $H^n(Y_T, O^\times_{Y_T}) = \text{Ker}(s_?^n) \oplus H^n(T, O^\times_T)$. Write $H^\bullet_{Y_T}$ for $H^\bullet_{\mathcal{G}_m/Y_T}$ and $H^\bullet_{Y_T}$ for $H^\bullet_{Y_T}$ for $H^\bullet_{Y_T}(T)$.

Since $\text{Pic}_{X/S}(T) = \text{Ker}(s^1_{1,T} : H^1(X_T, O^\times_{X_T}) \to H^1(T, O^\times_T))$

for the morphism $f : X \to S$ with a section [NMD, Proposition 8.1.4], from this spectral sequence, we have the following commutative diagram with exact rows:

$$
\begin{array}{cccccc}
H^1(H^0_{Y_T}) & \longrightarrow & H^1(T, O^\times_T) & \oplus & \text{Ker}(s^1_{1,T}) & \longrightarrow & H^0(X^\times_T, H^1(Gm_Y)) & \longrightarrow & H^2(H^0_{Y_T}) \\
\| & & \iota & & \| & & \| & & \\
\| & & ?_1 & & \downarrow & & \uparrow & & \\
(4.2) & \longrightarrow & \text{Pic}_T \oplus \text{Pic}_{Y/S}(T) & \longrightarrow & H^0(X^\times_T, \text{Pic}_Y(T)) & \longrightarrow & H^2(H^0_{Y_T}) \\
\| & & \uparrow & & \uparrow & & \| & & \\
?_1 & & \text{Pic}_T \oplus J_Y(T) & \longrightarrow & \text{Pic}_T \oplus H^0(X^\times_T, J_X(T)) & \longrightarrow & \| \\
\end{array}
$$

where we have written $J_T = \text{Pic}^0_{Y/S}$ (the identity connected component of $\text{Pic}_{Y/S}$). Here the horizontal exactness at the top two rows follows from the spectral sequence (4.1) (see [ECH, Appendix B]).

Take a correspondence $U \subset Y \times_S Y$ given by two flat finite projections $\pi_1, \pi_2 : U \to Y$ of constant degree (i.e., $\pi_j, O_U$ is locally free of finite rank $\text{deg}(\pi_j)$ over $O_Y$). Consider the pullback $U_X \subset X \times_S X$ given by the Cartesian diagram:

$$
\begin{array}{cccc}
U_X = U \times_{Y \times Y} (X \times_S X) & \longrightarrow & X \times_S X \\
\downarrow & & \downarrow \\
U & \longrightarrow & Y \times Y \\
\end{array}
$$

Let $\pi_{j,X} = \pi_j \times_S \pi : U_X \to X$ ($j = 1, 2$) be the projections.

Consider a new correspondence $U^{(q)}_X = U_{X \times X \times X \cdots \times X}$, whose projections are the iterated product

$$
\pi_{j,X}^{(q)} = \pi_{j,X} \times \cdots \times \pi_{j,X} : U^{(q)}_X \to X^{(q)} (j = 1, 2).
$$

Here is a first step to prove a result analogous to (3.6) for arithmetic cohomology.

**Lemma 4.1.** Let the notation and the assumption be as above. In particular, $\pi : X \to Y$ is a finite flat morphism of geometrically reduced proper schemes over $S = \text{Spec}(k)$ for a field $k$. Suppose that $X$ and $U_X$ are proper schemes over a field $k$ satisfying one of the following conditions:

1. $U_X$ is geometrically reduced, and for each geometrically connected component $X^\circ$ of $X$, its pull back to $U_X$ by $\pi_{2,X}$ is also connected; i.e., $\text{deg}(\pi_{2,X}) = 0$.

2. $(f \circ \pi_{2,X})_* O_{U_X} = f_* O_X$.

If $\pi_2 : U \to Y$ has constant degree $\text{deg}(\pi_2)$, then, for each $q > 0$, the action of $U^{(q)}_X$ on $H^0(X, O^\times_{X^{(q)}})$ factors through the multiplication by $\text{deg}(\pi_2) = \text{deg}(\pi_{2,X})$.

This result is given as [H15a, Lemma 3.1, Corollary 3.2].

To describe the correspondence action of $U$ on $H^0(X, O^\times_X)$ in down-to-earth terms, let us first recall the Čech cohomology: for a general $S$-scheme $T$,

$$
\begin{align*}
\bar{H}^q\left(\frac{X_T}{Y_T}, H^0(Gm/Y)\right) &= \{(c_{i_0, \ldots, i_q})_{i_0, \ldots, i_q} \in H^0(X^{(q+1)}_T, O^\times_{X^{(q+1)}_T}) \cup \prod_j (p_{i_0, \ldots, i_q})^{(-1)^j} = 1 \}
\end{align*}
$$

where

$$
\begin{align*}
\{db_{i_0, \ldots, i_q} = \prod_j (b_{i_0, \ldots, i_q} \circ p_{i_0, \ldots, i_q})^{(-1)^j} | b_{i_0, \ldots, i_q} \in H^0(X^{(q)}_T, O^\times_{X^{(q)}_T})\}
\end{align*}
$$
where we agree to put \( H^0(X_T^{(0)}, O_X^{(0)}) = 0 \) as a convention,

\[
X_T^{(q)} = X \times_Y X \times_Y \cdots \times_Y X \times_{S T} T, O_X^{(q)} = O_X \times_{O_Y} O_X \times_{O_Y} \cdots \times_{O_Y} O_X \times_{O_S} O_T,
\]

the identity \( \prod (c \circ p_{i_0, \ldots, i_{q+1}})^{-1} \) takes place in \( O_{X_T^{(q+2)}} \) and \( p_{i_0, \ldots, i_{q+1}} : X_T^{(q+2)} \to X_T^{(q+1)} \) is the projection to the product of \( X \) the \( j \)-th factor removed. Since \( T \times_T T \equiv T \) canonically, we have \( X_T^{(q)} \equiv X_T \times \cdots \times X_T \times T \) by transitivity of fiber product.

Consider \( \alpha \in H^0(X, O_X) \). Then we lift \( \pi_1^* X \alpha = \alpha \circ \pi_1, X \in H^0(U_X, O_{U_X}) \). Put \( \alpha_U := \pi_1^* X \alpha \). Note that \( \pi_2, X, O_{U_X} \) is locally free of rank \( d = \deg(X_2) \) over \( O_X \), the multiplication by \( \alpha_U \) has its characteristic polynomial \( P(T) \) of degree \( d \) with coefficients in \( O_X \). We define the norm \( N_U(\alpha_U) \) to be the constant term \( P(0) \). Since \( \alpha \) is a global section, \( N_U(\alpha_U) \) is a global section, as it is defined everywhere locally. If \( \alpha \in H^0(X, O_X^*) \), \( N_U(\alpha_U) \in H^0(X, O_X^*) \). Then define \( U(\alpha) = N_U(\alpha_U) \), and in this way, \( U \) acts on \( H^0(X, O_X^*) \).

For a degree \( q \) Čech cohomology class \( [\epsilon] \in \check{H}^q(X/Y, \check{H}^0(G_m/Y)) \) with a Čech q-cocycle \( c = (c_{i_0, \ldots, i_q}) \), \( U([\epsilon]) \) is given by the Čech cocycle class \( U(c) = (U(c_{i_0, \ldots, i_q})) \), where \( U(c_{i_0, \ldots, i_q}) \) is the image of the global section \( c_{i_0, \ldots, i_q} \) under \( U \). Indeed, \( (\pi_1^* X c_{i_0, \ldots, i_q}) \) plainly satisfies the cocycle condition, and \( (N_U(\pi_1^* X c_{i_0, \ldots, i_q})) \) is again a Čech cocycle as \( N_U \) is a multiplicative homomorphism. By the same token, this operation sends coboundaries to coboundaries, and define the action of \( U \) on the Čech cohomology group. We get the following vanishing result (cf. (3.6)):

**Proposition 4.2.** Suppose that \( S = \text{Spec}(k) \) for a field \( k \). Let \( \pi : X \to Y \) be a finite flat covering of (constant) degree \( d \) of geometrically reduced proper varieties over \( k \), and let \( \gamma : U \to U \) be two finite flat coverings of (constant degree) identifying the correspondence \( U \) with a closed subscheme \( Y \to Y \times_S Y \). Write \( \pi : U_X = U \times_Y X \to Y \) be the base-change to \( Y \). Suppose one of the conditions (1) and (2) of Lemma 4.1 for \( (X, U) \). Then

1. The correspondence \( U \subseteq Y \times_S Y \) sends \( \check{H}^q(H^0) \) into \( \deg(\pi_2)(\check{H}^q(H^0)) \) for all \( q > 0 \).
2. If \( d \) is a p-power and \( \deg(\pi_2) \) is divisible by \( p \), \( \check{H}^q(H^0) \) for \( q > 0 \) is killed by \( U^M \) if \( p^M \geq d \).
3. The cohomology \( \check{H}^q(H^0) \) with \( q > 0 \) is killed by \( d \).

This follows from Lemma 4.1, because on each Čech q-cocycle (whose value is a global section of iterated product \( X_T^{(q+1)} \)), the action of \( U \) is given by \( U^{(q+1)} \) by (4.3). See [H15a, Proposition 3.3] for a detailed proof. We can apply the above proposition to \( (U, X, Y) = (U(p), X_s, X_T^p) \) with \( U \) given by \( U(p) \subset X_T^p \times X_T^p \) corresponds to \( \pi \) \( \Gamma_p \) given by \( \Gamma = \Gamma_1(Np^r) \cap \Gamma_0(p)^{p+1} \) and \( U \) \( X_s \) is given by \( X(\Gamma) \) \( \Gamma' = \Gamma_1(Np^r) \cap \Gamma_0(p+1) \) both geometrically irreducible curves. In this case \( \pi \) is induced by \( z \to z^p \) on the upper complex plane and \( \pi_2 \) is the natural projection of degree \( p \). In this case, \( \deg(X_s/X_T^p) = p^{n-r} \) and \( \deg(\pi_2) = p \).

An easy criterion to see \( \pi^0(U_X^p) = \pi^0(X(q)) \) (which will not be used in this paper), we can offer

**Lemma 4.3.** For a finite flat covering \( V \to X \to Y \) of geometrically irreducible varieties over a field \( k \), if a fiber \( f \circ \pi \) of a k-closed point \( y \) of \( V \) is made of a single closed point \( v \) in \( V(k) \) (as a topological space), then \( V^{(q)} := V \times_Y V \times_Y \cdots \times_Y V \) and \( X^{(q)} \) are geometrically connected.

**Proof.** The q-fold tensor product of the stalks at \( v \) by

\[
\mathcal{O}_{V,v}^{(q)} := \mathcal{O}_{V,v} \otimes \mathcal{O}_{Y,v} \mathcal{O}_{V,v} \otimes \mathcal{O}_{Y,v} \cdots \otimes \mathcal{O}_{Y,v} \mathcal{O}_{V,v}
\]

is a local ring whose residue field is that of \( y \). This fact holds true for the base change \( V/k' \to X/k' \to Y/k' \) for any algebraic extension \( k'/k \); so, \( V^{(q)} \) and \( X^{(q)} \) are geometrically connected.
Assume that a finite group $G$ acts on $X/Y$ faithfully. Then we have a natural morphism $\phi : X \times G \to X \times_Y X$ given by $\phi(x, \sigma) = (x, \sigma(x))$. In other words, we have a commutative diagram

$$
\begin{array}{ccc}
X \times G & \xrightarrow{(x, \sigma) \mapsto x} & X \\
\downarrow & & \downarrow \\
X \quad & \longrightarrow & Y,
\end{array}
$$

which induces $\phi : X \times G \to X \times_Y X$ by the universality of the fiber product. Suppose that $\phi$ is surjective; for example, if $Y$ is a geometric quotient of $X$ by $G$; see [GME, §1.8.3]). Under this map, for any fppf abelian sheaf $F$, we have a natural map $\hat{H}^0(X/Y, F) \to H^0(G, F(X))$ sending a Čech 0-cocycle $c \in H^0(X, F) = F(X)$ (with $p_1^*c = p_2^*c$) to $c \in H^0(G, F(X))$. Obviously, by the surjectivity of $\phi$, the map $H^0(X/Y, F) \to H^0(G, F(X))$ is an isomorphism (e.g., [ECH, Example III.2.6, page 100]). Thus we get

**Lemma 4.4.** Let the notation be as above, and suppose that $\phi$ is surjective. For any scheme $T$ fppf over $S$, we have a canonical isomorphism: $\hat{H}^0(X_T/Y_T, F) \cong H^0(G, F(X_T))$.

We now assume $S = \text{Spec}(k)$ for a field $k$ and that $X$ and $Y$ are proper reduced connected curves. Then we have from the diagram (4.2) with the exact middle two columns and exact horizontal rows:

$$
\begin{array}{cccccc}
0 & \longrightarrow & Z & \longrightarrow & Z & \longrightarrow & 0 \\
\uparrow & & \text{deg} \uparrow \text{onto} & & \text{deg} \uparrow \text{onto} & & \uparrow \\
\hat{H}^1(H^0_F) & \longrightarrow & \text{Pic}_Y/S(T) & \xrightarrow{b} & \hat{H}^0(T, \text{Pic}_Y/S(T)) & \longrightarrow & \hat{H}^2(H^0_F) \\
\uparrow & & \cup & & \cup & & \uparrow \\
?_1 & \longrightarrow & J_Y(T) & \xrightarrow{c} & \hat{H}^0(T, J_X(T)) & \longrightarrow & ?_2,
\end{array}
$$

Thus we have $?_j = \hat{H}^j(H^0_F)$ ($j = 1, 2$).

By Proposition 4.2, if $q > 0$ and $X/Y$ is of degree $p$-power and $p \mid \text{deg}(\pi_2)$, $\hat{H}^q(H^0_F)$ is a $p$-group, killed by $U^M$ for $M \gg 0$. Taking $(X, Y, U)/S$ to be $(X_{s/Q}, X'_{s/Q}, U(p))/Q$ for $s > r \geq 1$, we get for the projection $\pi : X_s \to X'_s$

**Corollary 4.5.** Let $F$ be a number field or a finite extension of $\mathbb{Q}_l$ for a prime $l$. Then we have

(u) $\pi^* : J'_{s/Q}(F) \to \hat{H}^0(X_s, X'_s, J_{s/Q}(F)) \cong J_{s/Q}(F)[\gamma^{p^{r-1}} - 1]$ is a $U(p)$-isomorphism, where $J_{s/Q}(F)[\gamma^{p^{r-1}} - 1] = \text{Ker}(\gamma^{p^{r-1}} - 1 : J_s(F) \to J_s(F))$.

Here the identity at (u) follows from Lemma 4.4. The kernel $A \mapsto \text{Ker}(\gamma^{p^{r-1}} - 1 : J_s(A) \to J_s(A))$ is an abelian fppf sheaf (as the category of abelian fppf sheaves is abelian and regarding a sheaf as a presheaf is a left exact functor), and it is represented by the scheme theoretic kernel $J_{s/Q}[\gamma^{p^{r-1}} - 1]$ of the endomorphism $\gamma^{p^{r-1}} - 1$ of $J_{s/Q}$. From the exact sequence $0 \to J_s[\gamma^{p^{r-1}} - 1] \to J_s \xrightarrow{\gamma^{p^{r-1}} - 1} J_s$, we get another exact sequence

$$
0 \to J_s[\gamma^{p^{r-1}} - 1](F) \to J_s(F) \xrightarrow{\gamma^{p^{r-1}} - 1} J_s(F).
$$

Thus

$$
J_{s/Q}(F)[\gamma^{p^{r-1}} - 1] = J_{s/Q}[\gamma^{p^{r-1}} - 1](F).
$$

The above (u) combined with (u1) implies (u2) below:

(u2) $\pi^* : J_{r/Q} \to J_{s/Q}[\gamma^{p^{r-1}} - 1] = \text{Ker}(\gamma^{p^{r-1}} - 1 : J_{s/Q} \to J_{s/Q})$ is a $U(p)$-isomorphism.

Actually we can reformulate these facts as

**Lemma 4.6.** Then we have morphisms

$$
\iota^*_s : J_{s/Q}[\gamma^{p^{r-1}} - 1] \to J'_{s/Q} \quad \text{and} \quad \iota^*_s : J_{r/Q} \to J_{s/Q}/(\gamma^{p^{r-1}} - 1)(J_{s/Q})
$$
satisfying the following commutative diagrams:

\[
\begin{array}{ccc}
J'_s/Q & \xrightarrow{\pi'} & J_s/Q[\gamma^{p^{s-r}} - 1] \\
\downarrow u' & \nearrow \iota'_s & \downarrow u'' \\
J'_s/Q & \xrightarrow{\pi} & J_s/Q[\gamma^{p^{s-r}} - 1],
\end{array}
\]

(4.4)

and

\[
\begin{array}{ccc}
J'_s/Q & \xrightarrow{\pi'} & J_s/Q/(\gamma^{p^{s-r}} - 1)(J_s/Q) \\
\uparrow u^* & \nearrow \iota^*_s & \uparrow u''^* \\
J'_s/Q & \xrightarrow{\pi} & J_s/Q/(\gamma^{p^{s-r}} - 1)(J_s/Q),
\end{array}
\]

(4.5)

where \( u \) and \( u'' \) are \( U(p^{s-r}) = U(p)^{s-r} \) and \( u^* \) and \( u''^* \) are \( U^*(p^{s-r}) = U^*(p)^{s-r} \). In particular, for an fppf extension \( T/Q \), the evaluated map at \( T \): \( (J_s/Q/(\gamma^{p^{s-r}} - 1)(J_s/Q))(T) \xrightarrow{\pi} J'_s(T) \) (resp. \( J'_s(T) \xrightarrow{\pi} J_s[\gamma^{p^{s-r}} - 1](T) \)) is a \( U^*(p) \)-isomorphism (resp. \( U(p) \)-isomorphism).

Note here that the natural homomorphism:

\[ \frac{J_s(T)}{(\gamma^{p^{s-r}} - 1)(J_s(T))} \to (J_s/Q/(\gamma^{p^{s-r}} - 1)(J_s/Q))(T) \]

may have non-trivial kernel and cokernel which may not be killed by a power of \( U^*(p) \). In other words, the left-hand-side is an fppf presheaf of \( T \) and the right-hand-side is its sheafication. On the other hand, \( T \to J_s[\gamma^{p^{s-r}} - 1](T) \) is already an fppf abelian sheaf; so, \( J'_s(T) \xrightarrow{\pi} J_s[\gamma^{p^{s-r}} - 1](T) \) is a \( U(p) \)-isomorphism without ambiguity.

Proof. We first prove the assertion for \( \pi^* \). We note that the category of groups schemes fppf over a base \( S \) is a full subcategory of the category of abelian fppf sheaves. We may regard \( J'_s/Q \) and \( J_s[\gamma^{p^{s-r}} - 1]/Q \) as abelian fppf sheaves over \( Q \) in this proof. Since these sheaves are represented by (reduced) algebraic groups over \( Q \), we can check being \( U(p) \)-isomorphism by evaluating the sheaf at a field \( k \) of characteristic 0 (e.g., [EAI, Lemma 4.18]). By Proposition 4.2 (2) applied to \( X = X_s/k = X_s \times_Q k \) and \( Y = X'_s/k \) (with \( S = \text{Spec}(k) \) and \( s \geq r \)),

\[ \mathcal{K} := \text{Ker}(J'_s/Q \to J_s/Q[\gamma^{p^{s-r}} - 1]) \]

is killed by \( U(p)^{s-r} \) as \( d = p^{s-r} = \deg(X_s/X'_s) \). Thus we get

\[ \mathcal{K} \subset \text{Ker}(U(p)^{s-r} : J'_s/Q \to J_s/Q). \]

Since the category of fppf abelian sheaves is an abelian category (because of the existence of the sheafication functor from presheaves to sheaves under fppf topology described in [ECH, §II.2]), the above inclusion implies the existence of \( \iota^*_s \) with \( \pi^* \circ \iota^*_s = U(p)^{s-r} \) as a morphism of abelian fppf sheaves. Since the category of group schemes fppf over a base \( S \) is a full subcategory of the category of abelian fppf sheaves, all morphisms appearing in the identity \( \pi^* \circ \iota^*_s = U(p)^{s-r} \) are morphism of group schemes. This proves the assertion for \( \pi^* \).

Note that the second assertion is the dual of the first; so, it can be proven reversing all the arrows and replacing \( J_s[\gamma^{p^{s-r}} - 1]/Q \) (resp. \( \pi^*, U(p) \)) by the quotient \( J_s/(\gamma^{p^{s-r}} - 1)J_s \) as fppf abelian sheaves (resp. \( \pi^*, U^*(p) \)). Since \( J_s/(\gamma^{p^{s-r}} - 1)(J_s) \) and \( J'_s \) are abelian schemes over \( Q \), the quotient abelian scheme \( J_s/(\gamma^{p^{s-r}} - 1)(J_s) \) is the dual of \( J_s[\gamma^{p^{s-r}} - 1] \) and \( \iota^*_s \) is the dual of \( \iota^*_s \).

By the second diagram of the above lemma, we get

\[ (u^*) \frac{J_s/(\gamma^{p^{s-r}} - 1)(J_s)/Q}{J'_s/Q} \xrightarrow{\pi} \frac{J'_s/Q}{J_s/Q} \text{ is a } U^*(p) \text{-isomorphism of abelian fppf sheaves.} \]

As a summary, we have

**Corollary 4.7.** Then the morphism \( \pi : X_s \to X'_s \) induces an isogeny

\[ \pi_* : \frac{J_s/(\gamma^{p^{s-r}} - 1)(J_s)/Q}{J'_s/Q} \]

whose kernel is killed by a sufficiently large power of \( U^*(p) \), and the pull-back map \( \pi^* \) induces an isogeny \( \pi^*: J_s[\gamma^{pj} - 1] \rightarrow J_p^s \) whose kernel is killed by a high power of \( U(p) \). Moreover, for a finite extension \( F \) of \( \mathbb{Q} \) or \( \mathbb{Q}_l \) (for a prime \( l \) not necessarily equal to \( p \)), \( \pi^*: J_s[\gamma^{pj} - 1](F) \rightarrow J_p(F) \) is an \( U(p) \)-isomorphism.

Proof. Let \( C \subset \text{Aut}(X_s) \) be the cyclic group generated by the action of \( \gamma^{pj} \). Then \( X_s/\mathcal{X}_s \) is an étale covering with Galois group \( C \) (even unramified at cusps). Thus \( \text{Lie}(J_p') = H^1(X'_s, \mathcal{O}_{X'_s}) = H_0(C, H^1(X_s, \mathcal{O}_{X_s})) = H_0(C, \text{Lie}(J_s)) \). This shows that \( \pi_* \) is an isogeny over \( \mathcal{X}_s \) and hence over \( \mathbb{Q} \), which is a \( U^*(p) \)-isomorphism by Lemma 4.6. By taking dual, \( \pi^* \) is also an isogeny, which is a \( U(p) \)-isomorphism even after evaluating the fppf sheaves at \( F \) by Lemma 4.6 and the remark following the lemma. This proves the corollary.

Then we get
\[
(u^*2) \quad J_s/(\gamma^{pj} - 1)(J_s)/\mathbb{Q} \rightarrow J_p/\mathbb{Q} \text{ is a } U^*(p)\text{-isomorphism of abelian fppf sheaves.}
\]

We can prove \((u^*2)\) in a more elementary way. We describe the easier proof. Identify \( J_s(C) = H^1(X_s, \mathbb{C}) \) whose Pontryagin dual is given by \( H_1(X_s, \mathbb{Z}) \). If \( k = \mathbb{Q} \), we have the Pontryagin dual version of \((u^2)\):
\[
(4.6) \quad H_1(X_s, \mathbb{Z}) \xrightarrow{\pi_*} H_1(X_s, \mathbb{Z})(/\gamma^{pj} - 1)(H_1(X_s, \mathbb{Z})) \text{ is a } U^*(p)\text{-isomorphism.}
\]
Since \( J_s, \mathbb{Q}(\mathbb{C}) \cong H_1(X_s, \mathbb{R})/H_1(X_s, \mathbb{Z}) \) as Lie groups, we get
\[
(4.7) \quad J_s(\mathbb{C}) \xrightarrow{\pi_*} J_s(\mathbb{C})(/\gamma^{pj} - 1)(J_s(\mathbb{C})) \text{ is an } U^*(p)\text{-isomorphism.}
\]
This implies \((u^*2)\). By \((4.7)\), writing \( \mathcal{Q} \) for the algebraic closure of \( \mathbb{Q} \) in \( \mathbb{C} \) and taking algebraic points, we get
\[
(4.8) \quad J_s(\mathcal{Q}) \xrightarrow{\pi_*} (J_s(\mathcal{Q})(/\gamma^{pj} - 1)(J_s(\mathcal{Q})) = J_s(\mathcal{Q})(/\gamma^{pj} - 1)(J_s(\mathcal{Q})) \text{ is an } U^*(p)\text{-isomorphism.}
\]

Remark 4.8. The \( U(p) \)-isomorphisms of Jacobians do not kill the part associated to finite slope Hecke eigenforms. Thus the above information includes not just the information of \( p \)-ordinary forms but also those of finite slope Hecke eigenforms.

5. Control of \( \Lambda \)-MW groups as fppf sheaves

Let \( k \) be either a number field in \( \mathcal{Q} \) or a finite extension of \( \mathbb{Q}_l \) in \( \mathcal{Q}_l \) for a prime \( l \). Write \( O_k \) (resp. \( W \)) for the (resp. \( l \)-adic) integer ring of \( k \) if \( k \) is a number field (resp. a finite extension of \( \mathbb{Q} \)). For an abelian variety \( A/k \), we have \( \widehat{A}(\kappa):=\lim_{\kappa} A(\kappa)/p^n A(\kappa) \) for a finite field extension \( \kappa/k \) in \((2.2)\). A down-to-earth description of the value of \( \widehat{A}(\kappa) \) is given by \((S)\) just above \((2.2)\).

We study \( J_r(k) \) equipped with the topology \( J_r(k) \) induced from \( k \) (so, it is discrete if \( k \) is a number field and is \( l \)-adic if \( k \) is a finite extension of \( \mathbb{Q} \)). The \( p \)-adic limits \( e = \lim_{n \to \infty} U^*(p^n) \) and \( e^* = \lim_{n \to \infty} U^*(p^n) \) are well defined on \( J_r(k) \). The Albanese functoriality gives rise to a projective system \( \{J_s(k), \pi_{s,r,s}\}_s \) and we have
\[
\widehat{J}_{\infty}(k) = \lim_{r} \widehat{J}_r(k) \text{ (with projective limit of } p\text{-profinite compact topology)}
\]
on which the co-ordinary projector \( e^* = \lim_{n \to \infty} U^*(p^n) \) acts. As before, adding superscript or subscript “ord” (resp. “co-ord”), we indicate the image of \( e \) (resp. \( e^* \)) depending on the situation.

We study mainly in this paper the control theorems of the \( w \)-twisted version \( \widehat{J}_{\infty}(k)^{\text{ord}} \) (which we introduce in Section 6) of \( \widehat{J}_{\infty}(k)^{\text{co-ord}} \) under the action of \( \Gamma \) and Hecke operators, and we have studied \( J^{\infty}(k) \) in \([H15a]\) in a similar way. Here the word “\( w \)-twisting” means modifying the transition maps by the Weil involution at each step. As fppf sheaves, we have an isomorphism \( \iota : \widehat{J}_{\infty}(k)^{\text{ord}} \cong \widehat{J}_{\infty}(k)^{\text{co-ord}} \) but \( \iota \circ T(n) = T^*(n) \circ \iota \) for all \( n \). Hereafter, unless otherwise mentioned, once our fppf abelian sheaf is evaluated at \( k \), all morphisms are continuous with respect to the topology defined above (and we do not mention continuity often).
From (u1), we get
\[(5.1) \quad J_r(k) \xrightarrow{\pi} J'_r(k) \text{ is a } U(p)\text{-isomorphism (for the projection } \pi : X'_r \to X_r).\]

The dual version (following from \((u^*1)\) is
\[(5.2) \quad J'_r(k) \xrightarrow{\pi} J_r(k) \text{ is a } U^*(p)\text{-isomorphism, where } \pi_\ast \text{ is the dual of } \pi^*\.\]

From (5.1) and (5.2), we get

\[\text{Lemma 5.1. For a field } k \text{ as above, we have} \]
\[\pi_\ast : \widehat{J}_r^\ast(k)^{\text{co-ord}} \cong \widehat{J}_r(k)^{\text{co-ord}} \text{ and } \pi^* : \widehat{J}_r(k)^{\text{ord}} \cong \widehat{J}_r^\ast(k)^{\text{ord}} \]

\[\text{for all } 0 < r < s \text{ with the projection } \pi : X'_r \to X_r.\]

From Corollary 4.7 (or Lemma 4.6 combined with \((u^*2)\) and \((u2)\)), for any field \(k\), we get

(I) \(\pi^* : J_r(k) \to J_r[\gamma^{p^{r-s}} - 1](k) \text{ is a } U(p)\text{-isomorphism, and obviously, } \pi^* : J_r \to J_r[\gamma^{p^{r-s}} - 1] \text{ is a } U(p)\text{-isomorphism of abelian fpff sheaves.} \]

(P) \(\pi_\ast : J_r \to J_r/(\gamma^{p^{r-s}} - 1)J_r \text{ is a } U^*(p)\text{-isomorphism of fpff abelian sheaves.} \)

Note that (P) does not mean that \(\frac{\widehat{J}_r(k)}{(\gamma^{p^{r-s}} - 1)J_r(k)} \to \widehat{J}_r(k) \text{ is a } U^*(p)\text{-isomorphism (as the sheaf quotient } J_r/(\gamma^{p^{r-s}} - 1)J_r \text{ and the corresponding presheaf quotient could be different). We now claim} \]

\[\text{Lemma 5.2. For integers } 0 < r < s, \text{ we have isomorphisms of fpff abelian sheaves} \]
\[\pi^* : \widehat{J}_r^{\text{ord}} \cong \widehat{J}_s[\gamma^{p^{r-s}} - 1]^{\text{ord}} \text{ and } \pi_\ast : \frac{\widehat{J}_s}{(\gamma^{p^{r-s}} - 1)\widehat{J}_s} \cong \widehat{J}_r^{\text{co-ord}}. \]

The first isomorphism \(\pi^*\) induces an isomorphism: \(\widehat{J}_r^{\text{ord}}(T) \cong \widehat{J}_s[\gamma^{p^{r-s}} - 1]^{\text{ord}}(T) \text{ for any fpff extension } T/k\) but the morphism induced by the second one: \(\frac{\widehat{J}_s}{(\gamma^{p^{r-s}} - 1)\widehat{J}_s} \to \widehat{J}_r^{\text{co-ord}}\) may not be an isomorphism.

\[\text{Proof. By (I) above, } \widehat{J}_r^{\text{ord}} \cong \widehat{A}^{\text{ord}} \text{ for the abelian variety } A = J_s[\gamma^{p^{r-s}} - 1] \text{ and } \widehat{A} \text{ as in (S) above (2.2). We consider the following exact sequence} \]
\[0 \to A \to J_s[\gamma^{p^{r-s}} - 1] \to J_s. \]

This produces another exact sequence \(0 \to \widehat{A} \to \widehat{J}_s \xrightarrow{\gamma^{p^{r-s}} - 1} \widehat{J}_s; \text{ so, we get } \widehat{A} \cong \widehat{J}_s[\gamma^{p^{r-s}} - 1]. \) Taking ordinary part and combining with the identity: \(J_r^{\text{ord}} \cong \widehat{J}_r[\gamma^{p^{r-s}} - 1] \text{. Taking} \text{ this holds true after evaluation at } T \text{ as the presheaf-kernel of a sheaf morphism is still a sheaf. The second assertion is the dual of the first.} \]

Passing to the limit, Lemmas 5.1 and 5.2 tells us

\[\text{Theorem 5.3. Let } k \text{ be either a number field or a finite extension of } \mathbb{Q}_l. \text{ Then we have isomorphisms of fpff abelian sheaves over } k:} \]
\[(a) \quad \widehat{J}_\infty[\gamma^{p^{r-s}} - 1] \cong \widehat{J}_r^{\text{ord}}; \]
\[(b) \quad (\widehat{J}_\infty/(\gamma^{p^{r-s}} - 1)(\widehat{J}_\infty))^{\text{co-ord}} \cong \widehat{J}_r^{\text{co-ord}} \]

\[\text{where we put } \widehat{J}_\infty/(\gamma^{p^{r-s}} - 1)(\widehat{J}_\infty))^{\text{co-ord}} := \lim_{s \to \infty} J_s/(\gamma^{p^{r-s}} - 1)(J_s)^{\text{co-ord}} \text{ as an fpff sheaf.} \]

\[\text{Proof. The assertion (b) is just the projective limit of the corresponding statement in Lemma 5.2.} \]

We prove (a). Since injective limit always preserves exact sequences, we have
\[0 \to \widehat{J}_r(k)^{\text{ord}} \to \lim_{s \to \infty} J_s(k)^{\text{ord}} \xrightarrow{\gamma^{p^{r-s}} - 1} \lim_{s \to \infty} \widehat{J}_s(k)^{\text{ord}} \]

is exact. Taking \(p\)-adic completion of each term of this sequence is left exact, we find the exactness of \(0 \to \widehat{J}_r(k)^{\text{ord}} \to \widehat{J}_\infty(k)^{\text{ord}} \xrightarrow{\gamma^{p^{r-s}} - 1} \widehat{J}_\infty(k)^{\text{ord}}\), showing (a). \(\square\)
Remark 5.4. As is clear from the warning after (P), the isomorphism (b) does not mean that
\[ \lim_{s} \left\{ \frac{\tilde{J}_s(T)}{(\gamma p^{r-s} - 1)(\tilde{J}_s(T))} \right\} \to \tilde{J}_r(T) \]
for each fppf extension $T/k$ is an isomorphism. The kernel and the cokernel of this map will be studied in Section 9.

6. SHEAVES ASSOCIATED TO MODULAR JACOBIANS

We fix an element $\zeta \in \mathbb{Z}_p(1) = \lim_{\rightarrow n} \mu_{p^n}(\overline{\mathbb{Q}})$; so, $\zeta$ is a coherent sequence of generators $\zeta_p^n$ of $\mu_{p^n}(\overline{\mathbb{Q}})$ (i.e., $\zeta_{p^{n+1}} = \zeta_p^n$). We also fix a generator $\zeta_N$ of $\mu_N(\overline{\mathbb{Q}})$, and put $\zeta_{Np^r} := \zeta_N \zeta_{p^r}$.

Identify the étale group scheme $\mathbb{Z}/Np^n\mathbb{Z}/\mathbb{Q}[\zeta_N, \zeta_{p^n}]$ with $\mu_{Np^n}$ by sending $m \in \mathbb{Z}$ to $\zeta_{Np^n}^m$. Then for a couple $(E, \phi_{Np^r})$ of $\mathbb{Q}[\mu_{p^n}]$-algebra $K$, let $\phi^* : E[Np^r] \to \mathbb{Z}/Np^r\mathbb{Z}$ be the Cartier dual of $\phi_{Np^r}$. Then $\phi^*$ induces $E[Np^r]/\text{Im}(\phi_{Np^r}) \cong \mathbb{Z}/Np^r\mathbb{Z}$. Define $i : \mathbb{Z}/p^r\mathbb{Z} \cong (E/\text{Im}(\phi_{Np^r}))[Np^r]$ by the inverse of $\phi^*$. Then we define $\phi_{Np^r} : \mu_{Np^r} \to E/\text{Im}(\phi_{Np^r})$ by $\phi_{Np^r} : \mu_{Np^r} \cong \mathbb{Z}/p^r\mathbb{Z} \to (E/\text{Im}(\phi_{Np^r}))[p^r] \subset E/\text{Im}(\phi_{Np^r})$. This induces an automorphism $w_r$ of $\mathbb{X}_r$ defined over $\mathbb{Q}[[\mu_{Np^r}]$, which in turn induces an automorphism $w_r$ of $\mathbb{X}_r(\mathbb{Q})$.

We have the following well known commutative diagram (e.g., [MFM, Section 4.6]):

\[ \xymatrix{ J_r \ar[r]^{T(n)} & J_r \ar[d]^{w_r} \ar[l]_{J_r} \ar[d]^{T(n)} \ar[r] & J_r \ar[l]_{w_r} } \]

Let $P \in \text{Spec}(h)(\overline{\mathbb{Q}}_p)$ be an arithmetic point of weight 2. Then we have a p-stabilized Hecke eigenform form $f_P$ associated to $P$; i.e., $f_P|T(n) = P(T(n))f_P$ for all $n$. Then $f_P^r = w_r(f_P)$ is the dual eigenform of $T(n)$. If $f_P$ is new at every prime $l|pN$, $f_P^r$ is a constant multiple of the complex conjugate $f_P^r$ of $f_P$ (but otherwise, it is different).

We then define as described in (S) in Section 2 an fpf abelian sheaf $\mathbb{X}$ for any abelian variety quotient or subgroup variety $X$ of $J_s/k$ over the fpf site over $k = \mathbb{Q}$ and $\mathbb{Q}_l$ (note here the definition of $\mathbb{X}$ depends on $k$).

Pick an automorphism $\sigma \in \text{Gal}(\mathbb{Q}(\mu_{Np^r})/\mathbb{Q})$ with $\zeta_{Np^r}^\sigma = \zeta_{Np^r}$ for $z \in (\mathbb{Z}/Np^r\mathbb{Z})^\times$. Since $w_r^\sigma$ is defined with respect to $\zeta_{Np^r}^\sigma = \zeta_{Np^r}$, put $w_r^\sigma = z \circ w_r = w_r \circ (z)^{-1}$ (see [MW86, page 237] and [MW84, 2.5.6]). Here $(z)$ is the image of $z$ in $(\mathbb{Z}/Np^r\mathbb{Z})^\times = \text{Gal}(X_r/X_0(Np^r))$. Let $\pi_{s,r,s} : J_s \to J_s$ for $s > r$ be the morphism induced by the covering map $X_s \to X_r$ through Albanese functoriality. Then we define $\pi_{s,r,s}^\sigma = w_r \circ \pi_{s,r,s} \circ w_r$. Then $(\pi_{s,r}^\sigma)^* = w_r(z^{-1})\pi_{s,r,s}(z)w_r = \pi_{s,r}^\sigma$ for all $\sigma \in \text{Gal}(\mathbb{Q}(\mu_{Np^r})/\mathbb{Q})$; thus, $\pi_{s,r}^\sigma$ is well defined over $\mathbb{Q}$, and satisfies $T(n) \circ \pi_{s,r}^\sigma = \pi_{s,r}^\sigma \circ T(n)$ for all $n$ prime to $Np$ and $U(q)^\sigma = \pi_{s,r}^\sigma \circ U(q)$ for all $q|Np$. Since $w_r^2 = 1$, by this $w$-twisting, the projective system $\{J_s, \pi_{s,r,s}\}$ equivariant under $T^*(n)$ is transformed into isomorphic projective system $\{J_s, \pi_{s,r,s}^\sigma\}$ (of abelian varieties defined over $\mathbb{Q}$) which is a Hecke equivariant (i.e., $T(n)$ and $U(l)$-equivariant). Thus what we proved for the co-ordinary part of the projective system $\{J_s, \pi_{s,r,s}\}$ is valid for the ordinary part of the projective system $\{\tilde{J}_s, \pi_{s,r,s}^\sigma\}$. If $X_s$ is either an algebraic subgroup or an abelian variety quotient of $J_s$, and $\pi_{s,r}^\sigma$ produces a projective system $\{X_s\}_s$ we define $\tilde{X}_\infty := \lim_{\rightarrow s} \tilde{X}_s(R)$ for an fpf extension $R$ of $k = \mathbb{Q}$, $\mathbb{Q}_l$ (again the definition of $\tilde{X}_s$ and hence $\tilde{X}_\infty$ depends on $k$). For each ind-object $R = \lim_{\rightarrow i} R_i$ of fpf, smooth or étale algebras $R_i/k$, we define $\tilde{X}_\infty = \lim_{\rightarrow i} \tilde{X}_\infty(R_i)$.

Lemma 6.1. Let $K/k$ be the Galois extension as in Section 2. Then the $\text{Gal}(K/k)$-action on $\tilde{X}_\infty(K)$ is continuous under the discrete topology on $\tilde{X}_\infty(K)$. In particular, the Galois cohomology group $H^q(\tilde{X}_\infty(K)) := H^q(\text{Gal}(K/k), \tilde{X}_\infty(K))$ for $q > 0$ is a torsion $\mathbb{Z}_p$-module for any intermediate extension $K/k/k$. 

Proof. By definition, \( \hat{X}_\infty(K) = \bigcup_{K/F/k} \hat{X}_\infty(F) \), and \( \hat{X}_\infty(F) \subseteq H^0(\text{Gal}(K/F), \hat{X}_\infty(K)) \) for all finite intermediate extension \( K/F/k \). Thus \( \hat{X}_\infty(K) = \lim_{\rightarrow F} H^0(\text{Gal}(K/F), \hat{X}_\infty(K)) \), which implies the continuity of the action under the discrete topology. Then the torsion property follows from [MFG, Corollary 4.26].

Let \( \iota : C_{r/q} \subseteq J_{r/q} \) be an abelian subvariety stable under Hecke operators (including \( U(l) \)) and \( w_r \) and \( \iota : J_{r/q} \to tC_{r/q} \) be the dual abelian quotient. We then define \( \pi : J_r \to D_r \) by \( D_r := tC_r \) and \( \pi = w_r \circ \iota_r \circ w_r \) for the map \( \iota_r \in \text{Aut}(tC_{r/q}|_{\overline{p}}) \) dual to \( w_r \in \text{Aut}(C_{r/q}|_{\overline{p}}) \). Again \( \pi \) is defined over \( \mathbb{Q} \). Then \( \iota \) and \( \pi \) are Hecke equivariant. Let \( \iota_* : C_* \to C_r \) be the quotient abelian variety of \( J_s \) defined in the same way taking \( C_* \) in place of \( C_r \) (and replacing \( r \) by \( s \)). Put \( \pi_* : J_* \to D_s \) which is Hecke equivariant.

Since the two morphisms \( J_r \to J_*^s \) and \( J_s^r \to J_s[\gamma^{p^{r-s}} - 1] \) (Picard functoriality) are \( U(p) \)-isomorphism of fppf abelian sheaves by \((u1)\) and Corollary 4.5, we get the following two isomorphisms of fppf abelian sheaves for \( s > r > 0 \):

\[
\begin{align*}
\pi_*^r : C_r[p^{\infty}]_{\text{ord}} &\sim C_s[p^{\infty}]_{\text{ord}} & \text{and} & & \pi_*^r : \hat{C}_r[p^{\infty}]_{\text{ord}} &\sim \hat{C}_s[p^{\infty}]_{\text{ord}},
\end{align*}
\]

since \( \hat{C}_s[p^{\infty}]_{\text{ord}} \) is the isomorphic image of \( \hat{C}_r[p^{\infty}]_{\text{ord}} \subset J_r \). In the following diagram, we have

\[
\begin{align*}
\hat{D}_r(p^{\infty})_{\text{ord}} &\sim \hat{D}_s(p^{\infty})_{\text{ord}},
\end{align*}
\]

since \( \hat{D}_s(p^{\infty})_{\text{ord}} \) is the isomorphic image of \( \hat{D}_r(p^{\infty})_{\text{ord}} \subset J_r \). By \( w \)-twisted Cartier duality [H14, §4], we have

\[
\begin{align*}
D_s[p^{\infty}]_{\text{ord}} &\sim D_r[p^{\infty}]_{\text{ord}},
\end{align*}
\]

Thus by Kummer sequence in Lemma 2.1, we have the following commutative diagram

\[
\begin{align*}
\hat{D}_r(p^{\infty})_{\text{ord}} &\sim \hat{D}_s(p^{\infty})_{\text{ord}} \quad \text{and} \quad \hat{D}_r(p^{\infty})_{\text{ord}} &\sim \hat{D}_s(p^{\infty})_{\text{ord}},
\end{align*}
\]

Passing to the limit, we get

\[
\begin{align*}
\hat{D}_r(p^{\infty})_{\text{ord}} &\sim \hat{D}_s(p^{\infty})_{\text{ord}},
\end{align*}
\]

\[
\begin{align*}
\text{as fppf abelian sheaves. In short, we get}
\end{align*}
\]

**Lemma 6.2.** Suppose that \( \kappa \) is a field extension of finite type of either a number field or a finite extension of \( \mathbb{Q}_l \). Then we have the following isomorphism

\[
\begin{align*}
\hat{C}_r(\kappa)_{\text{ord}} &\sim \hat{C}_s(\kappa)_{\text{ord}} \quad \text{and} \quad \hat{D}_r(\kappa)_{\text{ord}} &\sim \hat{D}_s(\kappa)_{\text{ord}},
\end{align*}
\]

for all \( s > r \) including \( s = \infty \).

By computation, \( \pi_{r,s}^* \circ \pi_{s,r}^* = p^{s-r}U(p^{s-r}) \). To see this, as Hecke operators, \( \pi_{r,s}^* = [\Gamma_r^s], \pi_{r,s,r}^* = [\Gamma_r] \). Thus we have

\[
\begin{align*}
\pi_{r,s}^* \circ \pi_{s,r}^*(x) = x|\Gamma_r^s \cdot w_s \cdot [\Gamma_r] \cdot w_r = x|\Gamma_r^s \cdot w_s \cdot w_r \cdot [\Gamma_r] = x|\Gamma_r^s \cdot [\Gamma_r] \cdot [w_s \cdot w_r \cdot [\Gamma_r] = x|\Gamma_r^s \cdot [\Gamma_r \cdot [\frac{1}{0} \cdot 0 \cdot p^{s-r} \cdot \Gamma_r] = p^{s-r}(x|U(p^{s-r})).
\end{align*}
\]

**Corollary 6.3.** We have the following two commutative diagram for \( s' > s \)

\[
\begin{align*}
\hat{C}_s[p^{\infty}]_{\text{ord}} &\sim \hat{C}_s[p^{\infty}]_{\text{ord}},
\end{align*}
\]

\[
\begin{align*}
\text{for the map}
\end{align*}
\]

\[
\begin{align*}
\pi_{s,s'}^* &\sim \pi_{s,s'}^* \quad \text{and} \quad \hat{C}_s[p^{\infty}]_{\text{ord}} &\sim \hat{C}_s[p^{\infty}]_{\text{ord}},
\end{align*}
\]

\[
\begin{align*}
\pi_{s,s'}^* &\sim \pi_{s,s'}^* \quad \text{and} \quad \hat{C}_s[p^{\infty}]_{\text{ord}} &\sim \hat{C}_s[p^{\infty}]_{\text{ord}},
\end{align*}
\]

\[
\begin{align*}
\text{for the map}
\end{align*}
\]
and

\[ \tilde{\mathcal{D}}^\mathrm{ord}_{s'} \xrightarrow{\pi^r_{s',r}} \tilde{\mathcal{D}}^\mathrm{ord}_s. \]

\[ \pi^r_{s',r} \]

\[ \tilde{\mathcal{D}}^\mathrm{ord}_s \]

\[ \tilde{\mathcal{D}}^\mathrm{ord}_{s'}. \]

Proof. By \( \pi^r_{s,s'} \) (resp. \( \pi^r_s \)), we identify \( \tilde{\mathcal{C}}^\mathrm{ord}_r \) with \( \tilde{\mathcal{C}}^\mathrm{ord}_s \) as in Lemma 6.2. Then the above two diagrams follow from (6.4).

By (6.4), we have exact sequences

\[ 0 \to C_s[p^{s-r}]^\mathrm{ord} \to C_s[p^\infty]^\mathrm{ord} \to C_r[p^\infty]^\mathrm{ord} \to 0, \]

(6.5)

\[ 0 \to D_r[p^{s-r}]^\mathrm{ord} \to D_r[p^\infty]^\mathrm{ord} \to D_s[p^\infty]^\mathrm{ord} \to 0. \]

Applying (2.1) to the exact sequence \( \mathcal{K}^r_s(K) \to C_s(K) \to C_r(K) \) for \( \mathcal{K}^r_s(K) = \text{Ker}(\pi^r_s)(K) \) and \( \mathcal{K}_{r,s}(K) \to C_r(K) \to D_s(K) \) for \( \mathcal{K}_{r,s} = \text{Ker}(\pi^r_{s,s'}) \), we get the following exact sequence of fppf abelian sheaves:

\[ 0 \to \hat{\mathcal{K}}^r_s \to \tilde{\mathcal{C}}^r_s \xrightarrow{\pi^r_s} \tilde{\mathcal{C}}^r_r \to 0, \]

\[ 0 \to \hat{\mathcal{K}}_{r,s} \to \tilde{D}^r_s \xrightarrow{\pi^r_{s,s'}} \tilde{D}^r_r \to 0. \]

Taking the ordinary part, we confirm exactness of

\[ 0 \to C_s[p^{s-r}]^\mathrm{ord}(\kappa) \to \tilde{\mathcal{C}}^r_s^\mathrm{ord}(\kappa) \xrightarrow{\pi^r_s} \tilde{\mathcal{C}}^r_r^\mathrm{ord}(\kappa) \to H^1(C_s[p^{s-r}]^\mathrm{ord}), \]

(6.6)

\[ 0 \to D_r[p^{s-r}]^\mathrm{ord}(\kappa) \to \tilde{D}^r_s^\mathrm{ord}(\kappa) \xrightarrow{\pi^r_{s,s'}} \tilde{D}^r_r \to H^1(D_r[p^{s-r}]^\mathrm{ord}). \]

Write \( H^1(X) = H^1(\text{Gal}(K/\kappa), X) \) for an intermediate extension \( K/\kappa/k \) and \( \text{Gal}(K/k) \)-module \( X \) and \( H^1_s(X) = H^1_s(\text{Spec}(\kappa), X) \) for a smooth/fppf extension for \( ? = \text{smooth or fppf} \). Then taking the \( p \)-adic completion, we get the following exact sequences as parts of the long exact sequences associated to (6.6)

\[ 0 \to C_s[p^{s-r}]^\mathrm{ord}(\kappa) \to \hat{\mathcal{C}}^r_s^\mathrm{ord}(\kappa) \to \hat{\mathcal{C}}^r_r^\mathrm{ord}(\kappa) \to H^1(C_s[p^{s-r}]^\mathrm{ord}), \]

(6.7)

\[ 0 \to D_r[p^{s-r}]^\mathrm{ord}(\kappa) \to \hat{\mathcal{D}}^r_s^\mathrm{ord}(\kappa) \to \hat{\mathcal{D}}^r_r \to H^1(D_r[p^{s-r}]^\mathrm{ord}). \]

for \( ? = \text{fppf, sm (cohomology under smooth topology) or nothing (i.e., Galois cohomology equivalent to étale cohomology in this case). Here if } ? = \text{fppf} \), \( \kappa/k \) is an extension of finite type, if \( ? = \text{sm} \), \( \kappa/k \) is a smooth extension of finite type, and if \( ? \) is nothing, \( K/\kappa/k \) is an intermediate field.

By Lemma 6.2, we can rewrite the first exact sequence of (6.5) as

\[ 0 \to C_r[p^{s-r}]^\mathrm{ord}(\kappa) \xrightarrow{\pi^r_{s,s'}} \tilde{\mathcal{C}}^r_s^\mathrm{ord}(\kappa) \xrightarrow{\pi^r_s} \tilde{\mathcal{C}}^r_r^\mathrm{ord}(\kappa) \to H^1(C_r[p^{s-r}]^\mathrm{ord}). \]

This (combined with Corollary 6.3) induces the corresponding diagram for \( H^1 \), for any extension \( \kappa/k \) inside \( K \),

\[ H^1(C_s[p^{s-r}]^\mathrm{ord}) \xrightarrow{\pi^r_{s,s'}} H^1(C_r[p^{s-r}]^\mathrm{ord}) \xrightarrow{\pi^r_s} \left( \frac{C_r(\kappa)}{p^{s-r}C_r(\kappa)} \right)^\mathrm{ord} \]

\[ \xrightarrow{p^{s-r}U(p)^{s-r}} \]

\[ H^1(C_s[p^{s-r}]^\mathrm{ord}) \xrightarrow{\pi^r_{s,s'}} H^1(C_r[p^{s-r}]^\mathrm{ord}) \xrightarrow{\pi^r_s} \left( \frac{C_r(\kappa)}{p^{s-r}C_r(\kappa)} \right)^\mathrm{ord}. \]

The right square is the result of Kummer theory for \( C_r \). Passing to the projective limit with respect to \( s \), we get a sequence

\[ 0 \to \lim_{\to s} C_r[p^{s-r}]^\mathrm{ord}(\kappa) \xrightarrow{\pi^r_{s,s'}} \lim_{\to s} \tilde{\mathcal{C}}^r_s^\mathrm{ord}(\kappa) \xrightarrow{\pi^r_s} \tilde{\mathcal{C}}^r_r^\mathrm{ord}(\kappa) \to H^1(C_r[p^{s-r}]^\mathrm{ord}) \]
which is exact at left three terms up to the term $\hat{C}_r^{\text{ord}}(\kappa)$.

**Proposition 6.4.** Let $k$ be a finite extension field of $\mathbb{Q}$ or $\mathbb{Q}_l$ for a prime $l$. Assume (2.2) for $\kappa/k$. Then we have the following identity

$$\hat{C}_\infty(\kappa)^{\text{ord}} = \lim_{s} \hat{C}_s(\kappa)^{\text{ord}} \cong \lim_{s} C_r[p^{s-r}]^{\text{ord}}(\kappa) = 0$$

and an exact sequence for $K/k$ as in Section 2:

$$0 \to T_p C_r^{\text{ord}} \to \lim_{s} \hat{C}_s(K)^{\text{ord}} \to \hat{C}_r(K)^{\text{ord}} \to 0$$

$$0 \to T_p C_r^{\text{ord}} \to \lim_{s} C_s[p^\infty](K)^{\text{ord}} \to C_r[p^\infty](K)^{\text{ord}} \to 0.$$  

In the last sequence, we have $\lim_{s} C_s[p^\infty](K)^{\text{ord}} \cong T_p C_r^{\text{ord}} \otimes \mathbb{Q}$. By the first identity, $\hat{C}_\infty^{\text{ord}}$ as a smooth (resp. étale) sheaf vanishes if $k$ is a number field or a local field with residual characteristic $l \neq p$ (resp. a $p$-adic field).

**Proof.** By (6.9), we get a sequence which is exact at the first three left terms (up to the term $\hat{C}_r^{\text{ord}}(\kappa)$):

$$0 \to \lim_{s} C_r[p^{s-r}]^{\text{ord}}(\kappa) \to \hat{C}_\infty^{\text{ord}}(\kappa) \xrightarrow{\pi} \hat{C}_r(\kappa)^{\text{ord}} \to \lim_{s} H^1(C_s[p^{s-r}]^{\text{ord}}).$$

Since $\delta$ is injective by Lemma 2.1 under (2.2), we get the first two identities. The vanishing of $\lim_{s} C_r[p^{s-r}]^{\text{ord}}(\kappa)$ follows because $C_r[p^{s-r}]^{\text{ord}}(\kappa)$ is a finite $p$-torsion module if $\kappa/k$ is an extension of finite type.

If $\kappa = K$, we may again pass to the limit of the first exact sequence of (6.6) again noting $C_r[p^{s-r}]^{\text{ord}}(\kappa) = C_r[p^{s-r}]^{\text{ord}}(K)$. The limit keeps exactness (as $\{C_r[p^{s-r}]^{\text{ord}}(\kappa)\}_s$ is a surjective projective system), and we get the following exact sequence

$$0 \to TC_r[p^\infty](K)^{\text{ord}} \to \lim_{s} \hat{C}_s(K)^{\text{ord}} \xrightarrow{\pi} \hat{C}_r(K)^{\text{ord}} \to 0.$$  

The divisible version can be proven taking the limit of (6.5). Since $C_r[p^\infty](K)^{\text{ord}}$ is $p$-divisible and the projective system of the exact sequences $0 \to C_r[p](K)^{\text{ord}} \to C_r[p^\infty](K)^{\text{ord}} \xrightarrow{\alpha\delta}, C_r[p^\infty](K)^{\text{ord}} \to 0$ by the transition map $x \mapsto p^n U(p^n)(x)$ satisfies the Mittag–Leffler condition (as $C_r[p](K)^{\text{ord}}$ is finite), $\lim_{s} C_r[p^\infty](K)^{\text{ord}}$ is a $p$-divisible module. Thus by the exact sequence, we have $T_p C_r^{\text{ord}} \otimes \mathbb{Q} \subset \lim_{s} C_s[p^\infty](K)^{\text{ord}}$, which implies

$$T_p C_r^{\text{ord}} \otimes \mathbb{Q} \cong \lim_{s} C_s[p^\infty](K)^{\text{ord}}$$

as $T_p C_r^{\text{ord}} \otimes \mathbb{Q}/T_p C_r \cong C_r[p^\infty]^{\text{ord}}(K)$.

We insert here Shimura's definition of his abelian subvariety [IAT, Theorem 7.14] and abelian variety quotient [Sh73] of $J_s$ associated to a member $f_P$ of a $p$-adic analytic family. Shimura mainly considered these abelian varieties associated to a primitive Hecke eigenform. Since we need those associated to old Hecke eigenforms, we give some details.

Let $P \in \text{Spec}(h)(\mathbb{Q}_p)$ be an arithmetic point of weight 2. Then we have a $p$-stabilized Hecke eigenform form $f_P$ associated to $P$; i.e., $f_P(T(n)) = P(T(n))f_P$ for all $n$ (e.g., [GME, Section 3.2]). Then $f_P^* = w_r(f_P)$ is the dual common eigenform of $T^*(n)$. If $f_P$ is new at every prime $l|pN$, $f_P^*$ is a constant multiple of the complex conjugate $f_P$ of $f_P$ (but otherwise, they are different). Shimura's abelian subvariety $A_P^0$ (associated to $f_P$) is defined to be the identity connected component of $\bigcap_{\alpha \in P} J_r[\alpha]$ regarding $P$ as a prime ideal of $h_r(Z)$.

The Rosati involution (induced by the canonical polarization) brings $h_r(Z)$ to $h_r^*(Z) \subset \text{End}(J_r/Q)$ isomorphically, and $h$ acts on $J_\infty$ through $T(n) \mapsto T^*(n)$ via this isomorphism. Let $f_P^* T^*(n) = P(T(n))f_P^*$, and regard $P$ as an algebra homomorphism $P^* : h_r^*(Z) \to \overline{\mathbb{Q}}$ (so, $P^*(T^*(n)) = P(T(n))$). Identify $P^*$ with the prime ideal Ker$(P^*)$, and define $A_{P^*}$ to be the identity connected component of $J_r[P^*] := \bigcap_{\alpha \in P} J_r[\alpha]$. Then $A_{P^*} \cong A_{P^*}^0$ by $w_r$ over $\overline{\mathbb{Q}}(\mu_{NP^*})$. 


Assume that \( r = r(P) \) is the minimal exponent of \( p \) in the level of \( f_P \). For \( s > r \), we write \( A_s \) (resp. \( A^*_s \)) for the abelian variety associated to \( f_P \) regarded as an old form of level \( p^s \) (resp. \( w_s(f_P) \)). In other words, regarding \( P^* \) as an ideal of \( \mathfrak{h}_s^*(\mathbb{Z}) \) via the projection \( \mathfrak{h}_s^*(\mathbb{Z}) \to \mathfrak{h}_r^*(\mathbb{Z}) \), we define \( A^*_s \) by the identity connected component of \( J_s[P^*] \). The Albanese functoriality \( \pi : J_s \to J_r \) induces an isogeny \( A^*_s \to A^*_r = A^*_p \). Similarly, the Picard functoriality \( \pi^*: J_r \to J_s \) induces an isogeny \( A^*_p = A_r \to A_s \). Since \( f_P \) is the complex conjugate of \( f_P \), we may assume that \( f_P \) is new, \( A^*_p = A_r \) inside \( J_r \) (for \( r = r(P) \)). Since \( w_s : A_s/\mathbb{Q}[\mathfrak{h}_s^*] \cong A^*_s/\mathbb{Q}[\mathfrak{h}_r^*] \) and \( A_s \) and \( A^*_s \) are isogenous to \( A_P \) over \( \mathbb{Q} \), \( A_s \) and \( A^*_s \) are isomorphic over \( \mathbb{Q} \). Consider the dual quotient \( J_s \to B_s \) (resp. \( J_s \to B^*_s \)) of \( A_s \) (resp. \( A_s \to J_s \)). In the same manner as above, \( B_s \) and \( B^*_s \) are isomorphic over \( \mathbb{Q} \). Then \( B_s \) (resp. \( B^*_s \)) is stable under \( T(n) \) and \( U(p) \) (resp. \( T^*(n) \) and \( U^*(p) \)) and \( \Omega_{B_s/\mathbb{C}} \) (resp. \( \Omega_{B^*_s/\mathbb{C}} \)) is spanned by \( f_{p,n}dz \) (resp. \( g_{p,n}dz \) for \( p = w_s(f_P) \)) for \( n \) running over \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \). We mainly apply Corollary 6.3 and Proposition 6.4 taking \( C_s \) (resp. \( D_s \)) to be \( A_s \) (resp. \( B_s \)).

7. Abelian factors of modular Jacobians

Let \( k \) be a finite extension of \( \mathbb{Q} \) inside \( \overline{\mathbb{Q}} \) or a finite extension of \( \mathbb{Q}_p \) over \( \overline{\mathbb{Q}}_p \). We study the control theorem for \( J_s(k) \) which is not covered in \([H15a]\).

Let \( A_s \) be an abelian subvariety of \( J_r \) defined over \( k \). Write \( A_s \) (\( s \geq r \)) for the image of \( A_r \) in \( J_s \) under the morphism \( \pi^*: J_r \to J_s \) given by Picard functoriality from the projection \( \pi : X_r \to X_s \). Hereafter we assume

(A) We have \( \alpha \in h(N) \) such that \((\nu - 1) = ax \) with \( x \in h(N) \) and that \( h/\alpha \) is free of finite rank over \( \mathbb{Z}_p \). Write \( \alpha_s \) for the image in \( h_s(N) \) (\( s \geq r \)) and \( a_s = (\alpha_s, b_s) \cap h_s(\mathbb{Z}_p) \) and put \( A_s = J_s[\alpha_s] \) and \( B_s = J_s[\alpha_s^\prime] \), where \( \alpha_s J_s \) is an abelian variety defined over \( \mathbb{Q} \) of \( J_s \) with \( a_s J_s(\overline{\mathbb{Q}}) = \sum_{\alpha \in a_s} a(J_s(\mathbb{Q})) \subseteq J_s(\overline{\mathbb{Q}}) \).

Here for \( s > s' \), coherency of \( \alpha_s \) means the following commutative diagram:

\[
\begin{array}{ccc}
\hat{J}_{s'}^{\text{ord}} & \xrightarrow{\pi^*} & \hat{J}_{s}^{\text{ord}} \\
\downarrow{\alpha_{s'}} & & \downarrow{\alpha_s} \\
\hat{J}_{s'}^{\text{ord}} & \xrightarrow{\pi^*} & \hat{J}_{s}^{\text{ord}}
\end{array}
\]

which is equivalent (by the self-duality of \( J_s \)) to the commutativity of

\[
\begin{array}{ccc}
\hat{J}_{s}^{\text{co-ord}} & \xrightarrow{\pi^*} & \hat{J}_{s'}^{\text{co-ord}} \\
\downarrow{\alpha_s} & & \downarrow{\alpha_{s'}} \\
\hat{J}_{s}^{\text{co-ord}} & \xrightarrow{\pi^*} & \hat{J}_{s'}^{\text{co-ord}}
\end{array}
\]

The following fact is proven in \([H15c, \text{Lemma 5.1}]\):

Lemma 7.1. Assume (A). Then we have \( \hat{A}_s^{\text{ord}} = \hat{J}_s^{\text{ord}}[\alpha_s] \) and \( \hat{A}_s^{\text{s}} = \hat{A}_s \). The abelian variety \( A_s^{\text{s}} \) (\( s > r \)) is the image of \( A_s^r \) in \( J_s \) under the morphism \( \pi^* = \pi^*_{s,r} : J_r \to J_s \) induced by Picard functoriality from the projection \( \pi = \pi_{s,r} : X_s \to X_r \) and is \( \mathbb{Q} \)-isogenous to \( B_s \). The morphism \( J_s \to B_s \) factors through \( J_s \) \((\nu^* : J_s \to J_r) \to B_r \). In addition, the sequence

\[
0 \to \hat{A}_s^{\text{ord}} \to \hat{J}_s^{\text{ord}} \xrightarrow{\alpha_s} \hat{J}_s^{\text{ord}} \to \hat{B}_s^{\text{ord}} \to 0 \quad \text{for} \quad 0 < \nu \leq r \leq s < \infty
\]

is an exact sequence of fppf sheaves.

This implies

Corollary 7.2. Let \( R = k \) if \( k \) is local, and let \( R \) be the \( S \)-integer ring of \( k \) (i.e., primes in \( S \) is inverted in \( R \)) if \( k \) is a number field. Then the sheaf \( \alpha_s(\hat{J}_s^{\text{ord}}) \) is a \( p \)-divisible étale/fppf sheaf over \( \text{Spec}(R) \), and its \( p \)-torsion part \( \alpha_s(\hat{J}_s^{\text{ord}})[p^\infty] \) is a \( p \)-divisible Barsotti-Tate group over \( R \).

In particular, the Tate module \( T_p\alpha_s(\hat{J}_s^{\text{ord}}) \) is well defined free \( \mathbb{Z}_p \)-module of finite rank for all \( r \leq s < \infty \).
Proof. By the above lemma, the fppf sheaf $\alpha_s(J^{\text{ord}}_s) = \ker(\tilde{\text{ord}}_s)\, \rho_s, \tilde{\text{ord}}_s)$ fits into the following commutative diagram with exact rows:

\[
\begin{array}{cccc}
A_s[p^\infty]^{\text{ord}} & \hookrightarrow & J_s[p^\infty]^{\text{ord}} & \twoheadrightarrow & \alpha(J_s[p^\infty]^{\text{ord}}) \\
\cap & & \cap & & \cap \\
\tilde{A}_s^{\text{ord}} & \twoheadrightarrow & \tilde{J}_s^{\text{ord}} & \twoheadrightarrow & \alpha(\tilde{J}_s^{\text{ord}}) \\
\downarrow & & \downarrow & & \downarrow \\
\tilde{A}_s^{\text{ord}}/A_s[p^\infty]^{\text{ord}} & \twoheadrightarrow & \tilde{J}_s^{\text{ord}}/J_s[p^\infty]^{\text{ord}} & \twoheadrightarrow & \alpha(\tilde{J}_s^{\text{ord}})/\alpha(J_s[p^\infty]^{\text{ord}}).
\end{array}
\]

The first two terms of the bottom row are sheaves of $\mathbb{Q}_p$-vector spaces, so is the last term. Thus we conclude $\alpha(J_s[p^\infty]^{\text{ord}}) = \alpha(\tilde{J}_s^{\text{ord}})[p^\infty]$. Since $\tilde{A}_s = \tilde{A}_s^{\text{ord}}$, $\tilde{A}_s[p^\infty]^{\text{ord}}$ is a direct summand of the Barsotti–Tate group $J_s[p^\infty]^{\text{ord}}$. Therefore $\alpha(J_s[p^\infty]^{\text{ord}})$ is a Barsotti–Tate group as desired.

Alternatively, we can identify $\alpha_s(J_s^{\text{ord}})[p^\infty]$ with the Barsotti–Tate $p$-divisible group of the abelian variety quotient $J_s/A_s^{\text{ord}}$.

\[\square\]

The condition (A) is a mild condition. Here are sufficient conditions for $(\alpha, A_s, B_s)$ to satisfy (A) given in [H15c, Proposition 5.2]:

**Proposition 7.3.** Let Spec($T$) be a connected component of Spec($h$) and Spec($I$) be a primitive irreducible component of Spec($T$). Then the condition (A) holds for the following choices of $(\alpha, A_s, B_s)$:

1. Suppose that an eigen cusp form $f = f_p$ new at each prime $|N$ belongs to Spec($T$) and that $T = I$ is regular. Writing the level of $f_p$ as $Np^r$, the algebra homomorphism $\lambda : T \to \mathbb{Q}_p$ given by $f(T(l)) = \lambda(T(l))f$ gives rise to a height 1 prime ideal $P = \ker(\lambda)$, which is principal generated by $a \in T$. This $a$ has its image $a_s \in T_s = T \otimes A_s$ for $A_s = A_s^{\text{ord}}$. Write $h_s = h_s \otimes A_s = T_s + 1, h_s$ as an algebra direct sum for an idempotent $1_s$. Then, the element $\alpha_s = a_s + 1_s \in h_s$ for the identity $1_s$ of $X_s$ satisfies (A). In this case, $\alpha = \lim_{s} \alpha_s$.

2. More generally than (1), we pick a general connected component Spec($T$) of Spec($h$). Pick a (classical) Hecke eigenform $f = f_p$ (of weight 2) for $P \in$ Spec($T$). Assume that $h_s$ for every $s \geq r$ is reduced and $P = (a)$ for $a \in T$, and write $a_s$ for the image of $a$ in $h_s$. Then decomposing $h_s = T_s + 1, h_s$, $\alpha_s = a_s + 1_s$ satisfies (A).

3. Fix $r > 0$. Then $\alpha$ for a factor $\alpha(\gamma^p - 1)$ in $\Lambda$, satisfies (A) for $A_s = J_s[\alpha]$, (the identity connected component).

**Remark 7.4.** (i) Under (1), all arithmetic points $P$ of weight 2 in Spec($I$) satisfies (A).

(ii) For a given weight 2 Hecke eigenform $f$, for density 1 primes $p$ of Spec($f$) $f$ is ordinary at $p$ (i.e., $a(p, f) \neq 0$ mod $p$; see [H13, §7]). Except for finitely many primes $p$ as above, $f$ belongs to a connected component $T$ which is regular (e.g., [F02, §3.1] and [H15c, Theorem 5.3]); so, (1) is satisfied for such $T$.

8. **Mordell–Weil groups of modular abelian factors**

Consider the composite morphism $\varpi_s : A_s \hookrightarrow J_s \twoheadrightarrow B_s$ of fppf abelian sheaves for triples $(\alpha_s, A_s, B_s)$ as in (A), and apply the results in Section 6 to abelian varieties $C_s = A_s$ and $D_s = B_s$. Let $C_s^{\text{ord}} := (\ker(\varpi_s)) \otimes \mathbb{Z}_p^{\text{ord}}$ be the $p$-primary ordinary part of $\ker(\varpi_s)$. Recall we have written $\rho_s$ for the morphism $J_s \twoheadrightarrow B_s$. As before, $\kappa$ is an intermediate extension $K/\kappa/k$ finite over $k$. Define the error terms by

\[
E_s^1(\kappa) := \alpha(\tilde{J}_s^{\text{ord}})(\kappa)/\alpha(\tilde{J}_s^{\text{ord}})(\kappa) \quad \text{and} \quad E_s^2(\kappa) := \text{Coker}(\tilde{J}_s^{\text{ord}})(\kappa) \otimes \mathbb{Z}_p \otimes \tilde{\text{ord}}(\kappa)
\]

for $\rho_s : \tilde{J}_s^{\text{ord}} \twoheadrightarrow \tilde{B}_s^{\text{ord}}(\kappa)$. Note that $E_s^1(\kappa)(\kappa) \subsetneq H^1(A_s^{\text{ord}}) = H^1(A_s^{\text{ord}}) \otimes \mathbb{Z}_p$ and $E_s^2(\kappa) = B_s^{\text{ord}}(\kappa)/\rho_s(\tilde{J}_s^{\text{ord}})(\kappa)(\kappa) \subsetneq H^1(\alpha(\tilde{J}_s^{\text{ord}}))(\kappa)$ are $p$-torsion finite module as long as $s$ is finite.
Lemma 8.1. We have the following commutative diagram with exact rows and exact columns:

\[
\begin{array}{ccc}
E_1^s(\kappa) & \xrightarrow{\alpha} & H_1^r(\tilde{A}_s^{\text{ord}}) \\
\text{onto} & & \text{onto} \\
\alpha(\tilde{J}_s^{\text{ord}}(\kappa)) & \xrightarrow{\alpha(\alpha(\tilde{J}_s^{\text{ord}}(\kappa)))} & \alpha(\tilde{J}_s^{\text{ord}}(\kappa)) \xrightarrow{\rho_s} B_r^{\text{ord}}(\kappa) \xrightarrow{\pi_s} E_2^s(\kappa).
\end{array}
\]

(8.2)

Each term of the bottom two rows is a profinite module if either \(k\) is local or \(S\) is a finite set.

The last assertion follows as \(C_s\) is finite and \(B_r^{\text{ord}}(\kappa)\) is profinite. We will define each maps in the following proof. Since the proof is the same in any cohomology theory: \(H_1^r\) for \(\pi = \text{sm}, \text{fppf}, \text{étale}\) and Galois cohomology. Therefore, we prove the lemma for the Galois cohomology dropping \(\pi\) from the notation. This lemma is valid for the Galois cohomology for infinite \(S\) as is clear from the proof below.

Proof. The bottom row is from the definition of \(E_2^s(\kappa)\) and the left column is by the definition of \(E_1^s(\kappa)\). The middle column is a part of the long exact sequence attached to \(0 \to C_s^{\text{ord}} \to \tilde{A}_s^{\text{ord}} \to \tilde{B}_r^{\text{ord}} \to 0\), where \(\tilde{B}_r^{\text{ord}}\) is identified with \(\tilde{B}_r^{\text{ord}}\) by Lemma 6.2 applied to \(D_s = B_s\). The right column comes from the long exact sequence attached to \(0 \to \alpha(\tilde{J}_s^{\text{ord}}) \to \tilde{J}_s^{\text{ord}} \to B_r^{\text{ord}} \to 0\) again \(\tilde{B}_r^{\text{ord}}\) is identified with \(\tilde{B}_r^{\text{ord}}\). The top row comes from the long exact sequence of \(0 \to \tilde{A}_s^{\text{ord}} \to \tilde{J}_s^{\text{ord}} \xrightarrow{\alpha} \alpha(\tilde{J}_s^{\text{ord}}) \to 0\).

As for the middle row, we consider the following commutative diagram (with exact rows in the category of fppf abelian sheaves):

\[
\begin{array}{ccc}
\alpha(\tilde{J}_s^{\text{ord}}) & \xrightarrow{\alpha} & \tilde{J}_s^{\text{ord}} \\
\text{onto} & & \text{onto} \\
\tilde{J}_s^{\text{ord}} \xrightarrow{\rho_s} & B_r^{\text{ord}}(\kappa) & \xrightarrow{\pi_s} E_2^s(\kappa).
\end{array}
\]

(8.3)

Under this circumstance, we have \(\alpha(\tilde{J}_s^{\text{ord}}) \cap \tilde{A}_s^{\text{ord}} = \alpha(\tilde{J}_s^{\text{ord}}) \times_{\tilde{J}_s^{\text{ord}}} \tilde{A}_s^{\text{ord}} = \text{Ker}(\pi_s)\) which is a finite étale group scheme over \(\mathbb{Q}\). Since the \(p\)-primary part of \(\alpha(\tilde{J}_s^{\text{ord}}) \cap \tilde{A}_s^{\text{ord}}\) is equal to \(\alpha(\tilde{J}_s^{\text{ord}})(\kappa)\), we have \(C_s^{\text{ord}} = \alpha(\tilde{J}_s^{\text{ord}})(\kappa)\).

Note that \(\alpha^2(\tilde{J}_s^{\text{ord}}) = \alpha(\tilde{J}_s^{\text{ord}})\) as sheaves (as \(\alpha : \alpha(\tilde{J}_s^{\text{ord}}) \to \alpha(\tilde{J}_s^{\text{ord}})\) is an isogeny, and hence, \(\alpha(\alpha(\tilde{J}_s^{\text{ord}}(K))) = \alpha(\tilde{J}_s^{\text{ord}}(K))\)). Thus we have a short exact sequence under \(\pi\)-topology for \(\pi = \text{fppf}, \text{sm}\) and ét:

\[
0 \to C_s^{\text{ord}}(K) \xrightarrow{\alpha} \alpha(\tilde{J}_s^{\text{ord}}(K)) \xrightarrow{\alpha} \alpha(\tilde{J}_s^{\text{ord}}(K)) \to 0.
\]

Look into the associated long exact sequence

\[
0 \to \alpha(\tilde{J}_s^{\text{ord}}(k))/\alpha(\alpha(\tilde{J}_s^{\text{ord}}(k))) \to H^1(\alpha(\tilde{J}_s^{\text{ord}}(\kappa)) \to H^1(\alpha(\tilde{J}_s^{\text{ord}})) \xrightarrow{\alpha} H^1(\alpha^2(\tilde{J}_s^{\text{ord}}))
\]

which shows the exactness of the middle row, taking the \(p\)-primary parts (and then the ordinary parts).

In the diagram (8.2), we identify \(\tilde{A}_s^{\text{ord}}\) with \(\tilde{A}_r^{\text{ord}}\) by \(\pi_s^*: J_r \to J_s\) for the projection \(\pi_s^*: X_s \to X_r\) (Picard functoriality); so, the projective system \(\{\tilde{A}_r^{\text{ord}} = \tilde{A}_r^{\text{ord}}, \pi_s^*\}_s\) (Albanese functoriality) gives rise to the nontrivial maps \(\pi_s^*: A_r^{\text{ord}} = A_r^{\text{ord}} \to \tilde{A}_r^{\text{ord}}\) given by \(x \mapsto U(p^{x-v})\). If we write \(H^1(\tilde{A}_r^{\text{ord}}) = (\mathbb{Q}_p/\mathbb{Z}_p)^m \oplus \Delta_r\) for a finite \(p\)-torsion group \(\Delta_r\) by Lemma 2.3 (assuming that \(S\) is finite), we have

\[
\lim_{\pi_s^*: x \mapsto p^{x-v}U(p^{x-v})}(\mathbb{Q}_p/\mathbb{Z}_p)^m \oplus \Delta_r = \mathbb{Q}_p^m.
\]

(8.4)

We quote from [CNF, Corollary 2.7.6] the following fact (which is valid also for infinite \(S\)):
Lemma 8.2. We have \( \lim_s H^1(A_r[p^s]^{\text{ord}}) = H^1(T_r A_r^{\text{ord}}) \).

We give a proof here for the sake of completeness.

**Proof.** More generally, let \( \{M_n\}_n \) be a projective system of finite \( \text{Gal}(k^S/k) \)-modules with surjective transition maps. Let \( B(M_n) \) (resp. \( Z(M_n) \)) be the module of 1-coboundaries (resp. continuous 1-cocycles) \( G \rightarrow M_n \). Let \( B(M_n) \) (resp. \( Z(M_n) \)) be the module of 1-coboundaries (resp. inhomogeneous continuous 1-cocycles) \( G := \text{Gal}(k^S/k) \rightarrow M_n \). We have the exact sequence \( 0 \rightarrow B(M_n) \rightarrow Z(M_n) \rightarrow H^1(G, M_n) \rightarrow 0 \). Plainly for \( m > n \), the natural map \( B(M_m) \rightarrow B(M_n) \) is onto. Thus the above sequences satisfies the Mittag–Leffler condition, and plainly \( \lim_n ?(M_n) =? (\lim_n M_n) \) for \( ? = B, Z \), we have \( \lim_n H^1(k^S/k, M_n) = H^1(k^S/k, \lim_n M_n) \). \( \blacksquare \)

We have the following commutative diagram with exact rows:

\[
\begin{array}{ccc}
\mathcal{C}^{\text{ord}} & \longrightarrow & \widehat{A}^{\text{ord}} \longrightarrow \widehat{B}^{\text{ord}} \\
\downarrow & & \downarrow \iota \\
\mathcal{C}^{\text{ord}}_r & \longrightarrow & \widehat{A}^{\text{ord}}_r \longrightarrow \widehat{B}^{\text{ord}}_r.
\end{array}
\]

By the snake lemma applied to the above diagram, we get the following exact sequence:

\[
0 \rightarrow A_r[p^{s-r}]^{\text{ord}} \rightarrow \mathcal{C}^{\text{ord}} \rightarrow \mathcal{C}^{\text{ord}}_r \rightarrow 0.
\]

Passing to the limit, we have

\[
(8.5) \quad T_r A = \lim_s A_r[p^s]^{\text{ord}} = \lim_s \mathcal{C}^{\text{ord}} \quad \text{and} \quad H^1(T_r A_r^{\text{ord}}) = \lim_s H^1(A_r[p^s]^{\text{ord}}) = \lim_s H^1(\mathcal{C}^{\text{ord}}).
\]

9. **Control theorems**

Taking the projective limit of the exact sequence \( 0 \rightarrow \widehat{A}_s^{\text{ord}} \rightarrow \widehat{J}_s^{\text{ord}} \alpha \rightarrow \widehat{J}_s^{\text{ord}} \), by the vanishing \( \lim_s \widehat{A}_s^{\text{ord}}(\kappa) = 0 \) in Proposition 6.4 applied to \( \mathcal{C}_s = A_s \), we get the injectivity of \( \widehat{J}_s^{\text{ord}} \alpha \rightarrow \widehat{J}_s^{\text{ord}} \).

Since all the terms of the exact sequences: \( 0 \rightarrow \alpha(\widehat{J}_s^{\text{ord}})(\kappa) \rightarrow \widehat{J}_s^{\text{ord}}(\kappa) \rightarrow \frac{\widehat{J}_s^{\text{ord}}(\kappa)}{\alpha(\widehat{J}_s^{\text{ord}})(\kappa)} \rightarrow 0 \) are compact \( p \)-profinite groups, after taking the limit with respect to \( \pi_s^r \), we still have an exact sequence

\[
0 \rightarrow \lim_s \alpha(\widehat{J}_s^{\text{ord}})(\kappa) \rightarrow \lim_s \widehat{J}_s^{\text{ord}}(\kappa) \rightarrow \lim_s \frac{\widehat{J}_s^{\text{ord}}(\kappa)}{\alpha(\widehat{J}_s^{\text{ord}})(\kappa)} \rightarrow 0
\]

with \( \lim_s \frac{\widehat{J}_s^{\text{ord}}(\kappa)}{\alpha(\widehat{J}_s^{\text{ord}})(\kappa)} \hookrightarrow \widehat{B}_r^{\text{ord}}(\kappa) \). Thus

\[
\frac{\widehat{J}_s^{\text{ord}}(\kappa)}{\alpha(\widehat{J}_s^{\text{ord}})(\kappa))} := \lim_s \frac{\widehat{J}_s^{\text{ord}}(\kappa)}{\alpha(\widehat{J}_s^{\text{ord}})(\kappa))} \cong \lim_s \frac{\alpha(\widehat{J}_s^{\text{ord}})(\kappa))}{\alpha(\widehat{J}_s^{\text{ord}})(\kappa))}.
\]

Here the last isomorphism follows from the injectivity of \( \alpha \). By the same token, we have

\[
\alpha(\widehat{J}_s^{\text{ord}})(\kappa)) := \lim_s \alpha(\widehat{J}_s^{\text{ord}})(\kappa)) = \lim_s \frac{\alpha(\widehat{J}_s^{\text{ord}})(\kappa))}{\alpha(\widehat{J}_s^{\text{ord}})(\kappa))}.
\]

Writing \( E^\infty_1(\kappa) = \lim_s E^1_s(\kappa) \) and passing to projective limit of the diagram (8.2), we get the following commutative diagram with exact rows:

\[
\begin{array}{ccc}
E^\infty_1(\kappa) & \longrightarrow & \lim_{s: x \rightarrow p^{s-r} U(p^{s-r})(x)} H^1(\widehat{A}^{\text{ord}}_r) \longrightarrow \lim_{s} H^1(\widehat{J}^{\text{ord}}_r(K)) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & H^1(T_r A_r^{\text{ord}}) \longrightarrow \lim_{s} H^1(\alpha(\widehat{J}^{\text{ord}}_r)(K))[\alpha] \\
\uparrow \rho_s & & \uparrow \\
\frac{\widehat{J}^{\text{ord}}_s(\kappa)}{\alpha(\widehat{J}^{\text{ord}}_s(\kappa))} & \longrightarrow & \widehat{B}^{\text{ord}}(\kappa) \longrightarrow E^\infty_2(\kappa).
\end{array}
\]
The rows are exact since projective limit is left exact. The maps $a$ and $d$ are onto if either $S$ is finite or $k$ is local (as projective limit is exact for profinite modules). By the same token, the right and left columns are also exact. Therefore $E_j^\infty(\kappa)$ ($j = 1, 2$) is a torsion $\Lambda$-module of finite type. As already remarked, the middle column is exact; so, $\alpha$ is injective. Since $\hat{J}_\infty(\kappa)^{\text{ord}}[\alpha] = \hat{\Lambda}_{\infty}(\kappa) = 0$, $\alpha : \alpha(J^\text{ord}_\infty(\kappa)) \to \alpha(J^\text{ord}_\infty(\kappa))$ and $b : \bar{B}^\text{ord}(\kappa) \to H^1(T_p^\text{an},(K)^{\text{ord}})$ are injective.

This shows

**Lemma 9.1.** Let $\kappa$ be a field extension of $\mathbb{Q}$ or $\mathbb{Q}_l$ for a prime $l$, but we assume finiteness condition (2.2) for the extension $\kappa/k$. We allow an infinite set $S$ of places of $k$ when $k$ is finite extension of $\mathbb{Q}$. Let $\alpha$ be as in (A). Then we have the following exact sequences (of $p$-profinite $\Lambda$-modules) up to $\Lambda$-torsion error:

$$0 \to \hat{J}_\infty(\kappa)^{\text{ord}} \xrightarrow{\alpha} \alpha(J^\text{ord}_\infty(\kappa)) \to E_1^\infty(\kappa)^{\text{ord}} \to 0$$

and

$$0 \to \alpha(J^\text{ord}_\infty(\kappa)) \to \hat{J}^\text{ord}_\infty(\kappa) \xrightarrow{\rho^\infty} \bar{B}^\text{ord}_\infty(\kappa) \to E_2^\infty(\kappa) \to 0.$$  

Here $E_j^\infty(\kappa)$ is a $\Lambda$-torsion module of finite type. In particular, taking $\alpha = \gamma - 1$, we conclude that the compact module $\hat{J}_\infty(\kappa)$ is a $\Lambda$-module of finite type.

The statement of this lemma is independent of the set $S$ (though in the proof, we used Galois cohomology groups for finite $S$ if $k$ is global); therefore, the lemma is valid also for an infinite set $S$ of places of $k$ (as long as $S$ contains all $p$-adic and archimedean places and places over $N$).

The left column of (9.1) is made up of compact modules for which projective limit is an exact functor; so, left column is exact; in particular

$$\lim_s \frac{\alpha(J^\text{ord}_s(\kappa))}{\alpha(\alpha(J^\text{ord}_s(\kappa)))} \to E_1^\infty(\kappa) := \lim_s E_1^s(\kappa)$$

is onto.

Take the maximal $\Lambda$-torsion module $X$ inside $\hat{J}_\infty^\text{ord}(\kappa)$. Since $X$ is unique, it is an $h$-module. The module $\hat{J}_\infty^\text{ord}(\kappa)$ is pseudo-isomorphic to $X \oplus \Lambda^R$ for a positive integer $R$. Since $\alpha$ is injective on $\hat{J}_\infty^\text{ord}(\kappa)$, for the $\alpha$-localization $h_{(\alpha)}$, we have $X_{(\alpha)} = X \otimes_h h_{(\alpha)} = 0$. Thus $\hat{J}_\infty^\text{ord}(\kappa) \otimes_h h_{(\alpha)} = \Lambda_p$-free, where $P_{\alpha} = (\alpha) \cap \Lambda$. Thus $\alpha(J^\text{ord}_s(\kappa)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and $\bar{B}^\text{ord}_s(\kappa) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ have equal $\mathbb{Q}_p$-dimension. Therefore, by the injectivity of $\alpha$, $E_1^\infty(\kappa)$ is torsion. However by (8.4), this torsion module is embedded in a $\mathbb{Q}_p$-vector space by the top sequence of (9.1), we have $E_1^\infty(\kappa) = 0$. This shows

**Theorem 9.2.** Let $\alpha$ be as in (A) and $k$ be a finite field extension of either $\mathbb{Q}$ or $\mathbb{Q}_l$ for a prime $l$. Assume (2.2) for the extension $\kappa/k$. Then we have the following exact sequence (of $p$-profinite $\Lambda$-modules):

$$0 \to \hat{J}_\infty^\text{ord}(\kappa) \xrightarrow{\alpha} \hat{J}^\text{ord}_\infty(\kappa) \xrightarrow{\rho^\infty} \bar{B}^\text{ord}_\infty(\kappa) \to E_2^\infty(\kappa) \to 0.$$  

In particular, taking $\alpha = \gamma - 1$, we conclude that the $\Lambda$-module $\hat{J}_\infty(\kappa)$ is a $\Lambda$-module of finite type and that $\hat{J}_\infty(\kappa)$ does not have any pseudo-null $\Lambda$-submodule non null (i.e., $\hat{J}_\infty(\kappa)$ has $\Lambda$-homological dimension $\leq 1$).

By this theorem (applied to $\alpha = \gamma^s - 1$ for $s = 1, 2, \ldots$), the localization $\hat{J}_\infty(\kappa)_P$ at an arithmetic prime $P$ is $\Lambda_P$-free of finite rank, which also follows from [N06, Proposition 12.7.13.4] as $\hat{J}_\infty(\kappa)$ can be realized inside Nekovár’s Selmer group by the embedding of Lemma 2.1.

10. Error term and the second Tate–Shafarevich group

Now let $k$ be either a number field or an $l$-adic field. When $k$ is a number field, the set $S$ of places of $k$ we have chosen can be infinite and can be the set of all places of $k$, though we loose some properties special for finite $S$ (so, we proceed carefully). As before, we write $H^q(M)$ for $H^q(k,M)$ (resp. $H^q(k^S/k,M)$) if $k$ is local (resp. global).

For any abelian variety $X/k$, we have an exact sequence $\hat{X}(k) \to H^1(T_p,X) \to \lim_n H^1(X)[p^n]$ by Lemma 2.1. Similarly, by Corollary 7.2, Lemma 2.2 tells us that $\alpha(J^\text{ord}_s(\kappa)) \to H^1(T_p,\alpha(J^\text{ord}_s)(\kappa)) \to E_1^s(\kappa) \to 0$.
\[ \lim_{n} H^1(\alpha(J_{s}^{\text{ord}}))[p^n] \] is exact. Thus we have the following commutative diagram in which the first two columns and the first three rows are exact by Lemma 8.2 and left exactness of the formation of projective limits combined (the surjectivity of the three horizontal arrows \( c_j \) \( (j = 1, 2, 3) \) are valid if \( S \) is finite or \( k \) is local):

\[
\begin{array}{c}
\alpha(J_{s}^{\text{ord}})(k) \xrightarrow{\iota} H^1(T_p \alpha(J_{s}^{\text{ord}})) \xrightarrow{\lim_{n}} H^1(\alpha(J_{s}^{\text{ord}}))[p^n] \\
\cap \downarrow a \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \q
Proof. Applying the snake lemma to the first two rows of (10.1), we find that $b$ is injective.

If $k$ is local, for an abelian variety $X$ over $k$ with $X' := \text{Pic}_X^0$, $X'(k)$ is isomorphic to $\mathbb{Z}_p^m$ times a finite group; so, if $l \neq p$, $X'(k)$ is finite p-group. By [ADT, I.3.4], $H^1(k, X) \cong X'(k)^\vee$; so, $H^1(k, \widehat{E}_r)$ is a finite group; so, $H^1(k, J^\text{ord}_s)$ and $H^1(k, \widehat{B}^\text{ord}_r)$ are finite groups. Therefore $T_pH^1(k, J^\text{ord}) = T_pH^1(k, \widehat{B}^\text{ord}) = 0$. Since $b$ is injective, $T_pH^1(k, \alpha(J^\text{ord}_s)) = 0$; so, $\text{Ker}(h) = \text{Im}(b) = 0$.

We note the following fact: If $k$ is local non-archimedean, for an abelian variety $A$ over $k$,

$$H^2(k, \widehat{A}) = H^2(k, A) = 0$$

for any abelian variety $A$ over $k$.

This follows from [ADT, Theorem I.3.2], since $H^2(k, \widehat{A}) = H^2(k, A) \otimes_{\mathbb{Z}} \mathbb{Z}_p$.

**Proposition 10.3.** Suppose that $k$ is local. Then we have

1. If $k$ is a finite extension of $\mathbb{Q}_l$ with $l \neq p$, then $E^2_2(k) = 0$.
2. If $k$ is a finite extension of $\mathbb{Q}_p$, and $A_r$ does not have a factor having split multiplicative reduction over the integer ring $W_r$ of $k[\mu_p]$, then $E^2_2(k)$ is finite of bounded order. In particular, if $A_r$ has good reduction over $W$ and $p > 2$, we have $|E^2_2(k)| \leq |A_r[p^\infty]^{\text{ord}(\mathbb{Z})}|^2$.

Proof. We first prove the assertion (1). Since the left column of (10.1) by Lemma 10.2 if $l \neq p$, applying the snake lemma to the middle two exact rows of (10.1), we find an exact sequence

$$0 \to E^2_2(k) \xrightarrow{\subset \to} \text{Im}({\varpi}_s) \to \text{Coker}(h) \to 0.$$

This implies $E^2_2(k) \hookrightarrow \text{Im}({\varpi}_s)$.

Let $X/k$ be a $p$-divisible Barsotti–Tate group. We have $H^2(k, T_p(X)) = \varinjlim_n H^2(k, X[p^n])$ (e.g., [CNF, 2.7.6]). By Tate duality (e.g., [MFG, Theorem 4.43]), we have $H^2(k, X[p^n]) \cong X^t[p^n](k)^\vee$ for the Cartier dual $X^t := \text{Hom}(T_pX, \mu_{p^n})$ of $X$. Thus we have

$$H^2(k, T_pX) = \varinjlim(X[p^n](k)^\vee) \cong (\varinjlim_n H^0(k, X^t[p^n]))^\vee,$$

since we have a canonical pairing $X[p^n] \times X^t[p^n] \to \mu_{p^n}$ (i.e., $X^t[p^n](k)^\vee \cong X[p^n](-1)(k)$).

Apply this to the complement $X$ of $A_s[p^\infty]^{\text{ord}}$ in $J_s[p^\infty]^{\text{ord}}$, so, $X + A_s[p^\infty]^{\text{ord}} = J_s[p^\infty]^{\text{ord}}$ with finite $X \cap A_s[p^\infty]^{\text{ord}}$. Requiring $X$ to be stable under $h_s$, for $h_s(Q_p) = h_s \otimes_{\mathbb{Z}_p} Q_p$, $X$ is uniquely determined as $h_s(Q_p) = (h_s(Q_p)/\alpha_s h_s(Q_p)) \oplus 1_{\text{Id}} h_s(Q_p)$ for an idempotent $1_s$ (so, $X = 1_s J_s[p^\infty]^{\text{ord}}$).

By local Tate duality, we get $H^2(k, T_pX) \cong H^0(k, X[p^\infty]^{\vee})$ and conclude

$$H^2(k, T_pX) \cong \varinjlim_n H^0(k, \text{Hom}(X[p^n](k), \mu_{p^n}(k))) = \varinjlim_n X[p^n](-1)(k) = X[p^\infty](-1)(k).$$

Thus we conclude the injectivity:

$$H^2(k, T_pX) \cong X[p^\infty](-1)(k) \xrightarrow{\subset \to} J_s[p^\infty]^{\text{ord}}(-1)(k) \cong H^2(k, T_pJ_s)^{\text{ord}},$$

which is injective as $X \subset J_s[p^\infty]^{\text{ord}}$. By definition, we have $X + A_s[p^\infty]^{\text{ord}} = J_s[p^\infty]^{\text{ord}}$. By the assumption (A) and the definition of $X$, $X = \alpha_s(J_s[p^\infty]^{\text{ord}})$. Therefore we get an injection:

$$H^2(k, T_p\alpha(J_s^{\text{ord}})) \cong H^2(k, T_p\alpha(J_s[p^\infty]^{\text{ord}})) \cong \alpha(J_s[p^\infty]^{\text{ord}})(-1)(k) \xrightarrow{\varphi_2} J_s[p^\infty]^{\text{ord}}(-1)(k) \cong H^2(k, T_pJ_s)^{\text{ord}}.$$

We have an exact sequence

$$H^1(k, T_p\alpha(J_s^{\text{ord}})) \rightarrow H^2(k, T_pA_r)^{\text{ord}} \rightarrow H^2(k, T_pJ_s)^{\text{ord}}.$$

Since $\varphi_2$ is injective, we find $\text{Im}(\varphi_2) = 0$; so, $E^2_2(k) = 0$ if $k$ is l-adic with $l \neq p$.

We now prove (2). Write $k_s = k[\mu_p]$. As shown in [H15c, §17], the sequence $\widehat{A}^{\text{ord}}(k_s) \hookrightarrow \widehat{J}^{\text{ord}}(k_s)$ is exact. By the cohomology exact sequence, $E^2_2(k)$ injects into $H^1(k_s / k, \widehat{A}^{\text{ord}}(k_s))$. To see the finiteness of $H^1(k_s / k, \widehat{A}^{\text{ord}}(k_s))$, we look into the inflation-restriction exact sequence,

$$0 \rightarrow H^1(k_r / k, \widehat{A}^{\text{ord}}(k_r)) \rightarrow H^1(k_s / k, \widehat{A}^{\text{ord}}(k_s)) \rightarrow H^0(k_r / k, H^1(k_s / k, \widehat{A}^{\text{ord}}(k_s))).$$
The left term $H^1(k_r/k, \widehat{\Lambda}_{r_{\text{ord}}}(k_r))$ is plainly finite. The right term $H^1(k_{\infty}/k_r, \widehat{\Lambda}_{r_{\text{ord}}}(k_{\infty}))$ is finite by [Sc83, Proposition 2 and Lemma 3] and [Sc87, Theorem 1] as $A_r$ has semi-stable but not split multiplicative reduction over $W_r$ (see [H15c, §17] for detailed explanation on this point). In particular, if $A_r$ has good reduction over $W$ and $p > 2$, 

$$|H^1(k_{\infty}/k, \widehat{\Lambda}_{r_{\text{ord}}}(k_{\infty}))| = |A_r[p^\text{ord}]|^2$$

by [Sc83, Proposition 2 and Lemma 3] (or [Sc82, Proposition 2]). Strictly speaking, Schneider assumes in [Sc83, §7] that $A_r$ has ordinary good reduction, but his argument works well without change replacing $(A_r(p) := A_r[p_{\infty}], A_r)$ there by $(A_r[p^\text{ord}], \widehat{A}_r)$. \hfill $\square$

Return to a number field $k$. Choose a finite $S$ sufficiently large (containing all archimedean places and all places above $Np$). Applying now the snake lemma to the middle two exact rows of (10.1), we find that

$$(10.4) \quad E_2^s(k) \xrightarrow{\alpha} \text{Im}(\varpi_s) \subset H^2(k^S/k, T_p\alpha(\widehat{\Lambda}_{r_{\text{ord}}}))$$

is injective.

Consider the $A$-BT group $G = \lim_n G_s$ for $G_s = J_s[p^\text{ord}]$. Write $T\tilde{G}_s = \lim_n G_s[p^n](\overline{Q})$. Then \{$T\tilde{G}_s, \pi_s^*\}_s$ forms a projective system of Tate modules, and $T\tilde{G} = \lim_s T\tilde{G}_s$ is $A$-free of finite rank (see [H14, Section 4]). Thus for maximal ideal $m$ of $A$, we have $T\tilde{G} = \lim_n G[m^n](\overline{Q})$ and $G[m^n](\overline{Q})$ is a finite Galois module. By [CNF, Corollary 2.7.6], we have

$$H^2(k^S/k, T\tilde{G}) = \lim_{n} H^2(k^S/k, G[m^n]) = \lim_{n} H^2(k^S/k, T\tilde{G}_s) = H^2(k^S/k, T_p\alpha(\widehat{\Lambda}_{r_{\text{ord}}}))$$

Thus passing to the limit of (10.4), we get

**Theorem 10.4.** Suppose that $k$ is a number field and that $A_r$ has potentially good reduction at every $p$-adic place of $k$. Pick a finite set $S$ of places of $k$ containing all places over $Np$ and archimedean places. Then we have a Hecke-linear embedding $e_\infty : E_2^\infty(k) \hookrightarrow H^1(k^S/k, T\tilde{G})$ whose image is almost $\Pi_2^\ell(T\tilde{G})[\alpha]$, up to finite error. Here $\Pi_2^\ell(T\tilde{G}) = \text{Ker}(H^1(k^S/k, T\tilde{G}) \rightarrow \prod_{i \in S} H^1(k_i/k, T\tilde{G}_i))$ (i = 1, 2) for $i$ running over all prime ideals of $k$ and $\Pi_2^\ell(T\tilde{G})[\alpha] = \text{Ker}(\alpha : \Pi_2^\ell(T\tilde{G}) \xrightarrow{\alpha} \Pi_2^\ell(T\tilde{G}))$.

Nekovář computed the error term corresponding to $E_2^\infty(k)$ for his Selmer group in [N06, Proposition 12.7.13.4] replacing the Mordell–Weil group by his Selmer group $H_1^1(T\tilde{G})$, and his error term is controlled by $H^2(T\tilde{G})[\alpha]$ localized at $(\alpha)$. Thus this theorem is a Mordell–Weil version of Nekovář’s result.

**Proof.** By (10.4) (after passing to the limit), we get an inclusion

$$E_2^\infty(k) \subset H^2(\alpha(T\tilde{G})).$$

Since $E_2^\infty(k)$ is killed by $\alpha$, we get $E_2^\infty(k) \subset H^2(\alpha(T\tilde{G}))[\alpha]$. From the following commutative diagram with exact rows for $s' > s$:

$$
\begin{array}{cccc}
A_r[p^\text{ord}] & \xrightarrow{\alpha} & G_s & \xrightarrow{\alpha} & G_s & \xrightarrow{\alpha} & B_r[p^\text{ord}] \\
\downarrow p^{-s}U(p^{s'-s}) & & \downarrow \pi_s & & \downarrow \pi_s & & \downarrow \pi_s \\
A_r[p^\text{ord}] & \xrightarrow{\alpha} & G_{s'} & \xrightarrow{\alpha} & G_{s'} & \xrightarrow{\alpha} & B_r[p^\text{ord}].
\end{array}
$$

we get a commutative diagram of Tate modules with exact rows:

$$
\begin{array}{cccc}
T_pA_r & \xrightarrow{\alpha} & T\tilde{G}_s & \xrightarrow{\alpha} & T\tilde{G}_s & \xrightarrow{\alpha} & T_pB_r \\
\downarrow p^{-s}U(p^{s'-s}) & & \downarrow \pi_s & & \downarrow \pi_s & & \downarrow \pi_s \\
T_pA_{s'} & \xrightarrow{\alpha} & T\tilde{G}_{s'} & \xrightarrow{\alpha} & T\tilde{G}_{s'} & \xrightarrow{\alpha} & T_pB_{s'}.
\end{array}
$$

Since we know

$$\text{Coker}(p^{s'-s}U(p^{s'-s}) : T_pA_{s'} \rightarrow T_pA_r) \cong T_pA_r/p^{s'-s}T_pA_r,$$

\]
we conclude $\lim_{\alpha} T_p A^\alpha = 0$. Passing to the limit (as projective limit is an exact functor for compact projective systems), we get the following exact sequence

$$0 \rightarrow TG \xrightarrow{\alpha} TG \rightarrow T_p B^\alpha \rightarrow 0.$$  

This implies $\alpha(TG) \cong TG$ and $H^2(TG)[\alpha] \cong \text{Coker}(H^1(TG) \rightarrow H^1(T_p B^\alpha))$.

Pick a prime ideal $t$. Again by (10.4) (after passing to the limit), we get the following commutative diagram

$$E_2^\infty(k) \xrightarrow{e_\infty} H^2(k, \alpha(TG)) \xleftarrow{\sim_\alpha} H^2(k, TG)$$

$$\downarrow \quad \text{Res}_t \quad \downarrow \quad \text{Res}_t$$

$$E_2^\infty(k_t) \xrightarrow{e_\infty} H^2(k_t, \alpha(TG)) \xleftarrow{\sim_\alpha} H^2(k_t, TG)$$

From Proposition 10.3, we know $\prod_{l \in S} E_2^\infty(k_l) = \prod_{P \mid l} E_2^\infty(k_l)$ is finite. Thus $E_2^\infty(k)$ lands in $\prod_{l \in S} H^2(k_l, TG)[\alpha]$ up to finite error, as desired. □

In summary, we have proven the following theorem:

**Theorem 10.5.** Let $\alpha$ be as in (A) and $k$ be a finite field extension of $\mathbb{Q}$. Suppose that $A_\alpha$ has potentially good reduction at every $p$-adic places of $k$. Pick a finite set $S$ of places of $k$ which contains all archimedean places and places over $Np$. Then we have the following exact sequence (of $p$-profinite $\Lambda$-modules):

$$0 \rightarrow \hat{J}_\infty^\alpha(k) \xrightarrow{\alpha} \hat{J}_\infty^\alpha(k) \xrightarrow{\rho_{\infty}} B^\pi_{\infty} \xrightarrow{\pi_{\infty}} H^2(k^S/k, TG),$$

and $\text{Im}(\pi_{\infty})$ is contained in $\prod_{l \in S} H^2(k_l, TG)[\alpha]$ up to finite error. In particular, taking $\alpha$ to be a factor of $(t^P - 1)/(t - 1)$, we conclude that the $\Lambda$-module $\hat{J}_\infty^\alpha$ is a $\Lambda$-module of finite type and that $\hat{J}_\infty^\alpha$ does not have any pseudo-null $\Lambda$-submodule non null (i.e., $\hat{J}_\infty^\alpha$ has $\Lambda$-homological dimension $\leq 1$).

Applying the above theorem to $\alpha = (t^P - 1)/(t - 1)$, for any prime factor $P$ of $\alpha$ in $\Lambda$, the localization $\hat{J}_\infty^\alpha P$ of $\hat{J}_\infty^\alpha$ at $P$ is $\Lambda_P$-torsion-free and hence is a free $\Lambda_P$-module of finite rank, since $\Lambda_P$ is a discrete valuation ring. We hope to prove $\Lambda$-torsion-ness of $\prod_{l \in S} H^2(k_l, TG)$ for a sufficiently large $S$ in a forthcoming paper; so, the error term $E_2^\infty(k)$ should be a finite module for most $\alpha$.

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