

BIG GALOIS REPRESENTATIONS AND p -ADIC L -FUNCTIONS

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ABSTRACT. Let $p \geq 5$ be a prime. If an irreducible component of the spectrum of the ‘big’ ordinary Hecke algebra does not have complex multiplication, under mild assumptions, we prove that the image of its Galois representation contains, up to finite error, a principal congruence subgroup $\Gamma(L)$ of $SL_2(\mathbb{Z}_p[[T]])$ for a principal ideal $(L) \neq 0$ of $\mathbb{Z}_p[[T]]$ for the canonical “weight” variable T . If nontrivial (i.e., $L \neq 1$), the power series L is proven to be a factor of the Kubota-Leopoldt p -adic L -function or of the square of the anticyclotomic Katz p -adic L -function (or a power of $(1+T)^{p^m} - 1$).

Throughout the paper, we fix a prime $p \geq 3$, field embeddings $\mathbb{C} \xrightarrow{i_\infty} \overline{\mathbb{Q}} \xrightarrow{i_p} \overline{\mathbb{Q}_p} \subset \mathbb{C}_p$ and a positive integer N prime to p . Let χ be a Dirichlet character modulo Np^{r+1} . Consider the space of modular forms $\mathcal{M}_{k+1}(\Gamma_0(Np^{r+1}), \chi)$ with $(p \nmid N, r \geq 0)$ (containing Eisenstein series) and cusp forms $\mathcal{S}_{k+1}(\Gamma_0(Np^{r+1}), \chi)$. Here χ is the Neben-typus. Let $\mathbb{Z}[\chi] \subset \overline{\mathbb{Q}}$ and $\mathbb{Z}_p[\chi] \subset \overline{\mathbb{Q}_p}$ be the rings generated by the values χ over \mathbb{Z} and \mathbb{Z}_p , respectively. The Hecke algebra $H = H_{k+1}(\Gamma_0(Np^{r+1}), \chi; \mathbb{Z}[\chi])$ over $\mathbb{Z}[\chi]$ is

$$H = \mathbb{Z}[\chi][T(n) | n = 1, 2, \dots] \subset \text{End}(\mathcal{M}_{k+1}(\Gamma_0(Np^{r+1}), \chi)).$$

For any $\mathbb{Z}[\chi]$ -algebra $A \subset \mathbb{C}$, $H_{k+1}(\Gamma_0(Np^{r+1}), \chi; A) = H \otimes_{\mathbb{Z}[\chi]} A$ is actually the A -subalgebra of $\text{End}(\mathcal{M}_{k+1}(\Gamma_0(Np^{r+1}), \chi))$ generated over A by the $T(l)$ ’s. Then we put

$$H_{k+1, \chi} = H_{k+1, \chi/W} = H_{k+1}(\Gamma_0(Np^{r+1}), \chi; W) := H \otimes_{\mathbb{Z}[\chi]} W$$

for a p -adic complete discrete valuation ring $W \subset \mathbb{C}_p$ containing $\mathbb{Z}_p[\chi]$. Let $\Lambda = \mathbb{Z}_p[[T]]$ (resp. $\Lambda_W = W[[T]]$), and write Q (resp. Q_W) for the quotient field of Λ (resp. Λ_W).

We often write our $T(l)$ as $U(l)$ when N is divisible by l . The ordinary part $\mathbf{H}_{k+1, \chi/W} \subset H_{k+1, \chi/W}$ is then the maximal ring direct summand on which $U(p)$ is invertible. We write e for the idempotent of $\mathbf{H}_{k+1, \chi/W}$; so, e is the p -adic limit in $H_{k+1, \chi/W}$ of $U(p)^{n!}$ as $n \rightarrow \infty$. We write the image of the idempotent as \mathcal{M}_{k+1}^{ord} for modular forms and \mathcal{S}_{k+1}^{ord} for cusp forms. Let $\chi_1 =$ the N -part of $\chi \times$ the tame p -part of χ . Then, by [H86a] and [H86b] (and [GME] §3.2), we have a unique ‘big’ Hecke algebra $\mathbf{H} = \mathbf{H}_{\chi_1/W}$ such that

- (1) \mathbf{H} is free of finite rank over Λ_W equipped with $T(n) \in \mathbf{H}$ for all n ,
- (2) if $k \geq 1$ and $\varepsilon : \mathbb{Z}_p^\times \rightarrow \mu_{p^\infty}(W)$ is a character,

$$\mathbf{H}/(1 + T - \varepsilon(\gamma)\gamma^k)\mathbf{H} \cong \mathbf{H}_{k+1, \varepsilon\chi_k} \text{ for } \chi_k := \chi_1\omega^{1-k} \ (\gamma = 1 + p \in \mathbb{Z}_p^\times),$$

sending $T(n)$ to $T(n)$, where ω is the Teichmüller character.

The corresponding objects for cusp forms are denoted by the corresponding lower case characters; so, $h = \mathbb{Z}[\chi][T(n) | n = 1, 2, \dots] \subset \text{End}(\mathcal{S}_{k+1}(\Gamma_0(Np^{r+1}), \chi))$, $h_{k+1, \chi/W} = h_{k+1}(\Gamma_0(Np^{r+1}), \chi; W) := h \otimes_{\mathbb{Z}[\chi]} W$, the ordinary part $\mathbf{h}_{k+1, \chi} \subset h_{k+1, \chi}$ and the big cuspidal Hecke algebra \mathbf{h}_W . Replacing modular forms by cusp forms (and upper case symbols by lower case symbols), we can construct the “big” cuspidal Hecke algebra $\mathbf{h} = \mathbf{h}_{\chi_1}(N)$. Then, similarly to the case of modular forms, we have the following characterization of the cuspidal ‘big’ Hecke algebra $\mathbf{h} = \mathbf{h}_{\chi_1/W} = \mathbf{h}_{\chi_1}(N)$:

- (1) \mathbf{h} is free of finite rank over Λ_W equipped with $T(n) \in \mathbf{h}$,
- (2) $\mathbf{h}/(1 + T - \varepsilon(\gamma)\gamma^k)\mathbf{h} \cong \mathbf{h}_{k+1, \varepsilon\chi_k}$ sending $T(n)$ to $T(n)$, if $k \geq 1$.

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We have a surjective Λ_W -algebra homomorphism $\mathbf{H} \rightarrow \mathbf{h}$ sending $T(n)$ to $T(n)$. We have a 2-dimensional semi-simple Galois representation $\rho_{\mathbf{H}}$ (of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$) with coefficients in the total quotient ring $Q(\mathbf{H})$ of \mathbf{H} (see [H86b] and [GME] §4.3). Here the total quotient ring $Q(\mathbf{H})$ is the ring of fractions by the multiplicative set of all non-zero divisors; so, $Q(\mathbf{H}) = \mathbf{H} \otimes_{\Lambda} \mathbb{Q}$. This representation preserves an \mathbf{H} -lattice $\mathcal{L} \subset Q(\mathbf{H})^2$ (i.e., an \mathbf{H} -submodule of $Q(\mathbf{H})^2$ of finite type which span $Q(\mathbf{H})^2$ over $Q(\mathbf{H})$), and as a map of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ into the profinite group $\text{Aut}_{\mathbf{H}}(\mathcal{L})$, it is continuous. The representation $\rho_{\mathbf{H}}$ restricted to the p -decomposition group $D_p = \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ (associated to i_p) is isomorphic to an upper triangular representation with unramified rank 1 quotient. Write $\rho_{\mathbf{H}}^{\text{ss}}$ for the semi-simplification over D_p . As is well known now (e.g., [GME] §4.3.2), $\rho_{\mathbf{H}}$ satisfies

$$(\text{Gal}) \quad \text{Tr}(\rho_{\mathbf{H}}(\text{Frob}_l)) = T(l) \quad (l \nmid Np), \quad \rho_{\mathbf{H}}^{\text{ss}}([\gamma^s, \mathbb{Q}_p]) \sim \begin{pmatrix} (1+T)^s & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \rho_{\mathbf{H}}^{\text{ss}}([p, \mathbb{Q}_p]) \sim \begin{pmatrix} * & 0 \\ 0 & U(p) \end{pmatrix},$$

where $\gamma^s = (1+p)^s \in \mathbb{Z}_p^\times$ for $s \in \mathbb{Z}_p$ and $[x, \mathbb{Q}_p]$ is the local Artin symbol. For each prime $P \in \text{Spec}(\mathbf{H})$, let $\kappa(P)$ be the residue field of P . Then $\text{Tr}(\rho_{\mathbf{H}}) \bmod P$ has values in \mathbf{H}/P , and by the technique of pseudo representations (cf. [MFG] §2.2), we can construct a unique semi-simple Galois representation $\rho_P : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow GL_2(\kappa(P))$ such that $\text{Tr}(\rho_P(\text{Frob}_l)) = (T(l) \bmod P)$. For any ideal $\mathfrak{a} \subset \mathbf{H}$ with reduced \mathbf{H}/\mathfrak{a} , we write $\rho_{\mathfrak{a}} = \prod_P \rho_P : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow GL_2(Q(\mathbf{H}/\mathfrak{a}))$ for the total quotient ring $Q(\mathbf{H}/\mathfrak{a})$ of \mathbf{H}/\mathfrak{a} , where P runs over minimal primes of $\text{Spec}(\mathbf{H}/\mathfrak{a})$. If $\mathfrak{a} = \text{Ker}(\mathbf{H} \rightarrow \mathbb{I})$ (resp. $\mathfrak{a} = \text{Ker}(\mathbf{H} \rightarrow \mathbb{T}^{\text{red}})$) for an irreducible component $\text{Spec}(\mathbb{I}) \subset \text{Spec}(\mathbf{h})$ (resp. a connected component $\text{Spec}(\mathbb{T}) \subset \text{Spec}(\mathbf{h})$), we write $\rho_{\mathbb{I}}$ (resp. $\rho_{\mathbb{T}}$) for $\rho_{\mathfrak{a}}$, where \mathbb{T}^{red} is \mathbb{T} modulo its nilradical. If \mathbb{T} or its irreducible component $\text{Spec}(\mathbb{I}) \subset \text{Spec}(\mathbb{T})$ is fixed in the context, we write $\bar{\rho} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow GL_2(\mathbb{F})$ for $\rho_{\mathfrak{m}_{\mathbb{T}}} = \rho_{\mathfrak{m}_{\mathbb{I}}}$ for the maximal ideals $\mathfrak{m}_{\mathbb{T}}$ of \mathbb{T} and $\mathfrak{m}_{\mathbb{I}}$ of \mathbb{I} .

Let $\text{Spec}(\mathbb{I})$ be an irreducible (reduced) component of $\text{Spec}(\mathbf{H})$ and write its normalization as $\text{Spec}(\tilde{\mathbb{I}})$. We often call \mathbb{I} a component of \mathbf{H} . We denote by $Q(\mathbb{I})$ for the quotient field of \mathbb{I} . We call a prime ideal $P \subset R$ of a ring R a *prime divisor* if $\text{Spec}(R/P)$ has codimension 1 in $\text{Spec}(R)$. We call an ideal \mathfrak{D} of \mathbb{I} a *divisor* if $\mathfrak{D} = \bigcap_P P^{m_P}$ for finitely many prime divisors P . Write $a(n)$ for the image of $T(n)$ (n prime to Np) in \mathbb{I} and $a(l)$ for the image of $U(l)$ if $l \mid Np$. If a prime divisor P of $\text{Spec}(\mathbb{I})$ contains $(1+T - \varepsilon(\gamma)\gamma^k)$ with $k \geq 1$, by (2) we have a Hecke eigenform $f_P \in \mathcal{M}_{k+1}(\Gamma_0(Np^{r(P)+1}), \varepsilon\chi_k)$ such that its eigenvalue for $T(n)$ is given by $a_P(n) := (a(n) \bmod P) \in \overline{\mathbb{Q}_p}$ for all n . A prime divisor P with $P \cap \Lambda_W = (1+T - \varepsilon(\gamma)\gamma^k)$ with $k \geq 1$ and a character $\varepsilon : \mathbb{Z}_p^\times \rightarrow \mu_{p^\infty}(W)$ is called an *arithmetic point* (or prime), and we write $\varepsilon_P = \varepsilon$ and $k(P) = k \geq 1$ for an arithmetic P . Thus \mathbb{I} gives rise to an analytic family $\mathcal{F}_{\mathbb{I}} = \{f_P | \text{arithmetic points } P \text{ in } \text{Spec}(\mathbb{I})(\overline{\mathbb{Q}_p})\}$ of slope 0 classical Hecke eigenforms. A component \mathbb{I} (or the associated family) is called *cuspidal* if $\text{Spec}(\mathbb{I}) \subset \text{Spec}(\mathbf{h})$. A cuspidal component \mathbb{I} is called a *CM component* if there exists a nontrivial character $\xi : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \mathbb{I}^\times$ such that $\rho_{\mathbb{I}} \cong \rho_{\mathbb{I}} \otimes \xi$. If a cuspidal \mathbb{I} is not a CM component, we call it a non CM component.

Put $\Gamma(\mathfrak{a}) = \{x \in SL_2(\Lambda) | x \equiv 1 \pmod{\mathfrak{a}} \cdot M_2(\Lambda)\}$ for an ideal $\mathfrak{a} \subset \Lambda$, and write $\Gamma(L) = \Gamma(\mathfrak{a})$ if $\mathfrak{a} = (L)$ ($L \in \Lambda$). The representation $\rho_{\mathbb{I}} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow GL_2(Q(\mathbb{I}))$ leaves stable an $\tilde{\mathbb{I}}$ -module \mathcal{L} of finite type in $Q(\mathbb{I})^2$ with $Q \cdot \mathcal{L} = Q(\mathbb{I})^2$. We assume throughout the paper, after extending scalar W ,

$$(F) \quad \text{the representation } \rho_{\mathbb{I}} \text{ has values in } GL_2(\tilde{\mathbb{I}})$$

(i.e., we assume to be able to find an $\tilde{\mathbb{I}}$ -free \mathcal{L}).

If $\bar{\rho}$ is absolutely irreducible, by the technique of pseudo representation, (F) can be checked to be true. If \mathbb{I} is a unique factorization domain with $\text{Spec}(\mathbb{I})(W) \neq \emptyset$, in particular, if \mathbb{I} is regular (so far, there is no known non-regular example of $\tilde{\mathbb{I}}$), replacing \mathcal{L} by its reflexive closure (i.e., the intersection of all $\tilde{\mathbb{I}}$ -free modules in $Q(\mathbb{I})^2$ containing \mathcal{L}), \mathcal{L} is free of rank 2 over \mathbb{I} . By resolution of singularity of arithmetic surfaces (see [L]), we can find an injective local Λ -algebra homomorphism $\mathbb{I} \hookrightarrow \mathbb{J} \subset Q(\mathbb{I})$ for a regular two dimensional \mathbb{J} , though \mathbb{J} is not finite over \mathbb{I} . Thus replacing \mathbb{I} by \mathbb{J} (and extending scalar to achieve $\text{Spec}(\mathbb{J})(W) \neq \emptyset$), we have a model $\rho_{\mathbb{J}} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow GL_2(\mathbb{J})$ isomorphic to $\rho_{\mathbb{I}}$ over $Q(\mathbb{I})$. See Section 9 for details of these facts. Anyway, we assume (F) in this paper.

Actually we choose \mathcal{L} coming from the projective limit (relative to p -power level) of Tate modules of modular jacobians, and for this choice of \mathcal{L} , \mathbb{I} -freeness of \mathcal{L} is known if \mathbb{T} is Gorenstein (which in turn follows from irreducibility of $\bar{\rho}$ and the p -distinguished-ness condition (R) below). Thus most cases, we can choose the scalar extension to $\tilde{\mathbb{I}}$ of the canonical \mathcal{L} free over \mathbb{I} . Write $[\rho_{\mathbb{I}}]$ for the

isomorphism class of $\rho_{\mathbb{I}}$ over $Q(\mathbb{I})$. Pick and fix a **non** CM component \mathbb{I} of prime-to- p level N , and assume the following condition (R) throughout the paper

$$(R) \quad \bar{\rho}|_{D_p} \cong \begin{pmatrix} \bar{\epsilon} & * \\ 0 & \bar{\delta} \end{pmatrix} \text{ with } \bar{\delta} \text{ unramified and } \bar{\epsilon} \neq \bar{\delta}.$$

In addition to (R), consider the following conditions:

- (s) $\text{Im}(\rho_{\mathbb{I}})$ and $\rho_{\mathbb{I}}(D_p)$ are both normalized by an element $g \in GL_2(\mathbb{I})$ with $\bar{g} := (g \bmod \mathfrak{m}_{\mathbb{I}})$ having eigenvalues $\bar{\alpha}, \bar{\beta}$ in \mathbb{F}_p with $\bar{\alpha}^2 \neq \bar{\beta}^2$;
- (u) $\rho_{\mathbb{I}}(D_p)$ contains a non-trivial unipotent element $g \in GL_2(\mathbb{I})$;
- (v) $\rho_{\mathbb{I}}(D_p)$ contains a unipotent element $g \in GL_2(\mathbb{I})$ with $g \not\equiv 1 \pmod{\mathfrak{m}_{\mathbb{I}}}$.

Theorem I. *Let the notation be as above. Suppose $p \geq 3$ and one of the two conditions (s) and (u). Then there exist a representation $\rho \in [\rho_{\mathbb{I}}]$ with values in $GL_2(\tilde{\mathbb{I}})$ such that $G := \text{Im}(\rho) \cap SL_2(\Lambda)$ contains $\Gamma(\mathfrak{a})$ for an ideal $0 \neq \mathfrak{a} \subset \Lambda$. If \mathfrak{c} is the Λ -ideal maximal among \mathfrak{a} with $G \supset \Gamma(\mathfrak{a})$, the ideal $\mathfrak{c}_P \subset \Lambda_P$ localized at a prime divisor P of Λ only depends on the isomorphism class $[\rho_{\mathbb{I}}]$ as long as $\rho_{\mathfrak{P}}$ is absolutely irreducible for all prime divisors $\mathfrak{P}|P$ in $\tilde{\mathbb{I}}$; in particular, if $\bar{\rho} = \rho_{\mathfrak{m}}$ for the maximal ideal \mathfrak{m} of \mathbb{I} is absolutely irreducible, the reflexive closure (L) of \mathfrak{c} is independent of the choice of ρ with $G \supset \Gamma(\mathfrak{a}) \neq 1$.*

Here the reflexive closure $\tilde{\mathfrak{c}}$ of an ideal $\mathfrak{c} \subset \Lambda$ means the intersection $\bigcap_{(\lambda) \supset \mathfrak{c}} (\lambda) \subset \Lambda$ of all principal ideals (λ) containing \mathfrak{c} which is known to be a principal ideal. It can be also defined as the intersection $\tilde{\mathfrak{c}} = \bigcap_P \mathfrak{c}_P$ inside Q for P running over all prime divisors of Λ (see [BCM] Chapter 7 for these facts). We write $0 \neq L = L(\mathbb{I}) \in \Lambda$ for a generator of the ideal $\tilde{\mathfrak{c}}$.

The reason for assuming the conditions (R) and one of (s) and (u) is a bit technical. These conditions are used to show in a key lemma Lemma 2.9 that the Lie algebra \mathcal{M}^0 of $\text{Im}(\rho_{\mathbb{I}}) \cap SL_2(\Lambda)$ (in the sense of Pink [P] described in the following section) is large so that $\mathfrak{sl}_2(\Lambda)/\mathcal{M}^0$ is a Λ -torsion module.

Plainly the condition (v) implies (u). The condition (u) is always satisfied by $\rho_{\mathbb{I}}$; it was first proven in [GV] as Theorem 3 under (R) and absolute irreducibility of the residual representation $\bar{\rho}$ over $\mathbb{Q}[\mu_p]$. The two assumptions: (R) and absolute irreducibility of $\bar{\rho}$ assumed in [GV] is now eliminated for the validity of (u) by a method different from [GV] (see [H12b] and [Z]), and (u) holds unconditionally. The condition (s) is easy to check (for example, it is valid if $\bar{\epsilon}|_{I_p}$ has order ≥ 3 ; indeed, by local class field theory, we view $\bar{\epsilon}|_{I_p}$ as a character of \mathbb{Z}_p^\times , which has values in \mathbb{F}_p^\times , and hence, if $\zeta = \bar{\epsilon}(\sigma)$ has order ≥ 3 for $\sigma \in I_p$, the adjoint action $Ad(j)$ of $j = \bar{\rho}(\sigma)$ on $\mathfrak{sl}_2(\mathbb{I}/\mathfrak{m}_{\mathbb{I}})$ has three distinct eigenvalues $\zeta, 1, \zeta^{-1}$ in \mathbb{F}_p). In the condition (s), we may replace g by $\lim_{n \rightarrow \infty} g^{q^n}$ for a sufficiently large p -power q ; so, we may assume that g has eigenvalues in \mathbb{Z}_p . This theorem will be proven in Section 3. The proof is difficult if $\mathbb{I} \neq \Lambda$, and the easier case of $\mathbb{I} = \Lambda$ is treated in [GME] Theorems 4.3.21 and 4.3.23. When $\bar{\rho}$ is absolutely irreducible, we call $L = L(\mathbb{I})$ as in the theorem the *global level* of $\rho_{\mathbb{I}}$ or of \mathbb{I} . More generally, when ρ_P is absolutely irreducible, the localized ideal \mathfrak{c}_P is well determined by $\rho_{\mathbb{I}}$ (see Lemma 3.3). When ρ_P is reducible, there is a way of normalizing \mathfrak{c}_P as we will explain after Lemma 3.5 using the following standard choice $\mathcal{L}_{can}(\tilde{\mathbb{I}})$ of the lattice \mathcal{L} . Then we define $(L(\mathbb{I})) = \bigcap_P \mathfrak{c}_P$ using this normalized \mathfrak{c}_P . To describe this standard example of \mathcal{L} stable under the Galois action, we note that $\rho_{\mathfrak{h}}$ was constructed in [H86b] through the Galois action on the χ_1 -part J of $\varprojlim_n e \cdot (T_p J_1(Np^n) \otimes_{\mathbb{Z}_p} W)$ for the p -adic Tate module $T_p J_1(Np^n)$ of the jacobian $J_1(Np^n)_{/\mathbb{Q}}$ of the modular curve $X_1(Np^n)_{/\mathbb{Q}}$. Suppose that \mathbb{I} is cuspidal. Let $\mathcal{L}_{can}(\tilde{\mathbb{I}})$ (resp. $\mathcal{L}_{can}(\mathbb{I})$) be the image of $J \otimes_{\mathfrak{h}} \tilde{\mathbb{I}}$ (resp. $J \otimes_{\mathfrak{h}} \mathbb{I}$) in $J \otimes_{\mathfrak{h}} Q(\mathbb{I}) \cong Q(\mathbb{I})^2$ for $\mathfrak{h} = \mathfrak{h}_{\chi_1/\mathbb{Z}_p}$. Consider the following version of (F):

$$(F_{can}) \quad \mathcal{L}_{can}(\tilde{\mathbb{I}}) \text{ is free of rank 2 over } \tilde{\mathbb{I}}.$$

This condition holds under (R) and absolute irreducibility of $\bar{\rho}$ (see Section 7 for this fact). Under the condition (R) (and (F_{can})), the Galois module $\mathcal{L}_{can}(\tilde{\mathbb{I}})$ fits into the following canonical exact sequence of D_p -modules: $0 \rightarrow \tilde{\mathbb{I}} \rightarrow \mathcal{L}_{can}(\tilde{\mathbb{I}}) \rightarrow \tilde{\mathbb{I}} \rightarrow 0$ coming from the connected-étale exact sequence of the Tate modules $e \cdot T_p J_1(Np^n)$, and the assertion (Gal) is realized through this exact sequence. Thus assuming (R), we take the Galois representation $\rho_{\mathbb{I}} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow GL_2(\tilde{\mathbb{I}})$ realized on $\mathcal{L}_{can}(\tilde{\mathbb{I}})$.

If R is a p -profinite local ring, as we will describe in Section 10, any Galois representation $\rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow GL_2(R)$ ramified at finitely many primes has a well defined prime-to- p conductor $C(\rho)$. We call \mathbb{I} minimal, if $C(\rho_{\mathbb{I}})$ is minimal among $C(\rho_{\mathbb{I}} \otimes \xi)$ for ξ running all finite order character of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ unramified at p . Since the global level of $\rho_{\mathbb{I}}$ and $\rho_{\mathbb{I}} \otimes \xi$ are equal, to describe $L(\mathbb{I})$, we may assume that \mathbb{I} is minimal and primitive in the sense of [H86a] §3 page 252 (so, $f_P \in \mathcal{F}_{\mathbb{I}}$ is a p -stabilized N -new form). In many cases, we can relate the generator $L(\mathbb{I})$ with p -adic L -functions. The following is a summary of determination of $L(\mathbb{I})$:

Theorem II. *Suppose $p \geq 5$, (F_{can}) and (R) and one of the conditions (s) and (v). Take a non CM minimal primitive cuspidal component \mathbb{I} of prime-to- p cube-free level N .*

- (1) *If $\text{Im}(\overline{\rho})$ contains $SL_2(\mathbb{F}_p)$ and $p \geq 7$, then $L(\mathbb{I}) = 1$.*
- (2) *If the projected image of $\overline{\rho}$ in $PGL_2(\overline{\mathbb{F}}_p)$ is either a tetrahedral, an octahedral or an icosahedral group, then $T|L(\mathbb{I})|T^n$ for an integer $n > 0$.*
- (3) *Suppose that $\overline{\rho}$ is absolutely irreducible and $\overline{\rho} \cong \text{Ind}_M^{\mathbb{Q}} \overline{\psi}$ for a quadratic field M and a character $\overline{\psi} : \text{Gal}(\overline{\mathbb{Q}}/M) \rightarrow \overline{\mathbb{F}}_p^{\times}$. Write $\mathfrak{C}(\overline{\psi})$ for the prime-to- p part of the conductor of $\overline{\psi}$.*
 - (a) *If there is no other imaginary quadratic field M' such that $\overline{\rho} \cong \text{Ind}_{M'}^{\mathbb{Q}} \overline{\varphi}$ for a character $\overline{\varphi} : \text{Gal}(\overline{\mathbb{Q}}/M') \rightarrow \overline{\mathbb{F}}_p^{\times}$ and either M is real or p do not split in M , $L(\mathbb{I})$ is a factor of $((1+T)^{p^m} - 1)^2$ for an integer $m > 0$.*
 - (b) *Suppose $p \nmid \varphi(N)$ and $N = C(\overline{\rho})$. If M is an imaginary quadratic field in which p splits, $\overline{\psi}$ ramifies at a prime over p and there is no other quadratic field M' such that $\overline{\rho} \cong \text{Ind}_{M'}^{\mathbb{Q}} \overline{\varphi}$ for a character $\overline{\varphi} : \text{Gal}(\overline{\mathbb{Q}}/M') \rightarrow \overline{\mathbb{F}}_p^{\times}$, then $L(\mathbb{I})$ is a factor of the square of the product of the (primitive) anticyclotomic Katz p -adic L -functions (cf. [K]) of prime-to- p conductor $\mathfrak{C}(\overline{\psi}^-)$ whose branch character modulo p is the anticyclotomic projection $\overline{\psi}^-$ of $\overline{\psi}$. Here $\overline{\psi}^-$ given by $\sigma \mapsto \overline{\psi}(\sigma)\overline{\psi}(c\sigma c^{-1})^{-1}$ for complex conjugation c .*
- (4) *Suppose $p \nmid \varphi(N)$ and (F_{can}) . If $\overline{\rho} \cong \overline{\theta} \oplus \overline{\psi}$ (with $\overline{\theta}$ ramified at p and $\overline{\psi}$ unramified at p) and there is no quadratic field M' such that $\overline{\rho} \cong \text{Ind}_{M'}^{\mathbb{Q}} \overline{\varphi}$ for a character $\overline{\varphi} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \overline{\mathbb{F}}_p^{\times}$, then $L(\mathbb{I})$ is a factor of the primitive Kubota-Leopoldt p -adic L -function whose branch character modulo p is $\overline{\psi}^{-1}\overline{\theta}$ as long as the prime-to- p conductor of $\overline{\psi}^{-1}\overline{\theta}$ and N match.*

The product of (possibly imprimitive) p -adic L -functions mentioned in the theorem will be made precise in Section 8 depending on $\overline{\rho}$. The assertion (1) is a version of a result of Mazur–Wiles in [MW] and Fischman [F] where $\mathbb{I} = \Lambda$ is assumed (see Remark 8.3 for more account in this case). The assertion (3b) is the most difficult to prove, and a sketch and the strategy of the proof are given after Theorem 8.5 before giving its long detailed proof. Theorem 8.5 gives a result slightly stronger than (3b) (in particular, we do not need to assume that N is cube-free). The assertion (4) can be proven similarly to (3b), and Ohta’s determination [O2] of the congruence module between the Eisenstein component and a cuspidal component is crucial. Some more complicated cases missing from Theorem II are discussed in Section 9.

This type of results, asserting that the image of the modular Galois representation of each non CM Hecke eigenform contains, up to conjugation, an open subgroup of $SL_2(\mathbb{Z}_p)$ was proven in a paper by Ribet [R] (and [R1]) in 1975 and by Momose [Mo] in 1981. As we will see in Proposition 5.1, a CM component and a non CM component do not intersect at any arithmetic point, and therefore $\text{Im}(\rho_{\mathfrak{P}})$ contains, up to conjugation, an open subgroup of $SL_2(\mathbb{Z}_p)$ for arithmetic points $\mathfrak{P} \in \text{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$ as long as \mathbb{I} is a non CM component. An investigation of the image for Λ -adic Galois representations was first done in 1986 by Mazur–Wiles in [MW] (just after the representation was constructed in [H86b]). We call a prime divisor $\mathfrak{P} \in \text{Spec}(\mathbb{I})$ *full* (in a weak sense) if $\text{Im}(\rho_{\mathfrak{P}})$ contains, up to conjugation in $GL_2(\kappa(\mathfrak{P}))$ for the residue field $\kappa(\mathfrak{P}) = Q(\mathbb{I}/\mathfrak{P})$ of \mathfrak{P} , an open subgroup of either $SL_2(\mathbb{Z}_p)$ or $SL_2(\mathbb{F}_p[[T]])$. Fullness of most primes above $(p) \subset \Lambda$ is treated in [H12a] (see also [SkW], in particular, results about “nice” primes there). The existence of such a full prime divisor is a key ingredient of the proof of the above theorems.

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1. LIE ALGEBRAS OF p -PROFINITE SUBGROUPS OF $SL(2)$.

If A is a p -profinite ring of characteristic p , obviously the power series $\log(1 + X)$ and $\exp(X)$ do not make much sense to create the logarithm and the exponential map; so, the relation between closed subgroups in $GL_n(A)$ and Lie subalgebras of $\mathfrak{gl}_n(A)$ appears not very direct. The *principal congruence subgroup*

$$\Gamma_A(\mathfrak{a}) := SL_2(A) \cap (1 + \mathfrak{a} \cdot \mathfrak{gl}_2(A)) = \{x \in SL_2(A) \mid x \equiv 1 \pmod{\mathfrak{a}}\}$$

for an A -ideal \mathfrak{a} obviously plays an important role in this paper. To study a general p -profinite subgroup \mathcal{G} of $SL_2(A)$ for a general p -profinite ring A , we somehow want to have an explicit relation between p -profinite subgroups \mathcal{G} of the form $SL_2(A) \cap (1 + X)$ and a Lie \mathbb{Z}_p -subalgebra $X \subset \mathfrak{gl}_2(A)$. Under the condition that $p > 2$, Pink [P] found a functorial explicit relation between closed subgroups in $SL_2(A)$ and Lie subalgebras of $\mathfrak{gl}_2(A)$ (valid even for A of characteristic p). We call subgroups of the form $SL_2(A) \cap (1 + X)$ (for a p -profinite Lie \mathbb{Z}_p -subalgebra X of $\mathfrak{gl}_2(A)$) *basic subgroups* following Pink.

We prepare some notation to quote here the results in [P]. Let A be a semi-local p -profinite ring (not necessarily of characteristic p). Since Pink's result allows semi-local p -profinite algebra, we do not assume A to be local in the exposition of his result. We assume $p > 2$. Define maps $\Theta : SL_2(A) \rightarrow \mathfrak{sl}_2(A)$ and $\zeta : SL_2(A) \rightarrow Z(A)$ for the center $Z(A)$ of the algebra $M_2(A)$ by

$$\Theta(x) = x - \frac{1}{2}\mathrm{Tr}(x) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \zeta(x) = \frac{1}{2}(\mathrm{Tr}(x) - 2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

For each p -profinite subgroup \mathcal{G} of $SL_2(A)$, define L by the closed additive subgroup of $\mathfrak{sl}_2(A)$ topologically generated by $\Theta(x)$ for all $x \in \mathcal{G}$. Then we put $C = \mathrm{Tr}(L \cdot L)$. Here $L \cdot L$ is the closed additive subgroup of $M_2(A)$ generated by $\{xy \mid x, y \in L\}$ for the matrix product xy , similarly L^n is the closed additive subgroup generated by iterated products (n times) of elements in L . We then define $L_1 = L$ and inductively $L_{n+1} = [L, L_n]$; so, $L_2 = [L, L]$, where $[L, L_n]$ is the closed additive subgroup generated by Lie bracket $[x, y] = xy - yx$ for $x \in L$ and $y \in L_n$. Then by [P] Proposition 3.1, we have

$$(1.1) \quad [L, L] \subset L, \quad C \cdot L \subset L, \quad L = L_1 \supset \cdots \supset L_n \supset L_{n+1} \supset \cdots \quad \text{and} \quad \bigcap_{n \geq 1} L_n = \bigcap_{n \geq 1} L^n = 0.$$

In particular, L is a Lie \mathbb{Z}_p -subalgebra of $\mathfrak{sl}_2(A)$. Put $\mathcal{M}_n(\mathcal{G}) = C \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus L_n \subset M_2(A) = \mathfrak{gl}_2(A)$, which is a closed Lie \mathbb{Z}_p -subalgebra by (1.1). We write simply $\mathcal{M}(\mathcal{G})$ (resp. $\mathcal{M}^0(\mathcal{G})$) for $\mathcal{M}_2(\mathcal{G})$ (resp. $\mathcal{M}_2(\mathcal{G}) \cap \mathfrak{sl}_2(A) = [L, L]$). Define

$$\mathcal{H}_n = \{x \in SL_2(A) \mid \Theta(x) \in L_n, \mathrm{Tr}(x) - 2 \in C\} \quad \text{for } n \geq 1.$$

If $x \in \mathcal{H}_n$, then $x = \Theta(x) + \zeta(x) + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; thus, $\mathcal{H}_n \subset SL_2(A) \cap (1 + \mathcal{M}_n(\mathcal{G}))$. If we pick $x \in SL_2(A) \cap (1 + \mathcal{M}_n(\mathcal{G}))$, then $x = 1 + c \cdot 1 + y$ with $y \in L_n$ and $c \in C$. Thus $\mathrm{Tr}(x) - 2 = 2c \in C$ and

$\Theta(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + c \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + y - \frac{1}{2}(2 + 2c) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = y$. This shows

$$\mathcal{H}_n = SL_2(A) \cap (1 + \mathcal{M}_n(\mathcal{G})) \text{ in particular, } \mathcal{H}_2 = SL_2(A) \cap (1 + \mathcal{M}(\mathcal{G})).$$

Here is a result in Pink (Theorem 3.3 combined with Theorem 2.7 both in [P]):

Theorem 1.1 (Pink). *Let the notation be as above. Suppose $p > 2$, and let A be a noetherian semi-local p -profinite commutative ring with identity. Take a p -profinite subgroup $\mathcal{G} \subset SL_2(A)$. Then we have*

- (1) \mathcal{G} is a normal closed subgroup of \mathcal{H}_1 (defined as above for \mathcal{G}),
- (2) \mathcal{H}_n is a p -profinite subgroup of $SL_2(A)$ inductively given by $\mathcal{H}_{n+1} = \langle \mathcal{H}_1, \mathcal{H}_n \rangle$ which is the closed subgroup topologically generated by commutators (x, y) with $x \in \mathcal{H}_1$ and $y \in \mathcal{H}_n$,
- (3) $\{\mathcal{H}_n\}_{n \geq 2}$ coincides with the descending central series of $\{\mathcal{G}_n\}_{n \geq 2}$ of \mathcal{G} , where $\mathcal{G}_{n+1} = \langle \mathcal{G}, \mathcal{G}_n \rangle$ starting with $\mathcal{G}_1 = \mathcal{G}$.

In particular, we have

- (P) *The topological commutator subgroup \mathcal{G}' of \mathcal{G} is the subgroup given by $SL_2(A) \cap (1 + \mathcal{M}(\mathcal{G}'))$ for the closed additive subgroup $\mathcal{M}(\mathcal{G}') \subset M_2(A)$ as above.*

Put $\mathcal{M}_j^0(\mathcal{G}) = \mathcal{M}_j(\mathcal{G}) \cap \mathfrak{sl}_2(A)$. By the above expression, $\mathcal{G} \mapsto \mathcal{M}_j(\mathcal{G})$ (resp. $\mathcal{G} \mapsto \mathcal{M}_j^0(\mathcal{G})$) is a covariant functor from p -profinite subgroups of $SL_2(A)$ into closed Lie \mathbb{Z}_p -subalgebras of $\mathfrak{gl}_2(A)$ (resp. $\mathfrak{sl}_2(A)$). In particular, $\mathcal{M}_j(\mathcal{G})$ and $\mathcal{M}_j^0(\mathcal{G})$ are stable under the adjoint action $x \mapsto gxg^{-1}$ of \mathcal{G} . For an A -ideal \mathfrak{a} , writing $\overline{\mathcal{G}}_{\mathfrak{a}} = (\mathcal{G} \bmod \mathfrak{a}) = (\mathcal{G} \cdot \Gamma_A(\mathfrak{a})) / \Gamma_A(\mathfrak{a})$, $\mathcal{M}_j(\overline{\mathcal{G}}_{\mathfrak{a}}) \subset \mathfrak{gl}_2(A/\mathfrak{a})$ (resp. $\mathcal{M}_j^0(\overline{\mathcal{G}}_{\mathfrak{a}}) \subset \mathfrak{sl}_2(A/\mathfrak{a})$) is the surjective image of $\mathcal{M}_j(\mathcal{G})$ (resp. $\mathcal{M}_j^0(\mathcal{G})$) under the reduction map $x \mapsto (x \bmod \mathfrak{a})$. Since \mathcal{H}_1 is a basic subgroup with $\mathcal{H}_1/\mathcal{G}$ abelian, we call \mathcal{H}_1 the *basic closure* of \mathcal{G} . If \mathcal{G} is normalized by an element of $GL_2(A)$, by construction, the basic closure \mathcal{H}_1 is also normalized by the same element. Thus the normalizer of \mathcal{G} in $GL_2(A)$ is contained in the normalizer of \mathcal{H}_1 in $GL_2(A)$. By the above theorem, any p -profinite subgroup of $SL_2(A)$ is basic up to abelian error.

Lemma 1.2. *Assume $p > 2$. Let A be an integral domain finite flat either over $\mathbb{F}_p[[T]]$, Λ or \mathbb{Z}_p with quotient field $Q(A)$. If a subgroup $G \subset SL_2(A)$ contains a principal congruence subgroup $\Gamma_A(\mathfrak{c})$ for a non-zero A -ideal \mathfrak{c} , then $\alpha G \alpha^{-1}$ for $\alpha \in GL_2(Q(A))$ contains $\Gamma_A(\mathfrak{c}')$ for another non-zero A -ideal \mathfrak{c}' depending on α .*

Proof. For simplicity, we write $\Gamma(\mathfrak{c})$ for $\Gamma_A(\mathfrak{c})$. We may suppose that $G = \Gamma(\mathfrak{c})$ for $\mathfrak{c} \subset \mathfrak{m}_A$; so, G is p -profinite. Then $\mathcal{M}(G) \supset \mathfrak{c}^2 \cdot \mathfrak{L}$ for $\mathfrak{L} = M_2(A)$. Replacing α by $\xi \alpha$ for a suitable $\xi \in A \cap Q(A)^\times$ for the quotient field $Q(A)$ of A , we may assume that $\alpha \in M_2(A) \cap GL_2(Q(A))$. Then $(\alpha \mathfrak{L} \alpha^{-1} \cap \mathfrak{L}) \supset \alpha \mathfrak{L} \alpha'$ for $\alpha' = \det(\alpha) \alpha^{-1} \in M_2(A)$. Since \mathfrak{L} and $\alpha \mathfrak{L} \alpha'$ are both free A -module of rank 3, $\mathfrak{L} / \alpha \mathfrak{L} \alpha'$ is a torsion A -module finite type annihilated by a non-zero A -ideal \mathfrak{c}'' . Then $\mathcal{M}(\alpha \Gamma(\mathfrak{c}) \alpha^{-1} \cap SL_2(A)) \supset \mathfrak{c}^2 \cdot \alpha \mathfrak{L} \alpha^{-1} \supset \mathfrak{c}^2 \mathfrak{c}'' \mathfrak{L}$. Thus the ideal $\mathfrak{c}_\alpha = \mathfrak{c}^2 \mathfrak{c}''$ does the job. \square

Let $\mathcal{B}/\mathbb{Z}_p \subset GL(2)/\mathbb{Z}_p$ (resp. \mathcal{Z}/\mathbb{Z}_p) be the upper triangular Borel subgroup (resp. the center of $GL(2)/\mathbb{Z}_p$) as an algebraic group. Write \mathcal{U}/\mathbb{Z}_p for the unipotent radical of \mathcal{B}/\mathbb{Z}_p , and put $\mathcal{Z}\mathcal{U}(A) = \mathcal{Z}(A)\mathcal{U}(A) \subset GL_2(A)$. Let $\mathfrak{B}/\mathbb{Z}_p$ (resp. $\mathfrak{U}/\mathbb{Z}_p$) be the Lie algebra of \mathcal{B}/\mathbb{Z}_p (resp. \mathcal{U}/\mathbb{Z}_p). We write $\mathcal{B} = \mathbb{G}_m^2 \ltimes \mathcal{U}$ by the splitting $\mathbb{G}_m^2 \ni (t, t') \mapsto \begin{pmatrix} t & 0 \\ 0 & t' \end{pmatrix} \in \mathcal{B}$.

Lemma 1.3. *Suppose $p > 2$, and let \mathbb{I} be a domain finite flat over Λ . Let $\mathbb{G} \subset SL_2(\mathbb{I})$ be a p -profinite subgroup. Suppose the following two conditions:*

- (B) *The group \mathbb{G} contains a subgroup of $\mathcal{B}(\mathbb{I}) \cap SL_2(\mathbb{I})$ which is, under the projection: $\mathcal{B} \rightarrow \mathcal{B}/\mathcal{Z}\mathcal{U} = \mathbb{G}_m$, isomorphic to the image of*

$$\mathcal{T} = \left\{ t(s) := \begin{pmatrix} (1+T)^{s/2} & 0 \\ 0 & (1+T)^{-s/2} \end{pmatrix} \middle| s \in \mathbb{Z}_p \right\} \cong \Gamma.$$

- (U) *The subgroup $U = \left\{ \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \middle| u \in \Lambda \right\} \cap \mathbb{G}$ is non-trivial.*

We denote by \mathfrak{u} the ideal $\mathfrak{u} = \left\{ u \in \Lambda \mid \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \in U \right\}$. Let \mathbb{G}' be the topological commutator subgroup of \mathbb{G} and $U' = U \cap \mathbb{G}'$. The group \mathcal{T} acts on U and U' by the conjugate action. Then we have

- (1) *The action of $\mathbb{Z}_p[[T]]$ on U and U' coincides with the action of Λ via the isomorphism $\mathbb{Z}_p[[T]] \cong \Lambda$ sending $t(1)$ to $1 + T$ (under the notation in (B)). Under this identification, U/U' is torsion of finite type (as a Λ -module) killed by the ideal (T) of Λ .*

- (2) If moreover there exists $g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathcal{B}(\mathbb{I}) \cap \mathbb{G}$ whose image in $\mathcal{B}(\mathbb{I})/\mathcal{Z}\mathcal{U}(\mathbb{I})$ is non-trivial, U/U' is killed by $ad^{-1} - 1$. If further $ad^{-1} - 1$ is prime to T , U/U' is finite (i.e., pseudo-null as a Λ -module).

Replacing the pair (U, U') by $(\mathbb{U} = \mathbb{G} \cap \mathcal{U}(\mathbb{I}), \mathbb{U}' = \mathbb{G}' \cap \mathcal{U}(\mathbb{I}))$, the same assertions (1–2) hold under the condition $\mathbb{U} \neq 1$.

Proof. Since the proof is the same for (U, U') and $(\mathbb{U}, \mathbb{U}')$, we give a proof for the pair (U, U') . Often we identify \mathfrak{u} with the Lie subalgebra $\left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in \mathfrak{u} \right\}$ in $\mathfrak{sl}_2(\Lambda)$. Under this identification, by definition, we have $U = 1 + \mathfrak{u} \subset \mathbb{G}$. Since U and \mathbb{G}' is normalized by the adjoint (conjugation) action of $\mathbb{G} \cap \mathcal{B}(\Lambda)\mathcal{U}(\mathbb{I})$, $\mathcal{T} \hookrightarrow \mathcal{B}(\mathbb{I})/\mathcal{Z}\mathcal{U}(\mathbb{I})$ acts on U and U' . Then the \mathbb{Z}_p -module U/U' carries a continuous action of Γ via $\Gamma = \{(1+T)^s \mid s \in \mathbb{Z}_p\} \cong \mathcal{T}$. Note that $\mathbb{Z}_p[[\Gamma]] = \Lambda$. Since the Λ -module structure on U induced by the adjoint action of \mathcal{T} and the one induced by the isomorphism $\log : U \ni (1+u) \mapsto u \in \mathfrak{u}$ match, $U \cong \mathfrak{u} \subset \mathfrak{sl}_2(\Lambda)$ is a Λ -module of finite type (as Λ is noetherian). Thus U/U' is a Λ -module of finite type embedded in \mathbb{G}/\mathbb{G}' . Pick $\tau \in \mathcal{B}(\mathbb{I}) \cap \mathbb{G}$ whose image in \mathcal{T} is equal to $\begin{pmatrix} (1+T)^{1/2} & 0 \\ 0 & (1+T)^{-1/2} \end{pmatrix}$. Then $\tau - 1$ acts on U/U' by multiplication by T and kills \mathbb{G}/\mathbb{G}' . Thus T is in the annihilator $\text{Ann}(U/U')$ of U/U' as asserted in (1).

If we have further $g = \begin{pmatrix} a & * \\ 0 & d \end{pmatrix} \in \mathbb{G} \cap \mathcal{B}(\mathbb{I})$ as in (2), by the same argument, U/U' is killed by $ad^{-1} - 1 \neq 0$. Thus U/U' is a module over a finite extension $\Lambda[\theta] \subset \mathbb{I}$ of Λ for $\theta = ad^{-1} - 1$. Taking a minimal polynomial $\Phi(X)$ of θ over Q , we have $\Lambda[\theta] \cong \Lambda[X]/(\Phi(X))$; so, $\Lambda[\theta]$ is finite flat over Λ . If θ is prime to T , U/U' is killed by an open ideal (θ, T) of $\Lambda[\theta] \subset \mathbb{I}$. Since U/U' is a Λ -module of finite type, it is a finite pseudo null Λ -module. \square

Here is another easy remark:

Lemma 1.4. *Let the notation and the assumption be as in Lemma 1.3. In addition, we assume to have $j = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta' \end{pmatrix} \in GL_2(\mathbb{Z}_p)$ such that $j\mathbb{G}j^{-1} = \mathbb{G}$ and $\zeta - \zeta' \in \mathbb{Z}_p^\times$. Then the group \mathbb{G} contains the subgroup \mathcal{T} ; in particular, the semi-simple group $\mathcal{Z}(\Lambda)\mathcal{T} \subset SL_2(\Lambda)$ normalizes \mathbb{G} .*

Proof. Forgetting about the center $\mathcal{Z}(\Lambda)$, we only need to show that \mathcal{T} normalizes \mathbb{G} . By assumption, there exists $\tau \in \mathbb{G}$ of the form $\tau = \begin{pmatrix} a & u \\ 0 & a^{-1} \end{pmatrix}$ for $a = (1+T)^{1/2}$. By computation, for the commutator (τ, j) , we have $(\tau, j) = \begin{pmatrix} 1 & ua(1-\zeta\zeta'^{-1}) \\ 0 & 1 \end{pmatrix} \in \mathbb{G}$. Thus $\mathbb{U} = \mathcal{U}(\mathbb{I}) \cap \mathbb{G}$ contains $\begin{pmatrix} 1 & ua(1-\zeta\zeta'^{-1}) \\ 0 & 1 \end{pmatrix}$. Since \mathbb{U} is a \mathbb{Z}_p -module, we can divide elements in \mathbb{U} by the \mathbb{Z}_p -unit $(1 - \zeta\zeta'^{-1})$; so, \mathbb{U} contains $\begin{pmatrix} 1 & ua \\ 0 & 1 \end{pmatrix}$ and $\tau^{-1} \begin{pmatrix} 1 & ua \\ 0 & 1 \end{pmatrix} \tau = \begin{pmatrix} 1 & a^{-1}u \\ 0 & 1 \end{pmatrix} =: \beta \in \mathbb{U}$. Then \mathbb{G} contains $\tau\beta^{-1} = t(1) = \begin{pmatrix} (1+T)^{1/2} & 0 \\ 0 & (1+T)^{-1/2} \end{pmatrix}$, and \mathbb{G} contains $\mathcal{T} = \{t(1)^s \mid s \in \mathbb{Z}_p\}$ which in particular normalizes \mathbb{G} . \square

For a prime divisor P of $\text{Spec}(\Lambda)$, we write A_0 for the subring $\mathbb{Z}_p \subset \kappa(P)$ if $\kappa(P)$ is of characteristic 0, and if $P = (p)$, we put $A_0 = \mathbb{F}_p[[T]] = \Lambda/P \subset \kappa(P)$.

Lemma 1.5. *Let the notation and the assumption be as in Lemma 1.3. Put $G = \mathbb{G} \cap SL_2(\Lambda)$ and let G_U be the subgroup of G topologically generated by gUg^{-1} for all $g \in G$. If there exists a prime divisor $P \in \text{Spec}(\Lambda)$ such that the image of G_U in $SL_2(\Lambda/P)$ contains an open subgroup of $SL_2(A_0)$, then we can find a Λ -module $\mathfrak{L} \subset \mathcal{M}(G) \cap \mathfrak{sl}_2(\Lambda)$ such that $\mathfrak{sl}_2(\Lambda)/\mathfrak{L}$ is Λ -torsion with $\mathfrak{sl}_2(\Lambda_P)/\mathfrak{L}_P = 0$ after localization and $\mathfrak{L} \subset \mathcal{M}(G_U) \cap \mathfrak{sl}_2(\Lambda) \subset \mathcal{M}(G) \cap \mathfrak{sl}_2(\Lambda) \subset \mathfrak{sl}_2(\Lambda)$. Moreover G_U contains $\Gamma_\Lambda(\mathfrak{c})$ for a non-zero Λ -ideal $\mathfrak{c} = \{\lambda \in \Lambda \mid \lambda \cdot M_2(\Lambda) \subset \mathcal{M}(G_U)\}$ prime to P .*

Proof. As before, we identify \mathfrak{u} with the Lie subalgebra $\left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in \mathfrak{u} \right\}$ in $\mathfrak{sl}_2(\Lambda)$; then $U = 1 + \mathfrak{u}$. Let \overline{H} (resp. $\overline{\mathfrak{H}}$) be the image of a subgroup H (resp. a submodule \mathfrak{H}) of $SL_2(\Lambda)$ (resp. of $\mathfrak{sl}_2(\Lambda)$) in $SL_2(\Lambda/P)$ (resp. in $\mathfrak{sl}_2(\Lambda/P)$). If G'_U is the topological commutator subgroup of G_U , its image \overline{G}'_U in $SL_2(\Lambda/P)$ is the topological commutator subgroup of \overline{G}_U . Since \overline{G}_U contains an open subgroup of $SL_2(A_0)$, \overline{G}'_U contains an open subgroup of $SL_2(A_0)$. Since $G' \supset G'_U$, we find $U' = G' \cap U \supset U'' := G'_U \cap U$. In any case, we find $\overline{U} = 1 + \overline{\mathfrak{u}} \neq 1$, where $\overline{\mathfrak{u}}$ is the image of \mathfrak{u} in $\mathfrak{sl}_2(\Lambda/P)$ (so, $\overline{\mathfrak{u}} \neq 0$). By Lemma 1.3, $U' = U \cap G'$ is non-trivial, and if $P \nmid T$, $\mathfrak{u}' = \{u \in \mathfrak{sl}_2(\Lambda) \mid 1+u \in U'\}$ is a non-trivial Lie Λ -subalgebra of $\mathfrak{sl}_2(\Lambda)$ with nontrivial image $\overline{\mathfrak{u}'}$ in $\mathfrak{sl}_2(\Lambda/P)$. Even if $P|T$, since \overline{G}'_U contains an open subgroup of $SL_2(A_0)$, $\overline{U}'' \subset \overline{U}'$ is non-trivial; so, $\overline{\mathfrak{u}'} \neq 0$. Let $H \subset G'$ be the subgroup generated by $gU'g^{-1}$ for all $g \in G$. Let $\mathcal{M} = \mathcal{M}(G_U)$

and $\overline{\mathcal{M}} = \mathcal{M}(\overline{G}_U)$. Then we have a natural surjection $\pi : \mathcal{M} \rightarrow \overline{\mathcal{M}}$ given by $x \mapsto x \bmod P$ for $x \in M_2(\Lambda)$. Let $\mathfrak{L} = \sum_{g \in G_U} g u' g^{-1} \subset \mathcal{M} \cap \mathfrak{sl}_2(\Lambda)$ and $\overline{\mathfrak{L}} = \sum_{g \in \overline{G}_U} g \overline{u}' g^{-1} \subset \overline{\mathcal{M}} \cap \mathfrak{sl}_2(\Lambda/P)$. As seen in the proof of Lemma 1.3, u' is a torsion-free Λ -submodule of $\mathfrak{sl}_2(\Lambda)$; so, \mathfrak{L} is a torsion-free Λ -submodule of $\mathfrak{sl}_2(\Lambda)$. Note that \mathfrak{L} is stable under the adjoint action of G_U . Since \overline{G}_U contains an open subgroup of $SL_2(A_0)$, the adjoint action of \overline{G}_U on $\overline{\mathfrak{L}}$ is irreducible; so, $\overline{\mathfrak{L}} \otimes_{\Lambda} \kappa(P)$ has dimension 3 over $\kappa(P)$; so, $\overline{\mathfrak{L}} \otimes_{\Lambda} \kappa(P) = \mathfrak{sl}_2(\kappa(P))$. By Nakayama's lemma, we have $\mathfrak{L}_P = \mathfrak{sl}_2(\Lambda_P)$. In particular, $\mathfrak{sl}_2(\Lambda)/\mathfrak{L}$ is a Λ -torsion module of finite type. For L in (P) of Theorem 1.1 contains \mathfrak{L} , and hence $L \cdot L \supset \mathfrak{L} \cdot \mathfrak{L}$. For the annihilator \mathfrak{a} of $\mathfrak{sl}_2(\Lambda)/\mathfrak{L}$, by a simple computation \mathcal{M} contains $L^\circ = \mathfrak{a}^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \mathfrak{L} \subset M_2(\Lambda)$. Since $\mathfrak{L}_P = \mathfrak{sl}_2(\Lambda_P)$, \mathfrak{a} is prime to P . Thus for the maximal Λ -submodule \mathcal{M}° of \mathcal{M} , we conclude $\mathcal{M}^\circ \supset L^\circ$, and $M_2(\Lambda)/\mathcal{M}^\circ$ is a torsion Λ -module. Thus the annihilator ideal \mathfrak{c} of $M_2(\Lambda)/\mathcal{M}^\circ$ is prime to P and $G_U \supset \Gamma_\Lambda(\mathfrak{c})$. \square

Question 1.6. *Under the conditions (B) and (U) in Lemma 1.3, are $\mathcal{M}(G)$ and $\mathcal{M}(\mathbb{G})$ Λ -modules? It is likely to be the case up to finite error, and if they are, our argument in the rest of the paper could be simplified a lot. They are obviously stable under the adjoint action of \mathcal{T} .*

2. FULLNESS OF LIE ALGEBRA

We start with the following well known fact whose proof is left to the reader (e.g., [GME] §4.3.5):

Lemma 2.1. *Let K be a field of characteristic 0. If \mathfrak{L} is a nontrivial proper Lie subalgebra over K in $\mathfrak{sl}_2(K)$, then \mathfrak{L} is a conjugate in $\mathfrak{sl}_2(K)$ of one of the following Lie K -subalgebras:*

- (1) $\{x \in M \mid \text{Tr}_{M/\mathbb{Q}}(x) = 0\}$ as an abelian Lie subalgebra for a semi-simple quadratic extension M of K (Cartan subalgebra).
- (2) $\mathfrak{U}_K = \left\{ \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \mid x \in K \right\}$ (nilpotent subalgebra).
- (3) $\mathfrak{B}_K = \left\{ \begin{pmatrix} a & x \\ 0 & -a \end{pmatrix} \mid a, x \in K \right\}$ (Borel subalgebra).

Corollary 2.2. *Let K be a field of characteristic different from 2 and L/K be a field extension. Let $0 \neq \mathfrak{L} \subset \mathfrak{sl}_2(L)$ be a vector K -subspace stable under the adjoint action of $SL_2(K)$. Then there exists $g \in GL_2(L)$ such that $g\mathfrak{L}g^{-1} \supset \mathfrak{sl}_2(K)$. If \mathfrak{L} contains some non-zero elements in $\mathfrak{sl}_2(K)$, \mathfrak{L} contains $\mathfrak{sl}_2(K)$ without conjugation.*

Proof. Put $\mathfrak{n}(X) = \left\{ \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \in \mathfrak{sl}_2(X) \mid x \in X \right\}$ for any intermediate extension $L/X/K$. Taking any diagonal matrix $g = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ in $SL_2(K)$ with $a^2 \neq a^{-2}$, $\mathfrak{n}(X)$ is the eigenspace in $\mathfrak{sl}_2(X)$ of $Ad(g)$ with eigenvalue a^2 . Since adjoint action: $Y \mapsto gYg^{-1}$ ($Y \in \mathfrak{sl}_2(L)$) of $g \in SL_2(K)$ is absolutely irreducible (as K has characteristic $\neq 2$), we find \mathfrak{L} span $\mathfrak{sl}_2(L)$ over L , and hence the eigenspace $\mathfrak{L}(a^2)$ in \mathfrak{L} of $Ad(g)$ with eigenvalue a^2 is non-trivial. In particular, $\mathfrak{L} \cap \mathfrak{n}(L) = \mathfrak{L}(a^2) \neq 0$. Let T be the diagonal torus in GL_2 ; so, $T(X) = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in GL_2(X) \mid a, b \in K^\times \right\}$. Note that $T(X)$ acts transitively on $\mathfrak{n}(X) \setminus \{0\}$. Thus conjugating \mathfrak{L} by an element of $T(L)$, we may assume that $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{L}$. Since the adjoint action of $SL_2(K)$ on $\mathfrak{sl}_2(K)$ is absolutely irreducible, $\mathfrak{L} \cap \mathfrak{sl}_2(K) \neq \{0\}$ implies $\mathfrak{L} \supset \mathfrak{sl}_2(K)$, as desired. \square

Here is a well known corollary (whose proof can be found in [GME] Corollary 4.3.14).

Corollary 2.3. *If G is a closed subgroup of $SL_2(\mathbb{Z}_p)$ of infinite order, then G has one of the following four forms*

- (1) G is an open subgroup of $SL_2(\mathbb{Z}_p)$;
- (2) G is an open subgroup of the normalizer of $M^\times \cap SL_2(\mathbb{Q}_p)$ for a semi-simple quadratic extension $M/\mathbb{Q}_p \subset M_2(\mathbb{Q}_p)$;
- (3) G is isomorphic to an open subgroup of the upper triangular Borel subgroup $\mathcal{B}(\mathbb{Z}_p) \subset SL_2(\mathbb{Z}_p)$;
- (4) G is isomorphic to an open subgroup of the upper triangular unipotent subgroup $\mathcal{U}(\mathbb{Z}_p) \subset SL_2(\mathbb{Z}_p)$.

Lemma 2.4. *Suppose $p > 2$ and A be an integral domain finite flat over $\mathbb{F}_p[[T]]$. If a closed subgroup G of $SL_2(A)$ contains $\overline{\mathcal{T}} := \left\{ \begin{pmatrix} (1+T)^s & 0 \\ 0 & (1+T)^{-s} \end{pmatrix} \mid s \in \mathbb{Z}_p \right\}$ and non-trivial upper unipotent and lower unipotent subgroups, then G contains an open subgroup of $SL_2(\mathbb{F}_p[[T]])$, and if G is p -profinite, $\mathcal{M}(G)$ contains an open submodule of $M_2(\mathbb{F}_p[[T]])$.*

Proof. Replacing G by $G \cap \Gamma_A(\mathfrak{m}_A)$, we may assume that G is p -profinite. Writing $K = \mathbb{F}_p((T))$ and $L = A \otimes_{\mathbb{F}_p[[T]]} K$, L is a finite field extension of K . Consider the X -span \mathfrak{L}_X of $\mathcal{M}_1^0(G)$ for $X = K, L$. Then $\dim_L \mathfrak{L}_L = 3$; so, $\mathfrak{L}_L = \mathfrak{sl}_2(L)$. Thus up to conjugation, \mathfrak{L}_K contains $\mathfrak{sl}_2(K)$ by the existence of non-trivial unipotent elements. Thus we may assume that $A = \mathbb{F}_p[[T]]$. By conjugation action of \overline{T} , the unipotent groups $U = \mathcal{U}(\mathbb{F}_p[[T]]) \cap G$ and $U_t = {}^t\mathcal{U}(\mathbb{F}_p[[T]]) \cap G$ are non-zero $\mathbb{F}_p[[T]]$ -modules, thus $[\mathcal{U}(\mathbb{F}_p[[T]]) : U] < \infty$ and $[{}^t\mathcal{U}(\mathbb{F}_p[[T]]) : U_t] < \infty$. Let \mathfrak{u} (resp. \mathfrak{u}_t) be the Lie algebra of U (resp. U_t); so, for example, $\mathfrak{u} = \{u - 1 \in \mathfrak{sl}_2(\mathbb{F}_p[[T]]) | u \in U\}$. Thus we find that $[\mathfrak{u}, \mathfrak{u}_t] \neq 0$ is also an $\mathbb{F}_p[[T]]$ -module in $\mathcal{M}^0(G)$, and hence $\mathcal{M}^0(G)$ has rank 3 over $\mathbb{F}_p[[T]]$. Also $C = \text{Tr}(\mathcal{M}^0(G) \cdot \mathcal{M}^0(G))$ as in Theorem 1.1 contains $\mathfrak{u}\mathfrak{u}_t$ regarding \mathfrak{u} and \mathfrak{u}_t as an ideal of $\mathbb{F}_p[[T]]$ by an obvious isomorphism $\mathfrak{U}(\mathbb{F}_p[[T]]) \cong {}^t\mathfrak{U}(\mathbb{F}_p[[T]]) \cong \mathbb{F}_p[[T]]$. Then G contains $\Gamma_{\mathbb{F}_p[[T]]}(\mathfrak{u}\mathfrak{u}_t)$, and hence G is open in $SL_2(\mathbb{F}_p[[T]])$. Note that $\mathcal{M}(G)$ is well defined containing $C \cdot 1_2 \oplus \mathcal{M}^0(G)$ with $\text{rank}_{\mathbb{F}_p[[T]]} \mathcal{M}^0(G) = 3$ and $C \neq 0$. This implies that $\mathcal{M}(G)$ is open in $M_2(\mathbb{F}_p[[T]])$. \square

Lemma 2.5. *Let V be a local \mathbb{Z}_p -algebra and A be a flat V -algebra. Then for a subgroup B of $\mathcal{B}(V)$ containing $\begin{pmatrix} \delta & \beta \\ 0 & \delta' \end{pmatrix}$ with $\delta - \delta' \in V^\times$ and a unipotent element $\begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix}$ with $\varepsilon \in V^\times$, the group $\mathbb{B} = \{g \in \mathcal{B}(A) | gBg^{-1} \subset \mathcal{B}(V)\}$ is contained in $\mathcal{B}(V)$ modulo the center of $GL_2(A)$. If A is reduced, the same assertion holds replacing $\{g \in \mathcal{B}(A) | gBg^{-1} \subset \mathcal{B}(V)\}$ by $\{g \in GL_2(A) | gBg^{-1} \subset \mathcal{B}(V)\}$.*

Proof. Take $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(A)$ satisfying $gBg^{-1} \subset \mathcal{B}(V)$; so, we have $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ which immediately implies $c^2 = 0$. Thus if A is reduced $c = 0$, and $g \in \mathcal{B}(A)$. Thus we may assume either $g \in \mathcal{B}(A)$ or that A is reduced to continue. The identity

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^{-1} = \begin{pmatrix} 1 & (a/d)\varepsilon \\ 0 & 1 \end{pmatrix} \in \mathcal{B}(V)$$

implies $a/d \in V^\times$. The identity

$$\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \delta & \beta \\ 0 & \delta' \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \delta & u(\delta' - \delta) + \beta \\ 0 & 1 \end{pmatrix} \in \mathcal{B}(V)$$

implies $u \in V$. This finishes the proof. \square

Lemma 2.6. *Let L/K be a finite field extension and \mathfrak{s} be a Lie K -subalgebra of $\mathfrak{sl}_2(L)$ containing $\mathfrak{U}(K)$. Suppose that we have a diagonal matrix $j \in GL_2(K)$ such that \mathfrak{s} is stable under the adjoint action $Ad(j)$ on $\mathfrak{sl}_2(L)$ and $Ad(j)$ has three distinct eigenvalues on $\mathfrak{sl}_2(K)$. If the L -span $\mathfrak{s}_L := L \cdot \mathfrak{s}$ is equal to $\mathfrak{sl}_2(L)$, then \mathfrak{s} contains $\mathfrak{sl}_2(K)$.*

Proof. The nilradical \mathfrak{R} of \mathfrak{s} is in the nilradical of $\mathfrak{s}_L = \mathfrak{sl}_2(L)$; so, $\mathfrak{R} = 0$, and \mathfrak{s} is semi-simple. Thus $\mathfrak{s} = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_n$ for K -simple components \mathfrak{s}_j . Each \mathfrak{s}_j generates $\mathfrak{sl}_2(L)$ over L . Thus $\mathfrak{s}_j = \alpha_j \mathfrak{s}_1$ for $\alpha_j \in L$. Since $\mathfrak{s}_j = [\mathfrak{s}_j, \mathfrak{s}_j] = \alpha_j^2 [\mathfrak{s}_1, \mathfrak{s}_1] \subset \alpha_j^2 \mathfrak{s}_1$, we have $\alpha_j^2 \in \alpha_j K$. This implies $\alpha_j \in K$; so, \mathfrak{s}_K is K -simple. The centralizer of j (i.e., the subalgebra fixed by $Ad(j)$) in \mathfrak{s} is a Cartan subalgebra \mathfrak{h}_K split over K (as j is diagonal in $GL_2(K)$ with $Ad(j)$ having three distinct eigenvalues), and $\mathfrak{h}_0 = \mathfrak{h}_K \cap \mathfrak{sl}_2(K)$ is a split Cartan subalgebra of $\mathfrak{sl}_2(K)$ normalizing $\mathfrak{U}(K)$ in \mathfrak{s} . Thus \mathfrak{s}_K is a split K -simple algebra containing an isomorphic image of $\mathfrak{sl}_2(K)$. Therefore, for a subfield $K' \subset L$ containing K , \mathfrak{s} is an inner conjugate of $\mathfrak{sl}_2(K')$; i.e., $\mathfrak{s} = g \cdot \mathfrak{sl}_2(K')g^{-1}$ for $g \in GL_2(L)$. Since $g \cdot \mathfrak{sl}_2(K')g^{-1} = \mathfrak{s} \supset \mathfrak{h}_0 \mathfrak{U}(K) \subset \mathfrak{sl}_2(K)$, we have $g^{-1} \mathfrak{h}_0 \mathfrak{U}(K)g \subset \mathfrak{sl}_2(K')$. We can then find $h \in SL_2(K')$ such that $h^{-1}g^{-1} \mathfrak{h}_0 \mathfrak{U}(K)gh = \mathfrak{h}_0 \mathfrak{U}(K)$. Then by Lemma 2.5, $gh \in K'^\times \mathcal{B}(K)$. Note that $gh \cdot \mathfrak{sl}_2(K')(gh)^{-1} = g \cdot \mathfrak{sl}_2(K')g^{-1}$; so, we may assume that $g \in \mathcal{B}(K)$; so, in particular, $\mathfrak{s} \supset \mathfrak{sl}_2(K)$. This finishes the proof. \square

Lemma 2.7. *Let V be a p -profinite discrete valuation ring with quotient field K . If a closed subgroup $H \subset SL_2(V)^m$ has open image in each factor of $SL_2(V)$, a conjugate in $GL_2(K)^m$ of H contains an open subgroup of $SL_2(V)$ diagonally embedded in $SL_2(V)^m$.*

Proof. The p -profinite property of V implies that V has finite residue field. We may assume that H is p -profinite. Since the topological commutator subgroup of H still has open image in each factor of $SL_2(V)$ (cf. [GME] Lemma 4.3.8), replacing H by its commutator, we may assume that H is basic. The result in Theorem 1.1 can be applied to the semi-local ring $A = V^m$. Consider the closed Lie \mathbb{Z}_p -subalgebra $\mathcal{M}(H)$ of $\mathfrak{gl}_2(A)$ associated to H and its subalgebra $\mathcal{M}^0(H) = \mathcal{M}(H) \cap \mathfrak{sl}_2(A)$. Write \mathfrak{M} for the V -span of $\mathcal{M}^0(H)$. These Lie algebra \mathfrak{M} is stable under the adjoint action of H . Write

\mathfrak{M}_j (resp. H_j) for the projection to the j -th factor $\mathfrak{sl}_2(V)$ (resp. $SL_2(V)$) of \mathfrak{M} (resp. H). Then \mathfrak{M}_j is the V -span of $\mathcal{M}^0(H_j)$. Since $\mathcal{M}^0(H_j) \cap \mathfrak{n}(K)$ for the upper nilpotent Lie algebra $\mathfrak{n}(X)$ as in the proof of Corollary 2.2 is an open ideal of V , by the absolute irreducibility of the adjoint action of H_j on \mathfrak{M}_j , $\mathcal{M}^0(H_j)$ is an open Lie \mathbb{Z}_p -subalgebra of \mathfrak{M}_j ; i.e., $[\mathfrak{M}_j : \mathcal{M}^0(H_j)] < \infty$. Then, by our assumption and Corollary 2.2, \mathfrak{M}_j for each j has rank 3 over V . We proceed by induction on m . If the intersection of \mathfrak{M} with one factor $\mathfrak{sl}_2(V)$, say the first one, is non-trivial, the intersection is stable under the adjoint action of \mathfrak{M} and H . Then the intersection projected to the first component has to have an open image in the first component $\mathfrak{sl}_2(V)$ as $\mathfrak{sl}_2(K)$ is a simple Lie algebra. Thus the intersection is either trivial or contains an open subalgebra of $\mathfrak{sl}_2(V)$. If it contains an open subalgebra, we can project H and \mathfrak{M} to the complementary direct summand, and by induction, we get the job done. Thus we may assume that any intersection of \mathfrak{M} with a direct factor $\mathfrak{sl}_2(V)$ is trivial. Thus the projection to the complementary direct summand $\mathfrak{sl}_2(V)^{m-1}$ is an injection. By induction assumption, conjugating H by an element in $\mathcal{B}(K)^{m-1}$, we may assume that the image of \mathfrak{M} in $\mathfrak{sl}_2(V)^{m-1}$ is contained in the diagonal image $\Delta(\mathfrak{sl}_2(V))$ of $\mathfrak{sl}_2(V)$. Thus we are reduced to the case where $m = 2$ regarding $\mathfrak{M} \subset \mathfrak{sl}_2(V) \times \Delta(\mathfrak{sl}_2(V)) \cong \mathfrak{sl}_2(V)^2$. Then $K \cdot \mathfrak{M} \subset \mathfrak{sl}_2(K)^2$ is a graph of an isomorphism $L : \mathfrak{sl}_2(K) \rightarrow \mathfrak{sl}_2(K)$ of Lie K -algebras. As is well known, such an isomorphism is inner given by a conjugation by an element of $GL_2(K)$. This finishes the proof. \square

Corollary 2.8. *Let V be a p -profinite discrete valuation ring with quotient field K , and let $A_0 = \mathbb{Z}_p$ if K has characteristic 0 and $A_0 = \mathbb{F}_p[[T]] \subset V$ if K has characteristic $p > 0$ for an element $T \in V$ analytically independent over \mathbb{F}_p . If a closed subgroup $H \subset SL_2(V)^m$ has image in each factor of $SL_2(V)$ containing an open subgroup of $SL_2(A_0)$ up to conjugation in $GL_2(K)$, then a conjugate in $GL_2(K)^m$ of H contains an open subgroup of $SL_2(A_0)$ diagonally embedded in $SL_2(V)^m$.*

Proof. If a p -profinite subgroup G of $SL_2(V)$ contains up to conjugation an open subgroup of $SL_2(A_0)$, we have $K \cdot \mathcal{M}^0(G) = \mathfrak{sl}_2(K)$ as the adjoint action of G on the both side of the identity is absolutely irreducible; so, $V \cdot \mathcal{M}^0(G)$ is an open Lie subalgebra of $\mathfrak{sl}_2(V)$. We apply the argument which proves Lemma 2.7 to the Lie algebra $V \cdot \mathcal{M}^0(H)$ which has projection to each factor $\mathfrak{sl}_2(V)$ with open image. Then after conjugation, $V \cdot \mathcal{M}^0(H)$ contains the diagonal image of an open Lie V -subalgebra of $\mathfrak{sl}_2(V)$. Thus $\mathcal{M}^0(H)$ must contain an open Lie V -subalgebra of $\mathfrak{sl}_2(A_0)$, which implies that H contains an open subgroup of $SL_2(A_0)$ diagonally embedded into $SL_2(V)^m$. \square

Recall the quotient field Q of Λ . As before, we fix a domain \mathbb{I} finite flat over Λ . For $g \in GL_2(\mathbb{I})$ and $x \in \mathfrak{sl}_2(\mathbb{I})$, we write $Ad(g)(x) = gxg^{-1}$ (the adjoint action of g). Hereafter we assume $p > 2$.

Lemma 2.9. *Let $G = \mathbb{G} \cap SL_2(\Lambda)$ for a p -profinite subgroup \mathbb{G} of $SL_2(\mathbb{I})$ satisfying the condition (B) of Lemma 1.3. Let P be a prime divisor of Λ . Suppose that \mathbb{I} is an integrally closed domain flat over Λ and one of the following conditions on existence of elements j, v in $GL_2(\mathbb{I})$:*

- (1) *There exists $j \in \mathcal{B}(\mathbb{I})$ with $j\mathbb{G}j^{-1} = \mathbb{G}$ such that the three eigenvalues of $Ad(j)$ are in \mathbb{Z}_p distinct modulo $\mathfrak{m}_{\mathbb{I}}$;*
- (2) *There exist $j \in \mathcal{B}(\mathbb{I})$ with $j\mathbb{G}j^{-1} = \mathbb{G}$ and $v \in \mathbb{G} \cap \mathcal{U}(\mathbb{I})$ such that the two eigenvalues of j are in \mathbb{Z}_p distinct modulo $\mathfrak{m}_{\mathbb{I}}$ and that v is non-trivial modulo $\mathfrak{m}_{\mathbb{I}}$;*
- (3) *There exist $j \in \mathcal{B}(\mathbb{I})$ with $j\mathbb{G}j^{-1} = \mathbb{G}$ and $v \in \mathbb{G} \cap \mathcal{U}(\mathbb{I})$ such that the two eigenvalues of j are in \mathbb{Z}_p distinct modulo $\mathfrak{m}_{\mathbb{I}}$ and that v is non-trivial modulo \mathfrak{P} for all prime ideals $\mathfrak{P}|P$.*

If the image $\overline{\mathbb{G}}_{\mathfrak{P}}$ of \mathbb{G} in $SL_2(\mathbb{I}/\mathfrak{P})$ for every prime divisor $\mathfrak{P}|P$ in \mathbb{I} contains, up to conjugation, an open subgroup of $SL_2(A_0)$, then we find a nonzero ideal $\mathfrak{c} \subset \Lambda$ prime to P and $\alpha \in \mathcal{B}(\mathbb{I}/P)$ such that $\alpha \cdot G\alpha^{-1} \supset \Gamma_{\Lambda}(\mathfrak{c})$. In particular, the image $\overline{\alpha G \alpha^{-1}}$ of $\alpha G \alpha^{-1}$ in $SL_2(\mathbb{I}/P)$ contains an open subgroup of $SL_2(A_0)$, and replacing G by $\alpha \cdot G \alpha^{-1}$, $U = G \cap \mathcal{U}(\Lambda)$ and $U_t = G \cap {}^t\mathcal{U}(\Lambda)$ for the opposite unipotent group ${}^t\mathcal{U}$ are both non-trivial with non-zero image in $GL_2(\Lambda/P)$. If the assumption (2) holds, we can choose $\alpha \in \mathcal{B}(\mathbb{I})$.

Proof. Replacing j by $\lim_{n \rightarrow \infty} j^{p^n}$, we may assume that j has finite order with two eigenvalues in \mathbb{Z}_p distinct modulo \mathfrak{m}_{Λ} ; and hence is semi-simple. Write $j = \begin{pmatrix} \zeta & * \\ 0 & \zeta' \end{pmatrix} \in \mathcal{B}(\mathbb{I})$. Conjugating G by $\alpha_0 = \begin{pmatrix} 1 & * / (\zeta - \zeta') \\ 0 & \zeta' \end{pmatrix} \in \mathcal{U}(\mathbb{I})$, we assume that $j = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta' \end{pmatrix}$ normalizes $\alpha_0 \mathbb{G} \alpha_0^{-1} \subset GL_2(\Lambda)$. We replace \mathbb{G} by $\alpha_0 \mathbb{G} \alpha_0^{-1}$. Then by Lemma 1.4, we have the diagonal group \mathcal{T} contained in \mathbb{G} . Since α_0 commutes

with upper unipotent element, this does not affect v in the condition (2) or (3). Write $v = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$. We have

$$(2.1) \quad \begin{cases} u \in \mathbb{I}^\times & \text{under (2),} \\ u \in \mathbb{I}_P^\times & \text{under (3).} \end{cases}$$

Conjugating by $\alpha_1 = \begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix}$, under either (2) or (3), we assume $v = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in G$. Since $(t(s), v) = t(s)vt(s)^{-1}v^{-1} = \begin{pmatrix} 1 & (1+T)^{s-1} \\ 0 & 1 \end{pmatrix}$ for $t(s)$ in (B) of Lemma 1.3 for all $s \in \mathbb{Z}_p$, under either (2) or (3), we have $\mathcal{U}(\Lambda) \subset G$.

First, we assume $\mathbb{I} = \Lambda$ and (1) and prove the lemma. Thus $\overline{\mathbb{G}}_{\mathfrak{P}}$ contains an open subgroup of $SL_2(A_0)$. We have the adjoint operator $Ad(j)$ acting on $M_2(\mathbb{I})$, $\mathcal{M} = \mathcal{M}(\mathbb{G})$ and $\overline{\mathcal{M}} = \mathcal{M}(\overline{\mathbb{G}}_{\mathfrak{P}})$. Write three eigenvalues of $Ad(j)$ as $a = \zeta\zeta'^{-1}$, 1 and a^{-1} . Then for $X = M_2(\mathbb{I})$, \mathcal{M} and $\overline{\mathcal{M}}$, we have a decomposition $X = X[a] \oplus X[1] \oplus X[a^{-1}]$ into the direct sum of eigenspaces $X[\lambda]$ with eigenvalue $\lambda = a, 1, a^{-1}$. Thus the reduction map $\mathcal{M}[\lambda] \rightarrow \overline{\mathcal{M}}[\lambda]$ modulo \mathfrak{Q} is a surjective map for any prime $\mathfrak{Q} \in \text{Spec}(\mathbb{I})$. If $\overline{\mathbb{G}}_{\mathfrak{P}}$ contains an open subgroup of $SL_2(A_0)$, we find that $\overline{\mathcal{M}}[\lambda]$ is non-trivial for all eigenvalues λ , and hence $\mathcal{M}[\lambda] \neq 0$ surjects down to $\overline{\mathcal{M}}[\lambda]$. Since $\mathcal{M}[a] = \mathcal{M} \cap \mathfrak{u}(\mathbb{I})$, we find $U = 1 + \mathcal{M}[a] \subset \mathbb{G}'$ has surjective image $\overline{U} = 1 + \overline{\mathcal{M}}[a] \subset \overline{\mathbb{G}}'_{\mathfrak{P}}$. Similarly $U_t = 1 + \mathcal{M}[a^{-1}] \subset \mathbb{G}'$ has surjective image $\overline{U}_t = 1 + \overline{\mathcal{M}}[a^{-1}] \subset \overline{\mathbb{G}}'_{\mathfrak{P}}$. Since $\overline{\mathbb{G}}_{\mathfrak{P}}$ contains an open subgroup of $SL_2(A_0)$, the two eigenspaces $\overline{\mathcal{M}}[a]$ and $\overline{\mathcal{M}}[a^{-1}]$ are both non-trivial; so, $\overline{U} \neq 1$ and $\overline{U}_t \neq 0$. Since $\mathfrak{u} = \{b \in \mathbb{I} \mid \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in U\}$ and $\mathfrak{u}_t = \{c \in \mathbb{I} \mid \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \in U_t\}$ are non-zero \mathbb{I} -ideals, $\overline{U} \neq 1$ and $\overline{U}_t \neq 1$ implies \mathfrak{u} and \mathfrak{u}_t is prime to \mathfrak{P} . We often identify \mathfrak{u} (resp. \mathfrak{u}_t) with the corresponding Lie algebra $\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in \mathfrak{u} \} = \mathfrak{u}(\mathbb{I}) \cap \mathcal{M}_1(\mathbb{G})$ (resp. $\{ \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} \mid c \in \mathfrak{u}_t \} = {}^t\mathfrak{u}(\mathbb{I}) \cap \mathcal{M}_1(\mathbb{G})$). Therefore $\overline{\mathbb{G}}_{\mathfrak{P}}$ contains open subgroups \overline{U} of $\mathcal{U}(\mathbb{I}/\mathfrak{P})$ and \overline{U}_t of ${}^t\mathcal{U}(\mathbb{I}/\mathfrak{P})$. This implies that $\overline{\mathbb{G}}_{\mathfrak{P}}$ contains an open subgroup H of $SL_2(\mathbb{I}/\mathfrak{P})$ as \overline{U} and \overline{U}_t generate an open subgroup of $SL_2(\mathbb{I}/\mathfrak{P})$. Indeed, for $b \in \mathfrak{u}$ and $c \in \mathfrak{u}_t$, taking $X = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}$, we have in $\mathcal{M}(H)$ the following element

$$[X, Y] = XY - YX = \begin{pmatrix} bc & 0 \\ 0 & -bc \end{pmatrix}.$$

Similarly, by Theorem 1.1, $\mathcal{M}(H)$ contains $\text{Tr}(\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}) = ab$ as a central element; so, it contains $\mathfrak{u}_t M_2(\mathbb{I})$; i.e., $\overline{\mathbb{G}}_{\mathfrak{P}}$ contains an open subgroup $\Gamma_{\mathbb{I}}(\mathfrak{u}_t)/\Gamma_{\mathbb{I}}(\mathfrak{u}_t P)$ in $SL_2(\mathbb{I}/\mathfrak{P})$. Then for the closed subgroup $G_U \subset \mathbb{G}$ topologically generated by conjugates gUg^{-1} for all $g \in \mathbb{G}$, \overline{G}_U contains an open subgroup of $SL_2(A_0)$ by Corollary 2.3 and Lemma 2.4. Then by Lemma 1.5, we get the desired assertion.

Since the two conditions (2) and (3) are similar, the proof is basically the same, though we need to localize the argument at P under (3). As is clear from the above proof under (1), we only need to prove $P \nmid \mathfrak{u}_t$. We now give a proof supposing $\mathbb{I} = \Lambda$ and (2). Since the reduction map $\mathcal{M}[1] \rightarrow \overline{\mathcal{M}}[1]$ modulo \mathfrak{P} is surjective, \mathbb{G} has an element $g \in \mathcal{B}(\mathbb{I})$ with eigenvalues z, z' with $z \not\equiv z' \pmod{\mathfrak{P}}$. Then by Lemma 1.3 (2), U/U' is killed by $(z'z^{-1} - 1, T)$. In particular, we can find $c \in \mathbb{I} \cap \mathbb{I}_{\mathfrak{P}}^\times$ in the annihilator of U/U' . Let

$$\mathcal{M}_0 := \mathcal{M} \cap (\mathfrak{u}(\mathbb{I}) \oplus {}^t\mathfrak{u}(\mathbb{I})) \quad \text{and} \quad \overline{\mathcal{M}}_0 := \overline{\mathcal{M}} \cap (\mathfrak{u}(\mathbb{I}/\mathfrak{P}) \oplus {}^t\mathfrak{u}(\mathbb{I}/\mathfrak{P})).$$

Then, under the reduction map modulo \mathfrak{P} , \mathcal{M}_0 surjects down to $\overline{\mathcal{M}}_0$ which has non-trivial intersection ${}^t\mathfrak{u}(\mathbb{I}/\mathfrak{P})$. Thus we have an element $(a, b) \in \mathcal{M}_0$ with $a \in \mathfrak{u}(\mathbb{I})$ and $b \in {}^t\mathfrak{u}(\mathbb{I})$ with $b \pmod{\mathfrak{P}} \neq 0$. Since $\mathbb{G} \supset \mathfrak{u}(\Lambda)$, we have $(ca, 0) \in \mathcal{M}_0$; so, $(0, cb) = (ca, cb) - (ca, 0) \in \mathcal{M}_0$. Thus we find that U_t and \overline{U}_t cannot be trivial. Then by the same argument as above, we conclude the assertion. Under (3), we go exactly the same way, replacing \mathbb{I} and Λ in the above argument by \mathbb{I}_P and Λ_P .

Now we assume (1) and that $\mathbb{I} \supsetneq \Lambda$. Since the proof is long in this case, we proceed in steps: First we claim

- (a) Conjugating \mathbb{G} by an element in $\mathcal{B}(\mathbb{I}_P)$, we achieve that $\overline{\mathbb{G}}'_{\mathfrak{P}} \cap SL_2(A_0)$ is open in $SL_2(A_0)$ for all prime divisor $\mathfrak{P} \mid P$ in \mathbb{I} .

To see the claim, for each $\mathfrak{P} \mid P$, since $\overline{\mathbb{G}}_{\mathfrak{P}}$ contains an open subgroup of $SL_2(A_0)$, its derived group $\overline{\mathbb{G}}'_{\mathfrak{P}}$ contains an open subgroup of a conjugate of $SL_2(A_0)$. Thus the $\kappa(\mathfrak{P})$ -span of $\overline{\mathcal{M}}^0 = \mathcal{M}^0(\overline{\mathbb{G}}_{\mathfrak{P}})$ has dimension 3 (by the irreducibility of the adjoint action of (an open subgroup of) $SL_2(A_0)$). Thus a -eigenspace $\overline{\mathcal{M}}[a] = \mathfrak{u}(\kappa(\mathfrak{P})) \cap \overline{\mathcal{M}}$ under the action of $Ad(j)$ is non-trivial. Taking $0 \neq u_{\mathfrak{P}} \in \mathbb{I}/\mathfrak{P}$

such that $\begin{pmatrix} 0 & u_{\mathfrak{P}} \\ 0 & 0 \end{pmatrix} \in \overline{\mathcal{M}}[a]$ and putting $\alpha_{\mathfrak{P}} = \begin{pmatrix} u_{\mathfrak{P}}^{-1} & 0 \\ 0 & 1 \end{pmatrix}$, we have $\mathfrak{U}(A_0) \subset \mathcal{M}(\alpha_{\mathfrak{P}} \overline{\mathbb{G}}_{\mathfrak{P}} \alpha_{\mathfrak{P}}^{-1})$. By the approximation theorem (e.g., [BCM] VII.2.4) applied to the Dedekind ring \mathbb{I}_P , we find $\alpha \in \mathcal{B}(\mathbb{I}_P)$ such that $\alpha \bmod \mathfrak{P} = \alpha_{\mathfrak{P}}$ for all $\mathfrak{P}|P$; so, replacing \mathbb{G} by $\alpha \mathbb{G} \alpha^{-1}$, we start with \mathbb{G} with $\overline{\mathbb{G}}_{\mathfrak{P}}$ containing $\mathcal{U}(A_0)$ for all $\mathfrak{P}|P$. Let Q_0 be the quotient field of A_0 . Consider the Q_0 -span $\mathfrak{s}_{\mathfrak{P}}$ of $\mathcal{M}^0(\overline{\mathbb{G}}_{\mathfrak{P}})$. Then $\kappa(\mathfrak{P}) \cdot \mathfrak{s}_{\mathfrak{P}} = \mathfrak{sl}_2(\kappa(\mathfrak{P}))$ as the adjoint action of $SL_2(A_0)$ (more precisely its conjugate) is absolutely irreducible. Then by Lemma 2.6 applied to $L = \kappa(\mathfrak{P})$ and $K = Q_0$, $\mathfrak{s}_{\mathfrak{P}}$ contains $\mathfrak{sl}_2(Q_0)$, which implies the claim.

Next we show

(b) Conjugating \mathbb{G} by an element in $\mathcal{B}(\mathbb{I}_P)$, we achieve $U = \mathcal{U}(\Lambda) \cap \mathbb{G} \neq 0$ and $U' = \mathcal{U}(\Lambda) \cap \mathbb{G}' \neq 0$.

To see this, we use the same symbol introduced at the beginning of this proof. In particular, $j = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta' \end{pmatrix} \in \mathcal{B}(\mathbb{I})$ and $\overline{\mathbb{G}}_P$ (resp. $\overline{\mathbb{G}}'_P$) is the image of \mathbb{G} (resp. \mathbb{G}') in $SL_2(\mathbb{I}/P\mathbb{I})$. Since $\overline{\mathbb{G}}_{\mathfrak{P}}$ (and hence $\overline{\mathbb{G}}'_{\mathfrak{P}}$) for each $\mathfrak{P}|P$ contains an open subgroup of $SL_2(A_0)$, we find the image $\mathcal{M}(\overline{\mathbb{G}}_{\mathfrak{P}})[\lambda]$ of $\overline{\mathcal{M}}[\lambda]$ in $\mathfrak{gl}_2(\mathbb{I}/\mathfrak{P})$ is non-trivial for all eigenvalues λ of $Ad(j)$ and all $\mathfrak{P}|P$, we conclude, as before,

- (1) $\mathcal{M}[\lambda] \neq 0$ and $\mathcal{M}(\overline{\mathbb{G}}_{\mathfrak{P}})[\lambda] \neq 0$ for all $\mathfrak{P}|P$,
- (2) the \mathbb{I} -ideal \mathfrak{a} generated by $\mathfrak{n} = \{a \in \mathbb{I} \mid \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \in \mathcal{M}[a]\}$ is prime to P ($\Rightarrow \mathbb{U} \neq 0$),
- (3) the \mathbb{I} -ideal \mathfrak{a}_t generated by $\mathfrak{n}_t = \{a \in \mathbb{I} \mid \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} \in \mathcal{M}[a^{-1}]\}$ is prime to P ($\Rightarrow \mathbb{U}_t \neq 0$).

By (1), we can pick $u \in \mathfrak{n}$ prime to P such that $(u \bmod \mathfrak{P}) \in A_0$ for all $\mathfrak{P}|P$. Conjugating \mathbb{G} by $\alpha = \begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix} \in \mathcal{B}(\mathbb{I}_P)$ with image $\overline{\alpha} \in \mathcal{B}(Q(A_0))$, we may assume that $\mathfrak{u} = \mathfrak{n} \cap \Lambda = \Lambda$ and $\overline{\alpha} \overline{\mathbb{G}}'_{\mathfrak{P}} \overline{\alpha}^{-1}$ contains an open subgroup of $SL_2(A_0)$ for all $\mathfrak{P}|P$. Since α is diagonal, j still normalize \mathbb{G} and \mathbb{G}' . Just to have $\mathfrak{u} \neq 0$, we only need to choose any $0 \neq u \in Q(\mathbb{I})\mathfrak{n}$; so, we can assume that u is prime to any given finite set of primes.

We now claim

(c) $\overline{\mathbb{G}}'_P$ (and hence $\overline{\mathbb{G}}_P$) contains an open subgroup of $SL_2(A_0)$ regarding $A_0 \subset \mathbb{I}/P\mathbb{I}$.

To see this, recall $\mathcal{M}^0(\mathcal{G}) = \mathcal{M}(\mathcal{G}) \cap \mathfrak{sl}_2(A)$ for a p -profinite subgroup \mathcal{G} of $SL_2(A)$. Write $\mathcal{M}^0 = \mathcal{M}^0(\mathbb{G})$ and $\overline{\mathcal{M}}^0 = \mathcal{M}^0(\overline{\mathbb{G}}_P)$. Replacing \mathbb{I} by the integral closure of \mathbb{I} in the Galois closure of $Q(\mathbb{I})$ over Q , we may assume that $Q(\mathbb{I})$ is a Galois extension of Q . Then $P = \prod_{\mathfrak{P} \in \Sigma} \mathfrak{P}^e$ for the finite set Σ of primes \mathfrak{P} in \mathbb{I} over P . Then we may identify $\kappa(\mathfrak{P})$ with a finite extension k of $\kappa(P)$ for all $\mathfrak{P}|P$. Since $\overline{\mathbb{G}}_{\mathfrak{P}}$ contains an open subgroup of $SL_2(A_0)$ for every $\mathfrak{P} \in \Sigma$, by Corollary 2.8, the image $\overline{\mathbb{G}}_{\sqrt{P}}$ of \mathbb{G} in $SL_2(\mathbb{I}_P/\sqrt{P}\mathbb{I}_P) = \prod_{\mathfrak{P}|P} SL_2(k)$ contains an open subgroup H of $SL_2(A_0)$ diagonally embedded in $\prod_{\mathfrak{P}|P} SL_2(k)$, where $\sqrt{P} = \prod_{\mathfrak{P} \in \Sigma} \mathfrak{P}$ is the radical of P in \mathbb{I} . Then H acts on $\kappa(P) \cdot \mathcal{M}^0$ by the adjoint action. Since $\mathfrak{U}(\Lambda) \subset \Lambda_P \cdot \mathcal{M}^0$, under the action of the group algebra $\kappa(P)[H]$, $\mathfrak{U}(\kappa(P)) \subset \overline{\mathcal{M}}^0$ generates an irreducible subspace equal to $\mathfrak{sl}_2(\kappa(P))$ (since $\mathfrak{U}(\kappa(P)) \subset \mathfrak{sl}_2(\kappa(P))$ is the highest weight root space and the adjoint square is absolutely irreducible as $p > 2$). Therefore $\kappa(P) \cdot \mathcal{M}^0(\overline{\mathbb{G}}_P)$ contains $\mathfrak{sl}_2(\kappa(P))$, which implies that $\overline{\mathbb{G}}'_P$ contains an open subgroup of $SL_2(A_0)$ by Lemma 2.4 and Corollary 2.3.

Next we look into the A -span $\mathfrak{s}_A := A \cdot \mathcal{M}^0$ of \mathcal{M}^0 for a subalgebra A of $Q(\mathbb{I})$, which is a Lie A -subalgebra of $\mathfrak{sl}_2(Q(\mathbb{I}))$. Let $\mathfrak{u} = \mathcal{M}^0[a] \cap \mathfrak{sl}_2(\Lambda)$ and $\mathfrak{u}_t = \mathcal{M}^0[a^{-1}] \cap \mathfrak{sl}_2(\Lambda)$. We claim

(d) $\mathfrak{s}_Q = Q \cdot \mathcal{M}^0$ contains $\mathfrak{sl}_2(Q)$ for the quotient field Q of Λ , $\dim_Q Q \cdot \mathfrak{u} = \dim_Q Q \cdot \mathfrak{u}_t = 1$ and $P \nmid \mathfrak{u}$.

To see this, pick a prime factor $\mathfrak{P}|P$ of P in \mathbb{I} . Taking $A = \mathbb{I}_{\mathfrak{P}}$, the image \mathfrak{s}_A of \mathfrak{s}_A in $\mathfrak{sl}_2(\mathbb{I}_{\mathfrak{P}}/\mathfrak{P}\mathbb{I}_{\mathfrak{P}})$ contains a non-trivial upper nilpotent algebra $\overline{\mathfrak{u}}_{\mathfrak{P}}$ which is the image of the Lie algebra of U . Since the image $\overline{\mathbb{G}}_{\mathfrak{P}}$ of \mathbb{G} in $SL_2(\mathbb{I}/\mathfrak{P})$ contains an open subgroup of $SL_2(A_0)$, $\mathfrak{s}_A = \mathfrak{sl}_2(\kappa(\mathfrak{P}))$, which implies $\mathfrak{s}_A/\mathfrak{P}\mathfrak{s}_A = \mathfrak{sl}_2(\kappa(\mathfrak{P}))$. From Nakayama's lemma, we deduce $\mathfrak{s}_{\mathbb{I}_{\mathfrak{P}}} = \mathfrak{sl}_2(\mathbb{I}_{\mathfrak{P}})$. Thus \mathfrak{s}_Q spans over $Q(\mathbb{I})$ the entire $\mathfrak{sl}_2(Q(\mathbb{I}))$. Again applying Lemma 2.6 for \mathfrak{s}_Q , $K = Q$ and $L = Q(\mathbb{I})$, we get $\mathfrak{s}_Q \supset \mathfrak{sl}_2(Q)$. This proves that $\dim_Q Q \cdot \mathfrak{u} = \dim_Q Q \cdot \mathfrak{u}_t = 1$. In particular, $U = 1 + \mathfrak{u} \subset G$ and $U_t = 1 + \mathfrak{u}_t \subset G$ are non-trivial unipotent subgroup. Regarding \mathfrak{u} and \mathfrak{u}_t as ideals of Λ , we find $G \supset \Gamma_{\Lambda}(\mathfrak{u}\mathfrak{u}_t)$. By (c), the image $\overline{\mathfrak{u}}$ in $\mathfrak{sl}_2(\Lambda/P)$ is non-trivial; so, $P \nmid \mathfrak{u}$.

Therefore, to finish the proof under (1) and $\mathbb{I} \neq \Lambda$, we need to prove that

(e) $P \nmid \mathfrak{u}_t$.

Let $\overline{H} = \overline{\mathbb{G}}'_P \cap SL_2(A_0)$, which is an open subgroup of $SL_2(A_0)$. Put $H = \pi^{-1}(\overline{H})$ for the projection $\pi : \mathbb{G}' \rightarrow \overline{\mathbb{G}}'_P$. Note that H is still normalized by j . For the order $\mathbb{I}' = \Lambda + P\mathbb{I} \subset \mathbb{I}$, $H \subset GL_2(\mathbb{I}')$. Note that $P\mathbb{I}'$ is still a prime in \mathbb{I}' with residue field $\kappa(P)$. In the above argument, we replace \mathbb{G} by H , \mathbb{I} by \mathbb{I}' and \mathfrak{s}_Q by $\mathfrak{s} = \Lambda_P \cdot \mathcal{M}^0$. Then \mathfrak{s} is a Lie Λ_P -subalgebra of Q -simple Lie algebra $\mathfrak{s}_Q = \mathfrak{sl}_2(K)$. We consider the adjoint action of $\overline{\mathbb{G}}_P$, which is now an open subgroup of $SL_2(A_0)$. By our replacement of \mathbb{I} by \mathbb{I}' , P is a prime in \mathbb{I}' with $\kappa(P) = \kappa(P\mathbb{I})$. Consider the image $\overline{\mathfrak{s}}_P$ of \mathfrak{s} in $\mathfrak{sl}_2(\mathbb{I}/P\mathbb{I})$. Since $\overline{\mathfrak{s}}_P$ is generated by $\kappa(P) \cdot \overline{\mathbf{u}}$ under the action of the group algebra $\kappa(P)[\overline{\mathbb{G}}_P]$, noting $\overline{\mathbb{G}}_P$ is now an open subgroup of $SL_2(A_0)$, we have $\overline{\mathfrak{s}}_P = \mathfrak{sl}_2(\kappa(P))$. Thus $\overline{\mathbb{G}}_P$ acts on $\overline{\mathfrak{s}}_P = \mathfrak{sl}_2(\kappa(P))$ by the adjoint representation $Ad(\kappa(P))$. Therefore its a^{-1} -eigenspace $\overline{\mathfrak{s}}_P[a^{-1}]$ of $Ad(j)$ is non-trivial in $\kappa(P)[\overline{\mathbb{G}}] \cdot \overline{\mathbf{u}} = \mathfrak{sl}_2(\kappa(P))$. This shows $\Lambda_P[\overline{\mathbb{G}}] \cdot \overline{\mathbf{u}} = \mathfrak{sl}_2(\kappa(P))$. By Nakayama's lemma, we conclude $\mathbb{I}_P[\mathbb{G}] \cdot \mathbf{u} = \mathfrak{sl}_2(\mathbb{I}_P)$. Since $\Lambda_P[\mathbb{G}] \cdot \mathbf{u}$ spans $\mathfrak{sl}_2(\mathbb{I}_P)$ stable under the action of \mathbb{G} , it contains $\mathfrak{sl}_2(\Lambda_P)$. Thus $\mathfrak{s} \supset \Lambda_P[\mathbb{G}] \cdot \mathbf{u} \supset \mathfrak{sl}_2(\Lambda_P)$, and

$${}^t\mathfrak{u}(\Lambda_P) \supset \Lambda_P \cdot \mathbf{u}_t = \mathfrak{s} \cap {}^t\mathfrak{u}(\Lambda_P) \supset \mathfrak{sl}_2(\Lambda_P) \cap {}^t\mathfrak{u}(\Lambda_P) = {}^t\mathfrak{u}(\Lambda_P).$$

Thus $\Lambda_P \cdot \mathbf{u}_t = {}^t\mathfrak{u}(\Lambda_P)$, and we conclude $\overline{\mathbf{u}}_t \neq 0$; so, $P \nmid \mathbf{u}_t$.

We now assume (2) or (3) and $\mathbb{I} \neq \Lambda$. As we have seen, for $\mathbf{u} = \{b \in \Lambda \mid \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in \mathbb{G}\}$ has non-trivial image $\overline{\mathbf{u}}$ in $\mathfrak{sl}_2(\mathbb{I}/\mathfrak{P})$ for all primes $\mathfrak{P} \mid P$. We have shown above, from the non-triviality of $\overline{\mathbf{u}}$, $\mathfrak{s}_{\mathbb{I}_P} = \mathfrak{sl}_2(\mathbb{I}_P)$. After reaching this point, by the same argument as above, we conclude $\overline{\mathbf{u}}_t \neq 0$; so, we get the desired assertion. \square

Remark 2.10. We insert here another shorter proof of the above lemma under (1) assuming an extra assumption $P \nmid (p)$. We first replace \mathbb{I} by the integral closure of \mathbb{I} in the Galois closure of $Q(\mathbb{I})$ over Q . Write $\mathfrak{g} = \text{Gal}(Q(\mathbb{I})/Q)$ for the finite Galois group. Let H be as in Step (c) in the above proof. We then replace \mathbb{G} by $\mathbb{G} = \{h \in \mathbb{G} \mid \pi(h) \in \overline{H}\}$ for the reduction map $\pi : GL_2(\mathbb{I}_P) \rightarrow GL_2(\mathbb{I}_P/P\mathbb{I}_P)$. We now replace \mathbb{I} by $\Lambda + P\mathbb{I}$. By this, we lose normality of \mathbb{I} but \mathbb{I}_P becomes local with only one maximal ideal $P\mathbb{I}_P$ satisfying $\kappa(P) = \kappa(P\mathbb{I})$ and $\mathcal{M}(\overline{\mathbb{G}}_P) \subset \mathfrak{gl}_2(\mathbb{Z}_p)$ invariant under \mathfrak{g} . Recall

$$\mathcal{M}_0 = \mathcal{M} \cap (\mathfrak{u}(\mathbb{I}) \oplus {}^t\mathfrak{u}(\mathbb{I})).$$

As before, we find $\mathbb{I}_P \cdot \mathcal{M}^0 = \mathfrak{sl}_2(\mathbb{I}_P)$, and $\mathbb{I}_P \cdot \mathcal{M}_0 = \mathfrak{u}(\mathbb{I}_P) \oplus {}^t\mathfrak{u}(\mathbb{I}_P)$. Thus \mathfrak{g} acts on $\mathbb{I}_P \cdot \mathcal{M}^0$ and $\mathbb{I}_P \cdot \mathcal{M}_0$. The \mathfrak{g} -cohomology sequence attached to the exact sequence

$$0 \rightarrow P\mathbb{I}_P \cdot \mathcal{M}_0 \rightarrow \mathbb{I}_P \cdot \mathcal{M}_0 \rightarrow \kappa(P) \cdot \overline{\mathcal{M}}_0 \rightarrow 0$$

gives a short exact sequence

$$0 \rightarrow H^0(\mathfrak{g}, P\mathbb{I}_P \cdot \mathcal{M}_0) \rightarrow H^0(\mathfrak{g}, \mathbb{I}_P \cdot \mathcal{M}_0) \rightarrow \kappa(P) \cdot \overline{\mathcal{M}}_0 \rightarrow 0$$

since $H^1(\mathfrak{g}, P\mathbb{I}_P \cdot \mathcal{M}_0) = 0$ (as $P\mathbb{I}_P \cdot \mathcal{M}_0$ is a \mathbb{Q}_p -vector space). Similarly, \mathfrak{g} acts on $\mathbb{I}_P \cdot \mathcal{M}^0$ and we have a short exact sequence

$$0 \rightarrow H^0(\mathfrak{g}, P\mathbb{I}_P \cdot \mathcal{M}^0) \rightarrow H^0(\mathfrak{g}, \mathbb{I}_P \cdot \mathcal{M}^0) \rightarrow \kappa(P) \cdot \overline{\mathcal{M}}^0 \rightarrow 0.$$

Since \mathcal{M}_0 is a Λ -module, the quotient $X := H^0(\mathfrak{g}, \mathbb{I}_P \cdot \mathcal{M}_0)/\mathcal{M}_0$ is a Λ -module of finite type. The quotient $Y := H^0(\mathfrak{g}, \mathbb{I}_P \cdot \mathcal{M}^0)/\mathcal{M}^0$ contains X as a direct summand (i.e., $Y = X \oplus Z$ with $Ad(j)$ acting trivially on Z), and on Y the open subgroup $\overline{\mathbb{G}}_P$ of $SL_2(\mathbb{Z}_p)$ acts by the adjoint action. Consider the Λ_P -span $\Lambda_P \cdot Y = (\Lambda_P \cdot X) \oplus (\Lambda_P \cdot Z)$. Since $H^0(\mathfrak{g}, \Lambda_P \cdot \mathcal{M}[a]) = \mathfrak{u}(\Lambda_P) = H^0(\mathfrak{g}, \mathfrak{u}(\mathbb{I}_P))$, the $\overline{\mathbb{G}}_P$ -module $\overline{Y} := Y \otimes_{\Lambda_P} \kappa(P)$ does not have any highest weight vector with respect to $\mathcal{B}(\kappa(P)) \cap \overline{\mathbb{G}}_P$; so, $\overline{Y} = 0$. By Nakayama's lemma, we have $Y = 0$. In particular, we get $\Lambda_P \cdot X = 0$. This shows that $H^0(\mathfrak{g}, \mathbb{I}_P \cdot \mathcal{M}[a^{-1}])$ surjects down to ${}^t\mathfrak{u}(\kappa(P))$ under the reduction map modulo P , which implies $\mathbf{u}_t = \mathcal{M}[a^{-1}]$ has nontrivial image in ${}^t\mathfrak{u}(\kappa(P))$. In particular, $P \nmid \mathbf{u}_t$.

Remark 2.11. In Step (b) in the above proof, conjugation by $\alpha = \begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix}$ brings \mathbb{G} into a subgroup $\alpha\mathbb{G}\alpha^{-1}$ containing $\Gamma_\Lambda(\mathfrak{c})$ for a non-zero ideal \mathfrak{c} . To prove this fact, we only needed to make $\mathbf{u} \neq 0$. As remarked in the proof, allowing u to be another non-zero element u' in $Q(\mathbb{I}) \cdot \mathfrak{n}$, we achieve $\mathbf{u} \neq 0$; so, we get another conductor \mathfrak{c}' using this u' and $\alpha' = \begin{pmatrix} u'^{-1} & 0 \\ 0 & 1 \end{pmatrix}$; i.e., $\alpha'\mathbb{G}\alpha'^{-1} \supset \Gamma_\Lambda(\mathfrak{c}')$ maximally. For any other prime divisor $P' \neq P$, as long as $\rho_{\mathfrak{P}'}$ is irreducible for all $\mathfrak{P}' \mid P'$ (and $\alpha'\mathbb{G}\alpha'^{-1} \subset GL_2(\mathbb{I}_{P'})$), we will prove $\mathfrak{c}_{P'} = \mathfrak{c}'_{P'}$ in Lemma 3.3.

Theorem 2.12. *Suppose $p > 2$ and that \mathbb{I} is integrally closed. Let \mathbb{G} be a p -profinite subgroup \mathbb{G} of $SL_2(\mathbb{I})$ satisfying the condition (B) of Lemma 1.3 and one of the three conditions (1–3) of Lemma 2.9. Take a prime divisor P of Λ . Suppose that the projected image $\widetilde{\mathbb{G}}_{\mathfrak{P}} \hookrightarrow SL_2(\mathbb{I}/\mathfrak{P})$ of \mathbb{G} contains an open subgroup of $SL_2(A_0)$ for all prime factors $\mathfrak{P}|P$ in \mathbb{I} . Then there exists $\alpha \in \mathcal{B}(\mathbb{I}_P)$ such that, writing $\mathbb{G}_\alpha = \alpha\mathbb{G}\alpha^{-1}$ and $G_\alpha =$ for $\mathbb{G}_\alpha \cap SL_2(\Lambda)$,*

- (1) *the image $\mathcal{M}^0(G_\alpha)$ in $\mathfrak{sl}_2(\Lambda)$ spans over Q the entire Lie algebra $\mathfrak{sl}_2(Q)$.*
- (2) *there exist a unique non-zero ideal \mathfrak{c}_α of Λ prime to P (dependent on α) maximal among ideals $\mathfrak{a} \subset \Lambda$ such that $\mathbb{G}_\alpha \supset \Gamma_\Lambda(\mathfrak{a})$ ($\Leftrightarrow G_\alpha \supset \Gamma_\Lambda(\mathfrak{a})$).*
- (3) *the ideal $\bigcap_{(\lambda) \supset \mathfrak{c}_\alpha} (\lambda)$ (the intersection of all principal ideals containing \mathfrak{c}_α) is a principal ideal (L_α) , and $\Gamma_\Lambda(L_\alpha)/\Gamma_\Lambda(\mathfrak{c}_\alpha)$ is finite.*

The above ideal \mathfrak{c}_α will be called the *conductor* of G_α or \mathbb{G}_α . The assertion (1) follows from the above Lemma 2.9. The other two assertions are covered by Theorem 4.3.21 of [GME].

3. GLOBAL LEVEL OF $\rho_{\mathbb{I}}$

Take an irreducible component $\text{Spec}(\mathbb{I})$ of $\text{Spec}(\mathbf{h})$ with its normalization $\text{Spec}(\widetilde{\mathbb{I}})$. Take the representation $\rho_{\mathbb{I}} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow GL_2(\widetilde{\mathbb{I}})$ associated to \mathbb{I} . Here we list one more fundamental property of $\rho_{\mathbb{I}}$:

$$(\text{Det}) \quad \det(\rho_{\mathbb{I}})(\sigma) = (1 + T)^{\log_p(\mathcal{N}(\sigma))/\log_p(\gamma)} \chi_1(\sigma) \quad \text{for all } \sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}),$$

where χ is the Neben character as in the introduction and $\mathcal{N} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \mathbb{Z}_p^\times$ is the p -adic cyclotomic character (see the second edition of [GME] Theorem 4.3.1 for this fact).

Lemma 3.1. *Let the notation be as above. Then for each prime divisor \mathfrak{P} of $\widetilde{\mathbb{I}}$, the image of $\text{Im}(\rho_{\mathbb{I}}) \cap SL_2(\widetilde{\mathbb{I}})$ in $SL_2(\widetilde{\mathbb{I}}/\mathfrak{P})$ is an open subgroup of $\text{Im}(\rho_{\mathfrak{P}}) \cap SL_2(\widetilde{\mathbb{I}}/\mathfrak{P})$. In particular, the reduction map: $\mathbb{G} = \text{Im}(\rho_{\mathbb{I}}) \cap \Gamma_{\widetilde{\mathbb{I}}}(\mathfrak{m}_{\widetilde{\mathbb{I}}}) \rightarrow \text{Im}(\rho_{\mathfrak{P}}) \cap \Gamma_{\widetilde{\mathbb{I}}/\mathfrak{P}}(\mathfrak{m}_{\widetilde{\mathbb{I}}/\mathfrak{P}})$ given by $x \mapsto (x \bmod \mathfrak{P})$ has finite cokernel.*

Proof. Let

$$\mathbb{H} := \{g \in \text{Im}(\rho_{\mathbb{I}}) \mid \det(g) \in \Gamma \text{ and } g \equiv 1 \pmod{\mathfrak{m}_{\widetilde{\mathbb{I}}}}\} \text{ and } \mathbb{S} = \mathbb{H} \cap SL_2(\widetilde{\mathbb{I}})$$

for $\Gamma = \{(1+T)^s \mid s \in \mathbb{Z}_p\} \subset \Lambda^\times$. By (Gal), we find $\tau \in \rho_{\mathbb{I}}(D_p) \cap \mathbb{H}$ such that $\tau = \begin{pmatrix} (1+T) & * \\ 0 & 1 \end{pmatrix}$. By (Det), the group \mathbb{H} is an open subgroup of $\text{Im}(\rho_{\mathbb{I}})$. Thus we need to prove the image $\overline{\mathbb{S}}_{\mathfrak{P}}$ of \mathbb{S} in $SL_2(\widetilde{\mathbb{I}}/\mathfrak{P})$ is open in $\text{Im}(\rho_{\mathfrak{P}}) \cap SL_2(\widetilde{\mathbb{I}}/\mathfrak{P})$. Let $\overline{\mathbb{H}}_{\mathfrak{P}}$ be the image of \mathbb{H} in $SL_2(\widetilde{\mathbb{I}}/\mathfrak{P})$. Since \mathbb{H} is open in $\text{Im}(\rho_{\mathbb{I}})$, $\overline{\mathbb{H}}_{\mathfrak{P}}$ is open in $\text{Im}(\rho_{\mathfrak{P}})$. Put $\overline{\mathbb{S}}_{\mathfrak{P}} = \overline{\mathbb{H}}_{\mathfrak{P}} \cap SL_2(\mathbb{I}/\mathfrak{P})$, and set $\mathcal{T}' = \{\tau^s \mid s \in \mathbb{Z}_p\}$. Since $\overline{\mathbb{H}}_{\mathfrak{P}}$ is open in $\text{Im}(\rho_{\mathfrak{P}})$, $\overline{\mathbb{S}}_{\mathfrak{P}}$ is open in $\text{Im}(\rho_{\mathfrak{P}}) \cap SL_2(\widetilde{\mathbb{I}}/\mathfrak{P})$. On the other hand, we have $\mathbb{H} = \mathcal{T}' \rtimes \mathbb{S}$ since \mathcal{T}' projects down (under the determinant map) isomorphically to Γ . In the same way, for the image $\overline{\mathcal{T}}'_{\mathfrak{P}}$ of \mathcal{T}' in $GL_2(\mathbb{I}/\mathfrak{P})$, we have $\overline{\mathbb{H}}_{\mathfrak{P}} = \overline{\mathcal{T}}'_{\mathfrak{P}} \rtimes \overline{\mathbb{S}}_{\mathfrak{P}}$. For \overline{g} in $\overline{\mathbb{H}}_{\mathfrak{P}}$, lifting g to \mathbb{H} , $\det g \in \mathcal{T}'$, so taking $s \in \mathbb{Z}_p$ with $\tau^s = \det(g)$, we have $g = \tau^s g_1$ with $g_1 = \tau^{-s} g \in \mathbb{G}$. Then $\overline{g} = \overline{\tau^s} \overline{g}_1$ for $\overline{g}_1 = (g_1 \bmod \mathfrak{P})$ and $\overline{\tau^s} = (\tau^s \bmod \mathfrak{P})$. Thus we have $\overline{\mathbb{S}}_{\mathfrak{P}} = \overline{\mathbb{S}}'_{\mathfrak{P}}$. Then the assertion is clear from this identity. \square

Take a non CM component $\text{Spec}(\mathbb{I}) \subset \text{Spec}(\mathbf{h})$ with its Galois representation $\rho_{\mathbb{I}}$ realized on $\mathcal{L}_{\text{can}}(\widetilde{\mathbb{I}})$ as in the introduction. Thus, for the integral closure $\widetilde{\mathbb{I}}$ of \mathbb{I} in $Q(\mathbb{I})$, $\rho_{\mathbb{I}}$ has values $GL_2(\widetilde{\mathbb{I}})$ (this is the assumption (F)). In the isomorphism class $[\rho_{\mathbb{I}}]$ over $\widetilde{\mathbb{I}}$, we have $\rho_{\mathbb{I}}$ satisfying (Gal). Taking the matrix form of $\rho_{\mathbb{I}}$ whose restriction to D_p is upper triangular, we put $\mathbb{G} = \text{Im}(\rho_{\mathbb{I}}) \cap \Gamma_{\widetilde{\mathbb{I}}}(\mathfrak{m}_{\widetilde{\mathbb{I}}})$, which satisfies the condition (B) of Lemma 1.3 by (Gal).

Lemma 3.2. *Let the notation and the assumption be as above. In particular, we assume that \mathbb{I} is a non CM component. Let $P \in \text{Spec}(\Lambda)$ be an arithmetic point. If one of the three conditions (1–3) of Lemma 2.9 is satisfied for \mathbb{G} and P , there exists a representation $\rho \cong \rho_{\mathbb{I}}$ (over \mathbb{I}_P) such that the projected image $\text{Im}(\rho_{P\mathbb{I}})$ in $GL_2(\mathbb{I}_P/P\mathbb{I}_P)$ contains an open subgroup of $SL_2(\mathbb{Z}_p)$ and $\text{Im}(\rho)$ still satisfies the condition (B) of Lemma 1.3.*

Proof. We pick a prime divisor \mathfrak{P} of $\tilde{\mathbb{I}}$ over P , and consider the Hecke eigenform $f_{\mathfrak{P}}$ associated to \mathfrak{P} ; so, $f_{\mathfrak{P}}|T(l) = a_{\mathfrak{P}}(l)f_{\mathfrak{P}}$ for primes l , where $a_{\mathfrak{P}}(l) = (T(l) \bmod \mathfrak{P}) \in \overline{\mathbb{Q}}_p$. Let $f_{\mathfrak{P}}^{\circ}$ be the new form in the automorphic representation generated by $f_{\mathfrak{P}}$. By Proposition 5.1, $f_{\mathfrak{P}}^{\circ}$ does not have complex multiplication. Then by a result of Ribet [R1], the Galois representation $\rho_{\mathfrak{P}}$ associated to the non CM new form $f_{\mathfrak{P}}^{\circ}$ has image containing an open subgroup of $SL_2(\mathbb{Z}_p)$, up to conjugation by an element in $\mathcal{B}(\kappa(\mathfrak{P}))$ (because of the Iwasawa decomposition of $GL(2)$). Conjugating $\rho_{\mathfrak{P}}$ by an upper triangular matrix, we may assume that $\text{Im}(\rho_{\mathfrak{P}})$ contains an open subgroup of $SL_2(\mathbb{Z}_p)$ (and the condition (B) in Lemma 1.3 is intact). To show the lemma, we may replace \mathbb{G} by an open subgroup of \mathbb{G} as long as the replacement still satisfies one of the three conditions (1–3) of Lemma 2.9. We may thus replace \mathbb{G} by \mathbb{S} in the proof of Lemma 3.1. Hence the reduction map $\mathbb{S} \rightarrow \overline{\mathbb{S}}_{\mathfrak{P}}$ is surjective, and $\overline{\mathbb{S}}_{\mathfrak{P}}$ is open in $\text{Im}(\rho_{\mathfrak{P}})$. Thus $\overline{\mathbb{S}}_{\mathfrak{P}} \cap SL_2(\mathbb{Z}_p)$ is an open subgroup of $SL_2(\mathbb{Z}_p)$. This fullness of $\rho_{\mathfrak{P}}$ holds for all prime divisor $\mathfrak{P}|P$ in $\tilde{\mathbb{I}}$. Then the result follows from Lemma 2.9 applied to this \mathbb{S} . \square

By Theorem 2.12 combined with the above lemma, we can choose a representative $\rho_{\mathbb{I}}$ in its isomorphism class over $Q(\mathbb{I})$ so that we have a nontrivial ideal \mathfrak{c} with $G \supset \Gamma(\mathfrak{c})$ and an effective divisor $(L) \subset \text{Spec}(\Lambda)$ such that $(\Gamma(L) : \Gamma(\mathfrak{c}))$ is finite. This proves Theorem I in the introduction except for the uniqueness of (L) depending only on the isomorphism class of $\rho_{\mathbb{I}}$ in $GL_2(\tilde{\mathbb{I}})$ for the normalization $\tilde{\mathbb{I}}$ of \mathbb{I} . Let $(L)_P$ be the P -primary part of (L) for each divisor P of Λ . Here we prove the uniqueness of $\mathfrak{c}_P = (L)_P$ only assuming absolute irreducibility of $\rho_{\mathfrak{P}}$ for all prime divisors $\mathfrak{P}|P$ in $\tilde{\mathbb{I}}$ and existence of a non-zero Λ -ideal \mathfrak{a} with $\text{Im}(\rho_{\mathbb{I}}) \supset \Gamma_{\Lambda}(\mathfrak{a})$. Then we define $L := L(\mathbb{I})$ as a generator of the reflexive closure of the conductor of $\text{Im}(\rho_{\mathbb{I}}) \cap SL_2(\Lambda)$.

Lemma 3.3. *Let the notation be as above and P be a prime divisor of Λ . Take a non CM component \mathbb{I} of \mathfrak{h} with normalization $\tilde{\mathbb{I}}$, and suppose to have an associated Galois representation $\rho_{\mathbb{I}} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow GL_2(\tilde{\mathbb{I}})$ whose image contains $\Gamma_{\Lambda}(\mathfrak{a})$ for a non-zero Λ -ideal \mathfrak{a} . Let \mathfrak{c} be the conductor of the intersection $\text{Im}(\rho_{\mathbb{I}}) \cap SL_2(\Lambda)$. Assume that $\rho_{\mathbb{I}}$ leaves an $\tilde{\mathbb{I}}_P$ -lattice $\mathcal{L}_P \subset \tilde{\mathbb{I}}_P^2$ stable. Then the localization \mathfrak{c}_P depends only on \mathcal{L}_P up to scalars. If $\rho_{\mathfrak{P}}$ is absolutely irreducible for all prime divisors $\mathfrak{P} \in \text{Spec}(\tilde{\mathbb{I}})$ over P , then \mathfrak{c}_P is independent of the choice of \mathcal{L}_P .*

The point of this lemma is that whatever choice of $u \in GL_2(Q(\mathbb{I}))$ with $u\mathcal{L}_P = \mathcal{L}_P$, as long as $\text{Im}(u \cdot \rho_{\mathbb{I}}u^{-1})$ has nontrivial conductor, its localization at P is equal to \mathfrak{c}_P for the conductor \mathfrak{c} of the original choice $\rho_{\mathbb{I}}$. This lemma proves the uniqueness of (L) in Theorem I under absolute irreducibility of $\bar{\rho}$ (and finishes the proof of Theorem I).

Proof. Write $\mathfrak{c}_P = P^m$ in the discrete valuation ring Λ_P . We study dependence on \mathcal{L}_P . Write $G = \text{Im}(\rho_{\mathbb{I}}) \cap SL_2(\Lambda)$, where $\rho_{\mathbb{I}}$ is chosen in the isomorphism class of $\rho_{\tilde{\mathbb{I}}_P}$ over $\tilde{\mathbb{I}}_P$ so that G has nontrivial conductor \mathfrak{c} . This is just a choice of a starting lattice \mathcal{L} , and we want to first prove the ideal $\mathfrak{c}_P = \mathfrak{c}_P(\mathcal{L}) := \mathfrak{c}\Lambda_P$ (for localization Λ_P of Λ at P) is equal to $\mathfrak{c}_P(\mathcal{L}')$ for any other choice $\mathcal{L}' = z\mathcal{L}$ for a scalar matrix $z \in GL_2(Q(\mathbb{I}))$. Identifying $\mathcal{L}_P = \tilde{\mathbb{I}}_P^2$ and writing $\mathcal{L}'_P = \Lambda_P^2 \subset \mathcal{L}_P$, we find $\mathcal{L}_P = \mathcal{L}'_P \otimes_{\Lambda_P} \tilde{\mathbb{I}}_P$, and \mathcal{L}'_P is stable under G .

We have another $\rho'_{\mathbb{I}} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow GL_2(\tilde{\mathbb{I}}_P)$ realized on \mathcal{L}'_P with nontrivial conductor $\mathfrak{c}(\mathcal{L}')$; so, writing \mathcal{Z} for the center of the algebraic group $GL(2)$, we find $h \in \mathcal{Z}(Q(\tilde{\mathbb{I}}))GL_2(\tilde{\mathbb{I}}_P)$ such that $h(\mathcal{L}'_P) = \mathcal{L}_P$ and $h\rho'_{\mathbb{I}}h^{-1} = \rho_{\mathbb{I}}$. We put $H = \text{Im}(\rho'_{\mathbb{I}}) \cap SL_2(\Lambda)$. Then we have the conductor ideal $\mathfrak{c}' \neq 0$ of H . Thus $G \cap H \supset \Gamma(\mathfrak{c} \cap \mathfrak{c}')$ and $\mathfrak{c} \cap \mathfrak{c}' \neq 0$. This shows $\rho_{\mathbb{I}}$ and $\rho'_{\mathbb{I}}$ are both absolutely irreducible over $G \cap H$. For $\sigma \in H$ and $g \in G \cap H$, we have

$$h^{\sigma} \rho'_{\mathbb{I}}(g) h^{-\sigma} = (h\rho'_{\mathbb{I}}(g)h^{-1})^{\sigma} = \rho_{\mathbb{I}}(g)^{\sigma} = \rho_{\mathbb{I}}(g) = h\rho'_{\mathbb{I}}(g)h^{-1}.$$

Thus $h^{-1}h^{\sigma}$ commutes with $\rho_{\mathbb{I}}|_{G \cap H}$. By absolute irreducibility of $\rho_{\mathbb{I}}|_{G \cap H}$, $h^{-1}h^{\sigma}$ is a scalar $z_{\sigma} \in Q(\tilde{\mathbb{I}})^{\times}$. Thus $\sigma \mapsto z_{\sigma}$ is a 1-cocycle of H with values in $Q(\tilde{\mathbb{I}})^{\times}$. By Hilbert's theorem 90, $z_{\sigma} = \zeta^{-1}\zeta^{\sigma}$ for $\zeta \in Q(\tilde{\mathbb{I}})^{\times}$ independent of σ . Then replacing h by $h\zeta^{-1}$, we may assume that $h \in GL_2(Q)$. Note that $h = zu$ with $u \in GL_2(\tilde{\mathbb{I}}_P)$. Since $h \in GL_2(Q)$, the elementary divisor z of h can be chosen in $GL_2(Q)$; so, we may assume that z is a scalar matrix in $GL_2(Q)$; so, $u \in GL_2(\Lambda_P)$. Thus $G = h \cdot Hh^{-1} \subset SL_2(\Lambda_P)$, and $\Gamma(\mathfrak{c}) \subset G \supset h\Gamma(\mathfrak{c}')h^{-1} = z\Gamma(\mathfrak{c}')z^{-1} = \Gamma(\mathfrak{c}')$. Since z is scalar, this implies $\mathfrak{c}_P = \mathfrak{c}'_P$ and hence $(L)_P$ is independent of choice of \mathcal{L}_P up to scalar.

If $\rho_{\mathfrak{P}}$ is absolutely irreducible for all $\mathfrak{P}|P$, the P -adic completion $\widehat{\mathcal{L}}_P = \prod_{\mathfrak{P}|P} \widehat{\mathcal{L}}_{\mathfrak{P}}$ is unique up to scalar by a result of Serre/Carayol [C], where $\widehat{\mathcal{L}}_{\mathfrak{P}}$ is the \mathfrak{P} -adic completion of \mathcal{L}_P . So the independence of $(L)_P$ on \mathcal{L} follows. \square

Corollary 3.4. *Let \mathbb{I} be a non CM component and pick $\rho_{\mathbb{I}}$ with values in $GL_2(\widetilde{\mathbb{I}})$ realized on $\mathcal{L}_{can}(\widetilde{\mathbb{I}})$ (so, we assume (F_{can})). Let P be a prime divisor of Λ . Suppose one of the following three conditions:*

- (1) *We have $\sigma \in D_p$ such that $Ad(\overline{\rho}(\sigma))$ has three distinct eigenvalues in \mathbb{F}_p ;*
- (2) *We have $\sigma, v \in D_p$ such that $\overline{\rho}(\sigma)$ has two distinct eigenvalues in \mathbb{F}_p and $\overline{\rho}(v)$ is a nontrivial unipotent element in $SL_2(\mathbb{F})$;*
- (3) *We have $\sigma, v \in D_p$ such that $\overline{\rho}(\sigma)$ has two distinct eigenvalues in \mathbb{F}_p and $\rho_{\mathfrak{P}}(v)$ is a nontrivial unipotent element in $SL_2(\kappa(\mathfrak{P}))$ for all prime divisors $\mathfrak{P}|P$ of $\widetilde{\mathbb{I}}$.*

Assume that $\text{Im}(\rho_{\mathbb{I}})$ has conductor \mathfrak{c} . Then $P \nmid \mathfrak{c}_P$ for a prime divisor P of $\text{Spec}(\Lambda)$ if and only if $\overline{\mathbb{G}}_{\mathfrak{P}}$ for all prime divisors $\mathfrak{P}|P$ in $\widetilde{\mathbb{I}}$ contains an open subgroup of $SL_2(A_0)$, up to conjugation.

Proof. We may assume that $\overline{\rho}(\sigma)$ and $\rho_{\mathbb{I}}(v)$ are upper triangular by (F_{can}) . We replace \mathbb{G} by its open p -profinite subgroup $\{g \in \mathbb{G} \mid (g \bmod \mathfrak{m}_{\mathbb{I}}) \in \mathcal{U}(\widetilde{\mathbb{I}}/\mathfrak{m}_{\mathbb{I}})\}$. Then \mathbb{G} is p -profinite, and all the assumptions are intact. In particular, it is still normalized by $\rho(\sigma)$ in the assumptions (1–3) and contains $\rho(v)$ for v in the assumption (2) and (3). By this modification, the assumptions of Lemma 2.9 is satisfied. Indeed, we may take $j = \lim_{n \rightarrow \infty} \rho(\sigma)^{p^n}$, and the unipotent part of $\rho(v)$ (which is found in \mathbb{G} by a similar argument proving Lemma 1.4) does the job for v in Lemma 2.9 (2-3). Thus we can apply Lemma 2.9 in this setting. The direction (\Rightarrow) is plain; so, we assume that $\overline{\mathbb{G}}_{\mathfrak{P}}$ contains an open subgroup of $SL_2(A_0)$ for all $\mathfrak{P}|P$. Then by Lemma 3.2, $\overline{\mathbb{G}}_P$ contains an open subgroup of $SL_2(A_0)$. Then we find $\alpha \in \mathcal{B}(\mathbb{I}_P)$ such that for $\rho' := \alpha \rho_{\mathbb{I}} \alpha^{-1}$, $\text{Im}(\rho')$ has conductor \mathfrak{c}' prime to P . Since $\rho_{\mathbb{I}}$ and ρ' are equivalent under $GL_2(\mathbb{I}_P)$, by the above lemma, we find $\mathfrak{c}_P = \mathfrak{c}'_P$. Thus we get $P \nmid \mathfrak{c}$. \square

When $\rho_{\mathfrak{P}}$ is reducible for some $\mathfrak{P}|P$, we have $\overline{\rho} \cong \begin{pmatrix} \overline{\theta} & 0 \\ 0 & \overline{\psi} \end{pmatrix}$ for a character $\overline{\psi}$ unramified at p . In this case, we need to worry if we can define an optimal level $L(\mathbb{I})$. We start with the isomorphism class of $\rho_{\mathbb{I}}$ over $\widetilde{\mathbb{I}}_{\mathfrak{P}}$ given by $\mathcal{L}_{can} \otimes_{\mathbb{I}} \widetilde{\mathbb{I}}_{\mathfrak{P}}$. We call a prime divisor \mathfrak{P} of \mathbb{I} *reducible* if $\rho_{\mathfrak{P}}$ is reducible.

Lemma 3.5. *Let the notation be as above. Suppose one of the conditions (1–3) in Corollary 3.4. Let \mathbb{I} be a non-CM component with normalization $\widetilde{\mathbb{I}}$ and put $\mathbb{G} = SL_2(\widetilde{\mathbb{I}}) \cap \text{Im}(\rho_{\mathbb{I}})$, where $\rho_{\mathbb{I}} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow GL_2(\widetilde{\mathbb{I}})$ (normalized to satisfy the condition (B) of Lemma 1.3) comes from the lattice $\mathcal{L}_{can}(\widetilde{\mathbb{I}})$. Suppose that there is no quadratic field M/\mathbb{Q} such that $\overline{\rho}$ is isomorphic to an induced representation from $\text{Gal}(\overline{\mathbb{Q}}/M)$. Let P be a prime divisor of Λ with a reducible prime divisor \mathfrak{P} of $\widetilde{\mathbb{I}}$ above P . Then we can find $\alpha \in \mathcal{B}(Q(\mathbb{I}))$ such that $\alpha \mathbb{G} \alpha^{-1} \cap \mathcal{U}(\Lambda)$ is equal to $\begin{pmatrix} 1 & \mathfrak{u} \\ 0 & 1 \end{pmatrix}$ for a Λ -ideal \mathfrak{u} prime to P and $\alpha \mathbb{G} \alpha^{-1} \subset SL_2(\widetilde{\mathbb{I}}_P)$. Moreover, defining \mathfrak{c}_P by a Λ_P -ideal given by $\Lambda_P \cap \widetilde{\mathbb{I}}_P \cdot \mathfrak{n}_t^{(\alpha)}$ for $\mathfrak{n}_t^{(\alpha)} = \{u \in \widetilde{\mathbb{I}}_P \mid \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \in \alpha \mathbb{G} \alpha^{-1}\}$, the Λ_P -ideal $\mathfrak{c}_P \subset \Lambda_P$ is independent of the choice of α .*

Under the circumstances in the lemma, for each prime divisor P over which we have no reducible prime divisor $\mathfrak{P}|P$ of \mathbb{I} , we put $\mathfrak{c}_P = \Lambda_P$. Then, abusing the language slightly, we set $(L(\mathbb{I}))$ to be the principal ideal $\bigcap_P \mathfrak{c}_P$.

Non-existence of quadratic fields M as in the lemma is equivalent to the fact that $\overline{\theta}^{-1} \overline{\psi}$ is not a quadratic character (as we will see later in the proof of Theorem 8.8), and if $\overline{\theta}^{-1} \overline{\psi}|_{I_p}$ has order ≥ 3 , plainly the condition (1) of Corollary 3.4 is satisfied.

Proof. We may replace \mathbb{G} by $\{g \in \text{Im}(\rho_{\mathbb{I}}) \cap SL_2(\widetilde{\mathbb{I}}) \mid (g \bmod \mathfrak{m}_{\mathbb{I}}) \in \mathcal{U}(\widetilde{\mathbb{I}}/\mathfrak{m}_{\mathbb{I}})\}$. Then \mathbb{G} is a p -profinite subgroup of $SL_2(\widetilde{\mathbb{I}})$. We proceed as in the proof of Lemma 2.9 looking into both $\mathcal{M}(\mathbb{G})$ and $\mathcal{M}_1(\mathbb{G})$ described above Theorem 1.1.

Pick another prime divisor $\mathcal{P} \in \text{Spec}(\Lambda)$ such that the image of $\rho_{\mathcal{P}\mathbb{I}} = (\rho_{\mathbb{I}} \bmod \mathcal{P}\mathbb{I})$ contains an open subgroup of $SL_2(\mathbb{Z}_p)$ (any arithmetic prime does the job by Lemma 3.2). Then the derived group $\overline{\mathbb{G}}_{\mathcal{P}}$ of the image $\overline{\mathbb{G}}_{\mathcal{P}}$ in $SL_2(\widetilde{\mathbb{I}}/\mathcal{P}\widetilde{\mathbb{I}})$ of \mathbb{G} contains an open subgroups of $SL_2(\mathbb{Z}_p)$. In the same manner as in the proof of Lemma 2.9, we find that $\mathbb{U} := \mathbb{G} \cap \mathcal{U}(\widetilde{\mathbb{I}})$ contains the non-trivial image of $\mathbb{U}' = \mathbb{G}' \cap \mathcal{U}(\widetilde{\mathbb{I}})$ in $\mathcal{U}(\widetilde{\mathbb{I}}/\mathcal{P}\widetilde{\mathbb{I}})$, and for the Λ -module $\mathfrak{n}' = \{u \in \widetilde{\mathbb{I}} \mid \begin{pmatrix} 1 & \mathfrak{u} \\ 0 & 1 \end{pmatrix} \in \mathbb{G}\}$, the

$\tilde{\mathbb{I}}_P$ -ideal $\tilde{\mathbb{I}}_P \cdot \mathfrak{n}'$ is prime to \mathcal{P} . Then picking $b \in \mathfrak{n}'$ with non-trivial image in $\tilde{\mathbb{I}}_P/\mathcal{P}\tilde{\mathbb{I}}_P$, we find $\beta = \begin{pmatrix} a^{-1} & 0 \\ b & 1 \end{pmatrix} \in \mathcal{B}(\tilde{\mathbb{I}}_P)$ such that $\beta\mathbb{G}\beta^{-1} \cap \mathcal{U}(\Lambda)$ contains U with $\bar{U} \neq 1$, and $\beta\mathbb{G}\beta^{-1}$ has conductor $\mathfrak{c}_\beta \neq 0$. Put $\mathfrak{n}'_t = \{u \in Q(\mathbb{I}) \mid \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \in \beta\mathbb{G}\beta^{-1}\} \subset \tilde{\mathbb{I}}$.

We now work over the ring $D = \tilde{\mathbb{I}}_P \cap \tilde{\mathbb{I}}_P \subset Q(\mathbb{I})$. Let $\mathfrak{n}_0 = \{u \in \tilde{\mathbb{I}} \mid \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \in \mathbb{G}\}$. Since D is a Zariski ring whose localizations are all discrete valuation ring, it is a principal ideal domain with the same quotient field $Q(\mathbb{I})$ (cf. [BCM] VII.3.6). Then we may assume that $a \in D$ with $D \cdot \mathfrak{n}_0 = (a)$. We put $\alpha = \begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix} \in \mathcal{B}(Q(\mathbb{I}))$. Let $\mathfrak{N} = \mathbb{I} \cdot \mathfrak{n}_t$ for $\mathfrak{n}_t = \{u \in Q(\mathbb{I}) \mid \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \in \alpha\mathbb{G}\alpha^{-1}\}$, and put $\mathfrak{n} = \{u \in Q(\mathbb{I}) \mid \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \in \alpha\mathbb{G}\alpha^{-1}\}$ which is contained in D because $D \cdot \mathfrak{n}_0 = (a)$. In any case, $\alpha\mathbb{G}\alpha^{-1}$ has conductor $\mathfrak{c}_\alpha \neq 0$. If $\mathfrak{P}'|P$ and $\text{Im}(\rho_{\mathfrak{P}'})$ contains an open subgroup of $SL_2(A_0)$ (for A_0 with respect to $\kappa(\mathfrak{P}')$), then $a \in \tilde{\mathbb{I}}_{\mathfrak{P}'}$, and $\mathfrak{n}_{\mathfrak{P}'} = \tilde{\mathbb{I}}_{\mathfrak{P}'}$. If $\rho_{\mathfrak{P}'}$ is absolutely irreducible but not full, the only possibility is $\rho_{\mathfrak{P}'} \cong \text{Ind}_M^{\mathbb{Q}} \lambda$ (by Lemma 2.1 or Lemma 8.4), where M is a quadratic field and $\lambda : \text{Gal}(\mathbb{Q}/M) \rightarrow \kappa(\mathfrak{P}')^\times$ is a character. Then $\bar{\rho}$ must be an induced representation from $\text{Gal}(\mathbb{Q}/M)$, which is prohibited by our assumption. Thus the conjugation by α has the following effect:

- for $\mathfrak{P}'|P$ with irreducible $\rho_{\mathfrak{P}'}$, we have $\mathfrak{n}_{\mathfrak{P}'} = \tilde{\mathbb{I}}_{\mathfrak{P}'}$,
- for $\mathfrak{P}|P$ with reducible $\rho_{\mathfrak{P}}$, it maximizes $\mathfrak{n}_{\mathfrak{P}}$ to $\mathfrak{n}_{\mathfrak{P}} = \tilde{\mathbb{I}}_{\mathfrak{P}} \cdot \mathfrak{n} = \tilde{\mathbb{I}}_{\mathfrak{P}}$ for \mathfrak{P} , and \mathfrak{u} becomes prime to P ,
- for $\mathfrak{P}|P$ with reducible $\rho_{\mathfrak{P}}$, it minimizes $\mathfrak{n}_{t,\mathfrak{P}} = \tilde{\mathbb{I}}_{\mathfrak{P}} \cdot \mathfrak{n}_t$.

Thus $\mathfrak{u}, \mathfrak{n}, \mathfrak{n}_t \subset D$ with $D\mathfrak{u} = D$, and we have still $\alpha\mathbb{G}\alpha^{-1}$ in $SL_2(D) \subset SL_2(\tilde{\mathbb{I}}_P)$. Therefore $G := \alpha\mathbb{G}\alpha^{-1} \cap SL_2(\Lambda)$ has the maximal upper unipotent subgroup of the form

$$(3.1) \quad U(\mathfrak{u}) = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathfrak{u} \right\}$$

for an ideal $\mathfrak{u} \subset \Lambda$ prime to P . We put $\mathfrak{C}_\alpha = (\mathbb{I} \cdot \mathfrak{n}_t) \cap \Lambda$ for \mathfrak{n}_t defined for this $\alpha\mathbb{G}\alpha^{-1}$.

We want to show that $\mathfrak{C}_{\alpha,P} := \tilde{\mathbb{I}}_P \cdot \mathfrak{C}_\alpha$ is independent of the choice of a and \mathcal{P} . Choose another point \mathcal{P}' such that the image of $\rho_{\mathcal{P}'\mathbb{I}} = (\rho_{\mathbb{I}} \bmod \mathcal{P}'\mathbb{I})$ contains an open subgroup of $SL_2(\mathbb{Z}_p)$. Put $D' = \tilde{\mathbb{I}}_P \cap \tilde{\mathbb{I}}_{\mathcal{P}'} \cap \tilde{\mathbb{I}}_P$, and choose a generator $(b) = D' \cdot \mathfrak{n}$. Then for $\alpha' = \begin{pmatrix} b^{-1} & 0 \\ 0 & 1 \end{pmatrix}$, $\alpha'\alpha^{-1} \in \mathcal{B}(\tilde{\mathbb{I}}_P)$ as $a\tilde{\mathbb{I}}_P = \tilde{\mathbb{I}}_P \cdot \mathfrak{n} = b\tilde{\mathbb{I}}_P$. Since $\alpha'\alpha^{-1} \in \mathcal{B}(\tilde{\mathbb{I}}_P)$ is diagonal, we find $\mathfrak{C}_{\alpha,P} = \mathfrak{C}_{\alpha',P}$. Thus $\mathfrak{c}_P = \mathfrak{C}_{\alpha,P}$ is independent of the choice of (a, \mathcal{P}) . For any other prime divisor P' with absolutely irreducible $\rho_{\mathfrak{P}'}$ for all $\mathfrak{P}'|P'$, we choose $\gamma \in \mathcal{B}(\tilde{\mathbb{I}}_{P'})$ so that $\gamma\mathbb{G}\gamma^{-1}$ contains $\Gamma_\Lambda(\mathfrak{a}) \neq 1$. Let \mathfrak{c}_γ be the conductor ideal of $\gamma\mathbb{G}\gamma^{-1}$. Then, as we have seen, under non-existence of the quadratic field M , $\text{Im}(\rho_{\mathfrak{P}'})$ contains an open subgroup of $SL_2(A_0)$ for all $\mathfrak{P}'|P'$, and by Lemma 2.9, $\mathfrak{c}_{\gamma,P'} = \Lambda_{P'}$. Thus our definition of $\mathfrak{c}_{P'} = \Lambda_{P'}$ is legitimate, and $\mathfrak{c}_P = \mathfrak{c}_{\gamma,P'} = \Lambda_{P'}$. Then $(L(\mathbb{I}))$ is the principal ideal $\bigcap_{P'} \mathfrak{c}_{P'}$, which is thus independent of our choice of the pair (a, \mathcal{P}) . \square

In this reducible case, assuming that there is no quadratic field M/\mathbb{Q} such that $\bar{\rho}$ is isomorphic to an induced representation from $\text{Gal}(\mathbb{Q}/M)$, we define $(L(\mathbb{I}))$ as in the above proof. When χ has conductor divisible by N with $p \nmid \varphi(N)$ and $\bar{\psi} = 1$, the result in §3.3 of [O1] (which further developed an idea of Harder–Pink) shows that actually we can choose $\rho_{\mathbb{I}}$ as specified in the above lemma in the isomorphism class over $\tilde{\mathbb{I}}_{\mathfrak{P}}$ of the representation realized on $\mathcal{L}_{can}(\tilde{\mathbb{I}})$, and this is why we have the belief that the level should be defined directly by the Galois action on $\mathcal{L}_{can}(\tilde{\mathbb{I}})$.

Corollary 3.6. *Let the notation and the assumption be as in Lemma 3.5. Normalize $\rho_{\mathbb{I}}$ as in Lemma 3.5. Let $P \in \text{Spec}(\Lambda)$ be a prime divisor prime to (p) such that $\rho_{\mathfrak{P}}$ is reducible for some prime divisor $\mathfrak{P}|P$ of $\tilde{\mathbb{I}}$. Then we have*

- (1) *For \mathfrak{c} defined after Lemma 3.5, we have $\mathfrak{c}_P = \Lambda_P \cdot \mathfrak{u}_t$, where $\mathfrak{u}_t = \{c \in \Lambda \mid \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \in \text{Im}(\rho_{\mathbb{I}})\}$.*
- (2) *For the conductor \mathfrak{c}' of \mathbb{G} , we have $\mathfrak{c}_P = \mathfrak{c}'_P$.*
- (3) *The localization \mathfrak{c}_P is equal to $(\bigcap_{\mathfrak{P}|P} \mathfrak{a}_{\mathfrak{P}}) \cap \Lambda_P$ for the minimal $\tilde{\mathbb{I}}_{\mathfrak{P}}$ -ideal $\mathfrak{a}_{\mathfrak{P}}$ such that the reduction $(\rho_{\mathbb{I}} \bmod \mathfrak{a}_{\mathfrak{P}})$ has values in $\mathcal{B}(\tilde{\mathbb{I}}_{\mathfrak{P}}/\mathfrak{a}_{\mathfrak{P}})$ (up to conjugation).*

Proof. We first prove (1) and (3). By the proof of the above lemma, $\mathfrak{c}_P = (\tilde{\mathbb{I}}_P \cdot \mathfrak{n}_t) \cap \Lambda_P$, which is equal to $(\Lambda_P \cdot \mathfrak{n}_t) \cap \Lambda_P$ since \mathfrak{n}_t is a Λ -module. Thus $\tilde{\mathbb{I}}_P \mathfrak{c}_P \subset \bigcap_{\mathfrak{P}|P} \mathfrak{a}_{\mathfrak{P}}$. Let $\rho_i = (\rho_{\mathbb{I}} \bmod \mathfrak{i})$ for an ideal \mathfrak{i} of $\tilde{\mathbb{I}}_{\mathfrak{P}}$. Suppose that $\mathfrak{a}_{\mathfrak{P}} \supseteq \mathfrak{c}_{\mathfrak{P}} = \tilde{\mathbb{I}}_{\mathfrak{P}} \mathfrak{c}_P$, for $\mathfrak{i} = \mathfrak{a}_{\mathfrak{P}} \mathfrak{P}$, consider ρ_i . Note that \mathbb{G} acts on

$\mathfrak{s}_{\mathfrak{P}} = \widetilde{\mathbb{I}}_{\mathfrak{P}} \cdot \mathcal{M}^0(\mathbb{G}) \subset \mathfrak{s}'_{\mathfrak{P}} = \widetilde{\mathbb{I}}_{\mathfrak{P}} \cdot \mathcal{M}^0_1(\mathbb{G})$ by adjoint action. Consider the image $\overline{\mathfrak{s}}_{\mathfrak{b}}$ of $\mathfrak{s}_{\mathfrak{P}}$ in $\mathfrak{sl}_2(\widetilde{\mathbb{I}}_{\mathfrak{P}}/\mathfrak{b})$ for an $\widetilde{\mathbb{I}}_{\mathfrak{P}}$ -ideal \mathfrak{b} . Then we have an exact sequence of $\widetilde{\mathbb{I}}_{\mathfrak{P}}[\widetilde{\mathbb{G}}]$ -modules

$$0 \rightarrow \overline{V} \rightarrow \overline{\mathfrak{s}}_{\mathfrak{i}} \rightarrow \overline{\mathfrak{s}}_{\mathfrak{a}_{\mathfrak{P}}} \rightarrow 0$$

for $\widetilde{\mathbb{G}} = \text{Im}(\rho'_i)$. Then \overline{V} is a $\kappa(\mathfrak{P})$ vector space of dimension at most 3. If \overline{V} is made up of upper triangular matrices, $\overline{\mathfrak{s}}_{\mathfrak{i}} \subset \mathfrak{B}(\widetilde{\mathbb{I}}_{\mathfrak{P}}/\mathfrak{i})$, and ρ_i has values in $\mathcal{B}(\widetilde{\mathbb{I}}_{\mathfrak{P}}/\mathfrak{i})$, a contradiction to the minimality of $\mathfrak{a}_{\mathfrak{P}}$. Thus we have $0 \neq X = \begin{pmatrix} 0 & 0 \\ v & 0 \end{pmatrix} \in \overline{V} \cap {}^t\mathfrak{U}(\widetilde{\mathbb{I}}_{\mathfrak{P}}/\mathfrak{i})$. We also have $Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{n}$. Then $0 \neq [X, Y] = \begin{pmatrix} v & 0 \\ 0 & -v \end{pmatrix}$. This shows that $\dim_{\kappa(\mathfrak{P})} \overline{V} = 3$. By Nakayama's lemma, we find $V = \text{Ker}(\mathfrak{s}_{\mathfrak{P}} \rightarrow \overline{\mathfrak{s}}_{\mathfrak{a}_{\mathfrak{P}}}) = \mathfrak{a}_{\mathfrak{P}} \cdot \mathfrak{sl}_2(\widetilde{\mathbb{I}}_{\mathfrak{P}})$. This shows $\mathfrak{a}_{\mathfrak{P}} = \widetilde{\mathbb{I}}_{\mathfrak{P}}\mathfrak{n}_t \supset \widetilde{\mathbb{I}}_{\mathfrak{P}}\mathfrak{u}_t$ for $\mathfrak{u}_t = \mathfrak{n}_t \cap \Lambda$.

To show $\mathfrak{a}_{\mathfrak{P}} = \widetilde{\mathbb{I}}_{\mathfrak{P}}\mathfrak{u}_t$, replace $Q(\mathbb{I})$ by its Galois closure over Q and $\widetilde{\mathbb{I}}$ by the integral closure in the Galois closure. Thus $\widetilde{\mathbb{I}}/\Lambda$ is a Galois covering with finite Galois group \mathfrak{g} . Then $\mathfrak{a}_{\mathfrak{P}} = \mathfrak{P}^{\varepsilon(\mathfrak{P})}$. We take $\varepsilon = \max(\varepsilon(\mathfrak{P}))_{\mathfrak{P}|P}$ and write $P^{\varepsilon} = \mathfrak{P}^{\varepsilon} \cap \Lambda$ which is independent of the choice of \mathfrak{P} . Let $P^{\varepsilon}\widetilde{\mathbb{I}}_{\mathfrak{P}} = \mathfrak{P}^{\varepsilon}$. Replace \mathbb{G} by $\{g \in \mathbb{G} \mid (g \bmod P^{\varepsilon}) \in \mathcal{B}(\widetilde{\mathbb{I}}_P/P^{\varepsilon}\widetilde{\mathbb{I}}_P)\}$. Then by the above argument, for the kernel $V = \text{Ker}(\mathfrak{s}_{P^{\varepsilon}} \rightarrow \overline{\mathfrak{s}}_{P^{\varepsilon}})$ is given by $P^{\varepsilon} \cdot \mathfrak{sl}_2(\widetilde{\mathbb{I}}_P)$. Thus \mathfrak{n}_t has v with non-zero image in $\mathfrak{P}^{\varepsilon}/\mathfrak{P}^{\varepsilon+1}$ for all $\mathfrak{P}|P$. By Nakayama's lemma again, we have $V = \widetilde{\mathbb{I}}_P v = P^{\varepsilon} \cdot {}^t\mathfrak{U}(\widetilde{\mathbb{I}}_P)$; so, $\widetilde{\mathbb{I}}_P v$ is a \mathfrak{g} -module, and $\mathfrak{u}_t \subset H^0(\mathfrak{g}, \widetilde{\mathbb{I}}_P v) =: \mathfrak{u}'_t$. We have an exact sequence $0 \rightarrow PV \rightarrow V \rightarrow \overline{V} \rightarrow 0$. Taking \mathfrak{g} -invariant, we have another exact sequence $0 \rightarrow H^0(\mathfrak{g}, PV) \rightarrow H^0(\mathfrak{g}, V) \rightarrow H^0(\mathfrak{g}, \overline{V}) \rightarrow 0$ as V and \overline{V} is a \mathbb{Q}_p -vector space. This shows $\widetilde{\mathbb{I}}_P\mathfrak{u}'_t = P^{\varepsilon}\widetilde{\mathbb{I}}_P$. Thus $\mathfrak{u}'_t\widetilde{\mathbb{I}}_P \cap \Lambda_P = \mathfrak{n}_t\widetilde{\mathbb{I}}_P \cap \Lambda_P$, which implies $\mathfrak{u}'_t = \widetilde{\mathbb{I}}_P\mathfrak{n}_t \cap \Lambda_P = \Lambda_P\mathfrak{u}_t$, by definition. Then we have $\mathfrak{c}_P = \Lambda_P\mathfrak{u}_t = P^{\varepsilon}\Lambda_P$. By our construction, we get $\mathfrak{c}_P = P^{\varepsilon}\widetilde{\mathbb{I}}_P = \bigcap_{\mathfrak{P}|P} \mathfrak{a}_{\mathfrak{P}} \cap \Lambda_P$, proving (1) and (3).

We prove (2). Since we normalized \mathbb{G} by conjugating by α as in the proof of Lemma 3.5, by (3.1), for $G := \mathbb{G} \cap SL_2(\Lambda)$, we have

$$G \cap \mathcal{U}(\Lambda) = U(\mathfrak{u}) = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathfrak{u} \right\}$$

for an ideal $\mathfrak{u} \subset \Lambda$ prime to P . By a simple computation, for $c \in \mathfrak{u}_t$, we have

$$(3.2) \quad \begin{aligned} & \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} = \begin{pmatrix} 1+bc & b \\ c & 1 \end{pmatrix} \\ & \begin{pmatrix} 1+bc & b \\ c & 1 \end{pmatrix} \begin{pmatrix} 1 & -(1+bc)^{-1}b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+bc & 0 \\ c & 1-bc(1+bc)^{-1} \end{pmatrix} \\ & \begin{pmatrix} 1 & 0 \\ -c(1+bc)^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1+bc & 0 \\ c & 1-bc(1+bc)^{-1} \end{pmatrix} = \begin{pmatrix} 1+bc & 0 \\ 0 & (1+bc)^{-1} \end{pmatrix}. \end{aligned}$$

Thus $\mathcal{M}_1(G)$ contains

$$\mathfrak{A} := \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mid a \in (p) \cap \mathfrak{uu}_t, b \in (p) \cap \mathfrak{u}, c \in \mathfrak{u}_t \right\}.$$

This shows the existence of \mathfrak{c}'' with $\mathfrak{c}''_P = \mathfrak{u}_{t,P} = \mathfrak{c}_P$ such that $\Gamma_{\Lambda}(\mathfrak{c}'') \subset G$. Thus $\mathfrak{c}'' \subset \mathfrak{c}'$. By (1) and (3), it is clear to have $\mathfrak{c}''_P \supset \mathfrak{c}'_P$. This finishes the proof of (2). \square

4. EISENSTEIN COMPONENTS

We now describe explicitly the congruence between the Eisenstein component and the cuspidal component (a description of Eisenstein ideal in the big Hecke algebra) via the theory of Λ -adic form.

We first define the Eisenstein component of the space of Λ -adic forms. Let $\mathcal{M}(N, \chi_1; \Lambda_W)$ (resp. $\mathcal{S}(N, \chi_1; \Lambda_W)$) be the space of p -ordinary Λ_W -adic modular forms (resp. p -ordinary Λ_W -adic cusp forms). Thus $\mathcal{M}(N, \chi_1; \Lambda_W)$ (resp. $\mathcal{S}(N, \chi_1; \Lambda_W)$) is a collection of all formal q -expansions $F(q) = \sum_{n=0}^{\infty} a(n, F)(T)q^n \in \Lambda_W[[q]]$ such that $f_P = \sum_{n=0}^{\infty} (a(n, F) \bmod P)q^n$ gives rise to a modular form in $\mathcal{M}_{k+1}^{ord}(\Gamma_0(Np^{r(P)+1}), \varepsilon_P \chi_k(P))$ (resp. $\mathcal{S}_{k+1}^{ord}(\Gamma_0(Np^{r(P)+1}), \varepsilon_P \chi_k(P))$) for all arithmetic points P , where $p^{r(P)}$ is the order of ε_P . Again $F \mapsto f_P$ induces an isomorphism

$$\mathcal{M}(N, \chi_1; \Lambda_W) \otimes_{\Lambda_W} \Lambda_W/P \cong \mathcal{M}_{k+1}^{ord}(\Gamma_0(Np^{r(P)+1}), \chi_k \varepsilon_P; W[\varepsilon_P])$$

for all arithmetic points (see [GME] Theorem 3.2.15 and Corollary 3.2.18 or [LFE] §7.3). This implies that $\mathcal{M}(N, \chi_1; \Lambda_W)$ and $\mathcal{S}(N, \chi_1; \Lambda_W)$ are free of finite rank over Λ_W , and the Λ_W -module $\mathcal{M}(N, \chi_1; \Lambda_W)$ (resp. $\mathcal{S}(N, \chi_1; \Lambda_W)$) is naturally a faithful module over \mathbf{H} (resp. \mathbf{h}). The above specialization map is compatible with the Hecke operator action. Recall the quotient field Q of Λ , and take an algebraic closure \overline{Q} of Q . We can extend scalar to an extension A/Λ_W inside \overline{Q} to

define $\mathcal{S}(N, \chi_1; A) = \mathcal{S}(N, \chi_1; \Lambda_W) \otimes_{\Lambda_W} A$ and $\mathcal{M}(N, \chi_1; A) = \mathcal{M}(N, \chi_1; \Lambda_W) \otimes_{\Lambda_W} A$. If $A = \mathbb{I}$ is finite over Λ_W , associating the family $\{f_P\}_{P \in \text{Spec}(\mathbb{I})}$ to a form $F \in \mathcal{M}(N, \chi_1; \mathbb{I})$, we may regard these as spaces of “analytic families of slope 0 of modular forms” with coefficients in \mathbb{I} (we also call them the space of \mathbb{I} -adic p -ordinary cusp forms and the space of \mathbb{I} -adic p -ordinary modular forms, respectively). See [LFE] Chapter 7, [GME] Chapter 3 and [H86a] for these facts.

Let Q_W be the quotient field of Λ_W . Then there is a canonical decomposition

$$\mathcal{M}(N, \chi_1; Q_W) = \mathcal{S}(N, \chi_1; Q_W) \oplus \mathcal{E}(N, \chi_1; Q_W)$$

as modules over \mathbf{H} . The space $\mathcal{E}(N, \chi_1; Q_W)$ is spanned by Λ -adic Eisenstein series. Assuming that N is cube free, we make explicit the Eisenstein series: For any character $\psi : (\mathbb{Z}/M_1\mathbb{Z})^\times \rightarrow W^\times$, $\theta : (\mathbb{Z}/M_2\mathbb{Z})^\times \rightarrow W^\times$ with $\psi\theta = \chi_1$, $M_1M_2|Np$, $p|M_2$ and $p \nmid M_1$, there exists a unique Λ -adic Eisenstein series in $\mathcal{M}(N, \chi_1; Q_W)$ defined by its q -expansion:

$$a(\theta, \psi)(T) + \sum_{n=1}^{\infty} \left(\sum_{0 < d|n, p \nmid d} \theta(d)\psi\left(\frac{n}{d}\right)\langle d \rangle(T) \right) q^n,$$

where $\langle d \rangle(T) = (1+T)^{\log_p(d)/\log_p(\gamma)}$, $a(\theta, \psi) = 0$ if ψ is non-trivial, and otherwise, writing $\mathbf{1}_M$ for the trivial character modulo M , $a(\theta, \mathbf{1}_{M_1}) = \frac{1}{2}G(T) \in Q_W$ with

$$G(\gamma^k - 1) = (1 - \theta_{k+1}(p)p^k)L^{(M_1)}(-k, \theta_{k+1}) \text{ for all } 0 \leq k \in \mathbb{Z}.$$

As a convention, we put $\theta(d) = 0$ if d has a nontrivial common factor with M_2 and that $\psi(d) = 0$ if d has a nontrivial common factor with M_1 , and also θ_k is the character of $(\mathbb{Z}/M_2p\mathbb{Z})^\times$ given by $\theta_k = \theta\omega^{1-k}$. The L -function $L^{(M_1)}(s, \theta_{k+1}) = \sum_{n=1}^{\infty} \theta_{k+1}(n)n^{-s}$ for this possibly imprimitive θ_{k+1} . The existence of the above Eisenstein series is proven under $M_1M_2|Np$ (cf. [H86a] Theorem 7.2, see also [O2]). By counting number of pairs (θ, ψ) , we can prove that they span over Q_W a Hecke stable subspace $\mathcal{E}(N, \chi_1; Q_W)$ inside $\mathcal{M}(N, \chi_1; Q_W)$ complementary to $\mathcal{S}(N, \chi_1; Q_W)$ if N is cube free (e.g., [H86b] §5 and [O2] §1.4).

Our next goal is to extend Ohta’s construction of primitive Eisenstein series to imprimitive ones assuming that N is cube-free. In this way, we explicitly make a canonical Hecke eigenbasis of the Eisenstein component, which enables us to split Ohta’s residue exact sequence (4.1) in Proposition 4.2 and us to compute the characteristic power series of the Eisenstein congruence module in Corollary 4.3.

Let us prepare some notation to state Ohta’s exact sequence. For a profinite group G , we write $W[[G]] = \varprojlim_H W[G/H]$ for the continuous group algebra, where H runs over open subgroups of G . In particular, for the multiplicative group $\Gamma = 1 + p\mathbb{Z}_p \subset \mathbb{Z}_p^\times$, we can identify $W[[\Gamma]]$ with Λ_W by sending $\gamma \in \Gamma$ to $1 + T$. Let $C_r = C_r(N)$ be the set of all cusps of $X_r := X_1(Np^{r+1})(\mathbb{C})$, and consider the formal linear span

$$W[C_r] = \left\{ \sum_{s \in C_r} a_s s \mid a_s \in W \right\}.$$

Write simply $\Gamma_r := \Gamma_1(Np^{r+1})$. Since the Hecke correspondence $T_{r,s}(\alpha) \subset X_1(Np^r) \times X_1(Np^s)$ associated to the double coset $\Gamma_s \alpha \Gamma_r$ for $\alpha \in GL_2(\mathbb{Q})$ (with $\det(\alpha) > 0$) gives rise to a correspondence on $C_r \times C_s$ for $r, s \geq 0$, the Hecke correspondences acts on $W[C_r]$. In particular, $W[C_r]$ is equipped with the action of $T(l)$, $T(l, l)$ (in [IAT] Chapter 3) and $U(q)$ ($q|Np$), $\langle z \rangle = z_p \cdot [\Gamma_r \sigma_z \Gamma_r]$ for $z = (z_p, z_N) \in \mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$ with $z_p \in \mathbb{Z}_p^\times$, where $\sigma_z \in SL_2(\mathbb{Z})$ with $\sigma_z \equiv \begin{pmatrix} * & 0 \\ 0 & z \end{pmatrix} \pmod{Np^r}$. The coset $[q] = [\Gamma(Np^r) \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} \Gamma_1(Np^r/q)]$ for a prime $q|N$ gives rise to a linear map $[q] : W[C_r(N)] \rightarrow W[C_r(N/q)]$.

These operators are computed explicitly, choosing a standard representative set A_{Np^r}/\sim (see below for A_{Np^r}) for the cusps $C_r(N) := \Gamma_1(Np^r)\backslash\mathbf{P}^1(\mathbb{Q}) \subset X_r(\mathbb{C})$ in [O2] §2.1, where the action of $T(l)$ ($l \nmid Np$) is denoted by $T^*(l)$ and $U(q)$ ($q|Np$) is denoted by $T^*(q)$ in Ohta’s paper. The covering map $X_s \twoheadrightarrow X_r$ for $s > r$ induces a projection $\pi_{s,r} : W[C_s] \twoheadrightarrow W[C_r]$, and we define $W[[C_\infty(N)]] := \varprojlim_r W[C_r(N)]$. Since Hecke operators $T(l)$, $l \cdot T(l, l) = \langle l \rangle$ for $l \nmid Np$, $U(q)$, $\langle z \rangle$ and $[q]$ are compatible with the projection $\pi_{s,r}$, these operators act on $W[[C_\infty(N)]]$. We let the group $\mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$ act on $W[[C_\infty]]$ by the character $l \mapsto l \cdot T(l, l)$ for primes l diagonally embedded

in $\mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$. Thus $W[[C_\infty(N)]]$ is a module over $W[[\mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times]] = \varprojlim_r W[(\mathbb{Z}/Np^r\mathbb{Z})^\times]$ via the action of $\langle z \rangle$ and hence is a module over $\Lambda_W = W[[\Gamma]]$ as $\Gamma \subset \mathbb{Z}_p^\times$. Then we confirm $T(l), T(l, l), U(q) \in \text{End}_{W[[\Gamma]]}(W[[C_\infty(N)]])$ and that $[q] : W[[C_\infty(N)]] \rightarrow W[[C_\infty(N/q)]]$ are $W[[\Gamma]]$ -linear maps. Then the p -adic projector $e = \lim_{n \rightarrow \infty} U(p)^{n!}$ is well defined on $W[C_r]$ and hence on $W[[C_\infty]]$. Recalling the identification $W[[\Gamma]]$ with $\Lambda_W = W[[T]]$ by $\gamma = 1 + p \mapsto 1 + T$, we endow $e \cdot W[[C_\infty]]$ with a Λ_W -module structure. On $e \cdot W[[C_\infty]]$, Hecke operators acts Λ_W -linearly. As proved in [O] Proposition 4.3.14, $e \cdot W[[C_\infty]]$ is free of finite rank over Λ_W (and the rank is given explicitly there). Ohta's choice of the action of $\mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$ is one time twist of our action by the p -adic cyclotomic character; so, the definition of $E(\theta, \psi)$ looks a bit different from ours, but our definition is equivalent to that of [O2] with this twist. Supposing that $p \nmid \varphi(N) = |(\mathbb{Z}/N\mathbb{Z})^\times|$, we can decompose

$$e \cdot W[[C_\infty]] = \bigoplus_{\psi} e \cdot W[[C_\infty]][\psi],$$

where $e \cdot W[[C_\infty]][\psi]$ is the ψ -eigenspace of a character $\psi : (\mu_{p-1} \times (\mathbb{Z}/N\mathbb{Z})^\times) \rightarrow W^\times$ regarding $(\mu_{p-1} \times (\mathbb{Z}/N\mathbb{Z})^\times) \subset \mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$. Then from a result of Ohta [O2] (2.4.6), assuming $p \geq 5$, for a prime divisor $P \in \text{Spec}(\Lambda_W)$ prime to $(\varphi(N)) = \varphi(N)\Lambda_W$, we deduce a P -localized version of the canonical exact sequence of Hecke equivariant maps in [O2] (2.4.6):

$$(4.1) \quad 0 \rightarrow \mathcal{S}(N, \chi_1; \Lambda_W)_P \rightarrow \mathcal{M}(N, \chi_1; \Lambda_W)_P \xrightarrow{\text{Res}} e \cdot W[[C_\infty]][\chi_1]_P \rightarrow 0,$$

where the last map Res is canonical and called the residue map in [O2]. This sequence is valid without localization if $p \nmid \varphi(N)$. Thus as Hecke modules, $e \cdot W[[C_\infty]][\chi_1] \otimes_{\Lambda_W} Q \cong \mathcal{E}(N, \chi_1; Q)$.

We can extend this definition of $e \cdot W[[C_\infty]]$ slightly. Take a prime q outside pN and consider the \mathbb{C} -points of the elliptic Shimura curve $\mathbb{X}(N; q^j) = GL_2(\mathbb{Q}) \backslash (GL_2(\mathbb{A}^{(\infty)}) \times (\mathbb{C} - \mathbb{R})) / \Delta(Nq^j)$ and its connected component $X(N; q^j) = SL_2(\mathbb{Q}) \backslash (SL_2(\mathbb{A}^{(\infty)}) \times \mathfrak{H}) / \Delta(Nq^j) \cap SL_2(\mathbb{A}^{(\infty)})$, where $\Delta(Nq^j) = \widehat{\Gamma}_1(N) \cap \widehat{\Gamma}(q^j)$ for

$$(4.2) \quad \begin{aligned} \widehat{\Gamma}(q^j) &= \left\{ x \in GL_2(\widehat{\mathbb{Z}}) \mid x \equiv 1 \pmod{q^j M_2(\widehat{\mathbb{Z}})} \right\}, \\ \widehat{\Gamma}_1(N) &= \left\{ x \in GL_2(\widehat{\mathbb{Z}}) \mid x \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \pmod{NM_2(\widehat{\mathbb{Z}})} \right\}. \end{aligned}$$

Note that $\mathbb{X}(N; q^j)$ is isomorphic to a disjoint union of copies of $X(N; q^j)$ indexed by $(\mathbb{Z}/q^j\mathbb{Z})^\times$. We write $\mathbf{C}(N; q^j)$ (resp. $C(N; q^j)$) be the set of cusps of $\mathbb{X}(N; q^j)$ (resp. $X(N; q^j)$). Then we have $C(Np^r; q^j) \cong \{(A_N / \sim) \times A_{q^j}\} / \{\pm 1\}$, where

$$A_N = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in (\mathbb{Z}/N\mathbb{Z})^2 \mid x(\mathbb{Z}/N\mathbb{Z}) + y(\mathbb{Z}/N\mathbb{Z}) = \mathbb{Z}/N\mathbb{Z} \right\}$$

with $\begin{pmatrix} x \\ y \end{pmatrix} \sim \begin{pmatrix} x' \\ y' \end{pmatrix} \Leftrightarrow y = y'$ and $x \equiv x' \pmod{y(\mathbb{Z}/N\mathbb{Z})}$. If $\{\pm 1\}$ acts freely on A_N , we have

$$C(N; q^j) \cong ((A_N / \sim) \times A_{q^j}) / \{\pm 1\} \cong ((A_N / \sim) / \{\pm 1\}) \times A_{q^j}.$$

Replacing the auxiliary level N by Np^r for sufficiently large r (noting $p \geq 5$), we may assume that $\{\pm 1\}$ acts freely on A_{Np^r} . Thus for $r \gg 0$, we have

$$\mathbf{C}(Np^r; q^j) \cong \bigsqcup_{(\mathbb{Z}/q^j\mathbb{Z})^\times} ((A_{Np^r} / \sim) \times A_{q^j}) / \{\pm 1\} \cong \bigsqcup_{(\mathbb{Z}/q^j\mathbb{Z})^\times} ((A_{Np^r} / \sim) / \{\pm 1\}) \times A_{q^j}.$$

As before (see [O2] §2.1), $GL_2(\mathbb{Z}_q)$ acts on A_{q^j} by natural left multiplication on column vectors. Then $u \in GL_2(\mathbb{Z}_q)$ acts on $\bigsqcup_{(\mathbb{Z}/q^j\mathbb{Z})^\times} A_{q^j}$ via this multiplication but also permuting indices in $(\mathbb{Z}/q^j\mathbb{Z})^\times$ via multiplication by $\det(u)$. The set $\mathbf{C}(Np^r; q^j)$ of cusps inherits the $GL_2(\mathbb{Z}_q)$ -action from the curve $\mathbb{X}(Np^r; q^j)$, and this action is compatible with the action (including permutation of the components) on $\bigsqcup_{(\mathbb{Z}/q^j\mathbb{Z})^\times} A_{q^j}$. Consider $W[[\mathbf{C}(Np^\infty; q^j)]] = \varprojlim_r W[\mathbf{C}(Np^r; q^j)]$, which is naturally a Λ_W -module in the same manner as for $W[[C_\infty(N)]]$ through the action of \mathbb{Z}_p^\times on (A_{Np^r} / \sim) . Then we define $V_q = \varinjlim_j W[[\mathbf{C}(Np^\infty; q^j)]]$, where we regard

$$W[[\mathbf{C}(Np^\infty; q^j)]] = \bigoplus_{(\mathbb{Z}/q^j\mathbb{Z})^\times} W[[C_\infty(N)]] [A_{q^j}]$$

as a space of $W[[C_\infty]]$ -valued functions on $\bigsqcup_{(\mathbb{Z}/q^j\mathbb{Z})^\times} A_{q^j}$, and by the pull-back of the projection $\bigsqcup_{(\mathbb{Z}/q^{j+1}\mathbb{Z})^\times} A_{q^{j+1}} \rightarrow \bigsqcup_{(\mathbb{Z}/q^j\mathbb{Z})^\times} A_{q^j}$, we have taken the *inductive* limit. The idempotent e is well defined on V_q , and we have $e \cdot V_q = \varinjlim_j e \cdot W[[\mathbf{C}(Np^\infty; q^j)]]$.

On the pro-curve $\mathbb{X}(Np^r; q^\infty) = \varinjlim_j \mathbb{X}(Np^r; q^j) \cong GL_2(\mathbb{Q}) \backslash (GL_2(\mathbb{A}^{(\infty)}) \times (\mathbb{C} - \mathbb{R})) / \Delta(Nq^\infty)$ for $\Delta(Nq^\infty) = \bigcap_j \Delta(Nq^j) = \{x \in \Delta(N) | x_q = 1\}$, $GL_2(\mathbb{Q}_q)$ acts by right multiplication, which induces the (correspondence) action of $GL_2(\mathbb{Q}_q)$ on the cusps and induces a left action of $GL_2(\mathbb{Q}_q)$ on V_q . This $GL_2(\mathbb{Q}_q)$ -action induces the action of the maximal open compact subgroup $GL_2(\mathbb{Z}_q)$ already described on $\bigsqcup_{(\mathbb{Z}/q^j\mathbb{Z})^\times} A_{q^j}$. Plainly V_q is a smooth representation of $GL_2(\mathbb{Q}_q)$ with coefficients in Λ_W . At each finite q -level, $e \cdot W[[\mathbf{C}(Np^\infty; q^j)]]$ is free of finite rank over Λ_W as proved by Ohta in [O] §4.3. Though the curve $X_1(Np^r)$ is specifically dealt with in [O] §4.3, the argument for $\mathbb{X}(Np^r; q^j)$ is the same, or actually for a suitable choice of $g \in GL_2(\mathbb{Q}_q)$ (such that $g^{-1}\widehat{\Gamma}(q^j)_q \supset \widehat{\Gamma}_1(q^{2j})_q$ for the principal congruence subgroup $\widehat{\Gamma}(q^j)_q \subset GL_2(\mathbb{Z}_q)$), the right multiplication by g induces a Γ -equivariant covering $X_1(Np^r q^{2j}) \xrightarrow{g} \mathbb{X}(Np^r; q^j)^\circ$ for any geometrically connected component $\mathbb{X}(Np^r; q^j)^\circ$; so, Ohta's result actually implies this finiteness. We have

$$H^0(\widehat{\Gamma}(q^j)_q, e \cdot V_q) = e \cdot W[[\mathbf{C}(Np^\infty; q^j)]],$$

which is free of finite rank over Λ_W . Thus $e \cdot V_q \otimes_{\Lambda_W} Q_W$ is a finitely generated admissible smooth representation of $GL_2(\mathbb{Q}_q)$, and $e \cdot V_q$ is a Λ_W -lattice stable under the $GL_2(\mathbb{Q}_q)$ -action (see [HM] §4.4.1 for this type of Λ_W -adic automorphic representation).

Over the pairs of characters (θ, ψ) defined modulo M_1 and M_2 respectively, we confirm that $\mathcal{E}(N, \chi_1; Q)$ is a direct sum of Hecke eigenspaces spanned by $E(\theta, \psi)$. Let $P \in \text{Spec}(\Lambda_W)$ be a prime divisor. Assuming $p \nmid \varphi(N)$ if P is above $(p) \subset \Lambda$, it is easy to see that systems of the Hecke eigenvalues of $E(\theta, \psi)$ are distinct modulo the prime divisor P . Thus $e \cdot W[[C_\infty]][\chi_1]_P = \bigoplus_{(\theta, \psi)} \Lambda_{W, P} e(\theta, \psi)$ for an eigen basis $e(\theta, \psi)$ with the same eigenvalues as $E(\theta, \psi)$. Ohta showed in [O2] Theorem 2.4.10

$$(4.3) \quad \text{Res}(E(\theta, \psi)) = A(T; \theta, \psi) e(\theta, \psi) \quad \text{for } A(T; \theta, \psi) \in \Lambda_W,$$

where $A(T; \theta, \psi) \in \Lambda_W$ is given up to units in Λ_W as follows. Taking the power series $G(T; \xi) \in \Lambda_W$ so that $G(\gamma^s - 1; \xi) = L_p(-s, \xi\omega)$ ($\gamma = 1 + p$) for the Kubota–Leopoldt p -adic L -function $L_p(s, \xi)$ with a primitive even Dirichlet character ξ , $A(T; \theta, \psi)$ is given by, up to units in Λ_W ,

$$(4.4) \quad \begin{cases} t \cdot G(T; \theta\psi^{-1}\omega) \prod_{l|N} \{\omega(l)l^{-1}(\langle l \rangle(T) - \theta\psi^{-1}\omega(l)l^{-1})\} & \text{if } (\theta, \psi) = (\omega_{M_2}^{-1}, \mathbf{1}_{M_1}), \\ G(T; \theta\psi^{-1}\omega) \prod_{l|N, l \nmid \mathfrak{C}(\theta\psi^{-1})} \{\omega(l)l^{-1}(\langle l \rangle(T) - \theta\psi^{-1}\omega(l)l^{-1})\} & \text{otherwise,} \end{cases}$$

where $t = (1+T) - \gamma^{-1}$ and $\mathfrak{C}(\xi)$ is the conductor of the Dirichlet character ξ . Here ψ_{M_1} is a Dirichlet character modulo M_1 . In the exceptional case $(\theta, \psi) = (\omega^{-1}, \mathbf{1}_1)$ (which is equivalent to the case of $(\omega^{-2}, \mathbf{1}_1)$ in Ohta's paper), as is well known, the Eisenstein ideal is trivial and $A(T; \omega^{-1}, \mathbf{1}_1) \in \Lambda^\times$.

Definition 4.1. Let \mathbb{L}_{N, χ_1} be the product $\prod_{(\theta, \psi), \theta\psi = \chi_1} A(T; \theta, \psi)$ for the pairs (θ, ψ) running over all characters with $M_1 M_2 | Np$ except for the pair induced by $(\omega^{-1}, \mathbf{1}_1)$.

Letting $\sigma \in \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ act on $W[[T]]$ by $(\sum_n a_n T^n)^\sigma = \sum_n a_n^\sigma T^n$, we know that $A(T; \theta, \psi)^\sigma = A(T; \theta^\sigma, \psi^\sigma)$; so, \mathbb{L}_{N, χ_1} is Galois invariant, and hence $\mathbb{L}_{N, \chi_1} \in \Lambda = \mathbb{Z}_p[[T]]$. The following proposition is basically proven in [O2] Theorem 1.5.5. Since in the statement in [O2], he assumes $N | \mathfrak{C}(\theta)\mathfrak{C}(\psi)$, we give a proof under cube-freeness of N via the theory of admissible representation of $GL_2(\mathbb{Q}_q)$.

Proposition 4.2. Let the notation be as above. Suppose $p \geq 5$ and that N is cube free. Let P be a prime divisor of Λ_W prime to $(\mathbb{L}_{N, \chi_1}) \subset \Lambda_W$ and $\varphi(N)\Lambda_W$. After tensoring the localization $\Lambda_{W, P}$ at the prime P , Ohta's exact sequence (4.1) is split as a sequence of \mathbf{H}_P -modules.

Proof. By assumption,

$$\text{if } (\theta, \psi) \neq (\omega^{-1}, \mathbf{1}_1), \quad P \nmid A(T; \theta, \psi) \text{ in } W[[T]].$$

Thus $A(T; \theta, \psi) \in \Lambda_{W, P}^\times$. So, if $N = 1$, we can define Hecke equivariant section:

$$e \cdot W[[C_\infty]][\chi_1] \rightarrow \mathcal{M}(N, \chi_1; \Lambda_W) \quad \text{by } e(\theta, \psi) \mapsto E(\theta, \psi)$$

for $(\theta, \psi) \neq (\omega^{-1}, \mathbf{1}_1)$, and $e(\omega^{-1}, \mathbf{1}_1) \mapsto t \cdot E(\omega^{-1}, \mathbf{1}_1)$ otherwise ($t = (1+T) - \gamma^{-1}$). This gives rise to a section over $\Lambda_{W,P}$ of \mathbf{H}_P -modules.

We proceed by induction on the number of prime factors of N . Suppose we have a section:

$$e \cdot W[[C_\infty(N/q)]]_P \rightarrow \mathcal{M}(N/q, \chi_1; \Lambda_{W,P})$$

for $\mathcal{M}(N/q, \chi_1; \Lambda_{W,P}) = \mathcal{M}(N/q, \chi_1; \Lambda_W) \otimes_{\Lambda_W} \Lambda_{W,P}$. Take (θ, ψ) with $M_1 M_2 | N/q$. We claim that

(C) *the space V spanned by $e(\theta, \psi)|[q]$ and $e(\theta, \psi)$ in $W[[C_\infty(N)]]_P$ has rank 2 over $\Lambda_{W,P}$, and is a direct summand of $e \cdot W[[C_\infty(N)]]_P$.*

To prove this claim (C), we use the admissible representation $(e \cdot V_q) \otimes_\Lambda Q$ of $GL_2(\mathbb{Q}_q)$ defined for the prime to q -part $N^{(q)}$ of N (in place of N) whose detailed description is given just before stating the proposition. Then $e(\theta, \psi)$ generates a principal series representation $\pi_q \subset e \cdot V_q \otimes_{\Lambda_W} Q_W$ isomorphic to $\pi(\theta_q, \widehat{\psi}_q)$ over Q_W , where $\widehat{\psi}_q : \mathbb{Q}_q^\times \rightarrow \Lambda^\times$ is the unramified character sending the prime q to $\psi(q)\langle q \rangle$, and θ_q is just the $\theta|_{\mathbb{Q}_q^\times}$ regarding θ as an idele character. Then by the well known theory of admissible representations if $\kappa(P)$ has characteristic 0 and by Vigneras' modulo p representation theory of admissible representations (see [V] Theorem 3) if $P|(p)$, old-new congruence at q occurs only when the ratio of $(\widehat{\psi}_q/\theta_q)(q)$ is congruent to $q^{\pm 1}$ modulo P (for the maximal ideal P of $\Lambda_{W,P}$). This cannot happen if P is over (p) because $\langle q \rangle = \zeta(1+T)^{\log_p(q)/\log_p(\gamma)}$ for a root of unity ζ and $(1+T)^s \equiv (1+T^{p^m})^u = 1 + uT^{p^m} + \dots \pmod{p}$ if $s = p^m u$ with $u \in \mathbb{Z}_p^\times$.

If $\kappa(P)$ has characteristic 0, regarding $P \in \text{Hom}_{\mathbb{Z}_p\text{-alg}}(\mathbb{I}, \overline{\mathbb{Q}}_p)$, we have $P(1+T) = u \in \overline{\mathbb{Q}}_p$ with $|u-1|_p < 1$. Then $P(\widehat{\psi}_q \theta_q^{-1}(q) - q^{\pm 1}) = \psi \theta^{-1}(q) u^{\log_p(q)/\log_p(\gamma)} - q^{\pm 1} = 0$ implies $u = \gamma^{\pm 1} \zeta$ for a p -power root of unity ζ . Thus we have $P = ((1+T) - \gamma^{\pm 1} \zeta)$; so, $\psi \theta^{-1} \omega^{\mp 1}(q) q^{\pm 1} \zeta' - q^{\pm 1} = 0$ for another p -power root of unity ζ' ; hence, $\mu_{p^\infty} \ni \zeta' = \theta \psi^{-1} \omega^{\pm 1}(q)$. Since $P \nmid ((q)(T) - \theta \psi^{-1} \omega(l) q^{-1})$ (which is a factor of \mathbb{L}_{N, χ_1}) by our assumption, we find that $P = ((1+T) - \zeta' \gamma)$. Since P is now an arithmetic prime of weight 2 and N is cube-free, we know that \mathbf{H} and $\mathbf{H}_{2,\varepsilon}$ are reduced algebras by [H12a] Corollaries 1.2 and 1.3 (in [H12a], only the cuspidal Hecke algebra is dealt with, but the proof is the same for \mathbf{H}), Thus \mathbf{H}_P is an algebra direct sum of the Eisenstein part and the cuspidal part; so, the exact sequence has to split.

Thus hereafter, we may assume that

$$(\widehat{\psi}_q/\theta_q)(q) \not\equiv q^{\pm 1} \pmod{P}.$$

By the well known theory of admissible representations if $\kappa(P)$ has characteristic 0 and by Vigneras' modulo p representation theory of admissible representations (see [V] Theorem 3) if $P|(p)$, $\pi(\theta_q, \widehat{\psi}_q) \pmod{P\Lambda_{W,P}}$ is irreducible. The vectors $e(\theta, \psi)$ and $e(\theta, \psi)|[q]$ modulo $P\Lambda_{W,P}$ in the irreducible $\overline{\pi}_q := (\pi_q \pmod{P\Lambda_{W,P}})$ are linearly independent. This shows the above claim (C).

To make a section, first assume that q is prime to N/q . Letting (θ_1, ψ) be the pair with θ_1 which is θ regarded as a character modulo $M_1 q$, we have $e(\theta_1, \psi) = e(\theta, \psi) - \theta_q(q) e(\theta, \psi)|[q]$ up to units in $\Lambda_{W,P}$ by the argument in the previous section. Similarly $e(\theta, \psi_1) = e(\theta, \psi) - \widehat{\psi}_q(q) e(\theta, \psi)|[q]$ for ψ_1 which is ψ regarded as a character modulo $M_1 q$. Then

$$\begin{aligned} \text{Res}(E(\theta_1, \psi)) &= \text{Res}(E(\theta, \psi) - \theta_q(q) E(\theta, \psi)|[q]) = \text{Res}(E(\theta, \psi)) - \theta_q(q) \text{Res}(E(\theta, \psi)|[q]), \\ \text{Res}(E(\theta, \psi_1)) &= \text{Res}(E(\theta, \psi) - \widehat{\psi}_q(q) E(\theta, \psi)|[q]) = \text{Res}(E(\theta, \psi)) - \widehat{\psi}_q(q) \text{Res}(E(\theta, \psi)|[q]). \end{aligned}$$

Thus the section of level N/q extends to the level N .

Note that N is cube-free. Thus the remaining case is when $q^2 | N$. If $\mathfrak{C}(\theta)$ and $\mathfrak{C}(\psi)$ are both prime to q , by the irreducibility of $\overline{\pi}_q$,

$$e(\theta, \psi), e(\theta, \psi)|[q] \text{ and } e(\theta, \psi)|[q]^2$$

span three dimensional subspace in $\overline{\pi}_q$. Thus we have $e(\theta_1, \psi_1) = e(\theta, \psi_1) - \theta(q) e(\theta, \psi_1)$ which does not vanish in $\overline{\pi}_q$. Then $e(\theta_1, \psi_1) \mapsto E(\theta_1, \psi_1)$ gives a section on (θ_1, ψ_1) -eigenspace. If $q | \mathfrak{C}(\theta)$ but $q \nmid \mathfrak{C}(\psi)$, we define $e(\theta_1, \psi) = e(\theta, \psi_1) - \theta(q) e(\theta, \psi_1)$, and if $q \nmid \mathfrak{C}(\theta)$ but $q | \mathfrak{C}(\psi)$, $e(\theta, \psi_1) = e(\theta, \psi_1) - \widehat{\psi}(q) e(\theta, \psi_1)$, and the same argument works well. If $q | \mathfrak{C}(\theta)$ and $q | \mathfrak{C}(\psi)$ but one of the characters is imprimitive at another prime q' , we apply our argument to q' in place of q , and we get the section. The case where $N | \mathfrak{C}(\theta) \mathfrak{C}(\psi)$ is covered by Ohta's result explained at the beginning of the proof. This finishes the proof. \square

Let \mathbf{E} be the image of \mathbf{H} in $\text{End}_\Lambda(\mathcal{E}(N, \chi_1; \Lambda_W))$ and define $\mathcal{C}_{\mathbf{E}} = \mathbf{h} \otimes_{\mathbf{H}} \mathbf{E} \cong (\mathbf{h} \oplus \mathbf{E})/\mathbf{H}$ (the Eisenstein congruence module; see (7.3) in the text and [MFG] §5.3.3 for the definition of congruence modules).

Corollary 4.3. *Suppose $p \geq 5$ and that N is cube-free. Then the characteristic power series $\text{Char}(\mathcal{C}_{\mathbf{E}})$ of $\mathcal{C}_{\mathbf{E}}$ in $\Lambda_W[\frac{1}{\varphi(N)}]$ is equal to \mathbb{L}_{N, χ_1} up to Λ_W -units.*

Proof. We have a pairing $\mathbf{H} \times \mathcal{M}(N, \chi_1; \Lambda_W)$ given by $(h, f) = a(1, f|h)$ (see [GME] §3.2.4 and §3.2.6). If we define

$$\widetilde{\mathcal{M}}(N, \chi_1; \Lambda_W) = \{f \in \mathcal{M}(N, \chi_1; Q_W) \mid a(n, f) \in \Lambda_W \text{ for all } n > 0\}.$$

Then as is well known (see [H86a] §2), this pairing $\mathbf{H} \times \widetilde{\mathcal{M}}(N, \chi_1; \Lambda_W)$ is perfect; i.e., as Λ_W -modules, $\text{Hom}_{\Lambda_W}(\mathbf{H}, \Lambda_W) \cong \widetilde{\mathcal{M}}(N, \chi_1; \Lambda_W)$ and $\text{Hom}_{\Lambda_W}(\widetilde{\mathcal{M}}(N, \chi_1; \Lambda_W), \Lambda_W) \cong \mathbf{H}$ by sending the linear form: $\sum_n a(n, F)q^n \mapsto a(n, F)$ (indexed by n) to the Hecke operator $T(n)$. However, by definition, $\widetilde{\mathcal{M}}(N, \chi_1; \Lambda_W)/\mathcal{M}(N, \chi_1; \Lambda_W) \hookrightarrow Q_W/\Lambda_W$ by $f \mapsto a(0, f)$, and the inclusion $f \mapsto a(0, f)$ is Γ -equivariant. The group Γ acts on the constant term by the character: $\Gamma \ni z \mapsto z^{-1} \in W^\times$ (as by our choice of the action, weight 1 corresponds to the trivial action). This shows that after inverting $t = (1 + T) - \gamma^{-1}$, the pairing is perfect between $\mathcal{M}(N, \chi_1; A)$ and $\mathbf{H} \otimes_{\Lambda_W} A$ over the principal ideal domain $A := \Lambda_W[\frac{1}{t\varphi(N)}]$. The Λ_W -perfectness of the pairing on $\mathbf{h} \times \mathcal{S}(N, \chi_1; \Lambda_W)$ holds in the same way as in the case of \mathbf{H} without inverting t (or $\varphi(N)$). We have an integral \mathbf{H} -linear map $I : e \cdot W[[C_\infty]][\chi_1] \rightarrow \mathcal{M}(N, \chi_1; \Lambda_W)$ given by $I(e(\theta, \psi)) = E(\theta, \psi)$ if (θ, ψ) is not induced by $(\omega^{-1}, \mathbf{1}_1)$ and $I(e(\theta, \psi)) = tE(\theta, \psi)$ otherwise. By [O2] Lemma 1.4.9,

$$W[[C_\infty]][\frac{1}{\varphi(N)}] \cong \prod_{(\theta, \psi), \theta\psi = \chi_1} W[[T]][\frac{1}{\varphi(N)}]e(\theta, \psi) \cong \mathbf{E}[\frac{1}{\varphi(N)}]$$

as \mathbf{H} -modules. Then we have

$$\text{Res} \circ I(e \cdot W[[C_\infty]][\frac{1}{\varphi(N)}]) \cong \prod_{(\theta, \psi)} \frac{W[[T]][\frac{1}{\varphi(N)}]}{A(T; \theta, \psi)W[[T]][\frac{1}{\varphi(N)}]}e(\theta, \psi) =: \mathcal{C}_{\mathbf{E}}.$$

Putting $\mathcal{S} = \mathcal{S}(N, \chi_1; Q_W) \cap \mathcal{M}(N, \chi_1; A)$ and $\mathcal{E} = \mathcal{E}(N, \chi_1; Q_W) \cap \mathcal{M}(N, \chi_1; A)$ in $\mathcal{M}(N, \chi_1; Q_W)$, we have the following exact sequence of \mathbf{H} -modules:

$$(4.5) \quad 0 \rightarrow \mathcal{E} \oplus \mathcal{S} \rightarrow \mathcal{M}(N, \chi_1; \Lambda_W) \rightarrow C \rightarrow 0 \quad \text{with } C \cong \mathcal{C}_{\mathbf{E}} \text{ as } \mathbf{H}\text{-modules.}$$

Defining an A -dual module M^* by $M^* = \text{Hom}_A(M, Q_W/A)$ for any torsion A -module M of finite type, we have $M \cong M^*$ (non-canonically) as A -modules, by the following lemma applied to the principal ideal domain A . Noting that $\mathbf{H} \otimes_{\Lambda_W} A$ (resp. $(\mathbf{h} \oplus \mathbf{E}) \otimes_{\Lambda_W} A$) is the A -dual of $\mathcal{M}(N, \chi_1; A)$ (resp. $\mathcal{S} \oplus \mathcal{E}$) and again applying the following lemma to the exact sequence (4.5) tensored A over Λ_W , we have an \mathbf{H} -linear isomorphism

$$(\mathcal{C}_{\mathbf{E}})^* \cong \mathcal{C}_{\mathbf{E}} \otimes_{\Lambda_W} A.$$

From this identity, we get $\text{Char}_{\Lambda_W}(\mathcal{C}_{\mathbf{E}}) = \text{Char}_{\Lambda_W}(\mathcal{C}_{\mathbf{E}}) = \mathbb{L}_{N, \chi_1}$ up to units in A . Since non-divisibility $t \nmid \text{Char}_{\Lambda_W}(\mathcal{C}_{\mathbf{E}})$ $\text{Char}_{\Lambda_W}(\mathcal{C}_{\mathbf{E}})$ is known, we have $\text{Char}_{\Lambda_W}(\mathcal{C}_{\mathbf{E}}) = \text{Char}_{\Lambda_W}(\mathcal{C}_{\mathbf{E}}) = \mathbb{L}_{N, \chi_1}$ in $\Lambda_W[\frac{1}{\varphi(N)}]$ (up to units) as desired. \square

Lemma 4.4. *Let A be a principal ideal domain with quotient field K . For each A -module M , we define $M^* = \text{Hom}_A(M, K/A)$ and $M^\vee = \text{Hom}_A(M, A)$. For an exact sequence $0 \rightarrow M \rightarrow N \rightarrow T \rightarrow 0$ of A -free modules M and N of finite rank with A -torsion quotient T , we have a canonical exact sequence of A -modules*

$$0 \rightarrow N^\vee \rightarrow M^\vee \rightarrow T^* \rightarrow 0$$

and an isomorphism $T^* \cong T$ as A -modules.

Proof. Since A is a principal ideal domain, we have the following facts:

- (1) $M \mapsto M^*$ is a perfect duality with $M \cong (M^*)^*$ canonically for A -modules of finite type,
- (2) if an A -module T is torsion of finite type, $T \cong T^*$ as A -modules non-canonically,
- (3) if an A -module T is torsion of finite type, $\text{Ext}_A^1(K/A, T) \cong T$.

By perfect duality, we have an exact sequence $0 \rightarrow L^* \rightarrow N^* \rightarrow T^* \rightarrow 0$ of A -modules. Applying the covariant functor $X \mapsto \mathrm{Hom}_A(K/A, X)$ to this exact sequence and noting isomorphisms

$$\mathrm{Hom}_A(K/A, M^*) \cong \mathrm{Hom}_A(K/A, \mathrm{Hom}_A(M, K/A)) \cong \mathrm{Hom}_A(M \otimes_A K/A, K/A) \cong M^\vee$$

$$\text{and } \mathrm{Ext}_A^1(K/A, T^*) \cong T^*,$$

we get the exact sequence

$$0 \rightarrow N^\vee \rightarrow M^\vee \rightarrow T^* \rightarrow 0.$$

□

5. CM COMPONENTS

We study when a CM component of $\mathrm{Spec}(\mathbf{h})$ is a Gorenstein ring. The result is used to determine the characteristic ideal of the congruence module of the CM component and other non-CM components. The characteristic ideal is expected to give the level of non-CM components in the connected component containing the CM component. We first quote the following fact from [H12a] Section 3 (or [H11a] (CM1–3) in Section 1):

Proposition 5.1. *Let $\mathrm{Spec}(\mathbb{J})$ be a reduced irreducible component of $\mathrm{Spec}(\mathbf{h})$ as in the introduction. Write $\tilde{\mathbb{J}}$ be the integral closure of \mathbb{J} in its quotient field. The following five conditions are equivalent:*

- (CM1) \mathcal{F} is a CM family with $\rho_{\mathbb{J}} \cong \rho_{\mathbb{J}} \otimes \left(\frac{M/\mathbb{Q}}{\phantom{M/\mathbb{Q}}}\right)$ for a quadratic field M with discriminant D ;
- (CM2) The prime p splits in M , and we have $\rho_{\mathbb{J}} \cong \mathrm{Ind}_M^{\mathbb{Q}} \Psi_{\mathbb{J}}$ for a character $\Psi_{\mathbb{J}} : \mathrm{Gal}(\overline{\mathbb{Q}}/M) \rightarrow \tilde{\mathbb{J}}^\times$ with prime-to- p conductor $\mathfrak{C} = \mathfrak{C}(\Psi_{\mathbb{J}})$ unramified outside $\mathfrak{C}p$. We have $D \cdot N(\mathfrak{C})|N$;
- (CM3) For all arithmetic points P of $\mathrm{Spec}(\mathbb{J})(\overline{\mathbb{Q}}_p)$, f_P is a binary Hecke eigen theta series of the norm form of an imaginary quadratic extension M/\mathbb{Q} with prime-to- p conductor $N(\mathfrak{C})D$ (as a Hecke eigenform);
- (CM4) For some arithmetic point P of $\mathrm{Spec}(\mathbb{J})(\overline{\mathbb{Q}}_p)$, f_P is a binary Hecke eigen theta series of the norm form of an imaginary quadratic extension M/\mathbb{Q} with prime-to- p conductor $N(\mathfrak{C})D$ (as a Hecke eigenform);
- (CM5) For some arithmetic prime P , ρ_P is an induced representation of a character of $\mathrm{Gal}(\overline{\mathbb{Q}}/M)$ with prime-to- p conductor \mathfrak{C} , where M is a quadratic extension of \mathbb{Q} .

A binary Hecke eigen theta series of the norm form of an imaginary quadratic field is called a *CM theta series*.

See Section 10 for a description of the prime-to- p conductor of Galois representations with coefficients in a p -profinite ring R . We write $\mathfrak{C}(\xi)$ (resp. $C(\rho)$) for the prime-to- p conductor of a Galois character ξ with values in R^\times (resp. a two dimensional Galois representation ρ with coefficients in R). We say a Hecke eigenform f has conductor $C(f)$ if the automorphic representation generated by f has conductor $C(f)$; so, f itself could be an old form. Recall that the prime-to- p part C of this conductor $C(f)$ is equal to the prime-to- p conductor $C(\rho_f)$ of the p -adic Galois representation associated to f . We say that \mathbb{J} has CM (or is a CM component) by M if one of the above equivalent conditions is satisfied by an imaginary quadratic field M . In the rest of this section, we fix a CM component \mathbb{J} of \mathbf{h} having CM by an imaginary quadratic field M . For $\Psi_{\mathbb{J}}$ as in (CM2) and a complex conjugation $c \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, we put

$$(5.1) \quad \overline{\psi} = (\Psi_{\mathbb{J}} \pmod{\mathfrak{m}_{\mathbb{J}}}), \quad \overline{\psi}^c(\sigma) = \overline{\psi}(c\sigma c^{-1}), \quad \overline{\psi}^- = \overline{\psi}/\overline{\psi}^c$$

and

$$(5.2) \quad C = C(\rho_{\mathbb{J}}) = C(\mathrm{Ind}_M^{\mathbb{Q}} \Psi_{\mathbb{J}}) = N(\mathfrak{C}(\Psi_{\mathbb{J}}))D, \quad \mathfrak{C} = \mathfrak{C}(\Psi_{\mathbb{J}}), \quad \mathfrak{c} = \mathfrak{C}(\overline{\psi}^-), \quad \mathfrak{c}' = \mathfrak{C} \cap \overline{\mathfrak{C}}$$

for $\overline{\mathfrak{C}} = \mathfrak{C}^c$, where $N(\mathfrak{a})$ is the norm of a fractional ideal \mathfrak{a} of M and $C(\Psi_{\mathbb{J}})$ is the prime-to- p conductor. Then $\mathfrak{c} = \overline{\mathfrak{c}}$, and \mathfrak{c} is a factor of \mathfrak{c}' but may not be equal to \mathfrak{c}' .

Let $\mathrm{Spec}(\mathbf{h}_{cm}^M)$ be the minimal closed subscheme of $\mathrm{Spec}(\mathbf{h})$ containing all reduced irreducible components having CM by a fixed imaginary quadratic field M . We take the connected component $\mathrm{Spec}(\mathbb{T})$ of $\mathrm{Spec}(\mathbf{h})$ containing $\mathrm{Spec}(\mathbb{J})$. Let $\mathrm{Spec}(\mathbb{T}_{cm})$ be the union of all reduced CM components inside $\mathrm{Spec}(\mathbb{T})$. Note that $\mathrm{Spec}(\mathbb{T}_{cm})$ could contain components having CM by different imaginary

quadratic fields. We would like to know when \mathbb{T}_{cm} is a Gorenstein ring or more strongly a local complete intersection. This can be answered by proving \mathbb{T}_{cm} is isomorphic to the continuous group algebra $W[[Z_p]]$ for an appropriate ray class group Z_p of M (see Lemma 5.5). Such an identification could fail if either $\text{Spec}(\mathbb{T}_{cm})$ intersects with $\text{Spec}(\mathbf{h}_{cm}^M)$ and $\text{Spec}(\mathbf{h}_{cm}^K)$ for different fields K and M or $\text{Spec}(\mathbb{T}_{cm})$ contains a union of two copies of $\text{Spec}(W[[Z_p]])$; i.e., new and old (or old and old) CM components coming from a primitive CM component. Here the word ‘‘primitive’’ is used in the sense of [H86a] page 252 in §3. Thus we look for sufficient conditions to preclude these bad cases in terms of level and the prime-to- p conductor of $\bar{\rho}$. The answer will be given in Proposition 5.7. We start with a result which is useful in describing $\text{Spec}(\mathbf{h}_{cm}^M) \cap \text{Spec}(\mathbf{h}_{cm}^K)$:

Proposition 5.2. *Let A be a p -profinite local integral domain with quotient field $Q(A)$ for $p > 2$. Let M and K be two distinct quadratic fields in $\bar{\mathbb{Q}}$. Suppose that we have continuous characters $\varphi : \text{Gal}(\bar{\mathbb{Q}}/M) \rightarrow A^\times$ and $\phi : \text{Gal}(\bar{\mathbb{Q}}/K) \rightarrow A^\times$ with absolutely irreducible $\text{Ind}_M^{\mathbb{Q}} \varphi$ over $Q(A)$ such that $\text{Ind}_M^{\mathbb{Q}} \varphi \cong \text{Ind}_K^{\mathbb{Q}} \phi$. Write φ^σ for the character $\text{Gal}(\bar{\mathbb{Q}}/M) \ni \tau \mapsto \varphi(\sigma\tau\sigma^{-1})$ for $\sigma \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ inducing the generator of $\text{Gal}(M/\mathbb{Q})$. If the representations $\text{Ind}_M^{\mathbb{Q}} \varphi \cong \text{Ind}_K^{\mathbb{Q}} \phi$ are ordinary at p , then we have*

- (1) φ and ϕ are both of finite order,
- (2) We have $\varphi^\sigma = \left(\frac{MK/M}{\sigma}\right) \varphi$; so, φ^- given by $\varphi(\varphi^\sigma)^{-1} =: \varphi^{1-\sigma}$ is equal to $\left(\frac{MK/M}{\sigma}\right)$.
- (3) If p does not ramify in MK/\mathbb{Q} , φ and ϕ are both unramified at p .
- (4) If φ ramifies at a prime factor of p , then p splits in M , φ is unramified at another prime factor of p , p ramifies in K and ϕ is unramified at p .
- (5) If K is real and $\text{Ind}_M^{\mathbb{Q}} \varphi$ is odd, M has to be imaginary and ϕ ramifies at exactly one real place of K , where a two dimensional Galois representation is called odd if its determinant at complex conjugation is equal to -1 .

Conversely, if φ^- has order 2 and M is imaginary, we have a quadratic field $K \neq M$ and a character ϕ such that $\text{Ind}_M^{\mathbb{Q}} \varphi \cong \text{Ind}_K^{\mathbb{Q}} \phi$.

Here the word ‘‘ordinary’’ means that the representation restricted to a decomposition group at p is isomorphic to an upper triangular representation with an unramified 1-dimensional quotient. In our case, the restriction of, say, $\text{Ind}_M^{\mathbb{Q}} \varphi$ to a decomposition group at p is the direct sum $\varphi \oplus \varphi^\sigma$ (for σ as in (2)) or $\mathbf{1} \oplus \left(\frac{M/\mathbb{Q}}{\sigma}\right)$ for the identity character $\mathbf{1}$. Then ordinarity implies that φ is at least unramified at one prime in M over p .

Proof. Suppose $\text{Ind}_M^{\mathbb{Q}} \varphi \cong \text{Ind}_K^{\mathbb{Q}} \phi$. We first prove the assertion (2). Let N be the prime-to- p Artin conductor of $\text{Ind}_M^{\mathbb{Q}} \varphi$. For any prime l outside Np inert in K and split in M (such primes have positive density),

$$0 = \text{Tr}(\text{Ind}_K^{\mathbb{Q}} \phi(\text{Frob}_l)) = \text{Tr}(\text{Ind}_M^{\mathbb{Q}} \varphi(\text{Frob}_l)) = \varphi(l) + \varphi(l^\sigma)$$

for $\sigma \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ inducing a generator of $\text{Gal}(M/\mathbb{Q})$. Thus we have $\varphi^-(\text{Frob}_l) = -1$ if l is inert in M and split in K (note here that $-1 \neq 1$ because $p > 2$). In particular, for any other primes q outside Np inert in M and split in K , $\varphi^-(\text{Frob}_l) = -1 = \varphi^-(\text{Frob}_q)$. Since $\text{Frob}_l \text{Frob}_q^{-1}$ fix MK , by moving q , Chebotarev density tells us that φ^- factors through $\text{Gal}(MK/M)$. Since $\text{Ind}_M^{\mathbb{Q}} \varphi$ is absolutely irreducible, we have $\varphi \neq \varphi^\sigma$ (i.e., $\varphi^- \neq 1$). Thus we conclude $\varphi = \left(\frac{MK/M}{\sigma}\right) \varphi^\sigma$. This proves (2).

We now deal with the assertions (3) and (4). By the remark preceding this proof, we may assume that φ is unramified at one prime factor \mathfrak{p}^σ of p . If there is only one prime factor in M over p , this forces φ^σ to be unramified at p . If there are two factors of p in M , either φ is unramified also at \mathfrak{p} or K ramifies at p by (2). If K ramifies at p , there is only one prime factor in K over p , this forces ϕ to be unramified at p . Thus if MK/\mathbb{Q} is unramified at p , $\left(\frac{MK/M}{\sigma}\right)$ is unramified at p , and φ and ϕ are both unramified at p . This proves (3) and (4).

To show (1), first suppose that φ ramifies at a prime factor $\mathfrak{p}|p$. Thus p ramifies in K and splits in M . Then $\left(\frac{MK/M}{\sigma}\right)$ ramifies at two primes \mathfrak{p} and \mathfrak{p}^σ , and therefore φ has to be unramified at \mathfrak{p}^σ .

In short, φ ramifies at \mathfrak{p} and unramified at \mathfrak{p}^σ . Since p ramifies in K , ordinarity of $\text{Ind}_K^{\mathbb{Q}} \phi$ forces ϕ to be unramified at p ; so, ϕ factors through a finite ray class group $Cl_K(\mathfrak{f}')$ for an ideal \mathfrak{f}' prime to p . Thus $\text{Ind}_M^{\mathbb{Q}} \varphi \cong \text{Ind}_K^{\mathbb{Q}} \phi$ has finite image; so, φ has finite order.

Next suppose that φ is unramified at p . Then φ factors through the finite ray class group $Cl_M(\mathfrak{f})$ of M modulo \mathfrak{f} for the prime-to- p conductor \mathfrak{f} of φ . Now $\text{Ind}_M^{\mathbb{Q}} \varphi$ has finite image, and we conclude that ϕ is of finite order (this proves (1)).

To prove (5), now assume that M is imaginary and write $c \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ for complex conjugation. Since $\text{Ind}_M^{\mathbb{Q}} \varphi$ is automatically odd (as M is imaginary), we have $\text{Tr}(\text{Ind}_M^{\mathbb{Q}} \varphi(c)) = 0$. Regard ϕ as an idele character of $K_{\mathbb{A}}^\times$. Then

$$0 = \text{Tr}(\text{Ind}_M^{\mathbb{Q}} \varphi(c)) = \text{Tr}(\text{Ind}_K^{\mathbb{Q}} \phi(c)) = \begin{cases} \phi(-1_\infty) + \phi(-1_{\infty'}) & \text{if } K \text{ is real,} \\ 0 & \text{otherwise,} \end{cases}$$

where ∞ is an infinite place of K and ∞' is the other, and $1_{\infty'}$ is the identity of the ∞' -component $K_{\infty'}^\times = \mathbb{R} \subset K_{\mathbb{A}}^\times$. Thus ϕ ramifies at exactly one infinite place of K if K is real.

Since $\text{Ind}_M^{\mathbb{Q}} \varphi$ is an odd Galois representation, we again see that ϕ ramifies at exactly one infinite place of K if K is real. If M and K are both real, $\varphi^\sigma/\varphi = \left(\frac{MK/M}{\cdot}\right)$ is unramified at the two infinite places; so, either φ ramifies at the two infinite places or unramified at the two infinite places; so, this is impossible (finishing the proof of (5)).

Suppose now that φ^- has order 2 and that M is imaginary, to prove the converse. Then the splitting field of φ^- is a quadratic extension L/M . Since $(\varphi^-)^\sigma = (\varphi^-)^{-1} = \varphi^-$, $L^\sigma = L$; so, L/\mathbb{Q} is an abelian extension of degree 4. This also shows that $\text{Ind}_M^{\mathbb{Q}} \varphi^-$ is reducible: $\text{Ind}_M^{\mathbb{Q}} \varphi^- = \eta \oplus \xi$ for two characters $\eta, \xi : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ with $\eta/\xi = \left(\frac{M/\mathbb{Q}}{\cdot}\right)$. Since M is imaginary, for any complex conjugation c , $\eta(c) = \left(\frac{M/\mathbb{Q}}{c}\right) \xi(c) = -\xi(c)$. Since $\xi(c)$ and $\eta(c)$ are ± 1 , we conclude one of them is -1 , say, $\eta(c) = -1$. If one of ξ and η has order 4, the other also has order 4. Since $\xi^2 = \eta^2$ is an even character, its splitting field $K \subset L$ is a real quadratic field; so, $M \neq K$ and $L = MK$, a contradiction (as $\text{Gal}(L/\mathbb{Q})$ has to be cyclic of order 4). Thus L is not a cyclic extension; so, again it is a composite of two distinct quadratic fields M and K . Thus ξ and η have order 2. Let F be the splitting field of $\text{Ind}_M^{\mathbb{Q}} \varphi$. Thus $\text{Gal}(F/\mathbb{Q})$ is a dihedral group D_n fitting into the following exact sequence:

$$1 \rightarrow \mathcal{C} \rightarrow D_n \rightarrow \text{Gal}(M/\mathbb{Q}) \rightarrow 1$$

with cyclic $\mathcal{C} \cong \text{Gal}(F/M)$ of order n . Since the adjoint square representation $Ad(\text{Ind}_M^{\mathbb{Q}} \varphi)$ is isomorphic to $\left(\frac{M/\mathbb{Q}}{\cdot}\right) \oplus \text{Ind}_M^{\mathbb{Q}} \varphi^-$, the splitting field F contains the extension L/\mathbb{Q} with $\text{Gal}(L/\mathbb{Q}) \cong \{\pm 1\}^2$. Note that the maximal abelian quotient D_n^{ab} of D_n is isomorphic to $\{\pm 1\}^2$ or $\{\pm 1\}$ according as $|\mathcal{C}| = n$ is even or odd. Since D_n has quotient $\text{Gal}(L/\mathbb{Q}) \cong \{\pm 1\}^2$, n is an even number $2m$. Since $D_{2m} = \text{Gal}(M/\mathbb{Q}) \rtimes \mathcal{C}$ and $D_{2m}^{ab} \cong \{\pm 1\}^2$, the image of \mathcal{C} in D_{2m}^{ab} gives rise to one cyclic subgroup of order 2. The group $D_{2m}^{ab} \cong \{\pm 1\}^2$ has three nontrivial cyclic subgroups: the image $\overline{\mathcal{C}}$ of \mathcal{C} , the image of $\text{Gal}(M/\mathbb{Q})$ and $\overline{\mathcal{C}'}$. Then the pull back of $\overline{\mathcal{C}'}$ to D_n is a subgroup \mathcal{C}' of index two in D_n ; so, it is a normal subgroup, which is cyclic of order n . The fixed field K/\mathbb{Q} of \mathcal{C}' is a quadratic extension in L and $L = KM$. By construction, $\text{Gal}(F/K)$ has a character $\phi : \text{Gal}(F/K) = \mathcal{C}' \cong \mathcal{C} \xrightarrow{\varphi} \text{Im}(\varphi) \subset A^\times$, and $\text{Ind}_M^{\mathbb{Q}} \varphi \cong \text{Ind}_K^{\mathbb{Q}} \phi$. \square

Corollary 5.3. *Suppose $p > 2$. Let M and K be distinct imaginary quadratic fields in which p splits. If $P \in \text{Spec}(\mathbf{h}_{cm}^M) \cap \text{Spec}(\mathbf{h}_{cm}^K)$ is a prime divisor, we have $P \cap \mathbb{Z}_p[[T]] = (T)$.*

Proof. Since ρ_P has to be induced from M and K , we have $\text{Ind}_M^{\mathbb{Q}} \varphi \cong \text{Ind}_K^{\mathbb{Q}} \phi$. Since p has to be split both in K and M , ϕ and φ are unramified at p by (3) of the above lemma, and by the property (Gal), regarding $[\gamma, \mathbb{Q}_p]$ as an element of a p -inertia group in $\text{Gal}(\overline{\mathbb{Q}}/M)$, we have $1 + T = \varphi([\gamma, \mathbb{Q}_p]) = 1$; so, $T = 0$ in \mathbf{h}/P ; i.e, $T \in P$. \square

Let \mathbb{T}_P be the localization of \mathbb{T} at a prime divisor $P \in \text{Spec}(\mathbb{T})$ and write $\rho_{\mathbb{T}_P}$ for $\rho_{\mathfrak{a}}$ for $\mathfrak{a} = \text{Ker}(\mathbb{T} \rightarrow \mathbb{T}_P^{\text{red}})$. Let $u(q)$ for primes $q|Np$ be the image of $U(q)$ in \mathbb{T}_P . Similarly, we write $a(l) \in \mathbb{J}$

for the image of $T(l)$ or $U(l)$ in \mathbb{J} . We have $\kappa := \det(\rho_{\mathbb{T}_P}) : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow W[[T]]$. By [GME] Proposition 4.3.1, we have $\det(\rho_{\mathbb{T}_P})([u, \mathbb{Q}]) = \chi_1(u)(1+T)^{\log_p(u_p)/\log_p(\gamma)}$ for $u \in \widehat{\mathbb{Z}}^\times$. Then consider the projection $\langle \cdot \rangle : \text{Im}(\det(\rho_{\mathbb{T}_P})) \rightarrow \text{Im}(\det(\rho_{\mathbb{T}_P}))_p$ for the maximal p -profinite subgroup $\text{Im}(\det(\rho_{\mathbb{T}_P}))_p$ of $\text{Im}(\det(\rho_{\mathbb{T}_P}))$, and put $\langle \kappa \rangle = \langle \cdot \rangle \circ \kappa$; so, $\kappa = \chi_1 \langle \kappa \rangle$. Then we define $\rho'_{\mathbb{T}_P} = \rho_{\mathbb{T}_P} \otimes \sqrt{\langle \kappa \rangle}^{-1}$, where the square root is supposed to have values in the p -profinite part $\text{Im}(\det(\rho_{\mathbb{T}_P}))_p$. Note that $\sqrt{\langle \kappa \rangle}$ has values in $W[[T]]^\times$ and that $\rho'_{\mathbb{T}_P}$ has values in $GL_2(\mathbb{T}_P)$, since $p > 2$. We prepare

Lemma 5.4. *Let the notation be as above (in particular, P is a prime divisor of \mathbb{T}). Suppose $p > 2$. We assume that W is sufficiently large valuation ring finite flat over \mathbb{Z}_p . Let \mathbb{T}'_P be the subring of \mathbb{T}_P generated by $\{\text{Tr}(\rho_{\mathbb{T}_P}(\sigma))\}_{\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}$ over $W[[T]]_P$. Then \mathbb{T}'_P is generated by $\{\text{Tr}(\rho'_{\mathbb{T}_P}(\sigma))\}_{\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}$ over $W[[T]]_P$. Further suppose that the prime-to- p conductor of $\rho_{\mathbb{T}_P}$ coincides with the prime-to- p conductor $C(\rho_P)$ of ρ_P . If $N = C(\rho_P)$, we have*

- (1) \mathbb{T}_P is reduced, and if $C(\overline{\rho}) = N$, \mathbb{T} is reduced.
- (2) The total quotient rings $Q(\mathbb{T}_P)$ and $Q(\mathbb{T}'_P)$ coincide.
- (3) If $\kappa(P)$ has characteristic 0 or $p \nmid \varphi(N)$, we have $\mathbb{T}_P = \mathbb{T}'_P[u(p)]$ under absolute irreducibility of ρ_P , and
- (4) Suppose absolute irreducibility of ρ_P and that W is sufficiently large. We have $\mathbb{T}_P = \mathbb{T}'_P$ if one of the following conditions is satisfied
 - (a) $\kappa(P)$ has characteristic 0 and $u(p)^2 \not\equiv \chi^{(p)}(p) \pmod{P}$ for the prime to p -part $\chi^{(p)} = \chi|_{(\mathbb{Z}/N\mathbb{Z})^\times}$ of χ ,
 - (b) $\kappa(P)$ has characteristic 0 and $T \notin P$,
 - (c) $\chi_1|_{\mathbb{Z}_p^\times}$ is non-trivial.

Later we compute the congruence module of a CM component of the ring \mathbb{T}' in terms of anticyclotomic Katz p -adic L -functions. The relation between \mathbb{T} and \mathbb{T}' is clarified by this lemma.

Proof. For a continuous representation $\rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow GL_2(R)$ with a p -profinite local W -algebra R , let ξ be the unique square root character of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ with values in $1 + \mathfrak{m}_R$ of the projection of $\det(\rho)$ to $1 + \mathfrak{m}_R$. Let R_t (resp. R'_t) be the subring of R generated topologically over W by the value of $\text{Tr}(\rho)$ (resp. $\text{Tr}(\rho \otimes \xi^{-1})$). The subring R_t contains $2 \det(\rho(\sigma)) = \text{Tr}(\rho(\sigma))^2 - \text{Tr}(\rho(\sigma^2))$. Since $p > 2$, we have $\det(\rho(\sigma)) \in R_t$. Thus R_t contains the values of ξ , and hence $R'_t \subset R_t$. If further R'_t contains the value of ξ , we have $R'_t \supset R_t$ as $\text{Tr}(\rho \otimes \xi^{-1}) = \xi^{-1} \text{Tr}(\rho)$. Since $W[[T]]$ contains the value of $\sqrt{\langle \kappa \rangle}$, the subrings of \mathbb{T}_P generated over $W[[T]]_P$ by $\{\text{Tr}(\rho_{\mathbb{T}_P}(\sigma))\}_{\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}$ and $\{\text{Tr}(\rho'_{\mathbb{T}_P}(\sigma))\}_{\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}$ are the same. This shows that \mathbb{T}'_P is generated by $\{\text{Tr}(\rho'_{\mathbb{T}_P}(\sigma))\}_{\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}$ over $W[[T]]_P$.

Since the argument proving (1) is almost identical for \mathbb{T} and \mathbb{T}_P , we give here the one for \mathbb{T}_P . Since $N = C(\rho_P)|C(\rho_{\mathbb{T}})|N$ by Lemma 10.2 (1) and (4), we conclude $C(\rho_P) = C(\rho_{\mathbb{T}}) = N$. For any prime $P' \in \text{Spec}(\mathbb{T}_P)$, $P' \supset P$, and we have $N = C(\rho_P)|C(\rho_{P'})|N$; so, $C(\rho_{P'}) = N$. Since the nilradical of \mathbb{T} comes from q -old forms for $q|N$ (i.e., the nilradical acts faithfully on the space of q -old forms for $q|N$; see [H86a] Corollary 3.3), it has to be trivial. Thus we conclude the assertion (1) for \mathbb{T}_P .

We now look into the subring \mathbb{T}'_P of \mathbb{T}_P generated by $\{\text{Tr}(\rho_{\mathbb{T}_P}(\sigma))\}_{\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}$ over $W[[T]]_P$ more carefully. Since \mathbb{T}'_P contains $\text{Tr}(\rho_{\mathbb{T}_P})$ at the l -Frobenius element for all primes $l \nmid Np$, by Chebotarev density theorem, $\text{Tr}(\rho_{\mathbb{T}_P})$ has values in \mathbb{T}'_P . Thus, we have a representation $\tilde{\rho} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow GL_2(Q(\mathbb{T}'_P))$ with $\text{Tr}(\tilde{\rho}) = \text{Tr}(\rho_{\mathbb{T}_P})$ by the theory of pseudo representation. The projection of this representation to each simple factor of $Q(\mathbb{T}'_P)$ is absolutely irreducible. Since $u(q) = 0$ or a unit in each irreducible component of $\text{Spec}(\mathbb{T})$ (because of [MFM] Theorem 4.6.17), $u(q) = 0$ or a unit in the entire \mathbb{T} . Thus, as for (2), (3) and (4), we may assume that $u(q) \in \mathbb{T}^\times$. Under this assumption, for an arithmetic $P' \in \text{Spec}(\mathbb{T})$, $H^0(I_q, \rho_{P'}) \cong \kappa(P')$ (cf. [GME] Theorem 4.2.7). Thus $H_0(I_q, Q(\mathbb{T}'^{red})) \cong Q(\mathbb{T}'^{red})$, which implies $H_0(I_q, Q(\mathbb{T}'_P)) \cong Q(\mathbb{T}'_P)$. Take an element $\phi_q \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ which induces $[q, \mathbb{Q}_q]$ on the maximal abelian extension \mathbb{Q}_q^{ab} of \mathbb{Q}_q . Since $u(q)$ is the eigenvalue of $\tilde{\rho}(\phi_q)$ on $H_0(I_q, Q(\mathbb{T}'_P)) \cong Q(\mathbb{T}'_P)$, we have $u(q) \in Q(\mathbb{T}'_P)$. This proves (2).

Hereafter we assume absolute irreducibility of ρ_P . Then we have $\tilde{\rho}$ with values in $GL_2(\mathbb{T}'_P)$, and hereafter, we always take a \mathbb{T}'_P -free lattice $L(\tilde{\rho}) \subset Q(\mathbb{T}'_P)^2$ stable under $\tilde{\rho}$.

By definition, \mathbb{T}_P is generated over $W[[T]]_P$ by the image $t(l)$ of $T(l)$ for $l \nmid Np$ and the image $u(q)$ of $U(q)$ for $q \mid Np$. Since $t(l) = \text{Tr}(\rho_{\mathbb{T}_P}(Frob_l)) = \text{Tr}(\tilde{\rho}(Frob_l))$, by Chebotarev density, to show (3) and (4), we need to see if $u(q)$ is contained in \mathbb{T}'_P . Thus we may assume that $u(q) \in \mathbb{T}_P^\times$. then, under the assumption $N = C(\rho_P)$, we have $H_0(I_q, \rho_{\mathbb{T}_P})$ has rank 1 on which ϕ_q acts by $u(q)$ (e.g. [GME] Theorem 4.2.7 (2–3)).

Suppose $q \neq p$, and take any arithmetic prime P' of $\text{Spec}(\mathbb{T})$. Then, because of $u(q) \in \mathbb{T}^\times$, the local p -component of the automorphic representation $\pi_{P'}$ generated by $f_{P'}$ is either a Steinberg representation or in the principal series of the form $\pi(\alpha, \beta)$ with β unramified at q . In the Steinberg case, as $u(q) \not\equiv 0 \pmod{P'}$, χ_1 is unramified at q , and the q divides N exactly once. Then for any other arithmetic point P'' of $\text{Spec}(\mathbb{T})$, $\pi_{P''}$ is Steinberg at q and we have the identity $C_q(\rho_{P''}) = q$. We conclude that either the local component of $\pi_{P''}$ at p is Steinberg for all arithmetic $P'' \in \text{Spec}(\mathbb{T})$ (Steinberg case) or in principal series for all arithmetic $P'' \in \text{Spec}(\mathbb{T})$ (Principal case).

In the Steinberg case, we may write $\rho_{\mathbb{T}|D_q} \cong \begin{pmatrix} \epsilon & * \\ 0 & \delta \end{pmatrix}$ with $\epsilon/\delta = \mathcal{N}_q$ for the cyclotomic character $\mathcal{N}_q : \text{Gal}(\overline{\mathbb{Q}}_q/\mathbb{Q}_q) \rightarrow \mathbb{Z}_p^\times$. Thus we have $\kappa = \mathcal{N}_q \delta^2$ and $\epsilon/\delta(\phi_q) = q \neq 1$. Let $A := \tilde{\rho}(\phi_q) \langle \kappa \rangle^{-1/2}(\phi_q)$. Taking W sufficiently large (so that it contains \sqrt{q}), the operator A has two distinct eigenvalues $a = \sqrt{q}^{-1}, b = \sqrt{q}$ in W^\times . Note that $a \not\equiv b \pmod{P}$ (by the assumption that either $\kappa(P)$ has characteristic 0 or $p \nmid \varphi(N)$). In the principal case, since $u(q) \neq 0$, we may write $\rho_{\mathbb{T}|D_q} \cong \begin{pmatrix} \epsilon & 0 \\ 0 & \delta \end{pmatrix}$ with unramified δ , and $\epsilon|_{I_q} = \chi_1$ is non-trivial with conductor $\mathfrak{C}_q(\chi_1)$ dividing exactly N (by Lemma 10.2 (2) combined with [GME] Theorem 4.2.7 (3)). Thus we can find $\sigma \in I_q$ such that $\tilde{\rho}(\sigma)$ has two eigenvalues $a = 1, b$ in W^\times . Again we may assume $a \not\equiv b \pmod{P}$ by our assumption. Put $A := \tilde{\rho}(\sigma)$ in the principal series case. Write ρ'' for $\tilde{\rho} \otimes \langle \kappa \rangle^{-1/2}$ in the Steinberg case and for $\tilde{\rho}$ in the principal case. Now we argue in the two cases (the Steinberg case and the principal series case) at the same time. Take a \mathbb{T}'_P -free ρ'' -stable \mathbb{T}'_P -lattice $L(\rho'') \subset \rho''$. The matrix A acts on $L(\rho'')$ by two distinct eigenvalues a, b in W^\times with $a \not\equiv b \pmod{P}$. By adding “[a]”, we indicate the a -eigenspace of the operator A ; so, $L(\rho'')[a] \cong \mathbb{T}'_P \cong L(\rho'')[b]$. Then on the a -eigenspace $L(\rho'')[a] \cong \mathbb{T}'_P$, ϕ_q acts by $u(q) \langle \kappa \rangle^{-1/2}(\phi_q)$ in the Steinberg case and by $u(q)$ in the principal series case, and hence $u(q) \in \mathbb{T}'_P$. This shows (3).

It remains to prove (4). By (3), we have $\mathbb{T}_P = \mathbb{T}'_P[u(p)]$. Let $\mathbb{T}' \subset \mathbb{T}'_P$ be the p -profinite ring generated by the trace of $\rho_{\mathbb{T}_P}$ over $W[[T]]$. Now we have

$$\det(\rho_{\mathbb{T}_P})([p, \mathbb{Q}_p]) = \chi^{(p)}(p) \in W \subset \mathbb{T}'$$

as $\det(\rho_{\mathbb{T}_P})(\chi^{(p)})^{-1}$ factors through $\text{Gal}(\mathbb{Q}[\mu_{p^\infty}]/\mathbb{Q})$ in which $[p, \mathbb{Q}_p] = 1$. Thus we have $a := \text{Tr}(\rho_{\mathbb{T}_P}(\phi_p)) = u(p) + u(p)^{-1}\chi^{(p)}(p) \in \mathbb{T}' \subset \mathbb{T}'_P$, and $u(p)$ satisfies $X^2 - aX + \chi^{(p)}(p) = 0$. We conclude $u(p) \in \mathbb{T}'_P$ if $a^2 - 4\chi^{(p)}(p) = (u(p) - u(p)^{-1}\chi^{(p)}(p))^2$ is nonzero and a square in \mathbb{T}'_P . By [H11b], $u(p)$ is transcendental over W , and hence $a^2 - 4\chi^{(p)}(p) \neq 0$ always.

First assume $a^2 - 4\chi^{(p)}(p) \not\equiv 0 \pmod{P}$ and $\kappa(P)$ has characteristic 0. Since $u(p) \in Q(\mathbb{T}'_P)$, $u(p)$ is in the integral closure $\tilde{\mathbb{T}}'_P$ of \mathbb{T}'_P in $Q(\mathbb{T}'_P)$. Since $\tilde{\mathbb{T}}'_P/\mathbb{T}'_P$ is a torsion \mathbb{T}'_P -module of finite type, the support of $\tilde{\mathbb{T}}'_P/\mathbb{T}'_P$ in $\text{Spec}(\mathbb{T}'_P)$ is made up of only finitely many closed points. Thus by extending scalar, we may assume that $\tilde{\mathbb{T}}'_P/\tilde{P}' = K$ for $K = Q(W)$ for all maximal ideals \tilde{P}' of $\tilde{\mathbb{T}}'_P$ in the support of $\tilde{\mathbb{T}}'_P/\mathbb{T}'_P$. In other words, for any prime $\tilde{P}' \subset \tilde{\mathbb{T}}'_P$ over P' , $\tilde{\mathbb{T}}'_P/\tilde{P}' = \mathbb{T}'_P/P' = K$, and in particular, $\kappa(P) = \kappa(\tilde{P}')$. Let α be the image of $u(p) - u(p)^{-1}\chi^{(p)}(p)$ in $\kappa(P)$. By our assumption, $\alpha \in W \subset \kappa(P)$. Then regard $\alpha \in W \subset \mathbb{T}'$ and consider $\alpha^{-2}(u(p) - u(p)^{-1}\chi^{(p)}(p))^2 = \alpha^{-2}(a^2 - 4\chi^{(p)}(p)) \in \mathbb{T}'$, which is in $1 + (P \cap \mathbb{T}')$. Since $p > 2$, $1 + (P \cap \mathbb{T}')$ is p -profinite, and $(1 + (P \cap \mathbb{T}'))^2 = 1 + (P \cap \mathbb{T}')$. Thus $a^2 - 4\chi^{(p)}(p)$ is a square in \mathbb{T}'_P , which implies $u(p) \in \mathbb{T}'_P$.

Now assume that $a^2 - 4\chi^{(p)}(p) \equiv 0 \pmod{P}$. If $T \notin P$, by (Gal), we find $\sigma \in I_p$ such that the eigenvalue of $\tilde{\rho}(\sigma)$ is 1 and $z \in W^\times$ with $z \not\equiv 1 \pmod{P}$. Then if W is sufficiently large containing $z \pmod{P}$ in $\kappa(P)$, we can split the $\tilde{\rho}$ -representation module $(\mathbb{T}'_P)^2$ into the product of two eigenspaces of $\tilde{\rho}(\sigma)$. We have eigenspace decomposition $L(\tilde{\rho}) = L(\tilde{\rho})[1] \oplus L(\tilde{\rho})[z]$ under $\tilde{\rho}(\sigma)$. Then $u(p)$ acts on $L(\tilde{\rho})[1] = H^0(I_p, L(\tilde{\rho})) \cong \mathbb{T}'_P$ as a \mathbb{T}'_P -linear operator (the action of $Frob_p$); so, $u(p) \in \mathbb{T}'_P$.

If $\chi_1|_{\mathbb{Z}_p^\times}$ is non-trivial, we can find again $\sigma \in I_p$ such that the eigenvalue of $\tilde{\rho}(\sigma)$ is $a = 1$ and $b \in W^\times$ with $a \not\equiv b \pmod{P}$. Then under the notation introduced in the proof for $q \neq p$, we have

$L(\widehat{\rho})[a] \cong \mathbb{T}'_p \cong L(\widehat{\rho})[b]$. Since $u(p)$ is the eigenvalue of $Frob_p$ on $L(\rho')[a]$, we get $u(p) \in \mathbb{T}'_p$. This finishes the proof of the last assertion (4). \square

We will identify in Section 7 the characteristic ideal of the congruence module between the CM component $\text{Spec}(\mathbb{T}_{cm}) \subset \text{Spec}(\mathbb{T})$ and its complement with the ideal generated by the anticyclotomic Katz measure in [K] and [HT] (interpolating anticyclotomic Hecke L -values). Since the anticyclotomic Katz measure is a measure on the anticyclotomic class group, we need to relate class group $Z := Cl_M(\mathfrak{C}p^\infty)$ and its anticyclotomic counter part $Cl_M^-(\mathfrak{c}'p^\infty)$ ($\mathfrak{c}' = \mathfrak{C} \cap \overline{\mathfrak{C}}$). This is what we do now. Consider the ray class group $Cl_M(\mathfrak{C}p^r)$ modulo $\mathfrak{C}p^r$, and put

$$(5.3) \quad Z = \varprojlim_r Cl_M(\mathfrak{C}p^r), \quad \text{and} \quad \mathfrak{Z} = \varprojlim_r Cl_M(\mathfrak{c}'p^r).$$

On \mathfrak{Z} , complex conjugation c acts as an involution.

Let Z_p (resp. \mathfrak{Z}_p) be the Sylow p -part of Z (resp. \mathfrak{Z}). We have a natural inclusion $(\mathfrak{O}_p^\times \times \mathfrak{O}_{\overline{p}}^\times) / \mathfrak{O}^\times$ into \mathfrak{Z} . Let $Z^- = \mathfrak{Z} / \mathfrak{Z}^{1+c}$ (the maximal quotient on which c acts by -1). Then we have the two projections

$$\pi : \mathfrak{Z} \twoheadrightarrow Z \quad \text{and} \quad \pi^- : \mathfrak{Z} \rightarrow Z^-.$$

The projection π^- induces an isogeny $\mathfrak{Z}^{1-c} = \{zz^{-c} | z \in \mathfrak{Z}\} \rightarrow Z^-$ whose kernel and cokernel are killed by 2. In particular, assuming $p > 2$, π^- induces an isomorphism between the maximal p -profinite subgroups $Z_p^- \subset Z^-$ and $\mathfrak{Z}_p^{1-c} \subset \mathfrak{Z}^{1-c}$; namely, we have $\pi^- : \mathfrak{Z}_p^{1-c} \cong Z_p^-$ if $p > 2$. Similarly, π induces $\pi : \mathfrak{Z}_p^{1-c} \cong Z_p$ if $p > 2$. Assume now $p > 2$. Thus we have $\iota : Z_p \cong Z_p^-$ by first lifting $z \in Z_p$ to $\tilde{z} \in \mathfrak{Z}_p^{1-c}$ and taking its square root and then project down to $\pi^-(\tilde{z}^{1/2})$. The isomorphism ι identifies the maximal torsion free quotients of the two groups Z_p and Z_p^- which we write as Γ_M . This ι also induces W -algebra isomorphism $W[[Z_p]] \cong W[[Z_p^-]]$ which is again written by ι . Then we have $Z = Z^{(p)} \times Z_p$ with finite group $Z^{(p)}$ of order prime to p . Identify $Z_p = \text{Gal}(M_p/M)$ (resp. $Z^{(p)} = \text{Gal}(M_Z^{(p)}/M)$) for an abelian extension M_p/M (resp. $M_Z^{(p)}/M$) by the Artin symbol.

Lemma 5.5. *The algebra $W[[Z_p]]$ is a local complete intersection over Λ_W ; in particular, it is a Gorenstein ring.*

Proof. The natural map $\Gamma \subset \mathbb{Z}_p^\times \rightarrow Z$ induces a $W[[\Gamma]]$ -algebra structure on $W[[Z_p]]$. Identifying $W[[\Gamma]]$ with Λ_W by $\gamma \mapsto 1 + T$, we regard $W[[Z_p]]$ as a Λ_W -algebra. Writing Z_p/Γ as a product of cyclic groups $C_1 \times \cdots \times C_r$ with $|C_j| = q_j$ for a p -power q_j and picking $z_j \in Z_p$ whose image generates C_j , we have $z_j^{q_j} \in \Gamma$ which we regard as an element $[z_j^{q_j}]$ of $\Gamma \subset W[[\Gamma]] = \Lambda_W$. Then we have an isomorphism $\Lambda_W[T_1, \dots, T_r] / ((1 + T_j)^{q_j} - [z_j^{q_j}])_j$ for the polynomial ring $\Lambda_W[T_1, \dots, T_r]$ and its ideal $((1 + T_j)^{q_j} - [z_j^{q_j}])_j$ generated by $(1 + T_j)^{q_j} - [z_j^{q_j}] \in \Lambda_W[T_1, \dots, T_r]$ for $j = 1, 2, \dots, r$. This shows that $W[[Z_p]]$ is a p -profinite local complete intersection over the regular ring Λ_W (and is hence a Gorenstein Λ_W -algebra; see [CRT] Theorem 21.3). \square

We regard $W[[Z_p^-]]$ as a Λ_W -algebra by the isomorphism ι . Let \mathbb{T} be a connected component of \mathfrak{h} containing a CM component \mathbb{J} with $C(\rho_{\mathbb{J}}) = N$. Recall the character $\Psi_{\mathbb{J}} : \text{Gal}(\overline{\mathbb{Q}}/M) \rightarrow \widetilde{\mathbb{J}}^\times$ as in (CM2) in Proposition 5.1. By class field theory, we may regard $\Psi_{\mathbb{J}}$ as a character $\Psi_{\mathbb{J}} : Z \rightarrow \widetilde{\mathbb{J}}^\times$. Taking W sufficiently large so that $W = \widetilde{\mathbb{J}} \cap \overline{\mathbb{Q}}_p$. Then $\Psi_{\mathbb{J}}|_{Z^{(p)}}$ has values in $W^\times \subset \widetilde{\mathbb{J}}^\times$. Define $\Psi_{\mathbb{T}} : \text{Gal}(\overline{\mathbb{Q}}/M) \rightarrow W[[Z_p]]^\times$ by a homomorphism given by $\Psi_{\mathbb{T}}(\sigma) = \Psi_{\mathbb{J}}(\sigma|_{M_Z^{(p)}})\sigma|_{M_p} \in W[[Z_p]]$, where we regard $\Psi_{\mathbb{J}}(\sigma|_{M_Z^{(p)}}) \in W \subset W[[Z_p]]$. Then we define $\Psi'_{\mathbb{T}} : \text{Gal}(\overline{\mathbb{Q}}/M) \rightarrow W[[Z_p^-]]$ by $\Psi'_{\mathbb{T}} = \iota \circ \Psi_{\mathbb{T}}$. By the above Lemma 5.4 combined with [H86a] §7 (or [LFE] §7.6), there exist algebra homomorphisms $\Theta : \mathbb{T} \rightarrow W[[Z_p]]$ and $\Theta^- : \mathbb{T} \rightarrow W[[Z_p^-]]$ given by $\Theta(\text{Tr}(\rho_{\mathbb{T}}(Frob_l))) = \text{Tr}(\text{Ind}_M^{\mathbb{Q}} \Psi_{\mathbb{T}}(Frob_l))$ and $\Theta^-(\text{Tr}(\rho'_{\mathbb{T}}(Frob_l))) = \text{Tr}(\text{Ind}_M^{\mathbb{Q}} \Psi'_{\mathbb{T}}(Frob_l))$ for all primes l outside $N(\mathfrak{C})p$, where $\rho'_{\mathbb{T}} = \rho_{\mathbb{T}} \otimes \sqrt{\kappa}^{-1}$ as in Lemma 5.4. The above identities uniquely determines these homomorphisms by Lemma 5.4 (2). We check that Θ (and hence Θ^-) is a Λ_W -algebra homomorphism. We summarize what we remarked:

Lemma 5.6. *Let the notation be as above, and assume $p > 2$. Then*

- (1) $\Theta^- \circ \rho'_{\mathbb{T}} \cong \text{Ind}_M^{\mathbb{Q}} \Psi'_{\mathbb{T}}$ over $Q(\mathbb{T})$.

- (2) $\iota : Z_p \cong Z_p^-$ canonically,
(3) If $N = C(\rho_{\mathbb{J}})$ for a CM component \mathbb{J} , we have the following commutative diagram of Λ_W -algebra homomorphisms:

$$\begin{array}{ccc} \mathbb{T} & \xrightarrow{\Theta} & W[[Z_p]] \\ \parallel \downarrow & & \downarrow \iota \\ \mathbb{T} & \xrightarrow{\Theta^-} & W[[Z_p^-]]. \end{array}$$

Proof. The proof is essentially given above the lemma. Only fact we need to verify is the identity: $\rho'_{\mathbb{T}} \cong \text{Ind}_M^{\mathbb{Q}} \Psi'_{\mathbb{T}}$ over $Q(\mathbb{T})$. Let $\langle \Psi' \rangle$ be the projection of $\Psi'_{\mathbb{T}}$ to the p -profinite part of the image $\text{Im}(\Psi'_{\mathbb{T}})$. Since $\rho_{\mathbb{T}}|_{\text{Gal}(\overline{\mathbb{Q}}/M)} = \text{Ind}_M^{\mathbb{Q}} \Psi_{\mathbb{T}}|_{\text{Gal}(\overline{\mathbb{Q}}/M)} = \Psi \oplus \Psi^c$, we have $\kappa := \det(\rho_{\mathbb{T}}) = \det(\text{Ind}_M^{\mathbb{Q}} \Psi_{\mathbb{T}}) = \Psi^{1+c}$ over $\text{Gal}(\overline{\mathbb{Q}}/M)$, where $\Psi^{1+c}(\sigma) = \Psi(\sigma c \sigma^{-1})$. Thus $\langle \kappa \rangle|_{\text{Gal}(\overline{\mathbb{Q}}/M)}$ is equal to $\langle \Psi_{\mathbb{T}} \rangle^{1+c}$. Since $\rho'_{\mathbb{T}} = \rho_{\mathbb{T}} \otimes \sqrt{\langle \kappa \rangle}^{-1}$, we have

$$\Theta^- \circ \rho'_{\mathbb{T}} \cong \text{Ind}_M^{\mathbb{Q}} (\Psi_{\mathbb{T}} \langle \Psi_{\mathbb{T}} \rangle^{-(1+c)/2}) = \text{Ind}_M^{\mathbb{Q}} (\psi_{\mathbb{T}} \cdot \langle \Psi_{\mathbb{T}} \rangle^{(1-c)/2}),$$

where $\psi_{\mathbb{T}} = \Psi_{\mathbb{T}} / \langle \Psi_{\mathbb{T}} \rangle$ (the prime-to- p part of $\Psi_{\mathbb{T}}$). By the construction of ι and the definition $\Psi'_{\mathbb{T}} = \iota \circ \Psi_{\mathbb{T}}$, we confirm $\Psi'_{\mathbb{T}} = \psi_{\mathbb{T}} \cdot \langle \Psi_{\mathbb{T}} \rangle^{(1-c)/2}$. \square

Fix a CM irreducible component $\text{Spec}(\mathbb{J})$ of $\text{Spec}(\mathbf{h})$, and let $\text{Spec}(\mathbb{T})$ be the connected component of $\text{Spec}(\mathbf{h})$ containing $\text{Spec}(\mathbb{J})$. Let \mathfrak{C} be the prime-to- p conductor of the associated character $\Psi_{\mathbb{J}}$. Regard the character $\Psi_{\mathbb{J}}$ as \mathbb{Z}_p -algebra homomorphism of $\mathbb{Z}_p[[Z]]$ into \mathbb{J} . Then the algebra homomorphism $\Psi_{\mathbb{J}}$ restricted to $\mathbb{Z}_p[Z^{(p)}]$ has values in $\mathbb{J} \cap \overline{\mathbb{Q}}_p$ that is a discrete valuation ring finite flat over \mathbb{Z}_p . By extending scalar, we assume $\overline{\mathbb{Q}}_p \cap \mathbb{J} = W$.

Proposition 5.7. *Let \mathbb{J} be a CM component $\text{Spec}(\mathbb{J}) \subset \text{Spec}(\mathbb{T}_{cm})$, and let $\overline{\psi} = \Psi_{\mathbb{J}} \bmod \mathfrak{m}_{\mathbb{J}}$. Assume $p > 2$ and the following two conditions:*

- (i) $\overline{\psi}^-$ has order > 2 , and $\overline{\psi}$ is ramified at \mathfrak{p} (and unramified at \mathfrak{p}^c) with prime-to- p conductor \mathfrak{C} ,
(ii) $C(\overline{\rho}) = N$ for $\overline{\rho} = \rho_{\mathbb{J}} \bmod \mathfrak{m}_{\mathbb{J}} = \text{Ind}_M^{\mathbb{Q}} \overline{\psi}$.

Then we have

- (1) \mathbb{T} is a Gorenstein ring, and \mathbb{T}_{cm} is a local complete intersection canonically isomorphic to $W[[Z_p]]$ for the maximal p -profinite quotient Z_p of $Z = \varprojlim_n Cl_M(\mathfrak{C}\mathfrak{p}^n)$.
(2) Writing $\rho_{\mathbb{T}_{cm}} \cong \text{Ind}_M^{\mathbb{Q}} \psi$ (resp. $\overline{\rho} = \text{Ind}_M^{\mathbb{Q}} \overline{\psi}$) for a character $\psi : \text{Gal}(\overline{\mathbb{Q}}/M) \rightarrow W[[Z_p]]^{\times}$ (resp. $\overline{\psi} : \text{Gal}(\overline{\mathbb{Q}}/M) \rightarrow (W/\mathfrak{m}_W)^{\times}$), the ring \mathbb{T}_{cm} with universal character ψ is isomorphic to the universal deformation ring of $\overline{\psi}$ over W for characters unramified outside $\mathfrak{C}\mathfrak{p}$.
(3) Each CM component \mathbb{J} of \mathbb{T} is canonically isomorphic to $W[[\Gamma_M]]$ and hence $\widetilde{\mathbb{J}} = \mathbb{J}$, where Γ_M is the maximal torsion-free quotient of Z .

See the proof below the meaning of the universal deformation ring.

Proof. If $\overline{\rho} = \text{Ind}_M^{\mathbb{Q}} \overline{\psi}$, $\overline{\rho}$ determines the pair of characters $\{\overline{\psi}, \overline{\psi}^c\}$ as $\overline{\rho}|_{\text{Gal}(\overline{\mathbb{Q}}/M)} = \overline{\psi} \oplus \overline{\psi}^c$. By (1) (and Proposition 5.2), $\overline{\rho}$ is absolutely irreducible and is not isomorphic to any induced representations from any other quadratic field. Since $C(\overline{\rho}) = N(\mathfrak{C}(\overline{\psi}))D$, $N = C(\overline{\rho})$ and $N = N(\mathfrak{C})D$ implies $\mathfrak{C}(\overline{\psi}) = \mathfrak{C}$.

Let $(R, \tilde{\psi} : \text{Gal}(\overline{\mathbb{Q}}/M) \rightarrow R^{\times})$ be the universal couple with the universal character unramified outside $\mathfrak{p}\mathfrak{C}$ deforming $\overline{\psi}$ over W . This couple $(R, \tilde{\psi})$ is characterized by the following universal property: For any local pro-artinian W -algebra A with residue field \mathbb{F} and any character $\varphi : \text{Gal}(\overline{\mathbb{Q}}/M) \rightarrow A^{\times}$ unramified outside $\mathfrak{p}\mathfrak{C}$ with $\varphi \bmod \mathfrak{m}_A = \overline{\psi}$ (for the maximal ideal \mathfrak{m}_A of A), there exists a unique W -algebra homomorphism $\iota : R \rightarrow A$ such that $\varphi = \iota \circ \tilde{\psi}$. Such a pair (A, φ) is called a deformation of $\overline{\psi}$ (see [M] for general theory of Galois deformation).

We now show $R \cong W[[Z_p]]$ by class field theory. To see this, we pick a deformation $\varphi : \text{Gal}(\overline{\mathbb{Q}}/M) \rightarrow A^{\times}$ of $\overline{\psi}$ unramified outside $\mathfrak{p}\mathfrak{C}$; thus, A is a local artinian W -algebra sharing the residue field \mathbb{F} with W and $\varphi \bmod \mathfrak{m}_A = \overline{\psi}$ for the maximal ideal \mathfrak{m}_A of A . Let ψ be the Teichmüller

lift of $\bar{\psi}$; so, $\varphi' = \varphi\psi^{-1}$ has p -power order. For a prime $l \mid \mathfrak{C}$, by class field theory, the image I_l^{ab} of the inertia group $I_l \subset \text{Gal}(\bar{\mathbb{Q}}/M)$ in the Galois group of the maximal abelian extension of M over M is isomorphic to the multiplicative group \mathfrak{O}_l^\times of the l -adic integer ring of M_l . Since φ' has p -power order and $p \neq l$, φ' must be trivial on $1 + \mathfrak{O}_l \subset \mathfrak{O}_l^\times$. Thus l -conductor of φ' is at most l , and hence $\varphi = \varphi'\psi$ factors through Z . Thus φ' factors through the maximal p -profinite quotient Z_p and extends to a unique W -algebra homomorphism $\iota = \iota_\varphi : W[[Z_p]] \rightarrow A$ such that $\iota|_{Z_p} = \varphi'$. Since Z_p is the maximal p -profinite quotient of Z , by class field theory, we have the corresponding subfield \widetilde{M} of the ray class field modulo $\mathfrak{p}^\infty \mathfrak{C}$ such that $\text{Gal}(\widetilde{M}/M) \cong Z_p$ by Artin symbol. Writing the inclusion $Z_p \subset W[[Z_p]]$ as $\gamma \mapsto [\gamma]$ and identifying $\text{Gal}(\widetilde{M}/M) = Z_p$, define a character $\psi : \text{Gal}(\bar{\mathbb{Q}}/M) \rightarrow W[[Z_p]]$ by $\psi(\sigma) = \psi(\sigma)[\sigma|_{\widetilde{M}}]$. Then by our construction $\iota \circ \psi = \varphi$; so, $(W[[Z_p]], \psi)$ satisfies the universal property of $(R, \tilde{\psi})$ for deformations φ of $\bar{\psi}$.

For an ideal \mathfrak{a} of \mathbb{T} , write $\rho_{\mathfrak{a}} = \rho_{\mathbb{T}} \bmod \mathfrak{a}$ by abusing the symbol slightly. If $\rho_{\mathfrak{a}} \cong \text{Ind}_M^{\mathbb{Q}} \psi'$ for a character $\psi' : \text{Gal}(\bar{\mathbb{Q}}/M) \rightarrow A^\times$ for a local ring A containing \mathbb{T}/\mathfrak{a} , ψ' has values in $(\mathbb{T}/\mathfrak{a})^\times$ (by (i) and Hensel's lemma). Then we have $C(\bar{\rho})|C(\rho_{\mathfrak{a}})|C(\rho_{\mathbb{T}}) = C(\rho_{\mathbb{J}}) = C(\bar{\rho})$ by (ii). Thus $C(\bar{\rho}) = C(\rho_{\mathfrak{a}})$. Write \mathfrak{C}' for the prime-to- p conductor of ψ' . Then $N(\mathfrak{C}')D = C(\rho_{\mathfrak{a}}) = C(\bar{\rho}) = N(\mathfrak{C})D$. One of ψ' or ψ'^c must be a deformation of $\bar{\psi}$, and one of them ramifies at \mathfrak{p} . Let ψ' be the character ramifying at \mathfrak{p} . Then ramification of ψ' at \mathfrak{p} forces $\psi' \equiv \bar{\psi} \bmod \mathfrak{m}$, as $\bar{\psi}$ is the unique choice ramifying at \mathfrak{p} . Then we find $\mathfrak{C} = \mathfrak{C}(\bar{\psi})|\mathfrak{C}'$; so, $\mathfrak{C} = \mathfrak{C}'$. Thus ψ' factors through Z . By Proposition 5.2, (i) implies that the local ring of \mathbb{T} contains CM components of a single imaginary quadratic field M deforming $\bar{\psi}$. This shows that the reduced part \mathbb{T}_{cm}^{red} of \mathbb{T}_{cm} is the surjective image of $W[[Z_p]]$ for a canonical morphism $\pi : W[[Z_p]] \rightarrow \mathbb{T}_{cm}^{red}$ with $\text{Ind}_M^{\mathbb{Q}}(\pi \circ \psi) \cong \rho_{\mathbb{T}_{cm}^{red}}$. Since $N = C(\bar{\rho})$, \mathbb{T} kills any old forms of level N and hence \mathbb{T} is reduced (by Lemma 5.4 (1)). Thus $\text{Spec}(\mathbb{T}_{cm})$ is reduced, and hence \mathbb{T}_{cm} is the surjective image of $W[[Z_p]]$ under π .

Pick an irreducible component $\text{Spec}(\mathbb{J}) \subset \text{Spec}(W[[Z_p]])$. Then we have a continuous character $\Psi_{\mathbb{J}} : \text{Gal}(\bar{\mathbb{Q}}/M) \rightarrow \mathbb{J}^\times$ with $\Psi_{\mathbb{J}} \equiv \bar{\psi} \bmod \mathfrak{m}_{\mathbb{J}}$ such that $\rho_{\mathbb{J}} \cong \text{Ind}_M^{\mathbb{Q}} \Psi_{\mathbb{J}}$. From $\mathfrak{C} = \mathfrak{C}(\bar{\psi})|\mathfrak{C}(\Psi_{\mathbb{J}})|\mathfrak{C}$, we conclude $\mathfrak{C}(\Psi_{\mathbb{J}}) = \mathfrak{C}$. Thus ramification of $\Psi_{\mathbb{J}}$ is completely determined by $\bar{\psi}$; so, we have $W[[T]]$ -algebra homomorphism $\Theta : \mathbb{T} \rightarrow \mathbb{J}$ associated to $\Psi_{\mathbb{J}}$. Since Θ gives rise to a CM component, it factors through \mathbb{T}_{cm} and makes the following diagram commutative:

$$\begin{array}{ccc} \mathbb{T}_{cm} & \xrightarrow{\Theta} & \mathbb{J} \\ \uparrow & & \parallel \uparrow \\ W[[Z_p]] & \longrightarrow & \mathbb{J}. \end{array}$$

Thus $W[[Z_p]] \rightarrow \mathbb{T}_{cm}$ is nontrivial over all irreducible components of $\text{Spec}(W[[Z_p]])$; so, it is injective, and $\rho_{\mathbb{T}_{cm}} \cong \text{Ind}_M^{\mathbb{Q}} \psi$. This proves the assertion (2).

By Lemma 5.5, $\mathbb{T}_{cm} \cong W[[Z_p]]$ is a complete intersection. Each irreducible component of $\text{Spec}(W[[Z_p]])$ is given by $\text{Spec}(W[[\Gamma_M]])$, and hence any CM component of \mathbb{T} is canonically isomorphic to $W[[\Gamma_M]]$. Since $W[[\Gamma_M]]$ is integrally closed, we have $\widehat{\mathbb{J}} = \mathbb{J}$. This proves (3).

Taking inertia group $I_p = I_{\mathfrak{p}}$, Gorenstein-ness of \mathbb{T} follows Theorem 7.1 in the following section as $\bar{\rho}$ is absolutely irreducible and $\bar{\rho}|_{I_p} \cong \bar{\psi}|_{I_p} \oplus \bar{\psi}^c|_{I_p}$ with $\bar{\psi}$ ramified at \mathfrak{p} and $\bar{\psi}^c$ unramified at \mathfrak{p} . This finishes the proof of (1). \square

6. p -ADIC HECKE L -FUNCTIONS

In this section, we assume that W contains a Witt vector ring $W(\bar{\mathbb{F}}_p)$ for an algebraic closure $\bar{\mathbb{F}}_p$ of \mathbb{F}_p ; so, $\mathbb{F} = \bar{\mathbb{F}}_p$ in this section. We recall Katz's theory in [K] (and [HT]) of p -adic L -function. We fix a prime-to- p conductor ideal \mathfrak{C} of an imaginary quadratic field $M \subset \bar{\mathbb{Q}}$ in which p splits into $(p) = \mathfrak{p}\bar{\mathfrak{p}}$ ($\bar{\mathfrak{p}} = \mathfrak{p}^c$ for the generator c of $\text{Gal}(M/\mathbb{Q})$) for $\mathfrak{p} = \{\alpha \in \mathfrak{O} \mid |i_p(\alpha)|_p < 1\}$. We write the embedding $M \subset \bar{\mathbb{Q}}$ as $i : M \hookrightarrow \bar{\mathbb{Q}}$.

Let $\lambda : M_{\mathbb{A}}^\times/M^\times \rightarrow \mathbb{C}^\times$ be a type A_0 Hecke character (of conductor $\mathfrak{C}(\lambda)|\mathfrak{C}p^\infty$). Then λ has values in $\bar{\mathbb{Q}}$ on the finite part $M_{\mathbb{A}(\infty)}^\times$ of $M_{\mathbb{A}}^\times$. For the ray-class group \mathfrak{Z} modulo $\mathfrak{C}p^\infty$ of M , write $\hat{\lambda} : \mathfrak{Z} \rightarrow \bar{\mathbb{Q}}_p^\times$ for the p -adic avatar of λ . Let $-D$ be the discriminant of M so that $M = \mathbb{Q}[\sqrt{-D}]$, and

put $2\delta = \sqrt{-D}$. The alternating form $(x, y) = \text{Tr}_{M/\mathbb{Q}}(xy^c/\sqrt{-D})$ induces the principal polarization on the elliptic curve $E(\mathfrak{D})$ defined over $\mathcal{W} = i_p^{-1}(W)$ with complex multiplication by \mathfrak{D} with complex uniformization $E(\mathfrak{D})(\mathbb{C}) \cong \mathbb{C}/\mathfrak{D}$. A choice of Néron differential on $E(\mathfrak{D})/\mathcal{W}$ produces its complex period and p -adic period $(\Omega_\infty, \Omega_p) \in (\mathbb{C}^\times \times W^\times)$. Katz constructed in [K] (see also [HT] where the case $\mathfrak{C} \neq 1$ is treated) a measure φ with values in W on the ray-class group \mathfrak{Z} modulo $\mathfrak{C}p^\infty$ characterized by the following formula:

$$(6.1) \quad i_p^{-1} \left(\frac{\int_{\mathfrak{Z}} \widehat{\lambda} d\varphi}{\Omega_p^{k+2\kappa}} \right) = (\mathfrak{D}^\times : \mathbb{Z}^\times) \frac{c(\lambda)\pi^\kappa L(0, \lambda)}{\text{Im}(\delta)^\kappa \Omega_\infty^{k+2\kappa}} (1 - \lambda(\mathfrak{p}))(1 - \lambda(\overline{\mathfrak{p}})N(\mathfrak{p})^{-1}) \prod_{\mathfrak{L}|\mathfrak{C}} (1 - \lambda(\mathfrak{L})) \in W$$

for all Hecke characters λ modulo $\mathfrak{C}p^\infty$. Here $L(s, \lambda)$ is the primitive complex L -function of λ , and we use the convention that $\lambda(\mathfrak{L}) = 0$ for a prime \mathfrak{L} (of M) if \mathfrak{L} divides the conductor of λ , and if \mathfrak{L} is prime to the conductor of λ , $\lambda(\mathfrak{L})$ is the value of the primitive character associated to λ . Here the infinity type of λ is $ki + \kappa(i - c)$ for an integer k and κ with integer κ satisfying either $k > 0$ and $\kappa \geq 0$ or $k \leq 1$ and $\kappa \geq 1 - k$, $c(\lambda) \neq 0$ is a simple algebraic constant involving the root number of λ and the value of its Γ -factor as specified in [HT] Theorem 4.1. Identifying $W[[\mathfrak{Z}]]$ with the measure algebra under convolution product, we may regard $\varphi \in W[[\mathfrak{Z}]]$. Strictly speaking, the measure φ slightly depends on a choice of \mathfrak{F} in the following decomposition:

Definition 6.1. *We decompose \mathfrak{C} into a product $\mathfrak{F}\mathfrak{F}_c\mathfrak{I}$ such that \mathfrak{I} is a product of inert and ramified primes over \mathbb{Q} and $\mathfrak{F}\mathfrak{F}_c$ for a product of primes split over \mathbb{Q} with $\mathfrak{F} \subset \mathfrak{F}_c^c$ and $\mathfrak{F} + \mathfrak{F}_c = \mathfrak{D}$.*

By the interpolation formula (6.1) and the description of $c(\lambda)$ in [HT] Theorem 4.1, the measure is independent of \mathfrak{F} up to units in $W[[\mathfrak{Z}]]$.

Fix a CM component \mathbb{J} of \mathbf{h}_{cm}^M . Since we work under the assumptions of Proposition 5.7, we have $\widetilde{\mathbb{J}} = \mathbb{J}$. Then the associated character $\Psi_{\mathbb{J}}$ has values in \mathbb{J}^\times . Take its anticyclotomic projection $\Psi_{\mathbb{J}^-}$, and write \mathfrak{C} for the conductor of $\Psi_{\mathbb{J}^-}$, we may regard $\Psi_{\mathbb{J}^-}$ as a character $\Psi_{\mathbb{J}^-} : Z^- \rightarrow \mathbb{J}^\times$, which induces W -algebra homomorphism $\Psi_{\mathbb{J}^-} : W[[Z^-]] \rightarrow \mathbb{J}$. We then write $L_p(\Psi_{\mathbb{J}^-}) \in \mathbb{J}$ for the image under $\Psi_{\mathbb{J}^-} : W[[Z^-]] \rightarrow \mathbb{J}$ of $\varphi^- = \pi_*^-(\varphi)$. Decompose $Z^- = \Delta^- \times \Gamma_M^-$ for the maximal finite subgroup Δ^- and the maximal torsion-free quotient Γ_M^- . Via $\iota : Z_p \cong Z_p^-$, we may identify $\Gamma_M = \Gamma_M^-$. Then by this projection $\Psi_{\mathbb{J}^-} : W[[Z^-]] \rightarrow \mathbb{J}$, we may identify $\mathbb{J} = W[[\Gamma_M^-]] = W[[\Gamma_M]]$, and in this sense, $L_p(\Psi_{\mathbb{J}^-})$ is a branch of the anticyclotomic Katz measure $\pi_*^-(\varphi) = \varphi^- \in W[[Z^-]]$. We have a canonical decomposition $Z^- = Z_p^{(p)-} \times Z_p^-$ for the maximal finite subgroup $Z_p^{(p)-}$ of order prime to p . If we fix a character $\overline{\psi}^- : Z^- \rightarrow \mathbb{F}^\times$, its Teichmüller lift $\psi^- : Z^- \rightarrow W^\times$ factors through $Z^-/Z_p^- \cong Z_p^{(p)-}$. Then we have a ψ^- -projection $\pi_{\psi^-}^- : W[[Z^-]] \rightarrow W[[Z_p^-]]$ sending $(z^{(p)}, z_p) \in Z^- \subset W[[Z^-]]^\times$ to $\psi^-(z^{(p)})z_p \in W[[Z_p^-]]$. Then we put $L^-(\overline{\psi}^-) = \pi_{\psi^-}^- \circ \varphi^- = \pi_{\psi^-, *}\varphi^- \in W[[Z_p^-]] \xrightarrow{\iota^{-1}} W[[Z_p]]$. The projection of $L^-(\overline{\psi}^-)$ to each irreducible component \mathbb{J} of $\mathbb{T}_{cm}^M \cong W[[Z_p]] \cong W[[Z_p^-]]$ gives rise to $L_p(\Psi_{\mathbb{J}^-}) \in \mathbb{J}$.

7. CONGRUENCE MODULES

Let $\text{Spec}(\mathbb{T})$ be a reduced connected component of $\text{Spec}(\mathbf{h})$. Write $\rho_{\mathbb{T}} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow GL_2(Q(\mathbb{T}))$ for the Galois representation associated to this component. We quote the following result from [H12a] Theorem 4.1, which is essentially proven in [MW] Proposition 2 in §9:

Theorem 7.1. *Let \mathfrak{P} be a prime ideal in $\text{Spec}(\mathbb{T})$. If $\rho_{\mathfrak{P}}$ is absolutely irreducible and $\rho_{\mathfrak{P}}|_{I_p} \cong \begin{pmatrix} \delta & * \\ 0 & 1 \end{pmatrix}$ with $\delta \neq 1$ for the inertia group $I_p \subset \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ at p , then the localization $\mathbb{T}_{\mathfrak{P}}$ is a Gorenstein ring.*

Let J be as in the introduction (the ordinary part of the projective limit of the Tate modules of modular jacobians) on which \mathbf{h} acts, and write $J(\mathbb{T}) = \mathbb{T} \cdot J$. Then the connected-étale exact

sequence produces the following commutative diagram of exact rows:

$$\begin{array}{ccccc} J(\mathbb{T})^\circ & \xrightarrow{\hookrightarrow} & J(\mathbb{T}) & \xrightarrow{\twoheadrightarrow} & J(\mathbb{T})^{et} \\ \wr \downarrow & & \parallel \downarrow & & \wr \downarrow \\ \mathbb{T} & \xrightarrow{\hookrightarrow} & J(\mathbb{T}) & \xrightarrow{\twoheadrightarrow} & \mathrm{Hom}_\Lambda(J(\mathbb{T}), \Lambda). \end{array}$$

Here the vertical arrows are isomorphism of \mathbb{T} -modules. This is shown in [H86b] under the condition (R) in the introduction and in [O2] without assuming (R). Thus Gorenstein-ness of $\mathbb{T}_{\mathfrak{p}}$ implies freeness of $J(\mathbb{T})_{\mathfrak{p}}$ over $\mathbb{T}_{\mathfrak{p}}$. In particular, if (R) is satisfied and $\bar{\rho} = \rho_{\mathfrak{m}}$ is absolutely irreducible, $\mathcal{L}_{can}(\mathbb{I}) = J(\mathbb{T}) \otimes_{\mathbb{T}} \mathbb{I}$ is free of rank 2 as claimed in the introduction (so, in this case, (F_{can}) holds).

Let $\mathrm{Spec}(\mathbb{J}) \subset \mathrm{Spec}(\mathbf{h}_{cm}^M)$ be a CM irreducible component and $\mathrm{Spec}(\mathbb{T})$ be the connected component of $\mathrm{Spec}(\mathbf{h})$ with $\mathrm{Spec}(\mathbb{J}) \subset \mathrm{Spec}(\mathbb{T})$. Assume that $\mathrm{Spec}(\mathbb{T})$ is reduced, and write $\bar{\rho} = \rho_{\mathfrak{m}_{\mathbb{T}}} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow GL_2(\mathbb{F})$ for the mod p representation of the component \mathbb{T} . Then $\bar{\rho} \cong \mathrm{Ind}_M^{\mathbb{Q}} \bar{\psi}$ for a character $\bar{\psi} : \mathrm{Gal}(\overline{\mathbb{Q}}/M) \rightarrow \mathbb{F}^\times$. Let ψ be the Teichmüller lift of $\bar{\psi}$. Write $\mathfrak{C} = \mathfrak{C}(\Psi_{\mathbb{J}})$ for the prime-to- p conductor of the associated character $\Psi_{\mathbb{J}} : \mathrm{Gal}(\overline{\mathbb{Q}}/M) \rightarrow \tilde{\mathbb{J}}^\times$; so we assume $\bar{\psi} = (\Psi_{\mathbb{J}} \bmod \mathfrak{m}_{\mathbb{J}})$. Write $\mathrm{Spec}(\mathbb{T}_{cm}^M) = \mathrm{Spec}(\mathbf{h}_{cm}^M) \cap \mathrm{Spec}(\mathbb{T})$; so, $\mathrm{Spec}(\mathbb{T}_{cm}^M)$ is the minimal closed subscheme in $\mathrm{Spec}(\mathbb{T})$ containing all components with CM by M . We have therefore the projection maps

$$\mathbb{T} \twoheadrightarrow \mathbb{T}_{cm}^M \twoheadrightarrow \mathbb{J}$$

where all rings involved are Gorenstein rings if $\bar{\rho}$ is absolutely irreducible and $\bar{\psi}^-$ has order > 2 and is ramified at p (see Proposition 5.7). Recall S which is the set of split prime factors \mathfrak{q} in M of N but $\mathfrak{q} \nmid N(\mathfrak{C}(\Psi_{\mathbb{J}}^-))$. Consider

$$(7.1) \quad E_{1,N} = \prod_{\mathfrak{q} \in S} \{1 - \Psi_{\mathbb{J}}^-(\mathfrak{q})N(\mathfrak{q})^{-1}\}(1 - \Psi_{\mathbb{J}}^-(\bar{\mathfrak{q}})N(\bar{\mathfrak{q}})^{-1}) \in \mathbb{J} \text{ and } E_{1,N} = 1 \text{ if } S = \emptyset.$$

Note here that $E_{1,N}$ is the product of Euler factor at $\mathfrak{q} \in \Sigma_{\mathfrak{C}}$ of $L_p(\Psi_{\mathbb{J}}^-)$.

Hereafter in this section, we assume that $W \supset W(\overline{\mathbb{F}}_p)$ to have $L_p^-(\bar{\psi}^-) \in W[[Z_p^-]]$ as in Section 6.

Theorem 7.2. *Let the notation be as above. Suppose $W \supset W(\overline{\mathbb{F}}_p)$, $p \geq 5$, that $\mathrm{Spec}(\mathbb{T})$ contains a non-CM minimal primitive component $\mathrm{Spec}(\mathbb{I})$ and that $\bar{\rho} \cong \mathrm{Ind}_M^{\mathbb{Q}} \bar{\psi}$ for an imaginary quadratic field M in which p splits. Suppose further that $\bar{\psi}^-$ has order > 2 , $\bar{\psi}^-$ ramifies at \mathfrak{p} , and one of the following conditions:*

- (a) $p \nmid \varphi(N)$ and $C(\bar{\rho}) = N$;
- (b) $E_{1,N} \notin \mathfrak{m}_{\mathbb{J}}$ and $p \nmid \Phi(N)$ for the Euler function Φ of M (i.e., $\Phi(N) = N^2 \prod_{\mathfrak{q}|N} (1 - \frac{1}{N(\mathfrak{q})})$ for primes \mathfrak{q} in M).

Then \mathbb{T}_{cm}^M is canonically isomorphic to $W[[Z_p^-]]$ for the p -profinite part Z_p^- of the anticyclotomic ray class group of conductor $\mathfrak{C}(\bar{\psi}^-)p^\infty$. Writing $L^-(\bar{\psi}^-) \in W(\overline{\mathbb{F}}_p)[[Z_p^-]]$ for the anticyclotomic Katz measure of modulo p branch character $\bar{\psi}^-$ and $\mathrm{Spec}(\mathbb{T}_{cm}^\perp)$ for the complement of $\mathrm{Spec}(\mathbb{T}_{cm}^M)$ in $\mathrm{Spec}(\mathbb{T})$, we have $\mathbb{T}_{cm}^M \otimes_{\mathbb{T}} \mathbb{T}_{cm}^\perp \cong W[[Z_p^-]]/L^-(\bar{\psi}^-)W[[Z_p^-]]$.

Remark 7.3. We explain why we need to assume (a) or (b) in the above theorem. Since $C(\bar{\rho})|N$, by the existence of the Teichmüller lift of $\bar{\psi}$ and Galois deformation theory explained in Proposition 5.7 (2), \mathbb{T}_{cm}^M is non-trivial. Taking a component \mathbb{J} of \mathbb{T}_{cm}^M , the main reason for assuming (a) or (b) is to guarantee that $\rho_{\mathbb{J}}|_{\mathrm{Gal}(\overline{\mathbb{Q}}_q/\mathbb{Q}_q)}$ is minimal at primes $q|N$ split in M and that \mathbb{J} is primitive. In addition, the condition $E_{1,N} \notin \mathfrak{m}_{\mathbb{J}}$ in (b) (which is automatically satisfied under (a) as $E_{1,N} = 1$ in that case) is to guarantee that $\rho_{\mathbb{I}}|_{I_q}$ is never reducible indecomposable for any irreducible component $\mathrm{Spec}(\mathbb{I}')$ of $\mathrm{Spec}(\mathbb{T})$, where $I_q \subset \mathrm{Gal}(\overline{\mathbb{Q}}_q/\mathbb{Q}_q)$ is the inertia group.

Let us prove this fact. If ρ_P for $P \in \mathrm{Spec}(\mathbb{I}')$ is reducible indecomposable, as is well known (see Lemma 10.1 (4)), $\rho_P|_{\mathrm{Gal}(\overline{\mathbb{Q}}_q/\mathbb{Q}_q)}$ is isomorphic to $\begin{pmatrix} \eta^N & * \\ 0 & \eta \end{pmatrix}$ for the p -adic cyclotomic character \mathcal{N} acting on μ_{p^∞} (unramified at q). The character η restricted to I_q is of finite order (see Section 10). If η is ramified at q and $\eta|_{I_q} \not\equiv 1 \pmod{\mathfrak{m}_{\mathbb{I}'}}$, lifting $\eta \bmod \mathfrak{m}_{\mathbb{I}'}$ to a non-trivial character $\bar{\eta}$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$

with values in $\mathbb{T}/\mathfrak{m}_{\mathbb{T}}$ only ramified at q , the semi-simplification of $\bar{\rho} \otimes \bar{\eta}^{-1}$ is unramified; so, $\rho_{\mathbb{I}} \otimes \tilde{\eta}^{-1}$ has less conductor than $\rho_{\mathbb{I}}$ for the Teichmüller lift $\tilde{\eta}$ of $\bar{\eta}$. By the minimality of \mathbb{I} , this cannot happen; so, we conclude $\eta|_{I_q} \equiv 1 \pmod{\mathfrak{m}_{\mathbb{T}}}$. By local class field theory, we may regard $\eta|_{I_q}$ as a character of \mathbb{Z}_q^\times . Thus $\eta|_{I_q} \neq 1$ but $\bar{\eta}|_{I_q} = 1$ implies $q \equiv 1 \pmod{p}$, a contradiction against $p \nmid \varphi(N)$. Hence η is unramified. The \mathfrak{q} -factor or $\bar{\mathfrak{q}}$ -factor of $E_{1,N}$ is congruent to $1 - (\eta N / \eta)(\text{Frob}_q)q^{-1} = 0$ modulo $\mathfrak{m}_{\mathbb{J}}$. Since $\rho_{\mathbb{I}} \equiv \rho_{\mathbb{J}} \pmod{\mathfrak{m}_{\mathbb{T}}}$, we have $E_{1,N} \equiv 0 \pmod{\mathfrak{m}_{\mathbb{J}}}$ (contradicting against $E_{1,N} \notin \mathfrak{m}_{\mathbb{J}}$). Thus $\rho_P|_{I_q}$ for every prime $P \in \text{Spec}(\mathbb{T})$ is semi-simple for all primes $q|N$.

Once semi-simplicity of $\rho_P|_{I_q}$ is proven for all $q|N$, we can apply results in Section 10, and the following conditions for primes $q|N$ are equivalent:

- (1) $\rho_{\mathbb{I}}|_{\text{Gal}(\bar{\mathbb{Q}}_q/\mathbb{Q})}$ is absolutely irreducible.
- (2) $\bar{\rho}|_{\text{Gal}(\bar{\mathbb{Q}}_q/\mathbb{Q})}$ is absolutely irreducible.

Indeed, by Lemma 10.3 (2), under $p \nmid \varphi(N)$, (1) \Leftrightarrow (2) as $\bar{\rho} = \rho_{\mathbb{I}} \pmod{\mathfrak{m}_{\mathbb{I}}}$. Moreover from the minimality and primitiveness of $\rho_{\mathbb{I}}$, by Lemma 10.3 (4), under $p \nmid \Phi(N)$, $C(\bar{\rho}) = C(\rho_{\mathbb{I}}) = N$; thus (b) \Rightarrow (a). If $N = C(\bar{\rho})$, by Lemma 5.4 (1), \mathbb{T} is reduced. Hence \mathbb{T} is reduced under (a) or (b). Then the following condition is equivalent to (1) (or (2)):

- (3) $\rho_{\mathbb{J}}|_{\text{Gal}(\bar{\mathbb{Q}}_q/\mathbb{Q})}$ is absolutely irreducible.

Since $N = C(\bar{\rho})|C(\rho_{\mathbb{J}}) = N$, we conclude $N = C(\rho_{\mathbb{J}})$. Therefore, $\bar{\rho}$ and $\rho_{\mathbb{J}}$ must be minimal at prime q splits in M and \mathbb{J} is a primitive component. Then by Lemma 7.9 and Remark 7.8 below, the characteristic power series of the congruence module of \mathbb{T} with respect to $\lambda : \mathbb{T} \rightarrow \mathbb{J}$ can be computed exactly as a product of a certain ray class number of M and the Katz p -adic $L_p(\Psi_{\mathbb{J}})$, which is a key to reach the conclusion of the theorem.

We prepare some notation, four lemmas and a proposition for the proof of the theorem. The proof of the theorem will be given at the end of this section. For simplicity, we write the sequence $\mathbb{T} \twoheadrightarrow \mathbb{T}_{cm}^M \twoheadrightarrow \mathbb{J}$ as $R \xrightarrow{\theta} S \xrightarrow{\mu} A$ and we put $\lambda = \mu \circ \theta : R \rightarrow A$. Under the assumption of the theorem (and Remark 7.3), R, S, A are all Gorenstein rings (by Proposition 5.7). Thus we suppose Gorenstein-ness of R, S and A in this section. We write $B = \Lambda$. Since \mathbb{T} is reduced, the total quotient ring $Q(R)$ of R is a product of fields, and we have $Q(R) = Q_S \oplus Q(S)$ for the complementary semi-simple algebra Q_S . Let R_S be the projection of R in Q_S . We have the following (unique) decomposition

- (1) $\text{Spec}(R) = \text{Spec}(R_S) \cup \text{Spec}(S)$, union of closed subschemes inducing $R \hookrightarrow (R_S \oplus S)$ with Λ -torsion module $C_0(\theta, S) := (R_S \oplus S)/R$.

Similarly, we have $Q(S) = Q_A \oplus Q(A)$ and $Q(R) = Q'_A \oplus Q(A)$ as algebra direct sums. Write S_A (resp. R_A) for the projected image of S (resp. R) in Q_A (resp. Q'_A). Then we have

- (2) $\text{Spec}(S) = \text{Spec}(S_A) \cup \text{Spec}(A)$, union of closed subschemes inducing $S \hookrightarrow (S_A \oplus A)$ with Λ -torsion module $C_0(\mu, A) := (S_A \oplus A)/S$.
- (3) $\text{Spec}(R) = \text{Spec}(R_A) \cup \text{Spec}(A)$, union of closed subschemes inducing $R \hookrightarrow (R_A \oplus A)$ with Λ -torsion module $C_0(\lambda, A) := (R_A \oplus A)/R$.

Since \mathbb{T} is reduced, S is a reduced algebra, and by Gorenstein-ness, we have

$$(7.2) \quad \text{Hom}_B(R, B) \cong R, \text{Hom}_B(S, B) \cong S \text{ and } \text{Hom}_B(A, B) \cong A \text{ as } R\text{-modules.}$$

Write $\pi_S : R \twoheadrightarrow R_S$ and $\pi : R \rightarrow S$ for the two projections and $(\cdot, \cdot)_R : R \times R \rightarrow B$ and $(\cdot, \cdot)_S : S \times S \rightarrow B$ for the pairing giving the self-duality (7.2). We recall Lemma 1.6 of [H86c]:

Lemma 7.4. *The S -ideal $\mathfrak{b} := \text{Ker}(\pi_S : R \rightarrow R_S)$ is principal (and is S -free of rank 1).*

By [H88] Lemma 6.3 (or [MFG] §5.3.3), we get the following isomorphisms of R -modules:

$$(7.3) \quad C_0(\lambda; A) \cong R_A \otimes_R A, C_0(\theta; S) \cong R_S \otimes_R S \text{ and } C_0(\mu; A) \cong S_A \otimes_S A.$$

Recall the following fact first proved in [H88] as Theorem 6.6:

Lemma 7.5. *We have the following exact sequence of R -modules:*

$$0 \rightarrow C_0(\mu; A) \rightarrow C_0(\lambda; A) \rightarrow C_0(\theta; S) \otimes_S A \rightarrow 0.$$

By (7.3), the three congruence modules $C_0(\mu; A)$, $C_0(\lambda; A)$, $C_0(\theta; S) \otimes_R A$ are residue rings of R ; so, cyclic A -modules. Moreover they are the ring A modulo principal ideals. Write their generators as $Ac_\lambda = A \cap R \subset (R_A \oplus A)$, $Ac_\mu = A \cap S \subset (S_A \oplus A)$ and $Sc_\theta = S \cap R \subset (R_S \oplus S)$. Thus we have $C_0(\lambda; A) = A/c_\lambda A$, $C_0(\mu; A) = A/c_\mu A$ and $C_0(\theta; S) \otimes_S A = A/\bar{c}_\theta A$ for the image $\bar{c}_\theta \in A$ of $c_\theta \in S$. By the above lemma, we conclude the following result:

Corollary 7.6. *We have $\bar{c}_\theta \cdot c_\mu = c_\lambda$ up to units in A .*

We have a natural morphism $(\mathbb{Z}/(\mathfrak{C} \cap \mathbb{Z}))^\times \rightarrow Cl_M(\mathfrak{C})$ sending ideal (n) for an integer n prime to \mathfrak{C} to its class in $Cl_M(\mathfrak{C})$, and we write $h^-(\mathfrak{C})$ for the order of cokernel of this map. Write $l(\mathfrak{l})$ for the residual characteristic of \mathfrak{l} . By a simple computation, we have

Lemma 7.7. *Write \mathfrak{C} for $\mathfrak{C}(\Psi_{\mathbb{J}})$. Then the ratio*

$$\frac{h^-(\mathfrak{C})}{h(M) \cdot \prod_{\mathfrak{l}|\mathfrak{C}, \mathfrak{l}: \text{ inert prime}} (l(\mathfrak{l}) + 1) \prod_{\mathfrak{l}|\mathfrak{C}, \mathfrak{l}: \text{ split prime with } l(\mathfrak{l})|\mathfrak{C}} (l(\mathfrak{l}) - 1)}$$

is prime to p (if $p \nmid |\mathfrak{D}^\times|/2$ for the integer ring \mathfrak{D} of M), where $h(M)$ is the class number of M . Thus if \mathbb{J} is minimal primitive, $h^-(\mathfrak{C})$ is equal to

$$h_i(M/\mathbb{Q}) = h(M) \prod_{\mathfrak{l}|\mathfrak{C}, \mathfrak{l}: \text{ inert prime}} (l(\mathfrak{l}) + 1)$$

up to units in W .

Since $\rho_{\mathbb{J}}$ is minimal at primes $q|N$ split in M (see Remark 7.3), the q -part $\mathfrak{C}_q(\Psi_{\mathbb{J}})$ is minimal among $\mathfrak{C}_q(\Psi_{\mathbb{J}}\xi)$ for all finite order characters ξ of $\text{Gal}(\overline{\mathbb{Q}}_q/\mathbb{Q}_q)$; in particular, $\mathfrak{F}_c = \mathfrak{D}$ (by Lemma 10.4). Thus no rational prime split in M divides \mathfrak{C} .

Remark 7.8. The number $h_i(M/\mathbb{Q})$ is defined in [H09] §1, and $h_i(M/\mathbb{Q})L_p(\Psi_{\mathbb{J}}^-)$ (for the element $L_p(\Psi_{\mathbb{J}}^-) \in W(\overline{\mathbb{F}}_p)[[\Gamma_M]]$ giving the Katz p -adic L -function of $\Psi_{\mathbb{J}}^-$) is computed to be a factor the characteristic power series c_λ in [H09] Corollary 3.8 (or (A) in §1 of [H09]) assuming

- (1) $p \geq 5$,
- (2) primitiveness of \mathbb{J} (i.e., $N = N(\mathfrak{C}(\Psi_{\mathbb{J}}))D$)
- (3) local minimality at q of $\rho_{\mathbb{J}}$ as long as $\rho_{\mathbb{J}}|_{\text{Gal}(\overline{\mathbb{Q}}_q/\mathbb{Q}_q)}$ is reducible.

The reducibility of $\rho_{\mathbb{J}}|_{\text{Gal}(\overline{\mathbb{Q}}_q/\mathbb{Q}_q)}$ in the condition (3) is equivalent to the fact that the automorphic representation generated by $f_P \in \mathcal{F}_{\mathbb{J}}$ is in the principal series at q , and in this way, the result is stated in [H09].

Lemma 7.9. *Let the notation and the assumption be as in Theorem 7.2. Then under the assumption of Theorem 7.2, we have $c_\mu = h^-(\mathfrak{C}) = h_i(M/\mathbb{Q})$ up to units in \mathbb{J} . Here \mathfrak{C} is the prime-to- p conductor of $\Psi_{\mathbb{J}}$ in (CM2).*

Proof. As explained in Remark 7.3, we have $C(\bar{\rho}) = N$ under the assumptions of Theorem 7.2. If $C(\bar{\rho}) = N = C(\rho_{\mathbb{J}})$, by Proposition 5.7, without localization, $R = \mathbb{T}$, $S = \mathbb{T}_{cm}$ and $A = \mathbb{J}$ are Gorenstein rings. Since S is isomorphic to the group algebra $W[[Z_p]]$ by Proposition 5.7, the assertion follows from Lemma 1.9 and Lemma 1.11 in [H86c]. \square

Recall the anticyclotomic Katz p -adic L -function $L_p(\Psi_{\mathbb{J}}^-)$ as in Section 6. Identifying \mathbb{J} with $W[[\Gamma_M]]$, $\Psi_{\mathbb{J}}^- : Z^- \rightarrow \Gamma_M$ induces a surjective algebra homomorphism $W[[Z_p]] \rightarrow \mathbb{J}$ and $L_p(\Psi_{\mathbb{J}}^-)$ is the image of the measure $L^-(\bar{\psi}^-)$ in Theorem 7.2. We regard $L_p(\Psi_{\mathbb{J}}^-) \in \mathbb{J}$.

Proposition 7.10. *Under the assumption of Theorem 7.2, we have $c_\lambda = h^-(\mathfrak{C})L_p(\Psi_{\mathbb{J}}^-)$ up to units in \mathbb{J} for the prime-to- p conductor \mathfrak{C} of $\Psi_{\mathbb{J}}$ in (CM2).*

This is where we need the assumption $p \geq 5$ in Theorem 7.2.

Proof. The fixed field \widetilde{M}/M of $\text{Ker}(\Psi^-)$ for $\Psi = \Psi_{\mathbb{J}}$ has Galois group $\text{Gal}(\widetilde{M}/M) \cong \text{Im}(\Psi^-)$. The maximal torsion-free quotient Γ_M of $\text{Gal}(\widetilde{M}/M)$ is a \mathbb{Z}_p -free module of rank 1. Fix a decomposition $\text{Gal}(\widetilde{M}/M) = \Delta \times \Gamma_M$ for the maximal finite subgroup Δ of $\text{Gal}(\widetilde{M}/M)$. By Proposition 5.7, the

character Ψ^- induces an algebra isomorphism $\Psi_*^- : W[[\Gamma_M]] \cong \mathbb{J}$. Then the maximal p -abelian extension L/\widetilde{M} unramified outside \mathfrak{p} has Galois group X which is naturally a $W[[\text{Gal}(\widetilde{M}/M)]]$ -module (in the standard manner of Iwasawa's theory). Let $\psi^- := \Psi^-|_{\Delta}$ (which has values in W^\times), and put $X(\psi^-) = X \otimes_{W[[\Delta], \psi_*^-]} W$ which is the maximal quotient of X on which Δ acts by ψ^- . Thus $X(\psi^-)$ is naturally an \mathbb{J} -module via Ψ_*^- , and it is known to be a torsion \mathbb{J} -module of finite type. Let $\mathcal{F}^-(\psi^-)$ be the Iwasawa power series in \mathbb{J} of $X(\psi^-)$; i.e., the characteristic power series of $X(\psi^-)$ as a torsion \mathbb{J} -module of finite type (see [MFG] page 291 for the characteristic power series). By the proof of the main conjecture over M by K. Rubin [Ru] or the proof of its anticyclotomic version by Tilouine/Mazur ([T] and [MT]), we know $\mathcal{F}^-(\psi^-) = L_p(\Psi_{\mathbb{J}}^-)$ up to units in \mathbb{J} . By [H09] Corollary 3.8 (see also Remark 7.8), if $p \geq 5$ (and $N = N(\mathfrak{C})D$ which follows from the assumption of Theorem 7.2 as explained in Remark 7.3), we have $h^-(\mathfrak{C})L_p(\Psi_{\mathbb{J}}^-)|_{c_\lambda}$. By [MT] (and [HT1] Corollary 3.3.7), we also know $c_\lambda|h^-(\mathfrak{C})\mathcal{F}^-(\psi^-)$. Combining all of these, we conclude the equality of the proposition. Since the residual representation $\bar{\rho}$ is absolutely irreducible, actually, the above identity is proven in [H09] without using the solution of the main conjecture (and in this way, the anticyclotomic main conjecture is proven in [H09] for general CM fields). \square

Proof of Theorem 7.2. As explained in Remark 7.3, we have $C = N(\mathfrak{C})D = C(\rho_{\mathbb{J}}) = C(\bar{\rho})$ always under the assumption of the theorem. Then by Proposition 5.7, \mathbb{T} , \mathbb{T}_{cm} and \mathbb{J} are all Gorenstein. By Corollary 7.6, we find that $\bar{c}_\theta = c_\lambda/c_\mu$. By Proposition 7.10, $\bar{c}_\theta = L_p(\Psi_{\mathbb{J}}^-)$ up to units in \mathbb{J} . Since $L^-(\bar{\psi}^-)$ has image in \mathbb{J} given by $L_p(\Psi_{\mathbb{J}}^-)$ for all irreducible components $\text{Spec}(\mathbb{J}) \subset \text{Spec}(W[[Z_p]])$ with $\rho_{\mathbb{J}} \cong \text{Ind}_M^{\mathbb{Q}} \Psi_{\mathbb{J}}$. Thus we conclude $c_\theta = L^-(\bar{\psi}^-)$ up to units, proving (1). \square

8. LEVEL AND p -ADIC L -FUNCTIONS

Throughout this section we assume the condition (R) and one of the conditions (s) and (v) above Theorem I in the introduction, although some cases, the conditions follows from the specification of $\bar{\rho}$. Also, throughout this section, we take the base valuation ring W sufficiently large so that each irreducible component $\text{Spec}(\mathbb{I})$ of $\text{Spec}(\mathfrak{h})$ is geometrically irreducible over the quotient field $Q(W)$ of W .

Our proof heavily relies on Lemma 2.9; so, we first verify the assumptions of Lemma 2.9 under (R) and one of (s) and (v). When the condition (s) is satisfied, we replace g in (s) by $j = \lim_{n \rightarrow \infty} g^{p^n}$ and conjugating \mathbb{G} by an element in $\mathcal{B}(\mathbb{I})$, we may assume that $j = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta' \end{pmatrix}$ with $\zeta, \zeta' \in \mu_{p-1}$. If only (v) is satisfied, we take $\sigma \in D_p$ such that $\bar{\rho}(\sigma)$ with distinct two eigenvalues as in the condition (R), and put $j = \lim_{n \rightarrow \infty} g^{q^n}$ for $q = |\mathbb{F}|$; so, again we have $j = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta' \end{pmatrix}$ with $\zeta, \zeta' \in \mu_{q-1}$ normalizing \mathbb{G} and $\rho_{\mathbb{I}}(D_p)$. Hereafter, we exclusively use j to denote the above element in $\text{Im}(\rho_{\mathbb{I}})$.

Lemma 8.1. *Let $\mathbb{G} = \text{Im}(\rho_{\mathbb{I}}) \cap \Gamma_{\mathbb{I}}(\mathfrak{m}_{\mathbb{I}})$ for an irreducible component $\text{Spec}(\mathbb{I})$ of $\text{Spec}(\mathfrak{h})$, and write $\overline{\mathbb{G}}_{\mathfrak{P}}$ for the image of \mathbb{G} in $SL_2(\widetilde{\mathbb{I}}/\mathfrak{P})$ for each prime divisor $\mathfrak{P} \in \text{Spec}(\widetilde{\mathbb{I}})$. Then the $\kappa(\mathfrak{P})$ -span $\bar{\mathfrak{s}}_{\mathfrak{P}}$ of $\mathcal{M}_n^0(\overline{\mathbb{G}}_{\mathfrak{P}}) = \mathcal{M}_n(\overline{\mathbb{G}}_{\mathfrak{P}}) \cap \mathfrak{sl}_2(\widetilde{\mathbb{I}}/\mathfrak{P})$ is equal to $\mathfrak{sl}_2(\kappa(\mathfrak{P}))$ for some $n > 0$ if and only if $\overline{\mathbb{G}}_{\mathfrak{P}}$ contains an open subgroup of $SL_2(A_0)$.*

Proof. By (R) and one of (s) and (v), $\mathcal{M}_n(\mathbb{G}) \cap \mathcal{U}(\mathbb{I})$ surjects down to $\mathcal{M}_n(\overline{\mathbb{G}}_{\mathfrak{P}})$ for all $n > 0$. Since the proof is the same for any $n > 0$, we just assume that $n = 1$. Let $P = \mathfrak{P} \cap \Lambda$. Note that $\bar{\mathfrak{n}} = \mathcal{M}_1(\overline{\mathbb{G}}_{\mathfrak{P}}) \cap \mathcal{U}(\kappa(\mathfrak{P}))$ and $\bar{\mathfrak{n}}_t = \mathcal{M}_1(\overline{\mathbb{G}}_{\mathfrak{P}}) \cap {}^t\mathcal{U}(\kappa(\mathfrak{P}))$ are Λ/P -modules inside $\mathfrak{sl}_2(\widetilde{\mathbb{I}}/\mathfrak{P})$. Thus either $\bar{\mathfrak{n}} = 0$ or $\bar{\mathfrak{n}}$ is Λ/P -torsion-free of positive rank.

Suppose $\bar{\mathfrak{s}}_{\mathfrak{P}} = \mathfrak{sl}_2(\kappa(\mathfrak{P}))$. Then $\bar{\mathfrak{n}} \neq 0$ and $\bar{\mathfrak{n}}_t \neq 0$. This implies that $[\bar{\mathfrak{n}}, \bar{\mathfrak{n}}_t] \neq 0$ is a non-trivial torsion-free Λ/P -module of positive rank, and $Ad(j)$ acts trivially on $[\bar{\mathfrak{n}}, \bar{\mathfrak{n}}_t]$. Thus $\mathcal{M}_1^0(\overline{\mathbb{G}}_{\mathfrak{P}})$ must contain an open Lie-subalgebra of $\mathfrak{sl}_2(A_0)$ (see Section 2); so, $\bar{\mathfrak{s}}_{\mathfrak{P}} = \mathfrak{sl}_2(\kappa(\mathfrak{P}))$. Since

$$[\mathcal{M}_1^0(\overline{\mathbb{G}}_{\mathfrak{P}}), \mathcal{M}_1^0(\overline{\mathbb{G}}_{\mathfrak{P}})] \subset \mathcal{M}^0(\overline{\mathbb{G}}_{\mathfrak{P}}),$$

$\mathcal{M}^0(\overline{\mathbb{G}}_{\mathfrak{P}})$ (and actually $\mathcal{M}_n^0(\overline{\mathbb{G}}_{\mathfrak{P}})$ for each $n > 0$) span $\mathfrak{sl}_2(\kappa(\mathfrak{P}))$ over $\kappa(\mathfrak{P})$. Then $\overline{\mathbb{G}}_{\mathfrak{P}}' = SL_2(\kappa(\mathfrak{P})) \cap (1 + \mathcal{M}(\overline{\mathbb{G}}_{\mathfrak{P}}))$ contains an open subgroup of $SL_2(A_0)$. The converse is plain as $\mathfrak{sl}_2(A_0)$ contains a basis of $\mathfrak{sl}_2(\kappa(\mathfrak{P}))$ over $\kappa(\mathfrak{P})$. \square

Hereafter, suppose that \mathbb{I} is a non-CM component of \mathbf{h} . Let $\text{Spec}(\mathbb{T}) \subset \text{Spec}(\mathbf{h})$ be a connected component containing $\text{Spec}(\mathbb{I})$. Let $\rho_{\mathbb{T}} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow GL_2(Q(\mathbb{T}))$ be the associated Galois representation. We write $\overline{\rho} = \rho_{\mathfrak{m}_{\mathbb{T}}} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow GL_2(\mathbb{F})$ with $\mathbb{F} = \mathbb{T}/\mathfrak{m}_{\mathbb{T}}$ associated to the maximal ideal $\mathfrak{m}_{\mathbb{T}}$ of \mathbb{T} . We would like to relate the global level $L = L(\mathbb{I})$ of $\rho_{\mathbb{I}}$ (defined in Section 3) with a certain p -adic L -function. By a result of Ribet [R1] combined with Proposition 5.1 in the text, $\text{Im}(\rho_{\mathfrak{P}})$ contains an open subgroup of $SL_2(\mathbb{Z}_p)$ up to conjugation. Then by Theorem 2.12, we can pick a representation $\rho \in [\rho_{\mathbb{I}}]$ with values in $GL_2(\widetilde{\mathbb{I}})$ such that $\text{Im}(\rho) \supset \Gamma(\mathfrak{c})$ with nontrivial \mathfrak{c} . If $\overline{\rho}$ is absolutely irreducible, by Theorem 2.12, the global level $L = L(\mathbb{I})$ described just above Lemma 3.3 is well defined. If $\overline{\rho}$ is reducible, assuming the assumption of Lemma 3.5, we pick ρ in the \mathbb{I} -isomorphism class made out of $\mathcal{L}_{can}(\mathbb{I})$ and define $L(\mathbb{I})$ as described after the statement of Lemma 3.5 before its proof. We start with a version of results in [MW] §10 and A. Fischman [F]:

Theorem 8.2. *Suppose $\text{Im}(\overline{\rho})$ contains $SL_2(\mathbb{F}_p)$ for $p \geq 7$. Then the global level $L = L(\mathbb{I})$ of $\rho_{\mathbb{I}}$ for every irreducible component $\text{Spec}(\mathbb{I})$ of $\text{Spec}(\mathbb{T})$ is equal to 1.*

The assertion (1) in Theorem II in the introduction follows from this theorem. By the theory of pseudo representation, we can find a unique $\rho_{\mathbb{T}}$ with values in $GL_2(\mathbb{T})$ up to isomorphism. Thus we could assume that $\rho_{\mathbb{I}}$ has values in $GL_2(\mathbb{I})$, though we do not assume this.

Proof. Similarly to the proof of Lemma 3.1, writing \overline{g} for the image of $g \in \text{Im}(\rho_{\mathbb{I}})$ in $GL_2(\widetilde{\mathbb{I}}/\mathfrak{m}_{\overline{\mathbb{I}}})$, let

$$\mathbb{K} := \{g \in \text{Im}(\rho_{\mathbb{I}}) \mid \det(g) \in \Gamma\}, \mathbb{L} = \{g \in \mathbb{K} \mid \overline{g} \in \mathcal{U}(\mathbb{F})\} \text{ and } \mathbb{H} = \mathbb{K} \cap \Gamma_{\overline{\mathbb{I}}}(\mathfrak{m}_{\overline{\mathbb{I}}})$$

for $\Gamma = \{(1+T)^s \mid s \in \mathbb{Z}_p\} \subset \Lambda^\times$. By the existence of j , similarly to the proof of Lemma 1.4, from (Gal) , we find $\tau \in \rho_{\mathbb{I}}(D_p) \cap \mathbb{H}$ such that $\tau = \begin{pmatrix} 1+T & 0 \\ 0 & 1 \end{pmatrix}$. Then by (Det) in Section 3, the three groups $\text{Im}(\rho_{\mathbb{I}})/\mathbb{K}$, $\text{Im}(\rho_{\mathbb{I}})/\mathbb{L}$ and $\text{Im}(\rho_{\mathbb{I}})/\mathbb{H}$ are finite groups. Then for $\mathcal{T}' = \{\tau^s \mid s \in \mathbb{Z}_p\}$, we have $\mathbb{H} = \mathcal{T}' \rtimes \mathbb{G}$ for $\mathbb{G} = \mathbb{H} \cap SL_2(\widetilde{\mathbb{I}})$, $\mathbb{K} = \mathcal{T}' \rtimes \mathbb{K}^1$ for $\mathbb{K}^1 = \mathbb{K} \cap SL_2(\widetilde{\mathbb{I}})$ and $\mathbb{L} = \mathcal{T}' \rtimes \mathbb{L}^1$ for $\mathbb{L}^1 = \mathbb{L} \cap SL_2(\widetilde{\mathbb{I}})$ (see the proof of Lemma 3.1). Similarly, for the image $\overline{\mathcal{T}}'_a$ (resp. $\overline{\mathbb{H}}_a, \overline{\mathbb{K}}_a, \overline{\mathbb{L}}_a, \overline{\mathbb{L}}_a^1, \overline{\mathbb{G}}_a$ and $\overline{\mathbb{K}}_a^1$) of \mathcal{T}' (resp. $\mathbb{H}, \mathbb{K}, \mathbb{L}, \mathbb{L}^1, \mathbb{G}$ and \mathbb{K}^1) in $GL_2(\widetilde{\mathbb{I}}/\mathfrak{a})$ for any $\widetilde{\mathbb{I}}$ -ideal \mathfrak{a} , we have $\overline{\mathbb{H}}_a = \overline{\mathcal{T}}'_a \rtimes \overline{\mathbb{G}}_a$, $\overline{\mathbb{L}}_a = \overline{\mathcal{T}}'_a \rtimes \overline{\mathbb{L}}_a^1$ and $\overline{\mathbb{K}}_a = \overline{\mathcal{T}}'_a \rtimes \overline{\mathbb{K}}_a^1$. Thus the reduction maps $\mathbb{G} \rightarrow \overline{\mathbb{G}}_a$, $\mathbb{L}^1 \rightarrow \overline{\mathbb{L}}_a^1$ and $\mathbb{K}^1 \rightarrow \overline{\mathbb{K}}_a^1$ given by $g \mapsto (g \bmod \mathfrak{a})$ are all surjective. In particular, by our assumption, $\overline{\mathbb{K}}_a^1 := \mathbb{K}_{\mathfrak{m}_{\overline{\mathbb{I}}}}^1$ contains $SL_2(\mathbb{F}_p)$.

We prove $P \nmid L(\mathbb{I})$ for all prime divisors of Λ , which shows $L(\mathbb{I}) = 1$. Take a prime divisor of $\widetilde{\mathbb{I}}$ above P . Suppose that $\overline{\mathbb{K}}_{\mathfrak{P}}^1$ is a finite group. This is equivalent to assuming $\overline{\mathbb{G}}_{\mathfrak{P}}$ is finite since $\mathbb{K}^1/\mathbb{G} \cong \mathbb{K}/\mathbb{H} \hookrightarrow \text{Im}(\rho_{\mathbb{I}})/\mathbb{H}$ is finite. Thus $\overline{\mathbb{K}}_{\mathfrak{P}}^1$ is a finite group whose image modulo $\mathfrak{m}_{\overline{\mathbb{I}}}$ containing $SL_2(\mathbb{F}_p)$. By the classification of finite subgroups of $PGL_2(K)$ for a characteristic 0 field K , if $p \geq 7$ and $\kappa(P)$ has characteristic 0, there is no finite subgroup of $SL_2(\kappa(\mathfrak{P}))$ whose image in $SL_2(\mathbb{F})$ contains $SL_2(\mathbb{F}_p)$. This point is also plain if $p \geq 7$ as $SL_2(\mathbb{F}_p)$ with $p \geq 7$ does not have 2-dimensional representations over K (by a result of I. Schur [Sc] page 128). In any case, we conclude that $\overline{\mathbb{G}}_{\mathfrak{P}}$ is infinite if $p \geq 7$ and $\kappa(P)$ has characteristic 0. If $\kappa(P)$ has characteristic p , $\overline{\mathbb{L}}_{\mathfrak{P}}^1$ is still infinite. To see this, note that $\overline{\mathbb{L}}_{\mathfrak{m}_{\overline{\mathbb{I}}}}^1$ contains $\mathcal{U}(\mathbb{F}_p)$; so, $\overline{\mathbb{L}}_{\mathfrak{P}}^1$ contains an element whose reduction modulo $\mathfrak{m}_{\overline{\mathbb{I}}/\mathfrak{P}}$ is non-zero unipotent. Such an element under conjugation by $\overline{\mathcal{T}}'_{\mathfrak{P}}$ produces infinitely many elements. Then the open subgroup $\overline{\mathbb{G}}_{\mathfrak{P}}$ of $\overline{\mathbb{L}}_{\mathfrak{P}}^1$ has infinitely many elements. Therefore $\mathcal{M}_1^0(\overline{\mathbb{G}}_{\mathfrak{P}})$ is an infinite Lie algebra.

Let $\widetilde{\mathfrak{s}}_{\mathfrak{P}}$ be the Lie subalgebra of $\mathfrak{sl}_2(\widetilde{\mathbb{I}}/\mathfrak{P})$ generated by $\mathcal{M}_1^0(\overline{\mathbb{G}}_{\mathfrak{P}})$ over $\widetilde{\mathbb{I}}/\mathfrak{P}$. Since $\overline{\mathbb{G}}_{\mathfrak{P}}$ is infinite, $\widetilde{\mathfrak{s}}_{\mathfrak{P}}$ is nontrivial. Since $p \geq 5$, the adjoint representation of $SL_2(\mathbb{F}_p)$ on $\mathfrak{sl}_2(\mathbb{F}_p)$ is absolutely irreducible. Thus the quotient

$$\widetilde{\mathfrak{s}}_{\mathfrak{P}}/\mathfrak{m}_{\overline{\mathbb{I}}} \cdot \widetilde{\mathfrak{s}}_{\mathfrak{P}} = \widetilde{\mathfrak{s}}_{\mathfrak{P}} \otimes_{\mathbb{F}} \mathbb{F} \quad (\mathbb{F} = \widetilde{\mathbb{I}}/\mathfrak{m}_{\overline{\mathbb{I}}})$$

is isomorphic to a three dimensional irreducible subspace in $\mathfrak{sl}_2(\mathbb{F})$ over \mathbb{F} under the adjoint action of \mathbb{K}^1 . Then by Nakayama's lemma, $\widetilde{\mathfrak{s}}_{\mathfrak{P}}$ has at least rank 3 over $\widetilde{\mathbb{I}}/\mathfrak{P}$; so, the $\kappa(\mathfrak{P})$ -span $\widetilde{\mathfrak{s}}_{\mathfrak{P}} := \kappa(\mathfrak{P}) \cdot \widetilde{\mathfrak{s}}_{\mathfrak{P}}$ is equal to $\mathfrak{sl}_2(\kappa(\mathfrak{P}))$. By Lemma 8.1, $\overline{\mathbb{G}}_{\mathfrak{P}}$ contains an open subgroup of $SL_2(A_0)$ in $SL_2(\kappa(\mathfrak{P}))$. Hence, by Theorem 2.12 (2) and Corollary 3.4, we conclude $P \nmid L(\mathbb{I})$. \square

Remark 8.3. In the setting of the above theorem, assume $p = 5$. Again by Schur, the unique absolutely irreducible 2-dimensional representation over $\overline{\mathbb{Q}}_5$ of $SL_2(\mathbb{F}_5)$ can be only defined over the

integer ring of the field $\mathbb{Q}_5[\sqrt{5}]$. Since we have $\Lambda/P \cong \mathbb{Z}_5[\mu_5]$ for $P = \left(\frac{(1+T)^5-1}{T}\right) \subset \Lambda$, we have a subgroup H in $SL_2(\Lambda/P)$ whose reduction modulo the maximal ideal is isomorphic to $SL_2(\mathbb{F}_5)$. Therefore $G = \{x \in SL_2(\Lambda) \mid x \bmod P \in H\}$ has $(G \bmod \mathfrak{m}) = SL_2(\mathbb{F}_5)$ but the level of G is P .

We now deal with the case where the image of $\bar{\rho}$ does not contain $SL_2(\mathbb{F}_p)$. We start with the case of dihedral image of $\bar{\rho}$. Let κ be a local field. Write O for the maximal compact subring of κ . Let $\rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow GL_2(O)$ be a continuous Galois representation and put $G = \text{Im}(\rho) \cap \Gamma_O(\mathfrak{m}_O)$. We write \mathfrak{s} for the κ -span of the Lie algebra $\mathcal{M}_1^0(G) = \mathcal{M}_1(G) \cap \mathfrak{sl}_2(G)$.

Lemma 8.4. *Let the notation be as above. Suppose either that \mathfrak{s} is a Cartan subalgebra of $\mathfrak{sl}(2)$ or that $\text{Im}(\rho)$ modulo center is a finite dihedral group. If ρ is absolutely irreducible, there exists a quadratic field M/\mathbb{Q} and a character $\theta : \text{Gal}(\overline{\mathbb{Q}}/M) \rightarrow \overline{\kappa}^\times$ such that $\rho \cong \text{Ind}_M^{\mathbb{Q}} \theta$ and $\theta^c \neq \theta$, where $c \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ restricted to M is the generator of $\text{Gal}(M/\mathbb{Q})$ and $\theta^c(\sigma) = \theta(c\sigma c^{-1})$ for $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/M)$.*

This follows from Lemma 2.1 if κ has characteristic 0. We give here a different proof.

Proof. In any case, the group $\text{Im}(\rho)$ is in the normalizer of a Cartan subalgebra \mathfrak{H} ($\mathfrak{H} = \mathfrak{s}$ if \mathfrak{s} is a Cartan subalgebra). By extending scalar κ , we may assume that \mathfrak{H} is a split Cartan subalgebra. Then, we can find an open normal subgroup $H \subset \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that $\rho|_H$ is isomorphic to the direct sum of two abelian characters. Set $\rho_H = \rho|_H$; then, ρ_H is completely reducible. Write $\rho_H = \begin{pmatrix} \theta & 0 \\ 0 & \delta \end{pmatrix}$. Since ρ extends ρ_H , $g \mapsto \rho_H^h(g) := \rho_H(hgh^{-1}) = \rho(h)\rho_H(g)\rho(h)^{-1}$ is equivalent to ρ_H for all $h \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Thus $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $\{\delta, \theta\}$ by inner conjugation. Indeed,

$$(8.1) \quad \begin{pmatrix} \theta^h & 0 \\ 0 & \delta^h \end{pmatrix} \cong \rho(h) \begin{pmatrix} \theta & 0 \\ 0 & \delta \end{pmatrix} \rho(h)^{-1}.$$

Let $\Delta \subset \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ be the stabilizer of δ . Then $M = \overline{\mathbb{Q}}^\Delta$ is at most a quadratic extension of \mathbb{Q} . If $M = \mathbb{Q}$ and W is sufficiently large, the two characters extend to $\delta, \theta : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \mathbb{I}^\times$ (e.g. [GME] §5.1.1 or [MFG] §4.3.5), and $\rho^{ss} = \delta \oplus \theta$, which cannot happen as ρ is absolutely irreducible. Then $[\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) : \Delta] = 2$ and by Frobenius reciprocity, $\rho \cong \text{Ind}_M^{\mathbb{Q}} \delta \cong \text{Ind}_M^{\mathbb{Q}} \theta$ for the quadratic extension $M = \overline{\mathbb{Q}}^\Delta$ of \mathbb{Q} . We have therefore $\rho|_\Delta = \theta \oplus \theta^c$, and irreducibility of ρ implies $\theta \neq \theta^c$. \square

For a character φ of $\text{Gal}(\overline{\mathbb{Q}}/M)$ with an imaginary quadratic field M , we recall its anti-cyclotomic projection φ^- given by $\sigma \mapsto \varphi(\sigma)\varphi(c\sigma c^{-1})^{-1}$. Let \mathbb{I} be a minimal primitive non CM component of \mathfrak{h} with $\bar{\rho} \cong \text{Ind}_M^{\mathbb{Q}} \bar{\psi}$ for an imaginary quadratic field M in which p splits into $\mathfrak{p}\bar{\mathfrak{p}}$ and a character $\bar{\psi} : \text{Gal}(\overline{\mathbb{Q}}/M) \rightarrow \overline{\mathbb{F}}_p$ unramified at $\bar{\mathfrak{p}}$. Under the assumption (a) or (b) in Theorem 7.2, \mathbb{T}_{cm}^M is nontrivial. Pick one such CM component \mathbb{J} of \mathbb{T}_{cm}^M , and write \mathfrak{C} for the prime-to- p conductor of $\Psi_{\mathbb{J}}$. Let $\mathfrak{c}' = \mathfrak{C} \cap \overline{\mathfrak{C}}$, and write \mathfrak{c} for the prime-to- p conductor of $\bar{\psi}^-$ (so, $\mathfrak{c}|\mathfrak{c}'$). Assuming $W \supset W(\overline{\mathbb{F}}_p)$, we recall the anticyclotomic Katz measure $L^-(\bar{\psi}^-) \in W[[Z_p^-]]$ as in Theorem 7.2. The natural inclusion $\mathbb{Z}_p^\times \hookrightarrow \mathfrak{D}_{\mathfrak{p}}$ induces $\Gamma \hookrightarrow \Gamma_M \cong \Gamma_M^- \subset Z_p^-$, and hence $W[[Z_p^-]]$ is naturally a Λ_W -algebra for $\Lambda_W = W[[T]] = W[[\Gamma]]$. Since $W[[Z_p^-]]$ is free of finite rank r over Λ_W for the index $r = (Z_p^- : \Gamma)$, we have a regular representation Φ of $W[[Z_p^-]]$ into the $r \times r$ matrix algebra $M_r(\Lambda_W)$, and for $\alpha \in W[[Z_p^-]]$, we define its norm $N_{W[[Z_p^-]]/\Lambda_W}(\alpha) = \det(\Phi(\alpha))$. Define $L_{\bar{\psi}^-}^- = N_{W[[Z_p^-]]/\Lambda_W}(L^-(\bar{\psi}^-))$. This is the element we meant in Theorem II by the product of anticyclotomic Katz L -functions with a given mod p branch character. We also recall that the element $E_{1,N} \in \mathbb{J}$ defined in (7.1).

Theorem 8.5. *Let the notation be as above; in particular, let \mathbb{I} be a minimal primitive non CM component with $\bar{\rho} \cong \text{Ind}_M^{\mathbb{Q}} \bar{\psi}$ for an imaginary quadratic field M as above. Take a sufficiently large $W \supset W(\overline{\mathbb{F}}_p)$ so that each irreducible component of $\text{Spec}(\mathbb{T})$ is geometrically irreducible. Assume $p \geq 5$, and suppose further that $\bar{\psi}^-$ has order > 2 , $\bar{\psi}$ ramifies at \mathfrak{p} , and one of the following conditions:*

- (a) $p \nmid \varphi(N)$ and $C(\bar{\rho}) = N$;
- (b) $p \nmid \Phi(N)$ for the Euler function Φ of M and $E_{1,N} \notin \mathfrak{m}_{\mathbb{J}}$ for $E_{1,N}$ in (7.1).

Then for the product $L_{\bar{\psi}}^- = N_{W[[Z_p^-]]/\Lambda_W}(L^-(\bar{\psi}^-))$, the global level $L(\mathbb{I})$ of a non-CM component \mathbb{I} of \mathbb{T} is a factor of $(L_{\bar{\psi}}^-)^2$ in Λ_W . If $L_{\bar{\psi}}^-$ is a non unit in Λ_W , for any prime divisor P of $L_{\bar{\psi}}^-$, there exists a non CM component $\text{Spec}(\mathbb{I}) \subset \text{Spec}(\mathbb{T})$ such that $P_\Lambda | L(\mathbb{I})$ for $P_\Lambda = P \cap \Lambda$.

We have $C(\bar{\rho}) = N$ as explained in Remark 7.3, and therefore, this theorem proves the assertion (3b) of Theorem II in the introduction (where cube-freeness of N is assumed but it is not necessary in this residually induced case). Here are a sketch of the proof and a summary of how we use the listed assumptions in the proof. Since $\bar{\psi}$ ramifies at \mathfrak{p} , by Theorem 7.1, $\mathcal{L}_{can}(\mathbb{I}) \cong \mathbb{I}^2$; so, $\rho_{\mathbb{I}}$ realized on $\mathcal{L}_{can}(\mathbb{I})$ has values in $GL_2(\mathbb{I})$. Thus we do not need to take $\tilde{\mathbb{I}}$, and we work with \mathbb{I} instead of $\tilde{\mathbb{I}}$. Since $C(\bar{\rho})|N$, by Proposition 5.7, \mathbb{T}_{cm}^M is non-trivial. The condition that $\bar{\psi}^-$ has order > 2 ramified at p is equivalent to the fact that $\bar{\rho}$ is not isomorphic to $\text{Ind}_{M'}^{\mathbb{Q}} \bar{\psi}'$ for any quadratic fields M' other than M (Proposition 5.2); thus, we have $\mathbb{T}_{cm} = \mathbb{T}_{cm}^M$. The assumption (a) or (b) is used to identify \mathbb{T}_{cm} (or its localization) with (possibly a localization of) the group algebra $W[[Z_p]] \cong W[[Z_p^-]]$ that enables us to identify the congruence power series of \mathbb{J} inside $\text{Spec}(\mathbb{T}_{cm})$ with the class number $h^-(\mathcal{C})$ and that in $\text{Spec}(\mathbb{T})$ with $h^-(\mathcal{C})L_p(\Psi_{\mathbb{J}}^-)$ (see the later half of Section 7). As summarized in Remark 7.3,

- (1) minimality of \mathbb{I} implies minimality of $\rho_{\mathbb{J}}$ at primes q in N where $\rho_{\mathbb{J}}|_{\text{Gal}(\bar{\mathbb{Q}}_q/\mathbb{Q}_q)}$ is reducible.
- (2) the condition (b) actually implies (a), and \mathbb{T} is reduced by Lemma 5.4 (1).
- (3) any CM component \mathbb{J} of \mathbb{T}_{cm}^M is primitive.

In the computation of congruence modules (in Theorem 7.2) between \mathbb{J} and its complement \mathbb{J}^\perp and between \mathbb{T}_{cm} and \mathbb{T}_{cm}^\perp (i.e., determination of $\text{Spec}(\mathbb{J}) \cap \text{Spec}(\mathbb{J}^\perp)$ and $\text{Spec}(\mathbb{T}_{cm}) \cap \text{Spec}(\mathbb{T}_{cm}^\perp)$), we needed these properties (see Remark 7.8 for the necessity of these properties). Then by the relation in Corollary 7.6, we computed in Theorem 7.2 the characteristic element of $C := \mathbb{T}_{cm} \otimes_{\mathbb{T}} \mathbb{T}_{cm}^\perp$ in $\mathbb{T}_{cm} = W[[Z_p^-]]$ as the Katz measure without the class number factor. Hence, we are able to prove, by Galois deformation theory, that any (non CM) component \mathbb{I} in \mathbb{T}_{cm}^\perp has some points P having ρ_P isomorphic to an induced representation from M (i.e., $P|L(\mathbb{I})$) if and only if P is in $\text{Spec}(\mathbb{T}_{cm}) \cap \text{Spec}(\mathbb{T}_{cm}^\perp)$ (i.e., P is a factor of the Katz p -adic L -function $L_{\bar{\psi}}^-$).

Now, for simplicity, suppose $\mathbb{I} = \Lambda$. Then $\rho_P = (\rho_{\mathbb{I}} \bmod P)$ is isomorphic to an induced representation $\text{Ind}_M^{\mathbb{Q}} \theta$; thus, the adjoint square $Ad(\rho_P)$ of ρ_P is isomorphic to a reducible representation $\left(\frac{M/\mathbb{Q}}{\oplus}\right) \oplus \text{Ind}_M^{\mathbb{Q}} \theta^-$ with $\text{Ind}_M^{\mathbb{Q}} \theta^-$ absolutely irreducible. In this sketch, suppose for simplicity that P is exactly the annihilator $\text{Ann}_{\mathbb{I}_P}$ in \mathbb{I}_P of the \mathbb{I}_P -part $C \otimes_{\mathbb{T}} \mathbb{I}_P$ of the congruence module C (in other words, $C \otimes_{\mathbb{T}} \mathbb{I}_P \cong \mathbb{I}_P/P\mathbb{I}_P$). Then we show that $\rho_{P^2} := (\rho_{\mathbb{I}} \bmod P^2)$ is no longer an induced representation of a character of $\text{Gal}(\bar{\mathbb{Q}}/M)$ with values in $(\mathbb{I}/P^2)^\times$. Writing $V = \mathbb{I}_P^2$ for the space of $\rho_{\mathbb{I}}$ and $\mathfrak{sl}(V) = \{x \in \text{End}_{\mathbb{I}}(V) | \text{Tr}(x) = 0\}$, the \mathbb{I}_P -span \mathfrak{s}_P of $\mathcal{M}_1^0(\mathbb{G}) = \mathcal{M}_1(\bar{\mathbb{G}}_{\mathfrak{P}}) \cap \mathfrak{sl}_2(\kappa(\mathfrak{P}))$ (for $\mathbb{G} = \text{Im}(\rho_{\mathbb{I}}) \cap \Gamma_{\mathbb{I}}(\mathfrak{m}_{\mathbb{I}})$) is a Lie \mathbb{I}_P -subalgebra of $\mathfrak{sl}(V)$ stable under the adjoint action. Thus we have three possibility of the isomorphism class of the Galois module $P\mathfrak{s}_P/P^2\mathfrak{s}_P$ under the adjoint action of $\rho_{\mathbb{I}}$: (a) $Ad(\rho_P)$, (b) $\left(\frac{M/\mathbb{Q}}{\oplus}\right)$ or (c) $\text{Ind}_M^{\mathbb{Q}} \theta^-$. We will show that (b) is impossible, showing ρ_{P^2} in the case (b) is an induced representation (i.e., $P^2 \supset \text{Ann}_{\mathbb{I}_P}$ against our assumption: $\text{Ann}_{\mathbb{I}_P} = P\mathbb{I}_P$). In the case (a), the prime P divide exactly once $L(\mathbb{I})$ (i.e., $P \parallel L(\mathbb{I})$); so, the P -part of $L(\mathbb{I})$ is the factor of $L_{\bar{\psi}}^-$, and in the case (c), by a Lie algebra computation, we show that $P^2 \parallel L(\mathbb{I})$ (i.e., the P -part of $L(\mathbb{I})$ is the factor of $(L_{\bar{\psi}}^-)^2$). This is close to the conclusion of the theorem. We now give a detailed proof for general $\mathbb{I} \supset \Lambda$.

Proof. As explained above in the sketch, we have $\mathbb{T}_{cm} = \mathbb{T}_{cm}^M$ is nontrivial. Since \mathbb{I} is a non-CM component of \mathbb{T} , we have $\mathbb{T} \neq \mathbb{T}_{cm}$. Since $\bar{\rho}$ is absolutely irreducible, under (R), \mathbb{T} is Gorenstein, and hence $\rho_{\mathbb{T}}$ realized on $\mathcal{L}_{can}(\mathbb{T})$ has values in $GL_2(\mathbb{T})$; so, $\rho_{\mathbb{I}}$ realized on $\mathcal{L}_{can}(\mathbb{I})$ has values in $GL_2(\mathbb{I})$ and $\rho_{\mathbb{I}}|_{D_p} \subset \mathcal{B}(\mathbb{I})$ with (Gal) satisfied (thus (F_{can}) is satisfied). Let $\mathbb{G} = \text{Im}(\rho_{\mathbb{I}}) \cap \Gamma_{\mathbb{I}}(\mathfrak{m}_{\mathbb{I}})$, and write $\rho_{\mathfrak{P}} = (\rho_{\mathbb{I}} \bmod \mathfrak{P})$ for a prime $\mathfrak{P} \in \text{Spec}(\mathbb{I})$.

Pick now a prime divisor $P \in \text{Spec}(\Lambda)$ and a prime divisor $\mathfrak{P} \in \text{Spec}(\mathbb{I})$ above P . We consider the Lie algebra $\bar{\mathfrak{s}}_{\mathfrak{P}}$ of $\bar{\mathbb{G}}_{\mathfrak{P}} = (\mathbb{G} \bmod \mathfrak{P})$; i.e., we write $\bar{\mathfrak{s}}_{\mathfrak{P}}$ for the $\kappa(\mathfrak{P})$ -span of $\mathcal{M}_1^0(\bar{\mathbb{G}}_{\mathfrak{P}})$. There are the following five possibilities:

- (O) $\bar{\mathfrak{s}}_{\mathfrak{P}} = 0$.
- (C) $\bar{\mathfrak{s}}_{\mathfrak{P}}$ is a Cartan subalgebra \mathfrak{h} ,
- (N) $\bar{\mathfrak{s}}_{\mathfrak{P}}$ is a nonzero nilpotent subalgebra
- (B) $\bar{\mathfrak{s}}_{\mathfrak{P}}$ is a Borel subalgebra,
- (F) $\bar{\mathfrak{s}}_{\mathfrak{P}} = \mathfrak{sl}_2(\kappa(\mathfrak{P}))$,

If we are in Case (F) for all $\mathfrak{P}|P$, by Lemma 3.1 combined with Lemma 8.1, $\overline{\mathbb{G}}_{\mathfrak{P}}$ contains an open subgroup of $SL_2(A_0)$ for all $\mathfrak{P}|P$, and we have $\mathfrak{P} \nmid L(\mathbb{I})$ by Theorem 2.12 (2) and Corollary 3.4. If we are in Case (N) and (B) for some prime \mathfrak{P} , the group $\overline{\mathcal{G}}_{\mathfrak{P}} := \text{Im}(\rho_{\mathfrak{P}})$ normalizes $\bar{\mathfrak{s}}_{\mathfrak{P}}$. Since the normalizer of a (non-trivial) nilpotent or a Borel subalgebra is a Borel subgroup, $\rho_{\mathfrak{P}}$ has values in a Borel subgroup; so, $\rho_{\mathfrak{P}}$ is reducible, which is impossible by the absolute irreducibility of $\bar{\rho}$. Thus Cases (B) and (N) do not occur for any of $\mathfrak{P}|P$.

In the cases (O) and (C), we first show that $\rho_{\mathfrak{P}} \cong \text{Ind}_M^{\mathbb{Q}} \theta$ for a character $\theta : \text{Gal}(\overline{\mathbb{Q}}/M) \rightarrow (\mathbb{I}/\mathfrak{P})^{\times}$. Suppose first that we have some $\mathfrak{P}|P$ in Case (O). Then the basic closure of $\overline{\mathbb{G}}_{\mathfrak{P}}$ is contained in the center; so, $\overline{\mathbb{G}}_{\mathfrak{P}} \subset \{\pm 1\}$. Since $p > 2$, we have $\overline{\mathbb{G}}_{\mathfrak{P}} = 1$. Therefore under the notation in the proof of Lemma 8.1, we have $\overline{\mathbb{H}}_{\mathfrak{P}} = \overline{\mathcal{T}}'_{\mathfrak{P}} \rtimes \overline{\mathbb{G}}_{\mathfrak{P}} = \overline{\mathcal{T}}'_{\mathfrak{P}}$. Since $\text{Im}(\bar{\rho})$ is dihedral modulo center, taking $j \in \text{Im}(\rho_{\mathbb{I}})$ defined just before stating Lemma 8.1 (for \mathbb{I} in place of \mathbb{J} there), it contains an element j' of order 2 such that $j'jj'^{-1} = \begin{pmatrix} \zeta' & 0 \\ 0 & \zeta \end{pmatrix}$ (i.e., conjugation by j' interchanges the two distinct eigenvalues of j). This j' can be lifted to an element (still denoted by j') in $\text{Im}(\rho_{\mathbb{I}})$ keeping property of interchanging the two distinct eigenvalues of j (e.g., [CGP] § IV.3 or [TGP] § IV.7). Then it interchanges the eigenvalues of elements in $\overline{\mathcal{T}}'_{\mathfrak{P}}$; so, we have $j'\overline{\mathcal{T}}'_{\mathfrak{P}}j'^{-1} \subset \overline{\mathbb{H}}_{\mathfrak{P}} = \overline{\mathcal{T}}'_{\mathfrak{P}}$, which implies $\overline{\mathcal{T}}'_{\mathfrak{P}} = 1$. Thus we conclude $\mathfrak{P}|T$ and $\overline{\mathbb{H}}_{\mathfrak{P}} = 1$. Therefore $\text{Im}(\rho_{\mathfrak{P}})$ is isomorphically projected onto $\text{Im}(\bar{\rho})$, and hence we must have $\rho_{\mathfrak{P}} = \text{Ind}_M^{\mathbb{Q}} \theta$ for a character $\theta : \text{Gal}(\overline{\mathbb{Q}}/M) \rightarrow (\mathbb{I}/\mathfrak{P})^{\times}$.

Now we suppose that we have some $\mathfrak{P}|P$ in the remaining case (C). Since $\bar{\rho}$ is absolutely irreducible, $\rho_{\mathfrak{P}}$ is absolutely irreducible. Then, by Lemma 8.4, $\rho_{\mathfrak{P}} \cong \text{Ind}_K^{\mathbb{Q}} \theta$ for a quadratic extension K/\mathbb{Q} and a character θ of $\text{Gal}(\overline{\mathbb{Q}}/K)$. Since $\rho_{\mathfrak{P}}$ is ordinary and $\bar{\psi}$ ramifies at p , (p) must split in K/\mathbb{Q} as $(p) = \mathfrak{p}\mathfrak{p}^c$. Then we may assume that θ is ramified at \mathfrak{p} and unramified at \mathfrak{p}^c . By Proposition 5.2, K must be M , and ramification at \mathfrak{p} forces $\theta \bmod \mathfrak{m}_{\mathbb{I}} = \bar{\psi}$. By (Gal), if $P \neq (T)$ or $\bar{\psi}$ is ramified at p , θ is ramified over a decomposition group D_p at p , and the other θ^c is unramified at the decomposition group.

Hereafter we treat the two cases (O) and (C) at the same time writing $\rho_{\mathfrak{P}} \cong \text{Ind}_M^{\mathbb{Q}} \theta$. By primitiveness, $\mathcal{F}_{\mathbb{I}}$ is a family of N -new forms. Thus we have $C(\rho_{\mathfrak{P}}) = N = N(\mathfrak{C})D$ for $\mathfrak{C} = \mathfrak{C}(\Psi_{\mathbb{I}})$ (see Remark 7.3). We may also assume that W and \mathbb{I} have the same residue field \mathbb{F} . As before, let $Z = \varprojlim_n Cl_M(\mathbb{C}\mathfrak{p}^n)$ and Z_p be the maximal p -profinite quotient of Z . By Proposition 5.7 (2), there exist a character $\theta : \text{Gal}(\overline{\mathbb{Q}}/M) \rightarrow W[[Z_p]]^{\times}$ unramified outside $\mathfrak{C}\mathfrak{p}$ and a canonical isomorphism $\mathbb{T}_{cm} \cong W[[Z_p]]$ such that $\rho_{\mathbb{T}_{cm}} \cong \text{Ind}_M^{\mathbb{Q}} \theta$. Moreover, identifying $\mathbb{T}_{cm} = W[[Z_p]]$, $(\mathbb{T}_{cm}, \theta)$ is the universal couple over W among deformations of $\theta \bmod \mathfrak{m}_{\mathbb{I}}$. Thus the character $\theta : \text{Gal}(\overline{\mathbb{Q}}/M) \rightarrow \mathbb{T}_{cm}^{\times}$ satisfies

$$(8.2) \quad \theta = \theta \bmod \mathfrak{P}' \text{ for a prime } \mathfrak{P}' \in \text{Spec}(\mathbb{T}_{cm}).$$

In other words, taking $\iota : \mathbb{T}_{cm} \rightarrow \kappa(\mathfrak{P})$ such that $\iota \circ \theta = \theta$, we have $\mathfrak{P}' = \text{Ker}(\iota)$ for \mathfrak{P}' in (8.2). Since $N = N(\mathfrak{C})D$, the identification $\mathbb{T}_{cm} = W[[Z_p]]$ gives rise to the algebra homomorphism $\mathbb{T} \rightarrow W[[Z_p]]$ described in Lemma 5.6, which was written as Θ there.

We write ρ' for $\rho_{\mathbb{T}_{cm}}$, $\rho'_{\mathfrak{P}'}$ for $\rho' \bmod \mathfrak{P}'$. Then we have the identity of Galois representations $\rho_{\mathfrak{P}} \cong \rho'_{\mathfrak{P}'} = \text{Ind}_M^{\mathbb{Q}} \theta_{\mathfrak{P}'}$. This implies

$$(8.3) \quad \text{Tr}(\rho_{\mathfrak{P}}(\text{Frob}_l)) = \text{Tr}(\rho'_{\mathfrak{P}'}(\text{Frob}_l)) \text{ for all } l \text{ prime to } Np.$$

Let $\mathbb{T}' \subset \mathbb{T}$ be the Λ -subalgebra generated by the image of $T(l)$ for all l prime to Np . The identity (8.3) implies $\mathfrak{P}' \cap \mathbb{T}' = \mathfrak{P} \cap \mathbb{T}'$; i.e., the image of $\text{Spec}(\mathbb{T}_{cm})$ and $\text{Spec}(\mathbb{I})$ in $\text{Spec}(\mathbb{T}')$ intersects at $\mathfrak{P} \cap \mathbb{T}'$. We now show that $\text{Spec}(\mathbb{I})$ and $\text{Spec}(\mathbb{T}_{cm})$ intersect at the unique prime divisor $\mathfrak{P} = \mathfrak{P}'$ above $\mathfrak{P} \cap \mathbb{T}'$ in $\text{Spec}(\mathbb{T})$. Since $\bar{\psi}$ ramifies at p , we may assume that $\bar{\psi}$ is unramified at \mathfrak{p}^c . Then $\chi_1|_{\mathbb{Z}_p^{\times}} = \bar{\psi}|_{\mathfrak{D}_{\mathfrak{p}^c}^{\times}}$ (identifying $\mathbb{Z}_p = \mathfrak{D}_{\mathfrak{p}}$), which is non-trivial. As remarked in the sketch,

$C(\rho_{\mathfrak{P}}) = C(\rho_{\mathfrak{P}'}) = C(\bar{\rho}) = N$. Since $\bar{\psi}$ is ramified at \mathfrak{p} , $\chi_1|_{I_p} \bmod \mathfrak{m}_W = \bar{\psi}|_{I_p}$ is non-trivial. Thus the assumptions of Lemma 5.4 are met, and we conclude $\mathbb{T}'_{\mathfrak{P}} = \mathbb{T}_{\mathfrak{P}}$. Thus $\text{Spec}(\mathbb{T}_{cm})$ and $\text{Spec}(\mathbb{I})$ intersects at $\mathfrak{P}' = \mathfrak{P}$ in $\text{Spec}(\mathbb{I}) \cap \text{Spec}(\mathbb{T}_{cm})$.

By Proposition 5.2 (2), the assumption that the order of $\bar{\psi}^-$ is greater than 2 implies $\mathbb{T}_{cm}^M = \mathbb{T}_{cm}$. Write

$$\text{Spec}(\mathbb{T}) = \text{Spec}(\mathbb{T}_{cm}^{\perp}) \cup \text{Spec}(\mathbb{T}_{cm})$$

for the complementary union of irreducible components $\text{Spec}(\mathbb{T}_{cm}^{\perp}) \subset \text{Spec}(\mathbb{T})$. Note that

$$\text{Spec}(\mathbb{I}_P) \cap \text{Spec}(\mathbb{T}_{cm,P}) = \text{Spec}(\mathbb{I}_P \otimes_{\mathbb{T}} \mathbb{T}_{cm,P}) \subset \text{Spec}(\mathbb{T}_{cm,P}^{\perp} \otimes_{\mathbb{T}} \mathbb{T}_{cm,P}).$$

By Theorem 7.2, identifying \mathbb{T}_{cm} with $W[[Z_p^-]]$, we have closed immersions

$$\text{Spec}(\mathbb{T}_{cm,\mathfrak{P}}/L^-(\bar{\psi}^-)\mathbb{T}_{cm,\mathfrak{P}}) \subset \text{Spec}(\mathbb{T}_{cm,P}^{\perp} \otimes_{\mathbb{T}} \mathbb{T}_{cm,P}) \supset \text{Spec}(\mathbb{I}_{\mathfrak{P}} \otimes_{\mathbb{T}} \mathbb{T}_{cm,\mathfrak{P}}),$$

and we have, inside $\text{Spec}(\mathbb{T}_{cm,P}^{\perp} \otimes_{\mathbb{T}} \mathbb{T}_{cm,P})$,

$$(8.4) \quad \text{Spec}(\mathbb{I}_{\mathfrak{P}} \otimes_{\mathbb{T}} \mathbb{T}_{cm,\mathfrak{P}}) \subset \text{Spec}(\mathbb{T}_{cm,\mathfrak{P}}/L^-(\bar{\psi}^-)\mathbb{T}_{cm,\mathfrak{P}}).$$

Let

$$\mathfrak{b} := \text{Ann}_{\mathbb{T}_{cm,P}}(\mathbb{I}_{\mathfrak{P}} \otimes_{\mathbb{T}} \mathbb{T}_{cm,P}) \subset \mathbb{T}_{cm,P} \quad \text{and} \quad \mathfrak{a} := \text{Ann}_{\mathbb{I}_{\mathfrak{P}}}(\mathbb{I}_{\mathfrak{P}} \otimes_{\mathbb{T}} \mathbb{T}_{cm,P}) \subset \mathbb{I}_{\mathfrak{P}},$$

where $\text{Ann}_A(X)$ is the annihilator in the ring A of an A -module X . Put $\rho_{\mathfrak{b}} = (\rho_{\mathbb{T}_{cm}} \bmod \mathfrak{b})$ and $\rho_{\mathfrak{a}} = (\rho_{\mathbb{I}} \bmod \mathfrak{a})$. Thus $\text{Tr}(\rho_{\mathfrak{b}}) = \text{Tr}(\rho_{\mathfrak{a}})$, which implies $\rho_{\mathfrak{b}} \cong \rho_{\mathfrak{a}}$ (by a result of Carayol–Serre; e.g., [MFG] Proposition 2.13). Since the right-hand-side $\rho_{\mathfrak{b}}$ is an induced representation from M , the image $\text{Im}(\rho_{\mathfrak{a}})|_{\text{Gal}(\bar{\mathbb{Q}}/M)}$ (of the right-hand-side) is in the diagonal subgroup of $GL_2(\mathbb{I}/\mathfrak{a}\mathbb{I})$. Thus $(L(\mathbb{I}))_P \mathbb{I}_{\mathfrak{P}} \subset \mathfrak{a}$. By (8.4), \mathfrak{a} is a factor of $L_{\bar{\psi}^-} \Lambda_{W,P}$. This \mathfrak{a} depends on \mathfrak{P} , and $\mathfrak{a} \cap \Lambda$ is a power of P . We fix $\mathfrak{P}|P$ such that $\mathfrak{a} \cap \Lambda$ is the smallest. We would like to show $(\mathfrak{a} \cap \Lambda_P)^2 \subset (L(\mathbb{I}))_P$.

Suppose $(L(\mathbb{I}))_P \subsetneq \mathfrak{a}$. Let $\mathfrak{s}_{\mathfrak{P}} = \mathbb{I}_{\mathfrak{P}} \cdot \mathcal{M}_1^0(\mathbb{G})$. We consider the adjoint action of $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on

$$V := (\mathfrak{s}_{\mathfrak{P}} \cap \mathfrak{a} \cdot \mathfrak{sl}_2(\mathbb{I}_{\mathfrak{P}})) / (\mathfrak{s}_{\mathfrak{P}} \cap \mathfrak{a}' \cdot \mathfrak{sl}_2(\mathbb{I}_{\mathfrak{P}}))$$

for $\mathfrak{a}' := \mathfrak{a}\mathfrak{P} \supset (L(\mathbb{I}))_P$. Then the adjoint $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ -module V is isomorphic to a factor of $Ad(\rho_{\mathfrak{P}}) \cong \left(\frac{M/\mathbb{Q}}{\mathbb{Q}}\right) \oplus \text{Ind}_M^{\mathbb{Q}} \theta^-$. Since $\bar{\psi}^-$ is non-trivial, θ^- with $\theta^- \equiv \psi^- \bmod \mathfrak{m}_{\mathbb{I}}$ is non-trivial, and hence $\text{Ind}_M^{\mathbb{Q}} \theta^-$ is absolutely irreducible. If V contains the two irreducible factors, we have $\dim_{\kappa(\mathfrak{P})} V = 3$ and hence by Nakayama's lemma, we have $(\mathfrak{s}_{\mathfrak{P}} \cap \mathfrak{a} \cdot \mathfrak{sl}_2(\mathbb{I}_{\mathfrak{P}})) = \mathfrak{a} \cdot \mathfrak{sl}_2(\mathbb{I}_{\mathfrak{P}})$; so, $(L(\mathbb{I}))_P = \mathfrak{a}$, a contradiction. If V is made up of $\left(\frac{M/\mathbb{Q}}{\mathbb{Q}}\right)$, $\mathfrak{s}_{\mathfrak{P}}/(\mathfrak{s}_{\mathfrak{P}} \cap \mathfrak{a}' \cdot \mathfrak{sl}_2(\mathbb{I}_{\mathfrak{P}}))$ is diagonal; so, the image of \mathbb{G} in $SL_2(\mathbb{I}/\mathfrak{a}')$ is in the split diagonal torus. This implies $\rho_{\mathfrak{a}'}|_{\text{Gal}(\bar{\mathbb{Q}}/M)} = \theta' \oplus \theta''$ with $\theta' \cong \bar{\psi} \bmod \mathfrak{m}_{\mathbb{I}}$. By Frobenius reciprocity law (cf. [GME] §5.1.1), we conclude $\rho_{\mathfrak{a}'} \cong \text{Ind}_M^{\mathbb{Q}} \theta'$. This is impossible, as \mathfrak{a} is the minimal $\mathbb{I}_{\mathfrak{P}}$ -ideal so that $\rho_{\mathfrak{a}}$ is an induced representation from M . Thus V contains only $\text{Ind}_M^{\mathbb{Q}} \theta^-$. Then $\mathfrak{s}_{\mathfrak{P}}$ contains matrices $X := \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}$ and $Y := \begin{pmatrix} 0 & 0 \\ v & 0 \end{pmatrix}$ whose image in V are both nontrivial. This X is in $\mathcal{M}[a]$ and Y is in $\mathcal{M}[a^{-1}]$ in the proof of Lemma 2.9. In other words, for the Λ -module $\mathfrak{n} = \{x \in \mathbb{I} | \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \in \mathcal{M}(\mathbb{G})\}$ and its opposite \mathfrak{n}_t introduced in the proof of Lemma 2.9, $\mathfrak{a} = \mathbb{I}_{\mathfrak{P}}\mathfrak{n} = \mathbb{I}_{\mathfrak{P}}\mathfrak{n}_t$. Then

$$0 \neq [X, Y] = \begin{pmatrix} uv & 0 \\ 0 & -uv \end{pmatrix} \in (\mathfrak{s}_{\mathfrak{P}} \cap \mathfrak{a}^2 \cdot \mathfrak{sl}_2(\mathbb{I}_{\mathfrak{P}})) / (\mathfrak{s}_{\mathfrak{P}} \cap \mathfrak{a}^2 \mathfrak{P} \cdot \mathfrak{sl}_2(\mathbb{I}_{\mathfrak{P}})) =: V'.$$

The Lie algebra V' also has non-trivial image of λX and λY for any generator λ of \mathfrak{a} . This shows $\dim_{\kappa(\mathfrak{P})} V' = 3$, and by Nakayama's lemma $\mathfrak{s}_{\mathfrak{P}} \cap \mathfrak{a}^2 \cdot \mathfrak{sl}_2(\mathbb{I}_{\mathfrak{P}}) = \mathfrak{a}^2 \cdot \mathfrak{sl}_2(\mathbb{I}_{\mathfrak{P}})$. Thus $\mathfrak{a}^2 = \mathbb{I}_{\mathfrak{P}}\mathfrak{nn}_t$ and $\mathfrak{uu}_t \subset (L(\mathbb{I}))$, where as before we put $\mathfrak{u} = \mathfrak{n} \cap \Lambda$ and $\mathfrak{u}_t = \mathfrak{n}_t \cap \Lambda$.

We therefore need to show $\mathbb{I}_{\mathfrak{P}}\mathfrak{u} \cap \Lambda_P = \mathbb{I}_{\mathfrak{P}}\mathfrak{u}_t \cap \Lambda_P = \mathfrak{a} \cap \Lambda_P$. Recall we have chosen \mathfrak{P} so that $\mathfrak{a}_{\mathfrak{P}} \cap \Lambda = \Lambda_{W,P} \cap \Lambda$ (i.e., $\mathfrak{a}_{\mathfrak{P}} \cap \Lambda$ is divisible by P maximally). Take $\varepsilon \in \mathbb{I}_{\mathfrak{P}}^{\times}$ so that $\varepsilon \mathbb{I}_{\mathfrak{P}} \cap \Lambda_P = \mathfrak{a} \cap \Lambda_P$. Conjugating \mathbb{G} by $\alpha = \begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix}$, we may assume that $\mathfrak{u}_P = \Lambda_P \cap \mathfrak{n}_P = \mathfrak{a}_P \cap \Lambda_P$.

Since $\text{Im}(\rho_{\mathbb{I}})$ surjects down to a dihedral group $H := \text{Im}(\bar{\rho})$, in this case the condition (v) is not satisfied (as $\text{Im}(\bar{\rho})$ does not contain nontrivial unipotent elements); so, we are assuming (s) (which is satisfied if $\psi^-|_{D_p}$ has order ≥ 3). Note that $D := \bar{\rho}(D_p) \cap GL_2(\mathbb{F}_p)$ is made of diagonal matrices of order prime to p . Taking their Teichmüller lifts, we can lift D isomorphically onto $\tilde{D} \subset GL_2(\mathbb{Z}_p)$. By our construction, D is in the image of D_p . We can lift H isomorphically onto a

dihedral subgroup $\tilde{H} \subset \text{Im}(\rho_{\mathbb{I}})$ so that $\tilde{D} \subset \tilde{H}$ (e.g., Exercise 1 of § IV.3 in [CGP] or [TGP] § IV.7). Then, as explained in the proof of Corollary 3.4, $j \in GL_2(\mathbb{I})$ in the condition (1) of Lemma 2.9 is chosen in \tilde{D} . Thus we have $c \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ whose restriction to M is the complex conjugation such that $J = \rho_{\mathbb{I}}(c) \in \tilde{H}$. Then $JjJ^{-1} = \begin{pmatrix} \zeta' & 0 \\ 0 & \zeta \end{pmatrix}$ if $j = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta' \end{pmatrix}$; i.e., the conjugation of J interchanges the two eigenvalues. By Lemma 1.4, we have $\mathcal{T}' = \left\{ \begin{pmatrix} (1+T)^s & 0 \\ 0 & 1 \end{pmatrix} \mid s \in \mathbb{Z}_p \right\} \subset \rho_{\mathbb{I}}(D_{\mathfrak{p}}) \subset \text{Im}(\rho_{\mathbb{I}})$, and $J\mathcal{T}'J^{-1} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & (1+T)^{-s} \end{pmatrix} \mid s \in \mathbb{Z}_p \right\} \subset \rho_{\mathbb{I}}(cD_{\mathfrak{p}}c^{-1})$. We have chosen an eigen basis of $\tilde{\mathbb{I}}^2$ of j to write the matrix form of $\rho_{\mathbb{I}}$. Then to have \mathfrak{c} with $\Gamma_{\Lambda}(\mathfrak{c})$ inside $\text{Im}(\rho_{\mathbb{I}})$, we change the basis v of the ζ -eigenspace of j multiplying by an element in \mathbb{I} prime to P . Since $J = \rho_{\mathbb{I}}(c)$ interchanges the two eigenspaces of j , we choose the basis of the other ζ' -eigenspace to be given by $\rho_{\mathbb{I}}(c)v$. Then this J and \tilde{D} generates the dihedral subgroup of $\tilde{H} \subset GL_2(\mathbb{Z}_p) \cap \text{Im}(\rho_{\mathbb{I}})$ lifting H isomorphically, and J is equal to $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \text{Im}(\rho_{\mathbb{I}}) \cap GL_2(\Lambda)$. Hence we may assume $Y = JXJ^{-1}$, which implies $(\mathfrak{a} \cap \Lambda_P)^2 \subset (L(\mathbb{I}))_P$.

Note that the characteristic power series $\text{Char}_{\Lambda}(W[[Z_p^-]]/L^-(\overline{\psi}^-)W[[Z_p^-]])$ over Λ_W is given by $N_{W[[Z_p^-]]/\Lambda_W}(L^-(\overline{\psi}^-)) = L_{\overline{\psi}^-}^-$. Conversely if we start with $P|L_{\overline{\psi}^-}^-$, by Theorem 7.2, the intersection scheme $\text{Spec}(\mathbb{T}_{cm,P}^{\perp} \otimes_{\mathbb{T}} \mathbb{T}_{cm,P})$ is non-empty containing a prime divisor; so, we can find an irreducible component $\text{Spec}(\mathbb{I})$ of $\text{Spec}(\mathbb{T}_{cm}^{\perp})$ such that ρ_P is isomorphic to $\text{Ind}_M^{\mathbb{Q}} \lambda$ for a character λ . This implies $P|L(\mathbb{I})$. \square

Here is the result in the residually dihedral case not included in the above theorem.

Theorem 8.6. *Assume that $\overline{\rho}$ is absolutely irreducible with $\overline{\rho} \cong \text{Ind}_M^{\mathbb{Q}} \overline{\psi}$ for a quadratic field M and a character $\overline{\psi} : \text{Gal}(\overline{\mathbb{Q}}/M) \rightarrow \overline{\mathbb{F}}_p$ with $\overline{\psi}^-$ having order > 2 . Then for any non CM component \mathbb{I} of \mathbb{T} , we have*

- (1) *If M is real and p splits into $\mathfrak{p}\overline{\mathfrak{p}}$ in M , writing $1 + p^{m+1}\mathbb{Z}_p$ with $m \geq 0$ for the kernel of the natural map $\Gamma \rightarrow Cl_M(\mathfrak{p}^{\infty})$, then $L(\mathbb{I}) \supset ((1+T)^{p^m} - 1)^2$,*
- (2) *If p does not split in M , then $L(\mathbb{I}) \supset (T^2)$.*

This theorem settles the case (3a) of Theorem II in the introduction. Note also that $\overline{\psi}^-$ having order > 2 implies $|\mathbb{F}| \geq 4$.

Proof. Let P be a prime divisor of Λ , and write \mathfrak{P} any prime divisor of \mathbb{I} above P . Let \mathfrak{c} be the prime-to- p conductor of ψ . By the same argument as in the proof of the above Theorem 8.5, if $P|L(\mathbb{I})$, we have either $\overline{\mathfrak{s}}_{\mathfrak{P}} = 0$ or $\overline{\mathfrak{s}}_{\mathfrak{P}}$ is a Cartan subalgebra or $\overline{\mathfrak{s}}_{\mathfrak{P}} = \mathfrak{sl}(2)$. If $\overline{\mathfrak{s}}_{\mathfrak{P}} = \mathfrak{sl}(2)$ for all $\mathfrak{P}|P$, then $P \nmid L(\mathbb{I})$ by Lemma 8.1, Theorem 2.12 and Corollary 3.4 combined. If $\overline{\mathfrak{s}}_{\mathfrak{P}} \neq \mathfrak{sl}(2)$, by the same argument as in the proof of Theorem 8.5, $\rho_{\mathfrak{P}} = \text{Ind}_M^{\mathbb{Q}} \theta$ for a character θ . Then for the prime-to- p conductor \mathfrak{C} of θ , we may assume that $\theta : Z \rightarrow \kappa(P)$, where $Z = \varprojlim_n Cl_M(\mathfrak{C}\mathfrak{p}^{\infty})$. If M is real, Z is a finite group, and $\theta([\gamma, \mathbb{Q}_p]^{p^m}) = 1$, where we identify the inertia groups $I_{\mathfrak{p}}$ and $I_{\overline{\mathfrak{p}}}$ and $[\gamma, \mathbb{Q}_p] \in I_{\mathfrak{p}}$. This implies $P|(1+T)^{p^m} - 1$ by (Gal).

If M is imaginary and p is non-split, θ has to be unramified at p as θ or θ^c is unramified at p (note, by (s), that $\theta^-|_{D_p}$ has to have order ≥ 3). Then $\theta([\gamma, \mathbb{Q}_p]) = 1$, which implies $P|(T)$ by (Gal). Write Z for the class group of M .

To show $L(\mathbb{I})|(1+T)^{p^m} - 1$ for some $m \geq 0$, we deal with the two cases at the same time. Let Z_p be the p -part of Z ; so, $Z = Z_p \times Z'$ for Z' of prime-to- p order. Pick a prime \mathfrak{P} of \mathbb{I} for which $\rho_{\mathfrak{P}}$ is an induced representation from M . Let \mathfrak{a} be the minimal ideal of $\mathbb{I}_{\mathfrak{P}}$ such that $\rho_{\mathfrak{a}} = (\rho_{\mathbb{I}} \bmod \mathfrak{a})$ is an induced representation from M . Then $\rho_{\mathfrak{a}} \cong \text{Ind}_M^{\mathbb{Q}} \lambda$ and λ can be identified with a character of Z by class field theory. Thus we have a W -algebra homomorphism $W[Z] \rightarrow \mathbb{I}_{\mathfrak{P}}/\mathfrak{a}$ by the universality of the group algebra. This factors through a local ring of $W[Z]$ isomorphic to $W[Z_p]$. Since \mathbb{I} is generated topologically by $\text{Tr}(\rho_{\mathbb{I}})$ over $Q(W)$ and ψ^- has order > 3 , $\mathbb{I}_{\mathfrak{P}}/\mathfrak{a}\mathbb{I}_{\mathfrak{P}}$ is generated by the values of λ . Thus $\mathbb{I}_{\mathfrak{P}}/\mathfrak{a}$ is reduced; so, \mathfrak{a} is square-free, $\mathfrak{a} \cap \Lambda \supset ((1+T)^{p^m} - 1)$, and $(\mathfrak{a} \cap \Lambda_P) \supset (L(\mathbb{I}))_P$. Then by the same argument as in the proof of Theorem 8.5, we conclude $L(\mathbb{I})_P \supset (\mathfrak{a} \cap \Lambda)^2$. \square

Assume that $\overline{\rho}$ is absolutely irreducible whose projective image in $PGL_2(\mathbb{F})$ is one of the following three type of groups: a tetrahedral group, a octahedral group or an icosahedral group. These groups

cannot be a quotient of a Borel subgroup or a unipotent group or a dihedral group, if $\bar{\mathfrak{s}}_{\mathfrak{p}} \neq \mathfrak{sl}(2)$, we have $\bar{\mathfrak{s}}_{\mathfrak{p}} = 0$. Again under the notation of the proof of Lemma 3.1, $\bar{\mathbb{H}}_{\mathfrak{p}} = \bar{\mathcal{T}}'_{\mathfrak{p}} \times \bar{\mathbb{G}}_{\mathfrak{p}} = \bar{\mathcal{T}}'_{\mathfrak{p}}$ as $\bar{\mathbb{G}}_{\mathfrak{p}} = 1$. Then we have $\bar{\rho}(\sigma)$ whose projective image does not commute with the image of j , and we find $j' \in \text{Im}(\rho_{\mathfrak{p}})$ having the same effect on j projecting down to $\bar{\rho}(\sigma)$. Then $j'\bar{\mathcal{T}}'_{\mathfrak{p}}j'^{-1} = \bar{\mathbb{H}}_{\mathfrak{p}} = \bar{\mathcal{T}}'_{\mathfrak{p}}$, which implies $\bar{\mathcal{T}}'_{\mathfrak{p}} = 1$; so, $\mathfrak{P}|T$. Thus we get the following theorem which settles the case (2) of Theorem II in the introduction.

Theorem 8.7. *Assume that $\bar{\rho}$ is absolutely irreducible whose projective image is one of the following three type of groups: a tetrahedral group, an octahedral group or an icosahedral group. Then if $\text{Spec}(\mathbb{I})$ is an irreducible component of $\text{Spec}(\mathbb{T})$, we have $T|L(\mathbb{I})|T^n$ for sufficiently large $n > 0$.*

It is interesting to determine the minimal integer n depending on \mathbb{I} . The following theorem settles the last remaining case (4) of Theorem II. Recall the power series $A(T; \theta, \psi)$ in (4.4).

Theorem 8.8. *Suppose $p \geq 5$, $p \nmid \varphi(N)$ and that N is cube-free. Assume that $\bar{\rho}$ is reducible whose semi-simplification is a direct sum of two characters $\bar{\theta}$ and $\bar{\psi}$ with $\bar{\theta}$ ramified at p and $\bar{\psi}$ unramified at p and that $\bar{\theta}/\bar{\psi}$ has order > 2 . Let $\text{Spec}(\mathbb{I})$ be an irreducible component of $\text{Spec}(\mathbb{T})$. Then $L(\mathbb{I})$ is a factor of \mathbb{L}_{N, χ_1} given in Definition 4.1. Moreover for any prime divisor P of \mathbb{L}_{N, χ_1} , if $p \nmid \varphi(N)$, there exists an irreducible non-CM component $\text{Spec}(\mathbb{I}) \subset \text{Spec}(\mathbb{T})$ such that $P|L(\mathbb{I})$.*

Since $\bar{\rho}$ is an odd Galois representation, $\bar{\theta}/\bar{\psi}(c) = \bar{\theta}\bar{\psi}(c) = -1$, and hence $\bar{\theta}/\bar{\psi}$ has order at least 2. The strategy of proving this theorem is similar to the one we took for Theorem 8.5 replacing CM components by Eisenstein components in $\text{Spec}(\mathbf{H})$. As we computed the ideal of the intersection $\text{Spec}(\mathcal{C}_{\mathbf{E}}) = \text{Spec}(\mathbf{h}) \cap \text{Spec}(\mathbf{E})$ in Corollary 4.3, the argument goes through.

Proof. Let the notation be as in the proof of Theorem 8.5. In particular, P is a prime divisor of $\text{Spec}(\Lambda)$ and \mathfrak{P} is a prime divisor of $\text{Spec}(\mathbb{I})$ above P . Again there are the following five possibilities:

- (O) $\bar{\mathfrak{s}}_{\mathfrak{p}} = 0$.
- (C) $\bar{\mathfrak{s}}_{\mathfrak{p}}$ is a Cartan subalgebra \mathfrak{H} ,
- (N) $\bar{\mathfrak{s}}_{\mathfrak{p}}$ is a nonzero nilpotent subalgebra
- (B) $\bar{\mathfrak{s}}_{\mathfrak{p}}$ is a Borel subalgebra,
- (F) $\bar{\mathfrak{s}}_{\mathfrak{p}} = \mathfrak{sl}_2(\kappa(\mathfrak{P}))$.

We can forget about the case (F) for all $\mathfrak{P}|P$ as $P \nmid L(\mathbb{I})$ in Case (F). An induced representation $\text{Ind}_M^{\mathbb{Q}} \lambda$ for a quadratic extension M/\mathbb{Q} is reducible only when λ^- is trivial, and if $\lambda^- = \mathbf{1}$, λ extends to a character $\tilde{\lambda}$ of $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ and we have $\text{Ind}_M^{\mathbb{Q}} \lambda = \tilde{\lambda} \oplus \tilde{\lambda} \left(\frac{M/\mathbb{Q}}{\cdot} \right)$. By the assumption that $\bar{\theta}/\bar{\psi}$ has order ≥ 3 , we find that $\rho_{\mathfrak{p}} \cong \text{Ind}_M^{\mathbb{Q}} \lambda$ is impossible (so, the assumption of Lemma 3.5 is satisfied, and we have $(L(\mathbb{I}))$ well defined). In particular, any component of $\text{Spec}(\mathbb{T})$ does not have CM. Thus if $\bar{\mathfrak{s}}_{\mathfrak{p}} \neq \mathfrak{sl}(2)$ or $\bar{\mathfrak{s}}_{\mathfrak{p}} \neq 0$, we have $\rho_{\mathfrak{p}} \cong \begin{pmatrix} \theta_{\mathfrak{p}} & * \\ 0 & \psi_{\mathfrak{p}} \end{pmatrix}$ with $\theta_{\mathfrak{p}} \bmod \mathfrak{m}_{\mathbb{T}} = \bar{\theta}$ and $\psi_{\mathfrak{p}} \bmod \mathfrak{m}_{\mathbb{T}} = \bar{\psi}$ with $\psi_{\mathfrak{p}}$ unramified at p as $\bar{\theta}$ ramifies at p . If $\bar{\mathfrak{s}}_{\mathfrak{p}} = 0$, again we have $\bar{\mathbb{H}}_{\mathfrak{p}} = \bar{\mathcal{T}}'_{\mathfrak{p}}$, which is normalized by $\text{Im}(\rho_{\mathfrak{p}})$; so, if $\mathfrak{P} \nmid T$, $\rho_{\mathfrak{p}}$ is reducible. If $\mathfrak{P}|T$ and $\bar{\mathfrak{s}}_{\mathfrak{p}} = 0$, $\bar{\mathbb{H}}_{\mathfrak{p}} = 1$ and hence, $\text{Im}(\bar{\rho}_{\mathfrak{p}})$ surjects down onto $\text{Im}(\bar{\rho})$ with finite kernel K (the possible error term K is in the diagonal torus, which comes from the difference of $\det(\bar{\mathcal{T}}'_{\mathfrak{p}})$ and the p -profinite part of $\text{Im}(\det(\rho_{\mathfrak{p}}))$), which implies $\rho_{\mathfrak{p}}$ is reducible. Thus \mathfrak{P} is an Eisenstein ideal.

We now specify the Λ -adic Eisenstein component with which $\text{Spec}(\mathbb{I})$ intersects at \mathfrak{P} . Write

$$\rho_{\mathfrak{p}}^{ss} = \begin{pmatrix} \theta_{\mathfrak{p}} & 0 \\ 0 & \psi_{\mathfrak{p}} \end{pmatrix}.$$

The prime-to- p conductor of $\rho_{\mathfrak{p}}^{ss}$ is the product $\mathfrak{C}(\theta_{\mathfrak{p}})\mathfrak{C}(\psi_{\mathfrak{p}})$ of the prime-to- p conductors $\mathfrak{C}(\theta_{\mathfrak{p}})$ and $\mathfrak{C}(\psi_{\mathfrak{p}})$. Thus we have $\mathfrak{C}(\theta_{\mathfrak{p}})\mathfrak{C}(\psi_{\mathfrak{p}})|N$. By (Gal), we may assume that $\psi_{\mathfrak{p}}$ is unramified at p . Thus $\psi_{\mathfrak{p}}$ only (possibly) ramified at prime factors of N prime to p . By class field theory, the image of the inertia group I_l at l in the abelianization of the decomposition group D_l at l is isomorphic to the almost l -profinite group \mathbb{Z}_l^{\times} . Thus $\psi_{\mathfrak{p}}|_{I_l}$ with values in almost p -profinite subgroup of $\kappa(\mathfrak{P})^{\times}$ has to be of finite order. Then by global class field theory, $\psi_{\mathfrak{p}}$ is of finite order. If $\kappa(\mathfrak{P})$ has characteristic p , $\psi_{\mathfrak{p}}$ has values in $\bar{\mathbb{F}}_p^{\times}$ (so, $\psi_{\mathfrak{p}} = \bar{\psi}$), and we have a unique Teichmüller lift

$\psi : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow W^\times$ of $\psi_{\mathfrak{P}} = \overline{\psi}$. If $\kappa(\mathfrak{P})$ has characteristic 0, we put $\psi = \psi_{\mathfrak{P}}$. We may assume that ψ has values in W^\times , extending scalar if necessary. Now we consider the ray class group $Cl_{\mathbb{Q}}(Np^n)$ and $Y = \varprojlim_n Cl_{\mathbb{Q}}(Np^n) \cong (\mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times) / \{\pm 1\}$. By class field theory, for the maximal ray class field $\overline{\mathbb{Q}}[\mu_{Np^\infty}]/\mathbb{Q}$ modulo Np^∞ , we have a canonical isomorphism $\text{Gal}(\overline{\mathbb{Q}}[\mu_{Np^\infty}]/\mathbb{Q}) \cong Y$. We identify these two groups. Write Y_p for the Sylow p -profinite subgroup of Y ; so, $Y = Y^{(p)} \times Y_p$ canonically for the finite group $Y^{(p)}$ of order prime to p . We consider the group algebra $W[[Y_p]]$ and for $u \in Y$, we write $[u_p]$ for the group element in $Y_p \subset W[[Y_p]]^\times$ represented by the projection of u in Y_p . By the same deformation argument proving Proposition 5.7 (2) (and used in Theorem 8.5), for the Teichmüller lift $\theta_0 : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow W^\times$ of $\overline{\theta} = \theta_{\mathfrak{P}} \bmod \mathfrak{m}_{\mathbb{I}}$, if $p \nmid \varphi(N)$, $(W[[Y_p]], \theta)$ for $\theta([u, \mathbb{Q}_p]) = \theta_0(u)[u_p] \in W[[Y_p]]$ for $u \in Y$ is the universal couple among all deformations $(A, \epsilon : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow A^\times)$ of $(\mathbb{F}, \theta_0 \bmod \mathfrak{m}_W)$ with prime-to- p conductor $\mathfrak{C}(\epsilon)|N$. Let Y_t be the maximal torsion subgroup of Y_p . We may assume that any character: $Y_t \times Y^{(p)} \rightarrow \overline{\mathbb{Q}}_p^\times$ actually has values in W^\times by extending scalar if necessary. The maximal torsion free quotient of Y_p is canonically isomorphic to Γ , and we have a non-canonical decomposition $Y = Y_t \times \Gamma$ with an abelian p -group Y_t . We identify $W[[\Gamma]]$ with $W[[T]]$ by $\gamma \mapsto (1+T)$. Since $W[[Y_p]] = W[Y_t][[\Gamma]]$, geometrically irreducible components of $\text{Spec}(W[[Y_p]])$ are indexed by characters $\theta : Y_t \rightarrow \overline{\mathbb{Q}}_p^\times$ so that the component is given by the W -algebra projection $\theta_* : W[[Y_p]] \rightarrow W[[T]]$ sending $y \in Y_t$ to $\theta(y)$ and γ to $(1+T)$. We call this component the θ -component. Take θ such that $\theta_* \circ \theta \bmod \mathfrak{P} = \theta_{\mathfrak{P}}$ in \mathbb{I} . By (Gal), we have $\theta_* \circ \theta(\text{Frob}_l) = \theta(\text{Frob}_l)\langle l \rangle$ for all primes l outside Np . Since $\theta_{\mathfrak{P}}$ gives rise to a point \mathfrak{P}' of an irreducible component $\text{Spec}(W[[T]])$ of the universal deformation space $\text{Spec}(W[[Y_p]])$ so that $\theta_{\mathfrak{P}} \equiv \theta_* \circ \theta \bmod \mathfrak{P}'$ (with $\mathfrak{P}' = \mathfrak{P} \cap W[[T]]$). Consider the Λ -adic Eisenstein series $E(\theta, \psi)$. By our construction, $\rho_{\mathfrak{P}}$ is isomorphic to $\psi \oplus (\theta_* \circ \theta) \bmod \mathfrak{P}'$. Then in a way similar to the CM case, we can find a possibly “old” Eisenstein component \mathbb{I}' with Galois representation $\psi \oplus (\theta_* \circ \theta)$ which intersects with \mathbb{I} at P . Indeed, again by $l|C := \mathfrak{C}(\psi_{\mathfrak{P}})\mathfrak{C}(\theta_{\mathfrak{P}}) \Leftrightarrow l|N$, mismatch of $\dim H_0(I_l, \rho_{\mathfrak{P}})$ and $\dim H_0(I_l, \rho_{\mathbb{I}})$ could occur only when $l|(N/C)$ and $l|\mathfrak{C}(\xi)$ but $l \nmid \mathfrak{C}(\eta)$ for $\{\xi, \eta\} = \{\psi_{\mathfrak{P}}, \theta_{\mathfrak{P}}\}$. Then writing $\Xi(\eta)$ for the set of primes $l|(N/C)$ with the above divisibility/non-divisibility property, we consider the imprimitive characters ψ' (resp. θ') induced by ψ (resp. θ) modulo $M_1 := \mathfrak{C}(\psi) \prod_{l \in \Xi(\psi)} l$ (resp. $M_2 := p \cdot \mathfrak{C}(\theta) \prod_{l \in \Xi(\theta)} l$). Then the Eisenstein series $E(\theta', \psi')$ has congruence modulo P with the \mathbb{I} -adic form. Therefore $\mathbb{I}/(L(\mathbb{I}))\mathbb{I}$ is a surjective image of the Λ -submodule $\mathcal{C}_{\mathbb{E}} \otimes_{\mathbb{T}} \mathbb{I}$ of the Eisenstein congruence module $\mathcal{C}_{\mathbb{E}}$ defined just above Corollary 4.3. Therefore $(L(\mathbb{I}))_P \subset (\mathbb{L}_{N, \chi_1})_P$. Let $\mathfrak{a}_{\mathfrak{P}} = \text{Ann}_{\mathbb{I}_{\mathfrak{P}}}(\mathcal{C}_{\mathbb{E}} \otimes_{\mathbb{T}} \mathbb{I})$. Then $\mathfrak{a}_{\mathfrak{P}}$ is the minimal ideal so that $\rho_{\mathfrak{P}}$ is isomorphic to representation into $\mathcal{B}(\mathbb{I}_{\mathfrak{P}}/\mathfrak{a})$ and is a factor of $L(\overline{\theta}, \overline{\psi})$. If $P|(p)$, we know by [H12a] Theorem 6.2, $\rho_{\mathfrak{P}}$ is irreducible if $p \nmid \varphi(N)$; so, we may assume that $P \nmid (p)$. Then by Corollary 3.6, $\mathfrak{c}_P = \bigcap_{\mathfrak{P}|P} \mathfrak{a}_{\mathfrak{P}} \cap \Lambda_{W, P}$; so, $\mathfrak{c}_P | \mathbb{L}_{N, \chi_1}$.

The existence of \mathbb{I} with $\mathfrak{P}|L(\mathbb{I})$ for $P|\mathbb{L}_{N, \chi_1}$ follows from the definition of $\mathcal{C}_{\mathbb{E}}$. Indeed, there exists at least one component $\text{Spec}(\mathbb{I})$ containing $P \in \text{Spec}(\mathcal{C}_{\mathbb{E}})$. As already remarked, any component of $\text{Spec}(\mathbb{T})$ is not of CM type. \square

9. MIXED CASES

Pick a minimal primitive irreducible component $\text{Spec}(\mathbb{I})$ of $\text{Spec}(\mathfrak{h})$. Let $\text{Spec}(\mathbb{T})$ be the connected component of $\text{Spec}(\mathfrak{h})$ containing $\text{Spec}(\mathbb{I})$. We consider an $\tilde{\mathbb{I}}$ -lattice \mathcal{L} in $Q(\mathbb{I})^2$ stable under $\rho_{\mathbb{I}}$. Take its reflexive closure $\tilde{\mathcal{L}}$, which remains stable under $\rho_{\mathbb{I}}$. For any $0 \neq t \in \tilde{\mathbb{I}}$, the multiple $t\tilde{\mathcal{L}}$ remains $\tilde{\mathbb{I}}$ -reflexive. Thus by [BCM] VII.4.2 Proposition 7, the set of associated primes of $\tilde{\mathcal{L}}/t\tilde{\mathcal{L}}$ is made of prime divisors. Thus if $\tilde{\mathbb{I}}/t\tilde{\mathbb{I}} = W$, $\tilde{\mathcal{L}}/t\tilde{\mathcal{L}}$ has to be a free W -module, since W is a discrete valuation ring. Thus, we must have $\text{rank}_W \tilde{\mathcal{L}}/t\tilde{\mathcal{L}} = 2$, which implies that, by Nakayama’s lemma, $\tilde{\mathcal{L}}$ is free of rank 2 over $\tilde{\mathbb{I}}$. Thus if $\tilde{\mathbb{I}}$ is a unique factorization domain, the condition (F) in the introduction holds.

By resolution of singularity (see [L]), we have a dimension two complete regular local ring $\mathbb{I} \subset Q(\mathbb{I})$ containing \mathbb{I} . We may assume that $\pi : \text{Spec}(\mathbb{I}) \rightarrow \text{Spec}(\mathbb{I})$ is the minimal resolution. The non-flat locus of π in $\text{Spec}(\mathbb{I})$ is at most of codimension 2 (so, its support is the unique closed point \mathfrak{m} of $\text{Spec}(\mathbb{I})$). Writing E for the set of prime divisors of \mathbb{I} above the maximal ideal of \mathbb{I} (those of exceptional divisors over the singular point $\mathfrak{m}_{\mathbb{I}} \in \text{Spec}(\mathbb{I})$), the set of all prime divisors of \mathbb{I} is in

bijection to prime divisors of \mathbb{I} outside E as $\text{Spec}(\mathbb{I}) \setminus \{\mathfrak{m}_{\mathbb{I}}\} \cong \text{Spec}(\mathbb{I}) \setminus E$. Since \mathbb{I} is regular, it is a unique factorization domain (see [CRT] Theorem 20.3). Thus by the above argument, extending scalar W so that $\text{Spec}(\mathbb{I})(W) \neq \emptyset$, the reflexive \mathbb{I} -closure L of $\mathbb{I} \cdot \mathcal{L}_{can}(\mathbb{I}) \subset Q(\mathbb{I})^2$ is free of rank 2 over \mathbb{I} and is stable under $\rho_{\mathbb{I}}$. We write $\rho_{\mathbb{I}} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow GL_2(\mathbb{I})$ for the Galois representation realized on L . Though for simplicity, we assume the condition (F_{can}) in this section, all divisibility conjectures we make should hold for \mathbb{I} inverting primes of E taking $\rho_{\mathbb{I}}$ in place of $\rho_{\mathbb{I}}$ acting on $\mathcal{L}_{can}(\mathbb{I})$ without assuming (F_{can}) (we note here, $\tilde{\mathbb{I}}$ itself is regular in numerical examples so far known).

Throughout this section we assume (R) and (F_{can}) in addition to $p \geq 5$ and that N is cube-free. Then by Theorem I, we have the conductor \mathfrak{c} of $\text{Im}(\rho_{\mathbb{I}}) \cap SL_2(\Lambda)$. Here $\text{Im}(\rho_{\mathbb{I}})$ is taken in $GL_2(Q(\mathbb{I}))$. In order to determine the global level exactly, we need to know the characteristic power series of the congruence module between the CM part and the non CM part and also the Eisenstein and non Eisenstein parts of $\text{Spec}(\mathbb{T})$. A key ingredient of solving this question is Gorenstein-ness of each part (cf. Theorem 7.2). If different CM components or/and Eisenstein components are mixed, it is difficult to prove Gorenstein-ness of CM/Eisenstein part. Writing $\bar{\rho}$ for $\rho_{\mathfrak{m}_{\mathbb{T}}}$, let us describe this problem in more details. We separate our argument into the following five mixed cases (which cover all possible cases by Proposition 5.2):

- (EIS) $\bar{\rho} = \theta \oplus \psi$ with both $\bar{\theta}$ and $\bar{\psi}$ unramified at p with $\bar{\psi}/\bar{\theta}$ has order > 2 ;
- (UCM) absolutely irreducible $\bar{\rho} \cong \text{Ind}_M^{\mathbb{Q}} \bar{\psi}$ for an imaginary quadratic fields M with $\bar{\psi}$ unramified at p and $\bar{\psi}^-$ has order > 2 ;
- (SCM) absolutely irreducible $\bar{\rho} \cong \text{Ind}_M^{\mathbb{Q}} \bar{\varphi} \cong \text{Ind}_K^{\mathbb{Q}} \bar{\phi}$ for two p -splitting distinct imaginary quadratic fields M and K ; so, $\varphi^- = \left(\frac{MK/M}{}\right)$ and $\phi^- = \left(\frac{MK/K}{}\right)$;
- (HCM) absolutely irreducible $\bar{\rho} \cong \text{Ind}_M^{\mathbb{Q}} \bar{\varphi} \cong \text{Ind}_K^{\mathbb{Q}} \bar{\phi}$ for an imaginary quadratic fields M in which p splits and a quadratic field K which is real or in which p is not split; so, $\varphi^- = \left(\frac{MK/M}{}\right)$ and $\phi^- = \left(\frac{MK/K}{}\right)$.
- (ECM) $\bar{\rho} = \bar{\theta} \oplus \bar{\psi} \cong \text{Ind}_M^{\mathbb{Q}} \bar{\varphi}$ for a quadratic field M ; so, $\bar{\psi}/\bar{\theta} \equiv \left(\frac{M/\mathbb{Q}}{}\right) \pmod{\mathfrak{m}_{\mathbb{T}}}$.

The five cases are disjoint, and M in Case (ECM) is imaginary as $\bar{\psi}/\bar{\theta}$ is an odd character. Except for the case (ECM), we have well defined $L(\mathbb{I})$ (see Section 3). The difficulty of determining all possible cases of $\mathfrak{s}_{\mathfrak{p}} \neq \mathfrak{s}_{l_2}$ in these cases comes from the fact that some primes P with $\mathfrak{s}_{\mathfrak{p}} \neq \mathfrak{s}_{l_2}$ could be a prime of congruence between components of $U(p)$ -deprived Hecke algebra $\mathfrak{h}^{(p)} \subset \mathfrak{h}$ generated by $T(l)$ ($l \nmid Np$) and $U(q)$ for $q \neq p$ over Λ . In order to determine exact level $L(\mathbb{I})$, we need to show that the local component $\mathbb{T}^{(p)}$ of $\mathfrak{h}^{(p)}$ involved is Gorenstein up to finite error (which is not known and perhaps not expected in general either).

The Katz measure μ on \mathfrak{z} actually depends on the choice of p -adic CM type of the imaginary quadratic field M (i.e., a choice of (M, \mathfrak{p}) and $(M, \bar{\mathfrak{p}})$). Our choice is (M, \mathfrak{p}) for \mathfrak{p} corresponding to $i_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. If we change (M, \mathfrak{p}) by $(M, \bar{\mathfrak{p}})$, we get another measure, μ^* . The two measures are related by a functional equation (e.g., [H10] Introduction). We write $(L_{\psi^-}^-)^*$ for the product of the Katz p -adic L -function with modulo p branch character ψ^- with respect to $(M, \bar{\mathfrak{p}})$. We may conjecture the following outcome in the above cases:

Conjecture 9.1. *For the non CM component $\text{Spec}(\mathbb{I}) \subset \text{Spec}(\mathbb{T})$,*

- in Case (EIS), $L(\mathbb{I})$ is a factor of $L(\bar{\theta}, \bar{\psi}) \cdot L(\bar{\psi}, \bar{\theta})$;*
- in Case (UCM), $L(\mathbb{I})$ is a factor of $(L_{\bar{\psi}}^- \cdot (L_{\bar{\psi}^- \circ c}^-)^*)^2$;*
- in Case (SCM), $L(\mathbb{I})$ is a factor of $(L_{\bar{\varphi}}^- \cdot L_{\bar{\phi}}^- \cdot (L_{\bar{\varphi}^- \circ c}^-)^* \cdot (L_{\bar{\phi}^- \circ c}^-)^*)^2$;*
- in Case (HCM), $L(\mathbb{I})$ is a factor of*

$$\begin{cases} (L_{\bar{\varphi}}^-)^2 \cdot ((1+T)^{p^m} - 1)^2 & \text{if } \bar{\varphi}^- \text{ is ramified at } p, \\ \left(L_{\bar{\varphi}}^- \cdot (L_{\bar{\varphi}^- \circ c}^-)^* \cdot ((1+T)^{p^m} - 1) \right)^2 & \text{if } \bar{\varphi}^- \text{ is unramified at } p \end{cases}$$

for a sufficiently large integer $m > 0$;

in Case (ECM), for prime divisor $P \in \text{Spec}(\Lambda)$ not under the intersection of a CM and an Eisenstein component, we can define local conductor \mathfrak{c}_P as in Section 3. For P under the intersection of a CM and an Eisenstein component, further suppose that $\overline{\theta\psi}$ has prime-to- p conductor N . Then in the isomorphism class of $\rho_{\mathbb{I}}$ realized on $\mathcal{L}_{\text{can}}(\mathbb{I})$ over $\widetilde{\mathbb{I}}$, we can find ρ with maximal possible conductor \mathfrak{c} such that \mathfrak{c}_P coincides with the one given in Section 3 for P not under the intersection of a CM and an Eisenstein component, $\text{Im}(\rho) \supset \Gamma_{\Lambda}(\mathfrak{c})$, and $(L(\mathbb{I})) = \Lambda \cap \bigcap_P \mathfrak{c}_P$ is a factor of $L(\overline{\theta}, \overline{\psi}) \cdot L(\overline{\theta}, \overline{\psi}) \cdot (L_{\overline{\varphi}}^- \cdot (L_{\overline{\varphi}-\text{oc}}^-)^*)^2$.

To explain our heuristic reasoning supporting this conjecture, we pick Case (SCM). Then $\text{Spec}(\mathbb{T})$ could contain two CM components $\text{Spec}(\mathbb{T}_{\text{cm}}^K)$ and $\text{Spec}(\mathbb{T}_{\text{cm}}^M)$. After inverting T , by Corollary 5.3, the connected component S of $\text{Spec}(\mathbb{T}[\frac{1}{T}])$ containing $\text{Spec}(\mathbb{I}[\frac{1}{T}])$ can have non-trivial intersection only with one CM component, say, $\text{Spec}(\mathbb{T}_{\text{cm}}^M)$. Then our argument proving Theorem 8.5 relative to an irreducible component \mathbb{J} of \mathbb{T}_{cm}^M should go through after inverting T . Thus, outside an exceptional divisor (containing (T) and the zero divisor of $E_{1,N}$ for M and for K), quite possibly, the conjecture should follow. The real challenge would be the analysis at primes inside the exceptional divisor. All other cases should be similar in the sense that after inverting a suitable divisor, the assertion of the conjecture would be probable.

10. PRIME-TO- p CONDUCTOR OF p -ADIC GALOIS REPRESENTATION

We summarize facts on ramification at a prime $q \neq p$ of p -adic Galois representations we have used. Let R be a p -profinite local ring. Let $M \subset \overline{\mathbb{Q}}$ be a finite extension of \mathbb{Q} with integer ring \mathfrak{D} , and put $\widehat{\mathfrak{D}}^{(p)} = \prod_{l \neq p} (\mathfrak{D} \otimes_{\mathbb{Z}} \mathbb{Z}_l)$. For any continuous character $\psi : M^{\times} \backslash M_{\mathbb{A}}^{\times} \rightarrow R^{\times}$ unramified outside Np , the restriction $\psi : (\widehat{\mathfrak{D}}^{(p)})^{\times} \rightarrow R^{\times}$ has to be a finite order character, as ψ is ramified only at finitely many primes and R^{\times} is an almost p -profinite group. Thus we have an integral ideal $\mathfrak{C}(\psi)$ maximal among ideals \mathfrak{a} prime to p with $(1 + \mathfrak{a}\widehat{\mathfrak{D}}^{(p)}) \cap (\widehat{\mathfrak{D}}^{(p)})^{\times} \subset \text{Ker}(\psi)$. We call $\mathfrak{C}(\psi)$ the *prime-to- p conductor* of ψ . By local class field theory, a continuous character: $\text{Gal}(\overline{\mathbb{Q}}/M) \rightarrow R^{\times}$ can be viewed as an idele character ψ , and hence the definition of $\mathfrak{C}(\psi)$ applies also to Galois characters. For $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acting nontrivially on M leaving it stable, we define $\psi^{\sigma} : \text{Gal}(\overline{\mathbb{Q}}/M) \rightarrow R^{\times}$ by $\psi^{\sigma}(\tau) = \psi(\sigma\tau\sigma^{-1})$. The idele character corresponding to the Galois character ψ^{σ} is given by composing ψ with the action of σ on $M_{\mathbb{A}}^{\times}$. For a rational prime $q \neq p$, the q -primary part $\mathfrak{C}_q(\psi)$ of $\mathfrak{C}(\psi)$ is called the q -conductor of ψ . Obviously, $\mathfrak{C}_q(\psi)$ only depends on ψ restricted to the inertia group at q , and therefore, $\mathfrak{C}_q(\psi)$ is well defined for any finite order character ψ of the inertia group. If $M = \mathbb{Q}$, often we identify the ideal $\mathfrak{C}_q(\psi) = (q^e)$ with the positive integer q^e .

Now we recall the conductor $C_q(\rho)$ of a two-dimensional Galois representation $\rho : \text{Gal}(\overline{\mathbb{Q}}_q/\mathbb{Q}_q) \rightarrow GL_2(R)$ for a prime $q \neq p$ (e.g., [GME] Theorem 5.1.9). It only depends on the restriction of ρ to the inertia group $I_q \subset \text{Gal}(\overline{\mathbb{Q}}_q/\mathbb{Q}_q)$. Recall the exact sequence $1 \rightarrow I_q^w \rightarrow I_q \rightarrow I_q^t \rightarrow 0$ of the wild inertia group I_q^w and the tame inertia group $I_q^t \cong \widehat{\mathbb{Z}}^{(q)}$ which is an abelian group (e.g., [MFG] §3.2.5).

Lemma 10.1. *Let the notation be as above and $\rho : \text{Gal}(\overline{\mathbb{Q}}_q/\mathbb{Q}_q) \rightarrow GL_2(R)$ be a continuous representation for a reduced p -profinite noetherian local ring R . Suppose $q \neq p$.*

- (1) *Unless $\rho_{\mathfrak{p}}|_{I_q}$ is reducible indecomposable for some minimal prime \mathfrak{p} of R , $\rho|_{I_q}$ has finite image.*
- (2) *If there exists a prime ideal P_0 of the p -profinite ring R such that $\rho_0 = \rho \pmod{P_0}$ is absolutely irreducible over I_q^w , then for all prime ideal P of R , $\rho_P = (\rho \pmod{P})$ is absolutely irreducible over I_q^w .*
- (3) *Suppose that R is an integral domain. If $\rho|_{I_q^w}$ is reducible and ρ is absolutely irreducible, then $\rho \cong \text{Ind}_K^{\mathbb{Q}_q} \xi$ for a character ξ of $\text{Gal}(\overline{\mathbb{Q}}_q/\mathbb{Q}_q)$ of a quadratic extension K/\mathbb{Q}_q .*
- (4) *If R is an integral domain and $\rho|_{I_q}$ is reducible indecomposable, we have $\rho \cong \begin{pmatrix} \mathcal{N}\eta & * \\ 0 & \eta \end{pmatrix}$ with $\eta|_{I_q}$ having finite order, where \mathcal{N} is the unramified cyclotomic character acting on μ_{p^∞} .*

Proof. We first prove (1). Since $R \hookrightarrow \bigoplus_{\mathfrak{p}} R/\mathfrak{p}$ for finitely many minimal ideal \mathfrak{p} , replacing R by R/\mathfrak{p} , we may assume that R is an integral domain. Since $\rho|_{I_q^w}$ has finite image (factoring through

$GL_2(R/\mathfrak{m}_R)$ and $I_q = I_q^t \rtimes I_q^w$, by restricting ρ to $\text{Gal}(\overline{\mathbb{Q}}_q/K)$ for a finite extension K/\mathbb{Q}_q , we may assume that $\rho|_{\text{Gal}(\overline{\mathbb{Q}}_q/K)}$ is reducible to prove (3). Let $I_K^w = I_q^w \cap \text{Gal}(\overline{\mathbb{Q}}_q/K)$ and I_K^t be the image of $I_K = I_q \cap \text{Gal}(\overline{\mathbb{Q}}_q/K)$ in I_q^t . Then $1 \rightarrow I_K^w \rightarrow I_K \rightarrow I_K^t \rightarrow 1$ is exact. Since $\rho(I_K^w)$ cannot contain non-trivial unipotent element as $q \neq p$, $\rho|_{I_K^w} = \eta \oplus \xi$ for two finite order characters factoring through $(R/\mathfrak{m}_R)^\times$. Since I_K^t is cyclic, by [MFG] Corollary 4.37, either $\rho_M := \rho|_{\text{Gal}(\overline{\mathbb{Q}}_q/M)} \cong \tilde{\xi} \oplus \tilde{\eta}$ for extensions $\tilde{\xi}$ and $\tilde{\eta}$ of ξ and η to $\text{Gal}(\overline{\mathbb{Q}}_q/M)$ for an extension M/K with $[M : K] \leq 2$ or $\rho(I_K)$ contains non-trivial unipotent element of p -power order, which is excluded by our assumption; thus $\rho_M \cong \tilde{\xi} \oplus \tilde{\eta}$. Replacing M by its finite extension, we may assume that $\tilde{\xi}$ and $\tilde{\eta}$ factor through I_M^t on which an element $\phi \in \text{Gal}(\overline{\mathbb{Q}}_q/M)$ surjecting down to the Frobenius element over M acts by $\phi\sigma\phi^{-1} = \sigma^Q$ for a q -power Q . Defining the inner conjugate ξ^ϕ by $\xi^\phi(\sigma) = \xi(\phi\sigma\phi^{-1})$, we have $\tilde{\eta}^\phi = \tilde{\eta}^Q = \tilde{\eta}$. This implies $\tilde{\eta}$ is of finite order; so, $\rho|_{I_q}$ has finite image. This proves (1).

By p -profiniteness of R , the residue field R/\mathfrak{m}_R is finite for the maximal ideal \mathfrak{m}_R of R . Since $\Gamma_R(\mathfrak{m}_R)$ is p -profinite, $q \neq p$ implies that $\rho|_{I_q^w}$ factors through $GL_2(R/\mathfrak{m}_R)$. If $\rho_0|_{I_q^w}$ is absolutely irreducible, then $\bar{\rho}|_{I_q^w}$ is absolutely irreducible for $\bar{\rho} = \rho \pmod{\mathfrak{m}_R}$. Then $\rho_P|_{I_q^w}$ is absolutely irreducible for all $P \in \text{Spec}(R)$. This proves (2).

To prove (3), assume reducibility of $\rho|_{I_q^w}$. Then it is semi-simple as $q \neq p$; so, $\rho|_{I_q^w} \cong \xi \oplus \eta$. Then we have

$$\rho(\sigma) \begin{pmatrix} \xi & 0 \\ 0 & \eta \end{pmatrix} \rho(\sigma)^{-1} = \rho^\sigma \cong \begin{pmatrix} \xi^\sigma & 0 \\ 0 & \eta^\sigma \end{pmatrix}$$

for each $\sigma \in \text{Gal}(\overline{\mathbb{Q}}_q/\mathbb{Q}_q)$. If $\xi^\sigma = \xi$ for all $\sigma \in \text{Gal}(\overline{\mathbb{Q}}_q/\mathbb{Q}_q)$, absolutely irreducible ρ commutes with $\rho(I_q^w)$; so, we conclude $\xi = \eta$. We call this case Case Z. If $\xi^\sigma \cong \eta \neq \xi$, then its stabilizer is $\text{Gal}(\overline{\mathbb{Q}}_q/K)$ for a quadratic extension K/\mathbb{Q}_q . We call this case Case D.

In Case D, by [LRF] Proposition 24 in §8.1 (whose proof does not require $p \neq 0$ in R as long as $\rho|_{I_q^w}$ is semi-simple), ρ is isomorphic to an induced representation of a character of $\text{Gal}(\overline{\mathbb{Q}}_q/K)$ extending ξ as asserted.

Suppose that we are in Case Z. Then $\rho(I_q^w)$ is in the center of $\rho(I_q)$. Since I_q^t is abelian and $\rho(I_q) = \rho(I_q^w) \rtimes \rho(I_q^t)$, $\rho(I_q)$ is an abelian group. Thus $\rho|_{I_q}$ is reducible. Writing $\rho|_{I_q} = \xi \oplus \eta$, $\text{Gal}(\overline{\mathbb{Q}}_q/\mathbb{Q}_q)$ acts on ξ and η by inner conjugation. If the stabilizer of ξ is a proper subgroup of $\text{Gal}(\overline{\mathbb{Q}}_q/\mathbb{Q}_q)$, we find a quadratic extension K/\mathbb{Q}_q such that ρ is an induced representation as asserted. If the stabilizer is the entire group, $\xi = \eta$ and $\rho(I_q)$ is in the center of $\text{Im}(\rho)$. Since $\text{Im}(\rho) = \rho(I_q) \rtimes \langle \rho(\phi) \rangle$ for an element ϕ giving the Frobenius automorphism of the maximal unramified extension of \mathbb{Q}_q , $\text{Im}(\rho)$ is abelian, contradicting the absolute irreducibility of ρ . This finishes the proof of (3).

To prove (4), write $\rho|_{I_q} \cong \begin{pmatrix} \xi & u \\ 0 & \eta \end{pmatrix}$. Then $\rho^\phi|_{I_q} \cong \begin{pmatrix} \xi^\phi & u^\phi \\ 0 & \eta^\phi \end{pmatrix} \cong \rho|_{I_q}$ for a Frobenius $\phi \in \text{Gal}(\overline{\mathbb{Q}}_q/\mathbb{Q}_q)$. By indecomposability, we have $\eta^\phi = \eta^q = \eta$ over I_q ; so, $\eta|_{I_q}$ is of finite order, and $u^\phi = qu$, which shows $\eta/\xi = \mathcal{N}$ for the cyclotomic character \mathcal{N} giving the action of $\text{Gal}(\overline{\mathbb{Q}}_q/\mathbb{Q}_q)$ on μ_{p^∞} ; so, \mathcal{N} is unramified, and we get the desired result. \square

Suppose that R is an integral domain. Regarding ρ having values in $GL_2(Q(R))$ for the quotient field $Q(R)$ of R , we define $C_q(\rho) = q^e$ as follows. Let \mathbb{Q}_q^{ur} be the maximal unramified extension of \mathbb{Q}_q and K with integer ring V be the splitting field of $\rho|_{I_q}$. We put

$$I_i = \{\sigma \in \text{Gal}(K/\mathbb{Q}_q^{ur}) \mid \sigma(x) \equiv x \pmod{\mathfrak{m}_V^{i+1}}\}.$$

Then we define

$$e = \sum_{i=0}^{\infty} \frac{1}{[I_0 : I_i]} (2 - \dim_{Q(R)} H^0(I_i, \rho)).$$

If $P_1 \supset P_2$ are two primes of R , $\dim_{\kappa(P_1)} H^0(I_i, \rho_{P_1}) \geq \dim_{\kappa(P_2)} H^0(I_i, \rho_{P_2})$, we have $C_q(\rho_{P_1}) \leq C_q(\rho_{P_2})$ for $\rho_{P_j} = \rho \pmod{P_j}$. If R is not an integral domain, we define

$$C_q(\rho) = \text{Sup}_{P \in \text{Spec}(R)} C_q(\rho_P).$$

Here are some explicit identification of $C_q(\rho)$ (given in [GME] Theorem 5.1.9) when R is an integral domain. This covers all cases used in the proof of Theorem 8.5 (as $\bar{\rho}$ is an induced representation in

Theorem 8.5). If $\rho|_{I_q}$ is isomorphic to a representation $\begin{pmatrix} \eta & * \\ 0 & \eta \end{pmatrix}$ over $Q(R)$, we have

$$C_q(\rho) = \begin{cases} q & \text{if } \eta \text{ is unramified and } \rho \text{ is indecomposable,} \\ \mathfrak{C}_q(\eta)^2 & \text{if } \eta \text{ is ramified.} \end{cases}$$

If $\rho_P \cong \alpha \oplus \beta$ for two characters $\alpha, \beta : \text{Gal}(\overline{\mathbb{Q}}_q/\mathbb{Q}_q) \rightarrow R^\times$, the characters $\alpha|_{I_q}$ and $\beta|_{I_q}$ are of finite order. We then have $C_q(\rho) = \mathfrak{C}_q(\alpha)\mathfrak{C}_q(\beta)$. If ρ is absolutely irreducible and it has of the form $\rho \cong \text{Ind}_H^{I_q} \xi$ for an open subgroup H of I_q of index 2, writing (q^e) for the discriminant of the quadratic extension $\overline{\mathbb{Q}}_q^H/\overline{\mathbb{Q}}_q^{I_q}$, we have $C_q(\rho) = q^{e+f}$, where (q^f) is the norm relative to $\overline{\mathbb{Q}}_q^H/\overline{\mathbb{Q}}_q^{I_q}$ of the conductor $\mathfrak{C}_q(\xi)$ of ξ .

To an automorphic representation π generated by a holomorphic Hecke eigenform, we have a p -adic Galois representation ρ_f associated to f (cf. [GME] §4.2) whose restriction to I_q has the form we treated as above. Then $C_q(\rho_f)$ coincides with the q -part of the conductor $C_q(\pi)$ of π in the sense of [AAG] Theorem 4.24 (see also [C1] for $C_q(\rho_f) = C_q(\pi)$).

Then the following lemma is clear from our description of $C_q(\rho)$.

Lemma 10.2. *Suppose that R is a reduced p -profinite local ring. Let $\rho : \text{Gal}(\overline{\mathbb{Q}}_q/\mathbb{Q}_q) \rightarrow GL_2(R)$ be a continuous representation. Then*

- (1) *For any prime ideal P of R containing a minimal prime \mathfrak{p} , we have $C_q(\rho_P) \leq C_q(\rho)$.*
- (2) *Suppose that R is an integral domain in which $p \neq 0$. Then unless $\rho|_{I_q}$ is reducible indecomposable, for any point $P \in \text{Spec}(R[\frac{1}{p}])$, $C_q(\rho_P)$ is independent of P , in particular, $C_q(\rho_P) = C_q(\rho)$.*
- (3) *Suppose that $\text{Spec}(R) = \text{Spec}(I) \cup \text{Spec}(J)$ for two irreducible components $\text{Spec}(I)$ and $\text{Spec}(J)$. If $\text{Spec}(I[\frac{1}{p}]) \cap \text{Spec}(J[\frac{1}{p}])$ contain a prime P_0 and $\rho_{\mathfrak{p}}|_{I_q}$ is not reducible indecomposable for the two minimal prime ideals \mathfrak{p} of R , $C_q(\rho_P)$ is independent of $P \in \text{Spec}(R[\frac{1}{p}])$, in particular, $C_q(\rho_P) = C_q(\rho)$.*
- (4) *If P is a prime ideal of R with $\kappa(P)$ having characteristic 0 and $\rho_{\mathfrak{p}}|_{I_q}$ is not reducible indecomposable for the each minimal prime ideal \mathfrak{p} of R_P , then $C_q(\rho_{P'}) = C_q(\rho_P)$ for any prime ideal P' of the localization R_P .*

Proof. The first assertion follows directly from the definition, and the third is the special case of the second. For the assertions (2) and (3), we note that $\rho|_{I_q}$ has finite image under the assumption. If R is an integral domain and $p \notin P$ for $P \in \text{Spec}(R)$, $1 + PR = R \cap (1 + PR_P)$ is a torsion-free group; so, $\rho_P(I_q) \cong \rho(I_q)$. In particular, we have, for any subgroup $I \subset I_q$, $\dim_{Q(R)} H^0(I, \rho) = \dim_{\kappa(P)} H^0(I, \rho_P)$, which implies $C_q(\rho) = C_q(\rho_P)$, proving (2).

As for (3), writing $I = R/\mathfrak{p}$ and $J = R/\mathfrak{q}$, by (2), $C_q(\rho_P)$ is constant for all $P \in \text{Spec}(I[\frac{1}{p}])$ and $C_q(\rho_Q)$ is constant for all $Q \in \text{Spec}(J[\frac{1}{p}])$. Thus we have $C_q(\rho_P) = C_q(\rho_{\mathfrak{p}}) = C_q(\rho_{P_0}) = C_q(\rho_{\mathfrak{q}}) = C_q(\rho_Q)$.

The assertion (4) follows from (3). Note that $R_P[\frac{1}{p}] = R_P$ as $\kappa(P)$ has characteristic 0. For any two irreducible components $I = R/\mathfrak{p}$ and $J = R/\mathfrak{q}$, we have $P \in \text{Spec}(I) \cap \text{Spec}(J)$; so, $C_q(\rho_{\mathfrak{p}}) = C_q(\rho_{\mathfrak{q}}) = C_q(\rho_P)$. For any P' , taking a minimal prime ideal \mathfrak{q} contained in P' , we get $C_q(\rho_{P'}) = C_q(\rho_{\mathfrak{q}}) = C_q(\rho_P)$. This finishes the proof. \square

We call a representation $\rho : \text{Gal}(\overline{\mathbb{Q}}_q/\mathbb{Q}_q)$ minimal if $C_q(\rho \otimes \chi) \geq C_q(\rho)$ for any finite order character χ of $\text{Gal}(\overline{\mathbb{Q}}_q/\mathbb{Q}_q)$.

Lemma 10.3. *Let R be a p -profinite integral domain and $\rho : \text{Gal}(\overline{\mathbb{Q}}_q/\mathbb{Q}_q) \rightarrow GL_2(R)$ be a semi-simple representation. Let $\overline{\rho} = (\rho \bmod \mathfrak{m}_R)$.*

- (1) *If $\overline{\rho}|_{I_q} \cong \overline{\xi} \oplus \overline{\eta}$ for two characters $\overline{\xi}$ and $\overline{\eta}$ of I_q , then $\rho|_{I_q} \cong \xi \oplus \eta$ for two characters ξ and η . If further ρ is minimal and one of ξ and η extends to $\text{Gal}(\overline{\mathbb{Q}}_q/\mathbb{Q}_q)$, one of ξ and η is unramified, and $\overline{\rho}$ is minimal.*
- (2) *Suppose that $\overline{\rho} \cong \overline{\xi} \oplus \overline{\eta}$ for two characters $\overline{\xi}$ and $\overline{\eta}$ of $\text{Gal}(\overline{\mathbb{Q}}_q/\mathbb{Q}_q)$. If ρ is absolutely irreducible, $q \equiv 1 \pmod{p}$. If further ρ is minimal, $\overline{\rho}$ and $\overline{\xi}$ are unramified at q (so, $\overline{\rho}$ is minimal), and $\overline{\xi}/\overline{\eta}$ has order 2.*

- (3) Suppose that $\bar{\rho} \cong \bar{\xi} \oplus \bar{\eta}$ for two characters $\bar{\xi}$ and $\bar{\eta}$ of $\text{Gal}(\overline{\mathbb{Q}}_q/\mathbb{Q}_q)$. If ρ is reducible minimal isomorphic to $\xi \oplus \eta$, one of ξ and η is unramified, and $\bar{\rho}$ is minimal. If $\bar{\rho}$ is unramified but ρ is ramified, we have again $q \equiv 1 \pmod{p}$.
- (4) Assume that $p \neq 0$ in R . Suppose that either $\bar{\rho} \cong \bar{\eta} \oplus \bar{\xi}$ or $\bar{\rho} \cong \text{Ind}_K^{\mathbb{Q}_q} \bar{\xi}$ for a quadratic extension K/\mathbb{Q}_q . If $C_q(\bar{\rho}) < C_q(\rho)$ and ρ is minimal, we have $q^j \equiv 1 \pmod{p}$, where $j = 1$ if $\bar{\rho} \cong \bar{\eta} \oplus \bar{\xi}$ or K is ramified and $j = 2$ if K is unramified.

Proof. We first prove the assertion (1). If $\rho|_{I_q}$ is absolutely irreducible, by Lemma 10.1 (3), we have either (i) $\rho|_{I_q^w}$ is absolutely irreducible or (ii) $\rho \cong \text{Ind}_K^{\mathbb{Q}_q} \xi$ for a character ξ and a ramified quadratic extension K/\mathbb{Q}_q . Case (i) does not occur as $\rho|_{I_q^w}$ factors through $\bar{\rho}$. Suppose that we are in Case (ii) and that $\rho|_{I_q}$ is absolutely irreducible. Then we have $\xi^{1-\sigma} \equiv 1 \pmod{\mathfrak{m}_R}$ for $\sigma \in I_q$ non-trivial over K , as $\bar{\rho}|_{I_q}$ is reducible. Then by local class field theory, $\xi^{1-\sigma}$ can be regarded as a character of O^\times for the integer ring O of K . By irreducibility of $\rho|_{I_q}$, $\xi^{1-\sigma} \neq 1$ with $\xi^{1-\sigma} \equiv 1 \pmod{\mathfrak{m}_R}$. Thus $\xi^{1-\sigma}$ has p -power order. Since O^\times is a q -profinite group times \mathbb{F}_q^\times (as K is ramified over \mathbb{Q}_q), $\xi^{1-\sigma}$ factors through \mathbb{F}_q^\times . Any character of O^\times factoring through \mathbb{F}_q^\times is σ -invariant; so, $\xi^{1-\sigma}|_{I_q} = 1$. Thus $\text{Ind}_K^{\mathbb{Q}_q} \xi|_{I_q}$ is reducible, a contradiction. Thus $\rho|_{I_q} \cong \xi \oplus \eta$. If one of ξ and η extends to $\text{Gal}(\overline{\mathbb{Q}}_q/\mathbb{Q}_q)$, the two characters extend to $\text{Gal}(\overline{\mathbb{Q}}_q/\mathbb{Q}_q)$ as $\det(\rho)$ is a character of $\text{Gal}(\overline{\mathbb{Q}}_q/\mathbb{Q}_q)$. Then we must have $\rho = \xi \oplus \eta$ for suitable choice of extensions. By the minimality, one of η and ξ is unramified; so, one of $\bar{\xi}$ and $\bar{\eta}$ is unramified. This implies that $\bar{\rho}$ is minimal as well. This finishes the proof of the assertion (1).

We now prove (2). By (1), $\rho|_{I_q} \cong \xi \oplus \eta$ with $\bar{\xi} = \xi \pmod{\mathfrak{m}_R}$. Suppose first that ρ is absolutely irreducible. By Lemma 10.1 (3), we have $\rho \cong \text{Ind}_K^{\mathbb{Q}_q} \xi$ for a character ξ of $\text{Gal}(\overline{\mathbb{Q}}_q/K)$ extending the character ξ of $I_q \cap \text{Gal}(\overline{\mathbb{Q}}_q/K)$ for a quadratic extension K/\mathbb{Q}_q . If K/\mathbb{Q}_q is ramified, $\xi^\sigma = \xi$ for $\sigma \in I_q$ non-trivial on K ; so, ρ is reducible, a contradiction. Thus K/\mathbb{Q}_q is unramified. Take $\phi \in \text{Gal}(\overline{\mathbb{Q}}_q/\mathbb{Q}_q)$ giving rise to q -th power Frobenius modulo q . Then ϕ is nontrivial on K , we have $\xi^{1-\phi} \equiv 1 \pmod{\mathfrak{m}_R}$ and $\bar{\xi}/\bar{\eta} = \left(\frac{K/\mathbb{Q}_q}{\phi}\right)$ as $\bar{\rho} \cong \bar{\xi} \oplus \bar{\eta}$. Thus $\xi^{1-\phi}$ is a p -power order character. Note that $\xi|_{I_q}$ has finite order. Write $\xi|_{I_q} = \xi_p \xi^{(p)}$ so that ξ_p has p -power order and $\xi^{(p)}$ has order prime to p . Then $\xi_p^{1-\phi} = \xi^{1-\phi}$ and $(\xi^{(p)})^{1-\phi} = 1$, since $\xi^{1-\phi}$ has p -power order. Thus $\xi^{(p)}$ extends to a finite order character Ξ of $\text{Gal}(\overline{\mathbb{Q}}_q/\mathbb{Q}_q)$. Then $\rho \otimes \Xi^{-1}$ has less conductor than ρ . Since $\rho \otimes \Xi^{-1}$ is absolutely irreducible, it is ramified; so, ξ_p is non-trivial. Since ϕ acts on I_q^t by the cyclotomic character (e.g., [MFG] page 123), we have $\xi_p^{\phi-1} = \xi_p^{q-1} = 1$ which implies $q \equiv 1 \pmod{p}$. By minimality of ρ , we conclude $\Xi|_{I_q} = \xi^{(p)} = 1$, and $\xi|_{I_q}$ has order p -power. Thus $\bar{\xi}|_{I_q} = 1$ and hence, $\bar{\rho} = 1 \oplus \left(\frac{K/\mathbb{Q}_q}{\phi}\right)$, which is unramified, and $\bar{\xi}/\bar{\eta} = \left(\frac{K/\mathbb{Q}_q}{\phi}\right)$ has order 2. This finishes the proof of (2).

We prove (3). If ρ is reducible, by semi-simplicity of ρ , we have $\rho \cong \xi \oplus \eta$. By minimality of ρ , one of ξ and η is unramified, and hence $\bar{\rho}$ is minimal. If further $\bar{\rho}$ is unramified while ρ is ramified, one of the characters ξ and η non-trivial on I_q become trivial modulo \mathfrak{m}_R ; so, $q \equiv 1 \pmod{p}$.

To see (4), we note that under $p \nmid (q-1)$ and minimality of ρ , ρ is absolutely irreducible if and only if $\bar{\rho}$ is absolutely irreducible by (2). If $\rho \cong \eta \oplus \xi$ and $\bar{\rho} = \bar{\xi} \oplus \bar{\eta}$ with $\bar{\eta} = (\eta \pmod{\mathfrak{m}_R})$, for the Teichmüller lift $\tilde{\xi}$ of $\bar{\xi}$ and $\tilde{\eta}$ of $\bar{\eta}$, $C_q(\rho) > C_q(\bar{\rho})$ implies that one of $\xi\tilde{\xi}^{-1}$ and $\eta\tilde{\eta}^{-1}$, say $\xi\tilde{\xi}^{-1}$, is non-trivial over I_q^t of p -power order. Then $1 = (\xi\tilde{\xi}^{-1})^{1-\phi} = (\xi\tilde{\xi}^{-1})^{1-q}$ implies $q \equiv 1 \pmod{p}$. Now suppose that $\rho \cong \text{Ind}_K^{\mathbb{Q}_q} \xi$ is absolutely irreducible. Then for $\bar{\xi} = (\xi \pmod{\mathfrak{m}_R})$, $\bar{\xi}^{1-\sigma} \neq 1$ for $\sigma \in \text{Gal}(\overline{\mathbb{Q}}_q/\mathbb{Q}_q)$ non-trivial on the quadratic extension K . If K is ramified, again $C_q(\rho) > C_q(\bar{\rho})$ implies $1 = (\xi\tilde{\xi}^{-1})^{1-\phi} = (\xi\tilde{\xi}^{-1})^{1-q}$ as we can choose the Frobenius ϕ inducing identity on K . If K is unramified, by the same argument, $q^2 \equiv 1 \pmod{p}$ as the Frobenius over K acts on I_q^t by $\sigma \mapsto \sigma^{q^2}$.

By Lemma 10.1 (3), the remaining case is when $\rho|_{I_q^w}$ is irreducible but ρ is not induced. Since $\rho|_{I_q^w}$ factors through $\bar{\rho}$ which is induced, $\rho(I_q^w)$ is a dihedral group (as it $\rho|_{I_q^w}$ is irreducible). Since 2 is a factor of the order of the dihedral group $\rho(I_q^w)$, we conclude $q = 2$. Since $\rho(I_q^w)$ is dihedral, $\rho|_{I_2^w} \cong \text{Ind}_I^{\mathbb{Q}_q} \xi$ for an index 2 subgroup I of I_2^w . By [W], the projection G of $\text{Im}(\rho)$ in $PGL_2(R)$ is isomorphic either to S_4 or A_4 . We have an isomorphism $S_4/V \cong S_3$ for the unique $(2, 2)$ -subgroup V . Let L be the extension of \mathbb{Q}_2 such that $\text{Gal}(L/\mathbb{Q}_q) \cong G$ by ρ . Then L has subfield M with

$\text{Gal}(L/M) = V$. By [Kz] §5.1, $\rho|_{\text{Gal}(\overline{\mathbb{Q}}_2/M)} \cong \text{Ind}_{L'}^M \xi$ for any of three quadratic extensions L' of M in L . Since $p \neq 2$, $V \cap G$ has to injects into \overline{G} . Thus \overline{G} has to be isomorphic to S_4 or A_4 , a contradiction, since \overline{G} is dihedral. Therefore, this case cannot happen. This finishes the proof of (4). \square

For a global representation $\rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow GL_2(R)$, assuming that ρ only ramifies at finitely many places (so, $C_q(\rho|_{\text{Gal}(\overline{\mathbb{Q}}_q/\mathbb{Q}_q)}) = 1$ for almost all $q \neq p$), we define its prime-to- p conductor by $C(\rho) = \prod_{q \neq p} C_q(\rho|_{\text{Gal}(\overline{\mathbb{Q}}_q/\mathbb{Q}_q)})$. All Galois representations we studied in this paper ramify only at finitely many primes; so, it has well defined conductor. A global Galois representation ρ as above is called minimal if $C(\rho)$ is minimal among $C(\rho \otimes \xi)$ for all finite order Galois character ξ .

Lemma 10.4. *Let R be a p -profinite integral domain and $\Psi : \text{Gal}(\overline{\mathbb{Q}}/M) \rightarrow R^\times$ be a character with prime-to- p conductor \mathfrak{C} for an imaginary quadratic field M . If $\text{Ind}_M^{\mathbb{Q}} \Psi$ is minimal at primes q split in M , we have $\mathfrak{F}_c = \mathfrak{D}$ for the decomposition $\mathfrak{C} = \mathfrak{F}\mathfrak{F}_c\mathfrak{I}$ in Definition 6.1.*

Proof. If $\mathfrak{F}_c \neq \mathfrak{D}$, we have a rational prime q such that $(q)|\mathfrak{F}\mathfrak{F}_c$. Since $\mathfrak{F} + \mathfrak{F}_c = \mathfrak{D}$ and $\mathfrak{F} \subset \mathfrak{F}_c^c$, we can split $(q) = \mathfrak{Q}\mathfrak{Q}^c$ in M so that $\mathfrak{Q}|\mathfrak{F}$. Then identifying I_q with $I_{\mathfrak{Q}^c}$, we may regard $\Psi|_{I_{\mathfrak{Q}^c}}$ as a character of I_q . Since the image of I_q in $\text{Gal}(\mathbb{Q}_q^{ab}/\mathbb{Q}_q)$ for the maximal abelian extension $\mathbb{Q}_q^{ab}/\mathbb{Q}_q$ acts faithfully on $\text{Gal}(\mathbb{Q}_q[\mu_{q^\infty}]/\mathbb{Q}_q) \cong \mathbb{Z}_q^\times$, we have a global Galois character $\xi : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow R^\times$ unramified outside q and $\xi|_{I_q} = \Psi|_{I_q}$. Then we have, for q -primary parts,

$$C_q((\text{Ind}_M^{\mathbb{Q}} \Psi) \otimes \xi^{-1}) \supset \mathfrak{F}_q \supsetneq (\mathfrak{F}\mathfrak{F}_c)_q = C_q(\text{Ind}_M^{\mathbb{Q}} \Psi)$$

contradicting against minimality of $\text{Ind}_M^{\mathbb{Q}} \Psi$. \square

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