# PERMUTATION REPRESENTATIONS AND LANGLANDS BASE CHANGE

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## 1. Quaternion class sets

Pick a quaternion algebra $B$ over a totally real field $E$ and suppose that $B_\sigma = B \otimes_{E,\sigma} \mathbb{R} \cong \mathbb{H}$ (the Hamilton quaternion algebra) for all embeddings $\sigma : E \hookrightarrow \mathbb{R}$. To avoid complication from automorphically induced representations from $GL(1)$ over a CM quadratic extension over $E$, we assume that $B$ ramifies at some finite places. Take a totally real Galois extension $F/E$ with integer ring $O$ and Galois group $G = \mathrm{Gal}(F/E)$. Suppose we have a $G$-invariant maximal order $R_F$ of $B_F = B \otimes_{\mathbb{Q}} F$. An nonzero right $R_F$-submodule $\mathfrak{A} \subset B_F$ of finite type is called a fractional right $R_F$-ideal. We can think of an equivalence between right fractional $R_F$-ideals $\mathfrak{A}$ and $\mathfrak{B}$ of $B_F$. In other words, $\mathfrak{A} \sim \mathfrak{B} \Leftrightarrow \mathfrak{A} = \alpha\mathfrak{a}\mathfrak{B}$ for $\alpha \in B_F^\times$ and a $O$-ideal $\mathfrak{a}$. The resulting $G$-set $C_{0,F}$ of equivalence classes is a finite set (not a group, because $\mathfrak{A}^{-1}$ is a left ideal). We do not have a good definition of the norm map of $C_{0,F} \to C_{0,E}$ for $B_F = B \otimes_{\mathbb{Q}} F$ and $B_E = B \otimes_{\mathbb{Q}} E$. Since $B_E \subset B_F$ naturally, we still have a $G$-equivariant map $\iota_{F/E} : C_{0,E} \to C_{0,F}$ as long as we choose maximal orders compatibly: $R_E = R_F \cap B$.

## 2. Hecke operators

Write $\mathcal{M}_F$ for the space of functions $f : C_{0,F} \to K$ for a (fixed) algebraically closed field $K$. Then we have Hecke operators $T(\mathfrak{n})$ for nonzero $O$-ideals $\mathfrak{n}$ defined by

$$f|T(\mathfrak{n})(\mathfrak{a}) = \sum_{\mathfrak{x}} f(\mathfrak{a}\mathfrak{x}) \quad (\mathfrak{x} \text{ runs over right ideals with reduced norm } \mathfrak{n}).$$

The space $\mathcal{M}_F$ has inner product $\langle f, g\rangle = \sum_{x \in C_{0,F}} f(x)g(x)$ invariant under the action of $G$ and for which $T(\mathfrak{n})$ is self-adjoint. By the Jacquet-Langlands correspondence, $\mathcal{M}_F$ as a Hecke module is equivalent to a subspace of Hilbert modular forms over $F$ of weight 2.

We write $\mathbb{T}_F$ for the $K$-subalgebra of $\mathrm{End}(\mathcal{M}_F)$ generated by Hecke operators. If $K$ is of characteristic 0, $\mathbb{T}_F$ is a commutative semi-simple algebra, and $\mathcal{M}_F \cong \mathrm{Hom}_K(\mathbb{T}_F, K)$ as $\mathbb{T}_F$-modules (the multiplicity 1 theorem). Indeed, pick a $K$-linear form $\lambda : \mathcal{M}_F \to K$ which does not kill any common eigenform under the action of $\mathbb{T}_F$, the pairing $\langle T, f\rangle = \lambda(f|T)$ gives the duality. If $F/E$ is a Galois extension with Galois

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group $G$ and $R_F$ is $G$-invariant, we can make this isomorphism $G$-equivariant (taking $G$-invariant $\lambda$). Here $G$ acts on $\mathbb{T}_F$ by $T(\mathfrak{n}) \mapsto T(\mathfrak{n}^{\sigma^{-1}})$ and $f$ by $f(x) \mapsto f(x^\sigma)$. If $\mathbb{T}_F$ is semi-simple, by trace pairing, $\mathrm{Hom}_K(\mathbb{T}_F, K) \cong \mathbb{T}_F$ as $(\mathbb{T}_F, G)$-module; so, $K[C_{0,F}] \cong \mathcal{M}_F \cong \mathbb{T}_F$ as $K[G]$-modules.

In adelic language, regarding $B^\times$ as an algebraic group so that $B^\times(A) = (B \otimes_{\mathbb{A}} A)^\times$ for a $\mathbb{Q}$-algebra $A$, we can identify canonically $C_{0,F}$ with $B^\times(F)\backslash B^\times(F_{\mathbb{A}}^{(\infty)})/Z(F_{\mathbb{A}}^{(\infty)})\widehat{R}_F^\times$ for $\widehat{R}_F = R_F \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ with $\widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$ and the center $Z \subset B^\times$. Let $U = \prod_{\mathfrak{l}} U_{\mathfrak{l}} \subset \widehat{R}_F^\times$ be a subgroup of finite index ($U_{\mathfrak{l}} \subset B^\times F_{\mathfrak{l}}$), $\mathfrak{l}$ running over primes of $F$). We consider the ray $U$-class set $C_{0,F}(U) = B^\times(F)\backslash B^\times(F_{\mathbb{A}}^{(\infty)}/Z(F_{\mathbb{A}}^{(\infty)})U$ and a character $\varepsilon : U \cdot Z(F_{\mathbb{A}}^{(\infty)}) \to K^\times$ (with finite order $\varepsilon|U$). We have an isomorphism $C_{0,F} \cong C_{0,F}(\widehat{R}_F^\times)$.

Put $U_1 = \mathrm{Ker}(\varepsilon : U \to K^\times)$, and let $I = \mathrm{Hom}_{\mathrm{alg}}(O, K)$ and suppose $|I| = [F : \mathbb{Q}]$. Note that $R \otimes_O K \cong M_2(K)^I$ (and we fix this identification); so, we may choose a rational place $v$ so that $R_v \otimes_{O_v} K \cong M_2(K)^I$ and hence we have a natural homomorphism $\sigma : R_v \to M_2(K)$ extending $\sigma \in I$. Let $\mathbb{Z}[I]$ be the free module generated by $I$. For each $n = \sum_\sigma n_\sigma \sigma \in \mathbb{Z}[I]$ with $n_\sigma \geq 0$, we consider space $L(n; K)$ of $K$-polynomials in $(X_\sigma, Y_\sigma)_{\sigma \in I}$ homogeneous of degree $n_\sigma$ with respect to the pair $(X_\sigma, Y_\sigma)$. Regarding $f \in L(n; K)$ as a function on $W := (K^2)^I \ni (x_\sigma, y_\sigma) \mapsto f(x_\sigma, y_\sigma) \in K$, we consider the space $\mathcal{M}_{n,F}(U, \varepsilon)$ of functions

$$f : B^\times(F)\backslash B^\times(F_{\mathbb{A}}) \to L(n; K) \quad \text{with} \quad f(xu; (x_\sigma, y_\sigma)) = \varepsilon(zu) f(x; (x_\sigma, y_\sigma)\sigma(u_v)^{-1})$$

for all $u \in UB^\times(\mathbb{R})$ and $z \in Z(F_{\mathbb{A}})$. If $U$ is sufficiently small,

$$\dim_K \mathcal{M}_{n,F}(U, \varepsilon) = |C_{n,F}(U)|$$

for $C_{n,F}(U) = C_{0,F}(U) \times \{0 \leq j \leq n\}$, because $\dim_K L(n; K) = |\{0 \leq j \leq n\}|$. Decompose $U_1 g U_1 = \bigsqcup_i g_i U_1$, and define a linear operator on $\mathcal{M}_{n,F}(U, \varepsilon)$ by

$$f|[U_1 g U_1](x; w) = \sum_i f(x g_i; w).$$

This operator is a generalization of $T(\mathfrak{n})$.

For a prime $\mathfrak{l}$ of $F$, if $U_{\mathfrak{l}} \subsetneq (R_F \otimes_O O_{\mathfrak{l}})^\times$, we call $\mathfrak{l}$ is in the level of $U$, and write $S$ for the set of primes either in the level of $U$ or $\mathfrak{l}|v$. We write as $L(n, \varepsilon; K)$ the $Z(F_{\mathbb{A}})U$-module $L(n; K)$ with the action $zu \cdot f := \varepsilon(zu) f(x_\sigma, y_\sigma)\sigma(u_v)^{-1})$. We write $\mathbb{T}_{n,F}(U)$ for the subalgebra of $\mathrm{End}_K(\mathcal{M}_{n,F}(U, \varepsilon))$ generated over $K$ by the operators $[U_1 g U_1]$ for all $g \in B^\times(F_{\mathbb{A}}^{(S)})$ ($F_{\mathbb{A}}^{(S)}$: adeles outside $S$ and $\infty$). We write $T(\mathfrak{l}) = [U_1 g U_1]$ if $\mathfrak{l} \notin S$, $g \in R_{\mathfrak{l}}$ and $N(g)O = \mathfrak{l}$.

Let $B_1 = \mathrm{Ker}(N : B^\times \to \mathbb{G}_{m/E})$ for the reduced norm map $N$. In the above definition, we can replace every groups $(B^\times, U, Z)$ by $(B_1, B_1(F_{\mathbb{A}}^{(\infty)}) \cap U, Z \cap B_1)$ and obtain a $B_1$-version of corresponding spaces and Hecke algebras. We use same notation also for the $B_1$-version, and if necessary, we explicitly indicate with which version we are working.

## 3. A conjecture on Galois permutation representations

Suppose $F/E$ is a Galois extension with Galois group $G$. Then $G$ acts on $B_F = B \otimes_{\mathbb{Q}} F$ through the right factor $F$. We choose an open compact subgroup $U \subset \widehat{R}_F^\times$ so that it is stable under $G$. We suppose $\varepsilon(u^\tau) = \varepsilon(u)$ and $n\tau = n$ (through permutation of $I$). Then the Galois group $G$ naturally acts on the finite set $C_{n,F}(U)$ by

$$(x,j)^\sigma = (x^\sigma, j\sigma) \ \ (x \in C_{n,F}(U) \text{ and } 0 \le j \le n)$$

and on $\mathcal{M}_{n,F}(U,\varepsilon)$ by the pull back left action: $\tau \cdot f(x;(x_\sigma,y_\sigma)) = f(x^\tau;(x_{\sigma\tau},y_{\sigma\tau}))$. Since $\mathbb{T}_{n,F}(U)$ acts on $\mathcal{M}_{n,F}(U,\varepsilon)$, we can let $\tau \in G$ act on $\mathbb{T}_{n,F}(U)$ from the left by $[U_1 g U_1]^\sigma = [U_1 g^{\sigma^{-1}} U_1]$ (thus $T(\mathfrak{l})^\sigma = T(\mathfrak{l}^{\sigma^{-1}})$). Thus $G$ acts on the finite set $\mathrm{Spec}(\mathbb{T}_{n,F}(U))(K) = \mathrm{Hom}_{K\text{-alg}}(\mathbb{T}_{n,F}(U),K)$ from the right canonically. We have the following conjecture made long ago.

**Conjecture 3.1.** *Suppose $n\tau = n$ for all $\tau \in G$. If we are working with the $B^\times$-version of $\mathbb{T}_{n,F}$ and $C_{n,F}(U)$, we suppose that $|G|$ is odd (we do not suppose any condition on $G$ for the $B_1$-version). Then there exists a $G$-equivariant surjection $\iota : C_{n,F}(U) \twoheadrightarrow \mathrm{Spec}(\mathbb{T}_{n,F}(U))(K)$. If we suppose*

- (H1) $\mathbb{T}_{n,F}(U)$ *is semi-simple and* $\dim_K \mathbb{T}_{n,F}(U) = \dim_K \mathcal{M}_{n,F}(U,\varepsilon)$,
- (H2) $\Gamma_g = (U \cdot Z(F_{\mathbb{A}})) \cap g\mathcal{G}(F)g^{-1}$ *acts trivially on $L(n,\varepsilon;K)$ for all $g \in \mathcal{G}(F_{\mathbb{A}}^{(\infty)})$ for $\mathcal{G} = B^\times$ or $B_1$,*

*then $\iota$ is a bijection.*

The assumptions (H1) are the multiplicity one statement of the action of $\mathbb{T}_{n,F}(U)$. So if we choose $\varepsilon$ well so that the conductor of $\varepsilon$ match the level of $U$ of $\Gamma_0$-type, (H1) holds if $K$ is of characteristic 0 (by the classical multiplicity one theorem for $B^\times$ and by Ramakirishnan's multiplicity one theorem for $SL(2)$ if we work with the $B_1$-version). For such $(U,\varepsilon)$, (H1) holds for almost all characteristic $p > 0$. The assumption (H2) holds for all $U$ sufficiently small.

Hereafter, we **assume** (H1–2). Since $\mathbb{T}_{n,F}(U)$ is commutative semi-simple, $\mathbb{T}_{n,F}(U) \cong K[\mathrm{Spec}(\mathbb{T}_{n,F}(U))(K)]$ as $K[G]$-modules. As described in an exercise in Serre's book on linear representations of finite groups (II.13, Exercise 13.5), if $\mathrm{char}(K) = 0$,

$$\begin{aligned} &K[\mathrm{Spec}(\mathbb{T}_{n,F}(U))(K)] \cong \mathcal{M}_{n,F}(U,\varepsilon) = K[C_{n,F}(U)] \ \text{ as } G\text{-modules} \\ &\qquad \Rightarrow \mathrm{Spec}(\mathbb{T}_{n,F}(U))(K) \cong C_{n,F}(U) \ \text{ as } H\text{-sets for all cyclic subgroups } H \subset G. \end{aligned}$$

Since $\mathbb{T}_{n,F}(U) \cong \mathcal{M}_{n,F}(U,\varepsilon)$ as $G$-modules, as already stated, we have

**Theorem 3.2.** *Assume* $\mathrm{char}(K) = 0$ *or* $\mathrm{char}(K) > |C_{n,F}(U)|$. *If $G$ is cyclic, the conjecture holds.*

As for this theorem, even for the $B^\times$-version, we do not need to assume that $G$ has odd order.

## 4. Galois representations

Pick a sufficiently large prime $p$ and take $K = \overline{\mathbb{Q}}_p$. The prime $p$ is large enough so that any mod $p$ modular Galois representations lifts to $K$-representations (Wiles-Taylor). We suppose that $R_F$ is stable under $G$, which is equivalent to

(dd) if $B$ ramifies at a prime $\ell$ of $E$, $B_F$ ramifies at all prime factors of $\ell$.

Let $U_E = U^G$, and suppose that $U$ is of type $\Gamma_0(N)$ for an $O$-ideal $N$ outside ramified primes for $B$ (at each ramified primes for $B$, $U_{\mathfrak{l}}$ is the unique maximal compact subgroup). For simplicity,we suppose $\varepsilon(z\left(\begin{smallmatrix} a & b \\ cN & d \end{smallmatrix}\right)) = \varepsilon_+(z)\varepsilon(d)$ for $z \in Z(F_{\mathbb{A}})$ and $\left(\begin{smallmatrix} a & b \\ cN & d \end{smallmatrix}\right) \in \Gamma_0(N)_N$. We suppose $\varepsilon = \varepsilon_E \circ N_{F/E}$ for a similar character $\varepsilon_E$ of $U_E$.

Write $\mathbb{T}_E = \mathbb{T}_{n,E}(U_E)$ for $(U_E = U^G, \varepsilon_E)$, for a fixed $(U, \varepsilon)$ satisfying the assumptions (H1–2) of the conjecture. The assumption (H1) and (H2) in the conjecture implies $U_{\mathfrak{l}} = R_{F,\mathfrak{l}}^{\times}$ if $\mathfrak{l}$ ramifies in $B_F$.

For the moment we work with the $B^{\times}$-version. With each $P \in \mathrm{Spec}(\mathbb{T}_F)(K)$ regarded as a $K$-algebra homomorphism $P : \mathbb{T}_F \to K$, we can now attach a unique semi-simple $p$-adic Galois representation $\rho = \rho_P : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to GL_2(K)$ such that

(G1) $\rho$ is unramified outside $Np$ and {primes ramified in $B$}, crystalline of Hodge-Tate weight $(n_\sigma+1, 0)$ at each $p$-adic place $\sigma$ and $\det(\rho(Frob_{\mathfrak{l}})) = N(\mathfrak{l})\varepsilon_+(Frob_{\mathfrak{l}})$;
(G2) If $\mathfrak{l}$ ramifies in $B_F$, $\rho|_{D_{\mathfrak{l}}} \cong \left(\begin{smallmatrix} \alpha\mathcal{N} & * \\ 0 & \alpha \end{smallmatrix}\right)$, where $\mathcal{N}$ is the $p$-adic cyclotomic character;
(G3) $\rho|_{I_{\mathfrak{l}}} \cong \left(\begin{smallmatrix} 1 & 0 \\ 0 & \varepsilon_{\mathfrak{l}} \end{smallmatrix}\right)$ for all $\mathfrak{l}|N$.

For all primes $\mathfrak{l}$ in (G1), we have $\mathrm{Tr}(\rho_P(Frob_{\mathfrak{l}})) = P(T(\mathfrak{l}))$, and this characterizes $\rho_P$ by $P$. As for the $B_1$-version, $P \in \mathrm{Spec}(\mathbb{T}_F)(K)$ parameterizes a projective representation satisfying the conditions (G1–3) after taking "modulo center". We expect

($\mathrm{L}_F$) All irreducible $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to GL_2(K)$ (resp. $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to PGL_2(K)$ for the $B_1$-version) satisfying (G1–3) (resp. (G1–3) modulo center) are modular.

Based on this expectation, starting with $\rho_P$ for $P \in \mathrm{Spec}(\mathbb{T}_E)(K)$, Langlands predicted the existence of $\widehat{P} \in \mathrm{Spec}(\mathbb{T}_F)(K)$ such that $\rho_{\widehat{P}} \cong \rho_P|_{\mathrm{Gal}(\overline{\mathbb{Q}}/F)}$. To prove the existence of $\widehat{P}$ is the problem of base-change. Langlands solved this question if $G$ is soluble. By the solution of Serre's mod $p$ modularity conjecture, ($\mathrm{L}_{\mathbb{Q}}$) is valid.

Define the inner conjugate $\rho_P^\tau(\sigma) = \rho_P(\widetilde{\tau}\sigma\widetilde{\tau}^{-1})$ taking an extension $\widetilde{\tau}$ of $\tau \in G$. Then we have $\rho_{\tau(P)} \cong \rho_P^\tau$. For the $B^{\times}$-version, if $H^2(G, \mathbb{Z}/2\mathbb{Z})$ vanishes and $P$ is fixed by $G$, $\rho_P$ extends to a Galois representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ with determinant $\varepsilon_{E+}\mathcal{N}$ for the $p$-adic cyclotomic character $\mathcal{N}$ (I. Schur). The extension is unique, if $H^1(G, \mathbb{Z}/2\mathbb{Z}) = 0$. We call $G$ simply 2-connected if $H^j(G, \mathbb{Z}/2\mathbb{Z}) = 0$ for $j = 1, 2$. For the $B^{\times}$-version,, suppose that $G$ is simply 2-connected (for example, groups of odd order and $SL_2(\mathbb{F})$ for finite field $\mathbb{F}$ with $|\mathbb{F}| \geq 5$ is simply 2-connected). For the $B_1$-version, no condition on $G$ is necessary. Again by Schur, any projective $G$-invariant representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ extends uniquely to a projective representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ with prescribed determinant character module center.

Two finite $G$-sets $X$ and $Y$ are equivalent if and only if $|X^H| = |Y^H|$ for all subgroups $H$. We know, as explained quoting Exercise 13.5 of Serre's book, $|C_{n,F}(U)^H| = |\mathrm{Spec}(\mathbb{T}_F)(K)^H|$ for all cyclic subgroups $H$.

We can easily count $|C_{n,F}(U)^H| = |C_{n,F^H}(U_{F^H})|$ (up to an explicit 2-power), and assuming Conjecture 3.1, we get for $H \subset G$

$$|\mathrm{Spec}(\mathbb{T}_F)(K)^H| \overset{\text{Conjecture 3.1}}{=} |C_{n,F}(U)^H| = |C_{n,F^H}(U_{F^H})| = |\mathrm{Spec}(\mathbb{T}_{F^H})(K)| \tag{4.1}$$

up to an explicit 2-power. If $H \subset G$ is 2-simply connected, then

$$|\mathrm{Spec}(\mathbb{T}_F)(K)^H| = \#\{\rho_P : H\text{-invariant}\} = |\mathrm{Spec}(\mathbb{T}_{F^H})(K)| \tag{4.2}$$

up to explicit 2-power. The associated projective representation $\overline{\rho}_P : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to PGL_2(\overline{\mathbb{Q}}_p)$ (that is, $\rho_P$ modulo center) always extends to a unique projective representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Thus

**Theorem 4.1.** *Suppose* Conjecture 3.1 *and* $(\mathrm{L}_?)$ *for* $? = E$ *or* $F$ *for a sufficiently large* $p$. *Then* $\{\rho_P\}_{P \in \mathrm{Spec}(\mathbb{T}_E)(K)}$ *has a base-change lift to* $F$. *In particular, if* $G$ *is odd cyclic and* $E = \mathbb{Q}$, *we have base-change of* $\{\rho_P\}_{P\in\mathrm{Spec}(\mathbb{T}_\mathbb{Q})(K)}$ *to* $\{\rho_{\widehat{P}}\}_{\widehat{P}\in\mathrm{Spec}(\mathbb{T}_\mathbb{Q})(K)^G}$.

Of course, the second assertion is a theorem of Langlands. We can go reverse, since we know base-change by Langlands for soluble subgroups $H \subset G$.

**Theorem 4.2.** *If* $G$ *is soluble,* Conjecture 3.1 *holds for* $K$ *of characteristic 0.*

As a more concrete (but non-soluble) example, we can offer

**Theorem 4.3.** *Suppose that* $G = \mathrm{Gal}(F/\mathbb{Q}) \cong SL_2(\mathbb{F}_p)$ *or* $A_5$. *If* $B_{/\mathbb{Q}}$ *ramifies only at one prime in the set* $\{2,3,5,7,11,13,17,19,23\}$, *the conjecture for the* $B_1$*-version holds for* $(U,\varepsilon) = (\widehat{R}_F^\times, \mathbf{1})$ *and* $K$ *of characteristic* 0.

Here is a sketch of proof. For $A_5$ or $SL_2(\mathbb{F}_5)$, any proper subgroup $H \subsetneq G$ is soluble; so, $|\mathrm{Spec}(\mathbb{T}_F)(K)^H| = |C_{0,F}^H|$. Thus we need to show

$$|\mathrm{Spec}(\mathbb{T}_F)(K)^G| = |C_{0,F}^G| = |C_{0,\mathbb{Q}}|.$$

By $(\mathrm{L}_\mathbb{Q})$, $|\mathrm{Spec}(\mathbb{T}_F)(K)^G| \le |C_{0,\mathbb{Q}}|$ and

$$|C_{0,\mathbb{Q}}| = \begin{cases} 1 & \text{if } p = 2,3,5,7,13, \\ 2 & \text{if } p = 11,17,19 \\ 3 & \text{if } p = 23 \end{cases}$$

for $p$ as above. The constant function $\mathbf{1}$ on $B_1(F_\mathbb{A})$ gives rise to an element in $\mathrm{Spec}(\mathbb{T}_F)(K)^G$; so, $|\mathrm{Spec}(\mathbb{T}_F)(K)^G| \ge 1$, and this settles the case where $|C_{0,\mathbb{Q}}| = 1$. As for $p = 11$, we showed that $\Delta(z)^{1/12}\Delta(11z)^{1/12} \in S_2(\Gamma_0(11))$ can be lifted to any totally real field linearly disjoint from $\mathbb{Q}[\sqrt{131}]$ in my paper with Maeda (in the Orga Taussky-Todd memorial volume in 1998 from Pacific journal of mathematics), though we assumed that $F$ is unramified at $13 \cdot 131$ in the paper (but the method works just under the linear disjointness because of the progress (made after 1998) of the techniques used). Thus if $p = 11$, we have $2 \le |\mathrm{Spec}(\mathbb{T}_F)(K)^G| = |C_{0,F}^G| = |C_{0,\mathbb{Q}}| = 2$. The case of other primes listed above can be treated similarly.

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