* Hecke fields and $L$-invariant

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November 2010

The author is partially supported by the NSF grant: DMS 0753991 and DMS 0854949 and by Clay Mathematics institute as a senior scholar.
Each rational elliptic curve $E/\mathbb{Q}$ is associated to a Hecke eigenform $f_E$ of level $\Gamma_0(N)$ for a suitable positive integer $N$ in the following way (Shimura/Taniyama conjecture):

- Look at $T_l E = \lim \leftarrow \mathbb{E}[l^n]\mathbb{Q}$ (Tate module for a prime $l$), where $E[l^n]\mathbb{Q} = \{x \in E(\mathbb{Q}) | l^n x = 0\}$;

- The absolute Galois group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on it, and for the inertia group at another prime $p \neq l$,

  $$T_l E_{I_p} = \{x \in T_l E | \sigma x = x \forall \sigma \in I_p\} \cong \begin{cases} 0 \\ \mathbb{Z}_l \\ \mathbb{Z}_l^2 \end{cases}$$

  depending on $p$ with $\text{rank}_{\mathbb{Z}_l} T_l E_{I_p} = 2$ for almost all $p$.

- The Frobenius element $Frob_p$ has its characteristic polynomial independent of $l$

  $$\Phi_p(X) = \text{det}(1 - Frob_p|_{T_l E_{I_p}} X) \in \mathbb{Z}[X].$$
• Make the $H^1$–part of Hasse–Weil $L$-function $L(s, E)$ by

$$L(s, E) = \prod_p \Phi_p(p^{-s})^{-1} = \sum_{n=1}^{\infty} a_n n^{-s}.$$  

• Make a Fourier series convergent on $\mathcal{H} = \{ z \in \mathbb{C} | \text{Im}(z) > 0 \}$:

$$f(z) = \sum_{n=1}^{\infty} a_n \exp(2\pi i n z) \ (\text{Im}(z) > 0).$$

• Then $f\left(\frac{az+b}{cz+d}\right) = f(z)(cz + d)^2$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ with $N|c$ for a positive integer $N$ called the level (weight 2 modular form).
This can be generalized to any system \( \{ \rho_l \circ V(\rho_l) \cong F^2_l \}_l \) of \( l \)-adic odd Galois representations, where \( l \) runs over prime ideals of a number field \( F \). Then

\[
\prod_p \det(1 - \rho_l(Frob_p)\big|_{V(\rho_l)_{lp}})p^{-s})^{-1} = \sum_{n=1}^{\infty} a_n(\rho)n^{-s}
\]
gives rise to a modular form

\[
f_\rho = \sum_{n=1}^{\infty} a_n(\rho) \exp(2\pi inz)
\]

which satisfies \( f\left(\frac{az+b}{cz+d}\right) = \psi(d)f(z)(cz+d)^k \)
as before for an integer \( k > 0 \) and a Dirichlet character \( \psi \) modulo the level \( N \).

For a number field \( K \subset \overline{\mathbb{Q}} \), the field

\[
K(a_\rho(n)|n = 1, 2, \ldots)
\]
is called the Hecke field, which is one of the most mysterious series of number fields. One can simplify the question, fixing one prime \( p \), and consider a simple extension \( K(a_\rho(p)) \) or for a family of systems of \( l \)-adic Galois representations \( \mathcal{F} = \{ \rho \} \), we could think of \( K(a_\rho(p))_{\rho \in \mathcal{F}} \), and ask

how big this field is?
1. Notation

To define the family $\mathcal{F}$ we study, we introduce some notation. Fix

- An odd prime $p > 2$;
- a positive integer $N$ ($p \nmid N$);
- two field embeddings $\mathbb{C} \leftarrow \overline{\mathbb{Q}} \leftarrow \overline{\mathbb{Q}}_p$.

Consider the space of cusp form

$$S_{k+1, \psi} = S_{k+1}(\Gamma_0(Np^r+1), \psi) \ (r \geq 0)$$

of weight “$k + 1$” with Nebentypus $\psi$.

Let the rings

$$\mathbb{Z}[\psi] \subset \mathbb{C} \text{ and } \mathbb{Z}_p[\psi] \subset \overline{\mathbb{Q}}_p$$

be generated by $\psi(n) \ (n = 1, 2, \ldots)$ over $\mathbb{Z}$ and $\mathbb{Z}_p$.

The Hecke algebra over $\mathbb{Z}$ is

$$h = \mathbb{Z}[\psi][T(n) | n = 1, 2, \ldots] \subset \text{End}(S_{k+1, \psi}).$$

Put $h_{k+1, \psi} = h \otimes_{\mathbb{Z}[\psi]} \mathbb{Z}_p[\psi]$.

Sometimes our $T(p)$ is written as $U(p)$ as the level is divisible by $p$. 
§2. Big Hecke algebra

The ordinary part $h_{k+1,\psi}^{\text{ord}} \subset h_{k+1,\psi}$ is the maximal ring direct summand on which $U(p)$ is invertible; so,

$$h^{\text{ord}} = e \cdot h \quad \text{for} \quad e = \lim_{n \to \infty} U(p)^n!.$$

Let $\psi_1 = \psi_N \times$ the tame $p$-part of $\psi$. Then, we have a unique ‘big’ Hecke algebra $h = h_{\psi_1}$ such that

- $h$ is free of finite rank over $\mathbb{Z}_p[[T]]$ with $T(n) \in h$ ($n = 1, 2, \ldots; T(p) = U(p)$)
- Let $\gamma = 1 + p$. If $k \geq 1$ and $\varepsilon : \mathbb{Z}_p^\times \to \mu_{p\infty}$ is a character,

$$h/(1 + T - \psi(\gamma)\varepsilon(\gamma)\gamma^k)h \cong h_{k+1,\varepsilon\psi_k}^{\text{ord}}$$

for $\psi_k := \psi_1\omega^{1-k}$, sending $T(n)$ to $T(n)$, where $\omega$ is the Teichmüller character.
§3. Galois representation

Each irreducible component

\[ \text{Spec}(\mathcal{I}) \subset \text{Spec}(\mathfrak{h}) \]

has a Galois representation

\[ \rho_{\mathcal{I}} : \text{Gal}(\overline{Q}/Q) \to GL(2) \]

with coefficients in \( \mathcal{I} \) (or its quotient field) such that

\[ \text{Tr}(\rho_{\mathcal{I}}(\text{Frob}_l)) = a(l) \]

(for the image \( a(l) \) in \( \mathcal{I} \) of \( T(l) \)) for almost all primes \( \ell \). Usually \( \rho_{\mathcal{I}} \) has values in \( GL_2(\mathcal{I}) \), and we suppose this for simplicity.

We regard \( P \in \text{Spec}(\mathcal{I})(\overline{Q}_p) \) as an algebra homomorphism \( P : \mathcal{I} \to \overline{Q}_p \), and we put \( \rho_P = P \circ \rho_{\mathcal{I}} : \text{Gal}(\overline{Q}/Q) \to GL_2(\overline{Q}_p) \).
§4. Analytic family

A point $P$ of $\text{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$ is called

**arithmetic**

if $P(1 + T - \varepsilon \psi_k(\gamma)\gamma^k) = 0$ for $k \geq 1$ and $\varepsilon : \mathbb{Z}_p^\times \to \mu_p\infty$.

If $P$ is arithmetic, we have a Hecke eigenform $f_P \in S_{k+1}(\Gamma_0(Np^r(P)), \varepsilon \psi_k)$ such that

$$f_P|T(n) = a_P(n)f_P \quad (n = 1, 2, \ldots)$$

for $a_P(n) := P(a(n)) = (a(n) \mod P) \in \overline{\mathbb{Q}}_p$.

We write $\varepsilon_P = \varepsilon$ and $k(P) = k$ for such a $P$.

Thus $\mathbb{I}$ gives rise to an **analytic family**

$$\mathcal{F}_\mathbb{I} = \{f_P|\text{arithmetic } P \in \text{Spec}(\mathbb{I})\}.$$
§5. CM component and CM family

We call a Galois representation $\rho$ CM if there exists an open subgroup $G \subset \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that the semi-simplification $(\rho|_G)^{ss}$ has abelian image over $G$.

We call $I$ a CM component if $\rho_I$ is CM.

If $I$ is a CM component, it is known that for an imaginary quadratic field $M$ in which $p$ splits, there exists a Galois character $\phi : \text{Gal}(\overline{\mathbb{Q}}/M) \to \mathbb{I}^\times$ such that $\rho_I \cong \text{Ind}_M^\mathbb{Q} \phi$.

If $\rho_P \cong \text{Ind}_M^\mathbb{Q} \phi_P$ for some arithmetic point $P$, $I$ is a CM component.
§6. A theorem on Hecke fields

Pick an infinite set $A$ of arithmetic points $P$ with fixed weight $k(P) = k \geq 1$. Write $H_A(I) \subset \overline{Q}$ for the field generated over $Q(\mu_{\infty})$ by $\{a_P(p)\}_{P \in A}$. Here is what we can prove:

**Theorem 1** (H-theorem). The field $H_A(I)$ is a finite extension of $Q(\mu_{\infty})$ if and only if $I$ is CM. Moreover

$$\limsup_{P \in A} [Q(\mu_{\infty})(a_P(p)) : Q(\mu_{\infty})] = \infty.$$ 

Assume that $[H_A(I) : Q(\mu_{\infty})] < \infty$. We try to prove that $I$ has CM. The converse is an easy application of Galois deformation theory.
§7. Number of eigenforms bounded

We start preparing to give a proof of the theorem. Put $K(f_P) = K[a_P(n); n = 1, 2, \ldots]$ inside $\overline{\mathbb{Q}}$.

**Lemma 1** (Bounded degree). *The degree*

$$[\mathbb{Q}(\mu_p\infty)(f_P) : \mathbb{Q}(\mu_p\infty)(a_P(p))]$$

*for arithmetic* $P$ *with fixed* $k(P) \geq 1$ *is bounded* (basically by $\text{rank}_{\mathbb{Z}_p[[T]]} h$) independently of $P$.

For a prime $l$ outside $N_p$, let

$$A(l) = \text{a root of } \det(X - \rho_I(Frob_l)) = 0.$$  

Then $\alpha_{l,P} := P(A(l)) \in \overline{\mathbb{Q}}_p$ is a root of

$$X^2 - a_P(l)X + \psi_k(l)l^{k(P)} = 0.$$  

If $l = p$, we put $A(l) = a(l)$. Fix $l$. Extending $I$, we assume that $A(l) \in I$. By the lemma, if $[\mathbb{Q}(\mu_p\infty)(a_P(p)) : \mathbb{Q}(\mu_p\infty)] < B$ for $B$ independent of $P \in \mathcal{A}$, $K_P = \mathbb{Q}(\mu_p\infty)(\alpha_{l,P})$ has bounded degree over $\mathbb{Q}(\mu_p\infty)$ independent of $l$ and $P$ for all $P \in \mathcal{A}$. 
§8. Weil numbers

For a prime \( l \), a Weil \( l \)-number \( \alpha \in \mathbb{C} \) of integer weight \( k \geq 0 \) satisfies

1. \( \alpha \) is an algebraic integer;
2. \( |\alpha^\sigma| = l^{k/2} \) for all \( \sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \).

The number of Weil \( l \)-numbers of a given weight \( k \) in \( \mathbb{Q}(\mu_{p\infty}) \) is finite up to roots of unity, as we have only finitely many possibility of its prime factorization. Indeed, if \( l \) is either inert or ramified fully in \( \mathbb{Q}(\mu_{p\infty})/\mathbb{Q} \), weight \( k \) Weil \( l \)-number is (at most) \( l^{k/2} \) up to roots of unity.
§9. Finiteness proposition

Two nonzero numbers \(a\) and \(b\) equivalent if \(a/b\) is a root of unity. Let \(\mathcal{K}_d\) be the set of all extensions of \(\mathbb{Q}[\mu_{p^\infty}]\) of degree \(d < \infty\) inside \(\overline{\mathbb{Q}}\) whose ramification at \(l\) is tame. Here is a slight improvement:

**Proposition 1** (Finiteness Proposition). We have only finitely many Weil \(l\)-numbers of a given weight in the set-theoretic union \(\bigcup_{K \in \mathcal{K}_d} K\) up to equivalence.

The proof is an elementary but subtle analysis of prime decomposition of the Weil number. Tameness is assumed since in that case, there are only finitely many isomorphism classes of \(K \otimes_\mathbb{Q} \mathbb{Q}_l\) for \(K \in \mathcal{K}_d\), and one can consider the prime factorization in a fixed algebra \(K \otimes_\mathbb{Q} \mathbb{Q}_l\) picking one isomorphism class.
§10. A rigidity lemma

Let $W$ be a $p$-adic valuation ring finite flat over $\mathbb{Z}_p$ and $\Phi(T) \in W[[T]]$. Regard $\Phi$ as a function of $t = 1 + T$; so, $\Phi(1) = \Phi|_{T=0}$. We start with a lemma whose characteristic $p$ version was studied by Chai:

**Lemma 2** (Rigidity). Suppose that there is an infinite subset $\Omega \subset \mu_p(\bar{K})$ such that $\Phi(\Omega) \subset \mu_p$. Then there exist $\zeta_0 \in \mu_p$ and $s \in \mathbb{Z}_p$ such that $\zeta_0^{-1}\Phi(t) = t^s = \sum_{n=0}^{\infty} \binom{s}{n} T^n$.

Note here that if $Z \subset \widehat{\mathbb{G}}_m \times \widehat{\mathbb{G}}_m = \text{Spf}(W[t, t^{-1}, t', t'^{-1}])$ is a formal subtorus, it is defined by the equation $t = t'^s$ for $s \in \mathbb{Q}_p$. Thus we need to prove that the graph of the function $t \mapsto \Phi(t)$ in $\widehat{\mathbb{G}}_m \times \widehat{\mathbb{G}}_m$ is a **formal subtorus**. For simplicity, assume $\Phi(1) = 1$ (so $\zeta_0 = 1$) in the following proof.
§11. Proof of the rigidity lemma.

Step 1: Regard \( \Phi \) as a morphism of formal schemes \( \hat{\mathbb{G}}_m \rightarrow \hat{\mathbb{G}}_m \).

Step 2: For any \( \sigma \) in an open subgroup \( 1 + p^m \mathbb{Z}_p \subset \text{Gal}(W[\mu_p\infty]/W) \subset \mathbb{Z}_p^\times \), we have \( \Phi(\zeta^z) = \Phi(\zeta^\sigma) = \sigma(\Phi(\zeta)) = \Phi(\zeta)^z \); so,

\[
\Phi(t^z) = \Phi(t)^z
\]

if \( \zeta^\sigma = \zeta^z \) for the value \( z \in 1 + p^m \mathbb{Z}_p \) of the cyclotomic character at \( \sigma \).

Step 3: The graph \( Z \) of \( t \mapsto \Phi(t) \) in \( \hat{\mathbb{G}}_m \times \hat{\mathbb{G}}_m \) is therefore \textbf{stable} under \( (t, t') \mapsto (t^z, t'^z) \) for \( z = 1 + p^m \mathbb{Z}_p \).

Step 4: Pick a point \( (t_0, t'_0 = \Phi(t_0)) \) of infinite order in \( Z \), then

\[
(t_0^{1+p^mz}, t'_0^{1+p^mz}) = (t_0, t'_0)(t_0, t'_0)^{p^mz} \in Z
\]

for all \( z \in \mathbb{Z}_p \). Thus \( Z \) has to be a coset of a \textbf{formal subgroup} generated by \( (t_0, t'_0)^{p^m} \). Since \( (1, 1) \in Z \), \( Z \) is a formal torus, and we find \( s \in \mathbb{Z}_p \) with \( \Phi(t) = t^s \). \( \square \)
§12. Frobenius eigenvalues

Suppose \([H_A(\mathbb{I}) : \mathbb{Q}(\mu_{p^\infty})] < \infty\). This implies \(K_P = \mathbb{Q}(\mu_{p^\infty})(\alpha_{l,P})\) has bounded degree over \(\mathbb{Q}(\mu_{p^\infty})\); so, for primes \(l \gg 0\), \(l\) is tamely ramified in \(K_P\) (the tameness assumption in Finiteness Proposition).

**Proposition 2** (Eigenvalue formula). *There exists a Weil \(l\)-number \(\alpha_1\) of weight 1 and a root of unity \(\zeta_0\) such that

\[
A(P) = \alpha_{l,P} = \zeta_0 \langle \alpha_1 \rangle^{k(P)-1}
\]

for all arithmetic \(P\); in other words,

\[
A(T) = \zeta_0 (1 + T)^s
\]

for \(s = \frac{\log_p(\alpha_1)}{\log_p(\gamma)}\).
§13. Proof of eigenvalue formula

We give a sketch of a proof assuming $\mathbb{I} = W[[T]]$. By Finiteness proposition, we have only a finite number of Weil $l$-numbers of weight $k$ in $\bigcup_{P \in \mathcal{A}} K_P$ up to multiplication by roots of unity, and hence

$A(P)$ for $P \in \mathcal{A}$ hits one of such Weil $l$-number $\alpha$ of weight $k$ infinitely many times, up to roots of unity.

After a suitable variable change $T \mapsto Y = \gamma^{-k}(1 + T) - 1$ and division by a Weil number, $A(Y)$ satisfies the assumption of the rigidity lemma. We have

$$A(Y) = \zeta \alpha (1 + Y)^{s_1}$$

for $s_1 \in \mathbb{Z}_p$, and $A(T) = \zeta_0 (1 + T)^{s}$. From this, it is not difficult to determine $s$ as stated in the proposition. \qed
§14. Abelian image lemma

Consider the endomorphism \( \sigma_s : (1 + T) \mapsto (1 + T)^s = \sum_{n=0}^{\infty} \binom{s}{n} T^n \) of a power series ring \( W[[T]] \) for \( s \in \mathbb{Z}_p \). Let \( A \) be an integral domain over \( W[[T]] \) of characteristic different from 2. Assume that the endomorphism \( \sigma_2 \) on \( W[[T]] \) extends to an endomorphism \( \sigma \) of \( A \).

**Lemma 3** (Abelian image). Take a continuous representation \( \rho : \text{Gal}(\overline{\mathbb{Q}}/F) \rightarrow GL_2(A) \) for a field \( F \subset \overline{\mathbb{Q}}, \) and put \( \rho^\sigma := \sigma \circ \rho \). If \( \text{Tr}(\rho^\sigma) = \text{Tr}(\rho^2) \). Then \( \rho \) is absolutely reducible over the quotient field \( \mathbb{Q} \) of \( A \).

Heuristically, the assumption implies that the **square map**: \( \sigma \mapsto \rho^2(\sigma) \) is still a representation \( \rho^\sigma \); so, it has to have an abelian image. Since any automorphism of the quotient field \( \mathbb{Q} \) of \( \mathbb{Z}_p[[T]] \) extends to its algebraic closure \( \overline{\mathbb{Q}} \supset I \), we can apply the above lemma to \( \rho_I \).
§15. Proof of the theorem. Suppose \([H_A(\mathbb{I}) : \mathbb{Q}(\mu_p^\infty)] < \infty\).

**Step 1:** We have \([K_P : \mathbb{Q}(\mu_p^\infty)]\) bounded independent of \(l\); so, if \(l \gg 0\), \(K_P\) is at most tamely ramified.

**Step 2:** By Eigenvalue formula, we have
\[
\text{Tr}(\rho(Frob_l)) = \zeta(1 + T)^a + \zeta'(1 + T)^{a'}
\]
for two roots of unity \(\zeta, \zeta'\) and \(a, a' \in \mathbb{Q}_p\).

**Step 3:** Not too difficult to show that the order of \(\zeta, \zeta'\) is bounded independent of \(l\).

**Step 4:** Let \(m_N = m_\mathbb{I}^N + (T)\) and \(\overline{\rho} = \rho_\mathbb{I} \mod m_N\) for \(N \gg 0\) and \(F\) be the splitting field of \(\overline{\rho}\); so, taking \(N \gg 0\), we may assume
\[
\text{Tr}(\rho(Frob_l^f)) = (1 + T)^{f_a} + (1 + T)^{f_{a'}}
\]
for all \(l \gg 0\) as long as \(Frob_l^f \in \text{Gal}(\overline{\mathbb{Q}}/F')\).

**Step 5:** This shows
\[
\text{Tr}(\sigma_s \circ \rho) = \text{Tr}(\rho^s)
\]
over \(G = \text{Gal}(\overline{\mathbb{Q}}/F)\). Then by the above lemma, \(\rho^{ss}|_G\) is abelian, and hence \(\mathbb{I}\) is CM.
§16. $p$-Adic $L$-function

Recall of adjoint $L$-functions and $\mathcal{L}$-invariant to state an application.

We have one variable $L_p \in \mathbb{I}$ characterized by

$$L_p(1, \text{Ad}(\rho_{\mathbb{I}})) = L_p(1, \rho_{\mathbb{I}}^{\text{sym} \otimes 2} \otimes \text{det}(\rho_{\mathbb{I}})^{-1})$$

and

$$L_p(P) := P(L_p) = \frac{L(1, \text{Ad}(f_P))}{\text{period}}$$

for all arithmetic $P$. 
§17. Congruence criterion and $L$-invariant

If $\text{Spec}(h) = \text{Spec}(I) \cup \text{Spec}(X)$ for the complement $X$, (under a mild assumption)

$$\text{Spec}(I) \cap \text{Spec}(X) = \text{Spec}(I \otimes_h X) \cong \text{Spec}(\frac{I}{L_p})$$

Adding the cyclotomic variable, because of the modifying Euler $p$-factor,

$L_p(s, \text{Ad}(\rho_I))$ has **exceptional zero** at 1,

and for an analytic $L$-invariant $0 \neq \mathcal{L}^{an}(\text{Ad}(\rho_I))$ in $\mathbb{I}[\frac{1}{p}]$, we expect to have

$$L'_p(s, \text{Ad}(\rho_I)) \big|_{s=1} \overset{?}{=} \mathcal{L}^{an}(\text{Ad}(\rho_I))L_p.$$. 
§18. An application

The adjoint $p$-adic $L$-function $L_p(s, Ad(f_P))$ has an exceptional zero at $s = 1$ coming from modifying Euler $p$-factor. Greenberg proposed Galois cohomological definition of an $L$-invariant $L(Ad(f_P))$, and we have the following formula in IMRN 59 (2004) 3177–3189: for $c = -2 \log_p(\gamma)$ and $a = a(p)$,

$$\mathcal{L}(Ad(f_P)) = c \cdot a^{-1} t \frac{da}{dt} \bigg|_{t=\gamma^k(P) \varepsilon_P(\gamma)}.$$  

Thus $P \to \mathcal{L}(Ad(f_P))$ is interpolated over $\text{Spec}(\mathcal{O})$ as an analytic function.

**Theorem 2** (Constancy). $P \to \mathcal{L}(Ad(f_P))$ is **constant** if and only if $\rho_{\mathcal{O}}$ has CM.

By this, $P \mapsto \mathcal{L}(Ad(f_P))$ is non-constant for non CM component; so, **non-zero** except for finitely many $P$. 
§19. Proof of the constancy theorem
For simplicity, assume $\mathbb{I} = W[[T]]$. Fix weight $k(P) \geq 1$. Suppose $\mathcal{L}(\text{Ad}(f_P))$ is constant.

**Step 1:** By assumption, $a^{-1}t \frac{da}{dt} = s \in W$ for $a(t) = a(p)(t)$. Thus $t \frac{da}{dt} = s \cdot a$.

**Step 2:** Putting $b(x) = \log_p \circ a(\exp_p(x))$ (for $x = \log_p(t)$), as $dx = \frac{dt}{t}$, by chain rule,
\[
\frac{db}{dx} = \frac{da \ db}{dx \ da} = \frac{da \ d\log_p(a)}{dx \ da} = s \cdot a \cdot \frac{1}{a} = s.
\]

**Step 3:** $b$ is a linear function of $x$:
\[
\log_p(a) = sx + c \iff a = C \exp_p(s \cdot \log_p(t)) = Ct^s.
\]

**Step 4:** Taking $\Phi(t) := t^s$, we find $\Phi(t^z) = \Phi(t)^z$ for $z \in \mathbb{Z}_p$. By the rigidity lemma and its proof, we conclude $s \in \mathbb{Z}_p$.

**Step 5:** Thus $[H_A(\mathbb{I}) : \mathbb{Q}(\mu_{p^\infty})] < \infty$. Then by the H-theorem, we conclude that $\mathcal{F}$ is a CM family. The converse is easy.