3. Lecture 3: Non-vanishing modulo $p$ of $L$–values

We construct an $\mathbb{F}$–valued measure ($\mathbb{F} = \mathbb{F}_p$) over the anti-cyclotomic class group $Cl_\infty = \lim_{\leftarrow n} Cl_n$ modulo $\Gamma^\infty$ whose integral against a character $\chi$ is the Hecke $L$–value $L(0, \chi^{-1}\lambda)$ (up to a period). The idea is to use Hecke relation of the Eisenstein series translated into a distribution relation on the profinite group $Cl_\infty$ and density of CM points.

3.1. Arithmetic Hecke characters. Let $M$ be an imaginary quadratic field. Each integral linear combination $\alpha \cdot i_\infty + j \cdot c \in \mathbb{Z}[\text{Gal}(M/\mathbb{Q})]$ is regarded as a character of $T$ by $x \mapsto i_\infty(x)^k c(x)^j$. We fix an arithmetic Hecke character $\lambda$ of infinite type $(k \cdot i_\infty + \kappa(i_\infty - c))$ for integers $k$ and $\kappa$. Originally $\lambda$ is defined as an ideal character with $\lambda(\alpha) = \alpha^{-k-\kappa(1-c)}$ if $\alpha \equiv 1 \mod \mathfrak{c}$ for its conductor ideal $\mathfrak{c}$.

For each prime $\mathfrak{L}$ of $M$, we choose its local generator $\mathfrak{p}_\mathfrak{L} \in M$ so that $\mathfrak{L} \mathfrak{D}_\mathfrak{L} \equiv (\mathfrak{p}_\mathfrak{L})$, where

$$\mathfrak{D}_\mathfrak{L} = \left\{ \frac{b}{a} | a, b \in \mathfrak{D}, a \mathfrak{D} + \mathfrak{L} = \mathfrak{D} \right\}$$

(the localization at $\mathfrak{L}$).

We write $\mathfrak{D}_\mathfrak{L}$ for the completion $\mathfrak{D}_\mathfrak{L} = \lim_{\leftarrow n} \mathfrak{D}/\mathfrak{L}^n$ and $M_{\mathfrak{L}} = \mathfrak{D}_\mathfrak{L} \otimes \mathfrak{D} M$. Recall the adele ring of $M$ as an $M$-subalgebra in the product $\prod_\mathfrak{L} M_{\mathfrak{L}}$ made up of $x^\infty = (x_{\mathfrak{L}})_\mathfrak{L}$ with $x_{\mathfrak{L}} \in \mathfrak{D}_\mathfrak{L}$ except for finitely many prime ideals $\mathfrak{L}$, and $M_\mathfrak{A} = M_\mathfrak{A}^\infty \times \mathbb{C}$. Embedding $M \subset M_{\mathfrak{L}}$ naturally and in $\mathbb{C}$ by $i_\infty$, we regard $M_{\mathfrak{L}}$ as an $M$-algebra by the diagonal embedding

$$M \ni \xi \mapsto (\xi, \ldots, \xi, \ldots, \xi) \in M_{\mathfrak{L}}.$$  

The infinity component of $x \in M_{\mathfrak{L}}$ is written as $x_\infty$. Put

$$M_{\mathfrak{A}}^{(\infty)} = \{ x \in M_{\mathfrak{A}}^\infty | x_{\mathfrak{L}} = 1 \text{ if } \mathfrak{L} \supset \mathfrak{c} \},$$

and $U(\mathfrak{c}) = \{ x \in \hat{\mathfrak{D}}^\times | x \equiv 1 \mod \hat{\mathfrak{c}} \}$ for an ideal $0 \neq \mathfrak{c} \subset \mathfrak{D}$. Write $U(\mathfrak{c})^{(\mathfrak{c})} = U(\mathfrak{c}) \cap (M_{\mathfrak{A}}^{(\infty)})^\times$ and $U(\mathfrak{c}) = U(\mathfrak{c})_{\mathfrak{c}} \times U(\mathfrak{c})^{(\mathfrak{c})}$.

Exercise 3.1. Let $\mathcal{I}(\mathfrak{c})$ be group of fractional ideals of $M$ prime to $\mathfrak{c}$. Prove an isomorphism $\mathcal{I}(\mathfrak{c}) \cong (M_{\mathfrak{A}}^{(\infty)})^\times/U(\mathfrak{c})^{(\mathfrak{c})}$ of groups sending each prime ideal $\mathfrak{L}$ to the element in $(M_{\mathfrak{A}}^{(\infty)})^\times$, written still $\mathfrak{p}_{\mathfrak{L}}$, whose $\mathfrak{L}$-component is equal to $\mathfrak{p}_{\mathfrak{L}}$ and all other components are 1.

By the above exercise, identifying $(M_{\mathfrak{A}}^{(\infty)})^\times/U(\mathfrak{c})^{(\mathfrak{c})}$ with $\mathcal{I}(\mathfrak{c})$, we may regard $\lambda$ as a character of $(M_{\mathfrak{A}}^{(\infty)})^\times/U(\mathfrak{c})^{(\mathfrak{c})}$. We can then extend $\lambda$ to a character of $M^\times \setminus M_{\mathfrak{A}}^\times$ in the following way for a place $v$.

Definition 3.2. For a place $v$, put $\lambda(\alpha x u) = u^{k_i v + \kappa(i_v - c_i v)} \lambda(x) \in \mathbb{C}_v^\times$ for $\alpha \in M^\times$, $u \in U(\mathfrak{c}) M_{\infty}^\times$ and $x \in (M_{\mathfrak{A}}^{(\mathfrak{c})})^\times$, where $\mathbb{C}_v = \mathbb{C}$ or $\hat{\mathbb{Q}}_p$.
according as \( v = \infty \) or a prime \( p \). If \( v = p \), we write the character obtained as \( \hat{\lambda} : M^* \backslash M_\mathbb{A}^* \rightarrow \overline{\mathbb{Q}}_p^* \) and if \( v = \infty \), we simply write it as \( \lambda \).

Because of this definition, \( k \cdot i_v + \kappa(i_v - c) \) appears as the exponent at \( v \); so, it is called infinite-type of \( \lambda \) and \( p \)-type of \( \hat{\lambda} \). The \( p \)-adic character \( \hat{\lambda} \) is called the \( p \)-adic avatar of \( \lambda \) (and its construction was originally due to Weil).

**Exercise 3.3.** Let \( Cl(\mathfrak{C}p^\infty) = \lim_{\rightarrow n} Cl(\mathfrak{C}p^n) \) as profinite compact group and \( U(\mathfrak{C}p^\infty) = \bigcap_n U(\mathfrak{C}p^n) \). Then prove that (i) \( M^* \backslash M_\mathbb{A}^* / U(\mathfrak{C}p^n) \cong Cl(\mathfrak{C}p^n) \) as compact group if \( n \leq \infty \) and (ii) \( \hat{\lambda} \) factors through \( Cl(\mathfrak{C}p^\infty) \).

Regarding \( \lambda \) as an idele character of \( M_\mathbb{A}^* \), we assume \( \lambda(x_\infty) = x_\infty^{k_i + \kappa(i_i - c_\infty)} \) for \( x_\infty \in T(\mathbb{R}) \). As before fixing two odd primes \( p \neq l \), we assume the following three conditions for simplicity (more general cases are treated in [H07]):

1. \( (ct) \ k \geq 2 \) and \( \kappa \geq 0 \left( \Rightarrow \text{criticality at } s = 0 \text{ for } L(s, \lambda). \right) \)
2. \( (ct) \ The \ conjugator \( \mathfrak{C} \) of \( \lambda \) is trivial; i.e., \( \mathfrak{C} = 1 \) and \( p \geq 5 \).

**3.2. Degeneration operators.** Let \( \mathfrak{l} = (l) \) \( (l > 0) \) be a prime ideal in \( \mathbb{Z} \). Consider the covariant classification functors defined over the category of \( \mathbb{Z}[\frac{1}{l}] \)-algebras:

\[
P_{\Gamma_0(l)}(A) = [(E, C, \omega)_A] \quad \text{and} \quad \mathcal{P}(A) = [(E, \omega)_A],
\]

where \([\cdot] = \{\cdot\} / \cong \) and \( C \) is a cyclic subgroup in \( E \) of order \( l \). Since \( \mathfrak{l}A = A \), we may consider the following morphism of functors \( [\mathfrak{l}] : \mathcal{P}_{\Gamma_0(l)}(A) \rightarrow \mathcal{P}(A) \) sending \( (E, C, \omega)_A \) to \( (E/C, (\pi^*)^{-1}\omega)_A \) for the projection \( \pi : E \rightarrow E/C \). Plainly \([\mathfrak{l}] \) is a morphism of functors; so, by pull back, we get the degeneration morphism \( [\mathfrak{l}] : G_k(1; A) \rightarrow G_k(\Gamma_0(l); A) \) given by \( f[\mathfrak{l}](E, C, \omega) = f(E/C, (\pi^*)^{-1}\omega) \). Adding level \( p^\infty \)-structure \( \phi_p : \mu_{p^\infty} \hookrightarrow E \), we get the corresponding map \( [\mathfrak{l}] : V(1; B) \rightarrow V_{\Gamma_0(l)/B} \).

**Exercise 3.4.** Prove \( f[\mathfrak{l}](q) = f(\text{Tate}(q^l), \frac{1}{l}\omega_{\text{can}}) = l^k \cdot f(q^l) \).

**3.3. Hecke operators.** We define an operator \( T(\mathfrak{l}) : G_k(1; B) \rightarrow G_k(1; B) \) for a prime \( \mathfrak{l} = (l) \) with \( l > 0 \) invertible in \( B \) by

\[
f[T(\mathfrak{l})](E, \omega) = \frac{1}{l} \sum_C (E/C, (\pi^*)^{-1}\omega),
\]

where \( C \) runs over all cyclic subgroup of order \( l \). Similarly we define \( U(\mathfrak{l}) : G_k(\Gamma_0(l); B) \rightarrow G_k(\Gamma_0(l); B) \) by

\[
f[U(\mathfrak{l})](E, C', \omega) = \frac{1}{l} \sum_C (E/C, \pi(C') = (C + C')/C', (\pi^*)^{-1}\omega),
\]
where $C$ runs over all cyclic subgroup of order $l$ different from $C'$.

**Exercise 3.5.** Write $q$-expansion of modular forms $f$ at the infinity, i.e., at $(\text{Tate}(q), C_{\text{can}}, \omega_{\text{can}})$ as $f(q) = \sum_n a(n, f)q^n$. Then prove

$$a(n, f|T(l)) = a(nl, f) + l^{k-1}a\left(\frac{n}{l}, f\right) \quad \text{and} \quad a(n, f|U(l)) = a(nl, f).$$

### 3.4. Eisenstein series

We are going to define an optimal Eisenstein series whose special values at CM points interpolate the values $L(0, \lambda \chi)$ for anticyclotomic characters $\chi$ of finite order.

For any even positive integer $k > 0$, we can now define the Eisenstein series $E_k$. We define the value $E_k(L)$ for $L \in \text{Lat} = \{\mathbb{Z}w_1 + \mathbb{Z}w_2 \subset \mathbb{C} | \text{Im}(w_1/w_2) > 0\}$ by

$$E_k(L) = (-1)^k \Gamma(k+s) \sum'_{w \in L/\mathbb{Z}^\times} \frac{1}{w^k |w|^{2s}} \bigg|_{s=0}.$$  

Here "$\sum'$" indicates that we are excluding $w = 0$ from the summation. This type of series is convergent when the real part of $s$ is sufficiently large and can be continued to a meromorphic function well defined at $s = 0$ (as long as either $k \geq 4$; see [LFE] §2.5 for analytic continuation).

**Lemma 3.6.** If $4 \leq k \in 2\mathbb{Z}$, the function $E_k$ gives an element in $G_k(1; \mathbb{C})$, whose $q$-expansion at the cusp $\infty$ is given by

$$E_k(q) = 2^{-1} \zeta(1-k) + \sum_{0<n\in\mathbb{Z}} \sum_{(a,b)\in(\mathbb{Z}\times\mathbb{Z})/\mathbb{Z}^\times} \frac{a}{|a|} a^{k-1} q^n.$$  

When $k = 2$, $E_k(z)$ is non-holomorphic and its Fourier expansion contains an extra term $\frac{c}{2\text{Im}(z)}$ for a constant $c$ in addition to the above holomorphic $q$-expansion.

From the effect of $T(l)$ and $U(l)$ on $q$-expansion, we verify easily the following lemma.

**Lemma 3.7.** For a prime $l = (l)$, we have

$$E_k|T(l) = (1 + l^{k-1})E_k,$$  

On the elliptic curve side, $((l)(E, \omega)) = (E \otimes_{\mathbb{Z}} l, \omega')$, where as an fpfp abelian sheaf, $E \otimes_{\mathbb{Z}} l$ is the sheafification of $A \mapsto E \otimes_{\mathbb{Z}} l(R) = E(R) \otimes_{\mathbb{Z}} l$. Tensoring $E$ with the exact sequence $0 \to l \to \mathbb{Z} \to \mathbb{Z}/l \to 0$ (of constant abelian fpfp sheaves), since $E$ is a divisible fpfp abelian sheaf, $E \otimes_{\mathbb{Z}} \mathbb{Z}/l = 0$, and we get

$$0 \to \text{Tor}_1(E, \mathbb{Z}/l) \to E \otimes_{\mathbb{Z}} l \to E \to 0.$$
is exact. Since \( \mathbb{Z} \) is a principal ideal domain, \( \text{Tor}_1(E, \mathbb{Z}/l) \cong E[l] = E[l] \); so, we have a commutative diagram:

\[
\begin{array}{ccc}
\text{Tor}_1(E, \mathbb{Z}/l) & \longrightarrow & E \otimes \mathbb{Z}/l \\
\downarrow & & \downarrow \\
E[l] & \longrightarrow & E.
\end{array}
\]

Multiplication by \( l \) induces \( E \cong E \otimes \mathbb{Z}/l \), which acts on \( \omega \) by \( \omega \mapsto l\omega =: \omega' \). On the lattice side, \( \mathbb{C}/L \otimes \mathbb{Z}/l = \mathbb{C}/lL \); so, it is given by multiplication \( L \mapsto lL \). For modular form \( f \), we define \( f(l)(E, \omega) = f((l)(E, \omega)) \).

**Exercise 3.8.** Let \( l \) be a prime outside \( \mathfrak{f} \). Suppose that \( l = (l) \) for a positive \( l \in \mathbb{Q} \). Let \( \mathbb{E}'_k = E_k - l^{-1}E_k[1][l] \) and \( \mathbb{E}_k = E_k - E_k(l)[l] \). Then prove

1. \( \mathbb{E}'_k U(l) = \mathbb{E}'_k \).
2. \( \mathbb{E}_k U(l) = l^{k-1}\mathbb{E}_k \).
3. Even if \( E_2 \) is non-holomorphic, \( \mathbb{E}_2 \) is holomorphic.

**Remark 3.9.** By \( a(n, df) = n \cdot a(n, f) \), the Hecke operator \( T(l) \) and \( U(l) \) satisfies \( T(l) \circ d = l \cdot d \circ T(l) \) and \( T(l) \circ d = l \cdot d \circ T(l) \) for the Katz differential operator \( d \). Thus for \( \mathbb{E}(\lambda) = d^*\mathbb{E}_k \) and \( \mathbb{E}'(\lambda) = d^*\mathbb{E}'_k \), we have under the notation of Lemma 3.8

\[
(3.4) \quad \mathbb{E}'(\lambda)|U(l) = l^{k}\mathbb{E}'(\lambda), \quad \mathbb{E}(\lambda)|U(l) = l^{k-1+*}\mathbb{E}(\lambda).
\]

3.5. **Anticyclotomic Hecke \( L \)-functions.** Pick a prime \( l \) of \( \mathcal{O} \). Define the order \( \mathcal{O}_n = \mathbb{Z} + l^n\mathcal{O} \) of conductor \( l^n \). We determine the type of Hecke \( L \)-function obtained by values of Eisenstein series at CM points. The result (equivalent to the one presented here) is explained well in H. Yoshida [LAP] V.3.2.

**Exercise 3.10.** Prove the identity:

\[
\{\text{non-proper } \mathcal{O}_{n+1}-\text{ideals}\} = \{l|a|a \text{ is an } \mathcal{O}_n-\text{ideal}\}.
\]

We admit

**Proposition 3.11.** Let \( I_n \) be the group of all proper fractional \( \mathcal{O}_n-\)ideals. Associating to each \( \mathcal{O}_{n+1}-\text{ideal } a \) the \( \mathcal{O}_n-\text{ideal } \mathcal{O}_n a \), we get the following homomorphism of groups \( \pi_n : I_{n+1} \rightarrow I_n \). The homomorphism \( \pi \) is surjective, and the kernel of \( \pi \) is isomorphic to \( \mathcal{O}^\times_{n, l}/\mathcal{O}^\times_{n+1, l} \). We have the following exact sequence:

\[
1 \rightarrow \mathcal{O}^\times_{n, l}/\mathcal{O}^\times_{n+1, l} \mathcal{O}^\times_{n} \rightarrow \text{Cl}_{n+1} \rightarrow \text{Cl}_{n} \rightarrow 1.
\]
Let \( \chi \) be a character of the group of fractional proper ideals of \( \mathcal{O}_n \). By the above proposition, \( \chi \) gives rise to a unique character of the full group of fractional ideals of \( M \). Put \( N(a) = [\mathcal{O}_n : a] = [\mathcal{O} : \mathcal{O}a] \). We then define a formal \( L \)-function:

\[
L^n(s, \chi) = \sum_{a \in \mathcal{O}_n} \chi(a)N(a)^{-s},
\]

where \( a \) runs over all proper \( \mathcal{O}_n \)-ideals. We write \( L(s, \chi) \) for \( L^0(s, \chi) \), which is the classical Hecke \( L \)-function. This \( L \)-function depends on \( n \), because the set of proper \( \mathcal{O}_n \)-ideals depends on \( n \). However \( L^0 \) and \( L^n \) are different at Euler \( l \)-factor.

Since \( Cl_\infty \) is almost pro-\( l \) group, all finite order characters of \( Cl_\infty \) has values in \( W[\mu_{p^m}] \) if every element of the finite \( p \)-Sylow subgroup of \( Cl_\infty \) is killed by \( p^m \). Replacing \( W \) be \( W[\mu_{p^m}] = W(\overline{\mathbb{F}}_p)[\mu_{p^m}] \), we write hereafter \( W \) for \( \overline{\mathbb{F}}_p[\mu_{p^m}] \). We will prove, assuming \( \mathcal{C} = 1 \), the following theorem at the very end of this chapter after long preparation (a proof in the general case where \( \mathcal{C} \neq 1 \) can be found in [H07]):

**Theorem 3.12.** Let \( p \) be an odd prime splitting in \( M/\mathbb{Q} \). Let \( \lambda \) be a Hecke character of \( M \) of conductor 1 and of infinity type \( k+k(1-c) \) with \( k \geq 2 \). Suppose (ct) and (ol) in §3.1. Then \( \pi^k\Gamma(k+\kappa)L(0,0,\chi^{-1},\lambda) \) \( \in W \) for all finite order characters \( \chi : Cl_\infty \to W^\times \) with nontrivial conductor. Moreover, except for finitely many characters \( \chi \) in \( Cl_\infty \), we have

\[
\frac{\pi^k\Gamma(k+\kappa)L(0,0,\chi^{-1},\lambda)}{\Omega_{\kappa+2\kappa}^\infty} \not\equiv 0 \mod m_W.
\]

### 3.6. Values at CM points

We take a proper \( \mathcal{O}_{n+1} \)-ideal \( a \) for \( n \geq 0 \), and regard it as a lattice in \( \mathbb{C} \) by \( a \mapsto i_\infty(a) \). We suppose that \( a_p = \mathcal{O}_p = \mathcal{O}_p \oplus \mathcal{O}_p \). This implies \( H^0(E(\mathcal{O}), \Omega_{E(\mathcal{O})}/\mathcal{W}) = H^0(E(a), \Omega_{E(a)/\mathcal{W}}) \) as \( E(a) \) and \( E(\mathcal{O}) \) are isogenous by an isogeny of degree prime to \( p \). Then a generator \( \omega(\mathcal{O}) \) of \( H^0(E(\mathcal{O}), \Omega_{E(\mathcal{O})}/\mathcal{W}) \) gives \( \omega(a) \) which generates \( H^0(E(a), \Omega_{E(a)/\mathcal{W}}) \). We have fixed \( \phi_p(\mathcal{O}) : \mu_{p^\infty} \cong E(\mathcal{O})[p^\infty] \), and this identification \( a_p = \mathcal{O}_p = \mathcal{O}_p \oplus \mathcal{O}_p \) induces \( \phi_p(a) : \mu_{p^\infty} \cong E(a)[p^\infty] \). Then \( a_1 \cong \mathcal{O}_{n+1,1} = \mathbb{Z}_l + \mathbb{I}^{n+1} \mathcal{O}_l \), and hence \( a\mathcal{O}_n \supset a \). The subgroup \( C(a) = a\mathcal{O}_n/a \in E(a)(\mathbb{C})/\mathcal{C}/i_\infty(a) \) gives a canonical cyclic subgroup \( C(a) \subset E(a) \) of order \( l \) (defined over \( \mathcal{W} \)). Write \( \omega_\infty(a) = du \) for the variable \( u \in \mathbb{C} \). For a \( p \)-adic modular form \( f \) of the form \( d^u g \) for classical \( g \in G_k(\Gamma_0(l); \mathcal{W}) \), we have by Theorem 2.12

\[
\frac{d^u f(x(a), \phi_p(a))}{\Omega_p^{k+2\kappa}} = \delta_k f(x(a), \omega(a)) = \frac{d^u f(x(a), \omega_\infty(a))}{\Omega_\infty^{k+2\kappa}}.
\]

Here \( x(a) \) is the test object: \( x(a) = (E(a), C(a))/\mathcal{W} \in X_0(l)(\mathcal{W}) \).
We write \( c_0 = (-1)^k \frac{\pi^s \Gamma(\frac{k+s}{2})}{\Im(\delta) \sqrt{D} \Gamma(s)} \) with \( 2\delta = \sqrt{-D} \). Here \( \Gamma(s) \) is the Euler’s Gamma function. By definition, we find, for \( \epsilon \) example,

\[
E_n > (which is equal to 1 if \( n > 0 \)),
\]

\[
(c_0 e)^{-1} \delta_k E_k(x(a), \omega(a)) = \sum_{w \in a/\mathcal{O}_{n+1}^x} \frac{1}{w^{k+s(1-c)} N_M/Q(w)^s} |_{s=0}
\]

\[
= \sum_{w \in a/\mathcal{O}_{n+1}^x} \lambda(w)^{-1} N_M/Q(w)^s |_{s=0}
\]

\[
= \lambda(a) N_M/Q(a)^{-s} \sum_{w a^{-1} \in \mathcal{O}_{n+1}} \frac{\lambda(w a^{-1})}{N_M/Q(w a^{-1})^s} |_{s=0}
\]

\[
= \lambda(a) L_{[a^{-1}]}^{n+1}(0, \lambda),
\]

where for the ideal class \([a^{-1}] \in Cl_{n+1}\) represented by a proper \( \mathcal{O}_{n+1}^- \) ideal \( a^{-1} \),

\[
L_{[a^{-1}]}^{n+1}(s, \lambda) = \sum_{b \in [a^{-1}]} \lambda(b) N_M/Q(b)^{-s}
\]

is the partial \( L \)-function of the class \([a^{-1}]\) for \( b \) running over all \( \mathcal{O}_{n+1}^- \) proper integral ideals prime to \( \mathfrak{c} \) in the class \([a^{-1}]\). In the second line of (3.6), we regard \( \lambda \) as an idele character and in the other lines as an ideal character. For an idele \( a \) with \( a \mathfrak{D} = a \mathfrak{D} \), we have \( \lambda(a^{(n)}) = \lambda(a) \) and \( \hat{\lambda}(a^{(p)}) = \hat{\lambda}(a) \).

We put \( \mathbb{E}(\lambda) = d^s E_k \) and \( \mathbb{E}^\prime(\lambda) = d^s E_k^\prime \) as in Remark 3.9. We want to evaluate \( \mathbb{E}(\lambda) \) and \( \mathbb{E}^\prime(\lambda) \) at \( x = (x(a), \omega(a)) \). Thus we write, for example, \( \mathbb{E}(\lambda) \) and \( \mathbb{E}^\prime(\lambda) \) for \( \mathbb{E}(\lambda) \) and \( \mathbb{E}^\prime(\lambda) \). Then by definition and Theorem 2.12, we have for \( x = (x(a), \omega(a)) \)

\[
\mathbb{E}^\prime(\lambda)(x) = \delta_k^s E_k(x) - l^{-1} \delta_k^s E_k(x(a \mathfrak{D}_n), \omega(a \mathfrak{D}_n))
\]

\[
\mathbb{E}(\lambda)(x) = \delta_k^s E_k(x) - \delta_k^s E_k(x(a \mathfrak{D}_n), \omega(a \mathfrak{D}_n)).
\]

because \( C(a) = a \mathfrak{D}_n/a \) and hence \([l](x(a)) = x(a \mathfrak{D}_n)\).

To simplify the notation, write \( \phi([a]) = \hat{\lambda}(a)^{-1} \phi(x(a), \omega(a)) \). By Exercise 2.4, for \( \phi = \mathbb{E}(\lambda) \) and \( \mathbb{E}^\prime(\lambda) \), the value \( \phi([a]) \) only depends on the ideal class \([a] \) but not the individual \( a \). The formula (3.7) combined
with (3.6) shows, for a proper $\mathcal{O}_{n+1}$-ideal $a$,

\begin{equation}
-1E'(\lambda)([a]) = c_0 \left( L_{[a^{-1}]}^{n+1}(0, \lambda) = -1L_{[a^{-1}]}^n(0, \lambda) \right)
\end{equation}

(3.8)

\begin{equation}
-1E(\lambda)([a]) = c_0 \left( L_{[a^{-1}]}^{n+1}(0, \lambda) = \lambda(0)L_{[a^{-1}]}^n(0, \lambda) \right)
\end{equation}

(3.9)

where $e = [\mathcal{O}^\times : \mathbb{Z}^\times]$. For a primitive character $\chi : Cl_f \to W^\times$

\begin{equation}
L^n(s, \lambda \chi) = \sum_a \lambda \chi(a) N_{\mathbb{M}/\mathbb{Q}}(a)^{-s},
\end{equation}

(3.10)

where $a$ runs over all proper ideals in $\mathcal{O}_n$ prime to $I^f$ and $N_{\mathbb{M}/\mathbb{Q}}(a) = [\mathcal{O}_n : a]$. For each primitive character $\chi : Cl_f \to \overline{\mathbb{Q}}^\times$, taking $n = f$, by a similar but more involved computation using (3.7), we have

(3.11)

\begin{equation}
e^{-1} \sum_{[a] \in Cl_{n+1}} \chi(a)E'(\lambda)([a]) = c_0 \cdot (L_{[a^{-1}]}^{n+1}(0, \lambda \chi^{-1}) - L^n(0, \lambda \chi^{-1}))
\end{equation}

and up to $p$-units,

(3.12)

\begin{equation}
e^{-1} \sum_{[a] \in Cl_{n+1}} \chi(a)E'(\lambda)([a]) = c_0 \cdot L^0(0, \chi^{-1}),
\end{equation}

(3.13)

By Theorems 2.5 and 2.12, we have proven that all these values are algebraic in $\overline{\mathbb{Q}}$ and actually integral over $\mathcal{W}$:

**Theorem 3.13.** Let $c_0 = (-1)^k \frac{\pi \Gamma(k+\kappa)}{\Im(\delta)^{k+\kappa} \sqrt{D_{\mathcal{O}}}}$ for integers $k > 0$ and $\kappa \geq 0$. Then the value of (3.11) is in $\mathcal{W}$ if $f > 0$ and $p \geq 5$ or $\kappa > 0$.

This follows from the fact that $E(\lambda)$ has $\mathcal{W}$-integral $q$-expansion (i.e., no constant term) if either $k \neq 2$ or $\kappa > 0$. If $\kappa = 0$ and $k = 2$, the constant term $2^{-1}(1-l)\zeta(-1)$ of $E'(\lambda)$ is $p$-integral under the condition: $p \geq 5$. so, the result is clear from the formula (3.12) and Theorems 2.5 and 2.12.
3.7. Construction of a modular measure. Let $R = W$ or $\mathbb{F} = \mathbb{F}_p$.
Let $f \in V_{\Gamma_0(0)/R}$ be a normalized Hecke eigenform (here normalization means that $f|T(n) = a(n, f)f$, $f|U(l) = a(l, f)f$ and $f(q) = \sum_{n=0}^\infty a(n, f)q^n$). A typical example of $f$ can be given as follows: Take a modular form $g$ in $G_k(\Gamma_0(l); R)$ for $R = \mathbb{F}$. Put $f = d^\ast g$ for the differential operator $d^\ast$ in 2.2. We write $f(x(a))$ for the value of at $x(a)$.
The Hecke operator $U(l)$ takes the space $V(\Gamma_0(l); R)$ into $V(\Gamma_0(l); R)$. We regard $U(l)$ as an operator acting on $V(\Gamma_0(l); W)$. Suppose that $g|U(l) = a \cdot g$ with $a \in W^{x}$; so, $f|U(l) = l^\ast a \cdot f$ for the positive generator $l$ of $l$ (see Remark 3.9). The Eisenstein series $E(\lambda)$ satisfies this condition by Lemma 3.8.

Choosing a basis $w = (w_1, w_2)$ of $\widehat{\mathcal{O}} = \mathcal{O} \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$, identify the full Tate module $T(E(\mathcal{O}), \mathbb{Z}) = \lim_{\leftarrow} E(\mathcal{O})[N] = \widehat{\mathcal{O}}$ with $\hat{\mathbb{Z}}^2$ by $\hat{\mathbb{Z}} \ni (a, b) \mapsto aw_1 + bw_2 \in T(E(\mathcal{O}))$, getting a level structure: $\mathbb{Q}^2 \otimes_{\mathbb{Q}} \mathbb{A}^{(p, \infty)} \cong V(\mathbb{Q})(E(\mathcal{O})) := T(E(\mathcal{O}) \otimes_{\mathbb{Z}} \mathbb{Q})$ defined over $\mathcal{W}$. Elliptic curves $E_{/A}$ with such level structure $\eta^{(p)} : (\mathbb{A}^{(p, \infty)})^2 \cong V(\mathbb{Q})(E)$ is classified by $Sh^{(p)}(A) = \lim_{\leftarrow} \mathcal{W}(\mathbb{Q})[\mathbb{Z}]$ up to prime-to-$p$ isogenies; i.e., if $\phi : E \to E'_{/A}$ is an isogeny with degree prime-to-$p$ with $\phi \circ \eta^{(p)} = \eta'^{(p)}$ gives a unique point $x \in Sh^{(p)}(A)$ such that $(E, \eta^{(p)})(x) = (E, \eta^{(p)})(E')$ for the universal couple $(E, \eta^{(p)})_{/Sh^{(p)}}$. The Shimura curve $Sh^{(p)}_{/\mathcal{W}}$ has a right action of $GL_2(\mathbb{A}^{(p, \infty)})$ by $\eta^{(p)} \mapsto \eta^{(p)} \circ g$ (Shimura’s global reciprocity). Choose the basis $w$ satisfying the following two conditions:

(B) $w_{1,1} = 1$ and $\mathcal{O}_1 = \mathbb{Z}[w_{2,1}]$.

Let $a$ be a proper $\mathcal{O}_n$–ideal (for $\mathcal{O}_n = \mathbb{Z} + \Gamma^n \mathcal{O}$) prime to $l$. Write

$l_1 = (1, \ldots, 1, l, 1, \ldots, 1) \in \mathbb{A}^\times$. Then $(w_1, l^n w_2)$ is a base of $\widehat{\mathcal{O}}_n$ and gives a level structure $\eta^{(p)}(\mathcal{O}_n) : \mathbb{Q}^2 \otimes_{\mathbb{Q}} \mathbb{A}^{(p, \infty)} \cong V(\mathbb{Q})(E(\mathcal{O}_n))$. We also write $l_1$ for $l \in \mathbb{Z}_l$ (if we want to avoid confusion). We choose a complete representative set $A = \{a_1, \ldots, a_H\} \subset M^\times$ so that $M^\times = \bigsqcup_{j=1}^H M^\times a_j \widehat{\mathcal{O}}_n M^\times$. Then $\alpha \widehat{\mathcal{O}}_n = \alpha a_j \widehat{\mathcal{O}}_n$ for $\alpha \in M^\times$ for some index $j$. We then define $\eta^{(p)}(a) = \alpha a_j \eta^{(p)}(\mathcal{O}_n)$. The small ambiguity of the choice of $\alpha$ does not cause any trouble.

Write $x(a) = (E(a), C(a), \omega(a))/\mathcal{W}$ (for $C(a) = \eta^{(p)}(\mathcal{O}_{n-1}/a) = \mathcal{A}_n/a \subset E(a)$). This is a test object of level $\Gamma_0(l)$ and is the image of $x(a)$ in $X_0(l)$. We pick a subgroup $C \subset E(\mathcal{O}_n)$ such that $C \cong \mathbb{Z}/l^n$ ($m > 0$) but $C \cap C(\mathcal{O}_n) = \{0\}$. Since $\mathcal{W}$ is strictly henselian (i.e., $\mathcal{W}/\mathfrak{m}_\mathcal{W} = \mathbb{F}_p = \mathbb{F}$) and $l \nmid p$, $E(\mathcal{O}_n)[l^n]$ is a constant étale group scheme isomorphic to $(\mathbb{Z}/l)^2$; so, making the quotient $E(\mathcal{O}_n)/C$ is easy.
(see [GME] §1.8.3). Then we define \( x(\mathcal{O}_n)/C \) by
\[
\left( \frac{E(\mathcal{O}_n)}{C}, \frac{C + C(\mathcal{O}_n)}{C}, (\pi^*)^{-1}\omega(\mathcal{O}_n) \right)
\]
for the projection map \( \pi : E(\mathcal{O}_n) \to E(\mathcal{O}_n)/C \).

**Lemma 3.14.** We have
\[
x(\mathcal{O}_n)/C = x(a_C) \in \mathcal{M}_{\text{reg}(\mathcal{W})}
\]
for a proper \( \mathcal{O}_{n+m} \)-ideal \( a = a_C \supset \mathcal{O}_n \) with \( (aa^c) = 1^{-2m} \), and for \( u \in \mathbb{Z}_l^\times \) we have
\[
x(a_C) = x(\mathcal{O}_n)/C = x(\mathcal{O}_{m+n})\left( \begin{array}{cc} \frac{u}{l} & 0 \\ 0 & 1 \end{array} \right).
\]

**Proof.** Write simply \( \eta \) for \( \eta^{(p)} \). The base of \( \mathcal{O}_{n,l} \) is given by \( \alpha_{n,l}(1, w_2) \) for \( \alpha_n = \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \) with a prime element \( l_1 \) of \( \mathbb{Z}_l \). The action on level \( l \) structure \( \eta \mapsto \eta \circ g \) induces the action \( \widetilde{L}^{(p)} \mapsto g^{-1}\widetilde{L}^{(p)} \) for \( \mathbb{Z} \)-lattices, as \( \widetilde{L}^{(p)} = \eta^{-1}(TE^{(p)}) \mapsto (\eta \circ g)^{-1}(TE^{(p)}) = g^{-1}\widetilde{L}^{(p)} \). Thus we find that \( \alpha_{n,l}^{-1}(x(\mathcal{O})) = x(\mathcal{O}_n) \) and \( \alpha_{n,l}^{-1}(x(\mathcal{O}_{n-1})) = x(\mathcal{O}_n) \). Since the general case of \( m > 1 \) follows by iteration of the formula in the case of \( m = 1 \), we suppose \( m = 1 \). Then the formula becomes, for a suitable \( u \in \mathbb{Z}_l^\times \)
\[
l_1^{-1}(x(a)) = x(la) = x(\mathcal{O}_{n+1})\left( \begin{array}{c} \frac{u}{l} \\ 1 \end{array} \right).
\]
if \( x(a) = x(\mathcal{O}_n)/C \) for \( C \) as above. To see this, note that the base of \( l_1a_1 \) is given by
\[
\left( \begin{array}{c} 1 - \frac{u_1}{l_1}u_2 \\ l_1^{-1}u_2 \end{array} \right) = \left( \begin{array}{c} \frac{u_1}{l_1} \\ 0 \end{array} \right) \alpha_{n+1}(\frac{u_2}{l_1}).
\]
Thus \( a_1/\mathcal{O}_n \) is generated by \( 1 - \frac{u_1u_2}{l_1} \) mod \( \mathcal{O}_{n,l} \) which gives the subgroup \( C \) for a suitable choice of \( u \). Since \( l^{(l)} \in \mathbb{Z}_l^\times \), the action of \( l_1 \) is equivalent to the action of \( l \in \mathbb{Z}(\mathbb{Q}) \) which is trivial; so, we forget \( l_1 \) in (3.14). \( \square \)

For each proper \( \mathcal{O}_n \)-ideal \( a \), we have an embedding \( \rho_a : M^\times_{\mathbb{A}_{\mathbb{A}}} \to GL_2(\mathbb{A}^{(p)}_{\mathbb{A}}) \) given by \( \alpha \eta_{(p)}(a) = \eta_{(p)}(a) \circ \rho_a(\alpha) \). Since \( \det(\rho_a(\alpha)) = \alpha \alpha^c \gg 0 \), \( \alpha \in \mathcal{O}_{(p)}^\times \) acts on \( Sh^{(p)} \) through \( \rho_a(\alpha) \in G(\mathbb{A}) \). We have
\[
\rho_a(\alpha)(x(a)) = (E(a), \eta_{(p)}(a)\rho_a(\alpha)) = (E(\alpha a), \eta_{(p)}(\alpha a))
\]
for the prime-to-\( p \) isogeny \( \alpha \in \text{End}_Z(E(a)) = \mathcal{O}(a) \). Thus \( \mathcal{O}_{(p)}^\times \) acts on \( Sh^{(p)} \) fixing the point \( x(a) \). We find \( \rho(\alpha)^\times(\omega(a) = \alpha \omega(a) \) and
\[
g(x(\alpha a), \omega(a)) = g(\rho(\alpha)(x(a), \omega(a))) = \alpha^{-k}g(x(a), \omega(a)).
\]
From this, we conclude
\[
f(x(\alpha a), \omega(a)) = f(\rho(\alpha)(x(a), \omega(a))) = \alpha^{-k(1-c)}f(x(a), \omega(a)),
\]
because the effect of the differential operator $d$ is identical with that of $\delta$ at the CM point $x(a)$ by Theorem 2.12. Since

$$\hat{\lambda}(a \lambda) = \alpha^{-k-\kappa(1-c)}\hat{\lambda}(a),$$

the value $\hat{\lambda}(a)^{-1}f(x(a), \omega(a))$ is independent of the representative set $A = \{a_j\}$ for $Cl_n$. Defining, for a proper $\mathfrak{D}_n$-ideal $a$ prime to $p$,

$$f([a]) = \hat{\lambda}(a)^{-1}f(x(a), \omega(a)),$$

we find that $f([a])$ only depends on the proper ideal class $[a] \in Cl_n$.

We write $x(a_u) = x(a)|\alpha_1^{-1}(1 + \frac{1}{1})$. Then $a_u$ depends only on $u \mod \mathfrak{l}$, and $\{a_u\}_{u \mod \mathfrak{l}}$ gives a complete representative set for proper $\mathfrak{D}_{n+1}$-ideal classes which project down to the ideal class $[a] \in Cl_n$. Since $a_u \mathfrak{D}_n = l^{-1}a$, we find $\hat{\lambda}(a_u) = \hat{\lambda}(l)^{-1}\hat{\lambda}(a)$. Recalling $f|U(l) = l^ka \cdot f$, we have

$$l^ka \cdot f([a]) = \hat{\lambda}(a)^{-1}f|U(l)(x(a)) = \frac{1}{\hat{\lambda}(l)l} \sum_{u \mod \mathfrak{l}} f([a_u]).$$

**Definition 3.15.** For a continuous function $\phi : Cl_\infty \to \mathbb{F}$, taking $n > 0$ so that $\phi$ factors through $Cl_n$, we define a measure $\varphi_f$ on $Cl_\infty$ with values in $\mathbb{F}$ by

$$\int_{Cl_\infty} \phi d\varphi_f = b^{-n} \sum_{a \in Cl_n} \phi(a^{-1})f([a]) \quad (\text{for } b = l^ka \cdot f).$$

3.8. **Non-triviality of the modular measure.** The non-triviality of the measure $\varphi_f$ is proven in [H04] Theorems 3.2 and 3.3. To recall the result in [H04], we recall a functorial action (introduced earlier) on $p$-adic modular forms, commuting with $U(l)$. Let $q$ be a prime ideal of $\mathbb{Q}$ different from $\mathfrak{l}$. For a test object $(E, \eta)$ of level $\Gamma_0(q)$, the $q$-part $\eta_q$ of $\eta$ is a subgroup $C \cong \mathbb{Z}/q$ in $E$. Then we can construct canonically $[q](E, \eta) = (E', \eta')$ with $E' = E/C$. If $q$ splits into $\Omega\bar{\Omega}$ in $M/\mathbb{Q}$, choosing $\eta_q$ induced by $E(a)[q^\infty] \cong M_{\Omega}/\Omega_{\Omega} \times M_{\bar{\Omega}}/\bar{\Omega}_{\bar{\Omega}} \cong \mathbb{Q}_q/\mathbb{Z}_q \times \mathbb{Q}_q/\mathbb{Z}_q$, we always have a level $\Gamma_0(q)$-structure $C = E(a)[\Omega_n]$ for $\Omega_n = \Omega \cap \bar{\Omega}$ on $E(a)$ induced by the choice of the factor $\Omega$. Then $[q](E(a)) = E(a\Omega_n^{-1})$ for a proper $\mathfrak{D}_n$-ideal $a$, as $\Omega_n^{-1}a/a \cong C$ by $\eta_q$ (so, $E(a)/C = E(a)/(a\Omega_n^{-1}) = E(a\Omega_n^{-1})$). When $q$ ramifies in $M/\mathbb{Q}$ as $q = \Omega^2$, $E(a)$ has a subgroup $C = E(a)[\Omega_n]$ isomorphic to $\mathbb{Z}/q$; so, we can still define $[q](E(a)) = E(a\Omega_n^{-1})$. The effect of $[q]$ on the $q$–expansion at the infinity cusp is computed in $\S 3.2$ and is given by a unit multiple of the $q$–expansion of $f$ at the Tate curve Tate($q^\infty$) for a positive generator $\varpi$ of $q$. The operator $[q]$ corresponds to the action
of \( g = \begin{pmatrix} 1 & 0 \\ 0 & \omega_q^{-1} \end{pmatrix} \in GL_2(\mathbb{Q}_q) \). In §3.2, we saw that \([q]\) induces a linear map well defined on \( V_{\overline{\tau}(0)/R} \) into \( V_{\overline{\tau}(q)/R} \).

We fix a decomposition \( Cl_\infty = \Gamma \times \Delta \) for a finite group \( \Delta \) and a torsion-free subgroup \( \Gamma \). Since each fractional \( \mathcal{O} \)-ideal \( \mathfrak{A} \) prime to \( \mathfrak{I} \) defines a class \([\mathfrak{A}]\) in \( Cl_\infty \), we can embed the ideal group of fractional ideals prime to \( \mathfrak{I} \) into \( Cl_\infty \). We write \( Cl_{\text{alg}} \) for its image.

Exercise 3.16.  
(1) Complex conjugation acts on \( z \in Cl_\infty \) by \( z \mapsto z^{-1} \).

(2) The intersection \( \Delta_{\text{alg}} = \Delta \cap Cl_{\text{alg}} \) is represented by square-free products of prime ideals of \( M \) ramified over \( \mathbb{Q} \). In other words, \( \Delta_{\text{alg}} \) is isomorphic to the ambiguous class group of \( M \).

(3) The quotient \( Cl_\infty / \Gamma \Delta_{\text{alg}} \) has a complete representative set in the set of prime ideals split over \( \mathbb{Q} \) (prime to \( \mathfrak{I} \)).

(4) Write \([\mathfrak{Q}]_\Gamma \) (resp. \([\mathfrak{Q}]_\Delta \)) for the projection of \([\mathfrak{Q}] \in Cl_{\text{alg}} \) to \( \Gamma \) (resp. to \( \Delta \)). If \([\mathfrak{Q}]_\Delta \notin [\mathfrak{Q}]_\Delta \Delta_{\text{alg}} \), then \([\mathfrak{Q}]_\Gamma / [\mathfrak{Q}]_\Gamma \notin Cl_{\text{alg}} \).

We choose a complete representative set \( \{ \mathfrak{R}^{-1} \mathfrak{r} \in \mathcal{R} \} \) for \( \Delta_{\text{alg}} \) such that the set \( \mathcal{R} \) is a subset of the set of all square-free product of primes in \( \mathbb{Q} \) ramifying in \( M/\mathbb{Q} \), and \( \mathfrak{R} \) is a unique ideal in \( M \) with \( \mathfrak{R}^2 = \mathfrak{r} \). The set \( \{ \mathfrak{R} \mathfrak{r} \in \mathcal{R} \} \) is a complete representative set for 2–torsion elements in the class group \( Cl_0 \) of \( \mathcal{O} \) (i.e., the ambiguous classes). We fix a character \( \nu : \Delta \to \mathbb{F}_\times \), and define

\[
(3.18) \quad f_\nu = \sum_{\mathfrak{r} \in \mathcal{R}} \hat{\nu}^{-1} (\mathfrak{R}) f[\mathfrak{r}].
\]

Choose a complete representative set \( \mathcal{Q} \) for \( Cl_\infty / \Gamma \Delta_{\text{alg}} \) made of primes of \( M \) split over \( \mathbb{Q} \) outside \( pt \). Since \( Cl_{\text{alg}} \) is dense in \( Cl_\infty \), we can choose \( \mathfrak{Q} \in \mathcal{Q} \) whose projection to \( \Gamma \) is whatever close to \( 1 \) under the profinite topology (this remark will be useful later). We choose \( \eta_{\mathfrak{q}}^{(p)} \) out of the base \((w_1, w_2)\) of \( \mathcal{O}_n \) so that at \( \mathfrak{q} = \mathfrak{Q} \cap \mathbb{Q} \), \( w_1 = (1, 0) \in \mathcal{O}_\mathfrak{Q} \times \mathcal{O}_\mathfrak{Q} = \mathcal{O}_\mathfrak{q} \) and \( w_2 = (0, 1) \in \mathcal{O}_\mathfrak{Q} \times \mathcal{O}_\mathfrak{Q} = \mathcal{O}_\mathfrak{q} \). Since all operators \([q] \) and \([\mathfrak{r}] \) involved in this definition commutes with \( U(1) \), \( f_\nu [\mathfrak{q}] \) is still an eigenform of \( U(1) \) with the same eigenvalue as \( f \). Thus in particular, we have a measure \( \varphi_{f_\nu} \). We project it to \( \Gamma \) along \( \nu \) which produces a measure \( \varphi_f^\nu \) on \( \Gamma \) explicitly given by

\[
\int_G \phi d\varphi_f^\nu = \sum_{\mathfrak{Q} \in \mathcal{Q}} \hat{\lambda}_{\nu^{-1}} (\mathfrak{Q}) \int_G \phi |\mathfrak{Q}| d\varphi_{f_\nu}[\mathfrak{q}],
\]

where \( \phi |\mathfrak{Q}| (y) = \phi (y |\mathfrak{Q}|^{-1}) \) for the projection \([\mathfrak{Q}]_\Gamma \) in \( \Gamma \) of \([\mathfrak{Q}] \in Cl_\infty \).
Lemma 3.17. If $\chi : Cl_\infty \rightarrow F^\times$ is a character with $\chi|_\Delta = \nu$, we have

$$\int_{\Gamma} \chi d\varphi_f = \int_{Cl_\infty} \chi d\varphi_f.$$

Proof. Write $\Gamma_{f,n}$ for the image of $\Gamma$ in $Cl_n$. For proper $O_n$–ideal $a$, by the above definition of these operators,

$$f|[[r]]|[[q]]([[a]]) = \hat{\lambda}(a)^{-1}f(x(\mathcal{O}^{-1}\mathcal{R}^{-1}a), \omega(\mathcal{O}^{-1}\mathcal{R}^{-1}a)).$$

For sufficiently large $n$, $\chi$ factors through $Cl_n$. Since $\chi = \nu$ on $\Delta$, we have

$$\int_{\Gamma} \chi d\varphi_f \nu_f = \sum_{Q \in Q} \sum_{r \in R} \sum_{a \in \Gamma_{f,n}} \hat{\lambda}(a)^{-1} f([[r]]|[[q]]([[a]])\)

$$= \sum_{a, Q, r} \chi(\mathcal{O}^{-1}\mathcal{R}^{-1}a) f([[r]]|[[q]]([[a]]) = \int_{Cl_\infty} \chi d\varphi_f,$$

because $Cl_\infty = \bigsqcup_{Q, R} [\mathcal{O}^{-1}\mathcal{R}^{-1}] \Gamma$. \qed

In the next couple of sections, we prove the following result (given in [H04] as Theorems 3.2 and 3.3):

**Theorem 3.18.** Fix a character $\nu : \Delta \rightarrow F^\times$, and define $f_\nu$ as in (3.18). If $f$ satisfies the following condition:

(H) for any given integer $r > 0$ and any congruence class $u \in (\mathbb{Z}/l^r\mathbb{Z})^\times$, there exists $0 \leq \xi \in u$ such that $a(\xi, f_\nu) \neq 0$,

then non-vanishing $\int_{Cl_\infty} \nu \chi d\varphi_f \neq 0$ holds except for finitely many characters $\chi : \Gamma \rightarrow \mu_\infty(F)$.

3.9. Preliminary to the proof of Theorem 3.18. We regard $f$ as a function of $Cl(\infty) = \bigsqcup_n Cl_n$ (embedded into $Sh^{(p)}$ over $X_0(l)$ by $a \mapsto x(a)$). By (3.16), we have, for an integer $n > m$,

$$\sum_{[a] \in Cl_n, \ a \mapsto [a] \in Cl_m} f([a]) = (\hat{\lambda}(l)l)^{n-m} f(U(l^{n-m})([a]),$$

where $[a]$ runs over all classes in $Cl_n$ which project down to $[a] \in Cl_m$.

We suppose that $f|U(l) = (a/\hat{\lambda}(l)l) f$ with a unit $a \in A$. For each function $\phi : Cl_\infty \rightarrow A$ factoring through $Cl_m$, we define

$$\int_{Cl_\infty} \phi d\varphi_f = a^{-m} \sum_{a \in Cl_m} \phi(a^{-1}) f([a]).$$

Classical modular forms are actually defined over a number field; so, we assume that $f$ is defined over the localization $\mathcal{V}$ of the integer ring in a number field $K$ containing $M$ over which $E(a)$ for each class $[a] \in Cl_0$ is defined. We write $\mathcal{P}|p$ for the prime ideal of the $p$–integral
where $\phi$ is a character of $\text{Cl}_\infty$, for $\sigma \in \text{Gal}(\mathbb{F}/\mathbb{F}_{p^r})$ $(\mathbb{F}_{p^r} = \mathcal{V}/\mathcal{P} \cap \mathcal{V})$,

$$
\int_{\text{Cl}_\infty} \chi(x) d\varphi_f(x) = 0 \iff \int_{\text{Cl}_\infty} \sigma \circ \chi(x) d\varphi_f(x) = 0.
$$

Decompose $\text{Cl}_\infty$ into a product of the maximal torsion-free $l$–profinite subgroup $\Gamma$ and a finite group $\Delta$.

Let $\mathbb{F}_q$ be the finite subfield of $\mathbb{F}$ generated by all $l|\Delta|$–th roots of unity over the field $\mathbb{F}_{p^r}$ of rationality of $f$ and $\lambda$. For any finite extension $\mathbb{F}'/\mathbb{F}_q$, we consider the trace map: $\text{Tr}_{\mathbb{F}'/\mathbb{F}_q}(\xi) = \sum_{\sigma \in \text{Gal}(\mathbb{F}'/\mathbb{F}_q)} \sigma(\xi)$ for $\xi \in \mathbb{F}'$. If $\chi : \text{Cl}_n \to \mathbb{F}^\times$ is a character, $d := [\text{Im}(\chi) : \text{Im}(\chi) \cap \mathbb{F}_q^\times]$ is not divisible by $p$ (as $|\mathbb{F}_q^m| = p^m - 1 \equiv 0 \mod p$). Thus $d \in \mathbb{F}^\times$, and

$$
\int_{\text{Cl}_\infty} \text{Tr}_{\mathbb{F}_q(\chi)/\mathbb{F}_q} \circ \chi(y^{-1}x) d\varphi_f(x) = \frac{d}{a^\mu} \sum_{a \in \text{Cl}_\infty : \chi(a^{-1}y) \in \mathbb{F}_q} \chi(y^{-1}a)f([a]),
$$

because, by Exercise 1.5, for an $l$–power root of unity $\zeta \in \mu_{\mu_l} - \mu_l$,

$$
\text{Tr}_{\mathbb{F}_q(\mu_{\mu_l})/\mathbb{F}_q}(\zeta^s) = \begin{cases} 
\mu_{\mu_l}^{-m} \zeta^s & \text{if } \zeta^s \in \mathbb{F}_q \text{ and } \mathbb{F}_q \cap \mu_{\mu_l}(\mathbb{F}) = \mu_{\mu_l}(\mathbb{F}) \\
0 & \text{otherwise.}
\end{cases}
$$
Suppose \( \int_{C_{\infty}} \chi(x) d\varphi_f(x) = 0 \) for an infinite set \( \mathcal{X} \) of characters \( \chi \). For sufficiently large \( m \), we always find a character \( \chi \in \mathcal{X} \) such that \( \text{Ker}(\chi) \subset \Gamma^{m_m} \). Then writing \( \text{Ker}(\chi) = \Gamma^{m_n} \) for \( n \geq m \), we have the vanishing from (3.22)

\[
\int_{C_{\infty}} \sigma \circ \chi d\varphi_f = 0 \quad \text{for all } \sigma \in \text{Gal}(\mathbb{F}/\mathbb{F}_q).
\]

This combined with (3.23), we find \( \sum_{y \in \chi^{-1}(\mathbb{F}_q^x)} \chi(ya)f(y[a]) = 0 \) for all \( a \in \Gamma_n \), where \( \Gamma_n \) is the image of \( \Gamma \) in \( C_{\infty} \).

### 3.10. Proof of Theorem 3.18

We write \( \mathbb{F}_p[f] \) for the minimal field of definition of \( f \in V(\mathbb{F}) \) (i.e., the field generated by \( a(\xi, f) \in \mathbb{F} \) for all \( 0 \leq \xi \in \mathbb{Z} \)). Similarly \( \mathbb{F}_p[\lambda] \) (resp. \( \mathbb{F}_p[\nu] \)) is the subfield of \( \mathbb{F} \) generated by the values \( \tilde{\lambda}([a]) \mod \mathcal{P} \) (resp. \( \nu([a]) \)) for all \( [a] \in C^{\text{alg}} \). Define \( \mathbb{F}_p[f, \lambda, \nu] \) by the composite of these fields and \( \mathbb{F}_p[\mu] \). Note that \( \mathbb{F}_p[f, \lambda, \nu] \) is a finite extension of \( \mathbb{F}_p \) as \( f \) is mod \( p \) reduction of some classical modular form of some weight \( \geq 2 \). Define \( 1 \leq r = r(\nu) \in \mathbb{Z} \) by \( |\mu|_{\infty}(\mathbb{F}_p[f, \lambda, \nu]) = l^r \).

By definition, the projection \( \{ [\mathbb{Q}]_\Gamma \}_{\mathbb{Q} \in \mathbb{Q}} \) of \( [\mathbb{Q}] \) in \( \Gamma \) are all distinct in \( C_{\infty}/C^{\text{alg}} \). By Lemma 3.17, we need to prove that the integral \( \int_{\Gamma} \chi d\varphi_f^{\nu} \) vanishes only for finitely many characters \( \chi \in \Gamma \). Suppose by absurdity that the integral vanishes for characters \( \chi \in \text{an infinite set } \mathcal{X} \).

Let \( \Gamma(n) = \Gamma^{l_{n-r}}/\Gamma^{l_n} \) for \( r = r(\nu) \). By applying (3.23) to a character in \( \mathcal{X} \) with \( \text{Ker}(\chi) = \Gamma^{l_n} \), we find

\[
(3.24) \sum_{\Omega \in \mathbb{Q}} \nu(\Omega)^{-1} \sum_{a \in y\chi^{-1}(\mu_{l^r})} \chi(a)f(\mu_{l^r}(\Omega)^{-1}(\mathbb{Q})_{\Gamma}) = 0.
\]

Fix \( \Omega \in \mathbb{Q} \). By Lemma 3.14, \( \{ x(a)||a| \in y\chi^{-1}(\mu_{l^r}) \} \) is given by \( \alpha(\frac{a}{l^r})(x(a_0)) \) for any member \( a_0 \in y\chi^{-1}(\mu_{l^r}) \), where

\[
(3.25) \alpha(t) = (\frac{l}{l^r}).
\]

Actually \( a \mapsto u \mod l^r \) gives a bijection of \( y\chi^{-1}(\mu_{l^r}) \) onto \( O/l^r \). We write the element \( a \) corresponding to \( u \) as \( \alpha(\frac{a}{l^r})a_0 \). This shows, choosing a primitive \( l^r \)-th root of unity \( \zeta = \exp(2\pi i/l^r) \) and \( a_y \in y\chi^{-1}(\mu_{l^r}) \), so that \( \chi(\alpha(\frac{a}{l^r})a_y) = \zeta^{uv} \) for an integer \( 0 < v < l^r \) prime to \( l \) (independent of \( y \), the inner sum of (3.24) is equal to

\[
\sum_{u \mod l^r} \zeta^{uv}(f_{\nu}(\alpha(\frac{u}{l^r}))([a_y\Omega^{-1}](\mathbb{Q})_{\Gamma})).
\]

The choice of \( v \) depends on \( \chi \). Since \( \mathcal{X} \) is infinite, we can choose an infinite subset \( \mathcal{X}' \) of \( \mathcal{X} \) for which \( v \) is independent of the element in \( \mathcal{X}' \). Then write \( n_j \) for the integers given by \( \Gamma^{n_{j}} = \text{Ker}(\chi) \) for \( \chi \in \mathcal{X}' \) (in
increasing order), and define $\Xi$ to be the set of points $x(\mathfrak{a})$ for $\mathfrak{a} \in Cl_{n_j}$ with $[\mathfrak{a} \mathcal{O}_{n_1}] = [\mathcal{O}_{n_1}]$ in $Cl_{n_1}$. Define also $h_\mathcal{O} = \sum_{u \bmod \mathfrak{p}} \zeta^{uv} f_\nu |(\mathfrak{f})_r|$, because (3.24) is now the sum:

$$\sum_{\mathcal{O} \in \mathcal{O}} \nu(\mathcal{O})^{-1} h_\mathcal{O}|[\mathfrak{q}](|\mathfrak{a}||\mathcal{O}|_r) = 0,$$

where $\mathfrak{q} = \mathcal{O} \cap F$. If necessary, as we remarked already, we reselect the representative set $\mathcal{O}$ so that $[\mathcal{O}]_r \in \text{Ker}(Cl_{\infty} \to Cl_{n_1})$. This is possible because $\{|\mathcal{O} |_r \} / \mathfrak{q}$ for all split primes is dense by Chebotarev-density, where $\mathcal{O} \sim \mathfrak{A}$ means the class of $\mathcal{O}$ is equal to the class of $\mathfrak{A}$ in $Cl_{\infty}/\mathfrak{A}$. Take $\mathcal{O}, \mathcal{O}'$ in $\mathcal{O}$. Then by Exercise 3.16 (4), $[\mathcal{O}]_r/[\mathcal{O}']_r \in Cl_{\mathfrak{A}} \iff \mathcal{O} = \mathcal{O}'$. Thus we may apply Corollary 3.21 in the following section to the following set of functions: $\{|\mathfrak{a} |_r \} \to h_\mathcal{O}|[\mathfrak{q}](|\mathfrak{a}||\mathcal{O}|_r))$. By the corollary, if $h_\mathcal{O}|[\mathfrak{q}] \neq 0$ for one $\mathcal{O}$, the above sum is nonzero as a function of $|\mathfrak{a}|_r$; so, this implies that $h_\mathcal{O}|[\mathfrak{q}] = 0$. By $q$-expansion principle, we conclude $h_\mathcal{O} = 0$ (as $h||[\mathfrak{q}](q) = h(\overline{q})$ for the positive generator $\varpi$ of $\mathfrak{q}$).

However, since we have $f_\nu/(u_1 \overline{v}_1) = \sum_{0 \leq \xi \in \mathbb{Z}} a(\xi, f_\nu) \zeta^{\xi q^k}$ for $\zeta = \exp(\frac{2\pi i}{l^m})$, the $q$-expansion coefficient $a(\xi, h_\mathcal{O})$ of $h_\mathcal{O}$ is given by $a(\xi, f_\nu)$ if $\xi \equiv -v \mod l^m$ and vanishes otherwise. This is a contradiction against the assumption (H).

3.11. Linear independence. Fix a positive integer $n_1 > 0$. We create complete representative set $R_n$ for Ker($Cl_n \to Cl_{n_1}$) by $a(\frac{n}{l^m})(x(\mathcal{O}_{n_1}))$ (for $\alpha(t)$ as in (3.25)) by choosing suitable integers $0 < u < l^m$. Choose an infinite sequence $u := 0 < n_1 < n_2 < \cdots < n_m < \cdots$ of positive integers. Take a geometrically irreducible component $V/F \subset Sh(h)\mathbb{F}$ containing $x(\mathcal{O}_{n_1})$, where $Sh(h)\mathbb{F} = Sh(h)\mathbb{F} \times_W \mathbb{F}$. Since $V$ is affine, we can write $V = \text{Spec}(\mathcal{O}_V)$ for $\mathcal{O}_V = H^0(V, \mathcal{O}_V)$. Sometimes we just write $O = O_V$ if confusion is unlikely. Define

$$\Xi_{R_n} = \Xi = \bigcup_{j=1}^{\infty} \{x(\mathfrak{a}) \in V | \mathfrak{a} \in R_{n_j} \} \subset V.$$

Since $SL_2(\mathfrak{A}(\mathbb{A}) \setminus \mathcal{O}_{n_1})$ keeps $V$ (by Shimura’s global reciprocity), $x(\mathfrak{a})$ as above always resides in one component $V$.

Let $F = F_\Xi$ denote the $\mathbb{F}$-algebra of functions $\phi: \Xi \to \mathbb{P}^1(\mathbb{F}) = \mathbb{F} \cup \{\infty\}$ with $|\phi^{-1}(0)| < \infty$ and $|\phi^{-1}(\infty)| < \infty$. The profinite class group $C = C_{n_1} := \text{Ker}(Cl_{\infty} \to Cl_{n_1})$ acts on $F$ by translation: $f(x) \mapsto f(xy)$ for each $y \in C_{n_1}$. In particular, $\alpha \in \mathcal{O}_{(p)}$ with trivial $|(\alpha)| \in Cl_{n_1}$ acts on $\Xi$ and such $\alpha$ is $p$-adically dense in $\mathcal{O}_{(p)}^\times$. For $f \in F(V)^\times$, for each
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\[ x = x(a) \in \Xi, \text{ expanding } f \text{ into a Laurent series } f(t) = \sum_n a_n t^n \in \mathbb{F}[t][t^{-1}] \text{ with leading nonzero term } a_m t^m \ (m \in \mathbb{Z}), \text{ we may define} \]

\[ f(x(a)) = \begin{cases} 
\infty & \text{if } m < 0, \\
0 & \text{if } m = 0, \\
0 & \text{if } m > 0.
\end{cases} \]

By Zariski density of \( \Xi \) in \( V = V_{/F}^{(p)} \), we can embed into \( \mathcal{F} \) the function field \( \mathbb{F}(V) \) of \( V \).

Exercise 3.19. Why is \( \Xi \) Zariski dense in \( V ? \) Why does density imply injectivity of \( \mathbb{F}(V) \) into \( \mathcal{F} ? \)

We will prove the following analogue of Sinnott’s theorem later if time allows.

Proposition 3.20. Take a finite set \( \Delta = \{\gamma_1, \ldots, \gamma_m\} \subset C_{n_1} \) injecting into \( \text{Cl}_\infty/\text{Cl}^{\text{alg}} \). Then the subset \( \tilde{\Xi} := \{(x(\delta(a)))_{\delta \in \Delta} | x(a) \in \Xi\} \) is Zariski dense in the product \( V^{\Delta}_{/F} \) of \( \Delta \) copies of \( V_{/F} \). This implies that the fields \( \gamma_1(\mathbb{F}(V)), \ldots, \gamma_m(\mathbb{F}(V)) \) are linearly disjoint over \( \mathbb{F} \) in \( \mathcal{F}_{\Xi} \), where \( \gamma(\mathbb{F}(V)) \) is the image of \( \mathbb{F}(V) \subset \mathcal{F} \) under the action of \( \gamma \in C_{n_1} \). In other words, we have injectivity of the map

\[ \gamma_1 \otimes \cdots \otimes \gamma_m : O_V \otimes_{F} O_V \otimes_{F} \cdots \otimes_{F} O_V \to \mathcal{F} \text{ sending } f_1 \otimes \cdots \otimes f_m \text{ to an element in } \mathcal{F} \text{ given by } x(a) \mapsto \prod_j f_j(x(\gamma_j a)). \]

The linear independence applied to the global sections of a modular line bundle (regarded as sitting inside the function field) yields the following result:

Corollary 3.21. Let the notation and the assumption be as in Proposition 3.20. Let \( \omega^k \) be a modular line bundle over the Igusa tower \( I_{g/F} \) over \( V_{/F} \). Then for a finite set \( \Delta \subset \text{Cl}_\infty \) injecting into \( \text{Cl}_\infty/\text{Cl}^{\text{alg}} \) and a set \( \{s_\delta \in H^0(I, \omega^k)\}_{\delta \in \Delta} \) of non-constant global sections \( s_\delta \) of \( \omega^k \) finite at \( \Xi \), the functions \( s_\delta \circ \delta \) \( (\delta \in \Delta) \) are linearly independent in \( \mathcal{F}_{\Xi} \).

We can bring the situation in the case of modular functions which is taken care of by the above theorem.

3.12. \( \ell \)-Adic Eisenstein measure modulo \( p \). We apply Theorem 3.18 to the Eisenstein series \( E(\lambda) \) in (3.4) for the Hecke character \( \lambda \) fixed in 3.1. We can easily check (H) in Theorem 3.18 for \( f = E(\lambda) \mod m_W \) and get Theorem 3.12.