

VANISHING OF THE μ -INVARIANT OF p -ADIC HECKE L -FUNCTIONS

HARUZO HIDA

CONTENTS

1. Modular Curves	2
1.1. The tower of modular curves	2
1.2. Hecke invariant subvarieties	4
2. Geometric modular forms	8
2.1. Moduli Problem	8
2.2. Modular forms of $\Gamma_1(p^n)$	8
2.3. p -Adic modular forms	9
2.4. Hecke invariant subvarieties	9
3. p -adic Hecke L -functions and its μ -invariant	10
References	13

Date: December 2, 2008.

Two lectures at Kyoto December 1, 2008; the author is partially supported by the NSF grants: DMS 0244401, DMS 0456252 and DMS 0753991.

First Talk

This series of two lectures is an introductory discussion of problems concerning vanishing of the Iwasawa μ -invariant of p -adic L -functions. This type of results for Kubota-Leopoldt p -adic L has found good applications in divisibility problems of class numbers (see [W1], [FW] and [ICF] Chapter 7) and in proofs of the main conjectures in Iwasawa's theory. Recently, new methods of proving the vanishing emerged in the work of Vatsal, Finis and myself. In these two lectures, we describe a geometric method, which was started by Sinnott in [Si] and [Si1] and has been generalized in [H04], [H08] and [H07] via the theory of Shimura varieties. We rely on a general philosophical principle (proposed by Chai, Oort and others): "A Hecke invariant subvariety of a Shimura variety is a Shimura subvariety", and we can prove this for the Hilbert modular variety and its self-products. In this talk, we only deal with modular curves and their self-products for simplicity; however, any essential ingredients for the general facts shows up in this simpler case. Write $G = GL(2)_{/\mathbb{Z}}$ with center $Z \cong \mathbb{G}_{m/\mathbb{Z}}$.

1. MODULAR CURVES

We study subvariety of self product of modular curves stable under toric action.

1.1. The tower of modular curves. We study classification problem of elliptic curves $E_{/A}$ over a ring A (which is an algebra over a base ring B). In other words, we consider the following moduli functor of level $\Gamma(N)$,

$$\mathcal{E}_{\Gamma(N),\zeta}(A) = [(E, \phi_N : (\mathbb{Z}/N\mathbb{Z})^2 \cong E[N]) | \langle \phi_N(1, 0), \phi_N(0, 1) \rangle = \zeta],$$

for all $\mathbb{Z}[\frac{1}{N}, \zeta]$ -algebras A , where ζ is a generator of μ_N . Here $\langle \cdot, \cdot \rangle$ is the Weil pairing. We know classically $\mathcal{E}_{\Gamma(N),\zeta}(\mathbb{C}) \cong \Gamma(N) \backslash \mathfrak{H}$. If we remove the contribution upon ζ and consider the functor $\mathcal{E}_{\Gamma(N)}(A) = [(E, \phi_N)_{/A}]$ defined on the category of $\mathbb{Z}[\frac{1}{N}]$ -algebras, we have $\mathcal{E}_{\Gamma(N)} = \bigsqcup_{\zeta} \mathcal{E}_{\Gamma(N),\zeta}$, and this functor is represented by a geometrically non-connected scheme $\mathcal{M}_{\Gamma(N)}$ and $Y(N)$ defined over $\mathbb{Z}[\frac{1}{N}]$ if $N \geq 3$. Note that $Y(N)_{/\mathbb{Z}[\frac{1}{N}, \zeta_N]} = \bigsqcup_{\zeta \in \mu_N^\times} Y_\zeta(N)$ with $Y_\zeta(N)(\mathbb{C}) \cong \Gamma(N) \backslash \mathfrak{H}$.

We can let a constant group $\alpha \in SL_2(\mathbb{Z}/N\mathbb{Z})$ act on $Y_\zeta(N)$ (and hence on $X_\zeta(N)$) by $(E, \phi) \mapsto (E, \phi \circ \alpha)$. Since $\langle \phi \circ \alpha(1, 0), \phi \circ \alpha(0, 1) \rangle = \zeta_N^{\det(\alpha)}$, the same action of $\alpha \in G(\mathbb{Z}/N\mathbb{Z})$ induces an automorphism of $Y(N)_{/\mathbb{Z}[\frac{1}{N}]}$ (and $X(N)_{/\mathbb{Z}[\frac{1}{N}]}$), which is induced by the Galois action $\zeta_N \mapsto \zeta_N^{\det(\alpha)}$ on $\text{Spec}(\mathbb{Z}[\zeta_N])$. The group $G(\widehat{\mathbb{Z}}) = \varprojlim_N G(\mathbb{Z}/N\mathbb{Z})$ acts on the limit $Y = \varprojlim_N Y(N)$ (which is a pro-scheme defined over \mathbb{Q}), and $SL_2(\widehat{\mathbb{Z}})$ preserves the connected component $Y_{\zeta_\infty} = \varprojlim_N Y_{\zeta_N}(N)$.

A remarkable fact Shimura found is that this action of $G(\widehat{\mathbb{Z}})$ can be extended to the finite adèle group $G(\mathbb{A}^{(\infty)})$ (see [IAT] Chapter 6). An interpretation by Deligne of this fact is equally remarkable (see [PAF] 4.2.1): To explain Deligne's idea, we define the Tate module $T(E) = \varprojlim_N E[N]$ for an elliptic curve $E_{/A}$ for a \mathbb{Q} -algebra A . Strictly speaking, taking a geometric point $s = \text{Spec}(k) \in \text{Spec}(A)$ (for an algebraically closed field k) in each connected component of $\text{Spec}(A)$, we are thinking of the $\widehat{\mathbb{Z}}$ -module $T(E) = \varprojlim_N E[N](k)$ (as is well known, the choice of s does not matter; see [PAF] Chapter 4, Appendix). Then

$T(E) \cong \widehat{\mathbb{Z}}^2$ and $V(E) = T(E) \otimes_{\mathbb{Z}} \mathbb{Q} \cong (\mathbb{A}^{(\infty)})^2$. Deligne realized that Y represents the following functor defined over \mathbb{Q} - ALG :

$$\mathcal{E}^{(\infty)}(A) = \{(E, \eta : (\mathbb{A}^{(\infty)})^2 \cong V(E))_{/A}\} / \text{isogenies.}$$

Here $\mathbb{A}^{(\infty)}$ is the finite adèle ring. Then $g \in G(\mathbb{A})$ sends a point $(E, \eta)_{/A} \in \mathcal{E}^{(\infty)}(A)$ to $(E, \eta \circ g^{(\infty)})_{/A}$ for the projection $g^{(\infty)}$ of g to $\mathbb{A}^{(\infty)}$.

Take the prime-to- p quotient $Y^{(p)} = \varprojlim_{p \nmid N} Y(N) = Y/GL_2(\mathbb{Z}_p)$. Put $V^{(p)}(E) = T(E) \otimes_{\widehat{\mathbb{Z}}} \mathbb{A}^{(p\infty)}$, and consider the prime-to- p level structure $\eta^{(p)} : (\mathbb{A}^{(p\infty)})^2 \cong V^{(p)}(E)$. Then $Y^{(p)}$ represents the following functor defined over $\mathbb{Z}_{(p)}$ - ALG :

$$\mathcal{E}^{(p)}(A) = \{(E, \eta : (\mathbb{A}^{(p\infty)})^2 \cong V^{(p)}(E))_{/A}\} / \text{prime-to-}p \text{ isogenies,}$$

where an isogeny ϕ is prime to p if the order of the kernel of ϕ is prime to p . This pro-scheme $Y^{(p)}$ is defined over $\mathbb{Z}_{(p)}$; so, its fiber $Y_{/\overline{\mathbb{F}}_p}^{(p)}$ over $\text{Spec}(\overline{\mathbb{F}}_p)$ is the $\overline{\mathbb{F}}_p$ -scheme classifying $(E, \eta^{(p)})_{/A}$ for $\overline{\mathbb{F}}_p$ -algebras A . On $Y^{(p)}$, again $G(\mathbb{A}^{(p)})$ acts by $\eta^{(p)} \mapsto \eta^{(p)} \circ g$.

If we have a prime-to- p **non-central** endomorphism $\alpha : E \rightarrow E$, then E has complex multiplication by $M = \mathbb{Q}[\alpha]$, and we can write $\alpha \circ \eta = \eta \circ \rho(\alpha)$ for $\rho(\alpha) \in G(\mathbb{A}^{(\infty)})$. Thus if $x = (E, \eta) \in Y(A)$, we find that $\rho(\alpha)(x) = x$. For any elliptic curve E , we have $\mathbb{Q} \subset \text{End}(E) \otimes_{\mathbb{Z}} \mathbb{Q}$; so, the central element $\xi \in \mathbb{Q}^\times \subset G(\mathbb{A}^{(\infty)})$ acts trivially on Y . Thus, $x = (E, \eta)$ has CM by an imaginary quadratic field M if and only if $M^\times / \mathbb{Q}^\times$ is the stabilizer of x in $\text{Aut}(Y)$.

Pick an elliptic curve E with complex multiplication by the integer ring R of M . We suppose that p splits into $(p) = \mathfrak{p}\bar{\mathfrak{p}}$ in R . Consider $W = W(\overline{\mathbb{F}}_p)$ inside $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}_p}$. We put $\mathcal{W} = W \cap \overline{\mathbb{Q}}$ which is a strict henselization of $\mathbb{Z}_{(p)}$. We suppose that $\mathfrak{p} = R \cap \mathfrak{m}_{\mathcal{W}}$ and $E[\mathfrak{p}^\infty]$ is étale over \mathcal{W} . Assume $E(\mathbb{C}) = \mathbb{C}/R$. Since the isomorphism class of E over \mathbb{C} is determined by the principal ideal class of R , we write this elliptic curve as $X = X(R)$. Then we can pick a level p -structure $\eta_p^\circ : \mu_{p^\infty} \cong X[\mathfrak{p}^\infty]$ and $\eta_p^{et} : \mathbb{Q}_p/\mathbb{Z}_p \cong X[\bar{\mathfrak{p}}^\infty]$. Write $\eta_p = (\eta_p^\circ, \eta_p^{et}) : \mu_{p^\infty} \times \mathbb{Q}_p/\mathbb{Z}_p \cong X[\mathfrak{p}^\infty] \times X[\bar{\mathfrak{p}}^\infty]$, and define a homomorphism ρ_p of $R_{(p)}^\times$ into the diagonal torus of $G(\mathbb{Z}_p)$ by $\alpha\eta_p = \eta_p \circ \rho_p(\alpha)$ for $\alpha \in R_{(p)}$. Thus $\eta_p^\circ \circ \rho_p(\alpha) = \alpha\eta_p$ identifying $R_{\mathfrak{p}}$ with \mathbb{Z}_p and $\eta_p^{et} \circ \rho_p(\alpha) = \alpha^c \eta_p^{et}$, writing $c : R_{\mathfrak{p}} \cong R_{\bar{\mathfrak{p}}}$.

Fix a base w_1, w_2 of $\widehat{R}^{(p)}$ over $\widehat{\mathbb{Z}}^{(p)}$, and identify $M_{\mathbb{A}}^{(p\infty)}$ with $(\mathbb{A}^{(p\infty)})^2$. The choice induces prime-to- p level structure $\eta^{(p)} : (\mathbb{A}^{(p\infty)})^2 \cong R \otimes_{\mathbb{Z}} \mathbb{A}^{(p\infty)} = V^{(p)}(X)$. We put $\eta = \eta_p \times \eta^{(p)}$. Define $\rho : R_{(p)}^\times \rightarrow G(\mathbb{A}^{(p\infty)})$ by $\eta \circ \rho(\alpha) = \alpha \circ \eta$. Thus $\rho = \rho_p \times \rho^{(p)} : R_{(p)}^\times \rightarrow G(\mathbb{Z}_p \times \mathbb{A}^{(p\infty)})$. Since $\alpha \in R_{(p)}^\times$ induces an isogeny $\alpha : X \rightarrow X$ sending $\alpha\eta^{(p)} = \eta^{(p)}\rho^{(p)}(\alpha)$, the point $x(R) = (X(R), \eta^{(p)})$ is fixed by $\rho(\alpha)$.

Pick a fractional ideal $\mathfrak{a} \supset R$ prime to p . For any $a \in (M_{\mathbb{A}}^{(p\infty)})^\times$ with $a\widehat{R} \cap M = \mathfrak{a}$ (so, $a^{-1} \in \widehat{R}$), we define $x(\mathfrak{a}) = \rho^{(p)}(a)^{-1}(x(R)) = (X(R), \eta \circ \rho(a^{-1}))$, which gives another point on $Y^{(p)}$. By tensoring $X(R)$ with the exact sequence $0 \rightarrow R \rightarrow \mathfrak{a} \rightarrow \mathfrak{a}/R \rightarrow 0$, we get another exact sequence

$$0 \rightarrow X(\mathfrak{a})[\mathfrak{a}^{-1}]_{/\mathcal{W}} \rightarrow X(R)_{/\mathcal{W}} \xrightarrow[\text{étale}]{\pi} X(R)_{/\mathcal{W}} \otimes_R \mathfrak{a} = X(\mathfrak{a})_{/\mathcal{W}} \rightarrow 0,$$

where $X(\mathbf{a})(\mathbb{C}) = \mathbb{C}/\mathbf{a}$. Then, we have the following commutative diagram:

$$\begin{array}{ccc} \widehat{R}^{(p\infty)} & \xrightarrow[\eta^{(p)}]{\sim} & T^{(p)}(X(R)) \\ \parallel \downarrow & & \downarrow \pi \\ \widehat{R}^{(p\infty)} & \xrightarrow[\eta^{(p)} \circ \rho^{(p)}(\mathbf{a})]{\sim} & T^{(p)}(X(\mathbf{a})). \end{array}$$

Thus writing $x(\mathbf{a}) = (X(\mathbf{a}), \eta^{(p)})$, we have $\rho^{(p)}(a)(x(\mathbf{a})) = x(R)$. From this fact, if $a \notin R_{(p)}^\times$, the point $x(\mathbf{a})$ is different from $x(R)$.

Consider the formal completion $\widehat{Y} = \widehat{Y}_{x/W}$ of $Y_{/W}^{(p)}$ along $x = x(\mathbf{a}) \in Y^{(p)}(\overline{\mathbb{F}}_p)$. Then by the universality of Y , \widehat{Y} satisfies

$$\widehat{Y}(A) \cong \widehat{\mathcal{E}}(A),$$

where A runs through p -profinite local W -algebras with $A/\mathfrak{m}_A = W/\mathfrak{m}_W = \overline{\mathbb{F}}_p$ and $\widehat{\mathcal{E}}(A) = [E_{/A} | E \otimes_A \overline{\mathbb{F}}_p = X(\mathbf{a})_{/\overline{\mathbb{F}}_p}]$. Indeed \widehat{Y} classifies $(E, \eta^{(p)})_{/A}$ with $(E, \eta_E^{(p)}) \times_A \overline{\mathbb{F}}_p = (X(\mathbf{a}), \eta^{(p)})$, but $\eta^{(p)}$ uniquely determines $\eta_E^{(p)}$; so, we can drop the datum of the level structure. By the deformation theory of Serre–Tate, $\widehat{Y} \cong \widehat{\mathbb{G}}_m$ canonically. Indeed, first $E_{/A} \in \widehat{\mathcal{E}}(A)$ is determined by the extension $E[p^\infty]^\circ \hookrightarrow E[p^\infty] \twoheadrightarrow E[p^\infty]^{et}$ of the Barsotti–Tate groups. By Serre–Tate, such an extension over A is classified by

$$\mathrm{Hom}(E[p^\infty]^{et}, E[p^\infty]^\circ) \cong \mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, \mu_{p^\infty/A}) = \varprojlim_n \mu_{p^n}(A) = \widehat{\mathbb{G}}_m(A).$$

For this identification, we need to fix $\eta_p^\circ : \mu_{p^\infty} \cong X(\mathbf{a})[\mathfrak{p}^\infty]$ and its dual inverse $\eta_p^{et} : \mathbb{Q}_p/\mathbb{Z}_p \cong X(\mathbf{a})[\overline{\mathfrak{p}}^\infty]$. Since $a_p = 1$, the above identification is independent of a and \mathbf{a} . Since $\rho(\alpha)$ fix $x(\mathbf{a})$, it acts on \widehat{Y} . We admit the following lemma.

Lemma 1.1. *Identifying \widehat{Y} with $\widehat{\mathbb{G}}_m = \mathrm{Spf}(\varprojlim_n W[t, t^{-1}]/(t-1)^n)$, if $\alpha \in M^\times$, we have $\rho(\alpha)(t) = t^{\alpha^{1-c}}$ for complex conjugation c .*

Note that if $\alpha \in M^\times$ is not prime to p , the action of $\rho(\alpha)$ is an endomorphism of $Y^{(p)}$ not an automorphism. A proof of this can be found in [H08] as Proposition 3.4.

We regard $\rho(M^\times) \subset G(\mathbb{A}^{(p\infty)})$ as the group of \mathbb{Q} -rational points of a torus T_x . Then the action of $\rho(\alpha)$ on the Serre–Tate coordinate is given by $t \mapsto t^{\alpha^{1-c}}$ factoring through $G(\mathbb{A}^{(p\infty)})/Z(\mathbb{Q})$, since $Z(\mathbb{Q})$ acts trivially on the Shimura variety $Y^{(p)}$.

1.2. Hecke invariant subvarieties. Write $\mathbb{F} = \overline{\mathbb{F}}_p = W/\mathfrak{m}_W = \mathcal{W}/\mathfrak{m}_{\mathcal{W}}$. Let \mathbf{a} be a fractional ideal of M with $\mathbf{a}_p = R_p$. We write $I_{\mathbf{a}}$ for the irreducible component of $Y_{/\mathbb{F}}^{(p)} = Y^{(p)} \times_{\mathcal{W}} \mathbb{F}$ containing $x(\mathbf{a})$. Note the following fact ([H08] Proposition 3.8), which is obvious since $\dim Y_{/\mathbb{F}}^{(p)} = 1$, but not trivial in the Hilbert modular case:

Fact 1.1. *If $H \subset Y_{/\mathbb{F}}^{(p)}$ is an irreducible closed subscheme containing $x(\mathbf{a})$ stable under the action of a p -adic open subgroup of $T_x(\mathbb{Z}_{(p)})$, Then either $H = \{x(\mathbf{a})\}$ or $H = I_{\mathbf{a}}$.*

Let \mathfrak{a} and \mathfrak{b} be fractional ideals of M with $\mathfrak{a}_p = \mathfrak{b}_p = R_p$ with $\mathfrak{a}\mathfrak{b}^{-1} = zR$ for $z \in (M_{\mathbb{A}}^{(p\infty)})^\times$. Then $\rho(z\beta)$ gives a morphism of $I_{\mathfrak{a}}$ onto $I_{\mathfrak{b}}$ for any $\beta \in R_{(p)}^\times$, and we have a skew diagonal $\Delta_{\alpha,\beta} = \text{Im}(\rho(\alpha) \times \rho(z\beta)) \subset I_{\mathfrak{a}} \times_{\mathbb{F}} I_{\mathfrak{b}}$ for $\alpha, \beta \in R_{(p)}^\times$. We want to give a sketch of a proof of the following two theorems ([H08] Corollaries 3.16 and 3.19):

Theorem 1.2. *Suppose that $H \subset I_{\mathfrak{a}} \times_{\mathbb{F}} I_{\mathfrak{b}}$ is an irreducible closed subscheme containing $(x(\mathfrak{a}), x(\mathfrak{b})) \in I_{\mathfrak{a}/\mathbb{F}} \times_{\mathbb{F}} I_{\mathfrak{b}}$ stable under the action of a p -adic open subgroup of $T_x(\mathbb{Z}_{(p)})$. Then either $H = x(\mathfrak{a}) \times I_{\mathfrak{b}}$ or $H = I_{\mathfrak{a}} \times x(\mathfrak{b})$ or $H = \Delta_{\alpha,\beta}$ for $\alpha, \beta \in R_{(p)}^\times$.*

Theorem 1.3. *Let $H \subsetneq I_{\mathfrak{a}_1} \times_{\mathbb{F}} \cdots \times_{\mathbb{F}} I_{\mathfrak{a}_n}$ (with $n \geq 2$) be a proper closed irreducible subscheme with a dominant projection to the product of the first $n-1$ factor and to the last factor. If H is stable under the diagonal action of a p -adic open subgroup of $T_x(\mathbb{Z}_{(p)})$, up to permutations of the first $(n-1)$ factors, we have $H = I_{\mathfrak{a}_1} \times \cdots \times I_{\mathfrak{a}_{n-2}} \times \Delta_{\alpha,\beta}$, where $\Delta_{\alpha,\beta}$ is defined with respect to $(\mathfrak{a}, \mathfrak{b}) = (\mathfrak{a}_{n-1}, \mathfrak{a}_n)$.*

We start with general lemmas. We suppose that the two projections to $I_{\mathfrak{a}}$ and $I_{\mathfrak{b}}$ are dominant. Take $a_i \in M_{\mathbb{A}}^\times$ with $a_i R = \mathfrak{a}_i$ and $a_{i,p} = a_{i,\infty} = 1$. Replacing H by $(\rho(a_1) \times \cdots \times \rho(a_n))(H)$, we may assume that $\mathfrak{a}_1 = \mathfrak{a}_2 = \cdots = \mathfrak{a}_n = R$; so, we write $x = x(R)$, $I = I_R$ and $x^n = (x, x, \dots, x) \in I^n$. Let $T \subset T_x(R_{(p)})$ be the open subgroup (under p -adic topology) fixing H by the diagonal action of $\rho(\alpha) \times \cdots \times \rho(\alpha)$ ($\alpha \in T$). Then the formal completion \widehat{H} along x^n is also stable under T , since x^n is fixed by T . By the Serre–Tate theory, $\widehat{H} \subset \widehat{\mathbb{G}}_{m/\mathbb{F}}^n$. Let $X_*(\widehat{\mathbb{G}}_{m/\mathbb{F}}^n) \cong \mathbb{Z}_p^n$ be the formal cocharacter group. Then, by a rigidity result of Chai (e.g., [H08] Lemma 3.7),

$$(1.1) \quad \widehat{H} = \bigcup_L \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} L \subset \widehat{\mathbb{G}}_m^n = \widehat{Y}_x^n,$$

where L runs over (finitely many) \mathbb{Z}_p -direct summand of $X_*(\widehat{\mathbb{G}}_m^n)$ of rank $n-1$. Let $\mathcal{H} \rightarrow H$ be the normalization of H . Since H is an excellent irreducible scheme, \widehat{H} is reduced equidimensional; so, all L as above has rank $n-1$. Since H is irreducible, \mathcal{H} is irreducible. By (1.1), each point over x^n of \mathcal{H} is indexed by $\{L\}$, and for the point $y_L \in \mathcal{H}$ over x^n corresponding to L , $\widehat{\mathcal{H}}_{y_L}$ is étale over $\widehat{\mathbb{G}}_m \otimes L$. Write $I^m = I' \times I''$ for $I' = I^{m-1}$ and $I'' = I$ for the last component.

Lemma 1.4. *The scheme \mathcal{H} is finite flat over I' around x^{n-1} . In particular, each L is of rank $n-1$ and projects down to an open subgroup of $X_*(\widehat{\mathbb{G}}_m^{n-1}) \cong \mathbb{Z}_p^{n-1}$. If one of L surjects down onto $X_*(\widehat{I}')$ ($\widehat{I}' = \widehat{\mathbb{G}}_m^{n-1}$), all of L surjects down onto $X_*(\widehat{I}')$, and the projection $\mathcal{H} \rightarrow I'$ is étale finite around x^{n-1} .*

Proof. Since the projection of the first $(n-1)$ -factor $I' = I^{m-1}$ is dominant, at least one of L , call it L_0 , projects down to an open \mathbb{Z}_p -submodule of $X_*(\widehat{\mathbb{G}}_m^{n-1})$. If there is L with image in $X_*(\widehat{I}')$ of rank $< n-1$, the non-flat locus $\mathcal{H}^{nf} \subset \mathcal{H}$ of $\mathcal{H} \rightarrow I'$ is a nonempty proper closed subscheme of \mathcal{H} . Since $\dim \widehat{\mathbb{G}}_m \otimes L = \text{rank } L = n-1$, \mathcal{H}^{nf} has dimension $n-1$ equal $\dim \mathcal{H}$; so, \mathcal{H} has to be reducible, a contradiction. Thus $\mathcal{H} \rightarrow I'$ is finite flat around x^{m-1} via a faithfully flat descent from $\widehat{\mathcal{H}}/\widehat{I}'$ to \mathcal{H}/I' .

If one of L , call it L_0 , surjects down to $X_*(\widehat{I}')$ and another L_1 has image smaller than $X_*(\widehat{I}')$, the ramified locus \mathcal{H}^{ram} of $\mathcal{H} \rightarrow I'$ is nontrivial proper closed subscheme of

dimension $n - 1$, again a contradiction to the irreducibility of \mathcal{H} ; so, $\mathcal{H} \rightarrow I'$ is étale finite around x^{m-1} , again via a faithfully flat descent from $\widehat{\mathcal{H}}/\widehat{I}'$ to \mathcal{H}/I' . \square

When $n = 2$, by applying a power of the p -power Frobenius or its dual to H (that is, applying $\rho(\alpha)$ for α generating $\mathfrak{p}R_{(p)}$ or its dual $\rho(\bar{\alpha})$), we may assume that at least one L surjects down to $X_*(\widehat{I}')$; so, by the above lemma, all L surjects down to $X_*(\widehat{I}')$. Thus we may assume that $\mathcal{H} \rightarrow I'$ is étale finite around x^{m-1} . Now assume $n = 2$. Then, over an open dense subscheme $U \subset \mathcal{H}$ containing all points above x^2 , the two projections $\pi_L : U \rightarrow I' = I$ and $\pi_R : U \rightarrow I'' = I$ are étale finite.

We consider the universal elliptic curve $\mathbf{E}_{/I}$. We pull it back to \mathcal{H} : $\mathbf{A} = \pi_L^* \mathbf{E}$ and $\mathbf{B} = \pi_R^* \mathbf{E}$. Pick a point $y \in \mathcal{H}$ over x^2 . Then $\widehat{\mathcal{H}} := \widehat{\mathcal{H}}_y = \{(t^b, t^a) | t \in \widehat{I} = \widehat{\mathbb{G}}_m\} \subset \widehat{I} \times \widehat{I}$. Since $\pi_j : \mathcal{H} \rightarrow I$ is étale finite around y , we may assume that $a, b \in R_{\mathfrak{p}}^\times = \mathbb{Z}_p^\times$; so, we may assume that $b = 1$. Identifying $X = X(R)_{/\mathbb{F}}$ with the fibers $\mathbf{A}_y = \mathbf{B}_y$ of \mathbf{A} and \mathbf{B} at y , we regard the unit $a \in \text{End}(X[p^\infty])$ as a homomorphism $a : \mathbf{A}_y[p^\infty] = X[p^\infty] \rightarrow X[p^\infty] = \mathbf{B}_y[p^\infty]$. We note the following fact (see [H08] Proposition 3.15):

Lemma 1.5 (C.-L. Chai). *Further shrinking the open neighborhood U of y in \mathcal{H} , we may assume that the isomorphism $a : \mathbf{A}_y[p^\infty] = X[p^\infty] \rightarrow X[p^\infty] = \mathbf{B}_y[p^\infty]$ extends to $\tilde{a} : \mathbf{A}_{/U}[p^\infty] \rightarrow \mathbf{B}_{/U}[p^\infty]$. This implies that $\widehat{\mathcal{H}}_u \cong \widehat{\mathbb{G}}_m$ by $(t, t^a) \leftrightarrow t$ at any point $u \in U(\mathbb{F})$.*

Here is a sketch of a proof of the above lemma. Since a can be approximated p -adically by $\alpha_n \in R_{(p)}$ modulo p^n , a can be extended to $\rho(\alpha_n) : \mathbf{A}[p^n]_{\widehat{\mathcal{H}}} \rightarrow \mathbf{B}[p^n]_{\widehat{\mathcal{H}}}$. Passing to a limit, we have an extension $\widehat{a} : \mathbf{A}[p^\infty]_{\widehat{\mathcal{H}}} \rightarrow \mathbf{B}[p^\infty]_{\widehat{\mathcal{H}}}$. Write $\widehat{\mathcal{O}}$ for the stalk of $\mathcal{O}_{\mathcal{H}}$ at y ; so, $\widehat{\mathcal{H}} = \text{Spf}(\widehat{\mathcal{O}})$. Since \widehat{a} is determined by its restriction a to $\mathbf{A}_y[p^\infty]$, it is a unique extension of a . Since $\widehat{\mathcal{O}} \otimes_{\widehat{\mathcal{O}}} \widehat{\mathcal{O}}$ is reduced (because of excellency of \mathcal{O}), the pull back of \widehat{a} by two projections $\widehat{\mathcal{H}} \times_{\widehat{\mathcal{H}}} \widehat{\mathcal{H}} \rightarrow \widehat{\mathcal{H}}$ coincides; so, \widehat{a} satisfies the descent datum with respect to $\widehat{\mathcal{O}}/\widehat{\mathcal{O}}$, getting desired U . \square

Proof of Theorem 1.2. Since the two projections $\pi_j : \mathcal{H} \rightarrow I$ ($j = L, R$) are dominant, we have $\text{End}(\mathbf{A}) \otimes \mathbb{Q} = \text{End}(\mathbf{B}) \otimes \mathbb{Q} = \mathbb{Q}$. Let $\mathbf{Y}_{/\mathcal{H}} = \mathbf{A} \times_{\mathcal{H}} \mathbf{B}$. Thus there are only two possibilities of $\text{End}^{\mathbb{Q}}(\mathbf{Y}) = \text{End}(\mathbf{Y}_{/\mathcal{H}}) \otimes \mathbb{Q}$: Either $\text{End}^{\mathbb{Q}}(\mathbf{Y}) = \mathbb{Q} \times \mathbb{Q}$ or $\text{End}^{\mathbb{Q}}(\mathbf{Y}) = M_2(\mathbb{Q})$. Suppose that $\text{End}^{\mathbb{Q}}(\mathbf{Y}) = M_2(\mathbb{Q})$. By semi-simplicity of the category of abelian schemes, we have two commuting idempotent $e_j \in \text{End}^{\mathbb{Q}}(\mathbf{Y})$ such that $e_A(\mathbf{Y}) = \mathbf{A}$ and $e_B(\mathbf{Y}) = \mathbf{B}$. Since $\text{End}^{\mathbb{Q}}(\mathbf{Y}) = M_2(\mathbb{Q})$, we can find an invertible element $\tilde{\beta}$ in $GL_2(\mathcal{O}_{(p)}) \subset M_2(\mathbb{Q})$ such that $\tilde{\beta} \circ e_A = e_B$; so, $\tilde{\beta} : \mathbf{A} \rightarrow \mathbf{B}$ is an isogeny, whose specialization to the fiber of \mathbf{A} and \mathbf{B} at y gives rise to an endomorphism $\beta \in \text{End}(X(R)) \otimes \mathbb{Q}$. Thus the isogeny $\tilde{\beta}$ is induced by $\rho(\beta)$, and we conclude $\Delta_{1,\beta} = H$.

We suppose $\text{End}^{\mathbb{Q}}(\mathbf{Y}) = \mathbb{Q} \times \mathbb{Q}$ and try to get a contradiction (in order to prove that $\text{End}^{\mathbb{Q}}(\mathbf{Y}) = M_2(\mathbb{Q})$). We pick a sufficiently small open compact subgroup $K \subset G(\mathbb{A}^{(p^\infty)})$ maximal at p so that the normalization \mathcal{H}_K of $H_K \subset Y_K \times Y_K$ is smooth at the image of y . The variety Y_K is naturally defined over a finite extension $\mathbb{F}_q/\mathbb{F}_p$ as the solution of the moduli problem $\mathcal{E}^{(p)}/K$. The universal elliptic curve \mathbf{E}_K is therefore defined over I_{K/\mathbb{F}_q} , and \mathcal{H}_K is a variety of finite type over \mathbb{F}_q . Let η be the generic point of $\mathcal{H}_{K/\mathbb{F}_q}$, and write $\bar{\eta}$ for the geometric point over η and $\mathbb{F}_q(\bar{\eta})^{\text{sep}}$ for the separable algebraic closure $\mathbb{F}_q(\bar{\eta})^{\text{sep}}$ of $\mathbb{F}_q(\bar{\eta})$ in $\mathbb{F}_q(\bar{\eta})$. Take an odd prime $\ell \neq p$, and consider the ℓ -adic Tate

module $T_\ell(\mathbf{Y}_{\bar{\eta}})$ for the generic fiber $\mathbf{Y}_{\bar{\eta}}$ of \mathbf{Y} . We consider the image of the Galois action $\text{Im}(\text{Gal}(\mathbb{F}_q(\bar{\eta})^{sep}/\mathbb{F}_q(\eta)))$ in $GL_{O_\ell \times O_\ell}(T_\ell(\mathbf{Y}_{\bar{\eta}}))$. Then by a result of Zarhin ([DAV] Theorem V.4.7), the Zariski closure over \mathbb{Q} of $\text{Im}(\text{Gal}(\mathbb{F}_q(\bar{\eta})^{sep}/\mathbb{F}_q(\eta)))$ is a reductive subgroup \mathcal{G} of $GL_{\mathbb{Q}_\ell \times \mathbb{Q}_\ell}(T_\ell(\mathbf{Y}_{\bar{\eta}}) \otimes \mathbb{Q})$, and $\text{Im}(\text{Gal}(\mathbb{F}_q(\bar{\eta})^{sep}/\mathbb{F}_q(\eta)))$ is an open subgroup of $\mathcal{G}(\mathbb{Q}_\ell)$. Moreover, by Zarhin's theorem, the centralizer of \mathcal{G} in $\text{End}_{\mathbb{Q}_\ell \times \mathbb{Q}_\ell}(T_\ell(\mathbf{Y}_{\bar{\eta}}) \otimes \mathbb{Q})$ is $\text{End}(\mathbf{Y}) \otimes \mathbb{Q}_\ell$. Since the reductive subgroups of $GL(2)$ are either tori or contain $SL(2)$, the derived group $\mathcal{G}_1(\mathbb{Q}_\ell)$ of $\mathcal{G}(\mathbb{Q}_\ell)$ has to be $SL_2(\mathbb{Q}_\ell \times \mathbb{Q}_\ell)$. By Chebotarev's density, we can find a set of closed points $u \in \mathcal{H}_K(\mathbb{F})$ with positive density such that the Zariski closure in \mathcal{G} of the subgroup generated by the Frobenius element $Frob_u \in \text{Im}(\text{Gal}(\mathbb{F}_q(\bar{\eta})^{sep}/\mathbb{F}_q(\eta)))$ at u with $\pi_j(u) = u_j$ ($u_j \in I_K(\mathbb{F})$) is a torus containing a maximal torus $T_u = (T_{u_1} \times T_{u_2}) \cap \mathcal{G}_1$ of the derived group \mathcal{G}_1 of \mathcal{G} . In particular the centralizer of T_u in \mathcal{G}_1 is itself. Thus \mathbf{Y}_u is isogenous to a product of two non-isogenous elliptic curves $Y_1 = E_{u_1}$ and $Y_2 = E_{u_2}$ defined over a finite field. The endomorphism algebra $M_j = \text{End}^{\mathbb{Q}}(Y_j)$ is a CM quadratic extension of \mathbb{Q} generated over \mathbb{Q} by the relative Frobenius map ϕ_j induced by $Frob_u$. The relative Frobenius map $Frob_u$ acting on $X_*(\widehat{I}_{u_1}) \cong O_p$ has one eigenvalue: $\phi_1^{(1-c)\sigma}$ for the CM type $\Sigma_1 = \{\sigma\}$ of Y_1 , which differ from the eigenvalues of $\phi_2 \in \text{End}(Y_2)$ on $X_*(\widehat{I}_{u_2}) \cong O_p$. Since we have proven that over the open dense subscheme U of \mathcal{H} , the formal completion of U at $u \in U$ with $u = (u_1, u_2) \in X \subset V^2$ is canonically isomorphic to a formal subtorus $\widehat{Z} \subset \widehat{I}_{u_1} \times \widehat{I}_{u_2}$ with co-character group $X_*(\widehat{Z}) \cong \mathbb{Z}_p$, we may assume that our point $u = (u_1, u_2)$ as above is in the (open dense) image U_K of U in H_K . Projecting $X_*(\widehat{Z})$ down to the left and the right factors I_K , the projection map $X_*(\widehat{Z}) \rightarrow X_*(\widehat{I}_{u_j})$ is actually an injection commuting with the action of $Frob_u$. Thus $Frob_u$ has more than one distinct eigenvalues on $X_*(\widehat{Z})$, which is a contradiction. Thus we conclude that $\text{End}^{\mathbb{Q}}(\mathbf{Y}) = M_2(\mathbb{Q})$ for any choice of small open compact subgroups K maximal at p .

As we have remarked at the beginning, $\text{End}^{\mathbb{Q}}(\mathbf{Y}) = M_2(\mathbb{Q})$ implies that we have an isogeny $\tilde{\beta} : \mathbf{A} \rightarrow \mathbf{B}$ over \mathcal{H} . Writing $\eta_j^{(p)}$ ($j = A, B$) for the prime-to- p level structure of \mathbf{A} and \mathbf{B} inducing the prime-to- p level structure already chosen for $X(R) = \mathbf{A}_y = \mathbf{B}_y$, we find that $\tilde{\beta} \circ \eta_A^{(p)} = \eta_B^{(p)} \circ g$ for $g \in G(\mathbb{A}^{(p\infty)})$. Specializing at y , we have $g = \rho(\beta)$ for $\beta \in \text{End}^{\mathbb{Q}}(X(R)) = M$. Thus $\widehat{\mathcal{H}}_y \subset \widehat{H}_{x^2} \subset \widehat{Y} \times \widehat{Y}$ is given by $\{t, t^{\beta^{1-c}}\} | t \in \widehat{\mathbb{G}}_m$ for nonzero $\beta \in R_{(p)}$. Suppose that y corresponds to L ; so, $\widehat{\mathcal{H}}_y \subset \widehat{I} \times \widehat{I}$ coincides with $\widehat{\mathbb{G}}_m \otimes L$. On the other hand, we have the skew-diagonal $\Delta_\beta = \Delta_{1,\beta} = \{(z, \rho(\beta)(z)) | z \in I\} \subset I \times I$. The formal completion $\widehat{\Delta}_\beta$ along (x, x) therefore coincides with $\widehat{\mathcal{H}}_y$ and $\widehat{\mathbb{G}}_m \otimes L \subset \widehat{H}_{(x,x)}$ inside \widehat{I}^2 . Thus $\Delta_\beta \subset H$. By the irreducibility of H , we conclude $H = \Delta_\beta$. Since Δ_β is smooth, $\Delta_\beta = \mathcal{H}$, and hence H is smooth everywhere. \square

There are two way of proving Theorem 1.3. One is an induction reducing things to Theorem 1.2, and another is to prove that $\text{End}(\mathbf{Y}) \otimes \mathbb{Q} = M_2(\mathbb{Q}) \times \mathbb{Q}^{n-2}$ for $\mathbf{Y} = \prod_j \pi_j^* \mathbf{E}$ for the projection π_j of \mathcal{H} to j -th component I (after a permutation of the factors I).

Second Talk

In this second talk, we relate the result proven in the first talk with linear independence modulo p of classical and p -adic modular forms.

2. GEOMETRIC MODULAR FORMS

We study classification problem of elliptic curves E/A with a differential ω over a ring A .

2.1. Moduli Problem. Consider the following moduli functor of level $\Gamma(N)$,

$$\mathcal{P}_{\Gamma(N),\zeta}(A) = [(E, \phi_N, \omega) \mid H^0(E, \Omega_{E/A}) = A\omega, (E, \phi_N)_{/A} \in \mathcal{E}_{\Gamma(N),\zeta}(A)],$$

for all $\mathbb{Z}[\frac{1}{N}, \zeta]$ -algebras A . Again we remove the contribution upon ζ and consider the functor $\mathcal{P}_{\Gamma(N)}(A) = [(E, \phi_N, \omega)_{/A}]$ defined on the category of $\mathbb{Z}[\frac{1}{N}]$ -algebras, we have $\mathcal{P}_{\Gamma(N)} = \bigsqcup_{\zeta} \mathcal{P}_{\Gamma(N),\zeta}$, and this functor is represented by a geometrically non-connected affine scheme $\mathcal{M}(N) = \text{Spec}(\mathcal{R}_{\Gamma(N)})$ defined over $\mathbb{Z}[\frac{1}{N}]$ if $N \geq 3$. When $N = 1$, over $\mathbb{Z}[\frac{1}{6}]$, $\mathcal{P}_{\Gamma(1)}$ is represented by $\mathcal{M}(1)_{/\mathbb{Z}[\frac{1}{6}]} = \text{Spec}(\mathcal{R})$ for $\mathcal{R} = \mathbb{Z}[\frac{1}{6}, g_2, g_3, \frac{1}{\Delta}]$ for $\Delta = g_2^3 - 27g_3^2$, where g_2, g_3 are variables. So, for simplicity, we assume that our base ring B has $1/6$ in it.

The universal elliptic curve with universal differential is given by

$$(\mathbf{E}, \boldsymbol{\omega})_{/\mathcal{M}_1} = \left(\text{Proj}(\mathcal{R}[X, Y, Z]/(ZY^2 - 4X^3 + g_2XZ^2 + g_3Z^3)), \frac{dX}{Y} \right).$$

We can let $a \in \mathbb{G}_{m/\mathbb{Z}[\frac{1}{N}]}$ act on $\mathcal{P}_{\Gamma(N)}$ by $(E, \omega) \mapsto (E, a\omega)$; so, \mathcal{R} is naturally a graded algebra $\mathcal{R}_{\Gamma(N)} = \bigoplus_{n \in \mathbb{Z}} \mathcal{R}_n(\Gamma(N))$. In particular, g_j has degree $2j$ for $j = 2, 3$. Then $Y(N) = \text{Proj}(\mathcal{R}(\Gamma(N))) = \text{Spec}(\mathcal{R}_0(\Gamma(N)))$. Write $G = \mathbb{Z}[\frac{1}{6}, g_2, g_3] \subset \mathcal{R}$ and define $G(\Gamma(N))$ by the integral closure of G . Then $G(\Gamma(N))$ is naturally a graded algebra, and $X(N) = \text{Proj}(G(\Gamma(N)))$ is a smooth compactification of the affine curve $Y(N)$.

2.2. Modular forms of $\Gamma_1(p^n)$. Let B be a base $\mathbb{Z}_{(p)}$ -algebra, and A denote a general B -algebra. A level $\Gamma_1(p^n)$ -structure on $(E, \omega)_{/A}$ is an embedding $i : \mu_{p^n} \hookrightarrow E[p^n]$ defined over A . Consider the following Γ_1 -version of the moduli functor: $\mathcal{P}_{\Gamma_1(p^n)}(A) = [(E, i, \omega)_{/A}]$. Then $\mathcal{P}_{\Gamma_1(p^n)}$ is again representable by an affine scheme $\mathcal{R}(\Gamma_1(p^n))$. We take the following moduli theoretic definition of modular form on $\Gamma_1(p^n)$: a modular form $f \in G_k(\Gamma_1(p^n); B)$ is a functorial morphism for all B -algebras A ,

$$\mathcal{P}_{\Gamma_1(p^n)}(A) = [(E, i, \omega)_{/A}] \rightarrow \mathbf{A}^1(A) = A$$

satisfying the following conditions: Regarding f as a function of isomorphism classes of $(E, i, \omega)_{/A}$ with $f((E, i, \omega)_{/A}) \in A$ satisfying

(G1) $f((E, i, a\omega)_{/A}) = a^{-k} f((E, i, \omega)_{/A})$ for $a \in A^\times = \mathbb{G}_m(A)$;

(G2) If $\rho : A \rightarrow A'$ is a morphism of B -algebras, then we have

$$f((E, i, \omega)_{/A} \times_A A') = \rho(f((E, i, \omega)_{/A}));$$

(G3) The function has q -expansion in $B[[q^{1/d}]]$ ($0 < d|p^n$) at the Tate curves with level $\Gamma_1(p^n)$ -structure.

The direct sum $G(\Gamma_1(p^n); A) = \bigoplus_{0 \leq k \in \mathbb{Z}} G_k(\Gamma_1(p^n); A)$ is a normal graded algebra. In particular, when $n = 0$, we have $\mathbf{P}^1(j)_{/A} = \text{Proj}(G(\Gamma_1(1); A)) = \text{Proj}(G)$, and $I_n = \text{Proj}(G(\Gamma_1(p^n); \mathbb{F}_p)) / \mathbf{P}^1(j)_{/\mathbb{F}_p}$ is the Igusa tower. We can think of Γ_0 -version, replacing i by a subgroup $C \subset E[p^n]$ isomorphic to μ_{p^n} étale locally. Then elements in $G_k(\Gamma_0(p^n); A)$ are functorial rules $\mathcal{P}_{\Gamma_0(p^n)}(A) = [(E, C, \omega)_{/A}] \rightarrow \mathbf{A}^1(A) = A$ satisfying the conditions (G1–3) replacing $\Gamma_1(p^n)$ -level structures i by $\Gamma_0(p^n)$ -level structures C .

2.3. p -Adic modular forms. A p -adic modular form f over a p -adic base ring $B = \varprojlim_n B/p^n B$ is a functorial rule: $(E, i = \eta_p^\circ : \mu_{p^\infty} \hookrightarrow E[p^\infty])_{/A} \mapsto f(E, i) \in A$ for a p -adic B -algebra $A = \varprojlim_n A/p^n A$. In other words, f satisfies the following conditions:

- (P1) $f((E, ai)_{/A}) = a^{-k} f((E, i)_{/A})$ for $a \in A^\times = \mathbb{G}_m(A)$;
 (P2) If $\rho : A \rightarrow A'$ is a p -adically continuous morphism of p -adic B -algebras, then

$$f((E, i)_{/A} \times_A A') = \rho(f((E, i)_{/A}));$$

- (P3) We have the q -expansion $f(q)$ in $B[[q]]$ at the Tate curves with the canonical $i_{can} : \mu_{p^\infty} \rightarrow \text{Tate}(q)$ and the t -expansion $f(t) = f(\widehat{\mathbf{E}}, \eta_p^\circ) \in \widehat{W[t, t^{-1}]} = W[[T]]$ ($T = t^{-1}$) for the universal deformation $\widehat{\mathbf{E}}$ of $X(R)$ over $\widehat{Y}_x^{(p)} = \widehat{\mathbb{G}}_m$ ($x = x(R)$).

By definition, the functor $\mathcal{P}_{\Gamma_1(p^\infty)/B} : p\text{-adic } B\text{-ALG} \rightarrow SETS$ given by

$$\mathcal{P}_{\Gamma_1(p^\infty)}(A) = [(E, i : \mu_{p^\infty} \cong E[p^\infty]^\circ)_{/A}]$$

is pro-represented by the formal completion of $\widehat{\mathcal{M}}_\infty := \varprojlim_n \text{Spec}(\mathcal{R}_{\Gamma_1(p^n)/(B/p^n B)})$ along its mod p fiber. The level $\Gamma_1(p^\infty)$ -structure i induces an isomorphism of formal group $\widehat{i} : \widehat{\mathbb{G}}_m \cong \widehat{E}$, and $\omega_p := \widehat{i}_*(\frac{dt}{t})$ on E . Thus if f is a modular form on $\Gamma_0(p^n)$ as above, we can define $\widehat{f}(E, i) := f(E, i(\mu_{p^n}), \omega_p)$, and in this way, we may consider a modular form as a p -adic modular form. They have the same q -expansion and t -expansion. Write S_∞ for the formal completion of $Y^{(p)}$ along the ordinary locus of $Y_{/\mathbb{F}}$. Let $I_\infty \rightarrow S_\infty$ be the Igusa tower whose reduction mod p is the irreducible component containing $x = x(R)$ mod p . We have

$$(2.1) \quad f(q) \equiv 0 \pmod{p} \iff f(t) \equiv 0 \pmod{p} \iff f|_{I_\infty} \equiv 0 \pmod{p},$$

because of the irreducibility of the Igusa tower over $I_{/\mathbb{F}}$ (a result of Igusa).

2.4. Hecke invariant subvarieties. We recall the result we proved in the first lecture. Recall the imaginary quadratic field M in which the prime p splits. We write I for the irreducible component of $Y_{/\mathbb{F}}^{(p)} = Y^{(p)} \times_{\mathcal{W}} \mathbb{F}$ containing $x(R)$. Recall

Theorem 2.1. *Let $H \subset \overbrace{I \times_{\mathbb{F}} I \times_{\mathbb{F}} \cdots \times_{\mathbb{F}} I}^n$ (with $n \geq 2$) be a proper closed irreducible subscheme with a dominant projection to the product of the first $n - 1$ factor and to the last factor. If H is stable under the diagonal action of a p -adic open subgroup of $T_x(\mathbb{Z}_{(p)})$,*

up to permutations of the first $(n - 1)$ factors, we have $H = \overbrace{I \times \cdots \times I}^{n-2} \times \Delta_{\alpha, \beta}$, where $\Delta_{\alpha, \beta}$ is the skewed diagonal $\{(\rho(\alpha)(z), \rho(\beta)(z)) | z \in I\} \subset I \times I$.

Consider $a_1, \dots, a_n \in R_p^\times / \mathbb{Z}_p^\times$ distinct modulo $R_p^\times / \mathbb{Z}_p^\times = T_x(\mathbb{Z}_{(p)})$ (thus $a_i/a_j \notin M^\times$ if $i \neq j$).

Corollary 2.2. *Fix a weight $k > 0$. Let $h \in G_k(\Gamma_0(p^r); \mathcal{W})$ be a lift of the Hasse invariant. Prepare an $(m+1)$ -set $\{h, f_{i,1}, \dots, f_{i,m}\} \subset G_k(\Gamma_1(p^r); \mathcal{W})$ of modular forms linearly independent modulo $\mathfrak{m}_{\mathcal{W}}$ (for $m > 0$). Then the nm -set of mod p modular forms $\{f_{i,1}(t^{a_i}), \dots, f_{i,m}(t^{a_i})\}_{i,j}$ modulo $\mathfrak{m}_{\mathcal{W}}$ is linearly independent in $\widehat{\mathbb{F}[t, t^{-1}]}$.*

Proof. We consider the completion $\widehat{\mathcal{O}} = \widehat{\mathcal{O}}_{\widehat{Y^{(p)}}, x} = \widehat{\mathbb{F}[t, t^{-1}]}$ along $x = (x(R) \bmod \mathfrak{m}_{\mathcal{W}}) \in Y^{(p)}(\mathbb{F})$. Then R_p^\times acts on $\widehat{W[t, t^{-1}]}$ by $t \mapsto t^{a_j}$. Thus we can think of

$$\varphi : \overbrace{\mathcal{O}_{Y^{(p)}, x} \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} \mathcal{O}_{Y^{(p)}, x}}^n \rightarrow \widehat{\mathcal{O}}$$

given by $\varphi(f_1(t) \otimes \cdots \otimes f_n(t)) = f_1(t^{a_1})f_2(t^{a_2}) \cdots f_n(t^{a_n})$. Note that $\frac{f}{h}(t) = f(t)$ for $f \in G_k(\Gamma_1(p^r); \mathcal{W})$ (since we may assume that $h \equiv 1 \pmod{\mathfrak{m}_{\mathcal{W}}}$). If $\text{Ker}(\varphi) = 0$, we get the desired result, since $\{f_{i,1}/h, \dots, f_{i,m}/h\} \subset \mathcal{O}_{Y^{(p)}, x}$. If not, $\text{Ker}(\varphi)$ defines an irreducible

closed subscheme \mathcal{X} of $\text{Spec}(\overbrace{\mathcal{O}_{Y^{(p)}, x} \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} \mathcal{O}_{Y^{(p)}, x}}^n) = I^n$. Let X be the Zariski closure of \mathcal{X} in I^n . Then X is invariant under the action of $\alpha \in T_x(\mathbb{Z}_{(p)})$ since it acts by $t \mapsto t^{\alpha^{1-c}}$ commuting with φ . By induction on n , we may assume that the projection to the first $(n-1)$ -factor I' of I^n of X and the last factor is dominant. Then by the above theorem, permuting the indices $i < n$, we find $\alpha, \beta \in M^\times$ such that $X \subset I'^{n-2} \times \Delta_{\alpha, \beta}$, which implies $a_{n-1}/a_n = \alpha/\beta \in M^\times$ after permuting indices of a_j for $j < n$, a contradiction. Thus $\text{Ker}(\varphi) = 0$, and we are done. \square

3. p -ADIC HECKE L -FUNCTIONS AND ITS μ -INVARIANT

In 1899, Hurwitz studied an analogue of the Riemann zeta function:

$$L(4k) = \sum_{a+bi \in \mathbb{Z}[i] - \{0\}} \frac{1}{(a+bi)^{4k}}$$

of Gaussian integers $\mathbb{Z}[i]$ and showed that for positive integer k ,

$$\frac{L(4k)}{\Omega^{4k}} \in \mathbb{Q} \quad (\Omega = 2 \int_0^1 \frac{dx}{\sqrt{1-x^4}} = \int_\gamma \frac{dx}{y} : \text{period of the lemniscate}).$$

Nowadays, we regard this value as a special value of a Hecke L -function:

$$L(s, \lambda) = \sum_{\mathfrak{a}} \lambda(\mathfrak{a}) N(\mathfrak{a})^{-s} \quad \text{of the Gaussian field } \mathbb{Q}[i].$$

Here λ is a Hecke ideal character of $\mathbb{Q}[i]$ with $\lambda((\alpha)) = \alpha^{-4k}$ and $L(4k) = 4L(0, \lambda)$.

We shall give a sketch of a proof of non-vanishing modulo p of almost all Hecke L -values for general imaginary quadratic fields $M = \mathbb{Q}[\sqrt{-D}]$. The Hurwitz formulation is modular: For any lattice $L = \mathbb{Z}w_1 + \mathbb{Z}w_2 \subset \mathbb{C}$, we can think about

$$E_{2k}(L) = \frac{1}{2} \sum_{aw_1 + bw_2 \in L} \frac{1}{(aw_1 + bw_2)^{2k}} \quad (\text{Eisenstein series}),$$

which is a function of lattices satisfying $E_{2k}(\alpha L) = \alpha^{-2k} E_{2k}(L)$. The quotient \mathbb{C}/L gives rise to an elliptic curve $X(L) \subset \mathbf{P}^2$ by Weierstrass theory. Since $X(L)$ has a unique nowhere vanishing differential du for the variable u of \mathbb{C} and we can recover out of $(X(L), du)$ the lattice L as $\{\int_\gamma du \mid \gamma \in H_1(E, \mathbb{Z})\}$, we can think of E_{2k} as a function of the pairs (E, ω) of an elliptic curve E and a nowhere vanishing differential ω satisfying $E_{2k}(E, \alpha\omega) = \alpha^{-2k} E_{2k}(E, \omega)$. Note that $E_4(E, \omega) = \frac{1}{120} g_2(E, \omega)$ and $E_6(E, \omega) = \frac{1}{280} g_3(E, \omega)$. More generally, E_{2k} is a rational isobaric polynomial of g_2 and g_3 . Thus E_{2k} is a modular form f of weight $2k$.

Fix an imaginary quadratic field M . Write R for the integer ring of M . We assume that p splits into $p = \mathfrak{p}\bar{\mathfrak{p}}$ in R so that \mathfrak{p} gives rise to a p -adic place $i_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$. Recall the ring W of Witt vectors with coefficients in the algebraic closure $\overline{\mathbb{F}} := \overline{\mathbb{F}_p}$ of \mathbb{F}_p and regard it as a valuation subring of $\mathbb{C}_p \supset \overline{\mathbb{Q}_p}$ (in other words, W is the closure in \mathbb{C}_p of the p -adic integer ring of the maximal unramified extension of \mathbb{Q}_p). So $\mathcal{W} = i_p^{-1}(W)$ with maximal ideal $\mathfrak{P} = \mathfrak{m}_{\mathcal{W}}$. Important facts are

- (E1) For a lattice $\mathfrak{a} \subset R$ prime to p (\mathfrak{a} is prime to p if $[R : \mathfrak{a}]$ is prime to p), $X(\mathfrak{a})$ is p -integrally defined over $\mathcal{W} \subset \overline{\mathbb{Q}}$;
- (E2) E_{2k} (up to a simple constant which we ignore in this lecture) is defined over \mathbb{Q} (actually over $\mathbb{Z}_{(p)}$ for almost all k) by the q -expansion principle;
- (E3) Let $\omega = \Omega du$ be a generator of $H^0(X(R), \Omega_{X(R)/\mathcal{W}})$ ($\Leftrightarrow H^0(X(\mathfrak{a}), \Omega_{X(\mathfrak{a})/\mathcal{W}}) = \mathcal{W}\omega$, because \mathfrak{a} is prime to p). Thus $\Omega = \int_\gamma \omega$ for a 1-cycle γ with $R_{(p)}\gamma = H_1(X(\mathfrak{a}), \mathbb{Z}_{(p)})$. Then $\frac{E_{2k}(X(\mathfrak{a}), \omega)}{\lambda(\mathfrak{a})}$ is the partial L -value of the ideal class of \mathfrak{a}^{-1} :

$$\frac{E_{2k}(X(\mathfrak{a}), \omega)}{\lambda(\mathfrak{a})} = \frac{E_{2k}(X(\mathfrak{a}), \Omega du)}{\lambda(\mathfrak{a})} = \frac{E_{2k}(X(\mathfrak{a}), du)}{\Omega^{2k} \lambda(\mathfrak{a})} = \frac{L_{\mathfrak{a}^{-1}}(2k, \lambda)}{\Omega^{2k}} \in \overline{\mathbb{Q}},$$

because $\frac{E_{2k}(X(\mathfrak{a}), du)}{\lambda(\mathfrak{a})} = L_{\mathfrak{a}^{-1}}(2k, \lambda) = \sum_{\mathfrak{b} \sim \mathfrak{a}^{-1}} \lambda(\mathfrak{b}) N(\mathfrak{b})^{-2k}$ for a Hecke character λ with $\lambda((\alpha)) = \alpha^{-2k}$.

In order to get p -integrality and an $U(p)$ -eigenform, we modify the Eisenstein series E_{2k} into $\mathcal{E}_{2k}(z) = E_{2k}(z) - E_{2k}(pz)$. Then we have $\mathcal{E}_{2k}(X(\mathfrak{a}), \omega) \in \mathcal{W}$.

A ray class character χ modulo p^n with values in \mathbb{C}_p^\times is called p -anticyclotomic if $\chi(\mathfrak{a}^c) = \chi(\mathfrak{a})^{-1}$ for complex conjugation c . Let C_n be the ring class group of $R_n = \mathbb{Z} + p^n R$. Then χ as above is anticyclotomic if and only if χ factors through C_n . Let $C_\infty = \varprojlim_n C_n$. The profinite group C_∞ is isomorphic to $\Delta \times \Gamma$ for a finite group Δ and a multiplicative group Γ isomorphic to the additive group \mathbb{Z}_p .

By results of Hurwitz, Damerell, Weil, Shimura, Katz combined, we have

$$\frac{(2k-1)! L^{(p)}(0, \lambda\chi)}{\Omega^{2k}} \in \mathcal{W} \text{ for } p\text{-anticyclotomic } \chi,$$

where $L^{(p)}(0, \lambda\chi) = \{(1 - \lambda\chi(\mathfrak{p})p^{-1})(1 - \lambda\chi(\bar{\mathfrak{p}}))\} L(0, \lambda\chi)$. Katz and others constructed a p -adic measure $d\mu_\lambda$ on C_∞ such that

$$\int_{C_\infty} \chi d\mu_\lambda = \frac{(2k-1)! L^{(p)}(0, \lambda\chi)}{\Omega^{2k}} \in \mathcal{W} \text{ for } \ell\text{-anticyclotomic } \chi.$$

We project down the measure to Γ , getting the anticyclotomic p -adic Hecke L -function $L_\lambda^- \in W[[\Gamma]]$. Since $W[[\Gamma]] \cong W[[T]]$ is a UFD, we can decompose $L_p^- = p^\mu L$ for a power series $L \in W[[T]]$ prime to p . Then we have

Theorem 3.1. *If $p > 2$, we have $\mu = 0$.*

The result in the above imaginary quadratic cases are also treated by Finis (see [F1] and [F2]) under different assumptions by a different method (see also Vatsal's papers [V1], [V2] and [V3] for related topics).

Here we shall give a very brief sketch of the proof of this special case, assuming for simplicity that $\Gamma = 1 + p\mathbb{Z}_p$ and M/\mathbb{Q} ramifies only at one prime. If we take a representative set $\mathfrak{a}_1, \dots, \mathfrak{a}_h$ of ideals representing C_∞/Γ , under the above assumption that the projection A_j of \mathfrak{a}_j to Γ is independent modulo $R_{(p)}^\times$ in the following sense:

$$(3.1) \quad A_i/A_j \notin \{\alpha^{1-c} | \alpha \in M^\times\} \text{ if } i \neq j.$$

We can identify Γ with a subgroup of $R_p^\times/\mathbb{Z}_p^\times$ which is isomorphic to \mathbb{Z}_p^\times by $R_p^\times = R_{\mathfrak{p}}^\times \times R_{\mathfrak{p}}^\times \ni (a, b) \mapsto ab^{-1} \in \mathbb{Z}_p^\times$. Thus the image of $A_j \in \mathbb{Z}_p^\times$ satisfies $A_j^h = \alpha_j^{1-c}$ for $\alpha_j \equiv 1 \pmod{p}$ generating \mathfrak{a}_j^h . Choose $a_j \in M_{\mathbb{A}}^\times$ with $a_j R = \mathfrak{a}_j$ and $a_{j,p} = a_{j,\infty} = 1$.

Consider $\mathcal{E} = \mathcal{E}_{2k}$. We regard \mathcal{E} as a p -adic modular form. Recall $X = X(R)$ over \mathcal{W} and over $W = \varprojlim_n \mathcal{W}/p^n \mathcal{W}$. Its formal group \widehat{X} is isomorphic to $\widehat{\mathbb{G}}_m$ by the level structure $\eta_p = \eta_p^\circ : \mu_{p^\infty} \cong X[\mathfrak{p}^\infty]$ over \mathcal{W} , we have a canonical identification $\widehat{X}_j = \widehat{\mathbb{G}}_m$ induced by η_p° over W . Then X/W has a differential $\omega_p = \frac{ds}{s}$ writing $\widehat{\mathbb{G}}_m = \widehat{\text{Spf}(W[s, s^{-1}])}$, where $\widehat{W[s, s^{-1}]} = \varprojlim_n \widehat{W[s, s^{-1}]} / (s^{p^n} - 1)$. The modular form \mathcal{E} has t -expansion with respect to the Serre-Tate coordinate t around $x(R) \in I_\infty$.

Since $z = [(z, 1)] \in T_x(\mathbb{Z}_p)/Z(\mathbb{Z}_p) = (R_{\mathfrak{p}}^\times \times R_{\mathfrak{p}}^\times)/\mathbb{Z}_p^\times$ acts on $\widehat{Y}_x^{(p)} = \widehat{\mathbb{G}}_m$ (for $x = x(R)$) by $t \mapsto t^z$, we can think of $\mathcal{E}(t^z)$, which is only defined over $\widehat{\mathbb{G}}_m = \widehat{I}_x = \widehat{Y}_x^{(p)}$. Since $\rho(a)(x(\mathfrak{a})) = x(R)$ for an idele $a \in M_{\mathbb{A}}^\times$ with $a_p = a_\infty = 1$, we can think of $\mathcal{E}|\rho(a^{-1})(E, \eta^{(p)}, \omega) = \mathcal{E}(E, \eta^{(p)} \circ \rho(a^{-1}), \omega)$. Then $\mathcal{E}|\rho(a^{-1})$ is a modular form with

$$\mathcal{E}|\rho(a^{-1})(X(R), \eta(R), \omega(R)) = \mathcal{E}(X(\mathfrak{a}), \eta(\mathfrak{a}), \omega(\mathfrak{a})).$$

The automorphic proof of this theorem consists of the following two steps.

(Step1) *t-expansion*: Essentially, the t -expansion of $\sum_j \lambda(\mathfrak{a}_j)^{-1} \mathcal{E}|\rho(\mathfrak{a}_j^{-1})(t^{A_j^{-1}})$ gives the expansion of the p -adic L -function L_λ^- putting $t = 1 + T$. This follows from Katz's construction of the measure (mainly his explicit relation of the invariant differentiation on $\widehat{Y}_p^{(p)} = \widehat{\mathbb{G}}_m$ with the Maass-Shimura differential operator via the Gauss-Manin connection).

If we write $\omega_p = (\eta_p^\circ)_*(\frac{dt}{t}) = \Omega_p \omega$, by a result of Katz/Shimura, we have

$$\frac{\theta^n \mathcal{E}|_{t=1}}{\Omega_p^{2k+2n}} = \frac{\sum_i \lambda(\mathfrak{a}_i)^{-1} A_i^{-n} \delta_{2k}^n \mathcal{E}_{2k}(x(\mathfrak{a}_i))}{\Omega^{2k+2n}} = \frac{\sum_i \lambda(\mathfrak{a}_i)^{-1} \alpha_i^{-n(1-c)/h} \delta_{2k}^n \mathcal{E}_{2k}(x(\mathfrak{a}_i))}{\Omega^{2k+2n}} = \frac{L(0, \lambda_n)}{\Omega^{2k+2n}},$$

where $\theta = t \frac{d}{dt}$, δ_{2k}^n is the Maass-Shimura differential operator and λ_n is a Hecke character whose infinity type is $2k + n(1 - c)$ (defined modulo p) and $\lambda_n(\mathfrak{a}_i) = \lambda(\mathfrak{a}_i) A_i^n$. Thus the

n -th moment of the anticyclotomic Katz measure is basically given by the right-hand side of the above equation, and hence, $\mathcal{E}(t) \in W[[t-1]]$ is the power series associated the λ -branch of the Katz measure.

(Step2) *Linear independence*: By q -expansion principle (and computation), we find h and $\lambda(\mathfrak{a}_j)^{-1}\mathcal{E}|\rho(a_j^{-1})$ are linearly independent modulo $\mathfrak{m}_{\mathcal{W}}$ for a lift h of Hasse-invariant. By Lemma 2.2, we find $\{\lambda(\mathfrak{a}_j)^{-1}\mathcal{E}|\rho(a_j^{-1})(t^{A_j^{-1}})\}_j$ is linearly independent in $\widehat{\mathbb{F}[t, t^{-1}]}$. This implies the vanishing of the μ -invariant.

REFERENCES

Books

- [DAV] G. Faltings and C.-L. Chai, *Degeneration of Abelian Varieties*, Springer, 1990
- [IAT] G. Shimura, *Introduction to the Arithmetic Theory of Automorphic Functions*, Princeton University Press, Princeton, NJ, and Iwanami Shoten, Tokyo, 1971.
- [ICF] L. C. Washington, *Introduction to Cyclotomic Fields*, Graduate Text in Mathematics, **83**, Springer, New York, 1980.
- [PAF] H. Hida, *p -Adic Automorphic Forms on Shimura Varieties*, Springer Monographs in Mathematics, 2004, Springer

Articles

- [C] C.-L. Chai, Families of ordinary abelian varieties: canonical coordinates, p -adic monodromy, Tate-linear subvarieties and Hecke orbits, preprint 2003 (available at: www.math.upenn.edu/~chai)
- [FW] B. Ferrero and L. Washington, The Iwasawa invariant μ_p vanishes for abelian number fields, *Ann. of Math.* **109** (1979), 377–395
- [F1] T. Finis, Divisibility of anticyclotomic L -functions and theta functions with complex multiplication. *Ann. of Math.* **163** (2006), 767–807
- [F2] T. Finis, The μ -invariant of anticyclotomic L -functions of imaginary quadratic fields. *J. Reine Angew. Math.* **596** (2006), 131–152
- [H04] H. Hida, Non-vanishing modulo p of Hecke L -values, in: “*Geometric Aspects of Dwork’s Theory, II*” (edited by Alan Adolphson, Francesco Baldassarri, Pierre Berthelot, Nicholas Katz, and Francois Loeser), Walter de Gruyter, 2004, pp. 735–784 (preprint available at www.math.ucla.edu/~hida)
- [H07] H. Hida, Non-vanishing modulo p of Hecke L -values and application, In “ *L -functions and Galois representations*”, London Mathematical Society Lecture Note Series **320** (2007) 207–269 (preprint available at www.math.ucla.edu/~hida)
- [H08] H. Hida, The Iwasawa μ -invariant of p -adic Hecke L -functions, to appear in *Annals of Mathematics*, (preprint available at www.math.ucla.edu/~hida or at <http://annals.math.princeton.edu/issues/AcceptedPapers.html>)
- [Si] W. Sinnott, On the μ -invariant of the Γ -transform of a rational function, *Inventiones Math.* **75** (1984), 273–282
- [Si1] W. Sinnott, On a theorem of L. Washington, *Astérisque* **147-148** (1987), 209–224
- [V1] V. Vatsal, Uniform distribution of Heegner points, preprint, 2000, *Inventiones Math.* **148** (2002), 1–46
- [V2] V. Vatsal, Special values of anticyclotomic L -functions, *Duke Math J.*, **116** (2003), 219–261
- [V3] V. Vatsal, Special values of L -functions modulo p . *International Congress of Mathematicians. Vol. II*, 501–514, Eur. Math. Soc., Zürich, 2006
- [W1] L. Washington, The non- p -part of the class number in a cyclotomic \mathbb{Z}_p -extension, *Inventiones Math.* **49** (1978), 87–97