NEARLY ORDINARY HECKE ALGEBRAS AND GALOIS REPRESENTATIONS OF SEVERAL VARIABLES

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0. Introduction. The purpose of this paper is to supplement our previous papers [7] and [8] on Hecke algebras over totally real fields with a result on the canonical Galois representations into GL_2 with coefficients in the total quotient rings of the Hecke algebras. The construction of such Galois representations is automatic (as already done in the case of $\mathbf{Q}$ in [6]) from the known result on Galois representations already available in [11], [12], [13] and [1] in view of the result (or rather the proof of the result) in [8] if the degree of the base totally real field is odd, and even in the remaining case (i.e. the even degree case), after learning the ingenious method of Wiles in [17] (see also Mazur [9, 1.8] and Gouvêa [4, III.5]) of glueing together infinitely many residual representations into the bigger representation, we now know how to construct them from the knowledge of the Hecke algebra studied in [8]. In this paper, adopting the method of Wiles in our nearly ordinary case, we shall construct such representations. These representations give nontrivial (series of) examples of Galois representations into $GL_2(\mathbf{Z}_p[[X_0, \ldots, X_{d+s}]])$ for the degree d of the base field ($s \geq 0$, and if the Leopoldt conjecture holds for the base field and p, then $s = 0$). Moreover, the description similar to [17, Theorem 2] of the restriction to the decomposition group at p (see Theorem I (iv) below) is expected to have an important application in the Iwasawa theory of CM-fields (as already done in works of Mazur and Wiles [10], [16] for the cyclotomic $\mathbf{Z}_p$-extension over totally real fields and of Tilouine [15] for the anti-cyclotomic $\mathbf{Z}_p$-extension over imaginary quadratic fields). The author hopes to return to this problem in a near future. As naturally presumed from what we have already mentioned, our method of constructing Galois representations heavily relies on the Wiles' method in [17], the existence of Galois representations attached to classical cusp forms proven by Wiles [17] and

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Taylor [14] and the description of the big nearly ordinary Hecke algebra given in [8]. Actually, we will construct the Galois representation into GL_2 over the full universal Hecke algebra (not only for the nearly ordinary part; see Theorem II below). This result may be of some interest from the point of view of the theory of universal (or the deformation of) Galois representations due to Mazur [9], and we shall formulate our theorems this perspective in mind. In fact, Gouvêa [4, III.5.6] has already constructed the representation as in Theorem II when $F = \mathbf{Q}$ using Mazur's theory of deformation (under some restriction on p).

Now let us give a precise formulation of the result: We fix throughout the paper a rational prime p and a totally real field F of finite degree. We use the notation introduced in [8] without explaining them in detail. Let $\mathfrak{m}$ be an ideal of the integer ring $\mathfrak{r}$ of F. We consider the open subgroups $U_0(\mathfrak{m})$ and $U_1(\mathfrak{m})$ of $GL_2(\hat{\mathfrak{r}})$ defined by

$$U_0(\mathfrak{m}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\hat{\mathfrak{r}}) \middle| c \in \mathfrak{m}\hat{\mathfrak{r}} \right\},$$

$$U_1(\mathfrak{m}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\hat{\mathfrak{r}}) \middle| c \in \mathfrak{m}\hat{\mathfrak{r}} \text{ and } a \equiv 1 \bmod \mathfrak{m}\hat{\mathfrak{r}} \right\},$$

$$U(\mathfrak{m}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\hat{\mathfrak{r}}) \middle| c \in \mathfrak{m}\hat{\mathfrak{r}} \text{ and } a \equiv d \equiv 1 \bmod \mathfrak{m}\hat{\mathfrak{r}} \right\},$$

where $\hat{\mathfrak{r}} = \prod_{\mathfrak{p}} \mathfrak{r}_{\mathfrak{p}}$ is the product of the completion $\mathfrak{r}_{\mathfrak{p}}$ at $\mathfrak{p}$ over all prime ideals $\mathfrak{p}$ of $\mathfrak{r}$. We denote by $\bar{\mathbf{Q}}$ the field of all numbers algebraic over $\mathbf{Q}$ inside $\mathbf{C}$ and fix an algebraic closure $\bar{\mathbf{Q}}_p$ of the p-adic field $\mathbf{Q}_p$ and an embedding of $\bar{\mathbf{Q}}$ into $\bar{\mathbf{Q}}_p$ once and for all. We suppose throughout the paper that F is contained in $\bar{\mathbf{Q}}$. Let Φ be the subfield of $\bar{\mathbf{Q}}$ generated by all the conjugates of F. Let K be a finite extension of the closure $\hat{\Phi}$ inside $\bar{\mathbf{Q}}_p$ of Φ and $\mathcal{O}$ denote the p-adic integer ring of K. For each open compact subgroup S of $GL_2(\hat{\mathfrak{r}})$ containing $U_1(N)$ for an integral ideal N prime to p, we consider the full Hecke algebra $\mathbf{h}(S; \mathcal{O})$ of infinite p-power level and its nearly ordinary part $\mathbf{h}^{\text{n.ord}}(S; \mathcal{O})$ defined in [8, Section 2]: Let us recall their definition. We denote by I the set of all the embeddings of F into $\bar{\mathbf{Q}}$ and let $\mathbf{Z}[I]$ be the free module generated by the elements of I. We take an element $k \in \mathbf{Z}[I]$ with $k \geq 2t$ for $t = \Sigma_\sigma \sigma \in \mathbf{Z}[I]$. Put $n = k - 2t$ and suppose that there exists $0 \leq v \in \mathbf{Z}[I]$ such that $n + 2v = \mu t$ with $0 \leq \mu \in \mathbf{Z}$. Then we consider the space $\mathbf{S}_{k,w}(S(p^\alpha); \mathbf{C})$ of holomorphic cusp forms on $GL_2(F_\mathbf{A})$ in the sense of [7, Section 2] with respect to $S(p^\alpha) = S \cap U(p^\alpha)$ and with the automorphic factor at the infinity given by

$$\det\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)^{-w} (cz + d)^k,$$

where $w = v + k - t$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(F_\infty)$ and z is a variable on the product $\mathfrak{H}^I$ of copies of the upper half complex planes indexed by I. On $\mathbf{S}_{k,w}(S(p^\alpha); \mathbf{C})$, we have the following three type of operators: For each ideal $\mathfrak{n}$, the Hecke operator $T(\mathfrak{n})$, the action of the group $\mathbf{G}^\alpha = S_0(p^\alpha)\hat{\mathfrak{r}}^\times / S(p^\alpha)\mathfrak{r}^\times$ for $S_0(p^\alpha) = S \cap U_0(p^\alpha)$ and the action of the center $F_\mathbf{A}^\times$. For each ideal $\mathfrak{q}$ prime to Np, we choose $q \in F_\mathbf{A}^\times$ such that $q\mathfrak{r} = \mathfrak{q}$ and $q_{Np} = q_\infty = 1$ and define an operator $\langle \mathfrak{q} \rangle$ on $\mathbf{S}_{k,w}(S(p^\alpha); \mathbf{C})$ by the action of the double coset $S(p^\alpha)qS(p^\alpha)$ as in [7, Section 2]. Then the Hecke algebra $\mathfrak{h}_{k,w}(S(p^\alpha); \mathfrak{r}_\Phi)$ over the integer ring $\mathfrak{r}_\Phi$ of Φ is the subalgebra of $\mathrm{End}_\mathbf{C}(\mathbf{S}_{k,w}(S(p^\alpha); \mathbf{C}))$ generated over $\mathfrak{r}_\Phi$ by the action of the group $\mathbf{G}^\alpha$ the Hecke operators $T(\mathfrak{n})$ and $\langle \mathfrak{n} \rangle$ for $\mathfrak{n}$ and $p^{-v}T(p)$. The restriction of the operator in $\mathfrak{h}_{k,w}(S(p^\beta); \mathfrak{r}_\Phi)$ to the subspace $\mathbf{S}_{k,w}(S(p^\alpha); \mathbf{C})$ for $\beta \geq \alpha > 0$ yields a surjective algebra homomorphism over the group algebra $\mathfrak{r}_\Phi[\mathbf{G}^\alpha]$ of $\mathfrak{h}_{k,w}(S(p^\beta); \mathfrak{r}_\Phi)$ onto $\mathfrak{h}_{k,w}(S(p^\alpha); \mathfrak{r}_\Phi)$. After putting $\mathfrak{h}_{k,w}(S(p^\alpha); \mathcal{O}) = \mathfrak{h}_{k,w}(S(p^\alpha); \mathfrak{r}_\Phi) \otimes_{\mathfrak{r}_\Phi} \mathcal{O}$, we define

$$\mathbf{h}(S; \mathcal{O}) = \varprojlim_\alpha \mathfrak{h}_{k,w}(S(p^\alpha); \mathcal{O}).$$

We equip on $\mathbf{h}(S; \mathcal{O})$ the topology of the projective limit of the p-adic topology of $\mathfrak{h}_{k,w}(S(p^\alpha); \mathcal{O})$ which makes it into a compact ring. The nearly ordinary part $\mathbf{h}^{\text{n.ord}}(S; \mathcal{O})$ is the maximal algebra direct summand of $\mathbf{h}(S; \mathcal{O})$ on which the image of $p^{-v}T(p)$ is a unit. Then it is known (cf. [8, Theorem 2.3]) that these algebras $\mathbf{h}(S; \mathcal{O})$ and $\mathbf{h}^{\text{n.ord}}(S; \mathcal{O})$ are determined independently of the choice of the weight (k, w). Strictly speaking, the pair of $\mathbf{h}(S; \mathcal{O})$ and specified elements $T(\mathfrak{n})$ is determined independently of weight (k, w) because the isomorphism between the Hecke algebras of two different weights takes $T(\mathfrak{n})$ to $T(\mathfrak{n})$. Naturally $\mathbf{h}(S; \mathcal{O})$ becomes an algebra over the continuous group algebra $\mathcal{A} = \mathcal{O}[[\mathbf{G}]]$ for the profinite group $\mathbf{G} = \varprojlim_\alpha \mathbf{G}^\alpha$. Then $\mathbf{G}$ is isomorphic to the product of the following two groups $\mathfrak{r}_p^\times$ and $\bar{Z}_0 = \bar{Z}_0(S)$, which is the image of the center $\hat{\mathfrak{r}}^\times$ of $GL_2(\hat{\mathfrak{r}})$ in $\mathbf{G}$

([8, Lemma 2.1]). Let $\mathbf{W}$ be the torsion free part of $\mathbf{G}$ and write $\mathbf{A}$ for the continuous group algebra $\mathcal{O}[[\mathbf{W}]]$ of $\mathbf{W}$. Then $\mathbf{W}$ is free of finite rank $\geq d+1$ as $\mathbf{Z}_p$-module for $d = [F:\mathbf{Q}]$, and by [8, Theorem 2.4], $\mathbf{h}^{n.ord}(S;\mathcal{O})$ is of finite type and torsion free as $\mathbf{A}$-module. Let $\mathbf{L}$ be the quotient field of $\mathbf{A}$, and fix an algebraic closure of $\bar{\mathbf{L}}$ of $\mathbf{L}$. We denote by $\mathcal{C}$ the category of complete noetherian local $\mathcal{O}$-algebras with finite residue field. Any object A in $\mathcal{C}$ with maximal ideal $\mathfrak{m}$ is assumed to be complete under the $\mathfrak{m}$-adic topology. Let $A \in Ob(\mathcal{C})$ be an integral domain and Q be the quotient field of A. We say that a representation $\pi: \mathrm{Gal}(\bar{\mathbf{Q}}/F) \to GL_2(Q)$ is *continuous* if there exists a finitely generated A-submodule L of Q^2 stable under π and $L \otimes_A Q = Q^2$. Then for the maximal ideal $\mathfrak{m}$ of A, $L/\mathfrak{m}^r L$ has only finitely many elements, and $\pi: \mathrm{Gal}(\bar{\mathbf{Q}}/F) \to \mathrm{Aut}_A(L) = \varprojlim_{\alpha} \mathrm{Aut}_A(L/\mathfrak{m}^\alpha L)$ is continuous with respect to the $\mathfrak{m}$-adic topology of $\mathrm{Aut}_A(L)$; that is, the topology of projective limit of the finite groups $\mathrm{Aut}_A(L/\mathfrak{m}^\alpha L)$.

THEOREM I. *Let $A \in Ob(\mathcal{C})$ be an integral domain of characteristic different from 2 and $\lambda: \mathbf{h}^{n.ord}(S;\mathcal{O}) \to A$ be a continuous $\mathcal{O}$-algebra homomorphism. Let Q be the quotient field of A. Then there exists a unique semisimple Galois representation $\pi: \mathrm{Gal}(\bar{\mathbf{Q}}/F) \to GL_2(Q)$ such that:*

(i) *π is continuous;*
(ii) *π is unramified outside Np, where N is the level of S, i.e., the largest ideal prime to p with $S \supset U_1(N)$;*
(iii) *For the Frobenius element $\phi_\mathfrak{q}$ for each prime $\mathfrak{q}$ outside Np, we have*

$$\det(1 - \pi(\phi_\mathfrak{q})X) = 1 - \lambda(T(\mathfrak{q}))X + \lambda(\langle \mathfrak{q}\rangle)\mathfrak{N}_{F/\mathbf{Q}}(\mathfrak{q})X^2;$$

(iv) *Let $\mathfrak{p}$ be a prime factor of p and fix a decomposition group $D_\mathfrak{p}$ of $\mathfrak{p}$ in $\mathrm{Gal}(\bar{\mathbf{Q}}/F)$. Then there exist two characters ϵ, δ of $D_\mathfrak{p}$ with values in A such that the restriction of π to $D_\mathfrak{p}$ is, up to equivalence, of the following form:*

$$\pi(\sigma) = \begin{pmatrix} \epsilon(\sigma) & * \\ 0 & \delta(\sigma) \end{pmatrix} \quad \textit{for} \quad \sigma \in D_\mathfrak{p}.$$

Moreover if A is finite and torsion-free over $\mathbf{A}$ and λ is an $\mathbf{A}$-algebra homomorphism, then π is absolutely irreducible.

This theorem will be proven in Section 3. Here are some remarks about the theorem:

(i) Since $\mathbf{h}^{n.ord}(S;\mathcal{O})$ is of finite type and torsion-free as $\mathbf{A}$-module, any irreducible component of $\mathrm{Spec}(\mathbf{h}^{n.ord}(S;\mathcal{O}))$ is isomorphic to $\mathrm{Spec}(\mathbf{I})$ for an integral domain $\mathbf{I}$ finite and torsion-free over $\mathbf{A}$. Writing $\lambda: \mathbf{h}^{n.ord}(S;\mathcal{O}) \to \mathbf{I}$ for the projection morphism, we have the canonical Galois representation $\pi: \mathrm{Gal}(\bar{\mathbf{Q}}/F) \to GL_2(\mathbf{K})$ for the quotient field $\mathbf{K}$ of $\mathbf{I}$. Especially, if we fix an isomorphism: $\mathbf{W} \cong \mathbf{Z}_p^r$ with $r \geq d+1$, we can identify $\mathbf{A}$ with $\mathcal{O}[[X_1, \ldots, X_r]]$. (If the Leopoldt conjecture is true for F and p, $r = d+1$). Thus if $\mathbf{I}$ as above coincides with $\mathbf{A}$ and $p \neq 2$, we have a Galois representation of several variables:

$$\pi: \mathrm{Gal}(\bar{\mathbf{Q}}/F) \to GL_2(\mathcal{O}[[X_1, \ldots, X_r]])$$

(see [17, Remark in Section 2.2]).

(ii) One can even determine exactly the characters ϵ and δ by the values $\lambda(T(\mathfrak{p}))$ and $\lambda(g)$ for $g \in \mathbf{G}$ (see Proposition 2.3 in the text for details).

(iii) Let $\rho: \mathbf{G} \to \bar{Z}_0$ be the projection map and consider the induced algebra homomorphism $\rho: \mathfrak{A} = \mathcal{O}[[\mathbf{G}]] \to \mathcal{O}[[\bar{Z}_0]]$. We write $\mathfrak{p}$ for the kernel of ρ. As one can easily see, the ordinary Hecke algebra $\mathbf{h}^{ord}(S;\mathcal{O})$ (which is written for $S = U_1(N)$ as $\mathbf{h}_0^{ord}(N;\mathcal{O})$ in [7]) is isomorphic to the torsion free part of $\mathbf{h}^{n.ord}(S;\mathcal{O}) \otimes_{\mathfrak{A}} \mathfrak{A}/\mathfrak{p}$. Thus $\mathbf{h}^{ord}(S;\mathcal{O})$ is a residue algebra of $\mathbf{h}^{n.ord}(S;\mathcal{O})$ by an ideal $\mathcal{P}$. If $\lambda: \mathbf{h}^{n.ord}(S;\mathcal{O}) \to A$ factors through $\mathbf{h}^{ord}(S;\mathcal{O})$, then for the associated Galois representation π, it has been shown by Wiles [17] the restriction of π to $D_\mathfrak{p}$ is reducible and the character δ as in the theorem is unramified.

(iv) Since $\mathbf{G} = \mathfrak{r}_p^\times \times \bar{Z}_0$, we may consider $\mathcal{O}[[\mathbf{G}]] = \mathcal{O}[[\mathfrak{r}_p^\times]] \hat{\otimes}_\mathcal{O} \mathcal{O}[[\bar{Z}_0]]$. When $F = \mathbf{Q}$, $\mathbf{h}^{n.ord}(S:\mathcal{O})$ can be naturally considered as a subalgebra of $\mathbf{h}^{ord}(S;\mathcal{O}) \hat{\otimes}_\mathcal{O} \mathcal{O}[[\mathbf{Z}_p^\times]]$ with the same total quotient ring, because of the nonexistence of the modular forms of multiple weight. When $F = \mathbf{Q}$, we can also identify $\mathbf{Z}_p^\times$ with the Galois group of the cyclotomic extension of all p-power roots of unity by means of the cyclotomic character. Let ι denote the Galois character with values in $\mathcal{O}[[\mathbf{Z}_p^\times]]$ which is the composite of the cyclotomic character with the natural tautological inclusion of $\mathbf{Z}_{p^\times}$ into $\mathcal{O}[[\mathbf{Z}_p]]$. If $\lambda: \mathbf{h}^{ord}(S;\mathcal{O}) \to A$ and $\mu: \mathcal{O}[[\mathbf{Z}_p^\times]] \to B$ are continuous $\mathcal{O}$-algebra homomorphisms and $A \hat{\otimes}_\mathcal{O} B$ is an integral domain in $\mathcal{C}$, then for the ordinary Galois representation π attached to λ, the Galois representation attached to $\lambda \otimes \mu$ is given by $\pi \otimes (\mu \circ \iota)$ (see also [9, Proposition 15]). Thus, when $F = \mathbf{Q}$, there is nothing essentially new here, and Theorem I follows almost directly from [6, Theorem 2.1].

Now let us present a result for the full Hecke algebra $\mathbf{h}(S;\mathcal{O})$. Let $\mathbf{h}$ be the smallest closed subalgebra of $\mathbf{h}(S; \mathcal{O})$ containing $\mathbf{A}$ and $T(\mathfrak{n})$ for all $\mathfrak{n}$ prime to Np. Then it is known that $\mathbf{h}$ is a reduced compact algebra.

THEOREM II. *Let $A \in Ob(\mathcal{C})$ be an integral domain of characteristic different from 2 and Q be the quotient field of A. For any continuous $\mathcal{O}$-algebra homomorphism $\lambda: \mathbf{h} \to A$, there exists a unique Galois representation $\pi(\lambda): \mathrm{Gal}(\bar{\mathbf{Q}}/F) \to GL_2(Q)$ such that:*

(i) *$\pi(\lambda)$ is continuous and semi-simple;*
(ii) *$\pi(\lambda)$ is unramified outside Np;*
(iii) *For the Frobenius element $\phi_{\mathfrak{q}}$ for each prime $\mathfrak{q}$ outside Np, we have*

$$\det(1 - \pi(\lambda)(\phi_{\mathfrak{q}})X) = 1 - \lambda(T(\mathfrak{q}))X + \lambda(\langle \mathfrak{q} \rangle)\mathfrak{N}_{F/\mathbf{Q}}(\mathfrak{q})X^2.$$

This theorem will be proven in Section 3. In view of the duality between Hecke algebra $\mathbf{h}(S; \mathcal{O})$ and the space of p-adic modular forms (cf. [7, Section 5]), this theorem implies that one can associate a canonical Galois representations to any p-adic common eigenform of all Hecke operators including classical cusp forms of weight less than $2t$ (cf. [13] and [17]).

1. Pseudo-representations. Before proving Theorem I in Section 3, we shall explain Wiles' method (in [17] and [14, Section 2]) of patching together the residual representations modulo prime ideals for an integral domain R of characteristic 0 into a representation into R. Let R be a topological commutative ring in which 2 is not a zero divisor. Let G be a compact group, $c \in G$ be a specified element of G of order 2 and $\pi: G \to GL_2(R)$ be a continuous representation with $\det(\pi(c)) = -1$ (such a representation will be called an odd representation with respect to c). Since π is odd, replacing π by a suitable isomorphic representation over the localization $R[2^{-1}]$ of R, we may assume that $\pi: G \to GL_2(R[2^{-1}])$ satisfies the following conditions:

(1.1a) $\mathrm{Tr}(\pi(\sigma))$ and $\det(\pi(\sigma))$ belong to R for all $\sigma \in G$;

(1.1b) $$\pi(c) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

For each $\sigma \in G$, we write $\pi(\sigma) = \left(\begin{smallmatrix} a(\sigma) & b(\sigma) \\ c(\sigma) & d(\sigma) \end{smallmatrix}\right)$ and define continuous functions

$X: G \times G \to R$, $A: G \to R$ and $D: G \to R$ by $X(\sigma, \tau) = 4b(\sigma)c(\tau)$, $A(\sigma) = 2a(\sigma)$ and $D(\sigma) = 2d(\sigma)$. Note that by (1.1b), we see easily that

$$A(\sigma) = (\mathrm{Tr}(\pi(\sigma)) - \mathrm{Tr}(\pi(\sigma c))) \quad \text{and} \quad D(\sigma) = (\mathrm{Tr}(\pi(\sigma)) + \mathrm{Tr}(\pi(\sigma c)))$$

$$\text{and} \quad X(\sigma, \tau) = 2A(\sigma\tau) - A(\sigma)A(\tau).$$

This shows that these functions in fact have values in R. Then this triple $\pi' = (A, D, X)$ of functions satisfies the following properties:

(1.2a) As functions on G or G^2, A, D and X are continuous,

(1.2b) $2A(\sigma\tau) = A(\sigma)A(\tau) + X(\sigma, \tau)$, $\quad 2D(\sigma\tau) = D(\sigma)D(\tau) + X(\tau, \sigma)$

and $4X(\sigma\tau, \rho\gamma) = A(\sigma)A(\gamma)X(\tau, \rho) + A(\gamma)D(\tau)X(\sigma, \rho)$

$$+ A(\sigma)D(\rho)X(\tau, \gamma) + D(\tau)D(\rho)X(\sigma, \gamma),$$

(1.2c) $$A(1) = D(1) = D(c) = 2, \qquad A(c) = -2,$$

$$\text{and} \quad X(\sigma, \rho) = X(\rho, \tau) = 0 \quad \text{if} \quad \rho = 1 \quad \text{and} \quad c,$$

(1.2d) $$X(\sigma, \tau)X(\rho, \eta) = X(\sigma, \eta)X(\rho, \tau),$$

(1.2e) $$A(\sigma) + D(\sigma) \in 2R \quad \text{for all} \quad \sigma \in G.$$

The properties (1.2c–d) follow directly from the definition and the first half of (1.2b) can be proven by computing directly the multiplicative formula:

$$\begin{pmatrix} a(\sigma) & b(\sigma) \\ c(\sigma) & d(\sigma) \end{pmatrix}\begin{pmatrix} a(\tau) & b(\tau) \\ c(\tau) & d(\tau) \end{pmatrix} = \begin{pmatrix} a(\sigma\tau) & b(\sigma\tau) \\ c(\sigma\tau) & d(\sigma\tau) \end{pmatrix}.$$

In addition to the two first formulas of (1.2b), we also have

$$b(\sigma\tau) = a(\sigma)b(\tau) + b(\sigma)d(\tau) \quad \text{and} \quad c(\sigma\tau) = c(\sigma)a(\tau) + d(\sigma)c(\tau).$$

Thus we know that

$$4X(\sigma\tau, \rho\gamma) = 16b(\sigma\tau)c(\rho\gamma) = 16(a(\sigma)b(\tau) + b(\sigma)d(\tau))(c(\rho)a(\gamma)$$

$$+ d(\rho)c(\gamma))$$

$$= A(\sigma)A(\gamma)X(\tau, \rho) + A(\gamma)D(\tau)X(\sigma, \rho)$$

$$+ A(\sigma)D(\rho)X(\tau, \gamma) + D(\tau)D(\rho)X(\sigma, \gamma).$$

For any topological algebra R in which 2 is not a zero divisor, we now define, according to [17, Section 2.2], a *pseudo-representation* of G into R to be a triple $\pi' = (A, D, X)$ consisting of continuous functions on G or G^2 satisfying the conditions (1.2a–e) (when 2 is invertible in R, the condition (1.2e) is superfluous). We define the trace $\mathrm{Tr}(\pi')$ (resp. the determinant $\det(\pi')$) of the pseudo representation π' to be a function on G given by

$$\mathrm{Tr}(\pi')(\sigma) = 2^{-1}(A(\sigma) + D(\sigma))$$

$$(\text{resp. } \det(\pi')(\sigma) = 4^{-1}(A(\sigma)D(\sigma) - X(\sigma, \sigma)).$$

They are all continuous functions with values in R. The following propositions due to Wiles describes how to recover the representation into R with the same trace and determinant out of a pseudo-representation and how pseudo-representations behave in much more coherent way than the representations under the specialization process:

Proposition 1.1. *Let $\pi' = (A, D, X)$ be a pseudo-representation of G into an integral domain R with quotient field Q. Suppose that Q is not of characteristic 2. Then there exists a representation $\pi: G \to GL_2(Q)$ with the same trace and determinant as π'. Moreover if R is an object of $\mathcal{C}$, then π is continuous.*

Proof. We repeat here the proof in [17, Section 2.2]. We divide our argument into two cases:

Case 1. The case when there exists $\rho, \gamma \in G$ such that $X(\rho, \gamma) \neq 0$. Then we define $\pi(\sigma) = \begin{pmatrix} a(\sigma) & b(\sigma) \\ c(\sigma) & d(\sigma) \end{pmatrix}$ by

$$a(\sigma) = 2^{-1}A(\sigma), \qquad d(\sigma) = 2^{-1}D(\sigma), \qquad c(\sigma) = 4^{-1}X(\rho, \sigma)$$

$$\text{and} \quad b(\sigma) = X(\sigma, \gamma)/X(\rho, \gamma).$$

Write $x(\sigma, \tau)$ for $4^{-1}X(\sigma, \tau)$. Then

$$b(\sigma)c(\tau) = x(\rho, \sigma)x(\tau, \gamma)/x(\rho, \gamma) = x(\sigma, \tau)$$

by the property (1.2d) of pseudo representation. Thus we know that the entry of $\pi(\sigma)\pi(\tau)$ at the upper left corner is equal to, by (1.2b),

$$a(\sigma)a(\tau) + b(\sigma)c(\tau) = a(\sigma)a(\tau) + x(\sigma, \tau) = a(\sigma\tau).$$

Similarly the lower right corner of $\pi(\sigma)\pi(\tau)$ is equal to

$$d(\sigma)d(\tau) + b(\tau)c(\sigma) = d(\sigma)d(\tau) + x(\tau, \sigma) = d(\sigma\tau).$$

We now compute lower left corner of $\pi(\sigma)\pi(\tau)$, which is given by

$$c(\sigma)a(\tau) + d(\sigma)c(\tau) = x(\rho, \sigma)a(\tau) + d(\sigma)x(\rho, \tau).$$

By applying the last formula in (1.2b) to $(1, \rho, \sigma, \tau)$, we have

$$c(\sigma\tau) = x(\rho, \sigma\tau) = a(\tau)x(\rho, \sigma) + d(\sigma)x(\rho, \tau),$$

since $x(1, \sigma) = x(1, \tau) = 0$ by (1.2c). This shows that

$$c(\sigma\tau) = c(\sigma)a(\tau) + d(\sigma)c(\tau).$$

Similarly by applying the same formula in (1.2b) to $(\sigma, \tau, 1, \gamma)$, we have

$$b(\sigma\tau)x(\rho, \gamma) = x(\sigma\tau, \gamma) = a(\sigma)x(\tau, \gamma) + d(\tau)x(\sigma, \gamma)$$

$$= (a(\sigma)b(\tau) + d(\tau)b(\sigma))x(\rho, \gamma),$$

which finishes the proof of the formula $\pi(\sigma)\pi(\tau) = \pi(\sigma\tau)$. Obviously, by definition, $\pi(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and hence π is the desired representation.

Case 2. The case when $X(\sigma, \tau) = x(\sigma, \tau) = 0$ for all σ, τ in G. In this case, by (1.2b), we have $a(\sigma)a(\tau) = a(\sigma\tau)$ and $d(\sigma)d(\tau) = d(\sigma\tau)$ for all $\sigma, \tau \in G$. Then we simply put

$$\pi(\sigma) = \begin{pmatrix} a(\sigma) & 0 \\ 0 & d(\sigma) \end{pmatrix}$$

which does the job. Now assuming that $R \in Ob(\mathcal{C})$, we shall prove the continuity of π in Case 1 because in Case 2, the continuity is obvious. Let $\mathfrak{a}$ be the ideal of R generated by $c(\tau)$ for all $\tau \in G$. Since $b(\sigma)c(\tau) = x(\sigma, \tau) \in 4^{-1}R$ for all τ by our construction, $b(\sigma) \in 4^{-1}\mathfrak{a}^{-1}$ for all $\sigma \in G$. Thus $\mathrm{Im}(\pi)$ is contained in

$$\begin{pmatrix} 2^{-1}R & 4^{-1}\mathfrak{a}^{-1} \\ \mathfrak{a} & 2^{-1}R \end{pmatrix}.$$

Thus we put $L_0 = \left\{\binom{a}{b} \middle| a \in R \text{ and } b \in \mathfrak{a}\right\}$, then $L = \Sigma_{\sigma\in G}\, \pi(\sigma)L_0$ is contained in $4^{-1}L_0$. Since A is noetherian, L is finitely generated over A, and since L contains L_0, $L \otimes_A Q = Q^2$. By definition, L is stable under π. Thus, according to the definition of continuity given above Theorem I, π is continuous.

Remark 1.2. Let the notation be as in Proposition 1.1 and its proof. Assume that R is an object in $\mathcal{C}$ and the residual characteristic of R is different from 2. By the above proof, it is clear that if a pseudo-representation $\rho = (a, d, x)\colon G \to R$ has $\sigma, \tau \in G$ such that $x(\sigma, \tau)$ is a unit of R, then we can construct a representation $\pi\colon G \to GL_2(R)$ out of ρ with the same trace and determinant.

PROPOSITION 1.3. *Suppose that R be a product of finitely many objects in $\mathcal{C}$ (thus, R may not be local). Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of R. Let $\pi(\mathfrak{a})$ and $\pi(\mathfrak{b})$ be pseudo representations of G into $R/\mathfrak{a}$ and $R/\mathfrak{b}$, respectively. Suppose that $\pi(\mathfrak{a})$ and $\pi(\mathfrak{b})$ are compatible; namely, there exist functions* tr *and* det *on a dense subset Σ of G with values in $R/\mathfrak{a} \cap \mathfrak{b}$ such that for all $\sigma \in \Sigma$,*

$$\mathrm{Tr}(\pi(\mathfrak{a})(\sigma)) \equiv \mathrm{tr}(\sigma) \bmod \mathfrak{a} \quad \textit{and} \quad \mathrm{Tr}(\pi(\mathfrak{b})(\sigma)) \equiv \mathrm{tr}(\sigma) \bmod \mathfrak{b}$$

$$\det(\pi(\mathfrak{a})(\sigma)) \equiv \det(\sigma) \bmod \mathfrak{a} \quad \textit{and} \quad \det(\pi(\mathfrak{b})(\sigma)) \equiv \det(\sigma) \bmod \mathfrak{b}.$$

Then there exists a pseudo representation $\pi(\mathfrak{a} \cap \mathfrak{b})$ of G into $R/\mathfrak{a} \cap \mathfrak{b}$ such that $\mathrm{Tr}(\pi(\mathfrak{a} \cap \mathfrak{b})(\sigma)) = \mathrm{tr}(\sigma)$ *and* $\det(\pi(\mathfrak{a} \cap \mathfrak{b})(\sigma)) = \det(\sigma)$ *on Σ.*

Proof. We again repeat the proof given in [17]. We consider the exact sequence:

$$0 \to R/\mathfrak{a} \cap \mathfrak{b} \to R/\mathfrak{a} \oplus R/\mathfrak{b} \xrightarrow{\alpha} R/\mathfrak{a} + \mathfrak{b} \to 0$$

$$a \mapsto a \bmod \mathfrak{a} \oplus a \bmod \mathfrak{b}$$

$$a \oplus b \mapsto a - b \bmod \mathfrak{a} + \mathfrak{b}.$$

We consider the pseudo representation $\pi = \pi(\mathfrak{a}) \oplus \pi(\mathfrak{b})$ with values in $R/\mathfrak{a} \oplus R/\mathfrak{b}$. The function $\alpha\circ\mathrm{Tr}(\pi)$ vanishes constantly on Σ. Since this function is continuous on G and Σ is dense in G, $\alpha\circ\mathrm{Tr}(\pi)$ vanishes on G. Thus $\mathrm{Tr}(\pi)$ has values in $R/\mathfrak{a} \cap \mathfrak{b}$. If we write $\pi = (A, D, X)$, then $A(\sigma) = (\mathrm{Tr}(\pi(\sigma)) - \mathrm{Tr}(\pi(\sigma c)))$ and $D(\sigma) = (\mathrm{Tr}(\pi(\sigma)) + \mathrm{Tr}(\pi(\sigma c)))$ and $X(\sigma, \tau) = 2A(\sigma\tau) - A(\sigma)A(\tau)$. Thus π itself has values in $R/\mathfrak{a} \cap \mathfrak{b}$ and gives the desired pseudo representation.

THEOREM 1.4 (Wiles). *Let R be a topological $\mathcal{O}$-algebra and $\{\mathfrak{p}_i\}_{i=1}^{\infty}$ a countable set of ideals of R such that $R/\mathfrak{p}_i$ belongs to $\mathcal{C}$ for all i. Suppose that* (i) $R = \varprojlim_{\alpha} R/(\cap_{i=1}^{\alpha} \mathfrak{p}_i)$ *as a topological algebra and* (ii) *there exist a dense subset Σ of G, functions* $\mathrm{tr}\colon \Sigma \to R$ *and* $\det\colon \Sigma \to R$ *and a pseudo representation $\pi_i = (A_i, D_i, X_i)\colon G \to R/\mathfrak{p}_i$ such that*

$$\mathrm{Tr}(\pi_i(\sigma)) = \mathrm{tr}(\sigma) \bmod \mathfrak{p}_i \quad \textit{and} \quad \det(\pi_i(\sigma)) = \det(\sigma) \bmod \mathfrak{p}_i$$

for all $\sigma \in \Sigma$. Then there exists a unique pseudo-representation $\pi = (A, D, X)\colon G \to R$ such that $\pi(\sigma) \bmod \mathfrak{p}_i = \pi_i(\sigma)$ for all i on Σ. Moreover, if $\lambda\colon R \to A$ is a continuous algebra homomorphism into an integral domain A in $\mathcal{C}$ of characteristic different from 2, then there exists a semi-simple continuous Galois representation $\Pi\colon G \to GL_2(Q)$ for the quotient field Q of A such that $\det(1 - \Pi(\sigma)X) = 1 - \lambda(\mathrm{Tr}(\pi(\sigma))X + \lambda(\det(\pi(\sigma))X^2$ *for all $\sigma \in \Sigma$.*

Proof. Write $\mathcal{P}_\alpha$ for $\cap_{i=1}^{\alpha} \mathfrak{p}_i$. Then by our assumption, we know that $R = \varprojlim_{\alpha} R/\mathcal{P}_\alpha$ and $R/\mathcal{P}_\alpha$ satisfies the assumption of Proposition 1.2. By Proposition 1.3, we can lift inductively the pseudo representations π_i for $i = 1, \ldots, \alpha$ to a pseudo-representation $\pi_\alpha = (A_\alpha, D_\alpha, X_\alpha)\colon G \to R/\mathcal{P}_\alpha$ such that $\mathrm{Tr}(\pi_\alpha(\sigma)) = \mathrm{tr}(\sigma) \bmod \mathcal{P}_\alpha$ and $\det(\pi_\alpha(\sigma)) = \det(\sigma) \bmod \mathcal{P}_\alpha$. Since we have

$$A_\alpha(\sigma) = \mathrm{Tr}(\pi_\alpha(\sigma)) - \mathrm{Tr}(\pi_\alpha(\sigma c)), \qquad D_\alpha(\sigma) = \mathrm{Tr}(\pi_\alpha(\sigma)) + \mathrm{Tr}(\pi_\alpha(\sigma c))$$

and

$$X_\alpha(\sigma, \tau) = 2A_\alpha(\sigma\tau) - A_\alpha(\sigma)A_\alpha(\tau) \quad \text{on} \quad \Sigma,$$

we see that

$$\pi_\alpha(\sigma) \equiv \pi_\beta(\sigma) \bmod \mathcal{P}_\alpha \quad \text{on} \quad \Sigma \quad \text{if} \quad \alpha < \beta.$$

Since Σ is dense, the above identity holds on the whole G, and thus, defining $\pi(\sigma) = \varprojlim_\alpha \pi_\alpha(\sigma)$ for each $\sigma \in G$, we obtain the desired pseudo representation π. The last assertion follows from Proposition 1.1 applied to the pseudo representation $\lambda \circ \pi \colon G \to A$.

2. Representation theoretic preliminaries. In this section, we gather several representation theoretic results necessary to the proof of Theorems I and II.

Let $\mathfrak{p}$ be a prime ideal of $\mathfrak{r}$ and consider an admissible irreducible representation ρ of $GL_2(F_\mathfrak{p})$ on a infinite dimensional $\mathbf{C}$-vector space $\mathbf{U}$. We first quote some well known result on local representations:

PROPOSITION 2.1 (Casselman). *Suppose that ρ is irreducible. Let $c(\rho)$ be the largest ideal of $\mathfrak{r}_\mathfrak{p}$ such that the subspace V of all vectors fixed by*

$$U_0(c(\rho))_\mathfrak{p} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathfrak{r}_\mathfrak{p}) \middle| c \in c(\rho) \right\}$$

is nontrivial. Then this subspace V has dimension 1 and on it the Hecke operator $T(\mathfrak{p})$ acts as follows:

(i) *If ρ is absolutely cuspidal, then $c(\rho)$ is divisible by $\mathfrak{p}^2$ and $T(\mathfrak{p}) = 0$ on V;*

(ii) *Suppose that ρ is a principal series representation $\pi(\xi, \eta)$ with characters ξ and η of $F_\mathfrak{p}^\times$. Then $c(\rho)$ is the product of the conductor of ξ and η. If both the characters ξ and η are unramified, then $T(\mathfrak{p})$ acts on V by the multiplication of $\xi(\bar\omega) + \eta(\bar\omega)$ for a prime element $\bar\omega$ of $\mathfrak{r}_\mathfrak{p}$. If ξ is unramified and η is ramified, then $T(\mathfrak{p})$ acts on V by the multiplication of $\xi(\bar\omega)$ for a prime element $\bar\omega$ of $\mathfrak{r}_\mathfrak{p}$. If both the characters are ramified, then $T(\mathfrak{p})$ annihilates V;*

(iii) *Suppose that ρ is a special representation $\sigma(\xi, \xi\alpha^{-1})$ for characters ξ and α of $F_\mathfrak{p}^\times$ with $\alpha(x) = |x|_\mathfrak{p}$. Then, if ξ is unramified $c(\rho) = \mathfrak{p}$ and $T(\mathfrak{p})$ acts on V by the multiplication of $\xi(\bar\omega)$, and if ξ is ramified, then $c(\rho)$ is the square of the conductor of ξ and $T(\mathfrak{p})$ annihilates V.*

Here $T(\mathfrak{p})$ acts on $\mathbf{U}^U$ for $U = U_0(\mathfrak{p}^r)_\mathfrak{p}$ by $\mathbf{u}|T(\mathfrak{p}) = \Sigma_x \rho(x^{-1})\mathbf{u}$ for a representative set $\{x\}$ of left coset decomposition $U \backslash (U\left(\begin{smallmatrix} 1 & 0 \\ 0 & \bar\omega \end{smallmatrix}\right)U)$. Usually in the references, the representation $\pi(\xi, \eta)$ here is written as $\rho(\alpha^{-1/2}\xi^{-1}, \alpha^{-1/2}\eta^{-1})$ and is the induced representation of the character: $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \mapsto \alpha^{-1/2}(ad)\xi^{-1}(a)\eta^{-1}(d)$ from the standard Borel subgroup to $GL_2(F_\mathfrak{p})$ (cf. [1, 0.5]). Similar notational convention also applies to $\sigma(\xi, \xi\alpha^{-1})$. Under this convention, the above result is just an interpretation of the result shown in [2]. We now put for each integer r

$$U_r = U(\mathfrak{p}^r)_\mathfrak{p} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_0(\mathfrak{p}^r)_\mathfrak{p} \middle| a \equiv d \equiv 1 \bmod \mathfrak{p}^r\mathfrak{r}_\mathfrak{p} \right\}.$$

Then $\mathbf{G}_r = U_0(\mathfrak{p}^r)_\mathfrak{p}/U_r$ acts naturally on $V_r = \mathbf{U}^{U_r}$. We normalize the action of $\mathbf{G}_r$ as $\mathbf{u}|g = \rho(g^{-1})\mathbf{u}$, which is a right action. Similarly, by choosing a prime element $\bar\omega$ of $\mathfrak{r}_\mathfrak{p}$, we can define a Hecke operator $T(\bar\omega)$ by

$$\mathbf{u}|T(\bar\omega) = \sum_x \rho(x^{-1})\mathbf{u}$$

for a representative set $\{x\}$ of left coset decomposition $U_r \backslash (U_r\left(\begin{smallmatrix} 1 & 0 \\ 0 & \bar\omega \end{smallmatrix}\right)U_r)$.

COROLLARY 2.2. *Let the notation be as in Proposition 2.1 and suppose that ρ is irreducible and that $V_r \neq 0$ and $T(\bar\omega)$ on V_r is not nilpotent for some large r. Then,*

(i) *ρ is isomorphic to either a principal series representation $\pi(\xi, \eta)$ or a special representation $\sigma(\xi, \eta)$.*

(ii) *Suppose that $\rho = \pi(\xi, \eta)$ and write $\mathfrak{p}^i$ (resp. $\mathfrak{p}^j$, $\mathfrak{p}^k$) for the conductor of $\xi^{-1}\eta$ (resp. ξ, η). Then we have $r \geq \max(j, k) \geq i$. When $i = j = k = r = 0$, then $T(\bar\omega)$ acts on V_0 by the multiplication of $\xi(\bar\omega) + \eta(\bar\omega)$, $\dim_\mathbf{C} V_0 = 1$. When $r > 0$ and $\xi \neq \eta$, then we can decompose $V_r = V(\xi) \oplus V(\eta) \oplus N$ so that $\dim_\mathbf{C} V(\xi) = \dim_\mathbf{C} V(\eta) = 1$, and on $V(\xi)$ (resp. $V(\eta)$), $U_0(\mathfrak{p}^r)_\mathfrak{p}$ acts by $\mathbf{u}|\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \xi(a)\eta(d)\mathbf{u}$ (resp. $\mathbf{u}|\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \eta(a)\xi(d)\mathbf{u}$), and on N (resp. on $V(\xi)$ and $V(\eta)$), $T(\bar\omega)$ acts as a nilpotent operator (resp. by the multiplication of $\xi(\bar\omega)$ and $\eta(\bar\omega)$). When $r > 0$ and $\xi = \eta$, then we can decompose $V_r = V(\xi) \oplus N$ so that $\dim_\mathbf{C} V(\xi) = 2$, and on $V(\xi)$, $U_0(\mathfrak{p}^r)_\mathfrak{p}$ acts by $\mathbf{u}|\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \xi(ad)\mathbf{u}$ and on $V(\xi)$ (resp. on N), $T(\bar\omega)$ acts nonsemisimply with unique eigenvalue $\xi(\bar\omega)$ (resp. as a nilpotent operator).*

(iii) *Suppose that $\rho = \sigma(\xi, \xi\alpha^{-1})$ and write $\mathfrak{p}^j$ for the conductor of ξ. then we have $r \geq \max(j, 1)$, and we can decompose $V_r = V(\xi) \oplus N$ such that on V_r, $U_0(\mathfrak{p}^r)_\mathfrak{p}$ acts via $\mathbf{u}|\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \xi(ad)\mathbf{u}$ and $T(\bar\omega)$ acts on N (resp. on*

$V(\xi)$) as a nilpotent operator (resp. by the multiplication of $\xi(\varpi)$). We also have $\dim_{\mathbf{C}} V(\xi) = 1$.

Proof. Define a character ω by $\mathbf{u}|z = \rho(z^{-1})\mathbf{u} = \omega(z)\mathbf{u}$ for $z \in F_{\mathfrak{p}}^{\times}$. Then we can decompose $V_r = \oplus_{(\chi,\psi)} V(\chi, \psi)$ so that $\mathbf{G}_r$ acts on $V(\chi, \psi)$ by $\mathbf{u}|\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \chi(a)\psi(d)\mathbf{u}$. Then $\chi\psi = \omega$ on $\mathfrak{r}_{\mathfrak{p}}^{\times}$. Extending χ to a character of $F_{\mathfrak{p}}^{\times}$, we now modify the action of ρ and consider $\chi \otimes \rho$ whose action on $\mathbf{u} \in \mathbf{U}$ is given by $\chi \otimes \rho(x)\mathbf{u} = \chi(\det(x))\rho(x)\mathbf{u}$. Then, on $V(\chi, \psi)$, we see $\mathbf{u}|\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \chi \otimes \rho\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^{-1}\mathbf{u} = \psi\chi^{-1}(d)\mathbf{u}$. On the other hand, we have for $z \in F_{\mathfrak{p}}^{\times}$,

$$\mathbf{u}|z = \chi \otimes \rho(z^{-1})\mathbf{u} = \omega\chi^{-2}(z)\mathbf{u} = \psi\chi^{-1}(z)\mathbf{u}.$$

We write the subspace $V(\chi, \psi)$ under the action of ρ as $V_\rho(\chi, \psi)$ to avoid confusion. Then by the above argument, the identity map ι induces an isomorphism ι: $V_\rho(\chi, \psi) \cong V_{\chi\otimes\rho}(1, \psi\chi^{-1})$. We see easily, for the Hecke operator $T_\rho(\varpi)$ under the action of ρ,

$$(*) \qquad \iota(\mathbf{u}|T_\rho(\varpi)) = \chi(\varpi)\iota(\mathbf{u})|T_{\chi\otimes\rho}(\varpi).$$

Thus if ρ is super cuspidal, then $\chi \otimes \rho$ is also super cuspidal for any character χ by definition, and therefore, by Proposition 2.1, $T_{\chi\otimes\rho}(\varpi)$ is nilpotent on $V_{\chi\otimes\rho}(1, \psi\chi^{-1})$ (see [5, Lemma 3.3] and the argument below). Then by (*), $T_\rho(\varpi)$ is nilpotent on $V_\rho(\chi, \psi)$. Thus if $T(\varpi)$ is not nilpotent on V_r for some r, then ρ must be principal or special. Now assume that $\rho = \pi(\xi, \eta)$. Then $\chi \otimes \rho = \pi(\chi^{-1}\xi, \chi^{-1}\eta)$. Thus, by Proposition 2.1, $T_{\chi\otimes\rho}(\varpi)$ is not nilpotent on $V_{\chi\otimes\rho}(1, \psi\chi^{-1})$ if and only if either $\chi^{-1}\xi$ or $\chi^{-1}\eta$ is unramified. Thus we may assume that either $\chi = \xi$ or $\chi = \eta$. Since $\pi(\xi, \eta) \cong \pi(\eta, \xi)$, the argument is the same in either case and thus hereafter, we assume that $\chi = \xi$. Then $\psi = \eta$ and thus we have $r \geq \max(j, k) \geq i$. The assertion in the case $i = j = k = r = 0$ follows directly from Proposition 2.1, (ii). Now we treat the remaining case where $r > 0$. Since $\chi \otimes \rho = \pi(1, \xi^{-1}\eta)$, by Proposition 2.1, $V_{\chi\otimes\rho}(1, \xi^{-1}\eta)$ has dimension $r - i + 1 > 0$ if $\mathfrak{p}^i$ is the conductor of $\xi^{-1}\eta$. Let $\mathbf{u}_0$ be a nontrivial vector in $V_{\chi\otimes\rho}(1, \xi^{-1}\eta)$ fixed by $U_1(\mathfrak{p}^i)$. Then it is well known that

$$\mathbf{u}_n = \xi \otimes \rho\left(\begin{pmatrix} \varpi^{-n} & 0 \\ 0 & 1 \end{pmatrix}\right)\mathbf{u}_0$$

for $n = 0, \ldots, r - i$ gives a basis of $V_{\chi\otimes\rho}(1, \xi^{-1}\eta)$. When $i = 0$, we define inductively

$$\mathbf{w}_0 = \mathbf{u}_0 - \mathbf{u}_1, \qquad \mathbf{w}_1 = \mathbf{u}_0 - \xi\eta^{-1}(\varpi)\mathbf{u}_1,$$

$$\mathbf{w}_2 = \mathbf{w}_1 - \xi \otimes \rho\left(\begin{pmatrix} \varpi^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right)\mathbf{w}_1 \quad \text{and}$$

$$\mathbf{w}_n = \xi \otimes \rho\left(\begin{pmatrix} \varpi^{-n+2} & 0 \\ 0 & 1 \end{pmatrix}\right)\mathbf{w}_2 \quad \text{for} \quad 2 \leq n \leq r - i.$$

Then one see easily (e.g. [5, Lemma 3.3]) that if $\xi \neq \eta$, then $\mathbf{w}_n$'s form a basis of $V_{\chi\otimes\rho}(1, \xi^{-1}\eta)$ and

$$\mathbf{w}_0|T(\varpi) = \mathbf{w}_0, \qquad \mathbf{w}_1|T(\varpi) = \xi^{-1}\eta(\varpi)\mathbf{w}_1, \qquad \mathbf{w}_2|T(\varpi) = 0$$

$$\text{and} \quad \mathbf{w}_n|T(\varpi) = \mathbf{w}_{n-1} \quad \text{if} \quad n > 2.$$

When $i = 0$ and $\xi = \eta$, then $\mathbf{w}_0, \mathbf{u}_1, \mathbf{w}_2, \ldots, \mathbf{w}_{r-i}$ form a basis and $T(\varpi)$ acts on $\mathbf{C}\mathbf{u}_1 + \mathbf{C}\mathbf{w}_0$ via an unipotent matrix and on $\mathbf{C}\mathbf{w}_2 + \cdots + \mathbf{C}\mathbf{w}_{r-i}$ as a nilpotent operator. When $i > 0$, we similarly define

$$\mathbf{w}_0 = \mathbf{u}_0 - \mathbf{u}_1, \qquad \mathbf{w}_1 = \xi\eta^{-1}(\varpi)\mathbf{w}_0 - \xi \otimes \rho\left(\begin{pmatrix} \varpi^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right)\mathbf{w}_0$$

$$\text{and} \quad \mathbf{w}_n = \xi \otimes \rho\left(\begin{pmatrix} \varpi^{-n+2} & 0 \\ 0 & 1 \end{pmatrix}\right)\mathbf{w}_1.$$

Then again $\mathbf{w}_n$'s form a basis of $V_{\chi\otimes\rho}(1, \xi^{-1}\eta)$ and satisfies $\mathbf{w}_0|T(\varpi) = \mathbf{w}_0$ and $\mathbf{w}_1|T(\varpi) = 0$ and $\mathbf{w}_n|T(\varpi) = \mathbf{w}_{n-1}$ if $n > 1$. This shows the desired assertion by (*) for principal series representations. Similar argument shows the assertion also for special representations.

Now consider a common eigenform $\mathbf{f}$ in $\mathbf{S}_{2t,t}(S(p^\alpha); \mathbf{C})$ ($t = \Sigma_\sigma \sigma \in \mathbf{Z}[I]$) of all Hecke operators $T(\mathfrak{n})$ for $\mathfrak{n}$ prime to p and $T(\mathfrak{p})$ for all prime factors $\mathfrak{p}$ of p. The operators $T(\mathfrak{q})$ for primes $\mathfrak{q}$ are defined as follows: We choose a prime element ϖ of $\mathfrak{r}_{\mathfrak{q}}$ and consider it as an element of $F_{\mathbf{A}}^{\times}$ and define $T(\mathfrak{q})$ by the action of the double coset $S(p^\alpha)\left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi \end{smallmatrix}\right)S(p^\alpha)$ on $\mathbf{S}_{2t,t}(S(p^\alpha); \mathbf{C})$; namely, for a representative set $\{x\}$ for $S(p^\alpha)\backslash\left(S(p^\alpha)\left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi \end{smallmatrix}\right)S(p^\alpha)\right)$, $\mathbf{f}|T(\mathfrak{q})(y) = \Sigma_x \mathbf{f}(yx^{-1})$. We define the Hecke operator $T(p)$ by $\Pi_{\mathfrak{p}|p} T(\mathfrak{p})^{a(\mathfrak{p})}$ using the prime decomposition $p = \Pi_{\mathfrak{p}|p}\mathfrak{p}^{a(\mathfrak{p})}$ in $\mathfrak{r}$. The Hecke operator $T(\mathfrak{p})$ and $T(p)$ may depend on the choice of ϖ, but $T(\mathfrak{n})$ for $\mathfrak{n}$ prime to p is independent of the choice of $\bar{\omega}$. More generally, one can define a Hecke operator $T(x)$ for each element $0 \neq x \in \mathfrak{r}_\mathfrak{p}$ with nonnegative valuation by the action of double coset $S(p^\alpha)\left(\begin{smallmatrix}1 & 0\\ 0 & x\end{smallmatrix}\right)S(p^\alpha)$. The limit $e = e_\alpha = \lim_{n\to\infty} T(p)^{n!} = \lim_{n\to\infty} T(x)^{n!}$ for $0 \neq x \in p\mathfrak{r}_p$ exists in $\mathfrak{h}_{2t,t}(S(p^\alpha); \mathcal{O})$ and is independent of the choice of x. Then by definition,

$$\mathfrak{h}_{2t,t}{}^{\mathrm{n.ord}}(S(p^\alpha); \mathcal{O}) = e_\alpha \mathfrak{h}_{2t,t}(S(p^\alpha); \mathcal{O}).$$

Note that the idempotent e is in fact algebraic and gives an element of $\mathfrak{h}_{2t,t}(S(p^\alpha); \bar{\mathbf{Q}})$. Thus e acts on $\mathbf{S}_{2t,t}(S(p^\alpha); \mathbf{C})$ and we put $\mathbf{S}_{2t,t}{}^{\mathrm{n.ord}}(S(p^\alpha); \mathbf{C}) = e\mathbf{S}_{2t,t}(S(p^\alpha); \mathbf{C})$. Assume that $\mathfrak{f} \in \mathbf{S}_{2t,t}{}^{n.\mathrm{ord}}(S(p^\alpha); \mathbf{C})$ and let $\rho = \otimes_\mathfrak{q} \rho_\mathfrak{q}$ be the automorphic representation on the linear span of all right translations of $\mathfrak{f}$ by the elements of $GL_2(F_\mathbf{A})$. Then by the strong multiplicity one theorem [2, Theorem 2], ρ is irreducible and by Corollary 2.2, $\rho_\mathfrak{p}$ for each prime factor $\mathfrak{p}$ of p is principal or special because $T(\mathfrak{p})$ acts nontrivially on $\mathfrak{f}$ by near ordinarity. Thus we write $\rho_\mathfrak{p} = \pi(\xi_\mathfrak{p}, \eta_\mathfrak{p})$ if $\rho_\mathfrak{p}$ is principal and $\rho_\mathfrak{p} = \sigma(\xi_\mathfrak{p}, \eta_\mathfrak{p})$ if $\rho_\mathfrak{p}$ is special. For prime $\mathfrak{q}$ outside Np, $\rho_\mathfrak{q}$ is spherical and hence, we can write $\rho_\mathfrak{q} = \pi(\xi_\mathfrak{q}, \eta_\mathfrak{q})$ for all primes $\mathfrak{q}$ outside Np. We may assume by Corollary 2.2 that $|\xi_\mathfrak{p}(\bar{\omega}_\mathfrak{p})|_p = 1$ in $\bar{\mathbf{Q}}_p$. Then by Corollary 2.2, $\mathfrak{f}|T(x) = \xi_\mathfrak{p}(x)\mathfrak{f}$ for all $0 \neq x \in \mathfrak{p}\mathfrak{r}_\mathfrak{p}$. Now we want to find a finite order character χ of $F_\mathbf{A}^\times/F^\times$ such that $\chi_\mathfrak{p} = \xi_\mathfrak{p}$ on $\mathfrak{r}_\mathfrak{p}^\times$ for all prime factors $\mathfrak{p}$ of p and unramified at every infinite place. In fact, we consider the restriction $\xi_\mathfrak{p}: \mathfrak{r}^\times \to \mathbf{C}^\times$ and denote by m the order of the image $\xi_\mathfrak{p}(\mathfrak{r}^\times)$. Then by a theorem of Chevalley [3], we can find an integral ideal $\mathfrak{q}(\mathfrak{p})$, which is prime to pN for any given integer N, so that if $\epsilon \equiv 1 \bmod \mathfrak{q}(\mathfrak{p})$ for $\epsilon \in \mathfrak{r}^\times$, then ϵ is an m-th power in $\mathfrak{r}^\times$. Thus the kernel of $\xi_\mathfrak{p}$ on $\mathfrak{r}^\times$ contains $\{\delta \in \mathfrak{r}^\times | \delta \equiv 1 \bmod \mathfrak{q}(\mathfrak{p})\}$. Thus we can find a finite order character $\psi_\mathfrak{p}$ of $F_{\mathfrak{q}(\mathfrak{p})}^\times$ such that $\psi_\mathfrak{p}\xi_\mathfrak{p}$ is trivial on $\mathfrak{r}^\times$. Then we can extend $\psi_\mathfrak{p}\xi_\mathfrak{p}$ on $\mathfrak{r}_{\mathfrak{q}(\mathfrak{p})}^\times \times \mathfrak{r}_\mathfrak{p}^\times$ to a finite order character $\chi^\mathfrak{p}$ of $F_\mathbf{A}^\times/F^\times$ whose conductor is a product of a power of $\mathfrak{p}$ and $\mathfrak{q}(\mathfrak{p})$. We may take $\mathfrak{q}(\mathfrak{p})$'s for $\mathfrak{p}|p$ so that they are mutually prime each other. Then $\chi = \prod_{\mathfrak{p}|p} \chi^\mathfrak{p}$ satisfies the required property. Let A be a subring of $\bar{\mathbf{Q}}_p$ generated over $\mathcal{O}$ by the eigenvalue $\lambda(h)$ of $\mathfrak{f}$ for all $h \in \mathfrak{h}_{k,w}{}^{n.\mathrm{ord}}(S(p^\alpha); \mathcal{O})$ and Q be the quotient field of A. Let $\sigma = \sigma(\rho)$ be the Galois representation corresponding to ρ (see [14, Theorem 2]) of $GL_2(\bar{\mathbf{Q}}/F)$ into $GL_2(Q)$ such that

(2.1a) *σ is continuous, irreducible and unramified outside Np,*

(2.1b) *For the Frobenius element $\phi_\mathfrak{q}$ for prime $\mathfrak{q}$ outside Np,*

$$\det(1 - \sigma(\phi_\mathfrak{q})X) = (1 - \xi_\mathfrak{q}(\bar{\omega}_\mathfrak{q})X)(1 - \eta_\mathfrak{q}(\bar{\omega}_\mathfrak{q})X).$$

We define an algebra homomorphism $\lambda: \mathfrak{h}_{k,w}{}^{n.\mathrm{ord}}(S(p^\alpha); \mathcal{O}) \to A$ by $\mathfrak{f}|h = \lambda(h)\mathfrak{f}$. Then we have $\lambda(T(\mathfrak{q})) = \xi_\mathfrak{q}(\bar{\omega}_\mathfrak{q}) + \eta_\mathfrak{q}(\bar{\omega}_\mathfrak{q})$ and $\lambda(\langle \mathfrak{q}\rangle)\mathfrak{N}_{F/\mathbf{Q}}(\mathfrak{q}) = \xi_\mathfrak{q}(\bar{\omega}_\mathfrak{q})\eta_\mathfrak{q}(\bar{\omega}_\mathfrak{q})$. Thus this representation σ is the representation as in Theorem I associated to λ. We consider $\chi \otimes \rho$, which is again a cuspidal automorphic representation. Then its component at $\mathfrak{p}$ is given either $\pi(\chi_\mathfrak{p}{}^{-1}\xi_\mathfrak{p}, \chi_\mathfrak{p}{}^{-1}\eta_\mathfrak{p})$ or $\sigma(\chi_\mathfrak{p}{}^{-1}\xi_\mathfrak{p}, \chi_\mathfrak{p}{}^{-1}\eta_\mathfrak{p})$. Since $\mathfrak{f}$ is of weight $2t$, if $\chi_\mathfrak{p}{}^{-1}\eta_\mathfrak{p}$ is unramified and if $\rho_\mathfrak{p}$ is principal, then $\chi_\mathfrak{p}{}^{-2}\xi_\mathfrak{p}\eta_\mathfrak{p}(\bar{\omega}_\mathfrak{p}) = \mathfrak{N}_{F/\mathbf{Q}}(\mathfrak{p})\chi^{-2}\psi(\mathfrak{p})$ for the central character ψ (which is of finite order) of ρ. Thus $\chi_\mathfrak{p}{}^{-1}\xi_\mathfrak{p} \neq \chi_\mathfrak{p}{}^{-1}\eta_\mathfrak{p}$, and without assuming the unramifiedness of $\chi_\mathfrak{p}{}^{-1}\eta_\mathfrak{p}$, the subspace in the representation space of $\rho_\mathfrak{p}$ on which $T(\bar{\omega}_\mathfrak{p})$ has p-adic unit eigenvalue is one dimensional. Since χ is of finite order, its values are roots of unity and hence $|\chi_\mathfrak{p}{}^{-1}\xi_\mathfrak{p}(\bar{\omega}_\mathfrak{p})|_p = 1$ and hence by Proposition 2.1 (and [7, Lemma 12.2]), the primitive form $\mathfrak{f}_0$ belonging to $\chi \otimes \rho$ is ordinary; namely, for the global conductor $c(\chi \otimes \rho)$, the subspace of the representation space of $\chi \otimes \rho$ realized on automorphic forms on $GL_2(F_\mathbf{A})$ fixed by $U_1(c(\chi \otimes \rho))$ is one dimensional and on which $T(p)$ acts by a p-adic unit. Note that $\sigma(\chi \otimes \rho) = \chi^{-1} \otimes \sigma(\rho)$. Then, by a theorem of Wiles [17, Theorem 2], on $D_\mathfrak{p}$, we have, up to equivalence

$$\sigma(\chi \otimes \rho)(\tau) = \begin{pmatrix} \epsilon'(\tau) & * \\ 0 & \delta'(\tau) \end{pmatrix} \quad \text{for} \quad \tau \in D_\mathfrak{p},$$

where $\delta': D_\mathfrak{p} \to \bar{\mathbf{Q}}_p$ is unramified and for the Frobenius element $\phi_\mathfrak{p}$, we have $\delta'(\phi_\mathfrak{p}) = \chi_\mathfrak{p}{}^{-1}\xi_\mathfrak{p}(\bar{\omega}_\mathfrak{p})$. This combined with the formula $\sigma(\chi \otimes \rho) = \chi^{-1} \otimes \sigma(\rho)$ shows that

$$\sigma(\rho)(\tau) = \begin{pmatrix} \epsilon(\tau) & * \\ 0 & \delta(\tau) \end{pmatrix} \quad \text{for} \quad \tau \in D_\mathfrak{p},$$

where δ is a character of $D_\mathfrak{p}$ corresponding to $\xi_\mathfrak{p}$ by local class field theory at $\mathfrak{p}$. Thus δ has values in A by Corollary 2.2, since $\xi_\mathfrak{p}(\bar{\omega}) \in A$ and $\mathfrak{h}_{k,w}{}^{\mathrm{n.ord}}(S(p^\alpha); \mathcal{O})$ contains the action of $S_0(p^\alpha)/S(p^\alpha)$. Since $\epsilon\delta = \det(\sigma(\rho))$ has values in A, ϵ also has values in A over $D_\mathfrak{p}$. Thus we have

PROPOSITION 2.3. *Let A be a subring of $\bar{\mathbf{Q}}_p$ and let $\Lambda: \mathfrak{h}_{k,w}{}^{\mathrm{n.ord}}(S(p^\alpha); \mathcal{O}) \to A$ (for $n \geq 0$ and $v \geq 0$ with $n + 2v \in \mathbf{Z}t$) be an algebra homomorphism. Then there exists A-valued characters ϵ and δ of the decomposition group $D_\mathfrak{p}$ for each prime factor $\mathfrak{p}$ of p such that the restriction of the Galois* *representation attached to* λ *to the decomposition group* $D_{\mathfrak{p}}$ *at each prime factor* $\mathfrak{p}$ *of* p *is, up to equivalence, of the following form:*

$$\begin{pmatrix} \epsilon(\tau) & * \\ 0 & \delta(\tau) \end{pmatrix} \quad \textit{for} \quad \tau \in D_{\mathfrak{p}}.$$

Moreover if $\xi_{\mathfrak{p}}$ *denotes the character of* $F_{\mathfrak{p}}^{\times}$ *corresponding to* δ *by local class field theory, then* $\lambda(x^{-v}T(x)) = \xi_{\mathfrak{p}}(x)$ *for all* $0 \neq x \in \mathfrak{r}_{\mathfrak{p}}$.

When $(k, w) = (2t, t)$ (i.e. $n = v = 0$), this proposition follows from the above argument. The general case follows from [8, Theorem 3.3] and the proof of Theorem I in the following section, which only involves the result in the case where $n = v = 0$.

3. Proof of Theorems I and II. We take, as the group G in Section 1, the Galois group over F of the maximal extension of F unramified outside Np, where N is the ideal prime to p which is maximal under the condition $S \supset U_1(N)$. This ideal is called the level of S. Then define a subset Σ of G by

$$\Sigma = \cup_{\mathfrak{q}} \{\phi_{\mathfrak{q}} \in G | \phi_{\mathfrak{q}} \text{ is a Frobenius element of } \mathfrak{q}\}$$

where $\mathfrak{q}$ runs over all prime ideals outside Np. Then Σ is dense in G by the density theorem of Chebotarev. Let R be either the subalgebra $\mathbf{h}$ of $\mathbf{h}(S; \mathcal{O})$ generated over $\mathbf{A}$ by $T(\mathfrak{n})$'s for $\mathfrak{n}$ prime to Np or the reduced part of $\mathbf{h}^{\text{n.ord}}(S; \mathcal{O})$ (i.e. the quotient of $\mathbf{h}^{\text{n.ord}}(S; \mathcal{O})$ by its nilradical). Now we know from [8, Theorem 2.3] that

$$R = \varprojlim_{\alpha} R_{\alpha} \text{ as a topological algebra,}$$

where R_{α} is a subalgebra of $\mathfrak{h}_{2t,t}(S(p^{\alpha}); \mathcal{O})$ generated over $\mathcal{O}[G^{\alpha}]$ by $T(\mathfrak{n})$'s for $\mathfrak{n}$ prime to Np if $R = \mathbf{h}$ and in the other case, R_{α} is the reduced part of $\mathfrak{h}_{k,w}{}^{\text{n.ord}}(S(p^{\alpha}); \mathcal{O})$ under the notation in [8, Section 2]. Note that R_{α} is a reduced algebra free of finite rank over $\mathcal{O}$. Therefore, R_{α} is a product of finitely many objects in $\mathcal{C}$, and there exists finitely many minimal prime ideals $\mathfrak{p}_{\alpha,i}$ of R_i such that $R_i/\mathfrak{p}_{\alpha,i}$ is free of finite rank over $\mathcal{O}$ and $\cap_i \mathfrak{p}_{\alpha,i} = \{0\}$. We lift $\mathfrak{p}_{\alpha,i}$'s to R and consider them as prime ideals of R. If λ denotes the algebra homomorphism of R_i into $\bar{\mathbf{Q}}_p$ whose kernel coincides with $\mathfrak{p}_{\alpha,i}$, then there exists a common eigenform $\mathbf{f}$ of all Hecke operators in

$\mathbf{S}_{2t,t}(U_1(Np^{\beta}); \mathcal{O})$ for sufficiently large β such that $\mathbf{f}|T(\mathfrak{n}) = \lambda(T(\mathfrak{n}))\mathbf{f}$ (see the proof of Proposition 2.3). Thus by the result of Taylor [14], there exists a Galois representation $\pi(\lambda)\colon \mathrm{Gal}(\bar{\mathbf{Q}}/F) \to GL_2(\bar{\mathbf{Q}}_p)$ such that

(3.1a) $\pi(\lambda)$ is continuous, irreducible and unramified outside Np,
(3.1b) For each $\phi_{\mathfrak{q}} \in \Sigma$, we have

$$\det(1 - \pi(\lambda)(\phi_{\mathfrak{q}})X) = 1 - \lambda(T(\mathfrak{q}))X + \mathfrak{N}_{F/\mathbf{Q}}(\mathfrak{q})\lambda(\langle \mathfrak{n} \rangle)X^2.$$

Then by identifying $R/\mathfrak{p}_{\alpha,i}$ with $\mathrm{Im}(\lambda)$, we can construct a pseudo representation $\pi_{\alpha,i}$ into $R/\mathfrak{p}_{\alpha,i}$. As the element $c \in G$ as in Section 1, we take the natural image of complex conjugation. Then it is well known that $\det(\pi(\lambda)(c)) = -1$ and hence one can associate the pseudo representation $\pi_{\alpha,i}$ to each $\mathfrak{p}_{\alpha,i}$. Now define functions $\mathrm{tr}\colon \Sigma \to R$ and $\det\colon \Sigma \to R$ by $\mathrm{tr}(\phi_{\mathfrak{q}}) = T(\mathfrak{q})$ and $\det(\phi_{\mathfrak{q}}) = \mathfrak{N}_{F/\mathbf{Q}}(\mathfrak{q})\langle \mathfrak{q} \rangle$. Thus, we have pseudo representations $\{\pi_{\alpha,i}\}$, the set of ideals $\{\mathfrak{p}_{\alpha,i}\}$, the dense subset Σ of G, and functions tr and det satisfying the assumption of Theorem 1.4. Then the assertion of Theorems I (except for (iv)) and Theorem II follows from Theorem 1.4 and the result in [8], especially, Theorems 2.3 and 2.4 and Corollary 2.5. We shall now show the assertion (iv) in Theorem I. Let $\mathfrak{p}$ be a prime factor of p in $\mathfrak{r}$. For each element x with nonnegative valuation in $\mathfrak{r}_{\mathfrak{p}}$, we consider the Hecke operator $T_{\alpha}(x)$ corresponding to the double coset action of $S(p^{\alpha})\left(\begin{smallmatrix} 1 & 0 \\ 0 & x \end{smallmatrix}\right)S(p^{\alpha})$ in $\mathfrak{h}_{2t,t}(S(p^{\alpha}); \mathcal{O})$. Then the projection of $T(x) = \varprojlim_{\alpha} T_{\alpha}(x)$ in $\mathbf{h}^{\text{n.ord}}(S; \mathcal{O})$ is a unit. Since $T(xy) = T(x)T(y)$, we can define a character $\Delta\colon F_{\mathfrak{p}}^{\times} \to \mathbf{h}^{\text{n.ord}}(S; \mathcal{O})$ by $\Delta(x) = T(x)$. By local class field theory, we may regard Δ as a character of $D_{\mathfrak{p}}$. Similarly, we can define another Galois character $\iota\colon \mathrm{Gal}(\bar{\mathbf{Q}}/F) \to \mathbf{h}^{\text{n.ord}}(S; \mathcal{O})$ so that ι is unramified outside Np for the level N of S and $\iota(\phi_{\mathfrak{q}}) = \langle \mathfrak{q} \rangle \mathfrak{N}_{F/\mathbf{Q}}(\mathfrak{q})$ for each prime ideal $\mathfrak{q}$ outside Np. Then for each minimal prime ideal $\mathfrak{p}$ of $\mathbf{h}^{\text{n.ord}}(S; \mathcal{O})$ and the projection $\rho\colon \mathbf{h}^{\text{n.ord}}(S; \mathcal{O}) \to \mathbf{I} = \mathbf{h}^{\text{n.ord}}(S; \mathcal{O})/\mathfrak{p}$, $\det(\pi) = \rho \circ \iota$ for $\pi = \pi(\rho)$. We define a Galois representation π' into $GL_2(\mathbf{K})$ for the quotient field $\mathbf{K}$ of $\mathbf{I}$ by

$$\pi'(\sigma) = \begin{pmatrix} \rho \circ (\iota\Delta^{-1}) & 0 \\ 0 & \rho \circ \Delta \end{pmatrix}.$$

Then, for any prime ideal $\mathfrak{p}_{\alpha,i}$ as above containing $\mathfrak{p}$, the two representations $\pi'|_{D_{\mathfrak{p}}}$ and $\pi|_{D_{\mathfrak{p}}}$ modulo $\mathfrak{p}_{\alpha,i}$ have the same trace by Proposition 2.3. Since prime ideals $\mathfrak{p}_{\alpha,i}$ containing $\mathfrak{p}$ are Zariski dense in Spec(I), the semisimplification of $\pi|_{D_p}$ coincides with $\pi'|_{D_p}$. This shows the assertion (iv) because any algebra homomorphism $\lambda: \mathbf{h}^{n.ord}(S; \mathcal{O}) \to A$ as in Theorem I factors through some ρ as above.

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REFERENCES

[1] H. Carayol, Sur les représentations l-adiques associées aux formes modulaires de Hilbert, *Ann. Scient. Éc. Norm. Sup.*, 4ᵉ *série*, **19** (1986), 409–468.

[2] W. Casselman, On some results of Atkin and Lehner, *Math. Ann.* **201** (1973), 301–314.

[3] C. Chevalley, Deux théorèmes d'arithmétique, *J. Math. Soc. Japan*, **3** (1951), 36–44.

[4] F. Q. Gouvêa, Arithmetic of p-adic modular forms, *Lecture notes in Math.* 1304, Springer, 1988.

[5] H. Hida, A p-adic measure attached to the zeta functions associated with two elliptic modular forms, I. *Invent. Math.* **79** (1985), 159–195.

[6] ———, Galois representations into $GL_2(\mathbf{Z}_p[[X]])$ attached to ordinary cusp forms, *Invent. Math.* **85** (1986), 545–613.

[7] ———, On p-adic Hecke algebras for GL_2 over totally real fields, *Ann. of Math.* **128** (1988), 295–384.

[8] ———, On nearly ordinary Hecke algebras for GL_2 over totally real fields, *Advanced Studies in Pure Math.* **17** (1989), 139–169.

[9] B. Mazur, Deforming Galois representations, preprint.

[10] ——— and A. Wiles, Class fields of abelian extension of **Q**, *Invent. Math.* **76** (1984), 179–330.

[11] M. Ohta, On l-adic representations attached to automorphic forms, *Japan. J. Math.* **8** (1982), 1–47.

[12] ———, On the zeta function of an abelian scheme over the Shimura curve, *Japan. J. Math.* **9** (1983), 1–25.

[13] J. D. Rogawski and J. B. Tunnell, On Artin L-functions associated to Hilbert modular forms of weight one, *Invent. Math.* **74** (1983), 1–42.

[14] R. Taylor, On Galois representations associated to Hilbert modular forms, preprint.

[15] J. Tilouine, Theorie d'Iwasawa classique et de l'algèbre de Hecke ordinaire, *Compositio Math.* **65** (1988), 265–320.

[16] A. Wiles, On p-adic representations for totally real fields, *Ann. of Math.* **123** (1986), 407–456.

[17] ———, On ordinary λ-adic representations associated to modular forms, preprint.