Irreducibility of the Igusa tower over unitary Shimura varieties

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To Freydoon Shahidi on the occasion of his sixtieth anniversary

Contents

In this paper, we describe a proof (more concise than the treatments in [**PAF**] Chapter 8 and [**H06**]) of irreducibility of the modulo p Igusa tower over a (unitary) Shimura variety. We study the decomposition group of the mixed characteristic valuation associated to each irreducible component of the Igusa tower (so, the argument is closer to [**PAF**] Chapter 8 than the purely characteristic p argument in [**H06**]). The author hopes that the account here is easier to follow than the technical but more general treatment in [**H06**] and [**PAF**].

There are at least two ways of showing irreducibility: (i) the use of the automorphism group of the function field of the Shimura variety of characteristic 0 (cf. [**PAF**] Sections 6.4.3 and 8.4.4), which uses characteristic 0 results to prove the characteristic p assertion, and (ii) a purely characteristic p proof following a line close to (i) (see [**H06**]). There are some other arguments (of purely in characteristic p) to prove the same result (covering different families of reductive groups giving the Shimura variety) as sketched in [**C1**] for the Siegel modular variety.

Here is an axiomatic approach to prove irreducibility of an étale covering $\pi : I \to S$ of a smooth irreducible variety S over the algebraic closure $\mathbb{F}$ of $\mathbb{F}_p$. Write $\pi_0(I)$ for the set of connected components of I. We start with the following two axioms:

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(A1) A group $\mathcal{G} = \mathcal{M} \times \mathcal{G}_1$ acts on I and S compatibly so that $\mathcal{M} \subset \mathrm{Aut}(I/S)$, $\mathcal{G}_1 \subset \mathrm{Aut}(S)$ and $\mathcal{G}_1$ acts trivially on $\pi_0(I)$.

(A2) $\mathcal{M}$ acts on each fiber of I/S **transitively**; so, $\mathcal{M}$ acts transitively on $\pi_0(I)$.

Under (A1–2), we study the stabilizer subgroup T_x in $\mathcal{G}$ of a point x in a connected component I° of I and try to prove the following conclusion:

(C) $\{T_x\}_x$ for a good choice of a collection of points x and $\mathcal{G}_1$ generate a dense subgroup of $\mathcal{G}$.

Once we reach the conclusion (C), by the transitivity (A2), we obtain $I^\circ = I$ getting the irreducibility of I.

In the setting of the Shimura variety Sh of PEL type (of level away from a given finite set Σ of places), assuming to have a smooth integral compactification of Sh over a p-adic discrete valuation ring $\mathcal{W}$ (see [**ACS**] 6.4.1), we can easily verify the axioms (A1–2) for the following reasons: compatibility of the action in (A1) and and the transitivity in (A2) follow from the definition. In this case of a Shimura variety, S is the ordinary locus of the modulo p Shimura variety $Sh_{/\mathbb{F}}$ of level away from a given finite set Σ of places including p and ∞. Then for the adele ring $\mathbb{A}^{(\Sigma)}$ away from Σ, $\mathcal{G}_1$ is the adele group $G(\mathbb{A}^{(\Sigma)})$ for the semi-simple group $G_{/\mathbb{Q}}$ (which is the derived group of the starting reductive group in Shimura's data), and $\mathcal{M}$ is the $\mathbb{Z}_p$-points $M(\mathbb{Z}_p)$ of the reductive part M of a parabolic subgroup of G. If we choose Σ so that $G(\mathbb{Q}_\ell)$ is generated by unipotent elements for all $\ell \notin \Sigma$, $\mathcal{G}_1$ has no nontrivial finite quotient group (because unipotent groups over characteristic 0 field are uniquely divisible). For any finite subcovering I'/S of I, $\mathcal{G}_1$ acts on the finite set $\pi_0(I')$ through a finite quotient of $\mathcal{G}_1$; so, the action is trivial, getting (A1).

In the above discussion of how to verify (A1), a key ingredient is that $\mathcal{G}_1$ is large enough not to have finite (nontrivial) quotient. As we will do in this paper, this is deduced from the existence of smooth toroidal compactification (if the Shimura variety is not projective) and a characteristic 0 determination of the automorphism group of the Shimura variety. Alternatively, one can prove that $\mathcal{G}_1$ is large by showing that the ℓ-adic monodromy homomorphism for primes $\ell \neq p$ has large open image in $G(\mathbb{A}^{(\Sigma)})$. Indeed, C.-L. Chai [**C**] (in the symplectic case) has deduced the open image result via group theory from the semi-simplicity theorem of Grothendieck-Deligne of the ℓ-adic representation. The method in [**C**] should also work for $\ell \notin \Sigma$ (for an appropriate Σ) in our setting.

Let $I^\circ_{/\mathbb{F}}$ be an irreducible component of $I_{/\mathbb{F}}$. We want to prove $I^\circ = \pi^{-1}(S) = I$ (irreducibility). Then $\mathrm{Gal}(I^\circ/S) \subset \mathcal{M}$, and if $\mathcal{M} = \mathrm{Gal}(I^\circ/S)$, we get $I^\circ = \pi^{-1}(S)$. Let D be the stabilizer of $I^\circ \in \pi_0(I)$ in $\mathcal{G}$. Pick a point $x \in I$ (which can be a generic point), and look at the stabilizer $T_x \subset \mathcal{G}$ of x. Since $g_x(x) \in I^\circ$ ($g_x \in \mathcal{M}$) by the transitivity of the action, we have $g_x T_x g_x^{-1} \subset D$. Then we show that $\mathcal{M} = \mathcal{G}/\mathcal{G}_1$ is generated topologically by $\{g_x T_x g_x^{-1} | x \in I\}$, which implies $\mathcal{M} = \mathrm{Gal}(I^\circ/S)$ and the conclusion (C).

In the setting of the Igusa tower of a Shimura variety, we can have at least three choices of the points $x \in I$:

(∞) A cusp, assuming that the group $G = \mathrm{Res}_{F/\mathbb{Q}} G_0$ for a quasi-split group G_0 over a number field F (acting on a tube domain). This is the proof given in $GSp(2n)$ in [**DAV**].

(cm) A closed point $x \in I(\mathbb{F})$ is fixed by a maximal torus T_x of G anisotropic at ∞; so, $g_x T_x(\mathbb{Z}_{(p)}) g_x^{-1} \subset D$ for $\mathbb{Z}_{(p)} = \mathbb{Q} \cap \mathbb{Z}_p$). A well chosen finite set of closed points $X := \{x\}$ is enough to generate a dense subgroup of D by $\{g_x T_x(\mathbb{Z}_{(p)}) g_x^{-1}\}_{x \in X}$. This is Ribet's choice for Hilbert modular varieties and is also taken in [**H06**]. If one uses a CM point (the so-called "hyper-symmetric point") carrying a product of copies of CM elliptic curves, often one such point is sufficient (see Section 3.5);

(gn) Taking a coordinate system $T = (T_1, \ldots, T_d)$ around $x \in Sh(W)$ with (x mod p) $\in I^\circ(\mathbb{F})$ (so that $\widehat{\mathcal{O}}_{Sh,x} \cong W[[T_1, \ldots, T_d]]$) and take the valuation

$$v_x(\sum_\alpha c(\alpha, f) T^\alpha) = \mathrm{Inf}_\alpha \, \mathrm{ord}_p(c(\alpha, f)).$$

Then the decomposition group D of v_x contains T_x (for all $x \in I^\circ$), and D is the stabilizer of the generic point of I° containing x (this choice is taken in [**PAF**] 8.4.4). The valuation v_x corresponds to the generic point of I°. The point x can be a cusp as in (∞), and in the case of the modular curve (see Section 1.3), the Hilbert-Siegel modular variety and $U(n, n)$ Shimura variety, the choice of the infinity cusp works as well (cf. [**PAF**] 6.4.3).

Actually there is (at least) one more choice. Igusa completed his tower over modular curves adding super-singular points and used such points to prove his irreducibility theorem in the 1950s. Here we describe the method (gn), but the base point x we use is the infinity cusp in the elliptic modular case and a hyper-symmetric point in the unitary case.

Fix a prime p and an algebraic closure $\mathbb{F}$ of $\mathbb{F}_p$. We fix an algebraic closure $\overline{\mathbb{Q}}$ (resp. $\overline{\mathbb{Q}}_p$) of $\mathbb{Q}$ (resp. $\mathbb{Q}_p$), respectively. We fix field embeddings $i_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ and $i_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Throughout this paper, proofs of the results claimed are given assuming $p > 2$ (just for simplicity; see [**H06**] for the treatment in the case $p = 2$).

1. Elliptic modular Igusa tower

As an introduction to the subject, we first describe the simplest case: the modular curves by the method (gn).

1.1. Elliptic modular function fields. We consider a field $\mathcal{K}$ given by $\bigcup_{p \nmid N} \mathbb{Q}(\mu_N)$ inside $\overline{\mathbb{Q}}$; so, $\mathcal{K} \overset{i_p}{\hookrightarrow} \overline{\mathbb{Q}}_p$. Take a p-adic place $\mathfrak{P}$ of $\mathcal{K}$ given by i_p and write $\mathcal{W} \subset \mathcal{K}$ for the *discrete* valuation ring of $\mathfrak{P}$. We thus have a continuous embedding $i_p : \mathcal{W} \hookrightarrow \overline{\mathbb{Q}}_p$, and for the maximal ideal $\mathfrak{m}$ of $\mathcal{W}$, $\mathbb{F} = \mathcal{W}/\mathfrak{m}$ is an algebraic closure of $\mathbb{F}_p$. Put $\mathcal{G} = GL_1(\mathbb{Z}_p) \times SL_2(\mathbb{A}^{(p\infty)})$ and we embed diagonally $\mathbb{Z}_{(p)}$-points of the standard diagonal torus $M \subset SL(2)$ (of the upper triangular Borel subgroup $P = \left\{ \left(\begin{smallmatrix} a & * \\ 0 & a^{-1} \end{smallmatrix}\right) \middle| a \in GL(1) \right\}$ of $SL(2)$) into $\mathcal{G}$ so that $\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)$ is sent to $a \in GL_1(\mathbb{Z}_p)$ at p and $\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right) \in SL_2(\mathbb{Q}_\ell)$ at all primes $\ell \nmid p$.

We consider the modular curve $X(N)_{/\mathbb{Z}[\frac{1}{N}]}$ for an integer N prime to p which classifies pairs $(E, \phi_N)_{/A}$, where E is an elliptic over A and $\phi_N : (\mathbb{Z}/N\mathbb{Z})^2 \cong A[N] = \mathrm{Ker}(N : A \to A)$ is an isomorphism of finite flat group schemes over A. The level structure ϕ_N specify a primitive root of unity $\zeta_N \in \mu_N$ via the Weil pairing

$$\zeta_N := \langle \phi_N(1, 0), \phi_N(0, 1) \rangle.$$

Thus $X(N)$ has a scheme structure over $\mathbb{Z}[\mu_N, \frac{1}{N}]$ but we may consider it defined over $\mathbb{Z}[\frac{1}{N}]$ composing the morphism $\mathrm{Spec}(\mathbb{Z}[\mu_N, \frac{1}{N}]) \to \mathrm{Spec}(\mathbb{Z}[\frac{1}{N}])$. If we consider level p^m-structure ϕ_p of type $\Gamma = \Gamma_?(p^m)$ ($? = 0, 1$) given as follows: ϕ_p is a subgroup isomorphic to μ_{p^m} étale locally if $\Gamma = \Gamma_0(p^m)$ and $\phi_p : \mu_{p^m} \hookrightarrow E[p^m]$ (a closed immersion of finite flat group schemes) if $\Gamma = \Gamma_1(p^m)$ (and $Np^m \geq 4$), we can think of the fine moduli space $X(N,\Gamma)_{/B}$ over the base ring $B_{/\mathbb{Z}[\frac{1}{N}]}$ which classifies triples $(E, \phi_N, \phi_p)_{/A}$ over B-algebras A. As the ring B, we take one of $\mathcal{W}$, $\mathbb{F}$ and $\mathcal{K}$. As we observed, the open curves $X(N)$ (resp. $X(N,\Gamma)$) can be regarded as a scheme over $\mathrm{Spec}(\mathbb{Z}[\frac{1}{N}, \mu_N])$ (resp. over $\mathrm{Spec}(\mathbb{Q}[\mu_N])$). For N prime to p, $X(N)_{/\mathbb{Q}[\mu_N]}$ is geometrically irreducible.

We can think of the p-integral Shimura curve

$$Sh_{/\mathbb{Z}_{(p)}} = \varprojlim_{p \nmid N} X(N)_{/\mathbb{Z}_{(p)}},$$

and more generally over $\mathbb{Q}$,

$$Sh_{\Gamma/\mathbb{Q}} = \varprojlim_{p \nmid N} X(N,\Gamma)_{/\mathbb{Q}}$$

(regarding these schemes as $\mathbb{Z}_{(p)}$-schemes or $\mathbb{Q}$-schemes). Let

$$X(N,\Gamma)_{/B} = X(N,\Gamma)_{/\mathbb{Z}[\mu_N, \frac{1}{N}]} \times_{\mathbb{Z}[\mu_N]} B \ \text{ and } \ X(N)_{/\mathcal{W}} = X(N)_{/\mathbb{Z}[\mu_N, \frac{1}{N}]} \times_{\mathbb{Z}[\mu_N, \frac{1}{N}]} B.$$

The pro-schemes

$$X_{\Gamma/B} = \varprojlim_{N} X(N,\Gamma)_{/B} \ \text{ for } B = \mathcal{K} \ \text{ and } \ X^{(p)}_{/\mathcal{W}} = \varprojlim_{p \nmid N} X(N)_{/\mathcal{W}}$$

give geometrically irreducible components of $Sh_{\Gamma/\mathbb{Q}} \times_{\mathbb{Q}} \mathcal{K}$ and $Sh^{(p)}_{/\mathbb{Z}_{(p)}} \times_{\mathbb{Z}_{(p)}} \mathcal{W}$ (the neutral components). If convenient, we write $Sh_{\Gamma_1(p^0)/\mathbb{Z}_{(p)}}$ for $Sh_{/\mathbb{Z}_{(p)}}$ (abusing the notation). By the interpretation of Deligne–Kottwitz, we have

$$Sh_\Gamma(A) \cong \frac{\{(E, \eta : (\mathbb{A}^{(p\infty)})^2 \cong V(E), \phi_p)_{/A}\}}{\text{prime-to-}p \text{ isogenies}}, \tag{1.1}$$

where A runs over $\mathbb{Z}_{(p)}$-algebras if $\Gamma = \Gamma_1(p^0)$ and B-algebras ($B = \mathbb{F}$ or $\mathbb{Q}$) if $\Gamma = \Gamma_?(p^m)$ with $m > 0$ ($? = 0, 1$), $V(E) = \mathbb{A}^{(p\infty)} \otimes \varprojlim_{p \nmid N} E[N]$. Thus $(a, g) \in \mathcal{G}$ ($a \in GL_1(\mathbb{Z}_p)$ and $g \in SL_2(\mathbb{A}^{(p\infty)})$) acts on Sh_Γ by

$$(E, \eta, \phi_p) \mapsto (E, \eta \circ g, \phi_p \circ a),$$

where $a \in GL_1(\mathbb{Z}_p) \cong M(\mathbb{Z}_p)$. Write $\mathfrak{F}_\Gamma$ for the function field $\mathcal{K}(X_\Gamma)$ and $\mathfrak{F}^{(p)}$ for $\mathcal{K}(X^{(p)})$ (the arithmetic automorphic function fields). This action produces an embedding

$$\tau : \mathcal{G}/\{\pm 1\} \hookrightarrow \mathrm{Aut}(\mathfrak{F}_{\Gamma_1(p^\infty)}/\mathcal{K}) = \mathrm{Aut}(X_{\Gamma_1(p^\infty)/\mathcal{K}}).$$

The action of $\tau(a, g)$ on the function field $\mathfrak{F}_\Gamma$ is on the left and has the following property (by Shimura; e.g., [**IAT**] Theorem 6.23 or [**PAF**] Theorem 4.14): For $a \in GL_1(\mathbb{Z}_{(p)})$ (corresponding to $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ in $M(\mathbb{Z}_{(p)})$ diagonally embedded in $SL_2(\mathbb{A}^{(\infty)})$), we have for $f \in \mathcal{F}_\Gamma$

$$\tau(a)(f)(z) = f(a^{-2} z); \tag{1.2}$$

so, we have $\tau(\alpha)(f) = f(\alpha^{-1}(z))$ for $\alpha = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$. This formula is valid for general $\alpha \in GL_2(\mathbb{Z}_{(p)})$ if $f \in \mathfrak{F}^{(p)}$ (so, our normalization is different form Shimura's).

We define a valuation

$$v_\Gamma(f) = \inf_{\xi} \mathrm{ord}_p(c(\xi, f))$$

of modular functions $f = \sum_{\xi} c(\xi, f)q^{\xi} \in \mathfrak{F}_{\Gamma}$. We write v_m for v_{Γ} if $\Gamma = \Gamma_1(p^m)$. Thus the valuation $v_0 : \mathfrak{F}^{(p)} \to \mathbb{Z} \cup \{\infty\}$ has a standard unramified extension $v_{\Gamma} : \mathfrak{F}_{\Gamma} \twoheadrightarrow \mathbb{Z} \cup \{\infty\}$. Here are some easy facts:

LEMMA 1.1. (1) *If $a \in GL_1(\mathbb{Z}_{(p)}) \cong M(\mathbb{Z}_{(p)})$, then*

$$c(\xi, \tau(a)(f)) = c(a^2\xi, f).$$

In particular, the diagonally embedded $M(\mathbb{Z}_{(p)}) \subset \mathcal{G}$ preserves the valuation v_{Γ};

(2) *The vertical divisor $X^{(p)}_{/\mathbb{F}} := X^{(p)}_{/\mathcal{W}} \otimes_{\mathcal{W}} \mathbb{F}$ of $X^{(p)}_{/\mathcal{W}}$ is a prime divisor (geometrically irreducible) and gives rise to a* **unique** *valuation of $\mathfrak{F}^{(p)}$, whose explicit form is given by the valuation v_0.*

PROOF. The first assertion follows directly from (1.2). By the existence of a smooth compactification of $X^{(p)}$ over $\mathcal{W}$, Zariski's connectedness theorem tells us that $X^{(p)}_{/\mathbb{F}} = X^{(p)} \times_{\mathcal{W}} \mathbb{F}$ is irreducible. Thus the vertical Weil prime divisor $X^{(p)}_{/\mathbb{F}}$ on the smooth arithmetic surface $X^{(p)}_{/\mathcal{W}}$ gives rise to a unique valuation. By the irreducibility of $X^{(p)}_{/\mathbb{F}}$, a $\mathcal{W}$-integral modular form of level away from p vanishes on the divisor $X^{(p)}_{/\mathbb{F}}$ if and only if its q-expansion vanishes modulo p. Thus the valuation v_0 is the one corresponding to the vertical prime divisor $X^{(p)}_{/\mathbb{F}} \subset X^{(p)}_{/\mathcal{W}}$. □

1.2. mod p connected components and the valuation v_m. Let S be the ordinary locus $X^{(p)}[\frac{1}{H}]_{/\mathbb{F}}$ for the Hasse invariant H. Then S is an irreducible variety over $\mathbb{F}$, because H is a global section of the ample modular line bundle $\underline{\omega}^{\otimes(p-1)}$ of the compactification of $X^{(p)}_{/\mathbb{F}}$. Consider the valuation ring V of $\mathfrak{F}^{(p)}$ of the valuation v_0. Thus the residue field $V/\mathfrak{m}_V$ is the function field $\mathbb{F}(S)$ of S. Let $\mathbb{E}_{/X^{(p)}}$ be the universal elliptic curve. Then we consider the Cartesian diagram for $\mathbb{E}_V = \mathbb{E} \times_{X^{(p)}} \mathrm{Spec}(V)$:

$$\begin{array}{ccc} \mathbb{E}_V & \xrightarrow{\hookrightarrow} & \mathbb{E} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(V) & \xrightarrow[\hookrightarrow]{} & X^{(p)}. \end{array}$$

Since any lift of a power of H is inverted in V, $\mathbb{E}_{\widehat{V}} = \mathbb{E}_V \times_V \widehat{V}$ is an ordinary abelian scheme for the completed valuation ring $\widehat{V} = \varprojlim_n V/p^nV$. Thus we can think of the functor $I_{\widehat{V},m} = \mathrm{Isom}_{\widehat{V}}(\mu_{p^m}, \mathbb{E}_{\widehat{V}}[p^m])$ which assigns each p-adic $\widehat{V}$-algebra $R = \varprojlim_n R/p^nR$ the set of closed immersions: $\mu_{p^m/R} \to \mathbb{E}_{\widehat{V}}[p^m]_{/R}$ defined over R.

Since $\mathbb{E}_{\widehat{V}}[p^m]$ has a well defined connected component over $\widehat{V}$ isomorphic to μ_{p^m} étale locally ($\widehat{V}$ is a henselian local ring), we have canonical isomorphisms of formal schemes:

$$I_{\widehat{V},m} = \mathrm{Isom}_{\widehat{V}}(\mu_{p^m/\widehat{V}}, \mathbb{E}_{\widehat{V}}[p^m]^{\circ}) \overset{(*)}{\cong} \mathrm{Isom}_{\widehat{V}}((\mathbb{Z}/p^m\mathbb{Z})_{/\widehat{V}}, \mathbb{E}_{\widehat{V}}[p^m]^{et})$$
$$\overset{(**)}{\cong} \mathbb{E}_{\widehat{V}}[p^m]^{et} - \mathbb{E}_{\widehat{V}}[p^{m-1}]^{et},$$

where the identity $(*)$ is given by taking the inverse of the Cartier dual map and $(**)$ is given by $\phi \mapsto \phi(1)$ for $1 \in \mathbb{Z}/p^m\mathbb{Z}$ and $\phi \in \mathrm{Isom}_{\widehat{V}}((\mathbb{Z}/p^m\mathbb{Z})_{/\widehat{V}}, \mathbb{E}_{\widehat{V}}[p^m]^{et})$. Thus $I_{\widehat{V},m}/\mathrm{Spf}(\widehat{V})$ is étale finite. Note here $\mathbb{E}_{\widehat{V}}[p^m]^\circ$ is isomorphic to $\mu_{p^n/\widehat{V}}$ étale locally. By the second expression, $I_{\widehat{V},m}/\mathrm{Spf}(\widehat{V})$ is an étale finite covering over $\widehat{V}$, and $GL_1(\mathbb{Z}_p)$ naturally acts on $I_{\widehat{V},m}$. Since $I_{\widehat{V},m}$ is étale faithfully flat over $\mathrm{Spf}(\widehat{V})$, it is affine, and we may write $I_{\widehat{V},m} = \mathrm{Spf}(\widehat{V}_m)$. Then $\widehat{V}_m$ is a semi-local normal $\widehat{V}$-algebra étale finite over $\widehat{V}$; so, it is a product of complete discrete valuation rings whose maximal ideal is generated by the rational prime p. Write $W = \varprojlim_n \mathcal{W}/p^n\mathcal{W}$, and take a modular form E on $X^{(p)}_{/\mathcal{W}}$ lifting a positive power of the Hasse invariant H. Let $\widehat{X}^{(p)}$ be a formal completion of $X^{(p)}[\frac{1}{E}]_{/W}$ along S (the ordinary locus). The p-adic formal scheme $\widehat{X}^{(p)}$ does not depend on the choice of the lift E. Then we define a p-adic formal scheme $\widehat{X}_{\Gamma/W} = \mathrm{Isom}_{\widehat{X}^{(p)}}(\mu_{p^n}, \mathbb{E}) \cong \mathbb{E}[p^m]^{et} - \mathbb{E}[p^{m-1}]^{et}$ over $\widehat{X}^{(p)}$, which is étale finite over $\widehat{X}^{(p)}$. We may regard $\widehat{X}_{\Gamma/W}$ as the formal completion of $X_{\Gamma/\mathcal{W}}$ along $X_{\Gamma/\mathbb{F}}$. By definition, we have an open immersion

$$I_{\widehat{V},m} \hookrightarrow \widehat{X}_{\Gamma_1(p^m)/W} \times_{\widehat{X}^{(p)}_{/\mathcal{W}}} \mathrm{Spf}(\widehat{V}),$$

and $\widehat{V}_m$ is the product of the completions of valuation rings of $\mathfrak{F}_{\Gamma_1(p^m)}$ unramified over V. Thus $V_m = \widehat{V}_m \cap \mathfrak{F}_{\Gamma_1(p^m)}$ inside $\mathfrak{F}_{\Gamma_1(p^m)} \otimes_V \widehat{V}$ is a semi-local ring V_m with $\widehat{V}_m = \varprojlim_n V_m/p^nV_m = V_m \otimes_V \widehat{V}$.

We put $I_{V,m} = \mathrm{Spec}(V_m)$ and $X_{\Gamma/\mathbb{F}} = \varprojlim_{p\nmid N} X(N,\Gamma)_{/\mathbb{F}}$. Then

$$X_{\Gamma_1(p^m)/\mathbb{F}} = \mathrm{Isom}_S(\mu_{p^m}, \mathbb{E}[p^m]^\circ) =: I_m$$

gives rise to the Igusa tower $I \twoheadrightarrow \cdots \twoheadrightarrow I_m \twoheadrightarrow \cdots \twoheadrightarrow I_1 \twoheadrightarrow S$ over S. We may regard the moduli scheme $X(N,\Gamma)_{/\mathbb{F}}$ as a scheme over $X(N)[\frac{1}{H}]$ (forgetting the level p-structure). The set of generic points $\{\eta_{I^\circ_m} \in I^\circ_{m/\mathbb{F}} | I^\circ_{m/\mathbb{F}} \in \pi_0(I_{m/\mathbb{F}})\}$ is in bijection to $\pi_0(I_m)$, and

$$\widehat{V}_m \otimes_{\mathbb{Z}_p} \mathbb{F} = V_m \otimes_{\mathbb{Z}_{(p)}} \mathbb{F} = \prod_{I^\circ_m \in \pi_0(I_m)} \mathbb{F}(I^\circ_m) \ (\Leftrightarrow I_{V,m} \otimes_{\mathbb{Z}_{(p)}} \mathbb{F} = \bigsqcup_{I^\circ \in \pi_0(I_m)} \{\eta_{I^\circ_m}\}).$$

By the definition of the action of $(a,g) \in \mathcal{G}$:

$$(E, \eta^{(p)}, \phi_p) \mapsto (E, \eta^{(p)} \circ g, \phi_p \circ a),$$

$\mathcal{G} := GL_1(\mathbb{Z}_p) \times SL_2(\mathbb{A}^{(p\infty)})$ acts on $I_{\widehat{V},m}$ and hence on $I_{V,m}$ $(m = 1, 2, \ldots, \infty)$, $\mathrm{Spec}(V)$ (by Lemma 1.1 (2)), $\mathfrak{F}_\Gamma$, I_m, $X_{\Gamma/\mathbb{F}}$ and $X_{\Gamma/\mathcal{K}}$. Thus we can make the étale quotient $I_{\Gamma_0(p^m)} := I_{V,m}/GL_1(\mathbb{Z}/p^m\mathbb{Z})$. Again we have $I_{\Gamma_0(p^m)} = \mathrm{Spec}(V_{\Gamma_0(p^m)})$, and $V_{\Gamma_0(p^m)}$ is a valuation ring finite flat over V sharing the same residue field. Indeed, there is a unique connected subgroup of E (isomorphic to μ_{p^m} étale locally) if $(E, \phi_N)_{/A}$ gives rise to a unique A-point of $X(N, \Gamma_0(p^m))_{/\mathbb{F}}$. Thus for any $m > 0$, $S_{/\mathbb{F}} = \varprojlim_{p\nmid N} X(N, \Gamma_0(p^m))_{/\mathbb{F}}$. This shows that the residue field of $V_{\Gamma_0(p^m)}$ is the function field of S and that the quotient field of $V_{\Gamma_0(p^m)}$ is $\mathfrak{F}_{\Gamma_0(p^m)}$. Since V_m/V is étale, we have

$$\widehat{V}_{\Gamma_0(p^m)} = \varprojlim_m V_{\Gamma_0(p^m)}/p^m V_{\Gamma_0(p^m)} = \varprojlim_m V/p^mV = \widehat{V},$$

and V_m is étale finite over $V_{\Gamma_0(p^m)}$. This shows

LEMMA 1.2. *We have the following one-to-one onto correspondences:*

$$\left\{v : \mathfrak{F}_{\Gamma_1(p^m)} \to \mathbb{Z} \middle| v|_{\mathfrak{F}_{\Gamma_0(p^m)}} = v_{\Gamma_0(p^m)} \text{ unramified over } v_0\right\} \leftrightarrow \mathrm{Max}(V_m) \leftrightarrow \pi_0(I_m) \leftrightarrow \{\eta_{I_m^\circ}\},$$

where v is a p-adic valuation of $\mathfrak{F}_{\Gamma_1(p^m)}$ unramified (of degree 1) over v_0 and $\mathrm{Max}(V_m)$ *is the set of maximal ideals of V_m.*

The correspondence is given by

$$v \leftrightarrow \mathfrak{m}_v = \{x \in V_m | v(x) > 0\} \leftrightarrow I_m^\circ \quad \text{with } \mathbb{F}(I_m^\circ) = V_m/\mathfrak{m}_v.$$

LEMMA 1.3. *The action of $\mathcal{G}_1 := SL_2(\mathbb{A}^{(p\infty)})$ fixes $v_m = v_{\Gamma_1(p^m)}$ and each element of $\pi_0(I_m)$.*

PROOF. Since $\mathfrak{F}_{\Gamma_1(p^m)}/\mathfrak{F}_{\Gamma_0(p^m)}$ is a finite Galois extension, the set of extensions of $v_{\Gamma_0(p^m)}$ to $\mathfrak{F}_{\Gamma_1(p^m)}$ is a finite set, and by the above lemma, it is in bijection with $\pi_0(I_m)$. Thus the action of $SL_2(\mathbb{A}^{(p\infty)})$ on $\pi_0(I_m)$ gives a finite permutation representation of $SL_2(\mathbb{A}^{(p\infty)})$. Since $SL_2(k)$ of any field k of characteristic 0 does not have nontrivial finite quotient group (because it is generated by divisible unipotent subgroups), the action of $SL_2(\mathbb{A}^{(p\infty)})$ fixes every irreducible component of $\pi_0(I_m)$. □

1.3. Proof of irreducibility of elliptic Igusa tower. Let $v_\infty = v_{\Gamma_1(p^\infty)}$, and define

$$D = \left\{x \in (GL_1(\mathbb{Z}_p) \times SL_2(\mathbb{A}^{(p\infty)})) \middle| v_\infty \circ \tau(x) = v_\infty\right\}.$$

Since $M(\mathbb{Z}_{(p)})$ and $SL_2(\mathbb{A}^{(p\infty)})$ fixes v_∞ (Lemmas 1.1 and 1.3) and the subgroup $(M(\mathbb{Z}_{(p)})SL_2(\mathbb{A}^{(p\infty)}))$ is dense in $\mathcal{G} = GL_1(\mathbb{Z}_p) \times SL_2(\mathbb{A}^{(p\infty)})$, we conclude (C):

THEOREM 1.4. *We have $D = \mathcal{G}$.*

Let $K^{(p)}$ be an open compact subgroup of $SL_2(\mathbb{A}^{(p\infty)})$ and $K = K^{(p)} \times GL_2(\mathbb{Z}_p)$. Put $X_K = X^{(p)}/K^{(p)}$ (which is the level K modular curve). Let $I_K = I/K^{(p)}$, which is the Igusa tower over X_K. Since I is irreducible by

$$\mathrm{Aut}(I^\circ/S) = GL_1(\mathbb{Z}_p) \cong M(\mathbb{Z}_p) \quad \text{(the above theorem)},$$

I_K is irreducible. Thus we have reproved

COROLLARY 1.5 (Igusa). *The Igusa tower I_K over $X_{K/\mathbb{F}}$ is irreducible for $K = GL_2(\mathbb{Z}_p) \times K^{(p)}$ for any open compact subgroup $K^{(p)}$ of $SL_2(\mathbb{A}^{(p\infty)})$.*

2. Shimura varieties of unitary groups

We give an example S of smooth Shimura varieties for which irreducibility of the full Igusa tower is false but one can study the irreducible components explicitly. In other words, we construct a partial tower I°/S for which the axioms (A1–2) can be proved. Write W for the ring of Witt vectors of the algebraic closure $\mathbb{F}$ of $\mathbb{F}_p$ and embed W inside $\mathbb{C}_p$ (the p-adic completion of $\overline{\mathbb{Q}}_p$). Hereafter, we write $\mathcal{W}$ for the valuation ring $i_p^{-1}(W)$ and $\mathcal{K}$ for the field of fractions of $\mathcal{W}$. The (additive) valuation of W and $\mathcal{W}$ is written as ord_p; so, $\mathrm{ord}_p(p) = 1$. As before, we prove that $S_{/\mathcal{W}}$ is irreducible and smooth and that the Igusa tower $I_{/\mathbb{F}}$ is étale over $S_{/\mathbb{F}}$. Then for each point $x \in I(\mathcal{W})$, we take a coordinate $X_1, \ldots, X_d$ of I and define a valuation v_x of the function field of I by $v_x(\sum_\alpha c(\alpha)X^\alpha) = \mathrm{Inf}_\alpha \mathrm{ord}_p(c(\alpha))$ ($X^\alpha = X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_d^{\alpha_d}$). For any automorphism σ of $I_{/\mathcal{W}}$ fixing x, plainly $v_x \circ \sigma = v_x$. Then we conclude the irreducibility by showing that the stabilizers $\{T_x\}_{x \in I(\mathcal{W})}$ inside $\mathrm{Aut}(I_{/\mathcal{W}})$ of $x \in I(\mathcal{W})$ covers sufficiently many conjugacy classes of tori enough to prove (A1–2). Actually, in the simple case we study, a well chosen single point $x_0 \in I(\mathcal{W})$ is sufficient.

We first recall briefly the definition of unitary groups over an imaginary quadratic field F and the construction of the Shimura variety for the unitary groups. The main source of the information for this part is [**PAF**] Chapter 7. Then we prove the irreducibility of the Igusa tower.

Suppose that the imaginary quadratic field F is sitting inside $\overline{\mathbb{Q}}$, and write $1 : F \hookrightarrow \overline{\mathbb{Q}}$ for the identity embedding. Suppose for simplicity that the fixed prime p is *split* in F and that the embedding $1 : F \hookrightarrow \overline{\mathbb{Q}}$ composed with $i_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ gives the standard p-adic place $\mathfrak{p}$ of F. Write O for the integer ring of F.

2.1. Unitary groups. Write c for the generator of $\mathrm{Gal}(F/\mathbb{Q})$ (the complex conjugation on F). We fix a vector space V over F with c-Hermitian alternating form $\langle\ ,\ \rangle : V \times V \to \mathbb{Q}$. We assume having an O-submodule $L \subset V$ of finite type such that

(L1) $L \otimes_{\mathbb{Z}} \mathbb{Q} = V$;
(L2) $\langle\ ,\ \rangle$ induces $\mathrm{Hom}_{\mathbb{Z}_p}(L_p, \mathbb{Z}_p) \cong L_p$, where $L_p = L \otimes_{\mathbb{Z}} \mathbb{Z}_p$.

We fix an O-lattice L of V as above.

We identify V with the column vector space F^r by fixing a basis of V over F. Let $C = \mathrm{End}_F(V) = M_r(F)$. There exists an invertible matrix $s \in M_r(F)$ with ${}^t s^c = -s$ such that $\langle v, w\rangle = \mathrm{Tr}_{F/\mathbb{Q}}({}^t v s \cdot w^c)$, where $\mathrm{Tr}_{F/\mathbb{Q}}$ is the trace map: $F \to \mathbb{Q}$. On C, we have the involution ι given by $x^\iota = s^{-1}\,{}^t x^c s$. Define algebraic groups defined over $\mathbb{Q}$ by the following group functors from $\mathbb{Q}$-algebras R to groups:

$$\begin{aligned} GU(R) &= \left\{x \in C \otimes_{\mathbb{Q}} R \middle| x^\iota x \in R^\times\right\} \\ &= \left\{x \in C \otimes_{\mathbb{Q}} R \middle| {}^t x^c s \cdot x = \nu(x) s \ \text{ for } \ \nu(x) = x^\iota x \in R^\times\right\}, \\ U(R) &= \left\{x \in GU(R) \middle| x^\iota x = 1\right\},\ SU(R) = \left\{x \in U(R) \middle| \det(x) = 1\right\}, \end{aligned} \tag{2.1}$$

where $\det(x)$ is the determinant of x as an F-linear automorphism of V. Then SU is the derived group of GU and U. Let $Z \subset GU$ be the center; so, $Z(R) = (R \otimes_{\mathbb{Q}} F)^\times$ as a group functor. Since $F_{\mathbb{R}} = F \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{C}$ with $b^c = \overline{b}$ for complex conjugation $b \mapsto \overline{b}$, $S = \sqrt{-1}s \in M_r(F_{\mathbb{R}}) = M_r(\mathbb{C})$ is a Hermitian matrix. Thus $U(\mathbb{R})$ is the unitary group of S. We have $\mathrm{Hom}_{\text{field}}(F, \mathbb{C}) = \{1, c\}$ for the identity inclusion 1. Writing the signature of S as (m_1, m_c), we find $U(\mathbb{R}) \cong U_{m_1,m_c}(\mathbb{R}) = \left\{x \in GL_r(\mathbb{C}) \middle| {}^t\overline{x} I_{m_1,m_c} x = I_{m_1,m_c}\right\}$ for $I_{m_1,m_c} = \begin{pmatrix} 1_{m_1} & 0 \\ 0 & -1_{m_c}\end{pmatrix}$.

Example 2.1. For a $\mathbb{Q}$-algebra R,

(1) if $s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\iota = \begin{pmatrix} \overline{d} & -\overline{b} \\ -\overline{c} & \overline{a} \end{pmatrix}$ and $SU(R) = SL_2(R)$,

$$GL_2(R) = \left\{x \in GU(R) \middle| \det(x) = \nu(x)\right\};$$

(2) $GU(\mathbb{Q}) = GL_2(\mathbb{Q})Z(\mathbb{Q})^\times$ and $GU(R) = GL_2(R)Z(R)$.

2.2. Abelian schemes of hermitian type. To equip a complex structure with the real vector space $V_\infty = V \otimes_{\mathbb{Q}} \mathbb{R}$, we use an $\mathbb{R}$-algebra homomorphism $h : \mathbb{C} \hookrightarrow C_\infty = C \otimes_{\mathbb{Q}} \mathbb{R}$ with $h(\overline{z}) = h(z)^\iota$. We call such an algebra homomorphism an *ι-homomorphism.* Then $h(i)^\iota = -h(i)$ for $i = \sqrt{-1}$ and hence $x^\rho = h(i)^{-1} x^\iota h(i)$ is an involution of C_∞.

EXAMPLE 2.2. If $s = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, the morphism $a + bi \mapsto h(a+bi) = \left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right) \in M_2(\mathbb{R}) \subset C_\infty$ is an ι-homomorphism.

We suppose

(pos) The symmetric real bilinear form $(v, w) \mapsto \langle v, h(i)w\rangle$ on V_∞ is positive-definite.

It is easy to check h in Example 2.2 satisfies (pos).

By (pos), we have $0 < (xv, xv) = (v, (x^\rho x)v)$ for all $0 \neq v \in V_\infty$ and $x \in C_\infty$, and hence $x^\rho x$ only has positive eigenvalues; so, ρ is a positive involution of C (i.e., $\mathrm{Tr}_{C/\mathbb{Q}}(x^\rho x) > 0$ unless $x = 0$).

Fix one such $h := h_{\mathbf{0}} : \mathbb{C} \to C_\infty$, and define $\mathfrak{X}$ (resp. $\mathfrak{X}^+$) by the collection of all conjugates of $h_{\mathbf{0}}$ under $GU(\mathbb{R})$ (resp. under $SU(\mathbb{R})$). Any two homomorphisms satisfying (pos) are conjugates under $SU(\mathbb{R})$ (see [**PAF**] Lemma 7.3). Thus $\mathfrak{X}^+ = SU(\mathbb{R})/C_{\mathbf{0}}$ for the stabilizer $C_{\mathbf{0}}$ of $h_{\mathbf{0}}$ in $SU(\mathbb{R})$ is connected and is a connected component of $\mathfrak{X}$. On $\mathfrak{X}$, $GU(\mathbb{R})$ acts by conjugation (from the left), and by (pos) the stabilizer $C_{\mathbf{0}} \subset GU(\mathbb{R})$ of $h_{\mathbf{0}}$ is a maximal compact subgroup of $GU(\mathbb{R})$ modulo center.

EXAMPLE 2.3. Assume that $s = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and take $h_{\mathbf{0}}(a+bi) = \left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$. Since $h_{\mathbf{0}}(\mathbb{C}^\times)$ gives the stabilizer of $i \in \mathfrak{H} = \{z \in \mathbb{C} | \operatorname{Im}(z) > 0\}$, we have $\mathfrak{X}^+ \cong \mathfrak{H}$ by sending $g h_{\mathbf{0}} g^{-1}$ to $g(i)$. We also have $\mathfrak{X} \cong \mathfrak{H} \sqcup \overline{\mathfrak{H}} = (\mathbb{C} - \mathbb{R})$ in the same way.

Since $h : \mathbb{C} \to C_\infty$ is an $\mathbb{R}$-algebra homomorphism, we can split $V_{\mathbb{C}} = V \otimes_{\mathbb{Q}} \mathbb{C}$ into the direct sum of eigenspaces $V_{\mathbb{C}} = V_1 \oplus V_2$ so that $h(z)$ acts on V_1 (resp. V_2) through multiplication by z (resp. $\overline{z}$); thereby, we get a complex vector space structure on V_∞ by the projection $V_\infty \cong V_1$. Since $h(\mathbb{C}) \subset C_\infty$, $h(z)$ commutes with the action of F; so, V_j is stable under the action of $F_{\mathbb{C}} = F \otimes_{\mathbb{Q}} \mathbb{C}$. We get the representation $\rho_1 : F \hookrightarrow \mathrm{End}_{\mathbb{C}}(V_1)$. We define E for the subfield of $\mathbb{C}$ fixed by the open subgroup $\left\{\sigma \in \mathrm{Aut}(\mathbb{C}) \middle| \rho_1^\sigma \cong \rho_1\right\}$. If $h'(z) = g \cdot h(z) g^{-1}$ for $g \in GU(\mathbb{R})$, h' induces a similar decomposition $V_{\mathbb{C}} = V_1' \oplus V_2'$, and g induces a F-linear isomorphism between V_1 and V_1'; so, E is independent of the choice of h' in the $GU(\mathbb{R})$-conjugacy class of h. This field E is called the *reflex field* of $(GU, \mathfrak{X})$ (and is a canonical field of definition of our canonical models of the Shimura variety).

By the positivity (pos), the quotient complex torus $V_\infty/L = V_1/L$ has a Riemann form induced by $\langle \cdot, \cdot \rangle$. The theta functions with respect to the Hermitian form $\langle \cdot, \cdot \rangle$ give rise to global sections of an ample line bundle (e.g., [**ABV**] Chapter I) on V_1/L and hence embed V_1/L into a projective space over $\mathbb{C}$. The embedded image is the analytic space $A_h(\mathbb{C})$ associated with an abelian variety $A_{h/\mathbb{C}}$ by Chow's theorem (see [**ABV**] page 33). Multiplication by $b \in O$ on V_1/L induces an embedding $i : O \hookrightarrow \mathrm{End}(A_{h/\mathbb{C}})$ and $i : F \hookrightarrow \mathrm{End}^{\mathbb{Q}}(A_{h/\mathbb{C}}) = \mathrm{End}(A_{h/\mathbb{C}}) \otimes_{\mathbb{Z}} \mathbb{Q}$.

The representation ρ_1 is given by the action of F on the Lie algebra $Lie(A_h) = V_1$ at the origin of $A_h(\mathbb{C})$. Since A_h is projective, the field of definition of the abelian variety A_h is a field of finite type over $\mathbb{Q}$.

The reflex field E is the field of rationality of the representation of F on $Lie(A_h)$; so, the field of definition of (A_h, ι) always contains this field E. It would then be natural to expect that the moduli variety of triples (A, λ, ι) for an abelian variety A with F-linear isomorphism $Lie(A) \cong V_1$ is defined over E.

Since the isomorphism class of ρ_1 is determined by $\mathrm{Tr}(\rho_1)$ (see [**MFG**] Proposition 2.9), E is generated over $\mathbb{Q}$ by $\mathrm{Tr}(\rho_1(b))$ for all $b \in F$. Thus we have $E = F$ or $\mathbb{Q}$ and that $E = \mathbb{Q}$ implies $m_1 = m_c$, because $\mathrm{Tr}(\rho_1(\xi)) = m_1\xi + m_c\xi^c$ for $\xi \in F$. We write O_E for the integer ring of E. Let $\mathbb{Z}_{(p)} = \mathbb{Z}_p \cap \mathbb{Q}$, put $O_{(p)} = O \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$, $O_{E,(p)} = O_E \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$, and write $\mathcal{V}$ for the valuation ring $\mathcal{W} \cap E \supset O_{E,(p)}$ (so, $\mathcal{V}$ is the localization of O_E at $\mathfrak{p}$). More generally, for a finite set of places Σ, we write $\mathbb{Z}_\Sigma$ for the product of $\mathbb{Z}_\ell$ over finite places $\ell \in \Sigma$, and we put $\mathbb{Z}_{(\Sigma)} = \mathbb{Q} \cap \mathbb{Z}_\Sigma$ and $O_{(\Sigma)} = O \otimes_{\mathbb{Z}} \mathbb{Z}_{(\Sigma)}$. The ring $\mathcal{V}$ has residue field $\mathbb{F}_p$ since p is split in E because $E \subset F$.

2.3. Shimura variety for GU. We study the classification problem of the following quadruples $(A, \lambda, i, \overline{\eta}^{(p)})_{/R}$: A is a (projective) abelian scheme over a base R, ${}^tA = \mathrm{Pic}^0_{A/R}(A)$ is the dual abelian scheme of A, $\lambda : A \to {}^tA$ is a prime-to-p polarization (that is, an isogeny with degree prime to p fiber-by-fiber geometrically induced from an ample divisor), $i : O_{(p)} \hookrightarrow \mathrm{End}_R^{\mathbb{Z}_{(p)}}(A) = \mathrm{End}_R(A) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ is a $\mathbb{Z}_{(p)}$-algebra embedding (taking 1 to the identity of A) with $\lambda \circ i(\alpha^c) = {}^ti(\alpha) \circ \lambda$ for all $\alpha \in O$, and $\eta^{(p)}$ is a level structure. Regarding ${}^tA$ as a left O-module by $O \ni b \mapsto {}^ti(b^c) \in \mathrm{End}({}^tA)$, λ is F-linear. Hereafter we call λ *F-linear* in this sense. The base scheme R is assumed to be a scheme over $\mathrm{Spec}(\mathcal{V})$.

We clarify the meaning of the level structure $\eta^{(p)}$. Fix a base (geometric) point $s \in R$ and write A_s for the fiber of A at s. We consider the Tate module $\mathcal{T}(A_s) = \varprojlim_N A[N](k(s))$ and $V^{(p)}(A_s) = \mathcal{T}(A_s) \otimes_{\mathbb{Z}} \mathbb{A}^{(p\infty)}$, where N runs over all positive integers ordered by divisibility. The prime-to-p level structure $\eta^{(p)} : V(\mathbb{A}^{(p\infty)}) = V \otimes_{\mathbb{Q}} \mathbb{A}^{(p\infty)} \cong V^{(p)}(A_s)$ is an O-linear isomorphism. The duality pairing $e_N : A[N] \times {}^tA[N] \to \mu_N$ composed with λ gives, after taking the limit with respect to N, an alternating form $(\cdot,\cdot)_\lambda : V^{(p)}(A_s) \times V^{(p)}(A_s) \to \mathbb{A}^{(p\infty)}(1) := \lim_{p\nmid N} \mu_N$ satisfying the following conditions:

(P1) $(\alpha(x), y)_\lambda = (x, \alpha^c(y))_\lambda$ for $\alpha \in \mathrm{End}(A_{/B})$;
(P2) The pairing induces the self-duality: $A[p^n] \cong \mathrm{Hom}(A[p^n], \mu_{p^n})$ if $N = p^n$.

We require that $\eta^{(p)}$ send the alternating form $\langle\cdot,\cdot\rangle$ to $(\cdot,\cdot)_\lambda$ up to multiple of scalars in $(\mathbb{A}^{(p\infty)})^\times$. This is possible, because $\mathbb{A}^{(p\infty)}(1) \cong \mathbb{A}^{(p\infty)}$ up to scalar in $(\mathbb{A}^{(p\infty)})^\times$. Then $\eta^{(p)}$ is required to be an isomorphism of skew Hermitian F-modules with respect to the pairing $\langle\cdot,\cdot\rangle_\lambda$ on $V^{(p)}(A_s)$.

The algebraic fundamental group $\pi_1(R, s)$ acts on $V^{(p)}(A_s)$ preserving the skew Hermitian form $\langle\cdot,\cdot\rangle_\lambda$ up to scalar in $(\mathbb{A}^{(p\infty)})^\times$ (because it keeps the Weil e_N-pairing; see [**ABV**] Section 20). Take a closed subgroup $K^{(p)} \subset GU(\mathbb{A}^{(p\infty)})$. We write $\overline{\eta}^{(p)}$ for the orbit $\eta^{(p)} \circ K^{(p)}$. If $\sigma \circ \overline{\eta}^{(p)} = \overline{\eta}^{(p)}$ for all $\sigma \in \pi_1(R, s)$, we say the level structure $\overline{\eta}^{(p)}$ is defined over R. Even if we change the point $s \in R$, everything will be conjugated by an isomorphism; so, the definition does not depend on the choice of s as long as R is connected. For nonconnected R, we choose one geometric point at each connected component.

A quadruple $\underline{A}_{/R} = (A, \lambda, i, \overline{\eta}^{(p)})$ is *isomorphic* to $\underline{A}'_{/R} = (A', \lambda', i', \overline{\eta}'^{(p)})$ if we have an O-linear isogeny $\phi : A \to A'$ defined over R such that $p \nmid \deg(\phi)$,

$\phi^*\lambda' = {}^t\phi \circ \lambda' \circ \phi = \nu\lambda$ with $\nu \in \mathbb{Z}_{(p)+}^\times$, $\phi \circ i \circ \phi^{-1} = i'$, and $\overline{\eta}'^{(p)} = \phi \circ \overline{\eta}^{(p)}$. Here $\mathbb{Z}_{(p)+}^\times$ is the collection of all positive elements in $\mathbb{Z}_{(p)}^\times$. Thus ϕ brings the prime-to-p polarization class $\overline{\lambda}' = \{\nu\lambda' | \nu \in \mathbb{Z}_{(p)+}^\times\}$ of λ' to the class $\overline{\lambda}$ of λ: $\phi^*\overline{\lambda}' = \overline{\lambda}$. In this case, we write $A \approx A'$. We write $A \cong A'$ if the isogeny is an isomorphism of abelian schemes; that is, $\deg(\phi) = 1$.

We take the fibered category $\mathcal{C} = \mathcal{C}_{F,V}$ of the quadruples $(A, \lambda, i, \eta^{(p)})_{/R}$ over the category $\mathcal{V}$-SCH of $\mathcal{V}$-schemes and define

$$\text{(2.2)}\quad \mathrm{Hom}_{\mathcal{C}_{/R}}((A, \lambda, i, \eta^{(p)})_{/R}, (A', \lambda', i', \eta'^{(p)})_{/R}) = \left\{ \phi \in \mathrm{Hom}_R(A, A') \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \middle| \begin{array}{c} {}^t\phi \circ \lambda' \circ \phi = \nu\lambda \ \text{ with } \ 0 < \nu \in \mathbb{Z}_{(p)+}^\times, \\ \phi \circ i = i' \circ \phi \ \text{ and } \ \eta'^{(p)} = \phi \circ \eta^{(p)} \end{array} \right\}.$$

The representation ρ_1 is well defined over $\mathcal{V}$, since p splits in F; so, it is well defined over $\mathcal{O}_R$ for any $\mathcal{V}$-scheme R. We consider the functor $\mathcal{E}^{(p)} : \mathcal{V}\text{-}SCH \to SETS$ given by

$$\mathcal{E}^{(p)}(R) = \left\{ \underline{A}_{/R} = (A, \overline{\lambda}, i, \eta^{(p)})_{/R} \middle| Lie(A) \cong \rho_1 \text{ over } \mathcal{O}_R \right\} / \approx .$$

Since $A_{/R}$ is a group scheme, its tangent space at the zero section has a Lie algebra structure over $\mathcal{O}_R$. We write $Lie(A)$ for this Lie algebra. Since A is smooth over R, $Lie(A)$ is a locally free $\mathcal{O}_R$-module of rank $\dim_R A$. In our case, for a given quadruple $\underline{A} = (A, \lambda, i, \overline{\eta}^{(p)})_{/R}$, the Lie algebra $Lie(A)$ of A over $\mathcal{O}_R$ is an $O_{(p)}$-module via i. Since $Lie(A)$ is locally free of rank $\dim_R A$ over $\mathcal{O}_R$, we can think of an isomorphism $Lie(A) \cong \rho_1$ of $\mathcal{O}_R$-representations of $O_{(p)}$. One can find in [**PAF**] Chapter 7 a proof of the following theorem due to Shimura, Deligne and Kottwitz.

Theorem 2.1. *The functor $\mathcal{E}^{(p)}$ is representable by a quasi-projective smooth pro-scheme $Sh^{(p)}$ over $\mathcal{V}$. Letting $g \in GU(\mathbb{A}^{(p\infty)})$ act on $Sh^{(p)}$ by $\eta^{(p)} \mapsto \eta^{(p)} \circ g$, for any open compact subgroup $K \subset G(\mathbb{A}^{(p\infty)})$, the quotient scheme $Sh_K^{(p)} = Sh^{(p)}/K$ exists as a quasi-projective scheme of finite type over $\mathcal{V}$, and $Sh^{(p)} = \varprojlim_K Sh_K^{(p)}$. The Shimura variety $Sh_K^{(p)}$ is projective over $\mathcal{V}$ if the Hermitian pairing $\langle \cdot, \cdot \rangle$ is anisotropic.*

For a finite set of primes Σ containing p and ∞, we can think of the Shimura variety away from Σ as follows. Write $\Sigma = \{p, \infty\} \sqcup \Sigma'$. If $\Sigma' \neq \emptyset$, let $GU(\mathbb{Z}_{\Sigma'}) = \{g \in GU(\mathbb{Q}_{\Sigma'}) | gL_{\Sigma'} = L_{\Sigma'}\}$, and put $Sh^{(\Sigma)} = Sh^{(p)}/GU(\mathbb{Z}_{\Sigma'})$. It is known that $Sh_{/\mathcal{V}}^{(\Sigma)}$ is a smooth (quasi-projective) pro-scheme.

Recall the embedding $i_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ and the valuation ring $\mathcal{W}$ which is the pull-back by i_p of the p-adic integer ring of the maximal unramified extension of $\mathbb{Q}_p$. By our choice, $1 : F \hookrightarrow \overline{\mathbb{Q}} \overset{i_p}{\hookrightarrow} \overline{\mathbb{Q}}_p$ induces the valuation ring $\mathcal{V}$. Write $\mathcal{K}$ be the filed of fraction of $\mathcal{W}$. Let $Sh_{/\mathcal{W}}^{(\Sigma)} = Sh^{(\Sigma)} \times_{\mathrm{Spec}(\mathcal{V})} \mathrm{Spec}(\mathcal{W})$ and put $W = \varprojlim_n \mathcal{W}/p^n\mathcal{W}$. By the reduction map (see [**ACS**] Corollary 6.4.1.3), we have $\pi_0(Sh_{/\mathcal{K}}^{(\Sigma)}) \cong \pi_0(Sh_{/\mathbb{F}}^{(\Sigma)})$ for $Sh_{/\mathbb{F}}^{(\Sigma)} = Sh_{/\mathcal{W}}^{(\Sigma)} \times_{\mathcal{W}} \mathbb{F}$ by Zariski's connectedness theorem and the existence of smooth toroidal compactification of $Sh_{K/W}^{(p)}$, and $SU(\mathbb{A}^{(\Sigma)})$ leaves stable each irreducible component in $\pi_0(Sh_{/\mathcal{K}}^{(\Sigma)})$ because $\mathfrak{X}^+$ is a quotient of $SU(\mathbb{R})$. A proof of the existence of smooth toroidal compactification of $Sh^{(p)}_{K/W}$ can be also found in [**ACS**] 6.4.1. Thus, by the existence of a smooth toroidal compactification of $Sh^{(\Sigma)}_{/\mathcal{W}}$ (and Zariski's connectedness theorem), we get

PROPOSITION 2.2. *Geometrically irreducible components of $Sh^{(\Sigma)}_{/\mathcal{K}}$ are generic fibers of irreducible components of $Sh^{(\Sigma)}_{/\mathcal{W}}$. Each irreducible component of $Sh^{(\Sigma)}_{/\mathcal{W}}$ has irreducible special fiber over $\mathbb{F}$, and the group $SU(\mathbb{A}^{(\Sigma)})$ leaves stable each irreducible component of the Shimura variety $Sh^{(\Sigma)}_{/\mathbb{F}} = Sh^{(\Sigma)}_{/\mathcal{W}} \times_{\mathcal{W}} \mathbb{F}$.*

We can compute the stabilizer in $GU(\mathbb{A}^{(\Sigma)})$ of each point of $\pi_0(Sh^{(\Sigma)}_{/\mathbb{F}})$ explicitly ([**H06**] Lemma 1.1).

3. Igusa tower over unitary Shimura variety

We first define the Igusa tower over the GU Shimura variety and prove that the tower is not irreducible. Then we prove the irreducibility of the partial SU-tower. Let $G(\mathbb{Z}_p) = \{g \in G(\mathbb{Q}_p) | gL_p = L_p\}$ for $G = GU, U$ and SU. Let Σ be a finite set of rational places including p and ∞.

3.1. Unitary group over $\mathbb{Z}_p$. Recall our simplifying assumption: $p = \mathfrak{p}\overline{\mathfrak{p}}$ ($\mathfrak{p} \neq \overline{\mathfrak{p}}$) in O so that $\mathfrak{p}$ is induced by i_p. Since $O_p = O_{\mathfrak{p}} \times O_{\overline{\mathfrak{p}}} = \mathbb{Z}_p \times \mathbb{Z}_p$ on which c acts interchanging the coordinates: $(x,y)^c = (y,x)$ and $\xi \in O$ is sent to $(\xi, \xi^c) \in \mathbb{Z}_p \times \mathbb{Z}_p$, we have $GL_r(O_p) = GL_r(O_{\mathfrak{p}}) \times GL_r(O_{\overline{\mathfrak{p}}}) = GL_r(\mathbb{Z}_p) \times GL_r(\mathbb{Z}_p)$. Since $x^\iota = s^{-1}\,{}^t x^c s$ for the skew hermitian matrix $s = -{}^t s^c$, if $(x,y) \in U(\mathbb{Z}_p)$, we have

$$(x^{-1}, y^{-1}) = (x,y)^{-1} = x^\iota = (s, s^c)^{-1}({}^t y, {}^t x)(s, s^c) = (s^{-1}\,{}^t y s, s^{-c}\,{}^t x s^c)$$

and $y = {}^t s^{-1} x^{-1}\,{}^t s$. Thus, choosing a basis of L_p over O_p, we have $U(\mathbb{Z}_p) \cong GL_r(\mathbb{Z}_p)$ by sending $(x,y) \in U(\mathbb{Z}_p)$ to $x \in GL_r(\mathbb{Z}_p)$. Similarly, $SU(\mathbb{Z}_p) \cong SL_r(\mathbb{Z}_p)$ and $GU(\mathbb{Z}_p) \cong GL_r(\mathbb{Z}_p) \times GL_1(\mathbb{Z}_p)$ by $g = (x,y) \mapsto (x\nu(x,y)^{-1}, \nu(x,y))$.

3.2. The Igusa tower. Let $S_{/\mathcal{W}} = S^{(\Sigma)}_{/\mathcal{W}}$ be an irreducible component of the ordinary locus of $Sh^{(\Sigma)}_{/\mathcal{W}}$. Thus S is the subscheme obtained from $Sh^{(\Sigma)}_{/\mathcal{W}}$ removing the closed subscheme of non-ordinary locus at the special fiber at p. By $\langle \cdot, \cdot \rangle$, L_p is self dual. Since $O_p = O_{\mathfrak{p}} \oplus O_{\overline{\mathfrak{p}}}$, we have the corresponding decomposition $L_p = L_{\mathfrak{p}} \oplus L_{\overline{\mathfrak{p}}}$.

Let $\mathbf{A}_{/S}$ be the universal ordinary abelian scheme over S with its fiber A_x at $x \in S$. Pick a base point x_0 of $S(W)$ ($W = \varprojlim_n \mathcal{W}/p^n\mathcal{W}$) with reduction $\overline{x}_0 \in S(\mathbb{F})$ modulo p. We fix an identification: $L_{\mathfrak{p}} \cong T_p A_{x_0}[\mathfrak{p}^\infty]$ for the p-adic Tate module $T_p A_{x_0}[\mathfrak{p}^\infty]$ of the Barsotti-Tate group $A_{x_0}[\mathfrak{p}^\infty]$. Then over the formal completion $\widehat{S}$ along the special fiber, we have the reduction map $T_p A_{x_0}[\mathfrak{p}^\infty] \to T_p A_{\overline{x}_0}[\mathfrak{p}^\infty]^{et}$. The kernel of the reduction map gives rise to an $O_{\mathfrak{p}}$-direct summand $L_1 \subset L_{\mathfrak{p}}$. Since O acts on the tangent space at 0 via the identity inclusion into $\mathbb{Z}_p$ by multiplicity m_1 and the tangent space of $A[\mathfrak{p}]^\circ_{/x_0}$ is equal to this eigenspace in the tangent space of $A_{\overline{x}_0}$, we find that $L_1 \otimes_{O_{\mathfrak{p}}} \mathbb{F}_p \cong \mathbb{F}_p^{m_1}$; so, $L_1 \cong O_{\mathfrak{p}}^{m_1}$. Similarly, we define $L_c \subset L_{\overline{\mathfrak{p}}}$ using the reduction map on $\overline{\mathfrak{p}}$-torsion points of A_{x_0}. Then $L_c \cong O_{\overline{\mathfrak{p}}}^{m_c}$. Note that $L_{\mathfrak{p}}/L_1 \cong \mathrm{Hom}_{\mathbb{Z}_p}(L_c, \mathbb{Z}_p)$ and $L_{\overline{\mathfrak{p}}}/L_c \cong \mathrm{Hom}_{\mathbb{Z}_p}(L_1, \mathbb{Z}_p)$ by $\langle \cdot, \cdot \rangle$. Let $\mathcal{L} = (L_1 \oplus L_c)$ as O-modules.

We consider the functor $I_n = I_n^{(\Sigma)}$ from the category of $S_{/\mathcal{W}}$-schemes R into the category of sets taking R to the set of O-linear closed immersions of $\mathcal{L}\otimes_{\mathbb{Z}}\mu_{p^n/R}$ into $\mathbf{A}_{/R}[p^n]$, where $\mathbf{A}_{/R} = \mathbf{A}\times_S R$. Since the two schemes $\mathcal{L}\otimes_{\mathbb{Z}}\mu_{p^n}$ and $\mathbf{A}[p^n]$ are finite flat over S, by the theory of Hilbert scheme, this functor is representable by a scheme I_n. Then $I_{n/\mathcal{W}}$ classifies quintuples $(A, i, \overline{\lambda}, \eta^{(\Sigma)}, \phi_p)$ for an O-linear closed immersion $\phi_p : \mathcal{L}\otimes_{\mathbb{Z}}\mu_{p^n} \hookrightarrow A[p^n]$.

The formal completion $\widehat{S}$ along the special fiber $S_{/\mathbb{F}} = S\times_{\mathcal{W}}\mathbb{F}$ is a formal W-scheme. The connected component $\mathbf{A}[p^n]^\circ$ of $\mathbf{A}[p^n]$ is well defined over $\widehat{S}$, and hence the formal completion $\widehat{I}_{n/W}$ of I_n along its special fiber $I_{n/\mathbb{F}} = I_n\times_{\mathcal{W}}\mathbb{F}$ can be written as $\mathrm{Isom}_{\widehat{S}}(\mathcal{L}\otimes_{\mathbb{Z}}\mu_{p^n}, \mathbf{A}[p^n]^\circ)$. Then $\widehat{I}_{n/\widehat{S}}$ is isomorphic to the scheme $\mathrm{Isom}_{\widehat{S}}(\mathcal{L}^\vee/p^n\mathcal{L}^\vee, \mathbf{A}[p^n]^{et})$ étale finite over $\widehat{S}$, since by duality, $\phi_p : \mathcal{L}\otimes_{\mathbb{Z}}\mu_{p^n} \cong \mathbf{A}[p^n]^\circ$ gives rise to ${}^t\phi_p^{-1} : \mathcal{L}^\vee/p^n\mathcal{L}^\vee \cong \mathbf{A}[p^n]^{et}$ for $\mathcal{L}^\vee = L_p/\mathcal{L}$.

Let $I_{/\mathcal{W}} = \varprojlim_n I_{n/\mathcal{W}}$. Its special fiber

$$I_{/\mathcal{W}}\times_{\mathcal{W}}\mathbb{F} = \varprojlim_n I_{n/\mathbb{F}}$$

is called the Igusa tower over $S_{/\mathbb{F}}$. By the projection $L_p \twoheadrightarrow L_{\mathfrak{p}}$, we have $U(\mathbb{Z}_p) \cong GL(L_{\mathfrak{p}}) \cong GL_r(\mathbb{Z}_p)$. Consider the universal level structure $\boldsymbol{\phi}_p : \mathcal{L}\otimes\mu_{p^\infty} \hookrightarrow \mathbf{A}[p^\infty]$ over I. The group $GU(\mathbb{Z}_p)$ acts on L. Let

$$P(\mathbb{Z}_p) = \left\{g \in U(\mathbb{Z}_p) = GL(L_{\mathfrak{p}}) \middle| gL_1 = L_1\right\}.$$

Then $P(\mathbb{Z}_p)$ is a parabolic subgroup of $U(\mathbb{Z}_p) = GL_r(\mathbb{Z}_p)$ of the form, identifying $GL(L_1) = GL_{m_1}(\mathbb{Z}_p)$ and $GL(L_{\mathfrak{p}}/L_1) = GL_{m_c}(\mathbb{Z}_p)$,

$$\left\{\left(\begin{smallmatrix} a & b\\ 0 & d\end{smallmatrix}\right) \middle| (a,d) \in GL(L_1)\times GL(L_c^\vee)\right\} \cong \left\{\left(\begin{smallmatrix} a & b\\ 0 & d\end{smallmatrix}\right) \middle| (a,d) \in GL_{m_1}(\mathbb{Z}_p)\times GL_{m_c}(\mathbb{Z}_p)\right\}.$$

Here the action of $d \in GL(L_c^\vee)$ on $L_c^\vee = L_{\mathfrak{p}}/L_1$ is given by the matrix d and hence its acts on $L_c = \mathrm{Hom}(L_{\mathfrak{p}}/L_1, \mathbb{Z}_p)$ by the dual action (induced by $\langle\cdot,\cdot\rangle$) written as d^{-*}. Define $M(\mathbb{Z}_p) = GL(L_1)\times GL(L_c^\vee)$ for the reductive part of P. Put $M_1(\mathbb{Z}_p) = M(\mathbb{Z}_p)\cap SU(\mathbb{Z}_p)$. Then $M(\mathbb{Z}_p)$ acts on each fiber of I transitively, since $I/S_{/\mathbb{F}}$ is an $M(\mathbb{Z}_p)$-torsor by the action

$$(\phi_{\mathfrak{p}}, \phi_{\overline{\mathfrak{p}}})\circ(a,d) = (\phi_{\mathfrak{p}}\circ a, \phi_{\overline{\mathfrak{p}}}\circ d^{-*}),$$

where the original action of d on $L_{\mathfrak{p}}/L_1$ is dualized by the polarization pairing

$$\langle\cdot,\cdot\rangle_\lambda : \mathbf{A}[\mathfrak{p}^\infty]^{et}\times\varprojlim_n \mathbf{A}[\overline{\mathfrak{p}}^n]^\circ \to \mu_{p^\infty}.$$

3.3. Reducibility and irreducibility. First, we may assume that $S(\mathbb{C})$ is the image of $SU(\mathbb{A}^{(\Sigma)})\times\mathfrak{X}^+$ in $Sh^{(\Sigma)}(\mathbb{C}) = GU(\mathbb{Q})\backslash(GU(\mathbb{A}^{(\infty)})\times\mathfrak{X})/GU(\mathbb{Z}_\Sigma)\overline{Z(\mathbb{Q})}$, where $\mathbb{Z}_\Sigma = \prod_{\ell\in\Sigma-\{\infty\}}\mathbb{Z}_\ell$, $\mathbb{Q}_\Sigma = \prod_{\ell\in\Sigma-\{\infty\}}\mathbb{Q}_\ell$, $\mathbb{Z}_{(\Sigma)} = \mathbb{Q}\cap\mathbb{Z}_\Sigma$ in $\mathbb{Q}_\Sigma$ and $GU(\mathbb{Z}_\Sigma) = \{x \in GU(\mathbb{Q}_\Sigma) | xL_\Sigma = L_\Sigma\}$ for $L_\Sigma = L\otimes_{\mathbb{Z}}\mathbb{Z}_\Sigma$.

On S, the universal level structure $\boldsymbol{\eta}^{(\Sigma)} : V(\mathbb{A}^{(\Sigma)}) \cong V^{(\Sigma)}(\mathbf{A})$ induces the trivialization of étale $\mathbb{A}^{(\Sigma)}$-sheaf:

$$\det(\boldsymbol{\eta}^{(\Sigma)}) : \mathbb{A}^{(\Sigma)} \cong \bigwedge_{F_{\mathbb{A}^{(\Sigma)}}}^r V(\mathbb{A}^{(\Sigma)}) \cong \bigwedge_{F_{\mathbb{A}^{(\Sigma)}}}^r V^{(\Sigma)}(\mathbf{A}).$$

For any prime ℓ outside Σ, take an open compact subgroup K of $GU(\mathbb{A}^{(\Sigma)})$ such that $K = K_\ell\times K^{(\ell)}$ with $K_\ell = \{x \in GU(\mathbb{Z}_\ell) | xL_\ell = L_\ell\}$ and that $Sh^{(\Sigma)}/Sh_K^{(\Sigma)}$ ($Sh_K^{(\Sigma)} = Sh^{(\Sigma)}/K$) is an étale covering. Then for the principal congruence subgroup $K(\ell^n) \subset$

K modulo ℓ^n, $Sh_{K(\ell^n)/\mathcal{W}}$ is constructed as $\mathrm{Isom}_{Sh_K^{(\Sigma)}}(L/\ell^n L, \mathbf{A}_K[\ell^n])$ for the universal abelian scheme $\mathbf{A}_K$ over $Sh_K^{(\Sigma)}$. Let S_K be the image of S in $Sh_K^{(\Sigma)}$ and write again as x_0 the image of x_0 in S_K. By this expression, the action of $\pi_1(S_K, x_0)$ on the étale sheaf $\mathbf{A}_K[\ell^n]_{/S_K}$ factors through the action of $K_\ell \cap SU(\mathbb{Z}_\ell)$. In particular, its action on $\bigwedge^r_{O_\ell} \mathbf{A}_K[\ell^n]$ factors through $\det : K_\ell \cap SU(\mathbb{Z}_\ell) \to O_\ell^\times$ which is the trivial character by the definition of SU. Thus $\bigwedge^r_{O_\ell} \mathbf{A}_K[\ell^n]$ is a constant étale sheaf over $S_{K/\mathcal{W}}$. In other words, the action of $GU(\mathbb{A}^{(\Sigma)})$ on $\bigwedge^r_{F_{\mathbb{A}^{(\Sigma)}}} V(\mathbb{A}^{(\Sigma)})$ factors through determinant map, it is trivial on $SU(\mathbb{A}^{(\Sigma)})$, and $V^{(\Sigma)}(\mathbf{A})$ over the irreducible component $S_{/\mathcal{W}}$ is constant; so, the ℓ-adic sheaf $\bigwedge^r_{O_\ell} \mathcal{T}_\ell \mathbf{A}$ ($\mathcal{T}_\ell \mathbf{A} = \varprojlim_n \mathbf{A}[\ell^n]$) is identical to $\bigwedge^r_{O_\ell} \mathcal{T}_\ell A_x$ for the fiber of $\mathbf{A}$ at any closed point $x \in S(\mathcal{K})$.

For any exact sequence of free $\mathbb{Z}_p$-modules $X_1 \hookrightarrow X \twoheadrightarrow X_2$ with rank r_1, r and r_2 respectively, we have a natural direct summand $\bigwedge^{r_1} X_1 \otimes \bigwedge^{r_2} X_2$ in $\bigwedge^r X$, because ambiguity of lifting $x_2 \in X_2$ to $x \in X$ is killed by wedge product with $\bigwedge^{r_1} X_1$.

As for the fppf abelian sheaf $\bigwedge^{m_1}_{O_\mathfrak{p}} \mathbf{A}[\mathfrak{p}^n]^\circ_{/\widehat{S}_{/W}}$ over $\widehat{S}_{/W}$, it is isomorphic to $\bigwedge^{m_1}_{\mathbb{Z}_p}(O_\mathfrak{p} \otimes \mu_{p^n})^{m_1}$; so, its dual étale sheaf $\bigwedge^{m_1}_{O_{\overline{\mathfrak{p}}}} \mathbf{A}[\overline{\mathfrak{p}}^n]^{et}_{/\widehat{S}_{/\mathcal{W}}}$ is constant over $\widehat{S}_{/W}$. Similarly $\bigwedge^{m_c}_{O_\mathfrak{p}} \mathbf{A}[\mathfrak{p}^n]^{et}_{/\widehat{S}_{/\mathcal{W}}}$ is constant. Thus

$$\mathbb{E}[p^n] = \bigwedge^{m_c}_{\mathbb{Z}_p} \mathbf{A}[\mathfrak{p}^n]^{et} \otimes_{\mathbb{Z}_p} \bigwedge^{m_1}_{\mathbb{Z}_p} \mathbf{A}[\overline{\mathfrak{p}}^n]^{et}_{/\widehat{S}}$$

is isomorphic to the constant sheaf $\mathbb{Z}/p^n\mathbb{Z}$ over $\widehat{S}_{/W}$. Thus we have a morphism

$$\det : \widehat{I}_{n/\widehat{S}} = \mathrm{Isom}_{\widehat{S}}\Big(\frac{\mathcal{L}^\vee}{p^n \mathcal{L}^\vee}, \mathbf{A}[p^n]^{et}\Big) \to \mathrm{Isom}\Big(\bigwedge^{m_c} \frac{L_1^\vee}{p^n L_1^\vee} \otimes_{\mathbb{Z}_p} \bigwedge^{m_1} \frac{L_c^\vee}{p^n L_c^\vee}, \mathbb{E}[p^n]\Big) \cong (\mathbb{Z}/p^n\mathbb{Z})^\times$$

over $\widehat{S}$ taking ${}^t\phi_p^{-1} : \mathcal{L}^\vee/p^n\mathcal{L}^\vee \cong \mathbf{A}[p^n]^{et}$ to

$$\left(\bigwedge^{m_1} ({}^t\phi_p^{-1}|_{L_c^\vee/p^n L_c^\vee}) \otimes \bigwedge^{m_c} ({}^t\phi_p^{-1}|_{L_1^\vee/p^n L_1^\vee})\right).$$

Pick a generator

$$v \in \varprojlim_n \mathrm{Isom}\Big(\bigwedge^{m_c} \frac{L_1^\vee}{p^n L_1^\vee} \otimes_{\mathbb{Z}_p} \bigwedge^{m_1} \frac{L_c^\vee}{p^n L_c^\vee}, \mathbb{E}[p^n]\Big)$$

over $\mathbb{Z}_p$, and define $I_n^{SU} = I_n^{SU,(\Sigma)} = \det^{-1}(v \mod p^n)$ and $I^{SU} = I^{SU,(\Sigma)} = \varprojlim_n I_n^{SU,(\Sigma)}$. We claim

Theorem 3.1. *For each finite set Σ of rational places containing p and ∞, $I_n^{SU,(\Sigma)}/S$ is a geometrically irreducible component of $I_{n/S}$.*

3.4. Proof. By construction, $I_n^{SU,(\Sigma)}$ contains an irreducible component of $I^{(\Sigma)}_{n/S}$. Thus we need to prove irreducibility of $I_n^{SU,(\Sigma)}/S$ showing axioms (A1–2). For a point $x \in I^{SU,(\Sigma)}(\mathbb{F})$, consider the formal completion $\widehat{I}^{SU,(\Sigma)}_{x/W}$ along x. Then

$\mathcal{O}_{\widehat{I}^{SU,(\Sigma)}_{x/W}} \cong W[[X_1,\ldots,X_d]]$ for $d = \dim_{\mathcal{W}} S$ ($\Leftrightarrow \widehat{I}^{SU,(\Sigma)}_{x/W} \cong \mathrm{Spf}(W[[X_1,\ldots,X_d]])$). Define the valuation $v_x : \mathcal{O}_{\widehat{I}^{SU,(\Sigma)}_{x/W}} \to \mathbb{Z}\cup\{\infty\}$ as already mentioned:

$$v_x(\sum_\alpha c(\alpha)X^\alpha) = \mathrm{Inf}_\alpha\, \mathrm{ord}_p(c(\alpha))$$

where $X^\alpha = X_1^{\alpha_1}X_2^{\alpha_2}\cdots X_d^{\alpha_d}$. Then the stalk $\mathcal{O}_{I^{SU,(\Sigma)},x} \subset \mathcal{O}_{\widehat{I}^{SU,(\Sigma)}_{x/W}}$ inherits the valuation v_x and hence its function field $\mathfrak{F} = \mathcal{K}(I^{SU,(\Sigma)})$ gets the valuation v_x. The valuation v_x is unramified over the function field $\mathfrak{F}_{S^{(\Sigma)}} = \mathcal{K}(S^{(\Sigma)})$. Let D (resp. T_x) be the stabilizer of v_x (resp. x) in $M_1(\mathbb{Z}_p)\times SU(\mathbb{A}^{(\Sigma)})$. Then $T_x \subset D$.

First take Σ to be Σ_0 given by $\{p,\infty\}\cup\{\ell|SU \text{ is not quasi-split at } \ell\}$. Then $SU(\mathbb{A}^{(\Sigma)})$ does not have any finite quotient. In particular, $SU(\mathbb{A}^{(\Sigma)})$ fixes each connected component of I_n^{SU}, and $SU(\mathbb{A}^{(\Sigma)}) \subset D$. As will be seen in the following section, we can find one base point $x = x_0$ such that T_{x_0} has p-adically dense image in $M_1(\mathbb{Z}_p)$ under the projection: $SU(\mathbb{A}^{(\Sigma)})\times M_1(\mathbb{Z}_p) \to M_1(\mathbb{Z}_p)$. Thus $T_{x_0}\cdot SU(\mathbb{A}^{(\Sigma)})$ is dense in $SU(\mathbb{A}^{(\Sigma)})\times M_1(\mathbb{Z}_p)$. Since $D \supset T_{x_0}\cdot SU(\mathbb{A}^{(\Sigma)})$, D contains $SU(\mathbb{A}^{(\Sigma)})\times M_1(\mathbb{Z}_p)$ and in particular contains $M_1(\mathbb{Z}_p)$. This shows the irreducibility of $I^{SU,(\Sigma)}$.

If Σ_0 as above is bigger than the minimal choice $\sigma = \{p,\infty\}$, we note that $\mathbb{F}(S^{(\sigma)})$ and $\mathbb{F}(I^{SU,(\Sigma_0)})$ is linearly disjoint over $\mathbb{F}(S^{(\Sigma_0)})$. Indeed, we have

$$\mathbb{F}(S^{(\sigma)})\cap\mathbb{F}(I^{SU,(\Sigma_0)}) = \mathbb{F}(S^{(\Sigma_0)})$$

by construction, and the two extensions are Galois extension over $\mathbb{F}(S^{(\Sigma_0)})$. The quotient field $\mathbb{K}$ of the integral domain $\mathbb{F}(S^{(\sigma)})\otimes_{\mathbb{F}(S^{(\Sigma_0)})}\mathbb{F}(I_n^{SU,(\Sigma_0)})$ has degree equal to the covering degree $[I_n^{SU,(\sigma)} : S^{(\sigma)}]$ and $\mathbb{K}$ is an intermediate field of $\mathbb{F}(I_n^{SU,(\sigma)})/\mathbb{F}(S^{(\sigma)})$; so, $\mathbb{K}$ is the function field of the full Igusa tower $I^{SU,(\sigma)}_{n/S^{(\sigma)}}$. This shows that $I^{SU,(\sigma)}_{/S^{(\sigma)}}$ is still irreducible.

For an arbitrary $\Sigma \supsetneq \sigma$, the natural projection $I^{SU,(\sigma)} \to I^{SU,(\Sigma)}$ is surjective dominant; so, the irreducibility of $I^{SU,(\sigma)}$ implies the irreducibility of $I^{SU,(\Sigma)}$.

3.5. Finding the base point x_0. Here is how to find the point x_0 with p-adically dense image in $M_1(\mathbb{Z}_p)$. For simplicity, we assume that $p > 2$. The unitary group $GU_{/\mathbb{Q}}$ depends only on the hermitian vector space V not the lattice L. The unitary group $GU_{/\mathbb{Z}_{(p)}}$ depends on the hermitian form on $L_{(p)} = L\otimes_{\mathbb{Z}}\mathbb{Z}_{(p)}$, and $Sh^{(p)}_{/\mathcal{V}}$ only depends on $GU_{/\mathbb{Z}_{(p)}}$; so, we may change the lattice L without changing $L_{(p)}$. In particular, if necessary, replacing L keeping $L_{(p)}$ intact, we may assume that the hermitian matrix s is diagonalizable over L (if $p > 2$).

Since $g \in M(\mathbb{Z}_p)$ acts transitively on $\mathbb{E}[p^n]-\mathbb{E}[p^{n-1}] \cong (\mathbb{Z}/p^n\mathbb{Z})^\times$ by multiplication of $\det(g)$, we can change the element

$$v \in \varprojlim_n \mathrm{Isom}(\bigwedge^{m_c}\frac{L_1^\vee}{p^nL_1^\vee}\otimes_{\mathbb{Z}_p}\bigwedge^{m_1}\frac{L_c^\vee}{p^nL_c^\vee},\mathbb{E}[p^n])$$

(appearing in the definition of $I^{SU,(\Sigma_0)}$) at our will. Thus, changing v if necessary, we only need to find a hyper-symmetric point $x_0 \in I^{(\Sigma_0)}$ with $T_{x_0}\cdot SU(\mathbb{A}^{(\Sigma_0)})$ is dense in $SU(\mathbb{A}^{(\Sigma_0)})\times M_1(\mathbb{Z}_p)$. We may assume that $m_1m_c \neq 0$. Diagonalize the hermitian matrix s over L. By the self-duality of L_p, s has p-adic unit diagonal entries $s_1,\ldots,s_r \in F$ and $\mathrm{Im}(s_j) > 0 \Leftrightarrow j \le m_1$. Note that $|s_j|\sqrt{-1} = \pm s_j$ has positive imaginary part. Take an elliptic curve $E_{j/\mathcal{W}}$ with complex multiplication by F with Riemann form given by $F \times F \ni (v, w) \mapsto \mathrm{Tr}_{F/\mathbb{Q}}(v|s_j|\sqrt{-1}w^c)$. Since s_j is a p-adic unit, we may assume that $E_j(\mathbb{C}) \cong \mathbb{C}/\mathfrak{a}_j$ for a lattice $\mathfrak{a}_j$ in F with $\mathfrak{a}_{j,p} = O_p$. We identify $\mathrm{End}^{\mathbb{Q}}(E_j) := \mathrm{End}(E_j) \otimes_{\mathbb{Z}} \mathbb{Q}$ with F by sending $\xi \in F$ to the multiplication of ξ on $\mathbb{C}$. Take $A = E_1 \oplus E_2 \oplus \cdots \oplus E_r$. Embed F into $\mathrm{End}^{\mathbb{Q}}(A)$ so that $F \to \mathrm{End}^{\mathbb{Q}}(A) \twoheadrightarrow \mathrm{End}^{\mathbb{Q}}(E_j)$ is 1 if and only if $j \leq m_1$ (so, $F \to \mathrm{End}^{\mathbb{Q}}(A) \twoheadrightarrow \mathrm{End}^{\mathbb{Q}}(E_j)$ is complex conjugation c if and only if $j > m_1$). By our construction, we have an isomorphism $H_1(A(\mathbb{C}), \mathbb{Z}) \cong L$ which takes the Riemann form on $H_1(A(\mathbb{C}), \mathbb{Z})$ to $\langle \cdot, \cdot \rangle$ on L. The Hodge decomposition $H_1(A(\mathbb{C}), \mathbb{C}) = H^{-1,0} \oplus H^{0,-1}$ gives the decomposition $V \otimes_{\mathbb{Q}} \mathbb{C} = V_1 \oplus V_2$ and hence a point in $h_A \in \mathfrak{X}^+$.

Since p splits in F, E_j is ordinary; so, $h_A \in \mathfrak{X}^+$ projects down to a point $S(\mathcal{W})$. We have

$$\mathrm{End}_O^{(\Sigma_0)}(A_{/F}) := \mathrm{End}_O(A_{/F}) \otimes \mathbb{Z}_{(\Sigma_0)} = M_{m_1}(O_{(\Sigma_0)}) \times M_{m_c}(O_{(\Sigma_0)}).$$

Over the place $\mathfrak{p}$, $E_j[\mathfrak{p}^\infty]_{/\mathcal{W}} \cong \mu_{p^\infty/\mathcal{W}}$ if and only if $j \leq m_1$. We may identify $T_pE_j[\mathfrak{p}^\infty] \cong T_p(\mathfrak{a}_{j,\mathfrak{p}} \otimes \mu_{p^\infty}) = \mathbb{Z}_p(1)$ if $j \leq m_1$ and $T_pE_j[\overline{\mathfrak{p}}^\infty] \cong T_p(\mathfrak{a}_{j,\overline{\mathfrak{p}}} \otimes \mu_{p^\infty}) = \mathbb{Z}_p(1)$ if $j > m_1$. In this way we get $\phi_p : \mathcal{L} \cong T_pA[p^\infty]^\circ$. By duality, we get

$$\left(\bigoplus_{j=1}^{m_1} \mathfrak{a}_{j,\overline{\mathfrak{p}}}\right) \oplus \left(\bigoplus_{j=m_1+1}^{r} \mathfrak{a}_{j,\mathfrak{p}}\right) \overset{{}^t\phi_p^{-1}}{\cong} \left(\bigoplus_{j=1}^{m_1} T_pE_j[\overline{\mathfrak{p}}^\infty]^{et}_{/W}\right) \oplus \left(\bigoplus_{j=m_1+1}^{r} T_pE_j[\mathfrak{p}^\infty]^{et}_{/W}\right).$$

Then we put $\eta_p = \phi \oplus {}^t\phi_p^{-1} : L_p = \mathcal{L} \oplus \mathcal{L}^\vee \cong T_pA[p^\infty]^\circ \oplus T_pA[p^\infty]^{et} = T_pA[p^\infty]$. We choose $\eta^{(\Sigma_0)}$ of A defined over $\mathcal{W}$ so that $(A, \phi_p, \eta^{(\Sigma_0)})$ is over $x_0 \in I(\mathbb{F})$, and write $\eta = (\eta_p, \eta^{(\Sigma_0)})$. For each isogeny $\alpha \in \mathrm{End}_O^{(\Sigma_0)}(A_{/F})$ preserving polarization up to scalar and fixing the generator $v \in \varprojlim_n \mathrm{Isom}(\bigwedge^{m_c} \frac{L_1^\vee}{p^nL_1^\vee} \otimes_{\mathbb{Z}_p} \bigwedge^{m_1} \frac{L_c^\vee}{p^nL_c^\vee}, \mathbb{E}[p^n])$, we can define $\rho^{(\Sigma_0)}(\alpha) \in SU(\mathbb{A}^{(\Sigma_0)})$ by $\alpha \circ \eta^{(\Sigma_0)} = \eta^{(\Sigma_0)} \circ \rho^{(\Sigma_0)}(\alpha)$ and $\rho_p(\alpha) \in M(\mathbb{Z}_p)$ by $\alpha \circ \eta_p = \eta_p \circ \rho_p(\alpha)$. Then we embed α in $SU(\mathbb{A}^{(\Sigma_0)}) \times M(\mathbb{Z}_p)$ diagonally by $\alpha \mapsto (\rho^{(\Sigma_0)}(\alpha) \times \rho_p(\alpha))$. Note that $\alpha \circ v = v \Leftrightarrow \rho_p(\alpha) \in M_1(\mathbb{Z}_p)$. Since the abelian scheme above $\rho(\alpha)(x_0)$ is

$$(A, \eta \circ \rho(\alpha)) = (\mathrm{Im}(\alpha), \alpha \circ \eta) \overset{\alpha^{-1}}{\cong} (A, \eta),$$

we find that $\rho(\alpha)(x_0) = x_0$. By construction, the stabilizer of $x_0 \in I^{(\Sigma_0)}(\mathbb{F})$ contains the image $\mathrm{Im}(\rho)$ whose projection to $M_1(\mathbb{Z}_p)$ is the p-adically dense subgroup

$$(GL_{m_1}(O_{(\Sigma_0)}) \times GL_{m_c}(O_{(\Sigma_0)})) \cap SU(\mathbb{Q})$$

as desired.

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