Vanishing of the $\mu$-invariant of $p$-adic Hecke $L$-functions

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§1. Notation

Fix

- An odd prime $p > 2$;
- two field embeddings $\mathbb{C} \leftarrow \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p \subset \mathbb{C}_p$;
- a $p$-adic complete DVR $W \subset \mathbb{C}_p$ with residue field $\mathbb{F} = \overline{\mathbb{F}}_p$;
- a CM field $M \subset \overline{\mathbb{Q}}$ of degree $2d$ with complex conjugation $c$;
- a CM type $\Sigma \sqcup \Sigma^c = \text{Hom}_{\text{field}}(M, \overline{\mathbb{Q}}_p)$ as a set of $p$-adic places $\Sigma_p$ satisfying

$$\Sigma^c_p \cap \Sigma_p = \emptyset.$$ 

For each ideal $\mathfrak{a}$ of $M$ prime to $p$, taking its power $\mathfrak{a}^h = (\alpha)$ with $\alpha \in M$, define $\langle a^\sigma \rangle \in \mathbb{C}_p^\times$ by $\exp_p \left( \frac{1}{h} \log_p (\alpha^\sigma) \right)$ for the $p$-adic logarithm $\log_p$. Then, for $\kappa = \{\kappa_\sigma \geq 0\}_\sigma, k \in \mathbb{Z},$

$$\langle a^{-k\Sigma - \kappa(1-c)} \rangle := \prod_{\sigma \in \Sigma} \langle a^{-k\sigma - \kappa_\sigma \sigma(1-c)} \rangle$$

is independent of the choice of $\alpha$ and $h$. 

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§2. \(p\)-adic Hecke \(L\)-function of Katz

Consider an arithmetic Hecke character \(\lambda_{\kappa,k}^\prime\):
\[
\hat{\lambda}_{\kappa,k} : a \mapsto \langle a^{-k\Sigma-\kappa(1-c)} \rangle.
\]
We view the \(p\)-adic Hecke \(L\)-function of \(M\) as \(L_p(T_\sigma, X)_{\sigma \in \Sigma} \in W[[T_\sigma, X]]\) (a power series \(d+1\) variables).

For \(\kappa \geq 0(\iff \kappa_\sigma \geq 0 \ \forall \sigma \in \Sigma)\) and \(k > 0\), we have, for \(\gamma_\sigma = \gamma = 1 + p\),
\[
\frac{L_p(\hat{\lambda}_{\kappa,k})}{\Omega_p^{k\Sigma+2\kappa}} := \frac{L_p(\gamma_\sigma^{k_\sigma} - 1, \gamma^k - 1)}{\Omega_p^{k\Sigma+2\kappa}} = *E(\lambda_{\kappa,k}(0, \lambda_{\kappa,k})/\Omega_\infty^{k\Sigma+2\kappa}).
\]
Here \(\Omega_\Sigma = (\Omega_\Sigma, \sigma)_{\sigma \in \Sigma}\) is the \(p\)-adic/complex Néron period of the CM abelian variety of CM type \(\Sigma\) over \(\mathcal{W} = W \cap \overline{\mathbb{Q}}\), * is a simple constant including the \(\Gamma/\epsilon\)-factor, and
\[
E(\lambda) = \prod_{p \in \Sigma_p} (1 - \lambda(p^c))(1 - N(p)^{-1}\lambda(p)^{-1}).
\]
§3. $\mu$-invariant

Define $0 \leq \mu \in \mathbb{Z}$ by $p^\mu \mid L_p(T_\sigma, X)$ in $W[[T_\sigma, X]]_\sigma$. We describe a proof of

Theorem. $p \nmid L_p(T_\sigma, X) \iff \mu = 0$.

Since the proof is basically the same for any choice of $M$, for simplicity, we assume: $M$ is an imaginary quadratic field with integer ring $O$ having odd class number. For simplicity, we also assume

$$O^\times = \{\pm 1\} \quad \text{and} \quad p \geq 5.$$  

We start some preparation.
§4. Eisenstein series.

For a lattice $L = \mathbb{Z}w_1 + \mathbb{Z}w_2 \subset \mathbb{C}$, define

$$
\frac{(2\pi i)^k}{(k - 1)!} E_k(L) = \sum'_{\ell \in L/\mathbb{Z}^\times} \frac{1}{\ell^k} \quad \text{(Eisenstein series)},
$$

a function of lattices with $E_k(\alpha L) = \alpha^{-k} E_k(L)$.

By Weierstrass, $\mathbb{C}/L$ gives an elliptic curve $X = X(L) \subset \mathbb{P}^2$ with $H^0(X, \Omega_{X/\mathbb{C}}) = \mathbb{C} du$ for the variable $u$ of $\mathbb{C}$, we recover the lattice $L$ as $\{ \int_{\gamma} du | \gamma \in H_1(X, \mathbb{Z}) \}$ out of $(X, du)$.

For a given base ring $B$, a modular form $f$ of weight $k$ is a functorial rule assigning a number $f(X, \omega) \in A$ to (the isomorphism class of) a pair $(X, \omega)_A$ defined over $B$-algebras $A$ with $f(X, a \omega) = a^{-k} f(X, \omega)$, where $H^0(X, \Omega_{E/A}) = A\omega$. By $q$-expansion principle, $E_k$ is a modular form of weight $k$ in this sense defined over $B = \mathbb{Q}$. 
§5. \textit{p-Adic integrality}

Define $E_k(z) = \sum_{p \nmid n} a(n, E_k) q^n$ removing $p$-coefficients from the $q$-expansion $\sum_n a(n, E_k) q^n$ of $E_k$ ($\mathcal{E}_k$ is a linear combination of $E_k, E_k(pz)$ and $E_k(p^2z)$). Then $\mathcal{E}_k \in \mathbb{Z}[[q]]$, and

$$\mathcal{E}_k(X, \omega) \in \mathcal{W}$$

if $(X, \omega)$ is \textbf{rational} over $\mathcal{W} = W \cap \overline{\mathbb{Q}}$. Let $a$ be an ideal of $M$ prime to $p$. Take $\Omega_{\infty}$ independent of $a$ prime to $p$ such that

$$X(a) : y^2 = 4x^3 - g_2(\Omega_{\infty} a)x - g_3(\Omega_{\infty} a)$$

with $g_2(\Omega_{\infty} a), g_3(\Omega_{\infty} a) \in \mathcal{W}$ and $\omega(a) = \frac{dx}{y}$, where $g_2 = 120E_4$ and $g_3 = 280E_6$. Here $\Omega_{\infty}$ is the \textbf{period} $\int_0^1 \omega(O)$ of $\omega(O) = \frac{dx}{y}$ taking $a = O$. Then

$$\frac{\mathcal{E}_k(a)}{\Omega_{\infty}^k} = \mathcal{E}_k(\Omega_{\infty} a) = \mathcal{E}_k(X(a), \omega(a)) \in \mathcal{W}$$

and (strictly speaking if $(p - 1)|k$)

$$\frac{E_k(a)}{\lambda_{0,k}(a)} = \lambda_{0,k}(a)^{-1} \sum_{0 \neq \alpha \in a/O^\times} \alpha^{-k}$$

$$= \sum_{\alpha \in a/O^\times} \frac{\lambda_{0,k}(\alpha a^{-1})}{N(\alpha a^{-1})^s} \bigg|_{s=0} =: L_{a^{-1}}(0, \lambda_{0,k}).$$
§6. Shimura’s differential operator

We apply the invariant differential operator
\[
\delta_k = \frac{1}{2\pi i} \left( \frac{k}{2iy} + \frac{\partial}{\partial z} \right) \quad \text{and} \quad \delta_k^\kappa = \delta_{k+2\kappa-2} \cdots \delta_k.
\]

The effect of \( \delta_k^\kappa \) on each term \( \alpha^{-k} \) after evaluating at \( \alpha \) is \( \alpha^{-k} \mapsto \alpha^{-k-\kappa(1-c)} \), we have (strictly speaking if \( (p-1)|\kappa \))
\[
\frac{\delta_k^\kappa E_k(\alpha)}{\lambda_{0,k}(\alpha)} = \lambda_{0,k}(\alpha)^{-1} \sum_{\alpha \in \mathfrak{a}/\mathcal{O}^\times} \alpha^{-k-\kappa(1-c)}
\]
\[
= \langle \alpha^{1-c} \rangle^{-\kappa} L_{a-1}(0, \lambda_{\kappa,k}).
\]

We want to write \( L_{a-1}(0, \lambda_{\kappa,k}) \) exactly as the value \( f := \lambda_{0,k}(\alpha)^{-1} \delta_k^\kappa E_k \) at \( x(\alpha) := (X(\alpha), \omega(\alpha)) \) in order to compute \( L_p(T, X) \) as the \textbf{Taylor expansion} of \( f \) for a canonical \( p \)-adic parameter around the lattice \( \mathfrak{a} \in Y \) regarded as a point \( x(\alpha) \) of elliptic modular Shimura curve \( Y \) classifying elliptic curves with prime-to-\( p \) level structure. But we have many points \( x(\alpha) \in Y \) and a

\textbf{mismatch by the factor}: \( \langle \alpha^{1-c} \rangle^{-\kappa} \).
§7. Variable change to resolve mismatch

Let \( z_0 = w_1/w_2 \in \mathcal{H} \) with \( O = \mathbb{Z}w_1 + \mathbb{Z}w_2 \) projects down to \( x(O) \in Y \). The point \( z_0 \) is fixed by \( \alpha \in \mathbb{C}^\times \); i.e.,

For \( z_0 = w_1/w_2 \) with \( O = \mathbb{Z}w_1 + \mathbb{Z}w_2 \) and \( \rho(\alpha) \in M_2(\mathbb{R}) \) with \( \rho(\alpha) \begin{bmatrix} z_0 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha z_0 \\ \alpha \end{bmatrix} \),

\[ \rho(\alpha)(z_0) = z_0. \]

The the prime-to-\( p \) level Shimura curve has action of \( GL_2(\mathbb{A}(p^\infty)) \) and for \( a \in (M_2(\mathbb{A}(p^\infty)) \times \text{generating } \hat{a} \), the action of \( \rho(a) \) does

\[ \rho(a)(x(O)) = x(a) \iff x(O) = \rho(a^{-1})(x(a)). \]

Let \( E_a = E_k \circ \rho(a^{-1}) \) and \( E_a = E_k \circ \rho(a^{-1}) \).

Take parameter \( z \) around \( z_0 \) so that

\[ \rho(\alpha)(z) = \alpha^{1-c}z \text{ and } z(z_0) = 0. \]

Then we find, regarding \( \langle a \rangle \in \mathbb{C}^\times \)

\[
\delta_k^\kappa(E_a(\rho(\langle a \rangle)(z))) \big|_{z=0} \\
\lambda_{0,k}(a)
\]

\[ = \frac{\langle a^{1-c} \rangle^\kappa \delta_k^\kappa(E_a(z_0))}{\lambda_{0,k}(a)} \]

\[ = L_{a^{-1}}(0, \lambda_{\kappa,k}). \]
§8. Katz's $p$-adic differential operator

There is a canonical $p$-adic Serre–Tate parameter $t$ around $z_0 = x(O) \in Y$. Then $\log_p(t)$ behaves like $z$: $z = 0 \Leftrightarrow t = 1$ and $t \circ \rho(\alpha) = t^{\alpha^{1-c}}$. We have invariant differential operator $\theta := \frac{d}{dt}$, and by Katz and Shimura, for $E = \sum a \frac{E_a(t^{(a^{1-c})})}{\lambda_0, k(a)}$ for $E_a = E_k \circ \rho(a^{-1})$,

$$\frac{L_p(\hat{\lambda}_{k,k})}{\Omega_p^{k+2\kappa}} = \sum_a \frac{\theta^\kappa(E_a(t^{(a^{1-c})}))|_{t=1}}{\lambda_0, k(a)} = \theta^\kappa E|_{t=1}$$

$$= \sum_a \frac{\delta_k^\kappa(E_a(\rho(\langle a \rangle)(z))))}{\lambda_{k,k}(a)}|_{z=0}$$

$$= *E(\lambda_{k,k}) \frac{\pi^\kappa L(0, \lambda_{k,k})}{\Omega_\infty^{k+2\kappa}} \in \mathcal{W}$$

for $a$ running through ideal classes. Then $L_p(\gamma^k - 1, \gamma^k - 1)$ is the above value, and $L_p(T, \gamma^k - 1)$ is basically the Taylor expansion with respect to $T = t - 1$ of

$$E = \sum a \frac{E_a(t^{(a^{1-c})})}{\lambda_0, k(a)} \in W[[T]].$$
§9 $t$-Expansion principle

Each irreducible component $I/F$ of the reduction $Y/F$ modulo $p$ of the prime-to-$p$ level Shimura curve $Y$ has a cusp $s$.

Each modulo $p$ modular form on $I$ has $q$-expansion $f(q) = \sum_{n \geq 0} a(n, f)q^n$ around $s$.

(q-exp principle) $f(q) = 0 \iff f = 0$.

If $x(O) \in I$ (i.e., $X(O) \mod p$ gives rise to a point $x(O) \in I$), similarly,

(t-exp principle) $f(t) = 0 \iff f = 0$,

where

$$f(t) = \sum_{n \geq 0} a_nT^n \in \mathbb{F}[[T]] = \lim_{m \to \infty} \frac{\mathbb{F}[t]}{t^{p^m} - 1}$$

for $T = t - 1$. If $f$ is over $W$ (or its field of fractions $K$),

$f(t)$ is determined by derivatives $\{\theta^n f|_{t=1}\}$.
§10. Independence theorem

Suppose we could prove

"INDEPENDENCE THEOREM" For a non-constant mod $p$-modular form $f_a$ of weight $k$ indexed by ideal classes, \( \{ f_a(t^{(a^1-c)}) \}_{[a] \in \text{Cl}_M} \) are linearly independent over $\mathbb{F}$ in $\mathbb{F}[[T]]$.

Indeed, \( \{ t^{(a^1-c)} \}_a \) is algebraically independent over $\mathbb{F}(Y)$. We apply this to $f_a = (E_a \mod p)$ for an idele $a \in M^\times_A$ generating $\hat{a}$. Note that $E \mod p$ is a linear combination of $f_a(t^{(a^1-c)}) = E_a(t^{(a^1-c)}) \mod p$. Non-constancy of $f_a$ can be shown by $q$-expansion; so,

$$
\mu(L_p(T, \gamma^k - 1)) = \text{ord}_p(E_k),
$$

where $\text{ord}_p(f) = \min_n(\text{ord}_p(a(n, f)))$ writing the $q$-expansion of $f$ as $f(q) = \sum_n a(n, f)q^n$. This is by irreducibility of Igusa tower over prime-to-$p$ level $Y$ (i.e., the $q$-expansion determines the $t$-expansion). We prove

$$
\sup_k \text{ord}_p(E_k) = 0 \quad \text{and} \quad p \nmid L_p(T, X).
$$
§11. Modular Curves

To prove “INDEPENDENCE THEOREM”, we study subvariety of self product of modular curves stable under the diagonal “toric” action by \( \rho(\alpha) \) for \( \alpha \in M^\times \). Write \( G = GL(2)/\mathbb{Z} \).

We study classification problem of elliptic curves \( E_A \) over a ring \( A_B \) for \( B = \mathbb{Z}[\frac{1}{N}] \), looking into the following moduli functor of level \( \Gamma(N) \),

\[
\mathcal{E}_N(A) = \left[ (X, \phi_N : (\mathbb{Z}/N\mathbb{Z})^2 \cong E[N])_A \right],
\]

which is represented, if \( N > 3 \), by a geometrically reducible curve \( Y(N)_B \). Here \( [\cdot] = \{\cdot\}/\cong \).
§12. Shimura curve

We can let \( \alpha \in G(\mathbb{Z}/N\mathbb{Z}) \) act on \( Y(N) \) by \( (X, \phi) \mapsto (X, \phi \circ \alpha) \), and \( G(\hat{\mathbb{Z}}) = \varprojlim_N G(\mathbb{Z}/N\mathbb{Z}) \) acts on \( Y/\mathbb{Q} = \varprojlim_N Y(N)/\mathbb{Z}[[1/N]] \).

A remarkable fact Shimura found is that this action of \( G(\hat{\mathbb{Z}}) \) can be extended to the adele group \( G(\mathbb{A}^{(\infty)}) = G(\mathbb{A})/G(\mathbb{R}) \), and \( SL_2(\mathbb{A}^{(\infty)}) \) and \( G^+(\mathbb{Q}) \) preserve each geometrically irreducible component of \( Y \).

There is an interpretation by Deligne. Consider the Tate module \( T(X) = \varprojlim_N X[N] \) for an elliptic curve \( E_A \) for a \( \mathbb{Q} \)-algebra \( A \). Then \( T(X) \cong \hat{\mathbb{Z}}^2 \) and \( V(X) = T(X) \otimes \mathbb{Q} \cong (\mathbb{A}^{(\infty)})^2 \). Then \( Y \) represents the following functor: for the finite adeles \( \mathbb{A}^{(\infty)} \),

\[
\mathcal{E}^{(\infty)}(A) = \left\{ (X, \eta : (\mathbb{A}^{(\infty)})^2 \cong V(X))_A \right\} / \text{isogenies}.
\]

Then \( g \in G(\mathbb{A}) \) acts as

\[
Y(A) \ni (X, \eta)_A \mapsto (X, \eta \circ g^{(\infty)})_A \in Y(A).
\]

Take the quotient $Y \frac{(p)}{\mathbb{Z}(p)} = \lim_{p \nmid N} Y(N) = Y/G(\mathbb{Z}_p)$. Put $V^{(p)}(X) = T(X) \otimes \hat{\mathbb{Z}} A^{(p\infty)}$, and consider the prime-to-$p$ level structure $\eta^{(p)} : (\hat{\mathbb{A}}^{(p\infty)})^2 \cong V^{(p)}(X)$.

Then $Y^{(p)}$ extends to a smooth pro-curve over $\mathbb{Z}(p)$ and represents the following functor defined over $\mathbb{Z}(p)$-algebras $A$:

$$\mathcal{E}^{(p)}(A) = \frac{\{(X, \eta^{(p)} : (\hat{\mathbb{A}}^{(p\infty)})^2 \cong V^{(p)}(X)) \}/A}{\text{prime-to-$p$ isogenies}},$$

where an isogeny $\phi$ is prime to $p$ if $\deg(\phi)$ is prime to $p$.

On $Y^{(p)}$ and its $p$-fiber $Y^{(p)}_{/\mathbb{F}}$ over $\text{Spec}(\mathbb{F})$, again $g \in G(\hat{\mathbb{A}})$ acts by $\eta^{(p)} \mapsto \eta^{(p)} \circ g^{(p\infty)}$.

Define $\eta^{(p)}(O) : (\hat{\mathbb{Z}}(p))^2 \rightarrow TX(O)^{(p)} = \hat{O}^{(p)}$ by $(a, b) \mapsto aw_1 + bw_2 \ (O = \mathbb{Z}w_1 + \mathbb{Z}w_2)$. 
§14. Toric action

For \( \alpha \in O_{(p)}^\times \) with \( \rho(\alpha) \in G(\mathbb{A}^{(\infty)}) \), complex multiplication \( \alpha : X(O) \to X(O) \) induces
\[
\alpha \circ \eta^{(p)}(O) = \eta^{(p)}(O) \circ \rho(\alpha).
\]
For \( x(O) = (X(O), \eta^{(p)}(O)) \in Y^{(p)}(\mathcal{W}) \),
\[
\rho(\alpha)(x(O)) = x(O)
\]
The group \( \rho(O_{(p)}^\times) \) is the stabilizer of \( x(O) \).

We have \( \Sigma_p = \{ p \} \) and \( \Sigma^c_p = \{ \overline{p} \} \); \( X(O)[\overline{p}^{\infty}] \) is étale over \( \mathcal{W} \). Pick a level \( p \)-structure
\[
\eta_p^\circ : \mu_{p^{\infty}} \cong X[p^{\infty}] \quad \eta_p^{et} : \mathbb{Q}_p/\mathbb{Z}_p \cong X[\overline{p}^{\infty}]
\]
Write \( \eta_p = (\eta^\circ, \eta_p^{et}) \), and define a homomorphism \( \rho_p : O_{(p)}^\times \to G(\mathbb{Z}_p) \) by \( \alpha \eta_p = \eta_p \circ \rho_p(\alpha) \).
We have
\[
\eta_p^\circ \circ \rho_p(\alpha) = \alpha \eta_p, \quad \eta_p^{et} \circ \rho_p(\alpha) = \alpha^c \eta_p^{et}.
\]
For general ideal \( a \) with \( a\hat{\mathcal{O}} = \hat{a} \). Define,
\[
x(a) = (X(a), \eta^{(p)}(a)) = \rho(a)^{-1}(x(O)).
\]
Consider the formal completion $\hat{Y} = \hat{Y}_{x/W}$ of $Y^{(p)}_{/W}$ along $x \in Y^{(p)}(F)$ (as we can bring $x(a)$ to $x := x(O)$ by the action of $\rho(a)$). Then $A \mapsto \hat{Y}(A)$ is the deformation functor

$$\hat{Y}(A) := \{ E/A \mid E \otimes_A F = X(O)/F \} / \cong,$$

where $A$ runs through $p$-profinite local $W$-algebras with $A/m_A = W/m_W = F$.

First $E/A \in \hat{Y}(A)$ is determined by the extension $E[p^\infty]^\circ \hookrightarrow E[p^\infty] \to E[p^\infty]^{et}$ of the Barsotti-Tate groups.

By Serre–Tate, such an extension over $A$ is classified by

$$\text{Hom}(X[p^\infty]^{et}, E[p^\infty]) \cong \text{Hom}(\mathbb{Q}_p/\mathbb{Z}_p/A, \mu_{p^\infty}/A) = \lim_{\leftarrow n} \mu_{p^n}(A) = \hat{G}_m(A) \cong \hat{Y}(A).$$

Writing $\hat{G}_m = \text{Spf}(W[t, t^{-1}])$ for $W[t, t^{-1}] = \lim_{\leftarrow n} W[t]/(t^{p^n} - 1)$, we have

$$\rho(\alpha)(t) = t^{\alpha - c} \quad \text{for } \alpha \in O_{(p)}^\times.$$

Let $I$ be the irreducible component of $Y^{(p)}_{/F}$ containing $x(O)$. We have a skew diagonal $\Delta_{\alpha,\beta} = \{(\rho(\alpha)(z), \rho(\beta)(z))\} \subset I \times I$ for $\alpha, \beta \in O^\times_{(p)}$. Consider the product $\prod_{i=1}^{h} I \times \cdots \times I$ of copies of $I = I_{a_i}$ indexed by the ring class group $Cl_{M} \cong \{a_1, \ldots, a_h\}$ of conductor $p$.

Proposition. Let $H \subset I \times \cdots \times I$ (with $n \geq 1$) be a proper closed irreducible subscheme with a dominant projection to the product of the first $n-1$ factor and to the last factor. If $H$ is stable under the diagonal action of a $p$-adic open subgroup of $\rho(O^\times_{(p)})$, up to permutations of the first $(n-1)$ factors, we have $H = I_{a_1} \times \cdots \times I_{a_{n-2}} \times \Delta_{\alpha,\beta}$.

This is a proven case of the principle (of Oort–Chai; see recent Hida’s paper in Annals volume 172 for a proof):

**Hecke invariant subvariety of a mod $p$ Shimura variety**

is a Shimura subvariety.
§17. $E_a(t^{a_j^{1-c}}) \mod p$.

For simplicity, $M$ has odd class number. Let $a_1, \ldots, a_h$ be the representatives of class group of $M$. Regard $\langle a_j^{1-c} \rangle \in \mathbb{Z}_p^\times$ ($j = 1, 2, \ldots, h$). Regard $a = \langle a_j^{1-c} \rangle \in \text{Aut}(\hat{Y}) = \text{Aut}_{gp}(\hat{G}_m)$ given by $t \mapsto t^a$. Let $x = x(O)$ and $O_x$ for the stalk of $x \in Y(p) \mod p$. Suppose that the algebra homomorphism:

$$O_{x/\mathbb{F}}^\otimes h := \underbrace{O_x \otimes_\mathbb{F} O_x \otimes \cdots \otimes O_x}_h \rightarrow \mathbb{F}[[T]] = \hat{O}_{\hat{Y}/\mathbb{F}}$$

given by $f_1(t) \otimes \cdots \otimes f_h(t)$ to $\prod_j f_j(t^{a_j^{1-c}})$ has a nontrivial kernel $\mathcal{K}$. The Zariski closure $H$ of $\text{Spec}(O_{x/\mathbb{F}}^\otimes h/\mathcal{K})$ in $I_O^h$ is stable under the action of $\rho(O_{(p)}^\times)$. By Proposition, if $H \subsetneq I_O^h$, we have $i \neq j$ such that $\langle a_i^{1-c} \rangle / \langle a_j^{1-c} \rangle \in O_{(p)}^\times$, a contradiction; so,

$$H = I_O^h.$$

This implies, over $O_{x/\mathbb{F}}$,

$$\{t^{a_j^{1-c}}\}_j$$

are algebraically independent.
§18. Conclusion

Take an idele $a_j \hat{O} = \hat{a}_j$. We have

$$L_p(T, \gamma^k - 1) = \sum_{j=1}^{h} \hat{\lambda}_{0,k}(a_j)^{-1} \mathcal{E}_a(t^{\langle a_j^{1-c} \rangle}).$$

By $q$-expansion computation, $\mathcal{E}_a \mod p$ is non-constant and by algebraic independence of $\{t^{\langle a_j^{1-c} \rangle}\}_j$ over $\mathcal{O}_x$,

$$\{\mathcal{E}_a(t^{\langle a_j^{1-c} \rangle}) \mod p\}_j \text{ in } \mathbb{F}[[T]]$$

are linearly independent over $\mathbb{F}$ and hence $p \nmid L_p(T, \gamma^k - 1)$ in $W[[T]]$ and hence $p \nmid L_p(T, X)$ in $W[[T, X]]$. If $M$ is not an imaginary quadratic field, there is a rare case where $\mu(L_p(T_\sigma, \gamma^k - 1)) > 0$ for all $k$. Even in those special cases, proving that

$$0 \leq \mu(L_p(T_\sigma, \zeta \gamma^k - 1)) \leq \text{ord}_p(1 - \zeta) \approx \frac{1}{p^{n-1}}$$

with $p$-power roots of unity $\zeta \in \mu_{p^n} \setminus \mu_{p^{n-1}}$ for all $n > 0$, we conclude

$$p \nmid L_p(T_\sigma, X) \text{ in } W[[T_\sigma, X]]_{\sigma \in \Sigma}.$$