* Vanishing of the μ-invariant of p-adic Hecke L-functions

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§1. Notation

Fix

- An odd prime $p > 2$;
- two field embeddings $\mathbb{C} \hookleftarrow \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p \subset \mathbb{C}_p$;
- a p-adic complete DVR $W \subset \mathbb{C}_p$ with residue field $\mathbb{F} = \overline{\mathbb{F}}_p$;
- a CM field $M \subset \overline{\mathbb{Q}}$ of degree $2d$ with complex conjugation c;
- a CM type $\Sigma \sqcup \Sigma^c = \mathrm{Hom}_{\mathrm{field}}(M, \overline{\mathbb{Q}}_p)$ as a set of p-adic places Σ_p satisfying

$$\Sigma_p^c \cap \Sigma_p = \emptyset.$$

For each ideal $\mathfrak{a}$ of M prime to p, taking its power $\mathfrak{a}^h = (\alpha)$ with $\alpha \in M$, define $\langle \mathfrak{a}^\sigma \rangle \in \mathbb{C}_p^\times$ by $\exp_p(\frac{1}{h} \log_p(\alpha^\sigma))$ for the p-adic logarithm $\log_p$. Then, for $\kappa = \{\kappa_\sigma \geq 0\}_\sigma, k \in \mathbb{Z}$,

$$\left\langle \mathfrak{a}^{-k\Sigma-\kappa(1-c)} \right\rangle := \prod_{\sigma \in \Sigma} \left\langle \mathfrak{a}^{-k\sigma-\kappa_\sigma\sigma(1-c)} \right\rangle$$

is independent of the choice of α and h.

§2. p-adic Hecke L-function of Katz

Consider an arithmetic Hecke character $\lambda_{\kappa,k}$:

$$\widehat{\lambda}_{\kappa,k} : \mathfrak{a} \mapsto \langle \mathfrak{a}^{-k\Sigma-\kappa(1-c)} \rangle.$$

We view the p-adic Hecke L-function of M as $L_p(T_\sigma, X)_{\sigma\in\Sigma} \in W[[T_\sigma, X]]$ (a power series $d+1$ variables).

For $\kappa \geq 0 (\Leftrightarrow \kappa_\sigma \geq 0\ \forall\sigma \in \Sigma)$ and $k > 0$, we have, for $\gamma_\sigma = \gamma = 1+p$,

$$\frac{L_p(\widehat{\lambda}_{\kappa,k})}{\Omega_p^{k\Sigma+2\kappa}} := \frac{L_p(\gamma_\sigma^{\kappa_\sigma}-1, \gamma^k-1)}{\Omega_p^{k\Sigma+2\kappa}}$$
$$= *E(\lambda_{\kappa,k})\frac{\pi^\kappa L(0,\lambda_{\kappa,k})}{\Omega_\infty^{k\Sigma+2\kappa}}.$$

Here $\Omega_? = (\Omega_{?,\sigma})_{\sigma\in\Sigma}$ is the p-adic/complex Néron period of the CM abelian variety of CM type Σ over $\mathcal{W} = W \cap \overline{\mathbb{Q}}$, $*$ is a simple constant including the Γ/ϵ-factor, and

$$E(\lambda) = \prod_{\mathfrak{p}\in\Sigma_p} (1-\lambda(\mathfrak{p}^c))(1-N(\mathfrak{p})^{-1}\lambda(\mathfrak{p})^{-1}).$$

§3. μ-invariant

Define $0 \leq \mu \in \mathbb{Z}$ by $p^{\mu} \parallel L_p(T_\sigma, X)$ in $W[[T_\sigma, X]]_\sigma$. We describe a proof of

Theorem. $p \nmid L_p(T_\sigma, X)$ ($\Leftrightarrow \mu = 0$).

Since the proof is basically the same for any choice of M, for simplicity, we assume: M is an **imaginary quadratic field** with integer ring O having **odd class number**. For simplicity, we also assume

$$O^{\times} = \{\pm 1\} \text{ and } p \geq 5.$$

We start some preparation.

§4. Eisenstein series.

For a lattice $L = \mathbb{Z}w_1 + \mathbb{Z}w_2 \subset \mathbb{C}$, define

$$\frac{(2\pi i)^k}{(k-1)!}E_k(L) = \sum_{\ell \in L/\mathbb{Z}^\times}{}' \frac{1}{\ell^k} \text{ (Eisenstein series)},$$

a function of lattices with $E_k(\alpha L) = \alpha^{-k}E_k(L)$.

By Weierstrass, $\mathbb{C}/L$ gives an elliptic curve $X = X(L) \subset \mathbf{P}^2$ with $H^0(X, \Omega_{X/\mathbb{C}}) = \mathbb{C}du$ for the variable u of $\mathbb{C}$, we recover the lattice L as $\{\int_\gamma du | \gamma \in H_1(X, \mathbb{Z})\}$ out of (X, du).

For a given base ring B, a modular form f of weight k is a **functorial rule** assigning a number $f(X, \omega) \in A$ to (the isomorphism class of) a pair $(X, \omega)_{/A}$ defined over B-algebras A with $f(X, a\omega) = a^{-k}f(X, \omega)$, where $H^0(X, \Omega_{E/A}) = A\omega$. By q-**expansion principle**, E_k is a **modular form** of weight k in this sense defined over $B = \mathbb{Q}$.

§5. p-Adic integrality

Define $\mathcal{E}_k(z) = \sum_{p \nmid n} a(n, E_k) q^n$ removing p-coefficients from the q-expansion $\sum_n a(n, E_k) q^n$ of E_k ($\mathcal{E}_k$ is a linear combination of $E_k, E_k(pz)$ and $E_k(p^2 z)$). Then $\mathcal{E}_k \in \mathbb{Z}[[q]]$, and

$$\mathcal{E}_k(X, \omega) \in \mathcal{W}$$

if (X, ω) is **rational** over $\mathcal{W} = W \cap \overline{\mathbb{Q}}$. Let $\mathfrak{a}$ be an ideal of M prime to p. Take Ω_∞ independent of $\mathfrak{a}$ prime to p such that

$$X(\mathfrak{a}) : y^2 = 4x^3 - g_2(\Omega_\infty \mathfrak{a}) x - g_3(\Omega_\infty \mathfrak{a})$$

with $g_2(\Omega_\infty \mathfrak{a}), g_3(\Omega_\infty \mathfrak{a}) \in \mathcal{W}$ and $\omega(\mathfrak{a}) = \frac{dx}{y}$, where $g_2 = 120 E_4$ and $g_3 = 280 E_6$. Here Ω_∞ is the **period** $\int_0^1 \omega(O)$ of $\omega(O) = \frac{dx}{y}$ taking $\mathfrak{a} = O$. Then

$$\frac{\mathcal{E}_k(\mathfrak{a})}{\Omega_\infty^k} = \mathcal{E}_k(\Omega_\infty \mathfrak{a}) = \mathcal{E}_k(X(\mathfrak{a}), \omega(\mathfrak{a})) \in \mathcal{W}$$

and (strictly speaking if $(p-1)|k$)

$$\frac{E_k(\mathfrak{a})}{\lambda_{0,k}(\mathfrak{a})} = \lambda_{0,k}(\mathfrak{a})^{-1} \sum_{0 \neq \alpha \in \mathfrak{a}/O^\times} \alpha^{-k}$$

$$= \sum_{\alpha \in \mathfrak{a}/O^\times} \frac{\lambda_{0,k}(\alpha \mathfrak{a}^{-1})}{N(\alpha \mathfrak{a}^{-1})^s}\Big|_{s=0} =: L_{\mathfrak{a}^{-1}}(0, \lambda_{0,k}).$$

§6. Shimura's differential operator

We apply the invariant differential operator

$$\delta_k = \frac{1}{2\pi i}\left(\frac{k}{2iy} + \frac{\partial}{\partial z}\right) \text{ and } \delta_k^\kappa = \overbrace{\delta_{k+2\kappa-2}\cdots\delta_k}^{\kappa}.$$

The effect of δ_k^κ on each term α^{-k} after evaluating at $\mathfrak{a}$ is $\alpha^{-k} \mapsto \alpha^{-k-\kappa(1-c)}$, we have (strictly speaking if $(p-1)|\kappa$)

$$\frac{\delta_k^\kappa E_k(\mathfrak{a})}{\lambda_{0,k}(\mathfrak{a})} = \lambda_{0,k}(\mathfrak{a})^{-1} \sum_{\alpha\in\mathfrak{a}/O^\times} \alpha^{-k-\kappa(1-c)}$$
$$= \langle \mathfrak{a}^{1-c}\rangle^{-\kappa} L_{\mathfrak{a}^{-1}}(0, \lambda_{\kappa,k}).$$

We **want** to write $L_{\mathfrak{a}^{-1}}(0, \lambda_{\kappa,k})$ exactly as the value $f := \lambda_{0,k}(\mathfrak{a})^{-1}\delta_k^\kappa E_k$ at $x(\mathfrak{a}) := (X(\mathfrak{a}), \omega(\mathfrak{a}))$ in order to compute $L_p(T, X)$ as the **Taylor expansion** of f for a canonical p-adic parameter around the lattice $\mathfrak{a} \in Y$ regarded as a point $x(\mathfrak{a})$ of elliptic modular Shimura curve Y classifying elliptic curves with prime-to–p level structure. But we have **many** points $x(\mathfrak{a}) \in Y$ and a

mismatch by the factor: $\langle \mathfrak{a}^{1-c}\rangle^{-\kappa}$.

§7. Variable change to resolve mismatch

Let z_0 be the point $z_0 = w_1/w_2 \in \mathfrak{H}$ with $O = \mathbb{Z}w_1 + \mathbb{Z}w_2$ projects down to $x(O) \in Y$. The point z_0 is fixed by $\alpha \in \mathbb{C}^\times$; i.e., For $z_0 = w_1/w_2$ with $O = \mathbb{Z}w_1 + \mathbb{Z}w_2$ and $\rho(\alpha) \in M_2(\mathbb{R})$ with $\rho(\alpha)\begin{bmatrix} z_0 \\ 1 \end{bmatrix} = \left[\begin{smallmatrix} \alpha z_0 \\ \alpha \end{smallmatrix}\right]$,

$$\rho(\alpha)(z_0) = z_0.$$

The the prime-to-p level Shimura curve has action of $GL_2(\mathbb{A}^{(p\infty)})$ and for $a \in (M_{\mathbb{A}}^{(p\infty)})^\times$ generating $\widehat{\mathfrak{a}}$, the action of $\rho(a)$ does

$$\rho(a)(x(O)) = x(\mathfrak{a}) \Leftrightarrow x(O) = \rho(a^{-1})(x(\mathfrak{a})).$$

Let $\mathcal{E}_{\mathfrak{a}} = \mathcal{E}_k \circ \rho(a^{-1})$ and $E_{\mathfrak{a}} = E_k \circ \rho(a^{-1})$. Take parameter z around z_0 so that

$$\rho(\alpha)(z) = \alpha^{1-c} z \quad \text{and } z(z_0) = 0.$$

Then we find, regarding $\langle \mathfrak{a} \rangle \in \mathbb{C}^\times$

$$\frac{\delta_k^\kappa(E_{\mathfrak{a}}(\rho(\langle \mathfrak{a} \rangle)(z)))|_{z=0}}{\lambda_{0,k}(\mathfrak{a})} = \frac{\langle \mathfrak{a}^{1-c} \rangle^\kappa \delta_k^\kappa(E_{\mathfrak{a}}(z_0))}{\lambda_{0,k}(\mathfrak{a})} = L_{\mathfrak{a}^{-1}}(0, \lambda_{\kappa,k}).$$

§8. Katz's p-adic differential operator

There is a canonical p-adic Serre–Tate parameter t around $z_0 = x(O) \in Y$. Then $\log_p(t)$ behaves like z: $z = 0 \Leftrightarrow t = 1$ and $t \circ \rho(\alpha) = t^{\alpha^{1-c}}$. We have invariant differential operator $\theta := t\frac{d}{dt}$, and by Katz and Shimura, for $E = \sum_{\mathfrak{a}} \frac{\mathcal{E}_{\mathfrak{a}}(t^{\langle \mathfrak{a}^{1-c} \rangle})}{\lambda_{0,k}(\mathfrak{a})}$ for $\mathcal{E}_{\mathfrak{a}} = \mathcal{E}_k \circ \rho(a^{-1})$,

$$\frac{L_p(\widehat{\lambda}_{\kappa,k})}{\Omega_p^{k+2\kappa}} = \sum_{\mathfrak{a}} \frac{\theta^{\kappa}(\mathcal{E}_{\mathfrak{a}}(t^{\langle \mathfrak{a}^{1-c} \rangle}))|_{t=1}}{\lambda_{0,k}(\mathfrak{a})} = \theta^{\kappa} E|_{t=1}$$

$$= \sum_{\mathfrak{a}} \frac{\delta_k^{\kappa}(\mathcal{E}_{\mathfrak{a}}(\rho(\langle \mathfrak{a} \rangle)(z)))}{\lambda_{\kappa,k}(\mathfrak{a})}\Big|_{z=0}$$

$$= *E(\lambda_{\kappa,k}) \frac{\pi^{\kappa} L(0, \lambda_{\kappa,k})}{\Omega_{\infty}^{k+2\kappa}} \in \mathcal{W}$$

for $\mathfrak{a}$ running through ideal classes. Then $L_p(\gamma^{\kappa} - 1, \gamma^k - 1)$ is the above value, and $L_p(T, \gamma^k - 1)$ is **basically** the Taylor expansion with respect to $T = t - 1$ of

$$E = \sum_{\mathfrak{a}} \frac{\mathcal{E}_{\mathfrak{a}}(t^{\langle \mathfrak{a}^{1-c} \rangle})}{\lambda_{0,k}(\mathfrak{a})} \in W[[T]].$$

§9 t-Expansion principle

Each irreducible component $I_{/\mathbb{F}}$ of the reduction $Y_{/\mathbb{F}}$ modulo p of the prime-to–p level Shimura curve Y has a cusp s.

Each modulo p modular form on I has q-expansion $f(q) = \sum_{n \geq 0} a(n, f) q^n$ around s.

(q-exp principle) $f(q) = 0 \Leftrightarrow f = 0$.

If $x(O) \in I$ (i.e., $X(O) \mod p$ gives rise to a point $x(O) \in I$), similarly,

(t-exp principle) $f(t) = 0 \Leftrightarrow f = 0$,

where

$$f(t) = \sum_{n \geq 0} a_n T^n \in \mathbb{F}[[T]] = \varprojlim_m \frac{\mathbb{F}[t]}{(t^{p^m} - 1)}$$

for $T = t - 1$. If f is over W (or its field of fractions K),

$f(t)$ is **determined** by derivatives $\{\theta^n f|_{t=1}\}_n$.

§10. Independence theorem

Suppose we could prove

"INDEPENDENCE THEOREM" *For a* **non-constant** *mod p-modular form $f_{\mathfrak{a}}$ of weight k indexed by ideal classes, $\{f_{\mathfrak{a}}(t^{\langle \mathfrak{a}^{1-c}\rangle})\}_{[\mathfrak{a}]\in Cl_M}$ are linearly independent over $\mathbb{F}$ in $\mathbb{F}[[T]]$.*

Indeed, $\{t^{\langle \mathfrak{a}^{1-c}\rangle}\}_{\mathfrak{a}}$ is **algebraically independent** over $\mathbb{F}(Y)$. We apply this to $f_{\mathfrak{a}} = (\mathcal{E}_{\mathfrak{a}} \mod p)$ for an idele $a \in M_{\mathbb{A}}^{\times}$ generating $\widehat{\mathfrak{a}}$. Note that $E \mod p$ is a linear combination of $f_{\mathfrak{a}}(t^{\langle \mathfrak{a}^{1-c}\rangle}) = \mathcal{E}_{\mathfrak{a}}(t^{\langle \mathfrak{a}^{1-c}\rangle}) \mod p$. Non-constancy of $f_{\mathfrak{a}}$ can be shown by q-expansion; so,

$$\mu(L_p(T,\gamma^k-1)) = \mathrm{ord}_p(\mathcal{E}_k),$$

where $\mathrm{ord}_p(f) = \min_n(\mathrm{ord}_p(a(n,f)))$ writing the q-expansion of f as $f(q) = \sum_n a(n,f)q^n$. This is by **irreducibility of Igusa tower over prime-to-p level** Y (i.e., the q-expansion determines the t-expansion). We prove

$$\sup_k \mathrm{ord}_p(\mathcal{E}_k) = 0 \quad \text{and} \quad p \nmid L_p(T,X).$$

§11. Modular Curves

To prove "INDEPENDENCE THEOREM", we study subvariety of self product of modular curves stable under the diagonal "toric" action by $\rho(\alpha)$ for $\alpha \in M^\times$. Write $G = GL(2)_{/\mathbb{Z}}$.

We study classification problem of elliptic curves $E_{/A}$ over a ring $A_{/B}$ for $B = \mathbb{Z}[\frac{1}{N}]$, looking into the following moduli functor of level $\Gamma(N)$,

$$\mathcal{E}_N(A) = \left[(X, \phi_N : (\mathbb{Z}/N\mathbb{Z})^2 \cong E[N])_{/A}\right],$$

which is represented, if $N > 3$, by a geometrically reducible curve $Y(N)_{/B}$. Here $[\cdot] = \{\cdot\}/\cong$.

§12. Shimura curve

We can let $\alpha \in G(\mathbb{Z}/N\mathbb{Z})$ act on $Y(N)$ by $(X,\phi) \mapsto (X,\phi\circ\alpha)$, and $G(\widehat{\mathbb{Z}}) = \varprojlim_N G(\mathbb{Z}/N\mathbb{Z})$ acts on $Y_{/\mathbb{Q}} = \varprojlim_N Y(N)_{/\mathbb{Z}[\frac{1}{N}]}$.

A remarkable fact Shimura found is that this action of $G(\widehat{\mathbb{Z}})$ can be extended to the adele group $G(\mathbb{A}^{(\infty)}) = G(\mathbb{A})/G(\mathbb{R})$, and $SL_2(\mathbb{A}^{(\infty)})$ and $G^{+}(\mathbb{Q})$ preserve each **geometrically irreducible** component of Y.

There is an interpretation by Deligne. Consider the Tate module $T(X) = \varprojlim_N X[N]$ for an elliptic curve $E_{/A}$ for a $\mathbb{Q}$-algebra A. Then $T(X) \cong \widehat{\mathbb{Z}}^2$ and $V(X) = T(X)\otimes_{\mathbb{Z}}\mathbb{Q} \cong (\mathbb{A}^{(\infty)})^2$. Then Y represents the following functor: for the finite adeles $\mathbb{A}^{(\infty)}$,

$$\mathcal{E}^{(\infty)}(A) = \frac{\{(X,\eta : (\mathbb{A}^{(\infty)})^2 \cong V(X))_{/A}\}}{\text{isogenies}}.$$

Then $g \in G(\mathbb{A})$ acts as

$$Y(A) \ni (X,\eta)_{/A} \mapsto (X,\eta\circ g^{(\infty)})_{/A} \in Y(A).$$

§13. Shimura curve of prime-to-p level.

Take the quotient $Y^{(p)}_{/\mathbb{Z}_{(p)}} = \varprojlim_{p \nmid N} Y(N) = Y/G(\mathbb{Z}_p)$. Put $V^{(p)}(X) = T(X) \otimes_{\widehat{\mathbb{Z}}} \mathbb{A}^{(p\infty)}$, and consider the prime-to–p level structure $\eta^{(p)} : (\mathbb{A}^{(p\infty)})^2 \cong V^{(p)}(X)$.

Then $Y^{(p)}$ extends to a **smooth** pro-curve over $\mathbb{Z}_{(p)}$ and represents the following functor defined over $\mathbb{Z}_{(p)}$-algebras A:

$$\mathcal{E}^{(p)}(A) = \frac{\{(X, \eta^{(p)} : (\mathbb{A}^{(p\infty)})^2 \cong V^{(p)}(X))_{/A}\}}{\text{prime-to–}p\text{ isogenies}},$$

where an isogeny ϕ is prime to p if $\deg(\phi)$ is prime to p.

On $Y^{(p)}$ and its p-fiber $Y^{(p)}_{/\mathbb{F}}$ over Spec($\mathbb{F}$), again $g \in G(\mathbb{A})$ acts by $\eta^{(p)} \mapsto \eta^{(p)} \circ g^{(p\infty)}$.

Define $\eta^{(p)}(O) : (\widehat{\mathbb{Z}}^{(p)})^2 \to TX(O)^{(p)} = \widehat{O}^{(p)}$ by $(a, b) \mapsto aw_1 + bw_2$ $(O = \mathbb{Z}w_1 + \mathbb{Z}w_2)$.

§14. Toric action

For $\alpha \in O_{(p)}^{\times}$ with $\rho(\alpha) \in G(\mathbb{A}^{(\infty)})$, complex multiplication $\alpha : X(O) \to X(O)$ induces

$$\alpha \circ \eta^{(p)}(O) = \eta^{(p)}(O) \circ \rho(\alpha).$$

For $x(O) = (X(O), \eta^{(p)}(O)) \in Y^{(p)}(\mathcal{W})$,

$$\rho(\alpha)(x(O)) = x(O)$$

The group $\rho(O_{(p)}^{\times})$ is the stabilizer of $x(O)$.

We have $\Sigma_p = \{\mathfrak{p}\}$ and $\Sigma_p^c = \{\overline{\mathfrak{p}}\}$; $X(O)[\overline{\mathfrak{p}}^{\infty}]$ is étale over $\mathcal{W}$. Pick a level p-structure

$$\eta_p^{\circ} : \mu_{p^{\infty}} \cong X[\mathfrak{p}^{\infty}] \quad \eta_p^{et} : \mathbb{Q}_p/\mathbb{Z}_p \cong X[\overline{\mathfrak{p}}^{\infty}].$$

Write $\eta_p = (\eta^{\circ}, \eta_p^{et})$, and define a homomorphism $\rho_p : O_{(p)}^{\times} \to G(\mathbb{Z}_p)$ by $\alpha\eta_p = \eta_p \circ \rho_p(\alpha)$. We have

$$\eta_p^{\circ} \circ \rho_p(\alpha) = \alpha\eta_p, \quad \eta_p^{et} \circ \rho_p(\alpha) = \alpha^c \eta_p^{et}.$$

For general ideal $\mathfrak{a}$ with $a\widehat{O} = \widehat{\mathfrak{a}}$. Define,

$$x(\mathfrak{a}) = (X(\mathfrak{a}), \eta^{(p)}(\mathfrak{a})) = \rho(a)^{-1}(x(O)).$$

§15. Infinitesimal coordinate around $x(O)$

Consider the formal completion $\widehat{Y} = \widehat{Y}_{x/W}$ of $Y^{(p)}_{/\mathcal{W}}$ along $x \in Y^{(p)}(\mathbb{F})$ (as we can bring $x(\mathfrak{a})$ to $x := x(O)$ by the action of $\rho(a)$). Then $A \mapsto \widehat{Y}(A)$ is the deformation functor

$$\widehat{Y}(A) := \left\{E_{/A} \middle| E \otimes_A \mathbb{F} = X(O)_{/\mathbb{F}}\right\} / \cong,$$

where A runs through p-profinite local W-algebras with $A/\mathfrak{m}_A = W/\mathfrak{m}_W = \mathbb{F}$.

First $E_{/A} \in \widehat{Y}(A)$ is determined by the extension $E[p^\infty]^\circ \hookrightarrow E[p^\infty] \twoheadrightarrow E[p^\infty]^{et}$ of the Barsotti-Tate groups.

By Serre–Tate, such an extension over A is classified by

$$\begin{aligned}\mathrm{Hom}(X[p^\infty]^{et}, E[p^\infty]^\circ) &\cong \mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_{p/A}, \mu_{p^\infty/A}) \\ &= \varprojlim_n \mu_{p^n}(A) = \widehat{\mathbb{G}}_m(A) \cong \widehat{Y}(A).\end{aligned}$$

Writing $\widehat{\mathbb{G}}_m = \mathrm{Spf}(\widehat{W[t,t^{-1}]})$ for $\widehat{W[t,t^{-1}]} = \varprojlim_n W[t]/(t^{p^n}-1)$, we have

$$\rho(\alpha)(t) = t^{\alpha^{1-c}} \quad \text{for } \alpha \in O_{(p)}^\times.$$

§16. Hecke invariant subvarieties.

Let I be the irreducible component of $Y^{(p)}_{/\mathbb{F}}$ containing $x(O)$. We have a skew diagonal $\Delta_{\alpha,\beta} = \{(\rho(\alpha)(z), \rho(\beta)(z))\} \subset I \times I$ for $\alpha, \beta \in O^{\times}_{(p)}$. Consider the product $\overbrace{I \times \cdots \times I}^{h}$ of copies of $I = I_{\mathfrak{a}_i}$ indexed by the ring class group $Cl_M \cong \{\mathfrak{a}_1, \ldots, \mathfrak{a}_h\}$ of conductor p.

Proposition. *Let $H \subsetneq I \times \cdots \times I$ (with $n \geq 1$) be a proper closed irreducible subscheme with a dominant projection to the product of the first $n-1$ factor and to the last factor. If H is stable under the diagonal action of a p-adic open subgroup of $\rho(O^{\times}_{(p)})$, up to permutations of the first $(n-1)$ factors, we have $H = I_{\mathfrak{a}_1} \times \cdots \times I_{\mathfrak{a}_{n-2}} \times \Delta_{\alpha,\beta}$.*

This is a proven case of the principle (of Oort–Chai; see recent Hida's paper in Annals volume 172 for a proof):

Hecke invariant subvariety of a mod p Shimura variety is a Shimura subvariety.

§17. $\mathcal{E}_{\mathfrak{a}}(t^{\langle \mathfrak{a}_j^{1-c} \rangle})$ mod p.

For simplicity, M has odd class number. Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_h$ be the representatives of class group of M. Regard $\langle \mathfrak{a}_j^{1-c} \rangle \in \mathbb{Z}_p^\times$ ($j = 1, 2, \ldots, h$). Regard $a = \langle \mathfrak{a}_j^{1-c} \rangle \in \mathrm{Aut}(\widehat{Y}) = \mathrm{Aut}_{gp}(\widehat{\mathbb{G}}_m)$ given by $t \mapsto t^a$. Let $x = x(O)$ and $\mathcal{O}_x$ for the stalk of $x \in Y^{(p)}$ mod p. Suppose that the algebra homomorphism:

$$\mathcal{O}_{x/\mathbb{F}}^{\otimes h} := \overbrace{\mathcal{O}_x \otimes_{\mathbb{F}} \mathcal{O}_x \otimes \cdots \otimes \mathcal{O}_x}^{h} \to \mathbb{F}[[T]] = \mathcal{O}_{\widehat{Y}/\mathbb{F}}$$

given by $f_1(t) \otimes \cdots \otimes f_h(t)$ to $\prod_j f_j(t^{\langle \mathfrak{a}_j^{1-c} \rangle})$ has a nontrivial kernel $\mathfrak{K}$. The Zariski closure H of $\mathrm{Spec}(\mathcal{O}_x^{\otimes h}/\mathfrak{K})$ in I_O^h is stable under the action of $\rho(O_{(p)}^\times)$. By Proposition, if $H \subsetneq I_O^h$, we have $i \neq j$ such that $\langle \mathfrak{a}_i^{1-c} \rangle / \langle \mathfrak{a}_j^{1-c} \rangle \in O_{(p)}^\times$, a contradiction; so,

$$H = I_O^h.$$

This implies, over $\mathcal{O}_{x/\mathbb{F}}$,

$\{t^{\langle \mathfrak{a}_j^{1-c} \rangle}\}_j$ are **algebraically independent**.

§18. Conclusion

Take an idele $a_j\widehat{O} = \widehat{\mathfrak{a}}_j$. We have

$$L_p(T,\gamma^k - 1) = \sum_{j=1}^{h} \widehat{\lambda}_{0,k}(\mathfrak{a}_j)^{-1}\mathcal{E}_{\mathfrak{a}}(t^{\langle \mathfrak{a}_j^{1-c}\rangle}).$$

By q-expansion computation, $\mathcal{E}_{\mathfrak{a}} \mod p$ is non-constant and by algebraic independence of $\{t^{\langle \mathfrak{a}_j^{1-c}\rangle}\}_j$ over $\mathcal{O}_x$,

$$\{\mathcal{E}_{\mathfrak{a}}(t^{\langle \mathfrak{a}_j^{1-c}\rangle}) \mod p\}_j \text{ in } \mathbb{F}[[T]]$$

are linearly independent over $\mathbb{F}$

and hence $p \nmid L_p(T,\gamma^k - 1)$ in $W[[T]]$ and hence $p \nmid L_p(T,X)$ in $W[[T,X]]$. If M is not an imaginary quadratic field, there is a rare case where $\mu(L_p(T_\sigma,\gamma^k - 1)) > 0$ for all k. Even in those special cases, proving that

$$0 \le \mu(L_p(T_\sigma,\zeta\gamma^k - 1)) \le \mathrm{ord}_p(1-\zeta) \approx \frac{1}{p^{n-1}}$$

with p-power roots of unity $\zeta \in \mu_{p^n} \setminus \mu_{p^{n-1}}$ for all $n > 0$, we conclude

$$p \nmid L_p(T_\sigma, X) \text{ in } W[[T_\sigma, X]]_{\sigma\in\Sigma}.$$