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On Λ–adic forms of half integral weight for $SL(2)_{/\mathbb{Q}}$

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1. — Let $\widetilde{S}$ be the two–fold metaplectic cover of $S = SL(2)_{/\mathbb{Z}}$ and fix a prime $p \geq 5$. In this short note**, we want to describe a technique of lifting a family of complex automorphic representations of $\widetilde{S}(\mathbb{A})$ to a "Λ–adic automorphic" representation Π of $\widetilde{S}(\mathbb{A}^{(p\infty)})$, where Λ is a one variable power series ring over an appropriate p–adically complete discrete valuation ring, and $\mathbb{A}^{(p\infty)}$ is the adele ring $\mathbb{A}$ of $\mathbb{Q}$ the p and ∞–components removed. Then we will have a Λ–adic version of a result of Waldspurger [Wa2]. We begin with the study of p–adic cusp forms of half integral weight and prove in Section 3 that the classical cusp forms of weight $k + \frac{1}{2}$ is dense in the space of p–adic cusp forms of half integral weight if $k \geq 2$ (Theorem 1). Then we study Λ–adic forms of half integral weight in Section 4 by combining the techniques of Wiles [Wi] (introduced for integral weights) and the representation theoretic technique of Waldspurger [Wa1,2]. Taking the limit shrinking the congruence subgroup, we get the desired Λ–adic representation of $\widetilde{S}(\mathbb{A}^{(p\infty)})$ (Proposition 1). Then we prove the weak multiplicity one theorem for p–ordinary Λ–adic automorphic representations (Theorem 2 in Section 4). Although our construction is just the combination of these two existing techniques, we get a fairly strong result on p–adic standard L–functions of $G = GL(2)_{/\mathbb{Q}}$. That is, a certain ratio of the restriction of 2–variable p–adic standard L–functions [K] to the line interpolating

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** Some part of the work presented in this note was actually done in 1988 in order to construct a p–adic standard L–functions for $GL(2)$ restricted at the center critical line (Theorem 4). The construction of two variable p–adic standard L–functions was later done by K. Kitagawa [K] using a different method.

the central critical values is shown to be square in the field of fractions of the Iwasawa algebra Λ (Theorems 3 and 4), which is the Λ-adic version of a result of Waldspurger ([Wa2] Corollary 2) we alluded to. A further scrutinizing of the representation we constructed might bring us a sharpening of this result giving a Λ-adic version of the result in [V]. However to make our presentation short, we will not touch this subject in the present account. Another interesting point which awaits further study is the behavior of the specialization $\pi_{wt=2}$ of irreducible factors π of Π at weight 2. In [GS], Greenberg and Stevens gave an interesting limit formula of the derivative of the p-adic standard L-function at the center critical point, when the L-function has an exceptional zero at this point. This is the unique case where the specialized automorphic representation $\pi_{wt=2}$ of $\widetilde{S}(\mathbb{Z}^{p\infty})$ (supplemented with the p-component) becomes super cuspidal at p although the integral image of $\pi_{wt=2}$ under the Shimura correspondence is special and p-ordinary. Thus the study of the behavior of the other local components of $\pi_{wt=2}$ might cast some new insight upon the p-adic analog of the conjecture of Birch–Swinnerton Dyer formulated in [MTT]. Although I have only worked out here the result for $SL(2)$ defined over $\mathbb{Q}$, our idea works fine for $SL(2)$ over general number fields. However, in the general case, the many variable standard p-adic L-functions defined on the spectrum of the p-adic Hecke algebra are not yet constructed.

2. — Let Δ be a congruence subgroup of level prime to p. When we consider modular forms of half integral weight, we assume that Δ is contained in $\Gamma_0(4)$. We write $\Delta_1(p^\alpha) = \Delta \cap \Gamma_1(p^\alpha)$ and $\Delta(p^\alpha) = \Delta_1(p) \cap \Gamma_0(p^\alpha)$. We use the same notation introduced in [H1] Sections 1 and 2 for classical modular forms. In particular, for each integer k and an algebra A, $\mathcal{P}_{k+(1/2)}(\Delta_1(p^\alpha); A))$ stands for the space of A-integral cusp forms of half integral weight $k+\frac{1}{2}$ with respect to $\Delta_1(p^\alpha)$, while for each integer κ, $S_\kappa(\Delta_1(p^\alpha); A))$ stands for the space of A-integral cusp forms of integral weight κ. Here the A-integrality is given by the q-expansion at the cusp ∞. For each Dirichlet character χ modulo Np^α, $\mathcal{P}_{k+(1/2)}(\Gamma_0(Np^\alpha); \chi; A)$ consists of cusp forms g in $\mathcal{P}_{k+(1/2)}(\Gamma_1(Np^\alpha); A)$ with $g|_{k+(1/2)}\sigma = \chi(d)g$ for each $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, where $g|_{k+(1/2)}\sigma$ is the action of σ defined in [H1] (2.2a) which is a little different from the normalization of [Sh1] p. 447. Our normalization is :

$$g|_{k+(1/2)}\sigma(z) = g(\sigma(z))j(\sigma,z)^{-1}J(\sigma,z)^{-k} \text{ for } \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $J(\sigma,z) = (cz+d)$ and $j(\sigma,z) = \theta(\sigma(z))/\theta(z)$ for $\theta(z) = \sum_{n=-\infty}^{\infty} \exp(2\pi i n^2 z)$. By [H1] Theorem 2.2 or its proof, $\mathcal{P}_{k+(1/2)}(\Gamma_1(Np^\alpha); A)$ is stable under this action of $\sigma \in \Gamma_0(Np^\alpha)$.

We now give an interpretation in adelic language following [Wa2] III. We write S for the algebraic group $SL(2)_{/\mathbb{Z}}$. We write $\widetilde{S}$ for the two fold metaplectic cover of $SL(2)$ defined in [Wa2] II.4. Thus $\widetilde{S}(\mathbb{Q}_p)$, $\widetilde{S}(\mathbb{A})$ and $\widetilde{S}(\mathbb{R})$ have meaning, where $\mathbb{A}$ is the adele ring of $\mathbb{Q}$. In other words, we have a non-splitting exact sequence of groups :

$$1 \longrightarrow \{\pm 1\} \longrightarrow \widetilde{S}(A) \longrightarrow S(A) \longrightarrow 1,$$

where A is either $\mathbb{A}$, $\mathbb{Q}_p$ or $\mathbb{R}$. Now let us describe the 2-cocycle β giving the extension $\widetilde{S}(\mathbb{Q}_v)$ for a place v of $\mathbb{Q}$. For each $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we put $x(\sigma) = d$ or c according as $c = 0$ or $c \neq 0$. We also put

$$s_v(\sigma) = \begin{cases} (c,d)_v & \text{if } cd \neq 0,\ v \text{ is finite},\ \ v_p(c) \text{ is odd}, \\ 1 & \text{otherwise}, \end{cases}$$

where $(c,d)_v$ is the Hilbert symbol at v (that is, $\sqrt{c}^{(d,\mathbb{Q}_v)} = (c,d)_v\sqrt{c}$ for the Artin symbol $(d,\mathbb{Q}_v)$ of d). Then we have

$$\beta_v(\sigma,\sigma') = (x(\sigma),\ x(\sigma'))_v(-x(\sigma)x(\sigma'),\ x(\sigma\sigma'))_v s_v(\sigma)s_v(\sigma')s_v(\sigma\sigma').$$

For $\sigma \in S(\mathbb{Q})$, the product $s(\sigma) = \Pi_v s_v(\sigma_v)$ is well defined. Similarly we may define $\beta(\sigma,\sigma') = \Pi_v\beta(\sigma_v,\sigma'_v)$ for σ, $\sigma' \in S(\mathbb{A})$. Then we identify $\widetilde{S}(\mathbb{A})$ with $S(\mathbb{A}) \times \{\pm 1\}$ under the multiplication law given by $(g,\varepsilon)(h,\varepsilon') = (gh, \beta(g,h)\varepsilon\varepsilon')$. By the product formula of the Hilbert symbol, $\beta(\sigma,\sigma') = s(\sigma)s(\sigma')s(\sigma\sigma')$ for σ, $\sigma' \in S(\mathbb{Q})$. Thus $\sigma \longmapsto (\sigma, s(\sigma))$ gives a section : $S(\mathbb{Q}) \to \widetilde{S}(\mathbb{A})$. We identify $S(\mathbb{Q})$ with its image in $\widetilde{S}(\mathbb{A})$. We also identify the standard maximal compact subgroup $SO_2(\mathbb{R})$ with $\mathbb{R}/\mathbb{Z}$ by $\theta \longmapsto \begin{pmatrix} \cos 2\pi\theta & \sin 2\pi\theta \\ -\sin 2\pi\theta & \cos 2\pi\theta \end{pmatrix}$. Then the pull back image of $SO_2(\mathbb{R})$ in $\widetilde{S}(\mathbb{R})$ can be identified with $\mathbb{R}/2\mathbb{Z}$. We write the corresponding element $r(\theta)$ in $\widetilde{S}(\mathbb{R})$ and $C_\infty = \{r(\theta) \mid \theta \in \mathbb{R}/2\mathbb{Z}\}$. Then $r(\theta) \longmapsto e((k+\frac{1}{2})\theta)$ for an integer k and $e(\theta) = \exp(2\pi i\theta)$ is a character of C_∞. Via $(g,\varepsilon) \longmapsto g(i) \in H$, we have $\widetilde{S}(\mathbb{R})/C_\infty \cong H$ for the upper half complex plane H. Let $\mathbf{e} : \mathbb{A}/\mathbb{Q} \to \mathbb{C}$ be the standard additive character such that $e(x_\infty) = \exp(2\pi i x_\infty)$. We write $\mathbf{e}_v$ for the restriction of $\mathbf{e}$ to $\mathbb{Q}_v$ for each place v, and we define $\gamma_v(t)$ to be the Weil's constant with respect to $\mathbf{e}_v$ and the quadratic form tx^2 on $\mathbb{Q}_v$ [W] p. 161. We put, following [Wa2], $\tilde{\gamma}(t) = (t,t)_v\gamma_v(t)\gamma_v(1)^{-1}$. Then we have $\tilde{\gamma}_v(tt') = (t,t')_v\tilde{\gamma}_v(t)\tilde{\gamma}_v(t')$, $\tilde{\gamma}_v(t^2) = 1$ for arbitrary v, and $\tilde{\gamma}_\ell(t) = 1$ if $t \in \mathbb{Z}_\ell^\times$ and $\ell \neq 2$, $\tilde{\gamma}_2(1) = \tilde{\gamma}_2(5) = 1$ and $\tilde{\gamma}_2(3) = \tilde{\gamma}_2(7) = -i$. Let

$$U_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in S(\widehat{\mathbb{Z}}) \mid c \in N\widehat{\mathbb{Z}} \right\} \quad (\widehat{\mathbb{Z}} = \Pi_{\ell\,\mathrm{prime}}\mathbb{Z}_\ell)$$

and write $U_0(N)_\ell$ for the ℓ–component of $U_0(N)$. Defining for $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in S(\mathbb{Q}_2)$

$$\tilde{\varepsilon}_2(\sigma) = \begin{cases} \tilde{\gamma}_2(d)^{-1}(c,d)_2 s_2(\sigma) & \text{if } c \neq 0, \\ \tilde{\gamma}_2(d) & \text{if } c = 0, \end{cases}$$

we can check that $\tilde{\varepsilon}$ extends to a character of $U_0(4)_2 \times \{\pm 1\}$ in $\widetilde{S}(\mathbb{Q}_2)$ non–trivial on $\{\pm 1\}$. Let χ be a character of $(\mathbb{Z}/N\mathbb{Z})^\times$ with $\chi(-1) = 1$. For $(u,\varepsilon) \in U_0(N) \times \{\pm 1\}$, we define $\chi(u) = \chi(d)$ if $u = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We then consider the space of functions f satisfying :

(m1) $$f(\alpha x(u,\varepsilon)r(\theta)) = \tilde{\varepsilon}_2(u_2)\varepsilon\chi(u)f(x)e((k+\frac{1}{2})\theta)$$

for $\alpha \in S(\mathbb{Q})$, $(u,\varepsilon) \in U_0(N) \times \{\pm 1\}$ and $r(\theta) \in C_\infty$. We impose another condition at ∞ :

(m2) $Df = \left(\dfrac{k'(k'-2)}{2}\right) f$ for $k' = k + \dfrac{1}{2}$ for the Casimir operator D at ∞.

We write $P_{k+(1/2)}(N,\chi;\mathbb{C})$ for the space of functions satisfying $(m1-2)$ which are cusp forms. Writing $J(g,z) = cz+d$ for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ and $z \in H$, we can identify

$$\widetilde{S}(\mathbb{R}) = \{(g, t(g,z)) | g \in SL_2(\mathbb{R}),\ t(g,z) : \text{ holomorphic on } H \text{ with } t(g,z)^2 = J(g,z)\}.$$

The product is then given by $(g,t(g,z))(h,t(h,z)) = (gh, t(g,h(z))t(h,z))$. We have a natural inclusion map $\widetilde{S}(\mathbb{R}) \to \widetilde{S}(\mathbb{A})$ and $\widetilde{S}(\mathbb{Q}_p) \to \widetilde{S}(\mathbb{A})$. We have the theta series : $\theta(z) : \sum_{n=-\infty}^{\infty} \exp(2\pi i n^2 z)$ defined on H. As is well known, putting $j(\gamma,z) = \theta(\gamma(z))/\theta(z)$, $j(\gamma,z)^2 = \left(\frac{-1}{d}\right)J(\gamma,z)$ if $\gamma \in \Gamma_0(4)$. Thus $\gamma \longmapsto (\gamma, \varepsilon_2(\gamma)j(\gamma,z))$ defines an inclusion of $\Gamma_0(4)$ into $\widetilde{S}(\mathbb{R})$. It is known that the extension splits over $U_1(4) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in S(\widehat{\mathbb{Z}}) \mid c \in 4\widehat{\mathbb{Z}} \text{ and } a \equiv d \equiv 1 \mod 4\widehat{\mathbb{Z}} \right\}$. Thus we have by the strong approximation theorem that $\widetilde{S}(\mathbb{A}) = S(\mathbb{Q})U_1(4)\widetilde{S}(\mathbb{R})$. We can identify these two realizations by $\widetilde{S}(\mathbb{A}) \ni (g, t(g,z)) \longmapsto (g, t(g_\infty,z)J(g_\infty,z)^{-1/2})$ where the square root is taken so that $-\pi/2 < \arg(cz+d)^{1/2} \leq \pi/2$.

For each cusp form $f \in P_{k+(1/2)}(N,\chi;\mathbb{C})$, we define $F : H \to \mathbb{C}$ by $F(z) = f((g,1))J(g,i)^{k+(1/2)}$ for $z = g(i)$ $(g \in S(\mathbb{R}))$. Then as shown in [Wa2] Proposition 3, $f \longmapsto F$ induces an isomorphism :

(2.1) $$P_{k+(1/2)}(N,\chi;\mathbb{C}) \cong \mathcal{P}_{k+(1/2)}(\Gamma_0(N),\chi;\mathbb{C}).$$

When f is cuspidal, the holomorphy of F follows from $(m1-2)$. Let us prove the above isomorphism. We have put $F(z) = f((g,1))J(g,i)^{k+(1/2)}$ for $g \in S(\mathbb{R})$. Then

$$F(\gamma(z)) = f((\gamma_\infty g,1))J(\gamma g,i)^{k+(1/2)}.$$

Suppose $\gamma \in \Gamma_0(N)$. Then note that

$$\begin{aligned}(\gamma_\infty g,1) &= (\gamma g\gamma_f^{-1},1) = (\gamma,s(\gamma))(g\gamma_f^{-1},1)(1,s(\gamma^{-1})\beta(\gamma,g\gamma_f^{-1})) \\ &= (\gamma,s(\gamma))(g,1)(\gamma_f^{-1},1)(1,s(\gamma^{-1})\beta(g,\gamma_f^{-1})\beta(\gamma,g\gamma_f^{-1})).\end{aligned}$$

Since β is a 2–cocycle, $\beta(h,k)\beta(g,hk) = \beta(gh,k)\beta(g,h)$. This shows

$$(\gamma_\infty g,1) = (\gamma,s(\gamma))(g,1)(\gamma_f^{-1},1)(1,s(\gamma^{-1})\beta(\gamma g,\gamma_f^{-1})\beta(\gamma,g)).$$

Thus :

$$\begin{aligned}F(\gamma(g)) &= f((\gamma,s(\gamma))(g,1)(\gamma_f^{-1},1)(1,s(\gamma^{-1})\beta(\gamma g,\gamma_f^{-1})\beta(\gamma,g)))J(\gamma g,i)^{k+(1/2)} \\ &= f((g,1)(\gamma_f^{-1},1)(1,s(\gamma^{-1})\beta(\gamma g,\gamma_f^{-1})\beta(\gamma,g)))J(\gamma g,i)^{k+(1/2)} \\ &= s(\gamma^{-1})\beta(\gamma g,\gamma_f^{-1})\beta(\gamma,g)\tilde{\varepsilon}_2(\gamma_f^{-1})\chi(\gamma_f^{-1})f((g,1))J(\gamma g,i)^{k+(1/2)}.\end{aligned}$$

Since $J(\gamma g,i)^{1/2} = \beta(\gamma,g)J(\gamma,z)^{1/2}J(g,i)^{1/2}$ and $\chi(\gamma_f^{-1}) = \chi(d)$ if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we see :

$$F(\gamma(z)) = s(\gamma^{-1})\beta(\gamma g,\gamma_f^{-1})\tilde{\varepsilon}_2(\gamma_f^{-1})\chi(d)J(\gamma,z)^{k+(1/2)}f(g,1)J(g,i)^{k+(1/2)}$$
$$= s(\gamma^{-1})\beta(\gamma g,\gamma_f^{-1})\tilde{\varepsilon}_2(\gamma_f^{-1})\chi(d)F(z)J(\gamma,z)^{k+(1/2)}.$$

Thus we need to prove $s(\gamma^{-1})\beta(\gamma g,\gamma_f^{-1})\tilde{\varepsilon}_2(\gamma_f^{-1}) = \left(\frac{c}{d}\right)\tilde{\gamma}_2(d)$. If $c = 0$, the both sides are trivial. Thus we may assume that $c \neq 0$. The case $c \neq 0$ is treated in [Wa2] p. 388.

For any open subgroup U of $U_0(4)$, we write $\Gamma_U = S(\mathbb{Q}) \cap US(\mathbb{R})$. We write $P_{k+(1/2)}(U;\mathbb{C})$ for the space of holomorphic cusp forms on $\widetilde{S}(\mathbb{A})$ satisfying $(m2)$ and

(m′1)
$$f(\alpha x(u,\varepsilon)r(\theta)) = \tilde{\varepsilon}_2((u_2,\varepsilon))f(x)e((k+\frac{1}{2})\theta) \text{ for } u \in U \text{ and } \alpha \in S(\mathbb{Q}).$$

Then $\mathcal{P}_{k+(1/2)}(\Gamma_U;\mathbb{C}) \cong P_{k+(1/2)}(U;\mathbb{C})$. Thus we can transfer the rational structure from the classical side to the adelic side to have the spaces $P_{k+(1/2)}(U;A)$ for any subalgebra A of $\mathbb{C}$.

3. — In this section, we first prove the density theorem of low weight classical cusp forms in the space of p-adic cusp forms of half integral weight. Using this fact, we describe another way, much closer to Weil's original definition in [W] and due to Shimura [Sh2], to define $\widetilde{S}(\mathbb{A})$. By the strong approximation theorem, we have a bijection :

$$\{\text{congruence subgroups of } S(\mathbb{Z}) \text{ of level prime to } p\} \longleftrightarrow$$
$$\mathcal{Z} = \{\text{open subgroups of } S(\mathbb{Z}^{(p)})\}$$

$$\Delta = \widehat{\Delta} \cap S(\mathbb{Z}) \leftrightarrow \widehat{\Delta} : \text{the closure of } \Delta \text{ in } S(\mathbb{Z}^{(p)}),$$

where $\mathbb{Z}^{(p)} = \prod_{\ell \neq p} \mathbb{Z}_\ell$. We put

$$\mathcal{S}_\kappa(\widehat{\Delta};A) = \bigcup_\alpha \mathcal{S}_\kappa(\Delta_1(p^\alpha);A) \text{ and } \mathcal{P}_{k+(1/2)}(\widehat{\Delta};A) = \bigcup_\alpha \mathcal{P}_{k+(1/2)}(\Delta_1(p^\alpha);A).$$

Let O be the ring of Witt vectors with coefficients in an algebraic closure $\overline{\mathbb{F}}_p$ of $\mathbb{F}_p$ and K be the field of fractions of O. Let Ω_p be the completion of an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ under its standard p-adic norm $|\,|_p$. We take an embedding : $K \to \Omega_p = \widehat{\overline{\mathbb{Q}}}_p$ and fix two embeddings $\overline{\mathbb{Q}} \to \mathbb{C}$ and $\overline{\mathbb{Q}} \to \Omega_p$ for an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$. Put $A = O \cap \overline{\mathbb{Q}}$ and $\mathcal{S}_\kappa(\widehat{\Delta};O) = \mathcal{S}_\kappa(\widehat{\Delta};A) \otimes_A O$, $\mathcal{P}_{k+(1/2)}(\widehat{\Delta};O) = \mathcal{P}_{k+(1/2)}(\widehat{\Delta};A) \otimes_A O$. We write $\widehat{\mathcal{S}}(\widehat{\Delta};O)$ (resp. $\widehat{\mathcal{P}}(\widehat{\Delta};O)$) for the p-adic completion of $\mathcal{S}_\kappa(\widehat{\Delta};O)$ (resp. $\mathcal{P}_{k+(1/2)}(\widehat{\Delta};O)$), which is independent of κ (resp. k) if $\kappa \geq 2$ (resp. $k \geq 2$). This fact is proven in [H2] and [H6] for integral weight and is conjectured in [H1] for half integral weight. Now we can give a proof of this fact for half integral weight.

THEOREM 1. — *If $k \geq 2$ and $p > 3$, we have an isomorphism preserving q-expansions :*

$$\widehat{\mathcal{P}}_{k+(1/2)}(\widehat{\Delta};O) \cong \widehat{\mathcal{P}}_{k+(3/2)}(\widehat{\Delta};O).$$

Proof : let U be an open subgroup of $G(\widehat{\mathbb{Z}})$ $(G = GL(2)_{/\mathbb{Z}})$ and $Y(U)$ be the corresponding open modular curve. Suppose that $U \supset G(\mathbb{Z}_p)$ and we put

$$U(p^\alpha) = \left\{ s \in \mathbb{S} \mid s_p \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \bmod p^\alpha \right\}.$$

For each positive integer N, we put $\zeta_N = \exp\left(\frac{2\pi i}{N}\right)$. Then $Y_\alpha = Y(U(p^\alpha))$ has a model over $A = \mathbb{Z}[1/6N, \zeta_N]$ for the level N of U which is the moduli space parametrizing an elliptic curve E with U-structure and a Drinfeld style level structure at p; that is, a morphism $\phi : \mathbb{Z}/p^\alpha\mathbb{Z} \to E$ of group schemes such that $\sum_{P \in \mathbb{Z}/p^\alpha\mathbb{Z}} [\phi(P)]$ is of degree p^α as a relative Cartier divisor (see [KM] Chapter 1 or [H7]). Suppose that $U \subset U_0(4)$. We can compactify Y_α adding cusps to get the proper curve X_α, which is regular proper over $\mathbb{Z}_p$ [KM]. Let $\omega_{/Y_\alpha}$ be the invertible sheaf corresponding to weight 1 modular forms studied in [KM]. Let I_α be the Igusa curve containing the cusp ∞ which is the irreducible component of $X_\alpha \bmod p^\alpha$. If we consider the p-ordinary moduli problem $\phi : \mu_{p^\alpha} \subset E$ of generalized semi-stable elliptic curves, it gives an open subscheme U_α of X_α whose fiber at p is I_α–{super singular points}. Then there exists a unique invertible sheaf $\omega_{1/2}$ on U_α such that $\omega_{1/2}^{\otimes 2} = \omega$ and $\theta \in \Gamma(U_{\alpha/\mathbb{C}}, \omega_{1/2})$. By the q-expansion principle and $p > 3$, θ is a section defined over $\mathbb{Z}_p$. We first suppose that U is contained in the principal congruence subgroup of level 24. Then the Dedekind η function is a section of $H^0(U_\alpha, \omega_{1/2})$. Writing $\omega(2 + (k/2))$ for $\omega_{1/2}^{\otimes k} \otimes \omega^\circ\ |_{U_\alpha}$, we consider the following commutative diagram :

$$\begin{array}{ccc}
 & & 0 \\
 & & \downarrow \\
H^0(U_\alpha,\omega(k+\frac{1}{2}))\otimes\mathbb{Z}/p^\alpha\mathbb{Z} & \longrightarrow & H^0(U_\alpha,\omega(k+\frac{1}{2})\otimes\mathbb{Z}/p^\alpha\mathbb{Z}) \\
\downarrow \eta & & \downarrow \eta \\
H^0(U_\alpha,\omega_1(k+1))\otimes\mathbb{Z}/p^\alpha\mathbb{Z} & \longrightarrow & H^0(U_\alpha,\omega_1(k+1)\otimes\mathbb{Z}/p^\alpha\mathbb{Z}) \\
\downarrow & & \downarrow \rho \\
H^0(U_\alpha,O(D))\otimes\mathbb{Z}/p^\alpha\mathbb{Z} & \longrightarrow & H^0(U_\alpha,O(D)\otimes\mathbb{Z}/p^\alpha\mathbb{Z}) \\
\downarrow & & \\
0, & &
\end{array}$$

where D is a cuspidal divisor given by $\mathrm{div}(\eta) = \sum_{s\in U_\alpha,\mathrm{ord}_s(\eta)>0}(\mathrm{ord}_s(\eta))s$ and the first horizontal maps are given by the multiplication by η. Here we regard D as a closed subscheme of U_α in a natural way, and $O(D)$ is its structure sheaf. The first row is exact. When $k \geq 2$, $\deg\left(\omega\left(k+\frac{1}{2}\right)\otimes_A\mathbb{Q}\right) > \deg(\Omega_{X_\alpha/A}\otimes_A\mathbb{Q})$. Thus the Riemann–Roch theorem tells us the vanishing of $H^1\left(X_\gamma,\omega\left(k+\frac{1}{2}\right)\right)\otimes_A\mathbb{Q} = H^1\left(U_\gamma,\omega\left(k+\frac{1}{2}\right)\right)\otimes_A\mathbb{Q}$. Since $\omega\left(k+\frac{1}{2}\right)$ is A-flat, this shows the vanishing of $H^1\left(U_\gamma,\omega\left(k+\frac{1}{2}\right)\right)$ and the exactness of the second row. Since the vertical maps are injective, we have a commutative diagram whose rows are exact if $k \geq 2$

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
H^0(U_\gamma,\omega(k+\frac{3}{2}))\otimes\mathbb{Z}/p^\beta\mathbb{Z} & \xrightarrow{E_\alpha} & H^0(U_\gamma,\omega(k+\frac{1}{2}))\otimes\mathbb{Z}/p^\beta\mathbb{Z} \\
\downarrow & & \downarrow \\
H^0(U_\gamma,\omega(k+2))\otimes\mathbb{Z}/p^\beta\mathbb{Z} & \xrightarrow{E_\alpha} & H^0(U_\gamma,\omega(k+1))\otimes\mathbb{Z}/p^\beta\mathbb{Z} \\
\downarrow & & \downarrow \\
H^0(U_\gamma,O(D))\otimes\mathbb{Z}/p^\beta\mathbb{Z} & = & H^0(U_\gamma,O(D))\otimes\mathbb{Z}/p^\beta\mathbb{Z}
\end{array}$$

where $0 < \beta \leq \alpha \leq \gamma$ and E_α is the modular form on U_α of weight 1 with $E_\alpha \equiv 1 \bmod p^\alpha$. Taking injective limit with respect to γ, we write

$$H^0(U_\infty,\omega(\ell)) = \varinjlim_\gamma H^0(U_\gamma,\omega(\ell)).$$

Then we have by the p-adic density theorem of integral weight modular forms, if $k \geq 2$

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
H^0(U_\infty,\omega(k+\frac{3}{2}))\otimes\mathbb{Z}/p^\beta\mathbb{Z} & \xrightarrow{E_\alpha} & H^0(U_\infty,\omega(k+\frac{1}{2}))\otimes\mathbb{Z}/p^\beta\mathbb{Z} \\
\downarrow & & \downarrow \\
H^0(U_\infty,\omega(k+2))\otimes\mathbb{Z}/p^\beta\mathbb{Z} & \cong & H^0(U_\infty,\omega(k+1))\otimes\mathbb{Z}/p^\beta\mathbb{Z} \\
\downarrow & & \downarrow \\
H^0(U_\infty,O(D))\otimes\mathbb{Z}/p^\beta\mathbb{Z} & = & H^0(U_\infty,O(D))\otimes\mathbb{Z}/p^\beta\mathbb{Z}.
\end{array}$$

This shows the p-adic density theorem for half integral weight if $24 \mid N$. If not, we just use restriction and transfer maps and recover the result in general if $p > 3$.

Put

$$\mathcal{S}_\kappa(A) = \bigcup_{\widehat{\Delta}\in\mathcal{Z}} \mathcal{S}_\kappa(\widehat{\Delta};A) \quad \text{and } \mathcal{P}_\kappa(A) = \bigcup_{\widehat{\Delta}\in\mathcal{Z}} \mathcal{P}_\kappa(\widehat{\Delta};A),$$
$$\widehat{\mathcal{S}}(O) = \bigcup_{\widehat{\Delta}\in\mathcal{Z}} \widehat{\mathcal{S}}(\widehat{\Delta};O) \quad \text{and } \widehat{\mathcal{P}}(O) = \bigcup_{\widehat{\Delta}\in\mathcal{Z}} \widehat{\mathcal{P}}(\widehat{\Delta};O).$$

If $f \in \mathcal{S}_\kappa(A)$, one can find Γ such that $f \in \mathcal{S}_\kappa(\Gamma;A)$. Then for each $x \in S(\mathbb{A}^{(p\infty)})$ ($\mathbb{A}^{(p\infty)} = \{x \in \mathbb{A} \mid x_p = x_\infty = 0\}$), one can find $u \in \widehat{\Gamma} \subset S(\mathbb{A}^{(\infty)})$ and $\gamma \in S(\mathbb{Q})$ such that $x = u\gamma$, where $\widehat{\Gamma}$ is the closure of Γ in $S(\mathbb{A}^{(\infty)})$ ($\mathbb{A}^{(\infty)} = \{x \in \mathbb{A} \mid x_\infty = 0\}$). Some time ago, Shimura defined the action of $x \in S(\mathbb{A}^{(p\infty)})$ on f by $f^x = f \mid \gamma$ [Sh2]. Then he showed that the action is a smooth action of $S(\mathbb{A}^{(p\infty)})$ on $\mathcal{S}_k(\mathbb{Q}_{ab}^{(p)})$, where $\mathbb{Q}_{ab}^{(p)} = \mathbb{Q}[\zeta_N \mid (p,N) = 1]$ is the maximal abelian extension of $\mathbb{Q}$ unramified at p.

Using Katz's theory of p-adic modular forms (see [H7] Chapter 2), it is easy to check that the action of $S(\mathbb{A}^{(p\infty)})$ preserves $\mathcal{S}_\kappa(A)$ and extends to $\widehat{S}(O)$ by p–adic continuity. Note that the representation of $S(\mathbb{A}^{(p\infty)})$ we obtained is smooth, but not of finite type. I like to call this representation the p-adic automorphic representation of $S(\mathbb{A}^{(p\infty)})$.

According to Shimura [Sh2], we can give a definition of $\widetilde{S}(\mathbb{A}^{(p\infty)})$ as follows :

$$\widetilde{S}(\mathbb{A}^{(p\infty)})=\{(x,v)\in S(\mathbb{A}^{(p\infty)})\times GL(\widehat{\mathcal{P}}(O))\mid (f^v)^2=(f^2)^x \text{ for all } f\in\widehat{\mathcal{P}}(O)\}.$$

Then we have an exact sequence : $1\to\{\pm1\}\to\widetilde{S}(\mathbb{A}^{(p\infty)})\xrightarrow{\pi}S(\mathbb{A}^{(p\infty)})\to 1$. It is basically shown in [Sh2] that any $x\in S(\mathbb{A}^{(p\infty)})$ is liftable to an automorphism v of $\mathcal{P}_{k+(1/2)}(\mathbb{Q}_{ab}^{(p)})$. Since x preserves A–integrality, v keeps A–integrality and hence gives an automorphism of $\widehat{\mathcal{P}}(O)$. This shows the surjectivity of π. There is an alternative way of showing the surjectivity of π. One can check that the action of $S(\mathbb{Z}^{(p)})$ is liftable to half integral weight by multiplying half integral weight cusp forms by η (or θ), because the action of $S(\mathbb{Z}^{(p)})$ preserves A–integral structure of integral weight cusp forms. It is easy to check the liftability of the action of upper triangular matrices. Thus by the Iwasawa decomposition, every $x\in S(\mathbb{A}^{(p\infty)})$ is liftable. By definition, we have a smooth p–adic "automorphic" representation of $\widetilde{S}(\mathbb{A}^{(p\infty)})$ on $\widehat{\mathcal{P}}(O)$.

Although we do not have a good action of $\widetilde{S}(\mathbb{Q}_p)$ on $\widehat{\mathcal{P}}(O)$, we can at least define an action of the maximal split torus $T(\mathbb{Z}_p)=\mathbb{Z}_p^\times$ in $S(\mathbb{Z}_p)$. Take a subgroup Δ corresponding to $\widehat{\Delta}\in\mathcal{Z}$. Thus its level N is prime to p. We assume that $\Gamma_0(4)\supset\Delta$. When A is a $\mathbb{Z}_p$–algebra, we can show multiplying by θ as done in [H1] §3 that $\mathcal{P}_{k+(1/2)}(\Delta_1(p^r);A)$ is stable under the action of $Z_N=\mathbb{Z}_p^\times\times(\mathbb{Z}/N\mathbb{Z})^\times$ for the level N of Δ, which is given for $f\in\mathcal{P}_{k+(1/2)}(\Delta_1(p^r);A)$ by

$$(3.1)\qquad f\mid z=z_p^k f\mid\sigma_z \text{ for } \sigma_z\in SL_2(\mathbb{Z}) \text{ with } \sigma_z\equiv\begin{pmatrix} z^{-1} & 0\\ 0 & z\end{pmatrix} \bmod Np^r.$$

This action of $\mathbb{Z}_p^\times$ extends by continuity to $\widehat{\mathcal{P}}(O)$.

4. — We put $W=1+p\mathbb{Z}_p$ in $\mathbb{Z}_p^\times$. Then $W\cong\mathbb{Z}_p$ as topological groups, and $\mathbb{Z}_p^\times=W\times\mu$ for the subgroup μ of $(p-1)$–th roots of unity. Simplifying the notation, we write $\mathcal{P}_{k+(1/2)}(Np^\alpha;A)$ for $\mathcal{P}_{k+(1/2)}(\Gamma_1(Np^\alpha);A)$. We put, for $\Delta\leftrightarrow\widehat{\Delta}\in\mathcal{Z}$ and a character ε of W modulo p^α,

$$\mathcal{P}_{k+(1/2)}(\Delta(p^\alpha);\varepsilon;A)=\{f\in\mathcal{P}_{k+(1/2)}(\Delta_1(p^\alpha);A)\mid f\mid z=\varepsilon(z)z_p^k f \text{ for } z\in W\},$$

where $\Delta(p^\alpha)=\Delta_1(p)\cap\Gamma_0(p^\alpha)$ and A is a ring either in Ω_p or in $\mathbb{C}$ containing all the values of ε on W. We now consider the action of the Hecke operator $T(q^2)$ for each prime q on $\mathcal{P}_{k+(1/2)}(\Delta_1(p^\alpha);\mathbb{C})$. As shown in [Sh1] Theorem 1.7, we know

$$(4.1)\quad \begin{aligned} a(n,f|T(q^2))&=a(p^2n,f)+q^{-1}\left(\tfrac{n}{p}\right)a(n,f|q)+q^{-1}a(n/q^2,f|q^2) \text{ if } q\nmid Np^\alpha,\\ a(n,f|T(q^2))&=a(p^2n,f) \text{ if } q|Np^\alpha, \end{aligned}$$

where N is the level of Δ and $q\in Z_N$ $(=\mathbb{Z}_p^\times\times(\mathbb{Z}/N\mathbb{Z})^\times)$ acts on f as in (3.1). This combined with [H1] Theorem 2.2 shows that $\mathcal{P}_{k+(1/2)}(\Delta_1(p^\alpha);O)$ is stable under $T(q^2)$. In particular, we can define the idempotent e in $\mathrm{End}_O(\mathcal{P}_{k+(1/2)}(\Delta_1(p^\alpha);O))$ by taking the limit :

$$(4.2)\qquad e=\lim_{n\to\infty}T(p^2)^{n!}.$$

We write M^{ord} for eM for any module M with an action of e.

Hereafter we allow as a base ring a finite extension of the ring of Witt vectors with coefficients in $\overline{\mathbb{F}}_p$ and write the ring as O and its field of fractions as K. All the definitions we have given for the ring of Witt vectors carry over to this slightly general situation by extending scalar to O from the ring of Witt vectors. Write $\Lambda=O[[W]]$ for the completed group algebra of W. Then Λ is isomorphic to the one variable power series ring $O[[X]]$ via $u\longmapsto 1+X$ if we fix a generator $u\in W$. We fix an algebraic closure $\overline{\mathbb{L}}$ of the quotient field $\mathbb{L}$ of Λ and consider the algebraic closure of K in Ω_p as a subfield of $\overline{\mathbb{L}}$. For each normal integral domain $\mathbb{I}$ in $\overline{\mathbb{L}}$ finite over Λ, let $\mathcal{X}(\mathbb{I})=\mathrm{Hom}_{O\text{-alg}}(\mathbb{I},\Omega_p)$ be the space of all Ω_p–valued points of $\mathrm{Spec}(\mathbb{I})$ and $\mathcal{A}(\mathbb{I})$ be the subset of arithmetic points, that is, those O–algebra homomorphisms $P:\mathbb{I}\to\Omega_p$ such that $P(\gamma)=\gamma^{k(P)}$ for an integer $k(P)\geq 0$ on a neighborhood of the identity of W. Thus $\varepsilon_P(\gamma)=P(\gamma)\gamma^{-k(P)}$ defines a finite order character of W, whose order will be denoted by $P^{r(P)-1}$. We write $\mathcal{A}(\mathbb{I};O)=\{P\in\mathcal{A}(\mathbb{I})\mid O\supset P(\mathbb{I})\}$. For each congruence subgroup Δ (with level N) associated with $\widehat{\Delta}\in\mathcal{Z}$, let $\mathbb{P}(\Delta;\mathbb{I})$ be the space of $\mathbb{I}$–adic cusp forms. Thus $\mathbf{f}\in\mathbb{P}(\Delta;\mathbb{I})$ is a formal q–expansion :

$$\mathbf{f}=\sum_{n=0}^{\infty}\mathbf{a}(n/N,\mathbf{f})q^{n/N}\in\mathbb{I}[[q^{1/N}]]$$

whose specialization $\mathbf{f}(P)=\sum_{n=0}^{\infty}P(\mathbf{a}(n/N,\mathbf{f}))q^{n/N}\in P(\mathbb{I})[[q^{1/N}]]$ at $P\in\mathcal{A}(\mathbb{I})$ is a classical cusp form in $\mathcal{P}_{k(P)+(1/2)}(\Delta(p^{r(P)}),\varepsilon_p;\Omega_p)$ for all $P\in\mathcal{A}(\mathbb{I})$ with sufficiently large $k(P)\geq 0$. When $\Delta=\Gamma_1(N)$ $(4\mid N)$, we write $\mathbb{P}(N;\mathbb{I})$ for $\mathbb{P}(\Delta;\mathbb{I})$. Since Λ is a regular local ring of dimension 2, $\mathbb{I}$ is Λ–free. Fixing a base $\{\mathbf{i}_j\}$ of $\mathbb{I}$ over Λ, we can write formally that $\mathbf{f} = \Sigma_j \mathbf{f}_j \mathbf{i}_j$. Then it is easy to see that $\mathbf{f}_j$ is a Λ–adic form. Thus $\mathbb{P}(\Delta;\mathbb{I}) = \mathbb{P}(\Delta;\Lambda) \otimes_\Lambda \mathbb{I}$.

There is another interpretation of the above space of Λ–adic forms. We first identify Λ with the measure algebra on W having values in O. Then to each $\mathbf{f} \in \mathbb{P}(\Delta;\Lambda)$, we associate a p-adic measure $\phi \longmapsto \int_W \phi d\mathbf{f}$ on W having values in $O[[q^{1/N}]]$ by

$$\int_W \phi d\mathbf{f} = \sum_{n=1}^{\infty} \int_W \phi d\mathbf{a}(n/N, \mathbf{f}) q^{n/N} \in O[[q^{1/N}]] . \tag{4.3}$$

Writing $\chi_P(w) = \varepsilon_P(w) w^{k(P)}$ for each arithmetic point P (that is, the character of W corresponding to P), we have $\int_W \chi_P d\mathbf{f} = \mathbf{f}(P) \in \mathcal{P}_{k(P)+(1/2)}(\Delta(p^{r(P)}), \varepsilon_P; \Omega_p)$ for sufficiently large $k(P)$. Since $\{\chi_P \mid k(P) \gg 0\}$ spans a dense subspace of continuous functions on W having values in K, as a measure, $d\mathbf{f}$ has values in $\widehat{\mathcal{P}}(O)$. In particular, the new measure $\phi \longmapsto \int_W \phi d\mathbf{f} \mid s$ for $s \in \widetilde{S}(\mathbb{A}^{(p\infty)})$ again comes from a Λ–adic form $\mathbf{f} \mid s \in \mathbb{P}(\Delta_s;\Lambda)$ for a suitable congruence subgroup Δ_s corresponding to $\widehat{\Delta}_s \in \mathcal{Z}$. Thus, we have a natural action of $\widetilde{S}(\mathbb{A}^{(p\infty)})$ on $\mathbb{P}(\mathbb{I}) = \bigcup_{\widehat{\Delta}\in\mathcal{Z}} \mathbb{P}(\Delta;\mathbb{I})$.

Similarly, we have an action of Hecke operators $T(q^2)$ and the group $\mathbb{Z}_p^\times$ on $\mathbb{P}(N;\mathbb{I})$. Writing $\iota : w \longmapsto [w]$ for the tautological character of W into Λ, we know that $w \in W$ acts on $\mathbb{P}(N;\mathbb{I})$ via ι, that is, $\mathbf{f} \mid w = [w]\mathbf{f}$. Since the projector e naturally acts on $\mathcal{P}_{k+(1/2)}(O)$ and hence on $\widehat{\mathcal{P}}(O)$, e again acts on $\mathbb{P}(\Delta;\mathbb{I})$ and $\mathbb{P}(\mathbb{I})$. We note this fact as

Proposition 1. — *As long as q is prime to the level N of Δ, we have Hecke operators $T(q^2)$ given by* (4.1) *and the ordinary projector e on $\mathbb{P}(\Delta;\mathbb{I})$, and the metaplectic group $\widetilde{S}(\mathbb{A}^{(p\infty)})$ naturally acts on $\mathbb{P}(\mathbb{I})$ through a smooth representation.* Here the smoothness means that the stabilizer of each vector in the representation space is open in $\widetilde{S}(\mathbb{A}^{(p\infty)})$.

We can think of the corresponding notion of $\mathbb{I}$–adic cusp forms for integral weight modular forms (cf. [H5] Chapter 7). We briefly recall the definition. For $\widehat{\Delta} \in \mathcal{Z}$, a formal q–expansion $\mathbf{f} \in \mathbb{I}[[q^{1/N}]]$ is called an $\mathbb{I}$–adic cusp form of integral weight if $\mathbf{f}(P) \in \mathcal{S}_{k(P)}(\Delta(p^{r(P)}), \varepsilon_P; \Omega_p)$ whenever P is arithmetic and $k(P)$ is sufficiently large. We write $\mathbb{S}(\Delta;\mathbb{I})$ for the space of $\mathbb{I}$–adic cusp forms (of integral weight). Then similar to Proposition 1, we have Hecke operators $T(n)$ (cf. [H5] Chapter 7) and the ordinary projector e on $\mathbb{S}(\Delta;\mathbb{I})$. In this case, e is given on the space of p–adic cusp forms by $e = \lim_{n\to\infty} T(p)^{n!}$. The group $S(\mathbb{A}^{(p\infty)})$ naturally acts on $\bigcup_{\widehat{\Delta}\in\mathcal{Z}} \mathbb{S}(\Delta;\mathbb{I})$. We actually need to have $G(\mathbb{A}^{(p\infty)})$–action (recall $G = GL(2)_{/\mathbb{Z}}$). Note that

$G(\mathbb{A}) = G(\mathbb{Q})G(\widehat{\mathbb{Z}})G_+(\mathbb{R})$ for the identity connected component $G_+(\mathbb{R})$ of $G(\mathbb{R})$. For each open subgroup U of $G(\widehat{\mathbb{Z}})$, we consider cusp forms $f : G(\mathbb{A}) \to \mathbb{C}$ satisfying :

(M1) $\quad f(\alpha x u) = f(x)\det(u_\infty) J(u_\infty, i)^{-k}$ for $u \in UC_\infty\mathbb{R}^\times$;

(M2) $$Df = \left(\frac{k(k-2)}{2}\right) f ;$$

(M3) $$\int_{\mathbb{Q}\backslash\mathbb{A}} f\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} x\right) du = 0 \text{ for all } x \in G(\mathbb{A}) .$$

We write $S_k(U;\mathbb{C})$ for the space of functions f satisfying (M1–3). Choosing a complete representative set $R = R(U)$ for $G(\mathbb{Q})\backslash G(\mathbb{A})/UG_+(\mathbb{R})$ in $G(\widehat{\mathbb{Z}})$, we can define $F_t \in \mathcal{S}_k(\Gamma_{tUt^{-1}};\mathbb{C})$ ($\Gamma_{tUt^{-1}} = S(\mathbb{Q}) \cap tUt^{-1}S(\mathbb{R})$) for each $t \in R$ by $F_t(z) = f(tg)\det(g)^{-1}J(g,i)^k$, where $g \in G_+(\mathbb{R})$ such that $g(i) = z$. Then it is easy to see $S_k(U;\mathbb{C}) \cong \oplus_{t\in R}\mathcal{S}_k(\Gamma_{tUt^{-1}};\mathbb{C})$. We then define $S_k(U;A)$ by the image of $\oplus_{t\in R}\mathcal{S}_k(\Gamma_{tUt^{-1}};A)$. We can take R inside $\mathcal{R} = \left\{\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \mid a \in \mathbb{Z}^{(p)}\right\}$. We always choose R in this way. Then we have e and $T(p)$ well defined on $S_k(U;\Omega_p)$. Let

$$\mathcal{U} = \{U : \text{ open subgroup of } G(\mathbb{Z}^{(p)})\} .$$

Write $U_0 = U \times GL_2(\mathbb{Z}_p)$ for $U \in \mathcal{U}$. Taking $R(U_0)$ in $\mathcal{R}$ so that $R(U_0) \supset R(V_0)$ if $V \supset U$ for all U, $V \in \mathcal{U}$, we define $\mathbb{S}(U;\mathbb{I}) = \oplus_{t\in R(U_0)}\mathbb{S}(\Gamma_{tU_0t^{-1}};\mathbb{I})$ and $\mathbb{S}(\mathbb{I}) = \bigcup_{U\in\mathcal{U}} \mathbb{S}(U;\mathbb{I})$. Using the stability of $\bigcup_{\widehat{\Delta}\in\mathcal{Z}} \mathbb{S}(\Delta;\mathbb{I})$ under $S(\mathbb{A}^{(p\infty)})$, it is easy to check that $\mathbb{S}(\mathbb{I})$ is stable under $S(\mathbb{A}^{(p\infty)})$. Since $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ with $a \in \mathbb{A}^{(p\infty)}$ basically permutes the direct summands $\mathbb{S}(\Gamma_{tU_0t^{-1}};\mathbb{I})$ of $\mathbb{S}(U;\mathbb{I})$, $\mathbb{S}(\mathbb{I})$ is stable under $G(\mathbb{A}^{(p\infty)})$. We thus have

Proposition 2. — *The space $\mathbb{S}(U;\mathbb{I})$ has, as $\mathbb{I}$–linear endomorphisms, the ordinary projector e and the Hecke operators $T(q)$ for primes q prime to the level of U. The group $G(\mathbb{A}^{(p\infty)})$ acts on $\mathbb{S}(\mathbb{I})$ smoothly.*

5. — Before going into a hard work, we like to give a sketch of the theory. The first main result is

THEOREM 2. — *The automorphic representation of $\widetilde{S}(\mathbb{A}^{(p\infty)})$ on $\mathbb{P}^{\mathrm{ord}}(\mathbb{I})$ is smooth and, after having extended scalar to the field of fractions of $\mathbb{I}$, is a discrete direct sum of irreducible admissible representations with multiplicity at most 1.*

Putting off all the details to the end of this paper for attentive readers, we here give a sketch of the proof. It is well known that $\mathbb{S}^{\mathrm{ord}}(N;\Lambda) = \mathbb{S}^{\mathrm{ord}}(\Gamma_1(N);\Lambda)$ is free of finite rank over Λ (see [H5] Chapter 7), and if $k(P) \geq 2$ for $P \in \mathcal{A}(\Lambda;O)$, then

$$(*) \qquad \mathbb{S}^{\mathrm{ord}}(\Delta;\Lambda)/P\mathbb{S}^{\mathrm{ord}}(\Delta;\Lambda) \cong S_k^{\mathrm{ord}}(\Delta(p^{r(P)}),\varepsilon_P;O).$$

This implies that there are only finitely many, bounded independently of weights, of complex irreducible automorphic representations of $G(\mathbb{A})$ which is p-ordinary and of conductor dividing Np. On the other hand, one has the Shimura correspondence :

$$Sh : \{\text{irreducible holomorphic automorphic representations of } \widetilde{S}(\mathbb{A}) \text{ of weight } k+\frac{1}{2}\} \to \{\text{irreducible holomorphic automorphic representation of } G(\mathbb{A}) \text{ of weight } 2k\}.$$

By a result of Waldspurger, there exists a bound $M > 0$ such that

$$\text{(i)} \qquad \#Sh^{-1}(\pi) \leq M \text{ for all } k, \text{ if } C(\pi) \mid Np,$$

where $C(\pi)$ is the conductor of π. If $\widetilde{\pi}$ is p-ordinary (that is, the eigenvalue of $T(p^2)$ in $\widetilde{\pi}$ is a p-adic unit), $Sh(\widetilde{\pi})$ is p-ordinary. Moreover, if we write V for the space of $\widetilde{\pi}$, we have a positive bound M' independently of weights (but depending on $\widehat{\Delta}$) such that

$$\text{(ii)} \qquad \dim_C H^0(\widehat{\Delta}(p),V) < M'.$$

Then (i) + (ii) $\Rightarrow$ $\mathrm{rank}_O \mathcal{P}^{\mathrm{ord}}_{k+(1/2)}(\Delta(p);O) < M''(\widehat{\Delta})$ independently of k for a positive bound $M''(\widehat{\Delta})$. Take a subset $\{\phi_1,\ldots,\phi_m\}$ in $\mathbb{P}^{\mathrm{ord}}(\Delta;\Lambda)$ which is linearly independent over Λ. Then we can find m rational numbers $n_1,\ldots,n_m$ such that $D = \det(a(n_i,\phi_j)) \neq 0$. Therefore for arithmetic P with $k(P)$ sufficiently large and $\varepsilon_P = id$, $\phi_i(P)$ is and element of $\mathcal{P}^{\mathrm{ord}}_{k(P)+(1/2)}(\Delta(p);O)$ and $D(P) \neq 0$. In other words, $\{\phi_i(P)\}_i$ is linearly independent over O. Therefore $m < M''$. This implies that $\mathrm{rank}_\Lambda \mathbb{P}^{\mathrm{ord}}(\Delta;\Lambda) < M''$. As we will see later, $\mathbb{P}^{\mathrm{ord}}(\Delta;\Lambda)$ is actually free of finite rank over Λ. Then all the assertion follows from the weak multiplicity one theorem of Waldspurger by reducing the Λ-adic reprensentation modulo P.

Thus we have the Λ-adic Shimura correspondence :

$$Sh : \{\text{irreducible } \Lambda\text{-adic ordinary automorphic representations of } \widetilde{S}(\mathbb{A})\} \to \{\text{irreducible } \Lambda\text{-adic ordinary automorphic representations of } G(\mathbb{A})\}.$$

Suppose $\Pi = Sh(\widetilde{\Pi})$. We write $\widetilde{\pi}_P = \widetilde{\Pi} \bmod P$ and $\pi_P = Sh(\widetilde{\pi}_P)$. Then π_P for an arithmetic P is a scalar extension of classical representation if $k(P) \geq 2$. This means that one can supplement a (unique) local representation at p with π_P to get a complex automorphic representation if $k(P) \geq 2$, which we again write π_P. Similarly $\widetilde{\pi}_P$ is associated with a complex automorphic representation of the metaplectic group if $k(P)$ is sufficiently large, because we can only prove the metaplectic version of (*) under the assumption that $k(P)$ is sufficiently large. Here note that $\pi_P \neq \Pi \bmod P$ but $\pi_P = Sh(\widetilde{\pi}_P) = \Pi \bmod P^2$, because representations of weight $k + \frac{1}{2}$ correspond to those of weight $2k$. Here we used the group structure of $\mathrm{Spec}(\Lambda)(O) = \mathrm{Hom}_{gr}(W, O^\times)$ to define P^2. The above fact characterizes the Λ-adic Shimura correspondence. By (*), the prime to p-part $C(\Pi)$ of the conductor of π_P is independent of P. Moreover the central character of Π can be written as $\iota\psi^2$ for a finite order even character ψ modulo $4pC(\Pi)$, where ι is the tautological character of W into $\Lambda^\times$ composed with the "norm" character : $(\mathbb{A}^{(p\infty)})^\times \ni x \longmapsto |x|_{\mathbb{A}}^{-1}\omega^{-1}(x) \in \mathbb{Z}_p^\times$ for the Teichmüller character ω. We put $\psi_P = \varepsilon_P\psi\omega^{-k(P)}$ for each arithmetic P. As a striking consequence of his theory, Waldspurger expressed the square of a certain ratio of two Fourier coefficients of a cusp form of half integral weight by a ratio of L-values attached to the image under the Shimura correspondence. Applying this result, we get a Λ-adic version of his result :

THEOREM 3. — *For each pair (m,n) of positive square free integers with $m/n \in \Pi_{l|4Np}\mathbb{Q}_l^2$, we find two elements Φ and Ψ in $\mathbb{I}$ such that if $k(P) > 1$ or $\psi_P^2 \neq 1$, we have :*

$$\frac{\Phi(P)^2}{\Psi(P)^2} = \frac{L(\frac{1}{2},\pi_P \otimes \psi_P^{-1}\chi_n)\psi_P(n+m)(n/m)^{k-(1/2)}}{L(\frac{1}{2},\pi_P \otimes \psi_P^{-1}\chi_m)}$$

as long as

$$L(\tfrac{1}{2},\pi_P \otimes \psi_P^{-1}\chi_m) \neq 0,$$

where χ_t is the quadratic character associated with $\mathbb{Q}(\sqrt{t})$.

6. — We now start filling the details with the argument in Section 5. Fix a character ψ of $(\mathbb{Z}/Np\mathbb{Z})^\times$. For each arithmetic point $P \in \mathcal{A}(\Lambda)$, we define a character ψ_P of Z_N by $\psi_P(z) = \psi(z)\chi_P(<z>)z^{-k(P)} = \psi\varepsilon_P\omega^{-k(P)}(z)$, where $z \longmapsto <z>$ is the projection to W and ω is the Teichmüller character. We now prove

PROPOSITION 3. — *The dimension of $\mathcal{P}^{\mathrm{ord}}_{k+(1/2)}(\Delta_0(p^{r(P)}),\psi_P;\Omega_p))$ is bounded independent of $P \in \mathcal{A}(\Lambda)$ if $k(P) \geq 1$ (the dimension depends on $\widehat{\Delta} \in \mathcal{Z}$).*

To prove the proposition, we prepare several lemmas. Let ℓ be a prime and put

$$U_r = U_{r,\ell} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}_\ell) \mid c \equiv 0 \bmod \ell^r \right\},$$

For each character χ of $\mathbb{Z}_\ell^\times$ modulo ℓ^r and a U_r–module M, we write $M(\chi)$ for the χ–eigenspace. That is, $M(\chi) = \left\{ m \in M \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} m = \chi(d) m \text{ for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_r \right\}$. When the reference to the level ℓ^r is necessary, we write $M(\ell^r,\chi)$ in place of $M(\chi)$.

LEMMA 1. — *Let π be an irreducible admissible representation of the metaplectic covering group $\widetilde{S}(\mathbb{Q}_\ell)$ of $SL_2(\mathbb{Q}_\ell)$ and V denote its representation space. Let χ be a character of $\mathbb{Q}_\ell^\times$ modulo ℓ^r. Suppose that π appears as a local factor of a holomorphic automorphic representation of weight $k+(1/2)$ $(k \geq 2)$. Then the dimension of $V(\ell^r,\chi)$ is bounded independent of V and χ (but it depends on r).*

Proof : when π is special or principal, then we can realize π as a subquotient of the induced representation space $\mathcal{B}_\mu = \mathcal{B}_{\mu,\mathbf{e}_\ell}$ of a character μ of the standard Borel subgroup, as in [Wa1, II.2] and [Wa2, II], for the ℓ–part $\mathbf{e}_\ell$ of the standard additive character $\mathbf{e}$ of $\mathbb{A}/\mathbb{Q}$ and a quasi character μ of the standard Borel subgroup of $\widetilde{S}(\mathbb{Q}_\ell)$. Since the left translation by the upper triangular matrices of $\widetilde{S}(\mathbb{Q}_\ell)$ is already prescribed on $\mathcal{B}_\mu$, any function in $\mathcal{B}_\mu$ is determined by its restriction to $SL_2(\mathbb{Z}_\ell) \times \{\pm 1\}$. Then for each given open compact subgroup U of $SL_2(\mathbb{Z}_\ell) \times \{\pm 1\}$, the dimen- sion of $H^0(U,\mathcal{B}_\mu)$ is bounded by the index $2(SL_2(\mathbb{Z}_\ell):U)$. A more effective bound can be obtained using the explicit calculation of the space $\mathcal{B}_\mu(\ell^r,\chi)$ done in [Wa2] Proposition 9 (p. 417) (see also Lemma 3 in the text). We then have $\dim(\mathcal{B}_\mu(\ell^r,\chi)) \leq 2(r+1)$. This settles the problem in the case of non–super cuspidal representations. Let $\widetilde{\Pi}$ be a holomorphic automorphic representation of $\widetilde{S}(\mathbb{A})$ of weight $k+\frac{1}{2}$ $(k \geq 2)$ having π as its factor

at ℓ. Let W be the space of the ℓ–component of the automorphic representation of $GL_2(\mathbb{A})$ corresponding to $\widetilde{\Pi}$ by the Shimura correspondence. Using the notation of [Wa1] V.4 (p. 99), we mean by W the ℓ–component of $\mathcal{V}'(\mathbf{e},V) \otimes \chi$. We know from [C] that $\dim W(\ell^r,\chi^2) \leq r+1$. By [Wa2] V, Proposition 5 (p. 404), $V(\ell^r,\chi)$ is a subspace of the space spanned by, with the notation in [Wa2], $i_{\nu,\ell} \circ j_{\nu,\ell}(w)(f_{r,\nu})$ for r sufficiently large (if $r \geq \max(2v_\ell(2)+1,\ v_\ell(C(\chi)))$ for the conductor $C(\chi)$ of χ), where $w \in W(\ell^r,\chi)$ and $\nu \in (\mathbb{Q}_e ll^\times(V)/(\mathbb{Q}_e ll^\times)^2)$. Here $f_{r,\nu}$ is a Schwartz–Bruhat function on $H_\ell = \{x \in M_2(\mathbb{Q}_\ell) \mid Tr(x) = 0\}$ determined by (r,ν) as specified in [Wa2] Chapter V. The choice of $\nu \in \mathbb{Q}_e ll^\times$ is bounded by $\#(\mathbb{Q}_e ll^\times/(\mathbb{Q}_e ll^\times)^2)$ which is 4 if $\ell > 2$ and 8 if $\ell = 2$. Thus we have, for general V,

$$\dim(V(\ell^r,\chi)) \leq 8(r+1) \quad \text{for } r \text{ sufficiently large}.$$

This finishes the proof.

LEMMA 2. — *Let π be an irreducible admissible representation of $\widetilde{S}(\mathbb{Q}_\ell)$ with representation space V. Suppose that π is super cuspidal. Then, for sufficiently large m, $T(\ell^m)$ annihilates $V(\ell^r,\chi)$ if $r > 0$.*

Proof : note that

$$U_0(\ell^r)\left(\begin{pmatrix} \ell^m & 0 \\ 0 & \ell^{-m} \end{pmatrix},1\right)U_0(\ell^r) = \bigcup_{u \in \mathbb{Z}_\ell/\ell^m\mathbb{Z}_\ell} \left(\begin{pmatrix} \ell^m & u\ell^{-m} \\ 0 & \ell^{-m} \end{pmatrix},1\right)U_0(\ell^r)$$

and $$\left(\begin{pmatrix} \ell^m & u\ell^{-m} \\ 0 & \ell^{-m} \end{pmatrix},1\right) = \left(\begin{pmatrix} \ell^m & 0 \\ 0 & \ell^{-m} \end{pmatrix},1\right)\left(\begin{pmatrix} 1 & \ell^{-2m}u \\ 0 & 1 \end{pmatrix},1\right).$$

Thus for $v \in H^0(U_0(\ell^r),V)$, we define an operator $\widetilde{T}(\ell^m)$ by

$$v \mid \widetilde{T}(\ell^m) = \sum_{u \in \mathbb{Z}_\ell/\ell^m\mathbb{Z}_\ell} \left(\begin{pmatrix} \ell^m & u\ell^{-m} \\ 0 & \ell^{-m} \end{pmatrix},1\right)v.$$

The operator $\widetilde{T}(\ell^m)$ coincides with the Hecke operator $(\widetilde{T}'_\ell)^m$ acting on $V(\ell^r,\chi)$ defined in [Wa2] III.3, pp. 388–389. Then we have

$$v \mid \widetilde{T}(\ell^m) = \pi\left(\left(\begin{pmatrix} \ell^m & 0 \\ 0 & \ell^{-m} \end{pmatrix},1\right)\right)\int_{\ell^{-2m}\mathbb{Z}_\ell} \pi\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix},1\right)v\,du = 0$$

for sufficiently large m by the definition of super cuspidality. As shown in [Wa2] Lemma 4, p. 389, we know that $\widetilde{T}'_\ell = \ell^{(3-2k)/2}\tilde{\gamma}_\ell(\ell\chi(\ell)^{-1}T(\ell^2)$ for $T(\ell^2)$ defined in [Sh1], where $\tilde{\gamma}_\ell(t) = (t,t)_\ell\gamma_\ell(t)\gamma_\ell(1)^{-1}$. Thus we know the lemma from the above result. Here we should note that the definition of our space of modular forms of half integral weight is different by the character $\left(\frac{-1}{\cdot}\right)^k$ from that of [Wa2], and thus we do not replace χ by χ_0 as was done in [Wa2] for these formulas.

LEMMA 3 ([V]). — *Suppose that $\ell > 2$. Let $V = \mathcal{B}_\mu$ and let χ be a continuous character of $\mathbb{Q}_\ell^\times$ into $\mathbb{C}^\times$. We consider the Hecke operator $T(\ell^r) = \chi(\ell)\tilde{\gamma}_\ell(\ell)^{-1}\ell^{(2k-3)/2}\tilde{T}'_\ell$. Suppose that $r \geq \mathrm{Sup}(v_\ell(C(\chi)), v_\ell(C(\mu\chi)C(\mu\chi^{-1})))$, where $C(\chi)$ is the conductor of χ. Then we have the following assertions :*

(i) *If both $\mu\chi$ and $\mu\chi^{-1}$ are non-trivial on $\mathbb{Z}_e ll^\times$, then $T(\ell^2)$ is nilpotent on $V(\ell^r;\chi)$ for $r > 0$;*

(ii) *Suppose that $\mu\chi^{-1}$ is unramified but $\mu\chi$ is ramified. Then we can decompose $V(\ell^r;\chi) = N \oplus V(C(\chi);\chi)$ so that $T(\ell^2)$ is nilpotent on N and $V(C(\chi);\chi)$ is one dimensional on which $T(\ell^2)$ acts by scalar multiplication of $\chi(\ell)\ell^{(2k-1)/2}\mu(\ell^{-1})$;*

(iii) *Suppose that $\mu\chi$ is unramified but $\mu\chi^{-1}$ is ramified. Then we can decompose $V(\ell^r;\chi) = N \oplus V(C(\chi);\chi)$ so that $T(\ell^2)$ is nilpotent on N and $V(C(\chi);\chi)$ is one dimensional on which $T(\ell^2)$ acts by scalar multiplication of $\chi(\ell)\ell^{(2k-1)/2}\mu(\ell)$;*

(iv) *Suppose that both $\mu\chi$ and $\mu\chi^{-1}$ are unramified. Then we can decompose $V(\ell^r;\chi) = N \oplus V(\ell;\chi)$ so that (i) $T(\ell^2)$ is nilpotent on N, (ii) $V(\ell;\chi)$ is two dimensional, and (iii) we have a base $\{v_1, v_2\}$ of $V(\ell;\chi)$ such that $v_1|T(\ell^2) = \chi(\ell)\ell^{(2k-1)/2}\mu(\ell^{-1})v_1$ and $v_2|T(\ell^2) = \chi_\ell(\ell)\ell^{(2k-1)/2}\mu(\ell)v_2 + cv_1$ with some constant c.*

Proof : write $\nu(\varepsilon)$ for the exponent of ℓ in $C(\varepsilon)$ for any character of ε of $\mathbb{Z}_e ll^\times$. As shown in [Wa2] Proposition 9, p. 417, under the assumption of $r \geq \mathrm{Sup}(\nu(\chi), 1)$, $V(\ell^r;\chi) \neq 0$ if and only if $r \geq \nu(\mu\chi) + \nu(\mu\chi^{-1})$. As long as $r > \nu(\chi)$ and $r > 1$, $T(\ell^2)$ sends $V(\ell^r;\chi)$ to $V(\ell^{r-1};\chi)$ (cf. [Wa2] Lemma 7 or [H2] (8.6)). This shows that for sufficiently large m, $V(\ell^r;\chi)|T(\ell^{2m})$ is contained in $V(C(\chi);\chi)$ or $V(\ell;\chi)$ if $\nu(\chi) \leq 1$. Unless χ is quadratic, $\nu(\chi) = \nu(\chi^2)$ since $\ell > 2$. Thus if $\chi^2 \neq id$, then $\nu(\mu\chi) + \nu(\mu\chi^{-1}) \geq \mathrm{Max}(\nu(\mu\chi), \nu(\mu\chi^{-1})) \geq \nu(\chi)$. If moreover both $\nu(\mu\chi)$ and $\nu(\mu\chi^{-1})$ are positive, then $\nu(\mu\chi) + \nu(\mu\chi^{-1}) > \nu(\chi)$ and thus $V(C(\chi);\chi) = 0$. Therefore $T(\ell^2)$ is nilpotent on $V(\ell^r;\chi)$ if $\chi^2 \neq id$ and if both $\nu(\mu\chi)$ and $\nu(\mu\chi^{-1})$ are positive. Now suppose that $\chi^2 = id$ and both $\nu(\mu\chi)$ and $\nu(\mu\chi^{-1})$ are positive. Then if $\chi \neq id$, then $C(\chi) = \ell$ and $V(C(\chi);\chi) = 0$ because $\nu(\mu\chi) + \nu(\mu\chi^{-1}) > 1$. If $\chi = id$, then again $V(\ell;\chi) = 0$ because $\nu(\mu) + \nu(\mu) > 1$. Thus $T(\ell^2)$ is nilpotent if both $\nu(\mu\chi)$ and $\nu(\mu\chi^{-1})$ are positive. Now suppose that $\nu(\mu\chi^{-1}) = 0$ but $\nu(\mu\chi) > 0$. Then $\chi^2 \neq id$ because $\nu(\mu\chi) = \nu(\chi^2) > 0$ (and hence $\nu(\chi^2) = \nu(\chi)$), and $V(C(\chi);\chi)$ is one dimensional by [Wa2] Proposition 9. Moreover by [Wa2] Proposition 10, (ii), we know that $T(\ell^2)$ acts on $V(C(\chi);\chi)$ by the scalar multiplication of $\chi(\ell)\ell^{(k/2)-1}\mu(\ell^{-1})$. Thus we can decompose $V(\ell^r;\chi) = N \oplus V(C(\chi);\chi)$ such that on N, $T(\ell^2)$ is nilpotent, and on the one dimensional space $V(C(\chi);\chi)$, $T(\ell^2)$ acts via the multiplication

of $\chi(\ell)\ell^{(k/2)-1}\mu(\ell^{-1})$. Suppose $\nu(\mu\chi^{-1}) = \nu(\mu\chi) = 0$ and $\chi \neq id$. Then $\chi^2 = id$ because $\nu(\mu\chi) = \nu(\chi^2) = 0$. By [Wa2] Proposition 2, $V(\ell;\chi)$ is 2–dimensional, and there is a base $\{v_1, v_2\}$ of $V(\ell;\chi)$ such that

$$v_1 \mid T(\ell^2) = \chi(\ell)\ell^{(k/2)-1}\mu(\ell^{-1})v_1 \text{ and}$$
$$v_2 \mid T(\ell^2) = \chi(\ell)\ell^{(k/2)-1}\mu(\ell)v_2 + \ell^{(k/2)-2}\tilde{\gamma}_\ell(\ell)^{-1}\chi(\ell)(\ell-1)v_1 .$$

Thus we can decompose $V(\ell^r;\chi) = N \oplus V(\ell;\chi)$ such that on N, $T(\ell^2)$ is nilpotent and on the 2–dimensional space $V(\ell;\chi)$, it acts by the above formula. Next suppose that $\nu(\mu\chi^{-1}) = \nu(\mu\chi) = 0$ and $\chi = id$. Then $V(\ell;\chi)$ is 2–dimensional, and we can find a base $\{v_1, v_2\}$ by [Wa2] Proposition 10 such that $(v_1 + v_2) \in V(1;\chi)$ and

$$v_1 \mid T(\ell^2) = \chi(\ell)\ell^{(k/2)-1}\mu(\ell^{-1})v_1 \text{ and } v_2 \mid T(\ell^2) = \chi(\ell)\ell^{(k/2)-1}\mu(\ell)v_2 + cv_1$$

with some constant c. The value of c is given by [Wa2] p. 420. This shows that $V(\ell^r;\chi) = N \oplus V(\ell;\chi)$ such that on N, $T(\ell^2)$ is nilpotent, and on the 2–dimensional space $V(\ell;\chi)$, $T(\ell^2)$ is an automorphism described as above. Finally we assume that $\nu(\mu\chi) = 0$ but $\nu(\mu\chi^{-1}) > 0$. Then $\nu(\mu\chi^{-1}) = \nu(\chi^{-2}) > 0$ and hence $\nu(\chi^2) = \nu(\chi)$. Thus again by [Wa2] Propositions 9 and 10, $V(C(\chi);\chi)$ is one dimensional and $T(\ell^2)$ acts on it by the multiplication of $\chi(\ell)\ell^{(k/2)-1}\mu(\ell)$. Therefore, we can decompose $V(\ell^r;\chi)$ into $V(C(\chi);\chi) \oplus N$, where on N, $T(\ell^2)$ is nilpotent and on the one–dimensional space $V(C(\chi);\chi)$, it acts by the scalar $\chi(\ell)\ell^{(k/2)-1}\mu(\ell)$.

LEMMA 4 ([Wa1] Proposition 18, p. 68). — *Let ρ^* be an irreducible admissible representation of $PGL_2(\mathbb{Q}_\ell)$ and let ρ be the corresponding irreducible admissible representation of $\widetilde{S}(\mathbb{Q}_\ell)$ via Weil representation with respect to the additive character $\mathbf{e}_\ell^\xi$ ($\xi \in \mathbb{Q}_e ll^\times$). Then we have*

Equivalence class of ρ^*	*Equivalence class of ρ*
$\pi(\mu,\mu^{-1})$ ($\mu^2 \neq \alpha$)	$\tilde{\pi}_{\mu\chi_\xi}$
$\sigma(\mu,\mu^{-1})$ ($\mu^2 \neq \alpha$, $\mu \neq \alpha^{1/2}$)	$\tilde{\sigma}_{\mu\chi_\xi}$
$\sigma(\alpha^{1/2},\alpha^{-1/2})$	*Supercuspidal*
Supercuspidal	*Supercuspidal*

where $\mathbf{e}_\ell$ is the standard additive character of $\mathbb{Q}_\ell$ and $\mathbf{e}_\ell^\xi(x) = \mathbf{e}_\ell(\xi x)$ and we have used the notation of [Wa1] *Propositions* 1 *and* 2.

Here note that $\tilde{\pi}_{\mu\chi_\xi}$ (resp. $\tilde{\sigma}_{\mu\chi_\xi}$) with respect to $\mathbf{e}_\ell^\xi$ is isomorphic to $\tilde{\pi}_{\mu\chi_{\xi\eta}}$ (resp. $\tilde{\sigma}_{\mu\chi_{\xi\eta}}$) with respect to $\mathbf{e}_\ell^{\xi\eta}$, and hence the right-hand side is well defined independent of the additive character.

A cusp form $f \in S_\kappa(\Delta_1(p^r);\mathbb{C})$ is called *ordinary* at p if $f \mid T(p) = \lambda f$ and $|\lambda|_p = 1$. An automorphic representation π of $GL_2(\mathbb{A})$ spanned by a holomorphic primitive form f is called *ordinary* at p if f is ordinary at p.

Lemma 5 (e.g. [H3] §2). — *Let π be a unitary holomorphic automorphic representation of $GL_2(\mathbb{A})$. Suppose that π is irreducible and ordinary at p. Then the local component π_p of π is either a principal series representation $\pi(\alpha,\beta)$ with unramified α or a special representation $\sigma(\alpha,\beta)$ with unramified α. Let f be the primitive form of weight k on $GL_2(\mathbb{A})$ belonging to π and write μ for the central character of π and $\lambda(T(p))$ for the eigenvalue of $T(p)$ on f. Then if $\pi_p = \pi(\alpha,\beta)$ and α and β are both unramified, then $\alpha(p)+\beta(p) = p^{(1-k)/2}\lambda(T(p))$, $\alpha(p)\beta(p) = \mu(p)$ and $|\lambda(T(p))|_p = 1$. If $\pi_p = \pi(\alpha,\beta)$ and β is ramified, then $\alpha(p) = p^{(1-k)/2}\lambda(T(p))$, $\alpha(p)\beta(p) = \mu(p)$ and $|\lambda(T(p))|_p = 1$. If $\pi_p = \sigma(\alpha,\beta)$, then π_∞ is of weight 2 and $\lambda(T(p)) = \alpha(p)$.*

Lemma 6. — *Let F be a number field of finite degree. Let ρ be a cuspidal automorphic representation of $PGL_2(F_A)$ and let R be the set of all cuspidal automorphic representations of $\widetilde{S}(F_\mathbb{A})$. Define for each integral ideal N of F,*

$$R(\rho;N) = \{\pi \in R \mid \pi_v^* \cong \rho_v \text{ for all } v \text{ outside } N\},$$

where π_v^* denotes the corresponding representation of $PGL_2(F_v)$ via Weil representations defined in [Wa1, V.4] *(where it is written as : $T \longmapsto \mathcal{V}'(\mathbf{e},T)$; see Lemma 4). Then we have*

$$\#R(\rho;N) \le \#\{\Pi_{v|N}F_v^\times/(F_v^\times)^2\}.$$

Proof : we know from Lemma 4 (or the remark after the lemma) that if T_v is principal or special, then $\mathcal{V}'(\mathbf{e}_v^x,T_v) \cong \mathcal{V}'(\mathbf{e}_v,T_v)$. Moreover if $x/y \in (F_v^\times)^2$, then $\mathcal{V}'(\mathbf{e}_v^x,T_v) = \mathcal{V}'(\mathbf{e}_v^y,T_v)$ for all T_v by [Wa1] Theorem 2, p. 80, Proposition 28, p. 98 and [Wa2] Assertion 3, p. 394. Thus the number of isomorphism classes in $\{\mathcal{V}'(\mathbf{e}_v^x,T_v) \mid x \in F_v^\times\}$ for all v outside N are at most $\#\{\Pi_{v|N}F_v^\times/(F_v^\times)^2\}$. Then the weak multiplicity one theorem [Wa1] VI shows the result.

Proof of Proposition 3 : we only prove the assertion when $\Delta = \Gamma_1(N)$. The general case follows from this special case because any Δ contains a conjugate of $\Gamma_1(N)$ for a suitable N. We shall prove the boundedness for $P \in \mathcal{A}(\Lambda)$ with $k(P) \ge 1$. We write $\chi = \psi_P$ for a given $P \in \mathcal{A}(\Lambda)$ and $\psi : (\mathbb{Z}/Np\mathbb{Z})^\times \to \overline{\mathbb{Q}}_p^\times$ and consider χ as an idele character so that $\chi(\varpi) = \chi(\ell)$ for a prime element ϖ at any prime ℓ outside Np. Let V be the subspace of functions on $\widetilde{S}(\mathbb{A})$ spanned by right translations of elements in

$$\mathcal{P}^{\text{ord}}_{k(P)+(1/2)}(Np^{r(P)},\psi_P;\mathbb{C})$$

under the Hecke algebra of $\widetilde{S}(\mathbb{A})$. We decompose $V = \oplus_\rho V(\rho)$ into the sum of irreducible subspaces $V(\rho)$. Then by the weak multiplicity one theorem proven by [Wa1] p. 131, each irreducible representation ρ occurs at most once. Decompose $\rho = \otimes_\ell \rho_\ell$ into the tensor product of local representations. Then by Lemma 2, ρ_p is either $\tilde{\sigma}_\mu$ or $\tilde{\pi}_\mu$ for a quasi character $\mu : \mathbb{Q}_p{}^\times \to \mathbb{C}^\times$. By the Weil representation, $\tilde{\sigma}_\mu$ corresponds to $\sigma(\mu,\mu^{-1})$ and $\pi(\mu,\mu^{-1})$, which is a representation of $PGL_2(\mathbb{A})$ (Lemma 4 and [Wa1] Proposition 27 and Lemma 70). Then the Shimura correspondence is given locally by

$$\tilde{\sigma}_\mu \longmapsto \sigma(\mu\chi,\mu^{-1}\chi) \text{ and } \tilde{\pi}_\mu \longmapsto \pi(\mu\chi,\mu^{-1}\chi)$$

and globally by $\rho \longmapsto \rho^* \otimes \chi$, where $\rho \longmapsto \rho^*$ is given via the global Weil representation. The eigenvalue for $T(p^2)$ on $V(\rho_p)(p^r;\chi)$ ($r = r(P)$) is given as follows (Lemma 3) : if $\mu\chi$ is unramified, then it is $\mu\chi(p)p^{(2k-1)/2}$ ($k = k(P)$); if $\mu^{-1}\chi$ is unramified, then $\mu^{-1}\chi(p)p^{(2k-1)/2}$ and if both $\mu\chi$ and $\mu^{-1}\chi$ are ramified, it vanishes. On the other hand, these values are the eigenvalue of $T(p)$ on $V(\rho_P^*\otimes\chi)(p^r;\chi^2)$ by Lemma 5. Note that even if both $\mu\chi$ and $\mu^{-1}\chi$ are unramified, at most one eigenvalue in $\mu^{-1}\chi(p)p^{(2k-1)/2}$ and $\mu\chi(p)p^{(2k-1)/2}$ can be a p-adic unit in $\overline{\mathbb{Q}}_p$. Thus ρ corresponds to the ordinary ρ^* of character χ^2 and of level at most $Np^{r(P)}$. Then by [H5] Theorem 7.3.3, the number of such automorphic representations occurring in $\mathcal{S}^{\text{ord}}_{2k(P)}(Np^{r(P)},\chi^2)$ is bounded independent of P if $k(P) \ge 1$. Then by Lemmas 2, 4 and 6, we know the assertion of the proposition.

We say an element $\mathbf{f} \in \mathbb{P}(\Delta;\mathbb{I})$ is ordinary, if for all $P \in \mathcal{A}(\mathbb{I})$ with sufficiently large $k(P)$, $\mathbf{f}_P \in \mathcal{P}^{\text{ord}}_{k(P)+(1/2)}(\Delta(p^{r(P)}),\varepsilon_P;\Omega_p)$. We denote the space of all $\mathbb{I}$-adic ordinary cusp forms as $\mathbb{P}^{\text{ord}}(\Delta;\mathbb{I})$. Then $\mathbb{P}^{\text{ord}}(\Delta;\mathbb{I}) = e\mathbb{P}(\Delta;\mathbb{I})$.

Proposition 4. — *For each $\widetilde{\Delta} \in \mathcal{Z}$ with $\Delta \subset \Gamma_0(4)$, $\mathbb{P}^{\text{ord}}(\Delta;\mathbb{I})$ is free of finite rank over $\mathbb{I}$.*

Proof : we prove the assertion for $\mathbb{P}^{\text{ord}}(N;\mathbb{I})$ applying the argument of Wiles [Wi]. The other cases can be treated similarly. Let $\Delta = \Gamma_1(N)$. Let $\mathbb{K}$ be the quotient field of $\mathbb{I}$, which is a finite extension of $\mathbb{L}$. We put $\mathbb{P}^{\text{ord}}(N;\mathbb{K}) = \mathbb{P}^{\text{ord}}(N;\mathbb{I}) \otimes_\mathbb{I} \mathbb{K}$. Let $\mathbf{f}_1,\mathbf{f}_2,\ldots,\mathbf{f}_r$ be a finite set of linearly independent elements in $\mathbb{P}^{\text{ord}}(N;\mathbb{I})$ over $\mathbb{I}$. Then we can find positive integers

$n_1, \ldots, n_r$ so that

$$D = \det(\mathbf{a}(n_i, \mathbf{f}_j)) \neq 0.$$

We now choose $P \in \mathcal{A}(\mathbb{I})$ so that for all $i = 1, \ldots, r$,

$$\mathbf{f}_i(P) \in \mathcal{P}^{\mathrm{ord}}_{k(P)+(1/2)}(\Delta(p^{r(P)}), \varepsilon_P; \Omega_p) \text{ and } D(P) \neq 0.$$

Then $0 \neq D(P) = \det(a(n_i, \mathbf{f}_j(P))$ and thus $\mathbf{f}_i(P)$ are linearly independent. Namely, we have

$$r \leq \dim \mathcal{P}^{\mathrm{ord}}_{k(P)+(1/2)}(\Delta(p^{r(P)}), \varepsilon_P; \Omega_p),$$

which is bounded independently of P by Proposition 3. Thus there is a maximal set $\{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_r\}$ of linearly independent elements in $\mathbb{P}^{\mathrm{ord}}(N, \mathbb{I})$. That is, $\dim_{\mathbb{K}} \mathbb{P}^{\mathrm{ord}}(N; \mathbb{K}) = r < \infty$. For any $\mathbf{f}$ in $\mathbb{P}^{\mathrm{ord}}(N; \mathbb{I})$, we can write $\mathbf{f} = \sum_{i=1}^{r} c_i(\mathbf{f})\mathbf{f}_i$ and $Dc_i(\mathbf{f}) \in \mathbb{I}$. Thus $D^{-1}(\mathbb{I}\mathbf{f}_1 + \cdots + \mathbb{I}\mathbf{f}_r) \supset \mathbb{P}^{\mathrm{ord}}(N; \mathbb{I})$ and hence $\mathbb{P}^{\mathrm{ord}}(N; \mathbb{I})$ is of finite type over $\mathbb{I}$ as $\mathbb{I}$–module, because $\mathbb{I}$ is noetherian. Now we see by definition that $\mathbb{P}^{\mathrm{ord}}(N; \mathbb{I}) = \cap_P \mathbb{P}^{\mathrm{ord}}(N; \mathbb{I}_P)$ where P runs over all prime ideals of height 1, $\mathbb{I}_P$ is the localizaton at prime P and $\mathbb{P}^{\mathrm{ord}}(N; \mathbb{I}_P) = \mathbb{P}^{\mathrm{ord}}(N; \mathbb{I}) \otimes_{\mathbb{I}} \mathbb{I}_P$. This shows that $\mathbb{P}^{\mathrm{ord}}(N; \mathbb{I})$ is $\mathbb{I}$–reflexive and hence if $\mathbb{I} = \Lambda$, then $\mathbb{P}^{\mathrm{ord}}(N; \Lambda)$ is Λ–free of finite rank. Since we already know that $\mathbb{P}^{\mathrm{ord}}(N; \mathbb{I}) = \mathbb{P}^{\mathrm{ord}}(N; \Lambda) \otimes_\Lambda \mathbb{I}$, we conclude that $\mathbb{P}^{\mathrm{ord}}(N; \mathbb{I})$ is $\mathbb{I}$–free of finite rank.

Proposition 5. — *Let $P \in \mathcal{A}(\mathbb{I})$. Then each $f \in \mathcal{P}^{\mathrm{ord}}_{k(P)+(1/2)}(\Delta(p^{r(P)}), \varepsilon_P; O)$ can be lifted to an ordinary Λ–adic form $\mathbf{f} \in \mathbb{P}^{\mathrm{ord}}(\Delta; \mathbb{I})$ such that $\mathbf{f}(P) = f$.*

Proof : it is sufficient to prove the assertion for $\mathbb{I} = \Lambda$. Let $E(X) \in \Lambda[[q]]$ be the Λ–adic Eisenstein series (cf. [H5] §7.1) such that for the generator $w = 1 + p$ of W

$$E(Q) = (Q(w)-1)\Big\{L_p(1-k(Q), \varepsilon_Q \omega^{-k(Q)})/2 + \sum_{n=1}^{\infty}\Big(\sum_{0<d|n} Q(<d>)d^{-1}\Big)q^n\Big\}$$

in $\mathcal{M}_{k(Q)}(\Delta(p^{r(Q)}), \varepsilon_Q; \overline{\mathbb{Q}}_p)$ for all $Q \in \mathcal{A}(\Lambda)$. Then we see for the point P_0 of $\mathcal{X}(\Lambda)$ corresponding to the trivial character of W, $E(P_0) = (1-p)\log(w)/p$, which is a p–adic unit. We then put $F = E(P_0)^{-1}E$ and consider the product fF inside $\Lambda[[q]]$. Then $Ff(Q) = fF(Q) \in \mathcal{P}_{k(P)+k(Q)+(1/2)}(\Delta(p^{r(P)}), \varepsilon_P\varepsilon_Q; \Omega)$. We define a formal q–expansion $F * f(X)$ by $Ff(\varepsilon_P^{-1}(w)w^{-k}X + (\varepsilon_P^{-1}(w)w^{-k} - 1)$, which is a Λ–adic cusp form in $\mathbb{P}(N; \Lambda)$ ([H5] Lemma 7.1.1). Then we see that $F * f(P) = fF(P_0) = f$. Then $e(F * f)(P) = (F * f(P)) \mid e = f$ by Lemma 7 and the assertion of the theorem follows.

Corollary 1. — *For $P \in \mathcal{A}(\mathbb{I}; O)$ with sufficiently large $k(P)$ depending on Δ, we have*

$$\mathcal{P}^{\mathrm{ord}}_{k(P)+(1/2)}(\Delta(p^{r(P)}), \varepsilon_P; O) \cong \mathbb{P}^{\mathrm{ord}}(\Delta; \mathbb{I})/P\mathbb{P}^{\mathrm{ord}}(\Delta; \mathbb{I}).$$

Proof : Choose a base $\mathbf{f}_1, \ldots, \mathbf{f}_r$ of $\mathbb{P}^{\mathrm{ord}}(\Delta; \mathbb{I})$. We can find $a > 0$ so that

(i) $\mathbf{f}_i(P) \in \mathcal{P}^{\mathrm{ord}}_{k(P)+(1/2)}(\Delta(p^{r(P)}); \varepsilon_P; O)$ *for all i and all P with $k(P) > a$, and*

(ii) *there exist integers n_i such that $\det(\mathbf{a}(n_i/N, \mathbf{f}_j))(P) \neq 0$ if $k(P) > a$.*

Then $\mathbf{f}_i(P)$ are linearly independent over O. Thus $\mathbb{P}^{\mathrm{ord}}(\Delta; \mathbb{I})/P\mathbb{P}^{\mathrm{ord}}(\Delta, \mathbb{I})$ injects into $\mathcal{P}^{\mathrm{ord}}_{k(P)+(1/2)}(\Lambda(p^{r(P)}); \varepsilon_P; O)$. Surjectivity of the morphism follows from Proposition 5.

Corollary 2. — *Let $\mathbf{f}_1, \ldots, \mathbf{f}_r$ be a base of $\mathbb{P}^{\mathrm{ord}}(\Delta; \mathbb{I})$. Then we can find integers $n_1, \ldots, n_r$ so that $\det(\mathbf{a}(n_i/N, \mathbf{f}_j)) \in \mathbb{I}^\times$.*

Proof : let $f_1, \ldots, f_r$ be a base of $\mathcal{P}^{\mathrm{ord}}_{k(P)+(1/2)}(\Delta(p^{r(P)}), \varepsilon_P; O)$. let ϖ be a prime element of O. If $\det(a(n_i/N, f_j) \equiv 0 \bmod \varpi O$ for all choice of integers $n, \ldots, n_r$, then $\{f_i \bmod \varpi\}$ are linearly dependent and hence we can find $\lambda_i \in O$ not all divisible by ϖ such that $\Sigma_i \lambda_i f_i \equiv 0 \bmod \varpi O$. Then

$$\varpi^{-1}\Sigma_i \lambda_i f_i \in \mathcal{P}^{\mathrm{ord}}_{k(P)+(1/2)}(\Delta(p^{r(P)}), \varepsilon_P; O)$$

but $\varpi^{-1}\lambda_i$ are not all in O. This contradicts to the fact that $\{f_i\}$ forms a base. Thus we can find the n_i's so that $\det(a(n_i, f_j)) \in O^\times$. Now applying this argument to a base $\{\mathbf{f}_i(P)\}$ by choosing P with sufficiently large $k(P)$, we find that $\det(\mathbf{a}(n_i/N, \mathbf{f}_j))(P) \in O^\times$ which implies that $\det(\mathbf{a}(n_i/N, \mathbf{f}_j)) \in \mathbb{I}^\times$.

Analogs of all the assertion so far we proved in this paragraph holds for $\mathbb{S}^{\mathrm{ord}}(U; \mathbb{I})$ in an obvious sense (see [H5] Chapter 7). In particular, the statement corresponding to Corollary 1 for $\mathbb{S}^{\mathrm{ord}}(\Delta; \mathbb{I})$ holds if $k(P) \geq 2$.

7. — We now restate Theorem 3 in the language of p–adic Hecke algebras. Let $\mathbf{h}^{\mathrm{ord}}(N; O)$ be the p–adic ordinary Hecke algebra defined in [H5] §7.3. Let us recall the definition. The algebra $\mathbf{h}^{\mathrm{ord}}(N; O)$ is the Λ–subalgebra of $\mathrm{End}_\Lambda(\mathbb{S}^{\mathrm{ord}}(N; \Lambda))$ generated by $T(n)$ for all n. There is another description of the algebra. Writing $\mathbf{h}^{\mathrm{ord}}_k(Np^r; O)$ for the O–subalgebra of $\mathrm{End}_0(S^{\mathrm{ord}}_k(Np^r; O))$ generated by $T(n)$ for all n, we have a natural isomorphism : $\mathbf{h}^{\mathrm{ord}}(N; O) \cong \varprojlim_\alpha \mathbf{h}^{\mathrm{ord}}_k(Np^\alpha; O)$ if $k \geq 2$, which takes $T(n)$ to $T(n)$ [H2]. Under the natural pairing $< h, \mathbf{f} >= \mathbf{a}(1, \mathbf{f} \mid h)$, we know

$$\mathrm{Hom}_\Lambda(\mathbf{h}^{\mathrm{ord}}(N; O), \Lambda) \cong \mathbb{S}^{\mathrm{ord}}(N; \Lambda) \text{ and}$$
$$\mathrm{Hom}_\Lambda(\mathbb{S}^{\mathrm{ord}}(N; \Lambda), \Lambda) \cong \mathbf{h}^{\mathrm{ord}}(N; O). \tag{7.1}$$

We have a smooth representation of $G(\mathbb{A}^{(p\infty)})$ on $\mathbb{S}^{\mathrm{ord}}(\mathbb{I}) = \bigcup_{U\in\mathcal{U}} \mathbb{S}^{\mathrm{ord}}(U;\mathbb{I}) =$
$e\mathbb{S}(\mathbb{I})$ and $\mathbb{S}(\mathbb{I})$. Thus compactly supported smooth functions on $G(\mathbb{A}^{(p\infty)})$
with values in $\mathbb{I}$ act on $\mathbb{S}(\mathbb{I})$. We fix an algebraic closure $\overline{\mathbb{L}}$ of $\mathbb{L}$. We
then consider $\mathbb{S}^{\mathrm{ord}}(A) = \mathbb{S}^{\mathrm{ord}}(\Lambda) \otimes_\Lambda A$ as a $G(\mathbb{A}^{(p\infty)})$–module for any Λ–
subalgebra A in $\overline{\mathbb{L}}$. Each irreducible factor of the representation on $\mathbb{S}^{\mathrm{ord}}(\overline{\mathbb{L}})$
of $G(\mathbb{A}^{(p\infty)})$ is admissible by the control theorem ([H5] Theorem 7.3.3,
which is the integral weight counterpart of Corollary 1 and is valid for all
arithmetic points of weight $k \geq 2$). Pick an arithmetic point P with $k(P) \geq 2$
and consider the localization Λ_P at P. Then by the control theorem,
$\mathbb{S}^{\mathrm{ord}}(\Lambda_P) \otimes_\Lambda K(P)$ for $K(P) = \Lambda_P/P$ is a semi-simple $GL_2(\mathbb{A}^{(p\infty)})$–
module. Since there are Zariski dense arithmetic points in $\mathrm{Spec}(\Lambda)$ at
which the control theorem holds, we see that $\mathbb{S}^{\mathrm{ord}}(\overline{\mathbb{L}})$ is semi-simple as
a $G(\mathbb{A}^{(p\infty)})$–module. Thus $\mathbb{S}^{\mathrm{ord}}(\overline{\mathbb{L}})$ is a sum of irreducible subspaces. The
multiplicity is one by the control theorem combined with the multiplicity one
theorem in classical situation. Since the proof of the factorization theorem
in [JL] §9 is purely algebraic, it carries over to our situation, and each
irreducible factor π of $\mathbb{S}^{\mathrm{ord}}(\overline{\mathbb{L}})$ is factored into the tensor product of local
representations : $\pi = \otimes_{\ell\neq p}\pi_\ell$. Let $\lambda; \mathbf{h}^{\mathrm{ord}}(C;O) \to \mathbb{I}$ be a primitive Λ–
algebra homomorphism. Then by the control theorem, we have a unique
automorphic representation $\pi(P) = \otimes_\ell \pi_\ell(P)$ corresponding to λ mod P for
$P \in \mathcal{A}(\mathbb{I})$ with $k(P) \geq 2$. Thus λ corresponds a unique factor $\pi = \pi(\lambda)$
of $\mathbb{S}^{\mathrm{ord}}(\overline{\mathbb{L}})$, and $\pi_\ell(P) = \pi_\ell$ mod P. We write $V(\pi)$ for the subspace of
$\mathbb{S}^{\mathrm{ord}}(\overline{\mathbb{L}})$ on which $S(\mathbb{A}^{(p\infty)})$ acts via π. Thus for each arithmetic point P
with $k(P) \geq 2$, $\lambda_P(T(n)) = \lambda(T(n))(P)$ is an algebraic number. Then for
each Dirichlet character φ, we can define the complex L–function :

$$L(s, \lambda_P \otimes \varphi) = \sum_{n=1}^{\infty} \varphi(n)\lambda_P(T(n))n^{-s}.$$

Note that $L(s,\pi(P)) = L(s + \frac{k(P)-1}{2}, \lambda_P)$ is the standard L–function of
$\pi(P)$. As is well known, the L–function $L(s, \lambda_P \otimes \varphi)$ has a motivic inter-
pretation. Since $\mathbb{I}$ is an integral domain, we see that
$Z_C = \mathbb{Z}_p^\times \times (\mathbb{Z}/C\mathbb{Z})^\times \ni z \longmapsto \lambda(<z>) \in \mathbb{I}$ is a character, where $<z>$ is
the operator induced by the central action of $z \in (\mathbb{Z}^{(p)})^\times \subset G(\mathbb{A}^{(p\infty)})$ on
$\mathbb{S}(\mathbb{I})$ (see (2.1)). In particular, it restriction to $\mu_{p-1} \times (\mathbb{Z}/C\mathbb{Z})^\times$ gives a cha-
racter $\psi_0 : \mu_{p-1} \times (\mathbb{Z}/C\mathbb{Z})^\times \to \overline{\mathbb{Q}}_p^\times$. We regard this character as a character
of Z_C composing the projection : $Z_N \to \mu_{p-1} \times (\mathbb{Z}/C\mathbb{Z})^\times \cong (\mathbb{Z}/Cp\mathbb{Z})^\times$ and
call it the character of λ.

We now consider the following conditions on λ :

(H_ℓ). *Writing* $\pi_\ell \cong \pi(\alpha,\beta)$ *when* π_ℓ *is principal* ($\ell \neq p$), *we have* $\alpha(-1) = \beta(-1) = 1$;

(H_p). $\psi_0 = \psi^2$ *for an even character* ψ *modulo* N *for* N *divisible by* C *and* 4.

Under this condition, the automorphic representation associated to λ is in the image of the Shimura correspondence (see [Wa2] Proposition 2). Now we consider the automorphism σ_m of Λ which takes w to w^m ($w \in W$) for m prime to p. This ring automorphism extends to an automorphism σ_m of $\mathbb{I}$ if $\mathbb{I}$ is sufficiently large. For each $P \in \mathcal{A}(\mathbb{I})$, we denote P^2 for $P \circ \sigma_2$. Then $k(P^2) = 2k(P)$ and $\varepsilon_{P^2} = \varepsilon_P^2$. As constructed in [K] and [GS], for each character φ of $(\mathbb{Z}/Np\mathbb{Z})^\times$, there is a two variable p-adic L–function $\mathcal{L}_p(P,Q;\lambda\otimes\varphi)$ defined on $\mathcal{X}(\mathbb{I}) \times \mathcal{X}(\Lambda)$ interpolating the value $L(k(Q), \lambda_P \otimes \varepsilon_Q^{-1}\omega^{k(Q)}\varphi)$ for $(P,Q) \in \mathcal{A}(\mathbb{I}) \times \mathcal{A}(\Lambda)$ with $0 < k(Q) < k(P)$. Here is a result slightly stronger than Theorem 3 :

THEOREM 4. — *Let* $\lambda : \mathbf{h}^{\mathrm{ord}}(C;O) \to \mathbb{I}$ *be a primitive* Λ*–algebra homomorphism. Suppose* (H_ℓ) *and* (H_p). *Then for any pair* (m,n) *of two square free positive integers with* $m/n \in \prod_{l|Np} \mathbb{Q}_l^2$, *there exists an element* Φ *in* $\mathbb{K}$ *such that for all* $P \in \mathcal{A}(\mathbb{I})$ *with* $k(P) \geq 1$, *if* $\mathcal{L}_p(P^2, P; \lambda\otimes\psi^{-1}\chi_m) \neq 0$, *we have*

$$\Phi(P)^2 = \frac{\mathcal{L}_p(P^2, P; \lambda\otimes\psi^{-1}\chi_n)\psi_P(n/m)(n/m)^{k(P)-(1/2)}}{\mathcal{L}_p(P^2, P; \lambda\otimes\psi^{-1}\chi_m)},$$

where χ_t *is the quadratic character corresponding to* $\mathbb{Q}(\sqrt{t})$. Here note that under our assumption on (m,n), (m/n) is prime to Np.

Proof : we take $P \in \mathcal{A}(\mathbb{I})$ with $k(P)$ sufficiently large. Let φ be a Dirichlet character. Then for the least common multiple N' of C and the conductor of φ, we find $\lambda\otimes\varphi : \mathbf{h}(N';O) \to \mathbb{I}$ such that $\lambda\otimes\varphi(T(n)) = \varphi(n)\lambda(T(n))$. Then the character of $\lambda\otimes\varphi$ is given by $\psi^2\varphi^2$. Taking even φ with sufficiently large 2–power conductor, we may assume that the conductor C' of $\lambda\otimes\varphi$ is divisible by 16. If we replace λ by $\lambda\otimes\varphi$, the role of ψ will be replaced by $\psi\varphi$. Since $\lambda\otimes\psi^{-1}\chi_n = (\lambda\otimes\varphi)\otimes(\varphi\psi)^{-1}\chi_n$, the L–value appearing in the assertion of the theorem is unchanged even if we replace λ by $\lambda\otimes\varphi$. Thus we may assume that $16 \mid C$ (hence π satisfies the condition (H2) in [Wa2] p. 378). Let f be the cusp form in $\mathcal{P}^{\mathrm{ord}}_{k(P)+1/2}(\Gamma_0(N^2p^{r(P)}), \psi_P; \Omega_p)$ which is a linear combination of the base defined in [Wa2] Theorem 1 for $\pi(P^2)$. Let us take $\mathbf{f} \in \mathbb{P}^{\mathrm{ord}}(N^2;\mathbb{I})$ such that $\mathbf{f} \mid T(q^2) = \sigma_2 \circ \lambda(T(q))\mathbf{f}$ for all prime q outside Np and $\mathbf{f}(P) = cf$ with $0 \neq c \in O$. Such $\mathbf{f}$ exists by Corollary 1. Then by [Wa2] Corollary 2, for any $Q \in \mathcal{A}(\mathbb{I})$ such that $\mathbf{f}(Q)$ is classical, we have :

$$\mathbf{a}(m,\mathbf{f})^2(Q)L(k(Q), \lambda_{Q^2}\otimes\psi_Q^{-1}\chi_n)\psi_Q(n/m)(n/m)^{k(Q)-(1/2)}$$
$$= \mathbf{a}(n,\mathbf{f})^2(Q)L(k(Q), \lambda_{Q^2}\otimes\psi_Q^{-1}\chi_m).$$

To get the p–adic interpolation, we need to remove certain Euler factor at p and divide the special value by a certain period. However the Euler factor and the period are the same for n and m under the condition of the theorem. Thus using two variable p-adic L–functions, the above identity can be stated as :

$$\mathbf{a}(m,\mathbf{f})(Q)^2\mathcal{L}_p(Q^2,Q;\lambda\otimes\psi^{-1}\chi_n)\psi_Q(n/m)(n/m)^{k(Q)-(1/2)} = \mathbf{a}(n,\mathbf{f})(Q)^2\mathcal{L}_p(Q^2,Q;\lambda\otimes\psi^{-1}\chi_m).$$

If $\mathcal{L}_p(Q^2,Q;\lambda\otimes\psi^{-1}\chi_n)=0$ for all Q as above, the p–adic $\mathcal{L}$–function $\mathcal{L}_p(\lambda\otimes\psi^{-1}\chi_n)$ vanishes. Hence there is nothing to prove. If $\mathcal{L}_p(\lambda\otimes\psi^{-1}\chi_n)\neq 0$, by the assumption of the theorem, $\mathcal{L}_p(\lambda\otimes\psi^{-1}\chi_m)\neq 0$. Then we may assume that

$$\mathcal{L}_p(P^2,P;\lambda\otimes\psi^{-1}\chi_m)\mathcal{L}_p(P^2,P;\lambda\otimes\psi^{-1}\chi_n)\neq 0$$

by moving around P. Then we may assume by Theorem 1 of [Wa2] that the m–th and n–th Fourier coefficients of f are both non–zero. Therefore $\mathbf{a}(m;\mathbf{f})\mathbf{a}(n;\mathbf{f})\neq 0$. Thus we can take $\Phi=\mathbf{a}(n,\mathbf{f})/\mathbf{a}(m,\mathbf{f})$. Now we have the evaluation property of Φ described in the theorem for almost all P. Note that $\mathcal{L}_p(P,Q;\lambda\otimes\psi^{-1}\chi_n)$ for a fixed n is a p–adic analytic function of (P,Q) (see [K] and [GS]). Thus as long as the removed Euler factor does not vanish, we get the result. The only case where the Euler factor vanishes is the case where $k(P)=1$ and the character of $\pi(P^2)$ is trivial. However this case is excluded because of the vanishing of the p–adic L–function in the denominator at (P^2,P).

Since $\mathcal{L}_p(P^2,P;\lambda\otimes\psi^{-1}\chi_n)=0\iff L(k(P),\lambda_{P^2}\otimes\psi_P^{-1}\chi_n)=0$ if either $k(P)>1$ or $\psi_P^2\neq 1$, Theorem 3 follows from Theorem 4.

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Corrections to "On Λ-adic forms of half integral weight for $SL(2)_{/\mathbb{Q}}$"

by Haruzo Hida

in "Number Theory, Paris 1992-93" Lecture notes of LMS 215, page 139-166

page 145 line 15: "$U(p^\alpha) = \left\{ s \in \mathbb{S} \mid s_p \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \bmod p^\alpha \right\}$" should read

"$U(p^\alpha) = \left\{ s \in U \mid s_p \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \bmod p^\alpha \right\}$"

page 145 line 1 from the bottom: "for $\omega_{1/2}{}^{\otimes k} \otimes \omega^\circ |_{U_\alpha}$" should read "for $\omega_{1/2}{}^{\otimes k} \otimes \omega^\circ |_{U_\alpha}$, where ω° is the dualizing sheaf on $X_{\alpha/\mathbb{Z}_p}$"

The first diagram in page 146 should be replaced by the following:

$$\begin{array}{ccccc} 0 \to H^0(U_\alpha, \omega(k+\tfrac{1}{2}) \otimes \mathbb{Z}/p^\alpha\mathbb{Z}) & \xrightarrow{\eta} & H^0(U_\alpha, \omega_1(k+1) \otimes \mathbb{Z}/p^\alpha\mathbb{Z}) & \xrightarrow{\rho} & H^0(U_\alpha, \mathcal{O}(D) \otimes \mathbb{Z}/p^\alpha\mathbb{Z}) \\ \uparrow & & \uparrow & & \uparrow \\ H^0(U_\alpha, \omega(k+\tfrac{1}{2})) \otimes \mathbb{Z}/p^\alpha\mathbb{Z} & \underset{\eta}{\to} & H^0(U_\alpha, \omega_1(k+1)) \otimes \mathbb{Z}/p^\alpha\mathbb{Z} & \to & H^0(U_\alpha, \mathcal{O}(D)) \otimes \mathbb{Z}/p^\alpha\mathbb{Z} \to 0, \end{array}$$

The second diagram in page 146 should be replaced by the following:

$$\begin{array}{ccccc} 0 \to H^0(U_\gamma, \omega(k+\tfrac{1}{2})) \otimes \mathbb{Z}/p^\beta\mathbb{Z} & \to & H^0(U_\gamma, \omega(k+1)) \otimes \mathbb{Z}/p^\beta\mathbb{Z} & \to & H^0(U_\gamma, \mathcal{O}(D)) \otimes \mathbb{Z}/p^\beta\mathbb{Z} \\ \downarrow E_\alpha & & \downarrow E_\alpha & & \| \\ 0 \to H^0(U_\gamma, \omega(k+\tfrac{3}{2})) \otimes \mathbb{Z}/p^\beta\mathbb{Z} & \to & H^0(U_\gamma, \omega(k+2)) \otimes \mathbb{Z}/p^\beta\mathbb{Z} & \to & H^0(U_\gamma, \mathcal{O}(D)) \otimes \mathbb{Z}/p^\beta\mathbb{Z}, \end{array}$$

The diagram in page 147 should be replaced by the following:

$$\begin{array}{ccccc} 0 \to H^0(U_\infty, \omega(k+\tfrac{1}{2})) \otimes \mathbb{Z}/p^\beta\mathbb{Z} & \to & H^0(U_\infty, \omega(k+1)) \otimes \mathbb{Z}/p^\beta\mathbb{Z} & \to & H^0(U_\infty, \mathcal{O}(D)) \otimes \mathbb{Z}/p^\beta\mathbb{Z} \\ \downarrow E_\alpha & & \wr\| & & \| \\ 0 \to H^0(U_\infty, \omega(k+\tfrac{3}{2})) \otimes \mathbb{Z}/p^\beta\mathbb{Z} & \to & H^0(U_\infty, \omega(k+2)) \otimes \mathbb{Z}/p^\beta\mathbb{Z} & \to & H^0(U_\infty, \mathcal{O}(D)) \otimes \mathbb{Z}/p^\beta\mathbb{Z}. \end{array}$$

The second formula in (4.1): "$a(n,f \mid T(q^2)) = a(p^2n,f)$ if $q \mid Np^\alpha$," should read

"$a(n,f \mid T(q^2)) = a(q^2n,f)$ if $q \mid Np^\alpha$,"

page 149 line 9 from the bottom: "$P^{r(P)-1}$" should read "$p^{r(P)-1}$"

In the formula of Theorem 3 in page 153: "$\psi_P(n+m)$" should read "$\psi_P(n/m)$"

At several places in page 155-157, "$\mathbb{Q}_e ll$" should read "$\mathbb{Q}_\ell$"

In the proof of Lemma 3, (k/2) should read k+(1/2) (thus (k/2)−1 is replaced by (2k−1)/2)

page 157 line 5 from the bottom: "$\mu^2 \neq \alpha$" should read "$\mu^2 = \alpha$"