Katz $p$-adic $L$-functions, congruence modules and deformation of Galois representations

H. HIDA* AND J. TILOUINE

0. Although the two-variable main conjecture for imaginary quadratic fields has been successfully proven by Rubin [R] using brilliant ideas found by Thaine and Kolyvagin, we still have some interest in studying the new proof of a special case of the conjecture, i.e., the anticyclotomic case given by Mazur and the second named author of the present article ([M-T], [T1]). Its interest lies firstly in surprising amenability of the method to the case of CM fields in place of imaginary quadratic fields and secondly in its possible relevance for non-abelian cases. In this short note, we begin with a short summary of the result in [M-T] and [T1] concerning the Iwasawa theory for imaginary quadratic fields, and after that, we shall give a very brief sketch of how one can generalize every step of the proof to the general CM-case. At the end, coming back to the original imaginary quadratic case, we remove some restriction of one of the main result in [M-T]. The idea for this slight amelioration to [M-T] is to consider deformations of Galois representations not only over finite fields but over any finite extension of $\mathbb{Q}_p$. Throughout the paper, we assume that $p > 2$.

1. Let $M$ be an imaginary quadratic field and $p$ be an odd prime which splits in $M$; i.e., $p = \overline{p}(p \neq \overline{p})$. We always fix the algebraic closures $\overline{\mathbb{Q}}$ and $\overline{\mathbb{Q}}_p$ and embeddings of $\overline{\mathbb{Q}}$ into $\mathbb{C}$ and $\overline{\mathbb{Q}}_p$. Any algebraic number field will be considered to be inside $\overline{\mathbb{Q}}$. Suppose the factor $p$ of $p$ is compatible with this embedding $M$ into $\overline{\mathbb{Q}}_p$. The scheme of the new proof of the main conjecture for the anti-cyclotomic $\mathbb{Z}_p$-tower of $M$ consists in proving two divisibility theorems between the following three power series:

\begin{equation}
L^{-|H|Iw^{-}},
\end{equation}

where

* The first named author is supported in part by an NSF grant
\( L^- \) is the Katz-Yager \( p \)-adic \( L \)-function (which interpolates \( p \)-adically Hurwitz-Damerell numbers) projected to one branch of the anticyclicotomic line of the imaginary quadratic field \( M \);

(ii) \( H \) is the characteristic power series of the congruence module attached to \( M \) (and the branch in (i)) constructed via the theory of Hecke algebras for \( GL(2) \)/\( \mathbb{Q} \);

(iii) \( Iw^- \) is the characteristic power series (of the branch in (i)) of the maximal \( p \)-ramified extension of the anti-cyclotomic \( \mathbb{Z}_p^\times \)-tower over \( M \).

Once these divisibilities are assumed, the proof is fairly easy: Under a suitable branch condition, we know from the analytic class number formula that the \( \lambda \) and \( \mu \)-invariants of \( G \) and \( Iw^- \) are the same and hence

\[
Iw^- = L^- \quad \text{up to a unit power series}
\]
as the anticyclicotomic main conjecture predicts.

Strictly speaking, the equality (1.2) is proven in [M-T] and [T] under the assumption that the class number of \( M \) is equal to 1. In fact, if the class number \( h \) of \( M \) is divisible by \( p \), we need to modify (1.1) as

\[
h \cdot L^- | H | h \cdot Iw^- \quad \text{for the class number} \ h \ \text{of} \ M.
\]

In [M-T], the second divisibility assertion: \( H \mid Iw^- \) is proven under the milder assumption that \( h \) is prime to \( p \) but there is another assumption that the branch character \( \psi \) of \( L^- \) must be non-trivial on the inertia group \( I_p \) at \( p \). We will prove the divisibility (1.3) outside the trivial zero of \( L^- \) (if any) without hypothesis in Appendix.

2. In this section, we deal with the generalization of the first divisibility result: \( L^- \mid H \) in the general CM case. The second divisibility: \( H \mid Iw^- \) will be dealt with in the following paragraphs. To state the result precisely, we fix a prime \( p \) and write the fixed embeddings as \( \iota_p : \overline{\mathbb{Q}} \rightarrow \overline{\mathbb{Q}}_p \) and \( \iota_{\infty} : \overline{\mathbb{Q}} \rightarrow \mathbb{C} \). We consider \( \overline{\mathbb{Q}} \) as a subfield of \( \overline{\mathbb{Q}}_p \) and \( \mathbb{C} \) by these embeddings. Let \( F \) be a totally real number field with class number \( h(F) \) and \( M/F \) be a totally imaginary quadratic extension whose class number is denoted by \( h(M) \). Let \( c \) be the complex conjugation which induces the unique non-trivial automorphism of \( M \) over \( F \). We assume the following ordinarity condition:
Ordinariness hypothesis  All prime factor \( p \) of \( p \) in \( F \) splits in \( M \).

Thus we can write the set of prime factors of \( p \) in \( M \) as a disjoint union \( S \cup S^e \) of two subsets of prime ideals so that \( \mathfrak{p} \in S \) if and only if \( \mathfrak{p}^e \in S^e \). If \( a \) is the number of prime ideals in \( F \) over \( p \), there are \( 2^a \) choices of such subset \( S \). Such an \( S \) will be called a \( p \)-adic CM-type. Considering \( S \) as a set of \( p \)-adic places of \( M \), let \( \Sigma \) be the set of embeddings of \( M \) into \( \overline{Q} \) which give rise to places in \( S \) after combining with \( \iota_p \). Then \( \Sigma \cup \Sigma \circ c \) is the total set of embeddings of \( M \) into \( \overline{Q} \) and hence gives a complex CM-type of \( M \). Hereafter we fix a \( p \)-adic CM-type \( S \) and compatible complex CM-type \( \Sigma \). Let \( G \) be the Galois group of the maximal \( p \)-ramified abelian extension \( M^\infty \) of \( M \). Then we fix a decomposition \( G = G_{tor} \times W \) for a finite group \( G_{tor} \) and a \( \mathbb{Z}_p \)-free module \( W \). Let \( K/\mathbb{Q}_p \) be a \( p \)-adically complete extension in the \( p \)-adic completion \( \Omega \) of \( \mathbb{Q}_p \) containing all the images \( \sigma(M) \) for \( \sigma \in \Sigma \) and \( \mathcal{O} = \mathcal{O}_K \) be the \( p \)-adic integer ring of \( K \). We now consider the continuous group algebras \( \Lambda = \mathcal{O}[[W]] \) and \( \mathcal{O}[[G]] = \Lambda[G_{tor}] \). By choosing a basis of \( W \), we have \( W \cong \mathbb{Z}_p^r \) and \( \Lambda \cong \mathcal{O}[X_1, \ldots, X_r] \). Here \( r = [F : \mathbb{Q}] + 1 + \delta \), where \( \delta \) is the defect of the Leopoldt conjecture for \( F \); i.e., \( \delta \geq 0 \) and \( \delta = 0 \) if and only if the Leopoldt conjecture holds for \( F \) and \( p \). Fix a character \( \lambda : G_{tor} \to \mathcal{O}^\times \) and define the projection \( \lambda_\sigma : \mathcal{O}[[G]] \to \mathcal{O}[[W]] = \Lambda \) by \( \lambda(g, w) = \lambda(g)[w] \) for the group element \( [w] \) in \( \Lambda \) for \( w \in W \) and \( g \in G_{tor} \). We consider two anti-cyclotomic characters of \( G \) given by \( \lambda_\sigma = \lambda(\sigma)^{-1} \) and \( \alpha = \lambda_\sigma \lambda_\xi^{-1} \), where \( \lambda_\xi = \lambda(c \sigma^{-1}) \) and \( \lambda_\xi^* = \lambda_\sigma(c \sigma^{-1}) \). Let \( M^-_\lambda \) be the subfield of \( M^\infty \) fixed by \( \ker(\alpha) \). Let \( M_S^-/M^-_\lambda \) be the maximal \( p \)-abelian extension unramified outside \( S \). Naturally \( X_S = \text{Gal}(M_S^-/M^-_\lambda) \) is a continuous module over \( \mathbb{Z}_p[[H]] \) of \( H = \text{Im}(\alpha) \). We consider the \( \lambda^- \)-branch of \( X_S \) defined by

\[
X_S(\lambda^-) = X_S \otimes_{\mathbb{Z}_p[G_{tor}]} \mathcal{O}(\lambda^-)
\]

where \( \mathcal{O}(\lambda^-) \) is the \( \mathcal{O} \)-free module of rank one on which \( G_{tor} \) acts via \( \lambda^- \).

Once we are given a \( p \)-adic CM-type \( S \), we have the following 3 objects as in the imaginary quadratic case:

(i) The \( \lambda^- \)-branch of the projection \( L^- \) of the Katz \( p \)-adic \( L \)-function \( L \in \mathcal{O}[[G]] \) to the anti-cyclotomic tower \( M^-_\lambda \);

(ii) The congruence power series \( H \in \Lambda \) attached to the \( \lambda^- \)-branch of the nearly ordinary Hecke algebra of CM-type \( S \);

(iii) The characteristic power series \( Iw^- \) of \( X_S(\lambda^-) \) in \( \Lambda \).

Note that \( \ker(\alpha) \) contains \( G_+ = \{ x \in G \mid cx^{-1} = x \} \) and we can realize the quotient \( G/G_+ \) inside \( G \) by the subgroup of commutators \( [x, c] = xcx^{-1}c^{-1} \).
Especially the maximal torsion-free quotient $W^-$ of $H$ can be thought of a direct factor of $W$ via this map. For a technical reason (namely, $H$ resides in $\Lambda$), we regard $L^-$ and $Iw^-$ as elements in $\Lambda$ via this inclusion although they belong to $\Lambda^* = \mathcal{O}[[W^-]]$. Moreover, to have a non-zero $Iw^-$, we need to suppose a weak version of the Leopoldt conjecture (depending on $S$) for the anti-cyclotomic tower. This weak form of Leopoldt’s conjecture holds true if the CM field $M$ is abelian over $\mathbb{Q}$. On the other hand, one can prove unconditionally (i.e. without supposing the weak Leopoldt conjecture) the non-vanishing of the characteristic power series $Iw$ of the maximal $S$-ramified abelian extension over the full $\mathbb{Z}_p$-tower of $M$. Before giving the precise definition of $L^-$ and $H$, we state the first theorem:

**Theorem 2.1** $L^-$ divides $H$ in $\mathcal{O}_n[[W]] \otimes_\mathbb{Z} \mathbb{Q}$. Moreover if the $\mu$-invariant of every branch of the Katz $p$-adic $L$-function of $M$ vanishes, then we have the strong divisibility:

$$(h(M)/h(F))L^- | H \text{ in } \mathcal{O}_n[[W]].$$

The following conjecture is obviously motivated by (1.1):

**Conjecture 2.2** $H = (h(M)/h(F))L^-$ up to a unit in $\mathcal{O}_n[[W]]$ if $p > 2$, where $h(M)$ (resp. $h(F)$) is the class number of $M$ (resp. $F$).

This conjecture is known to be true if $F = \mathbb{Q}$, $p \geq 5$ and the class number $h(M)$ of $M$ is prime to $p$ under a certain branch condition.

First, let us explain the definition of $L^-$. Although we will not make the identification with the power series ring due to the lack of canonical coordinates of $W$, we may regard any element of $\Lambda$ as a $p$-adic analytic function of several variables. There are two different ways of viewing $\Phi \in \Lambda$ as an analytic object: For $G = G$ or $W$, let $\mathcal{X}(G)$ be the set of all continuous characters of $G$ with values in $\mathcal{O}_p$. If one fixes a $\mathbb{Z}_p$-basis $(w_i)$ of $W$, then each character $P \in \mathcal{X}(W)$ is determined by its value $P(w_i) \in D^*$, where $D = \{x \in \mathcal{O}_p \mid |x - 1|_p < 1\}$. Thus $\mathcal{X}(W) \cong D^*$. Each character $P : G \to \mathcal{O}_p^\times$ induces an $\mathcal{O}$-algebra homomorphism $P : \mathcal{O}[[G]] \to \mathcal{O}_p$ such that $P|_G$ is the original character of $G$. In this way, we get an isomorphism:

$$\mathcal{X}(G) \cong \text{Spec}(\mathcal{O}[[G]])(\mathcal{O}_p) = \text{Hom}_{\mathcal{O}_{\text{alg}}}(\mathcal{O}[[G]], \mathcal{O}_p).$$

Then

(A1) $\Phi$ is an analytic function on $\mathcal{X}(G)$ whose value at $P$ is $P(\Phi) \in \mathcal{O}_p$. 


On the other hand, we can view $\Lambda$ as a space of measures on $G$ in the sense of Mazur so that

\[(A2) \quad \int_G P(g) \, d\Phi(g) = P(\Phi) = \Phi(P).\]

By class field theory, we can identify, via the Artin symbol, the group $G$ with the quotient of the idele group $M_A^\times$. For a given $A_0$-type Hecke character $\varphi : M_A^\times / M^\times \to \mathbb{C}^\times$ of $p$-power conductor whose infinity type is given by

$$\varphi(x_\infty) = x_\infty^{-\xi} = \Pi_{\sigma \in \Sigma} (x_\infty^\sigma)^{-\xi_\sigma} \text{ for } \xi \in \Sigma \subset \mathbb{Z}^\Sigma,$$

as shown by A. Weil in 1955, $\varphi$ has values in $\overline{\mathbb{Q}}$ on finite ideles and we have a unique $p$-adic avatar $\varphi : G \to \overline{\mathbb{Q}}_p^\times$ which satisfies $\varphi(x) = \varphi(x)$ if $x_p = x_\infty = 1$, and if $x_p \in M_p^\times$ is close enough to 1, then

$$\varphi(x_p) = x_p^{-\xi} = \Pi_{\sigma \in \Sigma} (x_p^\sigma)^{-\xi_\sigma}.$$

In 1978, Katz showed in [K] the existence of a unique $p$-adic $L$-function given by an element $L$ of $\mathcal{O}_\Lambda[[G]]$ such that

$$L(\varphi)_{\text{suitable } p\text{-adic period}} = c(\varphi) L(0, \varphi)_{\text{suitable complex period}}$$

whenever $\varphi$ is critical at 0 (i.e. if either $\xi_{\sigma_p} \geq \xi_\sigma + \xi_{\sigma_p} + 1 \geq 0$ or $\xi_\sigma + 1 \leq \xi_\sigma + \xi_{\sigma_p} + 1 \leq 0$ for all $\sigma \in \Sigma$). Here, $c(\varphi)$ is a simple constant including a modifying Euler $p$-factor, local Gauss sum, $\Gamma$-factor and a power of $\pi$. See [K, (5.3.0), (5.7.8-9)] for details. To define $L^-$, we first project Katz's $L$ to $\Lambda$. Namely, we fix once and for all a finite order character $\chi : G_{\text{tor}} \to \mathcal{O}^\times$. Then we have a continuous character $\lambda_* : G = G_{\text{tor}} \times W \to \mathcal{O}[[W]]$ given by

$$\lambda_*(g, w) = \lambda(g, w) \in \mathcal{O}[[W]],$$

where we consider $\lambda(g)$ for $g \in G_{\text{tor}}$ as a scalar in $\mathcal{O}$ but $w$ as a group element in $W$. This character induces the projection to $\Lambda$

$$\lambda_* : \mathcal{O}[[G]] = \Lambda[G_{\text{tor}}] \to \Lambda.$$

Then for any point $P \in \mathfrak{X}(W)$, $\lambda_p = P \circ \lambda_* : G \to \overline{\mathbb{Q}}_p$ is a $p$-adic character of $G$. When $\lambda_p$ is the avatar of an $A_0$-type Hecke character, we say that $P$ is arithmetic (this notion of arithmeticity is independent of the choice of $\lambda$). Let $c$ denote the complex conjugation in $\text{Gal}(\overline{\mathbb{Q}}/F)$ and write $\lambda^c(x) = \lambda(cx^{-1})$. We then consider the anti-cyclotomic character $\alpha$ attached to $\lambda_*$ given by

$$\alpha(x) = \lambda_*^{-1} \lambda^c(x) = \lambda_* (cxe^{-1}x^{-1})$$
and the corresponding $\Lambda$-algebra homomorphism

$$\alpha_* : \mathcal{O}[[G]] \to \mathcal{O}[[W]].$$

This $\alpha_*$ actually has values in the anti-cyclotomic part $\mathcal{O}[[W^-]]$, where

$$W^- = \{ w \in W \mid w^c = cwc^{-1} = w^{-1} \}.$$

Then we define

$$L^- = \alpha_*(L) \in \mathcal{O}_\Lambda[[W^-]].$$

Although the divisibility of Theorem 2.1 is stated as taking place in the bigger ring $\mathcal{O}[[W]] \supset \mathcal{O}[[W^-]]$, actually the congruence power series $H$ itself also falls in the subring $\mathcal{O}[[W^-]]$. However we will know this fact after proving the second divisibility: $H \mid (h(M)/h(F))Lw^-$ and we do not know this fact a priori. Thus we continue to formulate our result using $\mathcal{O}_\Lambda[[W]]$ as the base ring. This power series $L^-$ satisfies the following interpolation property:

$$\frac{L^-(P)}{p\text{-adic period}} = c(\lambda_p^c\lambda_p^{-1}) \frac{L(0, \lambda_p^c\lambda_p^{-1})}{\text{complex period}}$$

whenever $P$ is arithmetic and $\lambda_p^c\lambda_p^{-1}$ is critical at $P$.

We now define the $p$-adic Hecke algebra and the congruence power series and then give a sketch of the proof of the theorem. To define Hecke algebra, we explain first a few things about Hilbert modular forms. Let $I$ be the set of all field embeddings of $F$ into $\overline{Q}$. The weight $k = (k_\sigma)_{\sigma \in I}$ of a modular form will be an element of the free $\mathbb{Z}$-module $\mathbb{Z}[I]$ generated by elements of $I$. Actually, our holomorphic modular forms have double digit weight $(k, v) \in \mathbb{Z}[I]^2$ associated to the following automorphic factor:

$$J_{k,v}(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z) = \Pi_{\sigma \in I} \{ \det(\gamma_\sigma)^{v_{2\sigma}-1} (c^\sigma z_\sigma + d^\sigma)^{k_\sigma} \},$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(F_\infty)$ ($F_\infty = F \otimes_Q \mathbb{R} = \mathbb{R}^I$) with totally positive determinant and $z = (z_\sigma)_{\sigma \in I} \in \mathfrak{h}^I$ is a variable on the product of copies of upper half complex planes $\mathfrak{h}^I$ indexed by $I$. For each open compact subgroup $V$ of the finite part of the adele group $GL_2(F_\Lambda)$, let $S_{k,v}(V)$ be the space of holomorphic cusp forms $f$ of weight $(k, v)$ defined in [H1, §2]. Namely $f$ is a function on $GL_2(F_\Lambda)$ satisfying

$$f(\alpha xu) = f(x)J_{k,v}(u,0)^{-1} \text{ for } \alpha \in GL_2(F) \text{ and } u \in V \times C,$$

where $C$ is the stabilizer of $z_0 = (\sqrt{-1}, \ldots, \sqrt{-1}) \in \mathfrak{h}^I$ in $GL_2(F_\infty)$, which is isomorphic to the product of the center ($\cong (\mathbb{R}^\times)^I$) of $GL_2(F_\infty)$ and $SO_2(\mathbb{R})^I$. 

We can associate to \( f \) and each finite idele \( t \in GL_2(F_{A_f}) \), a function \( f_t \) on \( \mathcal{H} \) by

\[
f_t(z) = f(t \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}) J_{x,\nu}(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}, z_0).
\]

It is easy to check that \( f_t \) satisfies the automorphic condition:

\[
f_t(\gamma(z)) = f_t(z) J_{x,\nu}(\gamma, z)
\]

for \( \gamma \in \Gamma_t = t^{-1}VtGL_2^+(F_\infty) \cap GL_2(F) \),

where \( GL_2^+(F_\infty) \) is the connected component of \( GL_2(F_\infty) \) with identity. Similarly we write \( F_\infty^+ \) for the connected component with identity of \( F_\infty^x \). Then we suppose for \( f \in S_{k,\nu}(V) \) that, for all \( t \in GL_2(F_{A_f}) \),

(i) \( f_t \) is holomorphic on \( \mathcal{H} \) (holomorphy),

(ii) \( f_t(z) \) has the following Fourier expansion:

\[
\sum_{\xi \in F} c(\xi, f_t) \exp(2\pi i \text{Tr}(\xi z))
\]

with \( c(\xi, f_t) = 0 \) unless \( \xi^\sigma > 0 \) for all \( \sigma \in I \) (cuspidality).

Let \( D \) be the relative discriminant of \( M/F \) and let \( \mathfrak{o} \) and \( \mathfrak{A} \) be the integer ring of \( F \) and \( M \), respectively. As the open compact subgroup \( V \), we take the group \( V_\alpha \) given by

\[
\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\hat{\mathfrak{o}}) \mid c \in D p^\sigma \hat{\mathfrak{o}}, \quad a \equiv 1 \mod p^\sigma \hat{\mathfrak{o}}, \quad d \equiv 1 \mod Dp^\sigma \hat{\mathfrak{o}} \right\},
\]

where \( \hat{\mathfrak{o}} \) is the integer ring of \( F \) and \( \hat{\mathfrak{o}} = \lim \mathfrak{o}/Nt \mathfrak{o} \) is the product of \( t \)-adic completion of \( \mathfrak{o} \) over all primes \( t \). Let \( \chi : F_\infty^x \rightarrow \mathbb{Z}_p^\times \) be the cyclotomic character. If \( c(\xi, f_t) \in \overline{\mathbb{Q}} \) for all \( t \in GL_2(F_{A_f}) \), we can associate to each \( f \) as above the following \( p \)-adic \( q \)-expansion (cf. [H4, §1]):

\[
f(y) = \Sigma_{0 < \xi, \xi \in F} a(\xi y d, f) q^\xi \quad \text{with} \quad a(\xi y d, f) \in \overline{\mathbb{Q}},
\]

where \( d \) is any differential idele of \( F \) (i.e., its ideal is the different of \( F/Q \)) and \( y \mapsto a(y, f) \) is a function on finite ideles, vanishing outside integral ideles, given by

\[
a(y, f) = c(\xi, f_t) y_p^{-\nu} \xi^\sigma \chi(\det(t)) \quad \text{for} \quad t = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}
\]

with \( y \in \xi \text{ad}(V_\alpha \cap F_\infty^x) F_\infty^x \).

Out of this \( q \)-expansion, we can recover the Fourier expansion of \( f \):

\[
f\left( \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \right) = |y|_A \left\{ \sum_{0 < \xi, \xi \in F} a(\xi y d, f) (\xi y \infty)_p^{-\nu} e_F(i \xi y \infty) e_F(\xi x) \right\}.
\]
Here note that $a(\xi y, f)(\xi dy)^{\tau}$ is an algebraic number which is considered to be a complex number via the fixed embedding of $\mathbb{Q}$ into $\mathbb{C}$. When $a(y, f)$ is algebraic for all $y$ with $y_p = 1$, $f$ is called algebraic (this is equivalent to asking that $c(\xi, f_t)$ are algebraic for all $t$). We consider the union $S(\mathbb{Q})$ of all algebraic forms of all weight $(k, \nu)$ inside the space of formal $q$-expansions. Then putting a $p$-adic uniform norm

$$|f|_p = \text{Sup}_y |a(y, f)|_p$$
onumber

on $S(\mathbb{Q})$, we define the space $S$ of $p$-adic modular forms by the completion of $S(\mathbb{Q})$ under the norm $| |_p$.

Now we define the Hecke operators. For each $x \in F_A^X$ with $x_\infty = 1$, we can define the Hecke operator $T(x) = T_\alpha(x)$ acting on $S_{k,\nu}(V_\alpha)$ as follows: First take the double coset $V_\alpha \left( \begin{array}{cc} x & 0 \\ 0 & 1 \end{array} \right) V_\alpha$ and decompose it into a disjoint union of finite right cosets $\cup_i x_i V_\alpha$. Then we define $T_\alpha(x)$ by

$$f \mid T_\alpha(x)(g) = \Sigma_i f(g x_i).$$

Since we have taken the average of right translation of $f$ on a double coset, we can check easily that $T_\alpha(x)$ is a linear operator acting on $S_{k,\nu}(V_\alpha)$. Especially the action of $T(u)$ for $u \in \mathbb{F}_p^X$ factors through $(\tau/p^\nu r)^X$. Similarly, the center $F_A^X$ acts on $S_{k,\nu}(V_\alpha)$ so that $f \mid z(g) = f(gz)$. This action factors through $Z = F_A^X/F^X U(D)^{(p)} F^X$ for

$$U(D)^{(p)} = \{ u \in \mathbb{F}_p^X \mid u \equiv 1 \mod D \tau \text{ and } u_p = 1 \}.$$ 

Thus $S_{k,\nu}(V_\alpha)$ has an action of the group $G = Z \times \mathbb{F}_p^X$ and Hecke operators $T(x)$. The group $G$ is a profinite group and we can decompose

$$G = G_{\text{tor}} \times W$$

so that $W \cong \mathbb{Z}_p^{[F_\infty: \mathbb{Q}] + 1 + s}$ and $G_{\text{tor}}$ is a finite group. Since $M_A \supset F_A$, we have a natural homomorphism of $Z$ into $G$. On the other hand, by our choice of $p$-adic CM-type, we can identify $\mathbb{F}_p = \tau \otimes_{\mathbb{Z}} \mathbb{Z}_p$ with $\mathfrak{m}_S = \Pi_{\mathfrak{p} \in S} \mathfrak{m}_\mathfrak{p}$. This identification gives an injection of $\mathbb{F}_p^X$ into $M_A^X$ and yields a homomorphism of $\mathbb{F}_p^X$ into $G$. Thus we have natural morphisms:

$$\iota : G = Z \times \mathbb{F}_p^X \to G \quad \text{and} \quad \iota_* : \mathcal{O}[[G]] \to \mathcal{O}[[G]].$$

We can easily check that $\iota$ takes $W$ into a subgroup of finite index of $W$ and $\iota_*$ is an $\mathcal{O}[[W]]$-algebra homomorphism.
We take the Galois closure $\Phi$ of $F$ in $\overline{Q}$ and let $\mathfrak{B}$ be the valuation ring of $\Phi$ corresponding to the embedding $\Phi$ into $\overline{Q}_p$. We pick an element $\omega_p$ for each prime factor $p$ of $F$ such that $\omega_p = \omega p a$ for an ideal $a$ prime to $p$. We consider $\omega_p$ as a prime element in $F_p$. Then the $p$-adic Hecke algebra $h_{k,v}(Dp^\alpha; \mathfrak{B})$ with coefficients in $\mathfrak{B}$ is by definition the $\mathfrak{B}$-subalgebra of $\text{End}_{\mathfrak{B}}(S_{k,v}(V_\alpha))$ generated by

(a) Hecke operators $T(x)$ for all $x \in \mathfrak{B} \cap F_{\Phi}$,
(b) the Hecke operator $\omega_p^{-v}T(\omega_p)(\omega_p \in \mathfrak{r}_p)$ for all $p \mid p$,
(c) the action of the group $G = Z \times \mathbb{T}$.

It is well known that $h_{k,v}(Dp^\alpha; \mathfrak{B})$ is free of finite rank over $\mathfrak{B}$ (cf [H1, Th.3.1]). Especially $T(\omega_p)$ is divisible by $\omega_p^\nu = \Pi_{\sigma \in I} \omega_p^{\alpha\sigma}$. For each extension $K$ of $\mathbb{Q}_p$ containing $\Phi$, let $\mathfrak{O}$ be the $p$-adic integer ring of $K$. Then the $p$-adic Hecke algebra of level $Dp^\alpha$ is defined by

$$h_{k,v}(Dp^\alpha; \mathfrak{O}) = h_{k,v}(Dp^\alpha; \mathfrak{B}) \otimes_{\mathfrak{B}} \mathfrak{O}.$$ 

By definition, the restriction of $T_\beta(x)$ acting on $S_{k,v}(V_\beta)$ to $S_{k,v}(V_\alpha)$ for $\beta > \alpha > 0$ coincides $T_\alpha(x)$. Thus the restriction induces a surjective $\mathfrak{O}$-algebra homomorphism:

$$h_{k,v}(Dp^\beta; \mathfrak{O}) \to h_{k,v}(Dp^\alpha; \mathfrak{O})$$

which takes $T_\beta(x)$ to $T_\alpha(x)$. Thus we can take the projective limit

$$h_{k,v}(Dp^\infty; \mathfrak{O}) = \lim_{\alpha} h_{k,v}(Dp^\alpha; \mathfrak{O}),$$

which is naturally an algebra over the continuous group algebra $\mathfrak{O}[[G]]$. For each $\alpha$, we can decompose

$$h_{k,v}(Dp^\alpha; \mathfrak{O}) = h_{k,v}^{n,\text{ord}}(Dp^\alpha; \mathfrak{O}) \times h_{k,v}^{t}(Dp^\alpha; \mathfrak{O})$$

so that $p^{-v}T(p)$ is a unit in $h_{k,v}^{n,\text{ord}}(Dp^\alpha; \mathfrak{O})$ and is topologically nilpotent in $h_{k,v}^{t}(Dp^\alpha; \mathfrak{O})$. Then basic known facts are (see [H2]):

(H1) The pair $(h_{k,v}(Dp^\infty; \mathfrak{O}), x_p^{-v}T(x))$ is independent of $(k, v)$ if $k \geq 2t$, where $t = \Sigma_{\sigma \in I} \sigma$.

In fact, the union $S_{k,v}(V_\infty) = \bigcup_\alpha S_{k,v}(V_\alpha; \overline{Q})$ of all algebraic modular forms of weight $(k, v)$ is dense in $S$ and thus the algebra $h_{k,v}(Dp^\infty; \mathfrak{O})$ can be considered as a subalgebra of $\text{End}(S)$ topologically generated by $x_p^{-v}T(x)$ and it is independent of $(k, v)$. Now we can remove the suffix $(k, v)$ from notation of the Hecke algebra and we write $h(D; \mathfrak{O})$ (resp. $h = h^{n,\text{ord}}(D; \mathfrak{O})$).
for \( h_{k,v}(Dp^\infty;\mathcal{O}) \) (resp. \( h_{k,v}^{\text{ord}}(Dp^\infty;\mathcal{O}) \)). In other words, there is a universal Hecke operator \( T(x) \in h(D;\mathcal{O}) \) which is sent to \( x_{\mathcal{O}}^{-1}T(x) \) under the isomorphism: \( h(D;\mathcal{O}) \cong h_{k,v}(Dp^\infty;\mathcal{O}) \).

(H2) \( h \) is of finite type and torsion-free as \( \mathcal{O}[[W]] \)-module.

(H3) There exists an \( \mathcal{O}[[W]] \)-algebra homomorphism \( \theta^* : h \to \mathcal{O}[G] \) such that for primes outside \( D_p \)

\[
\theta^*(T(q)) = \begin{cases} 
[\Omega] + [\Omega^c] & \text{if } q = \Omega \Omega^c, \\
0 & \text{if } q \text{ remains prime in } M
\end{cases}
\]

where \([\Omega]\) is the image of the prime ideal \( \Omega \) under the Artin symbol.

This statement is just an interpretation of the existence of theta series \( \theta(\varphi) \) for each \( A_0 \)-type Hecke character \( \varphi \) of \( G \) characterized by

\[
\theta(\varphi) | T(q) = \begin{cases} 
(\varphi(\Omega) + \varphi(\Omega^c))\theta(\varphi) & \text{if } q = \Omega \Omega^c, \\
0 & \text{if } q \text{ remains prime in } M
\end{cases}
\]

By (H3), we may consider the composite \( \lambda \circ \theta^* : h \to \Lambda \).

(H4) After tensoring the quotient field \( L \) of \( \Lambda \) over \( \Lambda_0 = \mathcal{O}[[W]] \), we have a \( \Lambda_0 \)-algebra decomposition

\[
h \otimes_{\Lambda_0} L \cong L \oplus B \quad \text{for a complementary summand } B,
\]

where the projection to the first factor is given by \( \lambda \circ \theta^* \).

Then the congruence module of \( \lambda \) is defined by

(H5) \( \mathcal{C}(\lambda;\Lambda) = \Lambda/(h \otimes_{\Lambda_0} \Lambda \cap L) \).

The congruence power series \( H \) is then defined by the characteristic power series of \( \mathcal{C}(\lambda;\Lambda) \). By definition, the principal ideal \( H\Lambda \) is the reflexive closure of the ideal \( h \otimes_{\Lambda_0} \Lambda \cap L \) in \( \Lambda \).

3. We now give a sketch of the proof of Theorem 2.1. The idea of the proof is the comparison of two \( p \)-adic interpolations of Hecke \( L \)-functions of \( M \). One is Katz's way and the other is the \( p \)-adic \( L \)-function attached to the Rankin product \( L \)-function of \( \theta(\lambda_P) \) and \( \theta(\mu_Q) \). Here \( \mu \) is another character of \( G_{\text{tor}} \) and we extend it to a character \( \mu_* : G \to \Lambda^* \) similarly to \( \lambda_* \). In fact, we can show by the method of \( p \)-adic Rankin convolution ([H4, Theorem I]) that there exists a power series \( \Delta \) in \( \mathcal{O}[[W \times W]] \) such that

\[
\frac{\Delta(P,Q)}{H(P)} = c(P,Q) \frac{D(1 + \frac{m(q)-m(p)}{2}, \theta(\lambda_P), \theta(\mu_Q \circ c))}{(\theta(\lambda_P), \theta(\lambda_P))}
\]
whenever both $P$ and $Q$ are arithmetic and both $\lambda_P^{-1} \mu_Q$ and $\lambda_P^{-1} (\mu_Q)^c$ are critical. Here $D(s, \theta(\lambda_P), \theta(\mu_Q)^c)$ is the Rankin product of $\theta(\lambda_P)$ and $\theta(\mu_Q)^c$, i.e., the standard $L$-function for $GL(2) \times GL(2)$ attached to the tensor product of automorphic representations spanned by $\theta(\lambda_P)$ and $\theta(\mu_Q)^c$; $(\theta(\lambda_P), \theta(\lambda_P))$ is the self Petersson inner product of $\theta(\lambda_P)$ and $c(P, Q)$ is a simple constant including the modifying Euler $p$-factor, Gauss sums, $\Gamma$-factors and a power of $\pi$. The integer $m(P)$ is given as follows: Write the infinity type of $\lambda_P$ as $\xi$ and $m(P) = \xi_\sigma + \xi_{ac}$ for $\sigma \in \Sigma$ (this value is independent of $\sigma$). Similarly $m(Q)$ is defined for $\mu_Q$. Now looking at the Euler product of $D$ and the functional equation of Hecke $L$-functions, we see

$$D(1 + \frac{m(Q) - m(P)}{2}, \theta(\lambda_P), \theta(\mu_Q)^c) \approx L(0, \lambda_P^{-1} \mu_Q) L(0, \lambda_P^{-1} (\mu_Q)^c).$$

It is also well known that, with a simple constant $c(P)$ similar to $c(P, Q)$,

$$(\theta(\lambda_P), \theta(\lambda_P)) = c(P) (2^{1-[F:Q]} h(M)/h(F)) L(0, \lambda_P^c) L(0, \lambda_P^{-1}).$$

Modifying the Katz measure $L$ in $\mathcal{O}[G]$, we can find two power series $L'$ and $L''$ in $\mathcal{O}[W \times W]$ interpolating $L(0, \lambda_P^{-1} \mu_Q)$ and $L(0, \lambda_P^{-1} (\mu_Q)^c)$, respectively. Then out of the above formulas, we get the following identity:

$$\frac{\Delta}{H} = U \frac{LL''}{2^{1-[F:Q]} h(M)/h(F) P(L') L''} \text{ in } \mathcal{O}[W \times W],$$

where $U$ is a unit in $\mathcal{O}[W \times W]$. Thus if $L'$ and $L''$ are prime to $L^-$ in $\mathcal{O}[W \times W] \otimes_{\mathbb{Z}} \mathbb{Q}$, we get the desired divisibility. Almost immediately from the construction of $L'$ and $L''$, we know that for any character $P \in \mathcal{X}(W)$ the half specialized power series $L'_P(X) = L'(P, X)$ and $L'_P(X)$ in $\mathcal{O}[W]$ have their $\mu$-invariants independent of $P$, equal to the $\mu$-invariant of the Katz measure along the irreducible component of $\lambda^{-1} \mu$ and $\lambda^{-1} (\mu)^c$. If (a characteristic 0) prime factor $P(Y)$ (in $\mathcal{O}_{\mathcal{Q}}[W]$) of $L^{-}(Y)$ divides $L'$, then by letting $P$ approach to a zero of $P$, we observe that the $\mu$-invariant of $L'_P$ goes to infinity, which contradicts the constancy of the $\mu$-invariant of $L'_P$. Thus $L'$ is prime to $L''$ in $\mathcal{O}[W \times W] \otimes_{\mathbb{Z}} \mathbb{Q}$, which shows the desired assertion. Especially if the $\mu$-invariant of the Katz measure vanishes, then we know the strong divisibility as in the theorem.

4. Now we explain briefly how one can show the other divisibility: $H | Iw^-$ by using Mazur’s theory of deformation of Galois representations. We keep the notations and assumptions introduced above. In particular, we assume the ordinarity hypothesis and fix a $p$-adic CM-type $S$. To the pair $(S, \lambda)$, where $\lambda$ is a given character of $\mathbb{G}_{tor}$, we attached a congruence module, with characteristic power series $H$. On the other hand, let $M_\infty$ be the maximal abelian...
extension of $M$ unramified outside $p$ of $M$; so, we have $G = \text{Gal}(M_\infty/M)$. We have defined a character $\lambda_* : G \to \Lambda^\times$ for $\Lambda = \mathcal{O}[[W]]$ for a fixed finite order character $\lambda : G_{\text{tor}} \to \mathcal{O}^\times$. In fact, on $G_{\text{tor}}$, $\lambda_*$ coincides with $\lambda$ and on $W$, it is the tautological inclusion of $W$ into $\Lambda$. Define the ‘−’ part of $\lambda_*$, which we write as $\alpha = \alpha\lambda$, by

$$\alpha = \lambda_*(\lambda_*)^{-1} \text{ for } \lambda_*(\sigma) = \lambda_*(c\sigma c^{-1}).$$

Let $M^- = M^-(\lambda_-)$ be the fixed part of $M_\infty$ by $\text{Ker}(\alpha)$, which contains $G^+ = \{\sigma \in G \mid c(\sigma) = \sigma\}$. We write $H = \text{Gal}(M^-(\lambda_-)/M) \cong \text{Im}(\alpha)$. Let $M_S(\lambda_-)$ be the maximal $p$-abelian extension of $M^-$ unramified outside $S$. One can prove, under a weak Leopoldt type assumption for the extension $M^-(\lambda_-)/M$ (for details of this assumption, see our forthcoming paper), that $X_S = \text{Gal}(M_S(\lambda_-)/M^-)$ is torsion over $\mathbb{Z}_p[[H]]$. The character $\lambda^- = \lambda/\lambda^\varepsilon : G_{\text{tor}} \to \mathcal{O}^\times$ factors through the torsion part $H_{\text{tor}}$ of $H$ and the characteristic power series of the $\lambda^-$-part $X_S(\lambda^-) = X_S \otimes_{\mathbb{Z}_p[H_{\text{tor}}]} \mathcal{O}(\lambda^-)$ of $X_S$ is nothing but $Iw^-$. Then the precise result, we can obtain at this date is as follows:

**Theorem 4.1** (i) If $[F : \mathbb{Q}] > 1$, $H$ divides $(h(M)/h(F))Iw^-$ in $\mathcal{O}[[W]]$.

(ii) If $M$ is imaginary quadratic, then $H$ divides $h(M)Iw^-$ in $\mathcal{O}[[W]]$ unless $\lambda^- = \lambda/\lambda^\varepsilon$ restricted to the decomposition group $D$ of $\mathfrak{p}$ in $G$ is congruent to 1 modulo the maximal ideal $\mathfrak{p}\mathcal{O}$ of $\mathcal{O}$. In this exceptional case, we need to exclude the trivial zero, i.e., the divisibility holds in $\mathcal{O}[[W]][1/\mathfrak{p}^1]$, where $P_\lambda$ is a generator of the unique height one prime ideal corresponding to the character $\tau_\lambda : H \to \mathcal{O}^\times$ such that $\tau_\lambda(D) = 1$ and $\tau_\lambda \mid H_{\text{tor}} = \lambda^-.$

**Comments** (a) By a base change argument in Iwasawa theory, one can probably include the ‘trivial zero’ $P_\lambda$. Nevertheless, the argument possibly needs a sort of multiplicity one result for ‘trivial zeros’ of the Katz–Yager $p$-adic $L$-function which needs to be verified.

(b) The reason why things become easier when $[F : \mathbb{Q}] > 1$ is contained in the following easy lemma. To state the lemma, let us recall the character $\lambda_* : G \to \mathcal{O}[[W]]$ given by

$$\lambda_*(g, w) = \lambda(g)w \in \mathcal{O}[[W]] \text{ for } g \in G_{\text{tor}}$$

and $w \in W$ in §2.

**Lemma 4.2** (i) If $[F : \mathbb{Q}] > 1$, the ideal $\mathcal{I}$ generated by the values $\lambda_*(\sigma) - \lambda_*(c(\sigma))$, $\sigma$ running over the decomposition group $D_\mathfrak{p}$ at $\mathfrak{p}$ in $G$ is of height greater than 1, i.e., is not contained in any prime of height one in $\Lambda = \mathcal{O}[[W]]$. (ii) If $F = \mathbb{Q}$, this ideal is contained in $P_\lambda\Lambda$. 


The outline of the proof of Theorem 4.1 runs as follows. Let us fix a prime $P$ of height one such that the restrictions of $\lambda$ and $\lambda^c$ to $D_p$ are not congruent modulo $P$ for all prime $p$ in $F$ over $p$ (by Lemma 4.2, this gives no restriction when $[F : \mathbb{Q}] \geq 2$). We consider the complete discrete valuation ring $\Lambda_P$ with residue field $k(P)$ and look at the residual representation

$$\bar{\rho}_0 : \text{Gal}(\overline{\mathbb{Q}}/F) \to GL_2(k(P))$$

given as the reduction modulo $P$ of the induced representation $\rho_0$ of $\lambda_* : \text{Gal}(\overline{\mathbb{Q}}/M) \to \Lambda^\times$. By the choice of $P$, $\lambda \not\equiv \lambda^c \mod P$, hence $\bar{\rho}_0$ is irreducible.

The main step in the proof of Theorem 4.1 is to relate $X_S(\lambda) \otimes_\Lambda \Lambda_P$ to a module of Kähler differentials attached to some deformation problem of $\bar{\rho}_0$ over $\Lambda_P$. Since $k(P)$ is not a finite field, the study of this deformation problem, though very similar to the one made by B. Mazur in [M], is slightly trickier.

To define this problem, we need to introduce some notations. First, let $N$ be the ray class field of $M$ of conductor $p$ (one has of course $N$ inside $M_{\infty}$) and $N^{(p)}$ be the maximal $p$-extension of $N$ unramified outside $p$. It is clear that $\bar{\rho}_0$ restricted to $\text{Gal}(\overline{\mathbb{Q}}/N)$ factors through $\Pi_N = \text{Gal}(N^{(p)}/N)$, so that, for the deformation problem of $\bar{\rho}_0$, we can restrict ourselves to representations $\rho$ of $\Pi = \text{Gal}(N^{(p)}/F)$. The great advantage of such a limitation in the choice of $\rho$'s is that $\Pi$ is topologically of finite type ($\Pi_N$ is a pro-$p$-group and its Frattini quotient is finite by Kummer theory over $N$). Now, let $\text{Art}$ be the category of local artinian $\Lambda_P$-algebras with residue field $k(P)$, $\text{Sets}$ the category of sets and $\mathcal{F}$ the covariant functor

$$\mathcal{F} : \text{Art} \to \text{Sets}$$

given for $A \in \text{Ob}(\text{Art})$ with maximal ideal $m_A$ by

$$\mathcal{F}(A) = \{ \rho : \Pi \to GL_2(A) \mid \rho \text{ is finitely continuous and } \rho \mod m_A = \bar{\rho}_0 \}/\approx.$$

Here (i) '$\approx'$ denotes the strict equivalence of representations, that is, conjugation by a matrix in $GL_2(A)$ congruent to 1 modulo $m_A$.

(ii) The phrase 'finitely continuous' means that there exists a $\Lambda$-submodule $L$ in $A^2$ of finite type stable by $\rho$ generating $A^2$ over $\Lambda_P$. The reason for this definition instead of usual $P$-adic continuity is that $\Lambda_P$ is not locally compact for the $P$-adic topology, but $\Pi$ is even compact. Hence a $P$-adically continuous representation should have a very small image, and in some sense, we look for representations with open image (over $\Lambda$). Note that a finitely continuous representation induces a continuous representation: $\Pi \to GL(L)$, $L$ being endowed with the usual $m$-adic topology for the maximal ideal $m$ of $\Lambda$. 
This notion of 'finite continuity' does not depend on the choice of the lattice $L$ by the Artin–Rees lemma. We can extend this notion of finite continuity to any map $u$ of $\Pi$ to an $A$-module $V$ requiring that $u$ having values in a $\Lambda$-finite submodule $L$ in $V$ and the induced map $u : \Pi \to L$ is continuous under the $m$-adic topology on $L$. This generalized notion will be used later to define finitely continuous cohomology.

By using the fact that $\overline{\rho}_0$ is induced from a finitely continuous character: $\Pi \to k(P)^\times$, it is not so difficult to check by Schlessinger's criterion the following fact:

**Theorem 4.3** The functor $\mathfrak{F}$ is pro-representable; that is, there exists a unique universal couple $(R', \rho')$ where $R'$ is a local noetherian complete $\Lambda_p$-algebra with residue field $k(P)$ and $\rho' \in \mathfrak{F}(R') = \lim_{\alpha} \mathfrak{F}(R'/m_{R'}^\alpha)$.

**Comments**

a) The ‘continuity’ property $\rho'$ enjoys should be called ‘profinite continuity’, meaning that for any artinian quotient $\varphi : R' \to A$ of $R'$, $\varphi \circ \rho'$ is finitely continuous. There is also an obvious notion of profinite continuity of maps from $\Pi$ to any $\Lambda_p$-module.

b) It is natural to ask for the pro-representability of this problem starting from an arbitrary irreducible finitely continuous representation $\overline{\rho}_0$. The answer is not known in general because of the lack of a cohomology theory adapted to finite continuous representations and subgroups of $GL_2(k(P))$. Such theory is available when $k(P)$ is a $p$-adic field, due to Lazard [L], and allows us to give a positive answer in this case. See Appendix below.

In fact, the universal ring we need is smaller than $R'$. It will pro-represent a subfunctor $\mathfrak{F}_S$ of $\mathfrak{F}$ requiring local conditions at primes of $F$ above $p$ (these conditions involve the choice we made of a $p$-adic CM-type $S$). We call this problem the $S$-nearly ordinary deformation problem of $\overline{\rho}_0$. For $\mathfrak{P}$ in $S$, recalling $p = \mathfrak{P} \cap F$, we choose $D_p$ so that $D_p$ is the decomposition group of $\mathfrak{P}$ in $\Pi_M = \text{Gal}(N^p(M)/M)$. A strict equivalence class $[\rho]$ in $\mathfrak{F}(A)$ belongs to $\mathfrak{F}_S(A)$ if and only if for any representative $\rho$, the following conditions are satisfied:

(4.1a) For each prime $p$ above $p$ in $F$, there exists a finitely continuous character $\delta_p : D_p \to \Lambda^\times$ such that $\rho$ restricted to $D_p$ is equivalent (but not necessarily strictly) to $\begin{pmatrix} * & * \\ 0 & \delta_p \end{pmatrix}$;
(4.1b) \( \delta_p \) is congruent modulo \( m_A \) to the restriction of \( \lambda^c \) to \( D_p \) and \( \delta_p \) restricted to the inertia subgroup \( I_p \) of \( D_p \) coincides with the restriction to \( I_p \) of \( \lambda^c \);

(4.1c) \( \det(\rho) = \det(\rho_0) \) (considered as having values in \( A \) via the structural morphism: \( \Lambda \to \Lambda_P \to A \)).

One can deduce from Theorem 4.3 that \( \mathfrak{F}_S \) is pro-representable. We denote by \( (R_S, \rho_S) \) the corresponding universal couple. Let us define a \( \Lambda_P \)-module \( \mathfrak{W}_P \) by

\[
\mathfrak{W}_P = \bigcup_{m=1}^{\infty} P^{-m} \Lambda_P/\Lambda_P = L/\Lambda_P,
\]

where \( L \) is the quotient field of \( \Lambda \). Then \( \mathfrak{W}_P \) is the injective envelope of \( k(P) \). We consider the algebra \( R_S[\mathfrak{W}_P] = R_S \oplus \mathfrak{W}_P \) with \( \mathfrak{W}^2_P = 0 \). One can consider, by abusing the notation, \( \mathfrak{F}_S(R_S[\mathfrak{W}_P]) \) is a set of profinitely continuous deformations of \( \bar{\rho} \) satisfying the above conditions (i), (ii) and (iii). Since \( \Pi \) is topologically finitely generated, by the profinite continuity, \( \rho \) has image in a noetherian subring \( R_m = R_S[P^{-m}\Lambda_P/\Lambda_P] \) for sufficiently large \( m \). Thus we have a local \( \Lambda \)-algebra homomorphism \( \varphi_s : R_S \to R_S[\mathfrak{W}_P] \) such that \( \rho \approx \varphi_s \circ \rho_S \). Now we consider the subset

\[
\mathfrak{F}_0(R_S[\mathfrak{W}_P]) = \{ \rho \in \mathfrak{F}_S(R_S[\mathfrak{W}_P]) \mid \rho \mod \mathfrak{W}_P = \rho_S \}.
\]

We also define \( \text{Sect}_\Lambda(R_S[\mathfrak{W}_P]/R_S) \) to be the set of continuous sections (under the \( m_{R_S} \)-adic topology) \( \varphi : R_S \to R_S[\mathfrak{W}_P] \) as \( R_S \)-algebras whose projection to \( \mathfrak{W}_P \) is contained in \( P^{-m}\Lambda_P/\Lambda_P \) for \( m \) sufficiently large. We put

\[
sl_2(\mathfrak{W}_P) = \{ x \in M_2(\mathfrak{W}_P) \mid \text{Tr}(x) = 0 \},
\]

which is a module over \( \Pi \) under the action: \( \sigma x = \rho_S(x)\sigma x \rho_S(x)^{-1} \). We consider the cohomology group \( H^1(\Pi, sl_2(\mathfrak{W}_P)) \), which is the quotient of the module of profinitely continuous 1-cocycles on \( \Pi \) having values in \( sl_2(P^{-m}\Lambda_P/\Lambda_P) \) for sufficiently large \( m \) modulo usual coboundaries. In fact, for each \( \rho \in \mathfrak{F}_S(R_S[\mathfrak{W}_P]) \), \( \varphi_s : R_S \to R_S[\mathfrak{W}_P] \) is a \( \Lambda \)-algebra homomorphism. If \( \rho \in \mathfrak{F}_0(R_S[\mathfrak{W}_P]) \), then by the fact that \( \rho \mod \mathfrak{W}_P = \rho_S \), \( \pi \circ \varphi = \text{id}_{R_S} \). Thus we have a morphism: \( \mathfrak{F}_0(R_S[\mathfrak{W}_P]) \to \text{Sect}_\Lambda(R_S[\mathfrak{W}_P]/R_S) \). This morphism is of course a surjective isomorphism because for \( \varphi \in \text{Sect}_\Lambda(R_S[\mathfrak{W}_P]/R_S) \), \( \varphi \circ \rho_S \) is an element of \( \mathfrak{F}_0(R_S[\mathfrak{W}_P]) \). Therefore we know that

\[
\mathfrak{F}_0(R_S[\mathfrak{W}_P]) \cong \text{Sect}_\Lambda(R_S[\mathfrak{W}_P]/R_S).
\]

For each \( p \in \Sigma \), we can find \( \alpha_p \in GL_2(R_S) \) such that

\[
\alpha_p \rho_S(\sigma) \alpha_p^{-1} = \begin{pmatrix} \ast & \delta_p^s(\sigma) \\ 0 & \ast \end{pmatrix} \quad \text{for all } \sigma \in D_p
\]

and \( \delta_p^s \equiv \lambda^c \mod m_{R_S} \).
We fix such a \( \alpha_p \) for each \( p \). Then we define the ordinary cohomology subgroup \( H^1_{\text{ord}}(\Pi, sl_2(\mathfrak{m}_p)) \) by the subgroup of cohomology classes of cocycle \( u \) satisfying, for every \( p \) dividing \( t \) in \( F \),

\[
\left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} \supset \alpha_p u(D_p)\alpha_p^{-1} \quad \text{and} \quad \left\{ \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \right\} \supset \alpha_p u(I_p)\alpha_p^{-1}.
\]

**Theorem 4.4** We have a canonical isomorphism:

\[
\text{Hom}_{R_S}(\Omega_{R_S/\Lambda_F}, \mathfrak{m}_p) \cong H^1_{\text{ord}}(\Pi, sl_2(\mathfrak{m}_p)),
\]

where the Kähler differential module \( \Omega_{R_S/\Lambda_F} \) is defined to be the module of continuous differentials, i.e., \( \Omega_{R_S/\Lambda_F} = I/I^2 \) for the kernel \( I \) of the multiplication map of the completed tensor product \( R_S \hat{\otimes}_{\Lambda_F} R_S \) (under the adic topology of the maximal ideal of \( R_S \hat{\otimes}_{\Lambda_F} R_S \)) to \( R_S \).

**Proof** For each \( \rho \in \mathfrak{g}_0(R_S[\mathfrak{m}_p]) \), we define \( u : \Pi \rightarrow M_2(\mathfrak{m}_p) \) by

\[
\rho(\sigma) = (1 \oplus u(\sigma))\rho_S(\sigma) \quad \text{for} \quad \sigma \in \Pi.
\]

Since \( \rho_S \) and \( \rho \) are both profinitely continuous, \( u \) has values in \( P^{-m}\Lambda_F/\Lambda_F \) for sufficiently large \( m \) and is profinitely continuous. Then by (6.1c), we know that \( \det(\rho_S) = \det(\rho) \). This shows that \( u \) has values in \( sl_2(\mathfrak{m}_p) \). Similarly by (6.1b), we know that \( \left\{ \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \right\} \supset \alpha_p u(I_p)\alpha_p^{-1} \). By the multiplicativity:

\[
\rho(\sigma)\rho(\tau) = \rho(\sigma\tau),
\]

we see easily that \( u \) is a cocycle and \( u \) is a coboundary if and only if \( \rho \approx \rho_S \). Thus the map \( \mathfrak{g}_0(R_S[\mathfrak{m}_p]) \rightarrow H^1_{\text{ord}}(\Pi, sl_2(\mathfrak{m}_p)) \) is injective. Surjectivity follows from the fact that we can recover a profinitely continuous representation out of a profinitely continuous cocycle by the above formula. Namely we know that

\[
\mathfrak{g}_0(R_S[\mathfrak{m}_p]) \cong H^1_{\text{ord}}(\Pi, sl_2(\mathfrak{m}_p)).
\]

If we have a section \( \varphi \in \text{Sect}_A(R_S[\mathfrak{m}_p]/R_S) \), we can write \( \varphi(r) = r \oplus d_\varphi(r) \). Then \( d_\varphi \in \text{Der}_A(R_S, \mathfrak{m}_p) = \text{Hom}_{R_S}(\Omega_{R_S/\Lambda_F}, \mathfrak{m}_p) \). It is easy that from any derivation \( d : R_S \rightarrow \mathfrak{m}_p \), we can reconstruct a section by the above formula. Thus we know that

\[
\text{Hom}_{R_S}(\Omega_{R_S/\Lambda_F}, \mathfrak{m}_p) \cong \text{Sect}_A(R_S[\mathfrak{m}_p]/R_S)
\]

which conclude the proof by (4.2).

We have an injection

\[
\text{res} : H^1(\Pi, sl_2(\mathfrak{m}_p)) \rightarrow H^1(\Pi_M, sl_2(\mathfrak{m}_p))^\text{Gal}(M/F).
\]
Note that as $\Pi_M$-module, $sl_2(\mathfrak{w}_P) \cong \mathfrak{w}_P(\alpha) \oplus \mathfrak{w}_P(\alpha^{-1}) \oplus \mathfrak{w}_P$, where $\alpha = \lambda(\mathfrak{c})^{-1}$ and $\mathfrak{w}_P(\alpha) \cong \mathfrak{w}_P$ as $\Lambda$-module but $\Pi$ acts via the one-dimensional abelian character $\alpha$. The action of $c$ interchanges $\mathfrak{w}_P(\alpha)$ and $\mathfrak{w}_P(\alpha^{-1})$ and acts by $-1$ on $\mathfrak{w}_P$. Thus we see
\[ H^1(\Pi_M, sl_2(\mathfrak{w}_P))^{{\text{Gal}(M/F)}} = H^1(\Pi_M, \mathfrak{w}_P(\alpha)) \oplus \text{Hom}_{\text{cont}}(G/(1 + c)G, \mathfrak{w}_P). \]
Recall that $M^{-}(\lambda_{-})/M$ is the extension corresponding to $\text{Ker}(\alpha)$. The inclusion of $H^1(\Pi_M, \mathfrak{w}_P(\alpha))$ into $H^1(\Pi_M, sl_2(\mathfrak{w}_P))^{{\text{Gal}(M/F)}}$ is given in terms of cocycle by the cocycle $U$ such that $U(\sigma) = \begin{pmatrix} 0 & u(\sigma) \\ 0 & -u(\sigma) \end{pmatrix}$ for $\sigma = c\sigma c^{-1}$. From this, it follows, by the ordinarity condition,
\[ \text{res}_{\Pi_M^-}(u)(c_{\mathfrak{p}}c^{-1}) = \text{res}_{\Pi_M^-}(u)(I_{\mathfrak{p}c}) = 0 \text{ for all } \mathfrak{p} \in S, \]
where $M^- = M^-(\lambda_{-})$. Namely $\text{res}_{\Pi_M^-}(u)$ is unramified outside $S$. Thus we have a natural map:
\[ \text{res} : H^1_{\text{ord}}(\Pi_M, \mathfrak{w}_P(\alpha)) \to \text{Hom}_H(X_S, \mathfrak{w}_P(\alpha)) = \text{Hom}_{\Lambda_{-}}(X_S(\lambda^{-}), \mathfrak{w}_P). \]

Comments We omitted $\alpha$ from the module of extreme right, because the $\Lambda_{-}$-module structure on $\mathfrak{w}_P$ given by $\alpha$ coincides with the natural structure given by the inclusion $\Lambda_{-} = \mathcal{O}[[\mathfrak{W}^{-}]]$ into $\Lambda$ through the $\Lambda$-module structure of $\mathfrak{w}_P$. Moreover we can write the extreme right as
\[ \text{Hom}_{\Lambda_{-}}(X_S(\lambda^{-}), \mathfrak{w}_P) = \text{Hom}_{\Lambda}(X_S(\lambda^{-}) \otimes_{\Lambda_{-}} \Lambda, \mathfrak{w}_P). \]
Thus the variable coming from the ‘$+$’ part $\mathfrak{W}^+$ in $\Lambda$ is just a ‘fake’ and the divisibility we will obtain is in fact the divisibility in $\Lambda_{-}$ although we have variables coming from $\mathfrak{W}^+$ inside the Hecke algebra. This is natural because $X_S(\lambda^{-})$ is a $\Lambda_{-}$-module. The use of ‘$+$’-variables is inevitable because we do not know a priori that the congruence power series belongs to the ‘$-$’ part. In the appendix, we prove that when $F = \mathbb{Q}$, the congruence power series belongs to the ordinary part of the Hecke algebra, which can be regarded as the ‘$-$’ part in our situation.

It is not difficult to show that the above map: $\text{res}$, is injective; namely,
\[ \text{Corollary 4.5} \quad \text{Hom}_{\Lambda}(X_S(\lambda^{-}) \otimes_{\Lambda_{-}} \Lambda, \mathfrak{w}_P) \ni H^1(\Pi_M, \mathfrak{w}_P(\alpha)). \]
Since the inclusion of $\text{Hom}_{\text{cont}}(G/(1 + c)G, \mathfrak{w}_P)$ into $H^1(\Pi_M, sl_2(I_P))^{{\text{Gal}(M/F)}}$ is given in terms of cocycle by
\[ \text{Hom}_{\text{cont}}(G/(1 + c)G, \mathfrak{w}_P) \ni u \mapsto U(\sigma) = \begin{pmatrix} u(\sigma) & 0 \\ 0 & -u(\sigma) \end{pmatrix}, \]
we know that if $U$ is ordinary, then $u$ is unramified everywhere. Let $\mathcal{C}l_{-}$ be the ‘$-$’ quotient of the ideal class group of $K$. We thus know that
Theorem 4.6 \( H_{\text{ord}}(\Pi, sl_2(\mathfrak{w}_P)) \cong \text{Hom}_{\Lambda}(\Omega_{R_S/\Lambda_P} \otimes_{R_S} \Lambda_P, \mathfrak{w}_P) \) injects naturally into
\[
\text{Hom}_{\Lambda}(X_S(\lambda^-) \otimes_{\Lambda} \Lambda, \mathfrak{w}_P) \oplus \text{Hom}(Cl^-, \mathfrak{w}_P)
\]
as \( \Lambda \)-module.

To relate \( X_S(\lambda^-) \) to the congruence power series, we recall the morphism \( \lambda_* \circ \theta^* : h \to \Lambda \) seen in §2, H3. Let \( R_0 \) be the local ring of \( h \) through which the above morphism factors. To make \( R_0 \) a \( \Lambda \)-algebra, we consider \( R = R_0 \otimes_{\Lambda_0} \Lambda \), which is still a complete local ring. Consider the module of differentials \( \mathcal{C}_1 = \Omega_{R/\Lambda} \otimes_{R} \Lambda \) introduced in [H1, p. 319], where the tensor product is taken via
\[
R \to \Lambda \otimes_{\Lambda_0} \Lambda \to \Lambda,
\]
which is \( \lambda_* \circ \theta^* \) composed with the multiplication on \( \Lambda \). Let \( R_P \) be the completion of the localization of \( R \) at \( P \). In [H3, Th.1], an \( S \)-nearly ordinary deformation \( \rho^{\text{mod}} : \Pi \to GL_2(R_P) \) of \( (k(P), \overline{\rho}_0) \) has been constructed. Especially \( R_P \) is generated over \( \Lambda_P \) by \( \text{Tr}(\rho^{\text{mod}}) \), and hence, the natural map \( \varphi : R_S \to R_P \) which induces the equality \([\varphi \circ \rho_S] = [\rho^{\text{mod}}] \) is surjective. Then \( \varphi \) induces another surjection
\[
\varphi_* : \Omega_{R_S/\Lambda_P} \otimes_{R_S} \Lambda_P \to \mathcal{C}_1 \otimes_{\Lambda} \Lambda_P.
\]
This combined with Theorem 4.6 yields

Theorem 4.7 We have a surjective homomorphism of \( \Lambda \)-modules:
\[
(X_S(\lambda^-) \otimes_{\Lambda} \Lambda') \oplus (Cl^- \otimes_{\mathcal{O}} \Lambda') \to \mathcal{C}_1,
\]
where \( \Lambda' \) is either \( \Lambda \) or \( \Lambda[\frac{1}{P}] \) in Lemme 4.2 according as \( F \neq \mathbb{Q} \) or \( F = \mathbb{Q} \) and \( \lambda_\text{mod} \pi \mathcal{O} \) is trivial on \( D_\pi \).

As explained in [T2], there is a divisibility theorem proven by M. Raynaud:

Theorem 4.8 \( H \) divides the characteristic power series of \( \mathcal{C}_1 \) in \( \Lambda \).

Then Theorems 4.7 and 4.8 prove Theorem 4.1.

Although we have concentrated to the anti-cyclotomic tower, there is a (hypothetical) way to include the case of the cyclotomic tower. To show the dependence on \( F \), we add subscript \( F \) to each notation, for example \( L_F^- \) for \( L^- \) over \( F \). Supposing the strong divisibility in \( \Lambda : L_{F_n}^- | Iw_{F_n}^- \) for the \( n \)th layer \( F_n \) of the cyclotomic \( \mathcal{Z}_p \)-extension of \( F \) for all \( n \), we hope that we could eventually get the full divisibility: \( L | Iw \) over \( F \)? But for the moment, this is still far away.
APPENDIX

Let $F/Q$ be a finite extension and fix an arbitrary finite Galois extension $N/F$. Let $N^{(p)}/N$ be the maximal $p$-profinite extension of $N$ unramified outside $p$ and $\infty$. Put $\Pi = \text{Gal}(N^{(p)}/F)$. In this appendix, we shall prove the existence of the universal deformation for any (continuous) absolutely irreducible Galois representation $\rho : \Pi \to GL_n(K)$ for a finite extension $K/\mathbb{Q}_p$ and then we prove the divisibility in $\Lambda'$ (as in Theorem 4.7) of $\chi(M)\text{Iw}^\dagger$ by $H$ when $M$ is an imaginary quadratic field. Let $\Lambda$ be a noetherian local ring with residue field $K$ and suppose that $\Lambda$ is complete under the $m$-adic topology for the maximal ideal $m$ of $\Lambda$. We consider the category $\text{Art}_\Lambda$ of artinian local $\Lambda$-algebras with residue field $K$. For any object $A$ in $\text{Art}_\Lambda$, the $p$-adic topology on $A$ gives a locally compact topology on $GL_n(A)$. We consider the covariant functor

$$\mathcal{F} : \text{Art}_\Lambda \to \text{Sets}$$

which associates to each object $A$ in $\text{Art}_\Lambda$ a set of strict equivalence classes of continuous representations $\rho : \Pi \to GL_n(A)$ such that $\rho \mod m_A = \overline{\rho}$. Then we have

**Theorem A.1** $\mathcal{F}$ is pro-representable on $\text{Art}_\Lambda$.

**Proof** We verify the Schlessinger's criterion $H_i$ ($i = 1, 2, \ldots, 4$) for pro-representability ([Sch]). The conditions $H_1$, $H_2$ and $H_4$ can be checked in exactly the same manner as in [M, 1.2]. We verify the finiteness of tangential dimension; i.e.,

$$H_3 : \text{dim}_K \mathcal{F}(K[\varepsilon]) \text{ is finite, where } K[\varepsilon] = K \oplus K\varepsilon \text{ with } \varepsilon^2 = 0.$$ 

If $\rho \in \mathcal{F}(K[\varepsilon])$, then we define a map $u = u_\rho : \Pi \to M_n(K)$ by $\rho(\sigma) = (1 \oplus u(\sigma)\varepsilon)\overline{\rho}(\sigma)$. Since $\rho$ is continuous, $u$ is a continuous 1-cocycle with values in the $\Pi$-module $M_n(K)$, where $\Pi$ acts on $M_n(K)$ by $\sigma x = \overline{\rho}(\sigma)x\overline{\rho}(\sigma)^{-1}$. On the other hand, if we have a continuous 1-cocycle $u$ as above, we construct a representation $\rho$ by $\rho(\sigma) = (1 \oplus u(\sigma)\varepsilon)\overline{\rho}(\sigma)$. As a map to $M_n(K)$, $\rho$ is continuous. Then $\rho$ is finitely continuous as a representation. Thus the map $\mathcal{F}(K[\varepsilon]) \to H_c^1(\Pi, M_n(K))$ is surjective. Here 'H' indicates the continuous cohomology. We see easily that $u(\sigma) = (\sigma - 1)m$ if and only if $(1 \oplus m)^{-1}\overline{\rho}(1 \oplus m) = \rho$ (i.e., $\rho$ is strictly equivalent to $\overline{\rho}$, which is the 'zero' element in $\mathcal{F}(K[\varepsilon])$). Thus we have

$$\mathcal{F}(K[\varepsilon]) \cong H_c^1(\Pi, M_n(K))$$

and

(A.1) $$H_c^1(\Pi, M_n(K)) \cong H_c^1(\Pi, sl_n(K)) \oplus \text{Hom}_c(\Pi, K).$$
By class field theory, $\text{Hom}_c(\Pi, K)$ is finite dimensional. We now claim
\[(A.2) \quad \dim_K H^1_c(\Pi, sl_n(K)) < +\infty.\]
Let us prove this. Let $F^\infty$ be the subfield of $N^{(p)}$ fixed by $\text{Ker}(\overline{\rho})$. Since cohomology groups of a finite group with coefficients in finite dimensional vector space over $K$ are finite dimensional, we may replace $\Pi$ by any normal subgroup of finite index because of the inflation-restriction sequence. First we may assume that $H = \text{Im}(\overline{\rho})$ is a pro-$p$-group without torsion and that $F^\infty/N$ is unramified outside $p$ and $\infty$. Then applying a theorem of Lazard [L, III.3.4.4.4], we know that $H$ has a subgroup of finite index which is pro-$p$-analytic. Hence we may even assume that $H$ itself is pro-$p$-analytic. By inflation-restriction sequence, the sequence:
\[(A.3) \quad 0 \to H^1_c(H, sl_n(K)) \to H^1_c(\Pi, sl_n(K)) \to \text{Hom}_H(\text{Ker}(\overline{\rho}), sl_n(K))\]
is exact. Let $M^\infty/F^\infty$ be the maximal $p$-abelian extension unramified outside $p$ and $\infty$ and $X$ be the Galois group $\text{Gal}(M^\infty/F^\infty)$. Let $A = \mathbb{Z}_p[[H]]$. Since $H$ is pro-$p$-analytic and is contained in the maximal compact subgroup of $GL_n(K)$, we know that $X$ is a $A$-module of finite type by [Ha, §3]. The maximal topological abelian quotient $\text{Ker}(\overline{\rho})^\text{ab}$ is a quotient of $X$ and hence of finite type over $A$. This proves that
\[(A.4) \quad \dim_K \text{Hom}_H(\text{Ker}(\overline{\rho}), sl_n(K)) < +\infty.\]
Thus we need to show the finite dimensionality of $H^1_c(H, sl_n(K))$. Let $\mathfrak{h}$ be the Lie algebra of $G \cap H$. Then again by a result of Lazard [L, V.2.4.10], we see
\[H^1_c(H, sl_n(K)) \cong H^0(H, H^1(\mathfrak{h}, sl_n(K))),\]
which is finite dimensional.

Let $h_0 = h_0^{\text{ord}}(D; \mathcal{O})$ be the ordinary Hecke algebra defined in [H1, Th.3.3] for any positive integer $D$ prime to $p$. In this case $G$ in §2 is just $\mathbb{Z} \times \mathbb{Z}_p^\times$ for $Z = ((\mathbb{Z}/D\mathbb{Z})^\times \times \mathbb{Z}_p^\times)/\{\pm 1\}$. Then we have

**Theorem A.2** Suppose that $p \geq 5$ and $F = \mathbb{Q}$. Let $\chi : \mathbb{A}^\times \to \mathbb{Z}_p^\times$ be the cyclotomic character such that $\chi(\omega_l) = l$ for the prime element $\omega_l$ in $\mathbb{Q}_l$ ($l \neq p$). Then we have an $\mathcal{O}[[G]]$-algebra isomorphism:
\[h \cong h_0 \hat{\otimes}_{\mathcal{O}} \mathcal{O}[[\mathbb{Z}_p^\times]],\]
which is given by $T(x) \mapsto T(x) \otimes [\chi(x)]$ for all $x \in \hat{\mathbb{A}} \cap A_f^\times$. Here $h_0 \hat{\otimes}_{\mathcal{O}} \mathcal{O}[[\mathbb{Z}_p^\times]]$ is the profinite completion of $h_0 \otimes_{\mathcal{O}} \mathcal{O}[[\mathbb{Z}_p^\times]]$, i.e., $m$-adic completion for the maximal ideal $m$ of $\Lambda_0$. 


Appendix

Proof Let $S(O) = \{ f \in S \mid a(y, f) \in O \}$ and $S = eS(O)$ for the idempotent $e$ of $h$ in $h(D; O)$. Let $S_0$ be the ordinary subspace of $S$ which is denoted by $S^{ord}_0(D; O)$ in [H1, p. 336]. Then it is known that the pairing given by

$$\langle h, f \rangle = a(1, f \mid h)$$
onumber

is perfect in the sense that $\text{Hom}_O(h, O) = S$ and vice versa [H4, Th.3.1]. For any character $\psi : \mathbb{Z}_p^* \rightarrow \overline{\mathbb{Q}}_p$ and $f \in S$, $f \otimes \psi(y)$ given by $a(y, f \otimes \psi) = \psi(\chi(y))a(y, f)$ is again an element in $S$ with $f \otimes \psi \mid e = f \otimes \psi$ (cf. [H4, §7.VII]). This shows that we have a natural $O[[G]]$-linear map $m : S_0 \hat{\otimes}_O \mathcal{C}(\mathbb{Z}_p^*; O) \rightarrow S$ given by

$$a(y, m(f \otimes \phi)) = \phi(\chi(y))a(y, f),$$

where $\mathcal{C}(\mathbb{Z}_p^*; O)$ is the Banach $O$-module of all continuous functions on $\mathbb{Z}_p^*$ into $O$ and $S_0 \hat{\otimes}_O \mathcal{C}(\mathbb{Z}_p^*; O)$ is the $p$-adic completion of $S_0 \otimes_O \mathcal{C}(\mathbb{Z}_p^*; O)$. Note that $\text{Hom}_O(S_0 \hat{\otimes}_O \mathcal{C}(\mathbb{Z}_p^*; O)) \cong h_0 \hat{\otimes}_O O[[\mathbb{Z}_p^*]]$. It is easy to verify that the dual map $m^* : h \rightarrow h_0 \hat{\otimes}_O O[[\mathbb{Z}_p^*]]$ is in fact an $O[[G]]$-algebra homomorphism. Since the projection map $h \rightarrow h_0$ is surjective by definition and since any $[z^{-1}] \in O[[\mathbb{Z}_p^*]]$ for $z \in \mathbb{Z}_p^*$ is the image of $T(z)$, $m^*$ is surjective. Note that $h_0 \hat{\otimes}_O O[[\mathbb{Z}_p^*]]$ is free of finite rank over $\Lambda_0$ by [H7, Th.3.1]. Since $h$ is torsion-free over $\Lambda_0$ and its generic rank is equal to that of $h_0 \hat{\otimes}_O O[[\mathbb{Z}_p^*]]$, we conclude that $m^*$ is an isomorphism.

Corollary A.3 The congruence power series $H$ can be chosen inside $\Lambda_-$. By this corollary, when $F = \mathbb{Q}$, it is sufficient to consider only ordinary Hecke algebras instead of nearly ordinary Hecke algebras and only ordinary deformations instead of nearly ordinary deformations. To make this fact more precise, let $M/\mathbb{Q}$ be an imaginary quadratic field of discriminant $D$ satisfying the ordinarity hypothesis: $p = \mathfrak{p}\mathfrak{p}^c$. We also assume that $p \geq 5$. Let $L$ (resp. $L^*$) be the maximal abelian extension of $M$ unramified outside $\mathfrak{p}$ (resp. $\mathfrak{p}^c$). Let $G_{cw} = \text{Gal}(L/M)$ and $G^*_{cw} = \text{Gal}(L^*/M)$ and $W_{cw}$ (resp. $W^*_{cw}$) be the maximal torsion-free quotient of $G_{cw}$ (resp. $G^*_{cw}$). Then the restriction map gives an isomorphism $W \cong W_{cw} \times W^*_{cw}$. Thus $\alpha : W_{cw} \ni w \mapsto wcw^{-1}c^{-1} \in W_-$ gives an isomorphism. Similarly, without losing generality, we may assume that $\lambda : G_{tor} \rightarrow O^*$ factors through $G_{cw}$. We decompose $G_{cw} = \Delta \times W_{cw}$. Let $\Lambda_- = O[[W_{cw}]]$ identifying $W_-$ with $W_{cw}$. We consider the character $\lambda_+ : G_{cw} \rightarrow \Lambda_-$ such that $\lambda_+(\delta, w) = \lambda(\delta)[w]$ for $\delta \in \Delta$ and $w \in W$. It is known that the $\mu$-invariant of $Iw_-$ and $L^-$ are both trivial [G]. Thus we only worry about height one primes $P$ (in $\Lambda_-$) of residual characteristic $0$. We take $N/\mathbb{Q}$ to be the ray class field of $M$ modulo $p$ and consider the Galois group $\Pi$ as in Theorem A.1. Let $K$ be the quotient
field of \( \Lambda_-/P \). Then \( K/Q_p \) is a finite extension and we consider the Galois representation:

\[
\rho_0 = \text{Ind}_{\Pi_K}^\Pi(\lambda_*) : \Pi \to GL_2(\Lambda_-), \quad \text{and}
\]

\[
\rho_P = \text{Ind}_{\Pi_K}^\Pi(\lambda_* \mod P) : \Pi \to GL_2(K).
\]

Suppose that \( P \neq P_\lambda \) as in Lemma 4.2. Then \( \rho_P \) is absolutely irreducible.

Let \( \Lambda \) be the \( P \)-adic completion of the localization of \( \Lambda_- \) at \( P \). Let \( \text{Art} \) be the category of artinian local \( \Lambda \)-algebras with residue field \( K \). Any object \( A \) in \( \text{Art} \) is a locally compact ring with respect to \( p \)-adic topology and thus we do not worry about 'finite continuity' etc. Let \( (R', \rho') \) be the universal couple representing the functor \( \mathcal{F} : \text{Art} \to \text{Sets} \) defined for \( \mathcal{P} = \rho_P \). We consider the subfunctor of \( \mathcal{F} \)

\[\mathcal{F}^{\text{ord}} : \text{Art} \to \text{Sets}\]

which associates to \( A \in \text{Ob}(\text{Art}) \) the set of strict equivalence class of representations \( \rho : \Pi \to GL_2(A) \) such that

(i) \( \rho \mod m_A = \rho_P \),

(ii) There exists a continuous character \( \delta : D_\mathcal{P} \to \mathbb{A}^\times \) such that \( \rho \) restricted to \( D_\mathcal{P} \) is equivalent (but not necessarily strictly) to \( \begin{pmatrix} * & * \\ 0 & \delta \end{pmatrix} \);

(iii) \( \delta \) is congruent modulo \( m_A \) to the restriction of \( \lambda_* \) to \( D_\mathcal{P} \) and \( \delta \) restricted to the inertia subgroup \( I_p \) of \( D_p \) coincides with the restriction to \( I_p \) of \( \lambda_* \) (i.e., \( \delta \) is unramified at \( \mathcal{P} \))

(iv) \( \det(\rho) = \det(\rho_0) \).

We say that an ideal \( a \) of \( R' \) is ordinary if \( \rho' \mod a \) satisfies (i), (ii), (iii) and (iv). Then it is an easy exercise to verify that if \( a \) and \( b \) are ordinary, then \( a + b \) and \( a \cap b \) are ordinary. Namely for \( \mathcal{I} = \cap_{\text{ordinary} a}, R^{\text{ord}} = R'/\mathcal{I} \) and \( \rho^{\text{ord}} = \rho' \mod \mathcal{I} \) represents \( \mathcal{F}^{\text{ord}} \). Then the same argument as in §4 prove that \( H \mid h(M)Iw^- \). From Theorem 2.1 and the vanishing of the \( \mu \)-invariant \([G]\), we conclude

Theorem A.4 Suppose \( p \geq 5 \) and that \( M \) is an imaginary quadratic field. Let \( \Lambda' = \Lambda_-[\frac{1}{P_\lambda}] \) if \( \lambda_- \mod \pi \mathcal{D} \) is trivial on \( D_\mathcal{P} \) and otherwise we put \( \Lambda' = \Lambda_- \). Then we have

\[ h(M)L^- \mid H \text{ in } \Lambda_- \quad \text{and} \quad H \mid h(M)Iw^- \text{ in } \Lambda'. \]

Although we confined ourselves to characters \( \lambda \) of \( p \)-power conductor, similar result holds for any character whose conductor is prime to its complex conjugate. We hope to prove the divisibility even at the 'trivial-zero' \( P_\lambda \) in our subsequent paper.
REFERENCES


