Growth of Hecke Fields Along a p-Adic Analytic Family of Modular Forms

Haruzo Hida

Abstract Fix a nearly ordinary non-CM p-adic analytic family of Hilbert modular Hecke eigenforms (over a totally real field F). We prove existence of a density one set Ξ of primes of the field F such that the degree of the field over $\mathbb{Q}(\mu_{p^\infty})$ generated by the Hecke eigenvalue of the Hecke operator $T(\mathfrak{l})$ grows indefinitely over the family for each prime $\mathfrak{l}$ in the set Ξ.

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We generalize in this paper all the principal results obtained in [H14] for the one variable cyclotomic p-ordinary Hecke algebra to the full nearly p-ordinary Hecke algebra of fixed central character. This algebra is finite flat over the m variable Iwasawa algebra for the degree m totally real base field F. The restriction coming from fixing a central character is essentially harmless as we can bring one central character to another by character twists (up to finite order character of bounded order).

Take the field $\overline{\mathbb{Q}}$ of all numbers in $\mathbb{C}$ algebraic over $\mathbb{Q}$. Fix a prime p and a field embedding $\overline{\mathbb{Q}} \overset{i_p}{\hookrightarrow} \overline{\mathbb{Q}}_p \subset \mathbb{C}_p$. Fix a totally real number field F (of degree m over $\mathbb{Q}$) inside $\overline{\mathbb{Q}}$ with integer ring O (as the base field for Hilbert modular forms). We use the symbol O exclusively for the integer ring of F, and for a general number field L, we write O_L for the integer ring of L. We choose and fix an O-ideal $\mathfrak{n}$ prime to p (as the level of modular form). We define an algebraic group G (resp. T_L) by $\mathrm{Res}_{O/\mathbb{Z}}\mathrm{GL}(2)$ (resp. $\mathrm{Res}_{O_L/\mathbb{Z}}\mathbb{G}_m$); so, $G(R) = \mathrm{GL}_2(R \otimes_{\mathbb{Z}} O)$ and $T_L(R) = (R \otimes_{\mathbb{Z}} O_L)^\times$. We write $T_F^\Delta \cong T_F^2$ for the diagonal torus of G; so, writing T^Δ for the diagonal torus of $\mathrm{GL}(2)_{/O}$, $T_F^\Delta = \mathrm{Res}_{O/\mathbb{Z}} T^\Delta$.

Let $S_\kappa(\mathfrak{n}, \epsilon; \mathbb{C})$ denote the space of weight κ adelic Hilbert cusp forms $\mathbf{f} : G(\mathbb{Q})\backslash G(\mathbb{A}) \to \mathbb{C}$ of level $\mathfrak{n}$ with character ϵ modulo $\mathfrak{n}$, where $\mathfrak{n}$ is a non-zero ideal of O. Here the weight $\kappa = (\kappa_1, \kappa_2)$ is the Hodge weight of the rank 2 pure

H. Hida (✉)
Department of Mathematics, UCLA, Los Angeles, CA 90095-1555, USA
e-mail: hida@math.ucla.edu

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motive $M(\mathbf{f})$ with coefficient in the Hecke field $\mathbb{Q}(\mathbf{f})$ associated with any Hecke eigenform $\mathbf{f} \in S_\kappa(\mathfrak{n}, \epsilon; \mathbb{C})$ (see [BR93]). Though $M(\mathbf{f})$ is possibly defined over a quadratic extension F' of F (depending on $\mathbf{f}$), the Hodge weight is well defined over F independent of the infinity places over a given place of F. For each field embedding $\sigma : F \hookrightarrow \overline{\mathbb{Q}}$, taking an extension $\tilde{\sigma}$ of σ to F', $M(\mathbf{f}) \otimes_{F', \iota_\infty \circ \tilde{\sigma}} \mathbb{C}$ has Hodge weights $(\kappa_{1,\sigma}, \kappa_{2,\sigma})$ and $(\kappa_{2,\sigma}, \kappa_{1,\sigma})$, and the motivic weight $[\kappa] := \kappa_{1,\sigma} + \kappa_{2,\sigma}$ is independent of σ. We normalize the weight imposing an inequality $\kappa_{1,\sigma} \leq \kappa_{2,\sigma}$. This normalization is the one in [HMI, (SA1–3)]. Writing I (resp. I_p) for the set of all field embeddings into $\overline{\mathbb{Q}}$ (resp. p-adic places) of F, we identify κ_j with $\sum_{\sigma \in I} \kappa_{j,\sigma}\sigma \in \mathbb{Z}[I]$. Sometimes we identify I_p and I regarding I_p as a set of p-adic places induced by $i_p \circ \sigma$ for $\sigma \in I$. Often we use I to denote $\sum_\sigma \sigma \in \mathbb{Z}[I]$. If the Hodge weight is given by $\kappa = (0, kI)$ for an integer $k \geq 1$, traditionally, the integer $k+1$ is called the weight [of the cusp forms in $S_\kappa(\mathfrak{n}, \epsilon; \mathbb{C})$] at all σ, but we use here the Hodge weight κ.

The "Neben character" ϵ we use is again not a traditional one (but the one introduced in [HMI]). It is a set of three characters $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_+)$, where $\epsilon_+ : F_{\mathbb{A}}^\times / F^\times \to \mathbb{C}^\times$ is the central character of the automorphic representation $\pi_{\mathbf{f}}$ of $G(\mathbb{A})$ generated by any Hecke eigenform $0 \neq \mathbf{f} \in S_\kappa(\mathfrak{n}, \epsilon; \mathbb{C})$. The character ϵ_+ has infinity type $I - \kappa_1 - \kappa_2$, and therefore its finite part has values in $\overline{\mathbb{Q}}^\times$. The finite order characters ϵ_j are $\overline{\mathbb{Q}}$-valued continuous characters of $\hat{O}^\times = \varprojlim_{0<N\in\mathbb{Z}} (O/NO)^\times$ with $\epsilon_1\epsilon_2 = \epsilon_+|_{\hat{O}^\times}$. These characters ϵ_j $(j = 1, 2)$ factor through $(O/\mathfrak{N})^\times$ for an integral ideal $\mathfrak{N}$. The two given data $\{\epsilon_1, \epsilon_2\}$ are purely local and may not extend to Hecke characters of the idele class group $F_{\mathbb{A}}^\times / F^\times$. Put $\epsilon^- := \epsilon_1\epsilon_2^{-1}$, and we assume that ϵ^- factors through $(O/\mathfrak{n})^\times$; so, the conductor of ϵ^- is a factor of $\mathfrak{n}$ and $\mathfrak{N}$ (which could be a proper factor of $\mathfrak{n}$). Then for the level group

$$U = U_0(\mathfrak{n}) = \{u = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(\widehat{\mathbb{Z}}) \text{ with } c \in \hat{\mathfrak{n}} = \mathfrak{n}\widehat{O}\},$$

we have $\mathbf{f}(gu) = \epsilon(u)\mathbf{f}(g)$ for all $g \in G(\mathbb{A})$ and $u \in U$, where

$$\epsilon(u) = \epsilon_2(\det(u))\epsilon^-(a_{\mathfrak{n}}) = \epsilon_1(\det(u))(\epsilon^-)^{-1}(d_{\mathfrak{n}})$$

for the projection $d_{\mathfrak{n}}$ of d to $\prod_{\mathfrak{l}|\mathfrak{n}} F_{\mathfrak{l}}$. The characters ϵ_j for $j = 1, 2$ factor through $(O/\mathfrak{n}_j)^\times$ for some multiple $\mathfrak{n}_j$ of $\mathfrak{n}$ but we do not insist on $\mathfrak{n} = \mathfrak{n}_j$. As long as the local component $\pi_{\mathfrak{l}}$ of $\pi_{\mathbf{f}}$ at a prime $\mathfrak{l}|\mathfrak{n}_j|\mathfrak{N}$ is principal of the form $\pi(\alpha, \beta)$ or Steinberg of the form $\sigma(\alpha, \beta)$, we choose the data so that $\{\epsilon_1, \epsilon_2\} = \{\alpha|_{O_{\mathfrak{l}}^\times}, \beta|_{O_{\mathfrak{l}}^\times}\}$ (see [H89, Sect. 2]). In other words, for a suitable choice of (ϵ_1, ϵ_2), we have a unique minimal form $\mathbf{f}^\circ \in S_\kappa(\mathfrak{n}^\circ, \epsilon; \mathbb{C})$ in $\pi_{\mathbf{f}}$ with minimal level $\mathfrak{n}^\circ|\mathfrak{n}$. This minimal level $\mathfrak{n}^\circ$ of $\pi_{\mathbf{f}}$ is a factor of the conductor of $\pi_{\mathbf{f}}$ but could be a **proper** factor of it. These minimal forms are $\mathfrak{p}$-adically interpolated (the interpolation property is not always true for new forms). A detailed description of cusp forms in $S_\kappa(\mathfrak{n}, \epsilon; \mathbb{C})$ will be recalled in Sect. 1.9 from [HMI].

Throughout the paper, $\mathfrak{n}$ denotes an O-ideal prime to p, and we work with cusp forms of (minimal) level $\mathfrak{n}p^{r+I_p}$ ($r = \sum_{\mathfrak{p}\in I_p} r_\mathfrak{p}\mathfrak{p} \in \mathbb{Z}[I_p]$ with $r_\mathfrak{p} \geq 0$ and $p^{r+I_p} = \prod_{\mathfrak{p}|p} \mathfrak{p}^{r_\mathfrak{p}+1}$, symbolically). Extend ϵ_j to a character of the finite adele group $(F_\mathbb{A}^{(\infty)})^\times$ (trivial outside the level $\mathfrak{n}_j$ and trivial at a choice of uniformizer $\varpi_\mathfrak{l}$ at each prime $\mathfrak{l}$), and extend the character ϵ of U to the semi-group

$$\Delta_0(\mathfrak{n}) = \left\{\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in G(\mathbb{A}^{(\infty)}) \cap M_2(\widehat{O})\middle| d\widehat{O} + \hat{\mathfrak{n}} = \widehat{O},\ c \in \hat{\mathfrak{n}}\right\}$$

by $\epsilon\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) = \epsilon_1(ad-bc)(\epsilon^-)^{-1}(d_\mathfrak{n})$. The Hecke operator $T(y)$ of the double coset $U\left(\begin{smallmatrix} y & 0 \\ 0 & 1\end{smallmatrix}\right)U = \bigsqcup_\delta \delta U$ is defined by $\mathbf{f}|T(y)(g) = \sum_\delta \epsilon(\delta)^{-1}\mathbf{f}(g\delta)$ [see (14)]. For a Hecke eigenform $\mathbf{f}$, the eigenvalue $a(y,\mathbf{f})$ of $T(y)$ depends only on the ideal $\mathfrak{y} = y\widehat{O} \cap F$ [see (19)]; so, for each prime $\mathfrak{l}$ of F, we write $a(\mathfrak{l},\mathbf{f})$ for $a(\varpi_\mathfrak{l},\mathbf{f})$ and put $T(\mathfrak{l}) := T(\varpi_\mathfrak{l})$, choosing a prime element $\varpi_\mathfrak{l}$ of the $\mathfrak{l}$-adic completion $O_\mathfrak{l}$. Therefore the yth Fourier coefficient $c(y,\mathbf{f})$ of $\mathbf{f}$ is $\epsilon_1(y)a(y,\mathbf{f})$ for each Hecke eigenform $\mathbf{f}$ normalized so that $c(1,\mathbf{f}) = 1$, and the Fourier coefficient depends on y (if $\epsilon_1 \neq 1$) not just on the ideal $\mathfrak{y}$. For $\mathfrak{l}|\mathfrak{n}p$, we often write $U(\mathfrak{l})$ for $T(\mathfrak{l})$. For a Hecke eigenform $\mathbf{f} \in S_\kappa(\mathfrak{n}p^{r+I_p}, \epsilon; \mathbb{C})$ ($p \nmid \mathfrak{n}$) and a subfield H of $\overline{\mathbb{Q}}$, the Hecke field $H(\mathbf{f})$ inside $\mathbb{C}$ is generated over H by the eigenvalues $a(\mathfrak{l},\mathbf{f})$ of $\mathbf{f}$ for the Hecke operators $T(\mathfrak{l})$ for all prime ideals $\mathfrak{l}$ and the values of ϵ over finite ideles. If $H \subset \overline{\mathbb{Q}}$, then $H(\mathbf{f})$ is a finite extension of H sitting inside $\overline{\mathbb{Q}}$.

Let W be a sufficiently large discrete valuation ring flat over $\mathbb{Z}_p$. Let $\mathbf{\Gamma} \cong \mathbb{Z}_p^m$ ($m = [F:\mathbb{Q}]$) be the maximal torsion-free quotient of $O_p^\times$ for $O_p = O \otimes_\mathbb{Z} \mathbb{Z}_p$. We use this symbol $\mathbf{\Gamma}$ exclusively for the base totally real field F. Later in Sect. 1.12, for a CM quadratic extension M/F, we write $\mathbf{\Gamma}_M$ for the maximal p-profinite torsion free quotient of the anti-cyclotomic quotient of the ray class group $Cl_M(p^\infty) = \varprojlim_n Cl_M(p^n)$ of M modulo p^∞ (i.e., the projective limit of the ray class group $Cl_M(p^n)$ modulo p^n). Here the anti-cyclotomic quotient is the maximal quotient on which the generator c of $\mathrm{Gal}(M/F)$ acts by -1. Note that we have a natural inclusion: $\mathbf{\Gamma} \to \mathbf{\Gamma}_M$ but it could have finite cokernel. We fix once and for all a splitting of the projection: $O_p^\times \twoheadrightarrow \mathbf{\Gamma}$ and decompose $O_p^\times = \mathbf{\Gamma} \times \Delta$ for a finite group Δ.

We fix a $\mathbb{Z}_p$-basis $\{\gamma_j\}_{j=1,\dots,m} \subset \mathbf{\Gamma}$ so that $\mathbf{\Gamma} = \prod_j \gamma_j^{\mathbb{Z}_p}$ and identify the Iwasawa algebra $\mathbf{\Lambda} = \mathbf{\Lambda}_W := W[[\mathbf{\Gamma}]]$ with the power series ring $W[[T]]$ ($T = \{T_j\}_{j=1,\dots,m}$) by $\mathbf{\Gamma} \ni \gamma_j \mapsto t_j := (1+T_j) \in \mathbf{\Lambda}$. We have $W[[T]] = \varprojlim_n W[t,t^{-1}]/(t^{p^n}-1)$, where $t = (t_j)_j$, $t^{-1} = (t_j^{-1})_j$ and $(t^{p^n}-1)$ is the ideal $(t_1^{p^n}-1,\dots,t_m^{p^n}-1)$ in $W[[T]]$. In this way, we identify the formal spectrum $\mathrm{Spf}(\mathbf{\Lambda})$ with $\widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \mathbf{\Gamma}^*$ for $\mathbf{\Gamma}^* = \mathrm{Hom}_{\mathbb{Z}_p}(\mathbf{\Gamma},\mathbb{Z}_p)$, as t_j giving the character of $\mathbf{\Gamma}^*$ corresponding $t_j(\gamma_i^*) = \delta_{ij}$ for the dual basis $\{\gamma_j^*\}_j$ of $\{\gamma_j\}_j$. Here $\widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \mathbf{\Gamma}^*$ sends a local p-profinite ring R to the p-profinite group $(1+\mathfrak{m}_R) \otimes_{\mathbb{Z}_p} \mathbf{\Gamma}^*$ as a group functor (for the maximal ideal $\mathfrak{m}_R$ of R).

A p-adic nearly ordinary analytic family of eigenforms $\mathcal{F} = \{\mathbf{f}_P | P \in \mathrm{Spec}(\mathbb{I})(\mathbb{C}_p)\}$ is indexed by points of $\mathrm{Spec}(\mathbb{I})(\mathbb{C}_p)$, where $\mathrm{Spec}(\mathbb{I})$ is an irreducible component of the spectrum of the big nearly ordinary Hecke algebra $\mathbf{h}$ and is a torsion-free domain of finite rank over $\mathbf{\Lambda}$ (in this sense, we call $\mathrm{Spec}(\mathbb{I})$ a finite torsion-free covering of $\mathrm{Spec}(\mathbf{\Lambda})$). For each $P \in \mathrm{Spec}(\mathbb{I})(\mathbb{C}_p)$, $\mathbf{f}_P$ is a p-adic Hecke eigenform of slope 0 and level $\mathfrak{n}p^\infty$ for a fixed prime to p-level $\mathfrak{n}$. The family is called analytic because $P \mapsto a(y, \mathbf{f}_P)$ is a p-adic analytic function on the rigid analytic space associated with the formal spectrum $\mathrm{Spf}(\mathbb{I})$ in the sense of Berthelot (cf. [dJ95, Sect. 7], see also [dJ98]). We call $P \in \mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$ *arithmetic* of weight $\kappa = \kappa(P) \in \mathbb{Z}[I]^2$ with character $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_+)$ if $\kappa_{2,\sigma} - \kappa_{1,\sigma} \geq 1$ for all $\sigma \in I$ (we write this condition as $\kappa_2 - \kappa_1 \geq I$), $\epsilon_2|_{\mathbf{\Gamma}}$ has values in $\mu_{p^\infty}(\overline{\mathbb{Q}}_p)$ and $P(t_j - \epsilon_2^{-1}(\gamma_j)\gamma_j^{\kappa_2}) = 0$ for all j (regarding P as a W-algebra homomorphism $P : \mathbb{I} \to \overline{\mathbb{Q}}_p$). Here $\gamma^k = \prod_{\sigma \in I} \sigma(\gamma)^{k_\sigma}$ for $\gamma \in O_p$ and $k = \sum_{\sigma \in I} k_\sigma \sigma$, and $k \geq I$ means $k_\sigma \geq 1$ for all $\sigma \in I$. If P is arithmetic, $\mathbf{f}_P$ is a classical p-stabilized Hecke eigenform (not just a p-adic modular form). In order to make the introduction succinct, we put off, to Sect. 1.9, recalling the theory of analytic families of eigenforms including the definition and necessary properties of CM families. We only remark that each universal nearly ordinary family comes from an irreducible component $\mathrm{Spec}(\mathbb{I})$ of the spectrum $\mathrm{Spec}(\mathbf{h})$ of the big nearly ordinary Hecke algebra $\mathbf{h}$, and we assume now that $\mathrm{Spec}(\mathbb{I})$ is one of such irreducible components.

In this paper, we prove the following theorem.

Theorem. *Let* $\mathrm{Spec}(\mathbb{I})$ *be an irreducible (reduced) component of* $\mathrm{Spec}(\mathbf{h})$ *and* $K = \mathbb{Q}(\mu_{p^\infty})$. *Then* $\mathbb{I}$ *is a non-CM component if there exists a prime* $\mathfrak{l}$ *of* F *and an infinite set* $\mathcal{A}$ *of arithmetic points in* $\mathrm{Spec}(\mathbb{I})$ *of a fixed weight* κ *with* $\kappa_2 - \kappa_1 \geq I$ *such that*

$$\limsup_{P \in \mathcal{A}} [K(a(\mathfrak{l}, \mathbf{f}_P)) : K] = \infty.$$

Indeed, if $\mathbb{I}$ *is a CM component, the degree* $[K(\mathbf{f}_P) : K]$ *is bounded independently of arithmetic* P. *Conversely, if* $\mathbb{I}$ *is a non-CM component, there exists a set of primes* Ξ *of* F *with Dirichlet density one such that for any infinite set* $\mathcal{A}$ *of arithmetic points in* $\mathrm{Spec}(\mathbb{I})$ *of a fixed weight* κ *with* $\kappa_2 - \kappa_1 \geq I$*, we have*

$$\limsup_{P \in \mathcal{A}} [K(a(\mathfrak{l}, \mathbf{f}_P)) : K] = \infty \ \textit{for each } \mathfrak{l} \in \Xi.$$

In particular, for any bound $B > 0$*, the set of arithmetic primes* P *of given weight* κ *in* $\mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$ *with* $[K(\mathbf{f}_P) : K] < B$ *is finite for a non-CM component* $\mathbb{I}$.

The first assertion and the boundedness of $[K(\mathbf{f}_P) : K]$ (for a CM component $\mathbb{I}$) independently of arithmetic P follow from the construction of CM families in Sects. 1.12 and 1.13 (see [H11, Corollary 4.2] for the argument for $F = \mathbb{Q}$ which holds without modification for general F). We prove in this paper a slightly stronger statement than the converse in the theorem. The formulation of Theorem 3.1 is a bit different from the above theorem asserting that boundedness of $[K(a(\mathfrak{l}, \mathbf{f}_P)) : K]$ ($P \in \mathcal{A}$) over $\mathfrak{l} \in \Sigma$ implies that $\mathbb{I}$ is a CM component as long as Σ has positive upper density.

We could have written the assertion of the theorem as $\lim_{P\in\mathcal{A}}[K(a(\mathfrak{l},\mathbf{f}_P)):K]=\infty$ for the "limit" with respect to the filter of $\mathcal{A}$ given by the complement of all finite subsets of $\mathcal{A}$ instead of $\limsup_{P\in\mathcal{A}}[K(a(\mathfrak{l},\mathbf{f}_P)):K]=\infty$. Hereafter we use this filter and use $\lim_{P\in\mathcal{A}}$ instead of $\limsup_{P\in\mathcal{A}}$. In [H11] (and [H13]), we proved a similar result for $K[a(\mathfrak{p},\mathbf{f}_P)]$ for $\mathfrak{p}|p$. Here the point is to study the same phenomena for $a(\mathfrak{l},\mathbf{f}_P)$ for $\mathfrak{l}$ outside $\mathfrak{n}p$. Indeed, we proved in [H14] the above fact replacing the nearly ordinary Hecke algebra by the smaller cyclotomic ordinary Hecke algebra of one variable. The many variable rigidity lemma (see Lemma 4.1) enabled us to extend our result in [H14] to the many variable setting here. We expect that, assuming $\kappa_2-\kappa_1\geq I$,

$$\lim_{P\in\mathcal{A}}[K(a(\mathfrak{l},\mathbf{f}_P)):K]=\infty \textit{ for any single } \mathfrak{l}\nmid\mathfrak{n}p \textit{ if } \mathbb{I} \textit{ is a non CM component}$$

as in the case of $\mathfrak{p}|p$ (see Conjecture 3.5). As in [H11], the proof of the above theorem is based on the elementary finiteness of Weil l-numbers of given weight in any extension of $\mathbb{Q}(\mu_{p^\infty})$ of bounded degree up to multiplication by roots of unity and rigidity lemmas (in Sect. 4) asserting that a geometrically connected formal subscheme in a formal split torus stable under the (central) action $t\mapsto t^z$ of z in an open subgroup of $\mathbb{Z}_p^\times$ is a formal subtorus. Another key tool is the determination by Rajan [Rj98] of compatible systems by trace of Frobeniai for primes of positive density (up to character twists).

Infinite growth of the absolute degree of Hecke fields (under different assumptions) was proven in [Se97] for growing level N, and Serre's analytic method is now generalized to (almost) an optimal form to automorphic representations of classical groups by Shin and Templier [ST13]). Our proof is purely algebraic, and the degree we study is over the infinite cyclotomic field $\mathbb{Q}[\mu_{p^\infty}]$ (while the above papers use non-elementary analytic trace formulas and Plancherel measures in representation theory). Our result applies to any thin infinite set $\mathcal{A}$ of slope 0 non-CM cusp forms, while in [Se97] and [ST13], they studied the set of all automorphic representations of given infinity type (and given central character), growing the level. Note here the Zariski closure of $\mathcal{A}$ could be a transcendental formal subscheme of $\widehat{\mathbb{G}}_m\otimes\mathbf{\Gamma}^*$ relative to the rational structure coming from T_F and could have the smallest positive dimension 1, while $\dim\widehat{\mathbb{G}}_m\otimes\mathbf{\Gamma}^*=m=[F:\mathbb{Q}]$. Another distinction from earlier works is that we are now able to prove that the entire $\mathbb{I}$ has CM if the degrees $[K(\mathbf{f}_P):K]$ are bounded only over arithmetic points P of a possibly very small closed subscheme in $\mathrm{Spf}(\mathbb{I})$.

To state transcendence results of Hecke operators, let L/F be a finite field extension inside $\mathbb{C}_p$ with integer ring O_L, and look into the torus $T_L=\mathrm{Res}_{O_L/\mathbb{Z}}\mathbb{G}_m$. Write $\mathbb{Z}_{(p)}=\mathbb{Q}\cap\mathbb{Z}_p$ and $O_{L,(p)}=O_L\otimes_{\mathbb{Z}}\mathbb{Z}_{(p)}\subset L^\times$. Consider an algebraic homomorphism $\nu\in\mathrm{Hom}_{\text{gp scheme}}(T_L,T_F)$. We regard $\nu: T_L(\mathbb{Z}_p)=O_{L,p}^\times=(O_L\otimes_{\mathbb{Z}}\mathbb{Z}_p)^\times\to T_F(\overline{\mathbb{Q}}_p)\supset T_F(\mathbb{Z}_p)=O_p^\times$. Project $\nu(T_L(\mathbb{Z}_p))\cap T_F(\mathbb{Z}_p)\subset O_p^\times$ to the maximal torsion free quotient $\mathbf{\Gamma}$ of $O_p^\times$. As an example of $\mathbb{Q}_p$-rational ν (so, $\nu(O_{L,(p)})\subset T_F(\mathbb{Z}_p)=O_p^\times$), we have the norm character $N_{L/\mathbb{Q}}$ or, if L is a CM field with a p-adic CM type Φ (in the sense of [HT93]), $\nu:(L\otimes_{\mathbb{Q}}\mathbb{Q}_p)^\times\to\mathbb{Q}_p^\times$ given by

$\nu(\xi) = \prod_{\varphi\in\Phi} \xi^{\varphi}$. Define an integral domain $R = R_\nu$ to be the subalgebra of $\mathbf{\Lambda}_{\mathbb{Z}_p}$ generated over $\mathbb{Z}_{(p)}$ by the projected image G of $\nu(T_L(\mathbb{Z}_{(p)})) \cap O_p^\times$ in $\mathbf{\Gamma}$. Note that for any $\xi \in R_\nu$ and any arithmetic point P, $P(\xi) = \xi_P$ is in $L^{\mathrm{gal}}(\mu_N, \mu_{p^\infty})$ for the Galois closure L^{gal} of $L/\mathbb{Q}$ and for a sufficiently large $0 < N \in \mathbb{Z}$ for which μ_N receives all the values of characters of Δ (e.g., $N = |\Delta|$). The field $L^{\mathrm{gal}}(\mu_N, \mu_{p^\infty})$ is a finite extension of $\mathbb{Q}(\mu_{p^\infty})$. For an integral domain A, write $Q(A)$ for the quotient field of A. By definition, R_ν is isomorphic to the group algebra $\mathbb{Z}_{(p)}[G]$ of the torsion-free group G. Unless $G = \{1\}$, the quotient field $Q(R_\nu)$ has infinite transcendental degree over $\mathbb{Q}$.

If the family (associated with $\mathbb{I}$) contains a theta series of weight κ with $\kappa_2 - \kappa_1 \geq I$ of the norm form of a quadratic extension $M_{/F}$, M is a CM field, and all forms indexed by $\mathrm{Spec}(\mathbb{I})$ have CM by the same CM field M (see Sects. 1.12 and 1.13). In particular, the ring generated over $\mathbb{Z}_{(p)}$ by $a(\mathfrak{l})$ for primes $\mathfrak{l}$ of F in any CM component is a finite extension of R_ν taking $L = M$ for ν given by a CM type of M; so, the Hecke field has bounded degree over $\mathbb{Q}(\mu_{p^\infty})$ for any CM component. Fix an algebraic closure $\overline{Q}$ of the quotient field $Q = Q(\mathbf{\Lambda}_{\mathbb{Z}_p})$ of $\mathbf{\Lambda}_{\mathbb{Z}_p}$. We regard $\mathbb{I}$ as a subring of $\overline{Q}$. As a corollary of Theorem 3.1, we prove

Corollary I. *Let the notation be as above; in particular,* $\mathrm{Spec}(\mathbb{I})$ *is an irreducible (reduced) component of* $\mathrm{Spec}(\mathbf{h})$. *We regard* $\mathbb{I} \subset \overline{Q}$ *as* $\mathbf{\Lambda}$*-algebras and* $R_\nu \subset \mathbf{\Lambda} \subset \overline{Q}$. *Take a set* Σ *of prime ideals of* F *prime to* $p\mathfrak{n}$. *Suppose that* Σ *has positive upper density. If* $Q(R_\nu)[a(\mathfrak{l})] \subset \overline{Q}$ *for all* $\mathfrak{l} \in \Sigma$ *is a finite extension of* $Q(R_\nu)$ *for the quotient field* $Q(R_\nu)$ *of* R_ν*, then* $\mathbb{I}$ *is a component having complex multiplication by a CM quadratic extension* $M_{/F}$.

An obvious consequence of the above corollary is

Corollary II. *Let the notation be as in the above theorem. If* $\mathbb{I}$ *is a non-CM component, for a density one subset* Ξ *of primes of* F*, the subring* $Q(R_\nu)[a(\mathfrak{l})]$ *of* $\overline{Q}$ *for all* $\mathfrak{l} \in \Xi$ *has transcendental degree* 1 *over* $Q(R_\nu)$.

We could have made a conjecture on a mod p version of the above corollary as was done in [H14, Sect. 0], but we do not have an explicit application (as discussed in [H14]) to the Iwasawa μ-invariant of the generalized version; so, we do not formulate formally the obvious conjecture. We denote by a Gothic letter an ideal of a number field (in particular, any lowercase Gothic letter denotes an ideal of F). The corresponding Roman letter denotes the residual characteristic if a Gothic letter is used for a prime ideal. Adding superscript "(∞)", we indicate finite adeles; so, for example, $(F_{\mathbb{A}}^{(\infty)})^\times = \{x \in F_{\mathbb{A}}^\times | x_\infty = 1\}$. Similarly, $\mathbb{A}^{(p\infty)}$ is made of adeles without p and ∞-components.

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1 Hilbert Modular Forms

We recall the arithmetic theory of Hilbert modular forms limiting ourselves to what we need later. The purpose of giving fair detail of the moduli theoretic interpretation of Hilbert modular forms here is twofold: (1) to make this article essentially self-contained and (2) because most account of this theory was written before the publication of the paper of Deligne–Pappas [DP94] and because there seems no detailed account available explaining that the correction to the original treatment in [Rp78] does not affect much the theory of p-adic modular forms.

Though most results in this section are used implicitly in the rest of the paper, the author also thought that it would be good to give a summary of the theory as this conference participants are very diverse and some of the people are quite far from the author's area of research. The reader who is familiar with the theory can go directly to Sect. 1.13 where a characterization of CM components is given (which is essential to the sequel). We keep the notation used in the introduction.

1.1 Abelian Varieties with Real Multiplication

Put $O^* = \{x \in F | \mathrm{Tr}(xO) \subset \mathbb{Z}\}$ (which is the different inverse $\mathfrak{d}^{-1}$). Recall the level ideal $\mathfrak{n}$, and fix a fractional ideal $\mathfrak{c}$ of F prime to $p\mathfrak{n}$. We write A for a fixed base commutative algebra with identity, in which the absolute norm $N(\mathfrak{c})$ and the prime-to-p part of $N(\mathfrak{n})$ are invertible. To include the case where p ramifies in the base field F, we use the moduli problem of Deligne–Pappas in [DP94] to define Hilbert modular varieties. As explained in [Z14, Sects. 2 and 3], if p is unramified in F, the resulting p-integral model of the Hilbert modular Shimura variety is canonically isomorphic to the one defined by Rapoport [Rp78] and Kottwitz [Ko92] (see also [PAF, Chap. 4]). Writing $\mathfrak{c}_+$ for the monoid of totally positive elements in $\mathfrak{c}$, giving data $(\mathfrak{c}, \mathfrak{c}_+)$ is equivalent to fix a strict ideal class of $\mathfrak{c}$. The Hilbert modular variety $\mathfrak{M} = \mathfrak{M}(\mathfrak{c}; \mathfrak{n})$ of level $\mathfrak{n}$ classifies triples $(X, \Lambda, i)_{/S}$ formed by

- An abelian scheme $\pi : X \to S$ of relative dimension $m = [F : \mathbb{Q}]$ over an A-scheme S (for the fixed algebra A) with an embedding: $O \hookrightarrow \mathrm{End}(X_{/S})$;
- An O-linear polarization $X^t := \mathrm{Pic}^0_{X/S} \underset{\sim}{\xrightarrow{\Lambda}} X \otimes \mathfrak{c}$ inducing an isomorphism $(\mathfrak{c}, \mathfrak{c}_+) \cong (\mathrm{Hom}_S^{Sym}(X_{/S}, X^t_{/S}), \mathcal{P}(X, X^t_{/S}))$, where $\mathrm{Hom}_S^{Sym}(X_{/S}, X^t_{/S})$ is the O-module of symmetric O-linear homomorphisms and $\mathcal{P}(X, X^t_{/S}) \subset \mathrm{Hom}_S^{Sym}(X_{/S}, X^t_{/S})$ is the positive cone made up of O-linear polarizations;
- A closed O-linear immersion $i = i_{\mathfrak{n}} : (\mathbb{G}_m \otimes_{\mathbb{Z}} O^*)[\mathfrak{n}] \hookrightarrow X$ for the group $(\mathbb{G}_m \otimes_{\mathbb{Z}} O^*)[\mathfrak{n}]$ of $\mathfrak{n}$-torsion points of the multiplicative O-module scheme $\mathbb{G}_m \otimes_{\mathbb{Z}} O^*$.

By Λ, we identify the O-module $\mathrm{Hom}_S^{Sym}(X_{/S}, X^t_{/S})$ of symmetric O-linear homomorphisms inside $\mathrm{Hom}_S(X_{/S}, X^t_{/S})$ with $\mathfrak{c}$. Then we require that the (multiplicative) monoid of symmetric O-linear isogenies induced locally by ample invertible sheaves be identified with the set of totally positive elements $\mathfrak{c}_+ \subset \mathfrak{c}$. The quasi-projective scheme $\mathfrak{M} = \mathfrak{M}(\mathfrak{c};\mathfrak{n})_{/A}$ is the coarse moduli scheme of the following functor $\wp$ from the category of A-schemes into the category *SETS*:

$$\wp(S) = \left[(X, \Lambda, i)_{/S}\right],$$

where $[\ ] = \{\ \}/\cong$ is the set of isomorphism classes of the objects inside the brackets, and we say $(X, \Lambda, i) \cong (X', \Lambda', i')$ if we have an O-linear isomorphism $\phi : X_{/S} \to X'_{/S}$ such that $\Lambda' = \phi \circ \Lambda \circ \phi^t$ and $i'^* \circ \phi = i^* (\Leftrightarrow \phi \circ i = i')$. The scheme $\mathfrak{M}$ is a fine moduli scheme if $\mathfrak{n}$ is sufficiently deep (see [DP94]).

1.2 Geometric Hilbert Modular Forms

In the definition of the functor $\wp$ in Sect. 1.1, we could impose local $\mathcal{O}_S \otimes_{\mathbb{Z}} O$-freeness of the $\mathcal{O}_S \otimes_{\mathbb{Z}} O$-module $\pi_*(\Omega_{X/S})$ as was done by Rapoport in [Rp78]. We consider an open subfunctor $\wp^R$ of $\wp$ which is defined by imposing local freeness of $\pi_*(\Omega_{X/S})$ over $\mathcal{O}_S \otimes_{\mathbb{Z}} O$. Over $\mathbb{Z}[\frac{1}{D_F}]$ for the discriminant D_F of F, the two functors $\wp^R$ and $\wp$ coincide (see [DP94]). We write $\mathfrak{M}^R(\mathfrak{c};\mathfrak{n})$ for the open subscheme of $\mathfrak{M}(\mathfrak{c};\mathfrak{n})$ representing $\wp^R$. For ω with $\pi_*(\Omega_{X/S}) = (\mathcal{O}_S \otimes_{\mathbb{Z}} O)\omega$, we consider the functor classifying quadruples (X, Λ, i, ω):

$$\mathcal{Q}(S) = \left[(X, \Lambda, i, \omega)_{/S}\right].$$

We let $a \in T_F(S) = H^0(S, (\mathcal{O}_S \otimes_{\mathbb{Z}} O)^\times)$ act on $\mathcal{Q}(S)$ by $(X, \Lambda, i, \omega) \mapsto (X, \Lambda, i, a\omega)$. By this action, $\mathcal{Q}$ is a T_F-torsor over the open subfunctor $\wp^R$ of $\wp$; so, $\mathcal{Q}$ is representable by an A-scheme $\mathcal{M} = \mathcal{M}(\mathfrak{c};\mathfrak{n})$ affine over $\mathfrak{M}^R = \mathfrak{M}^R(\mathfrak{c};\mathfrak{n})_{/A}$. For each weight $k \in X^*(T_F) = \mathrm{Hom}_{\text{gp-sch}}(T_F, \mathbb{G}_m)$, if $F \neq \mathbb{Q}$, the k^{-1}-eigenspace of $H^0(\mathcal{M}_{/A}, \mathcal{O}_{\mathcal{M}/A})$ is the space of modular forms of weight k integral over a ring A. We write $G_k(\mathfrak{c},\mathfrak{n}; A)$ for this space of A-integral modular forms, which is an A-module of finite type. Thus $f \in G_k(\mathfrak{c},\mathfrak{n}; A)$ is a functorial rule (i.e., a natural transformation $f : \mathcal{Q} \to \mathbb{G}_a$) assigning a value in B to each isomorphism class of $(X, \Lambda, i, \omega)_{/B}$ (defined over an A-algebra B) satisfying the following four conditions:

(G0) the value f at every cusp is finite (see below for its precise meaning);
(G1) $f(X, \Lambda, i, \omega) \in B$ if (X, Λ, i, ω) is defined over B;
(G2) $f((X, \Lambda, i, \omega) \otimes_B B') = \rho(f(X, \Lambda, i, \omega))$ for each morphism $\rho : B_{/A} \to B'_{/A}$;
(G3) $f(X, \Lambda, i, a\omega) = k(a)^{-1} f(X, \Lambda, i, \omega)$ for $a \in T_F(B)$.

Strictly speaking, the condition (G0) is only necessary when $F = \mathbb{Q}$ by the Koecher principle (see below at the end of this subsection for more details). By abusing the language, we consider f to be a function of isomorphism classes of test objects $(X, \Lambda, i, \omega)_{/B}$ hereafter. The sheaf of k^{-1}-eigenspace $\mathcal{O}_{\mathcal{M}}[k^{-1}]$ under the action of T_F is an invertible sheaf on $\mathfrak{M}^R_{/A}$. We write this sheaf as $\underline{\omega}^k$ (imposing (G0) when $F = \mathbb{Q}$). Then we have

$$G_k(\mathfrak{c},\mathfrak{n}; A) \cong H^0(\mathfrak{M}^R(\mathfrak{c};\mathfrak{n})_{/A}, \underline{\omega}^k_{/A}) \text{ canonically,}$$

as long as $\mathfrak{M}^R(\mathfrak{c};\mathfrak{n})$ is a fine moduli space. Writing $\underline{\mathbb{X}} := (\mathbb{X}, \boldsymbol{\lambda}, \mathbf{i}, \boldsymbol{\omega})$ for the universal abelian scheme over $\mathfrak{M}^R$, $s = f(\underline{\mathbb{X}})\boldsymbol{\omega}^k$ gives rise to the section of $\underline{\omega}^k$. Conversely, for any section $s \in H^0(\mathfrak{M}^R(\mathfrak{c};\mathfrak{n}), \underline{\omega}^k)$, taking the unique morphism $\phi : \mathrm{Spec}(B) \to \mathfrak{M}^R$ such that $\phi^*\underline{\mathbb{X}} = \underline{X}$ for $\underline{X} := (X, \Lambda, i, \omega)_{/B}$, we can define $f \in G_k$ by $\phi^* s = f(\underline{X})\omega^k$.

We suppose that the fractional ideal $\mathfrak{c}$ is prime to $\mathfrak{n}p$, and take two ideals $\mathfrak{a}$ and $\mathfrak{b}$ prime to $\mathfrak{n}p$ such that $\mathfrak{a}\mathfrak{b}^{-1} = \mathfrak{c}$. To $(\mathfrak{a}, \mathfrak{b})$, we attach the Tate AVRM $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)$ defined over the completed group ring $\mathbb{Z}((\mathfrak{a}\mathfrak{b}))$ made of formal series $f(q) = \sum_{\xi \gg -\infty} a(\xi)q^{\xi}$ ($a(\xi) \in \mathbb{Z}$). Here ξ runs over all elements in $\mathfrak{a}\mathfrak{b}$, and there exists a positive integer n (dependent on f) such that $a(\xi) = 0$ if $\sigma(\xi) < -n$ for some $\sigma \in I$. We write $A[[(\mathfrak{a}\mathfrak{b})_{\geq 0}]]$ for the subring of $A[[\mathfrak{a}\mathfrak{b}]]$ made of formal series f with $a(\xi) = 0$ for all ξ with $\sigma(\xi) < 0$ for at least one embedding $\sigma : F \hookrightarrow \mathbb{R}$. Actually, we skipped a step of introducing the toroidal compactification of $\mathfrak{M}^R$ and $\mathfrak{M}$ (done in [Rp78] and [DP94]), and the universal abelian scheme over the moduli scheme degenerates to $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)$ over the spectrum of (completed) stalk at the cusp corresponding to $(\mathfrak{a}, \mathfrak{b})$. The toroidal compactification of the scheme $\mathfrak{M}^R_{/A}$ is proper normal by Deligne and Pappas [DP94] and hence by Zariski's connected theorem, it is geometrically connected. Since $\mathfrak{M}^R$ is open dense in each fiber of $\mathfrak{M}$ (as shown by Deligne and Pappas [DP94]), it is geometrically connected. Therefore the q-expansion principle holds for $H^0(\mathfrak{M}^R(\mathfrak{c};\mathfrak{n}), \underline{\omega}^k)$. We refer details of these facts to [K78, Chap. I], [C90, DT04, Di03, DP94] [HT93, Sect. 1] and [PAF, Sect. 4.1.4]. The scheme $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)$ can be extended to a semi-abelian scheme over $\mathbb{Z}[[(\mathfrak{a}\mathfrak{b})_{\geq 0}]]$ adding the fiber $\mathbb{G}_m \otimes \mathfrak{a}^*$ over the augmentation ideal $\mathfrak{A}$. Since $\mathfrak{a}$ is prime to p, $\mathfrak{a}_p = O_p$. Thus if A is a $\mathbb{Z}_p$-algebra, we have the identity: $A \otimes_{\mathbb{Z}} \mathfrak{a}^* = A \otimes_{\mathbb{Z}_p} \mathfrak{a}_p^* = A \otimes_{\mathbb{Z}_p} O_p^* = A \otimes_{\mathbb{Z}} O^*$, and we have a canonical isomorphism:

$$Lie(\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q) \mod \mathfrak{A}) = Lie(\mathbb{G}_m \otimes \mathfrak{a}^*) \cong A \otimes_{\mathbb{Z}} \mathfrak{a}^* = A \otimes_{\mathbb{Z}} O^*.$$

By duality, we have $\Omega_{\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)/A[[(\mathfrak{a}\mathfrak{b})_{\geq 0}]]} \cong A[[(\mathfrak{a}\mathfrak{b})_{\geq 0}]]$. Indeed we have a canonical generator ω_{can} of $\Omega_{\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)}$ induced by $\frac{dt}{t} \otimes 1$ on $\mathbb{G}_m \otimes \mathfrak{a}^*$. Since $\mathfrak{a}$ is prime to $\mathfrak{n}$, we have a canonical inclusion $(\mathbb{G}_m \otimes O^*)[\mathfrak{n}] = (\mathbb{G}_m \otimes \mathfrak{a}^*)[\mathfrak{n}]$ into $\mathbb{G}_m \otimes \mathfrak{a}^*$, which induces a canonical closed immersion $i_{can} : (\mathbb{G}_m \otimes O^*)[\mathfrak{n}] \hookrightarrow \mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)$. As described in [K78, (1.1.14)] and [HT93, p. 204], $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)$ has a canonical $\mathfrak{c}$-polarization Λ_{can}. Thus we can evaluate $f \in G_k(\mathfrak{c}, \mathfrak{n}; A)$ at $(\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q), \Lambda_{can}, i_{can}, \omega_{can})$. The value $f(q) = f_{\mathfrak{a},\mathfrak{b}}(q)$ actually falls in $A[[(\mathfrak{a}\mathfrak{b})_{\geq 0}]]$ (if $F \neq \mathbb{Q}$: Koecher principle) and is called the q-expansion at the cusp $(\mathfrak{a}, \mathfrak{b})$. Finiteness at cusps in the condition (G0) can be stated as

(G0′) $f_{\mathfrak{a},\mathfrak{b}}(q) \in A[[(\mathfrak{a}\mathfrak{b})_{\geq 0}]]$ for all $(\mathfrak{a}, \mathfrak{b})$.

1.3 p-Adic Hilbert Modular Forms of Level $\mathfrak{n}p^{\infty}$

Suppose that $A = \varprojlim_n A/p^n A$ (such a ring is called a p-adic ring) and that $\mathfrak{n}$ is prime to p. We consider a functor into sets

$$\widehat{\wp}(A) = \left[(X, \Lambda, i_p, i_{\mathfrak{n}})_{/S}\right]$$

defined over the category of p-adic A-algebras $B = \varprojlim_n B/p^n B$. An important point is that we consider an embedding of ind-group schemes $i_p : \mu_{p^\infty} \otimes_{\mathbb{Z}_p} O_p^* \hookrightarrow X[p^\infty]$ (in place of a differential ω), which induces $\widehat{\mathbb{G}}_m \otimes O_p^* \cong \widehat{X}$ for the formal completion $\widehat{X}$ along the identity section of the characteristic p-fiber of the abelian scheme X over A.

We call an AVRM X over a characteristic p ring A p-ordinary if the Barsotti–Tate group $X[p^\infty]$ is ordinary; in other words, its (Frobenius) Newton polygon has only two slopes 0 and 1. In the moduli space $\mathfrak{M}(\mathfrak{c};\mathfrak{n})_{/\overline{\mathbb{F}}_p}$, locally under Zariski topology, the p-ordinary locus is an open subscheme of $\mathfrak{M}(\mathfrak{c};\mathfrak{n})$. Indeed, the locus is obtained by inverting the Hasse invariant (over $\mathfrak{M}(\mathfrak{c};\mathfrak{n})_{/\overline{\mathbb{F}}_p}$). So, the p-ordinary locus inside $\mathfrak{M}^R(\mathfrak{c};\mathfrak{n})$ is open in $\mathfrak{M}^R(\mathfrak{c};\mathfrak{n})$. In the same way as was done by Deligne–Ribet and Katz for the level p^∞-structure, we can prove that this functor is representable by the formal completion $\widehat{\mathfrak{M}}^R(\mathfrak{c};\mathfrak{n})$ of $\mathfrak{M}^R(\mathfrak{c};\mathfrak{n})$ along the p-ordinary locus of the modulo p fiber (e.g., [PAF, Sect. 4.1.9]).

Take a character $k \in \mathbb{Z}[I]$. A p-adic modular form $f_{/A}$ over a p-adic ring A is a function (strictly speaking, a functorial rule) of isomorphism classes of $(X, \Lambda, i_p, i_{\mathfrak{n}})_{/B}$ ($i_{\mathfrak{n}} : \mathbb{G}_m \otimes_{\mathbb{Z}} O^*[\mathfrak{n}] \hookrightarrow X$) satisfying the following three conditions:

(P1) $f(X, \Lambda, i_p, i_{\mathfrak{n}}) \in B$ if $(X, \Lambda, i_p, i_{\mathfrak{n}})$ is defined over B;
(P2) $f((X, \Lambda, i_p, i_{\mathfrak{n}}) \otimes_B B') = \rho(f(X, \Lambda, i_p, i_{\mathfrak{n}}))$ for each continuous A-algebra homomorphism $\rho : B \to B'$;
(P3) $f_{\mathfrak{a},\mathfrak{b}}(q) \in A[[(\mathfrak{ab})_{\geq 0}]]$ for all $(\mathfrak{a}, \mathfrak{b})$ prime to $\mathfrak{n}p$.

We write $V(\mathfrak{c}, \mathfrak{n}p^\infty; A)$ for the space of p-adic modular forms satisfying (P1-3). By definition, this space $V(\mathfrak{c}, \mathfrak{n}p^\infty; A)$ is a p-adically complete A-algebra.

The q-expansion principle is valid both for classical modular forms and p-adic modular forms f:

(q-exp) *The q-expansion: $f \mapsto f_{\mathfrak{a},\mathfrak{b}}(q) \in A[[(\mathfrak{ab})_{\geq 0}]]$ determines f uniquely.*

This follows from the irreducibility of the level p^∞ Igusa tower, which was proven in [DR80] (see also [PAF, Sect. 4.2.4] for another argument).

Fix a generator d of O_p^*. Since $\widehat{\mathbb{G}}_m \otimes O^*$ has a canonical invariant differential $\frac{dt}{t} \otimes d$, we have $\omega_p = i_{p,*}(\frac{dt}{t} \otimes d)$ on $X_{/B}$ [under the notation of (P1–3)]. This allows us to regard $f \in G_k(\mathfrak{c}, \mathfrak{n}; A)$ as a p-adic modular form by

$$f(X, \Lambda, i_p, i_{\mathfrak{n}}) := f(X, \Lambda, i_{\mathfrak{n}}, \omega_p).$$

By (q-exp), this gives an injection of $G_k(\mathfrak{c}, \mathfrak{n}; A)$ into $V(\mathfrak{c}, \mathfrak{n}p^\infty; A)$ preserving q-expansions.

1.4 Complex Analytic Hilbert Modular Forms

Over $\mathbb{C}$, the category of test objects (X, Λ, i, ω) is equivalent to the category of triples $(\mathcal{L}, \Lambda, i)$ made of the following data (by the theory of theta functions): $\mathcal{L}$ is an O-lattice in $O \otimes_{\mathbb{Z}} \mathbb{C} = \mathbb{C}^I$, an alternating pairing $\Lambda : \mathcal{L} \wedge_O \mathcal{L} \cong \mathfrak{c}^*$ and $i : \mathfrak{n}^*/O^* \hookrightarrow F\mathcal{L}/\mathcal{L}$. The alternating form Λ is supposed to be positive in the sense that $\Lambda(u, v)/\operatorname{Im}(uv^c)$ is totally positive definite. The differential ω can be recovered by $\iota : X(\mathbb{C}) = \mathbb{C}^I/\mathcal{L}$ so that $\omega = \iota^* du$ where $u = (u_\sigma)_{\sigma \in I}$ is the variable on $\mathbb{C}^I$. Conversely

$$\mathcal{L}_X = \left\{ \int_\gamma \omega \in O \otimes_{\mathbb{Z}} \mathbb{C} \middle| \gamma \in H_1(X(\mathbb{C}), \mathbb{Z}) \right\}$$

is a lattice in $\mathbb{C}^I$, and the polarization $\Lambda : X^t \cong X \otimes \mathfrak{c}$ induces $\mathcal{L}_X \wedge \mathcal{L}_X \cong \mathfrak{c}^*$.

Using this equivalence, we can relate our geometric definition of Hilbert modular forms with the classical analytic definition. Define $\mathfrak{Z}$ by the product of I copies of the upper half complex plane $\mathfrak{H}$. We regard $\mathfrak{Z} \subset F \otimes_{\mathbb{Q}} \mathbb{C} = \mathbb{C}^I$. For each $z \in \mathfrak{Z}$, we define

$$\mathcal{L}_z = 2\pi\sqrt{-1}(\mathfrak{b}z + \mathfrak{a}^*),\ \Lambda_z(2\pi\sqrt{-1}(az+b), 2\pi\sqrt{-1}(cz+d)) = -(ad-bc) \in \mathfrak{c}^*$$

with $i_z : \mathfrak{n}^*/O^* \to \mathbb{C}^I/\mathcal{L}_z$ given by $i_z(a \mod O^*) = (2\pi\sqrt{-1}a \mod \mathcal{L}_z)$.

Consider the following congruence subgroup $\Gamma_1^1(\mathfrak{n}; \mathfrak{a}, \mathfrak{b})$ given by

$$\left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(F) \middle| a, d \in O,\ b \in (\mathfrak{a}\mathfrak{b})^*,\ c \in \mathfrak{n}\mathfrak{a}\mathfrak{b}\mathfrak{d} \text{ and } d - 1 \in \mathfrak{n} \right\}.$$

Write $\Gamma_1^1(\mathfrak{c}; \mathfrak{n}) = \Gamma_1^1(\mathfrak{n}; O, \mathfrak{c}^{-1})$. We let $g = (g_\sigma) \in SL_2(F \otimes_{\mathbb{Q}} \mathbb{R}) = SL_2(\mathbb{R})^I$ act on $\mathfrak{Z}$ by linear fractional transformation of g_σ on each component z_σ. It is easy to verify

$$(\mathcal{L}_z, \Lambda_z, i_z) \cong (\mathcal{L}_w, \Lambda_w, i_w) \iff w = \gamma(z) \text{ for } \gamma \in \Gamma_1^1(\mathfrak{n}; \mathfrak{a}, \mathfrak{b}).$$

The set of pairs $(\mathfrak{a}, \mathfrak{b})$ with $\mathfrak{a}\mathfrak{b}^{-1} = \mathfrak{c}$ is in bijection with the set of cusps (unramified over ∞) of $\Gamma_1^1(\mathfrak{n}; \mathfrak{a}, \mathfrak{b})$. Two cusps are equivalent if they transform to each other by an element in $\Gamma_1^1(\mathfrak{n}; \mathfrak{a}, \mathfrak{b})$. The standard choice of the cusp is $(O, \mathfrak{c}^{-1})$, which we call the infinity cusp of $\mathfrak{M}(\mathfrak{c}; \mathfrak{n})$. For each ideal $\mathfrak{t}$, $(\mathfrak{t}, \mathfrak{t}\mathfrak{c}^{-1})$ gives another cusp. The two cusps $(\mathfrak{t}, \mathfrak{t}\mathfrak{c}^{-1})$ and $(\mathfrak{s}, \mathfrak{s}\mathfrak{c}^{-1})$ are equivalent under $\Gamma_1^1(\mathfrak{c}; \mathfrak{n})$ if $\mathfrak{t} = \alpha\mathfrak{s}$ for an element $\alpha \in F^\times$ with $\alpha \equiv 1 \mod \mathfrak{n}$ in $F_{\mathfrak{n}}^\times$. We have

$$\mathfrak{M}(\mathfrak{c}; \mathfrak{n})(\mathbb{C}) \cong \Gamma_1^1(\mathfrak{c}; \mathfrak{n}) \backslash \mathfrak{Z}, \text{ canonically.}$$

Recall $G := \operatorname{Res}_{O/\mathbb{Z}} \mathrm{GL}(2)$. Take the following open compact subgroup of $G(\mathbb{A}^{(\infty)})$:

$$U_1^1(\mathfrak{n}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(\widehat{\mathbb{Z}}) \middle| c \in \mathfrak{n}\widehat{O} \text{ and } a \equiv d \equiv 1 \mod \mathfrak{n}\widehat{O} \right\},$$

and put $K = K_1^1(\mathfrak{n}) = \left(\begin{smallmatrix} d & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1} U_1^1(\mathfrak{n}) \left(\begin{smallmatrix} d & 0 \\ 0 & 1 \end{smallmatrix}\right)$ for an idele d with $d\widehat{O} = \widehat{\mathfrak{d}}$ and $d_{\mathfrak{d}} = 1$. Here for an idele and an O-ideal $\mathfrak{a} \neq 0$, we write $x_{\mathfrak{a}}$ for the projection of x to $\prod_{\mathfrak{l}|\mathfrak{a}} F_{\mathfrak{l}}^{\times}$ and $x^{(\mathfrak{a})} = xx_{\mathfrak{a}}^{-1}$. Then taking an idele c with $c\widehat{O} = \widehat{\mathfrak{c}}$ and $c_{\mathfrak{c}} = 1$, we see that

$$\Gamma_1^1(\mathfrak{c};\mathfrak{n}) \subset \left(\left(\begin{smallmatrix} c & 0 \\ 0 & 1 \end{smallmatrix}\right) K \left(\begin{smallmatrix} c & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1} \cap G(\mathbb{Q})_+\right)$$

for $G(\mathbb{Q})_+$ made up of all elements in $G(\mathbb{Q})$ with totally positive determinant. Choosing a complete representative set $\{c\} \subset F_{\mathbb{A}}^{\times}$ for the strict ray class group $Cl_F^+(\mathfrak{n})$ modulo $\mathfrak{n}$, we find by the approximation theorem that

$$G(\mathbb{A}) = \bigsqcup_{c \in Cl_F^+(\mathfrak{n})} G(\mathbb{Q}) \left(\begin{smallmatrix} c & 0 \\ 0 & 1 \end{smallmatrix}\right) K \cdot G(\mathbb{R})^+$$

for the identity connected component $G(\mathbb{R})^+$ of the Lie group $G(\mathbb{R})$. This shows

$$G(\mathbb{Q})\backslash G(\mathbb{A})/KC_{\mathbf{i}} \cong G(\mathbb{Q})_+\backslash G(\mathbb{A})_+/KC_{\mathbf{i}} \cong \bigsqcup_{\mathfrak{c} \in Cl_F^+(\mathfrak{n})} \mathfrak{M}(\mathfrak{c};\mathfrak{n})(\mathbb{C}), \tag{1}$$

where $G(\mathbb{A})_+ = G(\mathbb{A}^{(\infty)})G(\mathbb{R})^+$ and $C_{\mathbf{i}}$ is the stabilizer of $\mathbf{i} = (\sqrt{-1}, \dots, \sqrt{-1}) \in \mathfrak{Z}$ in $G(\mathbb{R})^+$. By (1), a $Cl_F^+(\mathfrak{n})$-tuple $(f_{\mathfrak{c}})_{\mathfrak{c}}$ with $f_{\mathfrak{c}} \in G_k(\mathfrak{c},\mathfrak{n};\mathbb{C})$ can be viewed as a single automorphic form defined on $G(\mathbb{A})$.

Recall the identification $X^*(T_F)$ with $\mathbb{Z}[I]$ so that $k(x) = \prod_\sigma \sigma(x)^{k_\sigma}$ for $k = \sum_\sigma k_\sigma \sigma \in \mathbb{Z}[I]$. Regarding $f \in G_k(\mathfrak{c},\mathfrak{n};\mathbb{C})$ as a function of $z \in \mathfrak{Z}$ by $f(z) = f(\mathcal{L}_z, \Lambda_z, i_z)$, it satisfies the following automorphic property:

$$f(\gamma(z)) = f(z) \prod_\sigma (c^\sigma z_\sigma + d^\sigma)^{k_\sigma} \text{ for all } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1^1(\mathfrak{c};\mathfrak{n}). \tag{2}$$

The holomorphy of f follows from the functoriality (G2). The function f has the Fourier expansion

$$f(z) = \sum_{\xi \in (\mathfrak{ab})_{\geq 0}} a(\xi)\mathbf{e}_F(\xi z)$$

at the cusp corresponding to $(\mathfrak{a},\mathfrak{b})$. Here $\mathbf{e}_F(\xi z) = \exp(2\pi\sqrt{-1}\sum_\sigma \xi^\sigma z_\sigma)$. This Fourier expansion gives the q-expansion $f_{\mathfrak{a},\mathfrak{b}}(q)$ substituting q^ξ for $\mathbf{e}_F(\xi z)$.

1.5 Γ_0*-Level Structure and Hecke Operators*

We now assume that the base algebra A is a W-algebra. Choose a prime $\mathfrak{q}$ of F. We are going to define Hecke operators $U(\mathfrak{q}^n)$ and $T(1,\mathfrak{q}^n)$ assuming for simplicity that $\mathfrak{q} \nmid p\mathfrak{n}$, though we may extend the definition to arbitrary $\mathfrak{q}$ (see [PAF, Sect. 4.1.10]).

Then $X[\mathfrak{q}^r]$ is an étale group scheme over B if X is an abelian scheme over an A-algebra B. We call a subgroup $C \subset X$ cyclic of order $\mathfrak{q}^r$ if $C \cong O/\mathfrak{q}^r$ over an étale faithfully flat extension of B.

We can think of quintuples $(X, \Lambda, i, C, \omega)_{/S}$ adding an additional information C of a cyclic subgroup scheme $C \subset X$ cyclic of order $\mathfrak{q}^r$. We define the space of classical modular forms $G_k(\mathfrak{c}, \mathfrak{n}, \Gamma_0(\mathfrak{q}^r); A)$ (resp. the space $V(\mathfrak{c}, \mathfrak{n}p^\infty, \Gamma_0(\mathfrak{q}^r); A)$ of p-adic modular forms) of prime-to-p level $(\mathfrak{n}, \Gamma_0(\mathfrak{q}^r))$ by (G0-3) [resp. (P1-3)] replacing test objects (X, Λ, i, ω) [resp. $(X, \Lambda, i_\mathfrak{n}, i_p)$] by $(X, \Lambda, i, C, \omega)$ [resp. $(X, \Lambda, i_\mathfrak{n}, C, i_p)$].

Our Hecke operators are defined on the space of prime-to-p level $(\mathfrak{n}, \Gamma_0(\mathfrak{q}^r))$. The operator $U(\mathfrak{q}^n)$ is defined only when $r > 0$ and $T(1, \mathfrak{q}^n)$ is defined only when $r = 0$. For a cyclic subgroup C' of $X_{/B}$ of order $\mathfrak{q}^n$, we can define the quotient abelian scheme X/C' with projection $\pi : X \to X/C'$. The polarization Λ and the differential ω induce a polarization $\pi_*\Lambda$ and a differential $(\pi^*)^{-1}\omega$ on X/C'. If $C' \cap C = \{0\}$ (in this case, we call that C' and C are *disjoint*), $\pi(C)$ gives rise to the level $\Gamma_0(\mathfrak{q}^r)$-structure on X/C'. Then we define $U(\mathfrak{q})$-operators acting on $f \in V(\mathfrak{c}\mathfrak{q}^n; \mathfrak{n}p^\infty, \Gamma_0(\mathfrak{q}^r); A)$ by

$$f|U(\mathfrak{q}^n)(X, \Lambda, C, i_\mathfrak{n}, C, i_p) = \frac{1}{N(\mathfrak{q}^n)} \sum_{C'} f(X/C', \pi_*\Lambda, \pi \circ i_\mathfrak{n}, \pi(C), \pi \circ i_p) \qquad (3)$$

where C' runs over all cyclic subgroups of order $\mathfrak{q}^n$ *disjoint* from C. Since $\pi_*\Lambda = \pi \circ \Lambda \circ \pi^t$ is a $\mathfrak{c}\mathfrak{q}^n$-polarization, the modular form f has to be defined for abelian varieties with $\mathfrak{c}\mathfrak{q}^n$-polarization.

As for $T(1, \mathfrak{q}^n)$, since $\mathfrak{q} \nmid \mathfrak{n}$, forgetting the $\Gamma_0(\mathfrak{q}^n)$-structure, we define $T(1, \mathfrak{q}^n)$ acting on $f \in V(\mathfrak{c}\mathfrak{q}^n; \mathfrak{n}p^\infty; A)$ by

$$f|T(1, \mathfrak{q}^n)(X, \Lambda, i_\mathfrak{n}, i_p) = \frac{1}{N(\mathfrak{q}^n)} \sum_{C'} f(X/C', \pi_*\Lambda, \pi \circ i_\mathfrak{n}, \pi \circ i_p) \text{ if } f \in V(A), \qquad (4)$$

where C' runs over all cyclic subgroups of order $\mathfrak{q}^n$. We check that $f|U(\mathfrak{q}^n)$ [resp. $T(1, \mathfrak{q}^n)$] belongs to $V(\mathfrak{c}\mathfrak{q}^n; \mathfrak{n}p^\infty, \Gamma_0(\mathfrak{q}^r); A)$ [resp. $V(\mathfrak{c}\mathfrak{q}^n; \mathfrak{n}p^\infty; A)$], and compatible with the natural inclusion $G_k(\mathfrak{c}, \mathfrak{n}, \Gamma_0(\mathfrak{q}^r); A) \hookrightarrow V(\mathfrak{c}\mathfrak{q}^n; \mathfrak{n}p^\infty, \Gamma_0(\mathfrak{q}^r); A)$ [resp. $G_k(\mathfrak{c}, \mathfrak{n}; A) \hookrightarrow V(\mathfrak{c}\mathfrak{q}^n; \mathfrak{n}p^\infty; A)$] defined at the end of Sect. 1.3; so, these Hecke operators preserve classicality. We have

$$U(\mathfrak{q}^n) = U(\mathfrak{q})^n.$$

1.6 Hilbert Modular Shimura Varieties

We extend the level structure i limited to $\mathfrak{n}$-torsion points to far bigger structure $\eta^{(p)}$ including all prime-to-p torsion points. Let $\mathbb{Z}_{(p)} = \mathbb{Q} \cap \mathbb{Z}_p$ (the localization of $\mathbb{Z}$ at (p)). Triples $(X, \overline{\Lambda}, \eta^{(p)})_{/S}$ for $\mathbb{Z}_{(p)}$-schemes S are classified by an integral model $Sh^{(p)}_{/\mathbb{Z}_{(p)}}$ (cf. [Ko92]) of the Shimura variety $Sh_{/\mathbb{Q}}$ associated with the algebraic $\mathbb{Z}_{(p)}$-group G (in the sense of Deligne [D71, 4.22] interpreting Shimura's original definition in [Sh70] as a moduli of abelian schemes up to isogenies). Here the classification is up to prime-to-p isogenies, and $\overline{\Lambda}$ is an equivalence class of polarizations up to multiplication by totally positive elements in F prime to p.

To give a description of the functor represented by $Sh^{(p)}$, we introduce some more notations. We consider the fiber category $\mathcal{A}^{(p)}_F$ over schemes defined by

(Object) abelian schemes X with real multiplication by O;
(Morphism) $\mathrm{Hom}_{\mathcal{A}^{(p)}_F}(X,Y) = \mathrm{Hom}(X,Y)\otimes_{\mathbb{Z}}\mathbb{Z}_{(p)}$.

Isomorphisms in this category are isogenies with degree prime to p (called "prime-to-p isogenies"), and hence the degree of polarization Λ is supposed to be also prime to p. Two polarizations are equivalent if $\Lambda = c\Lambda' = \Lambda'\circ i(c)$ for a totally positive c prime to p. We fix an O-lattice $L \subset V = F^2$ with O-hermitian alternating pairing $\langle\cdot,\cdot\rangle$ inducing a self-duality on $L_p = L\otimes_{\mathbb{Z}}\mathbb{Z}_p$.

For an open-compact subgroup K of $G(\mathbb{A}^{(\infty)})$ *maximal at p* (i.e., $K = G(\mathbb{Z}_p)\times K^{(p)}$), we consider the following functor from $\mathbb{Z}_{(p)}$-schemes into *SETS*:

$$\wp^{(p)}_K(S) = \Big[(X,\overline{\Lambda},\overline{\eta}^{(p)})_{/S} \text{ with (det)}\Big]. \tag{5}$$

Here $\overline{\eta}^{(p)} : L\otimes_{\mathbb{Z}}\mathbb{A}^{(p\infty)} \cong V^{(p)}(X) = T(X)\otimes_{\mathbb{Z}}\mathbb{A}^{(p\infty)}$ is an equivalence class of $\eta^{(p)}$ modulo multiplication $\eta^{(p)} \mapsto \eta^{(p)}\circ k$ by $k\in K^{(p)}$ for the Tate module $T(X) = \varprojlim_{\mathfrak{n}} X[\mathfrak{n}]$ (in the sheafified sense that $\eta^{(p)} \equiv (\eta')^{(p)} \mod K$ étale-locally), and a $\Lambda\in\overline{\Lambda}$ induces the self-duality on L_p. As long as $K^{(p)}$ is sufficiently small, $\wp^{(p)}_K$ is representable over any $\mathbb{Z}_{(p)}$-algebra A (cf. [Ko92, DP94] and [Z14, Sect. 3]) by a scheme $Sh_{K/A} = Sh/K$, which is smooth over $\mathrm{Spec}(\mathbb{Z}_{(p)})$ if p is unramified in $F_{/\mathbb{Q}}$ and singular if $p|D_F$ but is smooth outside a closed subscheme of codimension 2 in the p-fiber $Sh^{(p)}\times_{\mathbb{Z}_{(p)}}\mathbb{F}_p$ by the result of [DP94]. We let $g\in G(\mathbb{A}^{(p\infty)})$ act on $Sh^{(p)}_{/\mathbb{Z}_{(p)}}$ by

$$x = (X,\overline{\Lambda},\eta) \mapsto g(x) = (X,\overline{\Lambda},\eta\circ g),$$

which gives a right action of $G(\mathbb{A})$ on $Sh^{(p)}$ through the projection $G(\mathbb{A}) \twoheadrightarrow G(\mathbb{A}^{(p\infty)})$.

By the universality, we have a morphism $\mathfrak{M}(\mathfrak{c};\mathfrak{n}) \to Sh^{(p)}/\widehat{\Gamma}^1_1(\mathfrak{c};\mathfrak{n})$ for the open compact subgroup: $\widehat{\Gamma}^1_1(\mathfrak{c};\mathfrak{n}) = \left(\begin{smallmatrix} c & 0\\ 0 & 1\end{smallmatrix}\right) K^1_1(\mathfrak{n}) \left(\begin{smallmatrix} c & 0\\ 0 & 1\end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} cd^{-1} & 0\\ 0 & 1\end{smallmatrix}\right) U^1_1(\mathfrak{n}) \left(\begin{smallmatrix} cd^{-1} & 0\\ 0 & 1\end{smallmatrix}\right)^{-1}$ maximal at p. The image of $\mathfrak{M}(\mathfrak{c};\mathfrak{n})$ gives a geometrically irreducible component of $Sh^{(p)}/\widehat{\Gamma}^1_1(\mathfrak{c};\mathfrak{n})$. If $\mathfrak{n}$ is sufficiently deep, we can identify $\mathfrak{M}(\mathfrak{c};\mathfrak{n})$ with its image in $Sh^{(p)}/\widehat{\Gamma}^1_1(\mathfrak{c};\mathfrak{n})$. By the action on the polarization $\Lambda \mapsto \alpha\Lambda$ for a suitable totally positive $\alpha\in F$, we can bring $\mathfrak{M}(\mathfrak{c};\mathfrak{n})$ into $\mathfrak{M}(\alpha\mathfrak{c};\mathfrak{n})$; so, the image of $\varprojlim_{\mathfrak{n}}\mathfrak{M}(\mathfrak{c};\mathfrak{n})$ in $Sh^{(p)}$ only depends on the strict ideal class of $\mathfrak{c}$ in $\varprojlim_{\mathfrak{n}:\mathfrak{n}+(p)=O} Cl^+_F(\mathfrak{n})$.

1.7 Level Structure with "Neben" Character

In order to make a good link between classical modular forms and adelic automorphic forms (which we will describe in the following subsection), we would like to introduce "Neben" characters. We fix an integral ideal $\mathfrak{n}' \subset O$. We think of the following level structure on an AVRM X:

$$i : (\mathbb{G}_m \otimes O^*)[\mathfrak{n}'] \hookrightarrow X[\mathfrak{n}'] \text{ and } i' : X[\mathfrak{n}'] \twoheadrightarrow O/\mathfrak{n}', \tag{6}$$

where the sequence

$$1 \to (\mathbb{G}_m \otimes O^*)[\mathfrak{n}'] \xrightarrow{i} X[\mathfrak{n}'] \xrightarrow{i'} O/\mathfrak{n}' \to 0 \tag{7}$$

is exact and is required to induce a canonical duality between $(\mathbb{G}_m \otimes O^*)[\mathfrak{n}']$ and $O/\mathfrak{n}'$ under the polarization Λ. Here, if $\mathfrak{n}' = (N)$ for an integer $N > 0$, a canonical duality pairing

$$\langle \cdot, \cdot \rangle : (\mathbb{G}_m \otimes O^*)[N] \times O/N \to \mu_N$$

is given by $\langle \zeta \otimes \alpha, m \otimes \beta \rangle = \zeta^{m\mathrm{Tr}(\alpha\beta)}$ for $(\alpha, \beta) \in O^* \times O$ and $(\zeta, m) \in \mu_N \times \mathbb{Z}/N$ identifying $(\mathbb{G}_m \otimes O^*)[N] = \mu_N \otimes O^*$ and $O/N = (\mathbb{Z}/N\mathbb{Z}) \otimes_{\mathbb{Z}} O$. In general, taking an integer $0 < N \in \mathfrak{n}'$, the canonical pairing between $(\mathbb{G}_m \otimes O^*)[\mathfrak{n}']$ and $O/\mathfrak{n}'$ is induced by the one for (N) via the canonical inclusion $(\mathbb{G}_m \otimes O^*)[\mathfrak{n}'] \hookrightarrow (\mathbb{G}_m \otimes O^*)[N]$ and the quotient map $O/(N) \twoheadrightarrow O/\mathfrak{n}'$.

We fix two characters $\epsilon_1 : (O/\mathfrak{n}')^\times \to A^\times$ and $\epsilon_2 : (O/\mathfrak{n}')^\times \to A^\times$, and we insist for $f \in G_k(\mathfrak{c}, \mathfrak{n}; A)$ on the version of (G0-3) for quintuples $(X, \Lambda, i \cdot a, d \cdot i', \omega)$ and the equivariancy:

$$f(X, \overline{\Lambda}, i \cdot d, a \cdot i', \omega) = \epsilon_1(d)\epsilon_2(a) f(X, \overline{\Lambda}, i, i', \omega) \text{ for } a, d \in (O/\mathfrak{n})^\times. \tag{Neben}$$

Here the order $\epsilon_1(d)\epsilon_2(a)$ is correct as the diagonal matrix $\left(\begin{smallmatrix} d & 0 \\ 0 & a \end{smallmatrix}\right)$ in $T^\Delta(O/\mathfrak{n}') \subset \mathrm{GL}_2(O/\mathfrak{n}')$ acts on the quotient $O/\mathfrak{n}'$ by a and the submodule $(\mathbb{G}_m \otimes O^*)[\mathfrak{n}']$ by d. The ordering of ϵ_1, ϵ_2 is normalized with respect to the Galois representation local at p of f (when f is a p-ordinary Hecke eigenform so that ϵ_1 as a Galois character corresponds to the quotient character of the local Galois representation; see (Ram) in Sect. 1.11). Here $\overline{\Lambda}$ is the polarization class modulo equivalence relation given by multiplication by totally positive numbers in F prime to p. We write $G_k(\mathfrak{c}, \Gamma_0(\mathfrak{n}), \epsilon; A)$ $(\epsilon = (\epsilon_1, \epsilon_2))$ for the A-module of geometric modular forms satisfying these conditions.

1.8 Adelic Hilbert Modular Forms

Let us interpret what we have said so far in automorphic language and give a definition of the adelic Hilbert modular forms and their Hecke algebra of level $\mathfrak{n}$ (cf. [H96, Sects. 2.2–4] and [PAF, Sects. 4.2.8–4.2.12]).

We consider the following open compact subgroup of $G(\mathbb{A}^{(\infty)})$:

$$U_0(\mathfrak{n}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(\widehat{\mathbb{Z}}) \middle| c \equiv 0 \mod \mathfrak{n}\widehat{O} \right\},$$
$$U_1^1(\mathfrak{n}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in U_0(\mathfrak{n}) \middle| a \equiv d \equiv 1 \mod \mathfrak{n}\widehat{O} \right\}, \tag{8}$$

where $\widehat{O} = O \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ and $\widehat{\mathbb{Z}} = \prod_\ell \mathbb{Z}_\ell$. Then we introduce the following semi-group

$$\Delta_0(\mathfrak{n}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(\mathbb{A}^{(\infty)}) \cap M_2(\widehat{O}) \middle| c \equiv 0 \mod \mathfrak{n}\widehat{O}, d_{\mathfrak{n}} \in O_{\mathfrak{n}}^\times \right\}, \tag{9}$$

where $d_{\mathfrak{n}}$ is the projection of $d \in \widehat{O}$ to $O_{\mathfrak{n}} := \prod_{\mathfrak{q}|\mathfrak{n}} O_{\mathfrak{q}}$ for prime ideals $\mathfrak{q}$. Recall the maximal diagonal torus T^Δ of $\mathrm{GL}(2)_{/O}$. Putting

$$D_0 = \left\{ \mathrm{diag}[a, d] = \left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) \in T^\Delta(F_{\mathbb{A}^{(\infty)}}) \cap M_2(\widehat{O}) \middle| d_{\mathfrak{n}} = 1 \right\}, \tag{10}$$

we have (e.g., [MFG, 3.1.6] and [PAF, Sect. 5.1])

$$\Delta_0(\mathfrak{n}) = U_0(\mathfrak{n}) D_0 U_0(\mathfrak{n}). \tag{11}$$

In this section, the group U is assumed to be a subgroup of $U_0(\mathfrak{n}p^\alpha)$ with $U \supset U_1^1(\mathfrak{n}p^\alpha)$ for some $0 < \alpha \le \infty$. Formal finite linear combinations $\sum_\delta c_\delta U\delta U$ of double cosets of U in $\Delta_0(\mathfrak{n}p^\alpha)$ form a ring $R(U, \Delta_0(\mathfrak{n}p^\alpha))$ under convolution product (see [IAT, Chap. 3] or [MFG, Sect. 3.1.6]). Recall the prime element $\varpi_{\mathfrak{q}}$ of $O_{\mathfrak{q}}$ for each prime $\mathfrak{q}$ fixed in the introduction. The algebra is commutative and is isomorphic to the polynomial ring over the group algebra $\mathbb{Z}[U_0(\mathfrak{n}p^\alpha)/U]$ with variables $\{T(\mathfrak{q}), T(\mathfrak{q}, \mathfrak{q})\}_{\mathfrak{q}}$. Here $T(\mathfrak{q})$ (resp. $T(\mathfrak{q}, \mathfrak{q})$ for primes $\mathfrak{q} \nmid \mathfrak{n}p^\alpha$) corresponds to the double coset $U \left(\begin{smallmatrix} \varpi_{\mathfrak{q}} & 0 \\ 0 & 1 \end{smallmatrix}\right) U$ (resp. $U_0 \varpi_{\mathfrak{q}} U$). The group element $u \in U_0(\mathfrak{n}p^\alpha)/U$ in the group algebra $\mathbb{Z}[U_0(\mathfrak{n}p^\alpha)/U]$ corresponds to the double coset UuU (cf. [H95, Sect. 2]).

As in the introduction, we extend ϵ_j to a character of $(F_{\mathbb{A}}^{(\infty)})^\times \subset \widehat{O}^\times \times \prod_{\mathfrak{q}} \varpi_{\mathfrak{q}}^{\mathbb{Z}}$ trivial on the factor $\prod_{\mathfrak{q}} \varpi_{\mathfrak{q}}^{\mathbb{Z}}$, and denote the extended character by the same symbol ϵ_j. In [HMI, (ex0–3)], ϵ_2 is extended as above, but the extension of ϵ_1 taken there is to keep the identity $\epsilon_+ = \epsilon_1\epsilon_2$ over $(F_{\mathbb{A}}^{(\infty)})^\times$. The present extension is more convenient in this paper.

The double coset ring $R(U, \Delta_0(\mathfrak{n}p^\alpha))$ naturally acts on the space of modular forms on U. We now recall the action (which is a slight simplification of the action of $[UxU]$ given in [HMI, (2.3.14)]). Recall the diagonal torus T^Δ of $\mathrm{GL}(2)_{/O}$; so,

$T^{\Delta} = \mathbb{G}^2_{m/O}$. Since $T^{\Delta}(O/\mathfrak{n}')$ is canonically a quotient of $U_0(\mathfrak{n}')$ for an ideal $\mathfrak{n}'$, a character $\epsilon : T^{\Delta}(O/\mathfrak{n}') \to \mathbb{C}^{\times}$ can be considered as a character of $U_0(\mathfrak{n}')$. If ϵ_j is defined modulo $\mathfrak{n}_j$, we can take $\mathfrak{n}'$ to be any multiple of $\mathfrak{n}_1 \cap \mathfrak{n}_2$. Writing $\epsilon\left(\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)\right) = \epsilon_1(a)\epsilon_2(d)$, if $\epsilon^- = \epsilon_1\epsilon_2^{-1}$ factors through $(O/\mathfrak{n})^{\times}$ for for an ideal $\mathfrak{n}|\mathfrak{n}'$, then we can extend the character ϵ of $U_0(\mathfrak{n}')$ to $\Delta_0(\mathfrak{n})$ by putting $\epsilon(\delta) = \epsilon_1(\det(\delta))(\epsilon^-)^{-1}(d_{\mathfrak{n}})$ for $\delta = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Delta_0(\mathfrak{n})$ (as before). In this sense, we hereafter assume that ϵ is defined modulo $\mathfrak{n}$ and regard ϵ as a character of the group $U_0(\mathfrak{n})$ and the semi-group $\Delta_0(\mathfrak{n})$. Recall that $\epsilon_+ : F^{\times}_{\mathbb{A}} \to \mathbb{C}^{\times}$ is a Hecke character trivial on $F^{\times}$ with infinity type $(1 - [\kappa])I$ (for an integer $[\kappa]$) such that $\epsilon_+(z) = \epsilon_1(z)\epsilon_2(z)$ for $z \in \widehat{O}^{\times}$.

Recall the set I of all embeddings of F into $\overline{\mathbb{Q}}$ and T^{Δ}_F for $\mathrm{Res}_{O/\mathbb{Z}}T^{\Delta}$ (the diagonal torus of G). Then the group of geometric characters $X^*(T^{\Delta}_F)$ is isomorphic to $\mathbb{Z}[I]^2$ so that $(m, n) \in \mathbb{Z}[I]^2$ send $\mathrm{diag}[x, y] \in T^{\Delta}_F$ to $x^m y^n = \prod_{\sigma\in I}(\sigma(x)^{m_\sigma}\sigma(y)^{n_\sigma})$. Taking $\kappa = (\kappa_1, \kappa_2) \in \mathbb{Z}[I]^2$, we assume $[\kappa]I = \kappa_1 + \kappa_2$, and we associate with κ a factor of automorphy:

$$J_\kappa(g, \tau) = \det(g_\infty)^{\kappa_1 - I} j(g_\infty, \tau)^{\kappa_2-\kappa_1+I} \quad \text{for } g \in G(\mathbb{A}) \text{ and } \tau \in \mathfrak{Z}. \tag{12}$$

We define $S_\kappa(U, \epsilon; \mathbb{C})$ for an open subgroup $U \subset U_0(\mathfrak{n})$ by the space of functions $\mathbf{f} : G(\mathbb{A}) \to \mathbb{C}$ satisfying the following three conditions (e.g., [HMI, (SA1–3)] and [PAF, Sect. 4.3.1]):

(S1) $\mathbf{f}(\alpha x u z) = \epsilon(u)\epsilon_+(z)\mathbf{f}(x)J_\kappa(u, \mathbf{i})^{-1}$ for $\alpha \in G(\mathbb{Q})$, $u \in U \cdot C_{\mathbf{i}}$ and $z \in Z(\mathbb{A})$.
(S2) Choose $u \in G(\mathbb{R})$ with $u(\mathbf{i}) = \tau$ for $\tau \in \mathfrak{Z}$, and put $\mathbf{f}_x(\tau) = \mathbf{f}(xu)J_\kappa(u, \mathbf{i})$ for each $x \in G(\mathbb{A}^{(\infty)})$ (which only depends on τ). Then $\mathbf{f}_x$ is a holomorphic function on $\mathfrak{Z}$ for all x.
(S3) $\mathbf{f}_x(\tau)$ for each x is rapidly decreasing as $\eta_\sigma \to \infty$ ($\tau = \xi + \mathbf{i}\eta$) for all $\sigma \in I$ uniformly.

If we replace the expression "rapidly decreasing" in (S3) by "slowly increasing," we get the definition of the space $G_\kappa(U, \epsilon; \mathbb{C})$. It is easy to check (e.g., [HMI, (2.3.5)] that the function $\mathbf{f}_x$ in (S2) satisfies

$$f(\gamma(\tau)) = \epsilon^{-1}(x^{-1}\gamma x)f(\tau)J_\kappa(\gamma, \tau) \quad \text{for all } \gamma \in \Gamma_x(U), \tag{13}$$

where $\Gamma_x(U) = xUx^{-1}G(\mathbb{R})^+ \cap G(\mathbb{Q})$. Also by (S3), $\mathbf{f}_x$ is rapidly decreasing towards all cusps of Γ_x; so, it is a cusp form. If we restrict $\mathbf{f}$ as above to $SL_2(F_{\mathbb{A}})$, the determinant factor $\det(g)^{\kappa_1 - I}$ in the factor $J_\kappa(g, \tau)$ disappears, and the automorphy factor becomes only dependent on $k = \kappa_2 - \kappa_1 + I \in \mathbb{Z}[I]$; so, the classical modular form in G_k has single digit weight $k \in \mathbb{Z}[I]$. Via (1), we have an embedding of $S_\kappa(U_0(\mathfrak{n}'), \epsilon; \mathbb{C})$ into $G_k(\Gamma_0(\mathfrak{n}'), \epsilon; \mathbb{C}) = \bigoplus_{[\mathfrak{c}]\in Cl^+_F} G_k(\mathfrak{c}, \Gamma_0(\mathfrak{n}'), \epsilon; \mathbb{C})$ ($\mathfrak{c}$ running over a complete representative set prime to $\mathfrak{n}'$ for the strict ideal class group Cl^+_F) bringing $\mathbf{f}$ into $(\mathbf{f}_{\mathfrak{c}})_{[\mathfrak{c}]}$ for $\mathbf{f}_{\mathfrak{c}} = \mathbf{f}_x$ [as in (S3)] with $x = \left(\begin{smallmatrix} cd^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)$ (for $d \in F^{\times}_{\mathbb{A}}$ with $d\widehat{O} = \widehat{\mathfrak{d}}$). The cusp form $\mathbf{f}_{\mathfrak{c}}$ is determined by the restriction of $\mathbf{f}$ to $x \cdot SL_2(F_{\mathbb{A}})$. Though in (13), ϵ^{-1} shows up, the Neben character of the direct factor $G_k(\mathfrak{c}, \Gamma_0(\mathfrak{n}'), \epsilon; \mathbb{C})$ is given by ϵ, since in (Neben), the order of (a, d) is reversed to have $\epsilon_1(d)\epsilon_2(a)$. If we vary the weight κ keeping $k = \kappa_2 - \kappa_1 + I$, the image of S_κ in $G_k(\Gamma_0(\mathfrak{n}'), \epsilon; \mathbb{C})$ transforms accordingly. By this identification, the Hecke operator $T(\mathfrak{q})$ for non-principal $\mathfrak{q}$ makes sense as an operator acting on a single space $G_\kappa(U, \epsilon; \mathbb{C})$, and its action depends on the choice of κ.

It is known that $G_\kappa = 0$ unless $\kappa_1 + \kappa_2 = [\kappa_1 + \kappa_2]I$ for $[\kappa_1 + \kappa_2] \in \mathbb{Z}$, because $I - (\kappa_1 + \kappa_2)$ is the infinity type of the central character of automorphic representations generated by G_κ. We write simply $[\kappa]$ for $[\kappa_1 + \kappa_2] \in \mathbb{Z}$ assuming $G_\kappa \neq 0$. The $SL(2)$-weight of the central character of an irreducible automorphic representation π generated by $\mathbf{f} \in G_\kappa(U, \epsilon; \mathbb{C})$ is given by k (which specifies the infinity type of π_∞ as a discrete series representation of $SL_2(F_\mathbb{R})$).

In the introduction, we have extended ϵ_j to $(F_\mathbb{A}^{(\infty)})^\times$ and ϵ to $\Delta_0(\mathfrak{n})$ (as long as ϵ^- is defined modulo $\mathfrak{n}$), and we have $\epsilon(\delta) = \epsilon_1(\det(\delta))(\epsilon^-)^{-1}(d_\mathfrak{n})$ for $\delta = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Delta_0(\mathfrak{n})$. Let $\mathcal{U}$ be the unipotent algebraic subgroup of $\mathrm{GL}(2)_{/O}$ defined by $\mathcal{U}(A) = \left\{\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right) \middle| a \in A\right\}$. Note here that $\mathcal{U}(\widehat{O}) \subset \mathrm{Ker}(\epsilon)$; so, $\epsilon(tu) = \epsilon(t)$ if $t \in D_0$ and $u \in \mathcal{U}(\widehat{O})$. For each $UyU \in R(U, \Delta_0(\mathfrak{n}p^\alpha))$, we decompose $UyU = \bigsqcup_{t \in D_0, u \in \mathcal{U}(\widehat{O})} utU$ for finitely many u and t (see [IAT, Chap. 3] or [MFG, Sect. 3.1.6]) and define

$$\mathbf{f}|[UyU](x) = \sum_{t,u} \epsilon(t)^{-1}\mathbf{f}(xut). \tag{14}$$

We check that this operator preserves the spaces of automorphic forms: $G_\kappa(\mathfrak{n}, \epsilon; \mathbb{C})$ and $S_\kappa(\mathfrak{n}, \epsilon; \mathbb{C})$, and depends only on UyU not the choice of y as long as $y \in D_0$. However it depends on the choice of $\varpi_\mathfrak{q}$ as the character ϵ (extended to $\Delta_0(\mathfrak{n})$) depends on $\varpi_\mathfrak{q}$. This action for y with $y_\mathfrak{n} = 1$ is independent of the choice of the extension of ϵ to $T^\Delta(F_\mathbb{A})$. When $y_\mathfrak{n} \neq 1$, we may assume that $y_\mathfrak{n} \in D_0 \subset T^\Delta(F_\mathbb{A})$, and in this case, t can be chosen so that $t_\mathfrak{n} = y_\mathfrak{n}$ (so $t_\mathfrak{n}$ is independent of single right cosets in the double coset). If we extend ϵ to $T^\Delta(F_\mathbb{A}^{(\infty)})$ by choosing another prime element $\varpi'_\mathfrak{q}$ and write the extension as ϵ', then we have

$$\epsilon(t_\mathfrak{n})[UyU] = \epsilon'(t_\mathfrak{n})[UyU]',$$

where the operator on the right-hand side is defined with respect to ϵ'. Thus the sole difference is the root of unity $\epsilon(t_\mathfrak{n})/\epsilon'(t_\mathfrak{n}) \in \mathrm{Im}(\epsilon|_{T^\Delta(O/\mathfrak{n}')})$. Since it depends on the choice of $\varpi_\mathfrak{q}$, we make the choice once and for all, and write $T(\mathfrak{q})$ for $\left[U\left(\begin{smallmatrix} \varpi_\mathfrak{q} & 0 \\ 0 & 1 \end{smallmatrix}\right)U\right]$ (if $\mathfrak{q} \nmid \mathfrak{n}$), which coincides with $T(1, \mathfrak{q})$ in (4) if $\mathfrak{q} \nmid \mathfrak{n}'$. By linearity, these actions of double cosets extend to the ring action of the double coset ring $R(U, \Delta_0(\mathfrak{n}p^\alpha))$.

To introduce rationality of modular forms, we recall Fourier expansion of adelic modular forms (cf. [HMI, Proposition 2.26]). Recall the embedding $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, and identify $\overline{\mathbb{Q}}$ with the image of ι_∞. Recall also the differential idele $d \in F_\mathbb{A}^\times$ with $d^{(\mathfrak{d})} = 1$ and $d\widehat{O} = \mathfrak{d}\widehat{O}$. Each member $\mathbf{f}$ of $S_\kappa(U, \epsilon; \mathbb{C})$ has its Fourier expansion:

$$\mathbf{f}\left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right) = |y|_\mathbb{A} \sum_{0 \ll \xi \in F} c(\xi y d, \mathbf{f})(\xi y_\infty)^{-\kappa_1}\mathbf{e}_F(i\xi y_\infty)\mathbf{e}_F(\xi x), \tag{15}$$

where $\mathbf{e}_F : F_{\mathbb{A}}/F \to \mathbb{C}^\times$ is the additive character with $\mathbf{e}_F(x_\infty) = \exp(2\pi i \sum_{\sigma\in I} x_\sigma)$ for $x_\infty = (x_\sigma)_\sigma \in \mathbb{R}^I = F \otimes_{\mathbb{Q}} \mathbb{R}$. Here $y \mapsto c(y, \mathbf{f})$ is a function defined on $y \in F_{\mathbb{A}}^\times$ only depending on its finite part $y^{(\infty)}$. The function $c(y, \mathbf{f})$ is supported by the set $(\widehat{O} \times F_\infty) \cap F_{\mathbb{A}}^\times$ of *integral* ideles.

Let $F[\kappa]$ be the field fixed by $\{\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F) | \kappa\sigma = \kappa\}$, over which the character $\kappa \in X^*(T_F^\Delta)$ is rational. Write $O[\kappa]$ for the integer ring of $F[\kappa]$. We also define $O[\kappa, \epsilon]$ for the integer ring of the field $F[\kappa, \epsilon]$ generated by the values of ϵ over $F[\kappa]$. For any $F[\kappa, \epsilon]$-algebra A inside $\mathbb{C}$, we define

$$S_\kappa(U, \epsilon; A) = \left\{\mathbf{f} \in S_\kappa(U, \epsilon; \mathbb{C}) \middle| c(y, \mathbf{f}) \in A \text{ as long as } y \text{ is integral}\right\}. \tag{16}$$

As we have seen, we can interpret $S_\kappa(U, \epsilon; A)$ as the space of A-rational global sections of a line bundle of a variety defined over A; so, by the flat base-change theorem (e.g., [GME, Lemma 1.10.2]),

$$S_\kappa(\mathfrak{n}, \epsilon; A) \otimes_A \mathbb{C} = S_\kappa(\mathfrak{n}, \epsilon; \mathbb{C}). \tag{17}$$

The Hecke operators preserve A-rational modular forms (cf. (23) below). We define the Hecke algebra $h_\kappa(U, \epsilon; A) \subset \mathrm{End}_A(S_\kappa(U, \epsilon; A))$ by the A-subalgebra generated by the Hecke operators of $R(U, \Delta_0(\mathfrak{n}p^\alpha))$. Thus for any $\overline{\mathbb{Q}}_p$-algebras A, we may consistently define

$$S_\kappa(U, \epsilon; A) = S_\kappa(U, \epsilon; \overline{\mathbb{Q}}) \otimes_{\overline{\mathbb{Q}}, \iota_p} A. \tag{18}$$

By linearity, $y \mapsto c(y, \mathbf{f})$ extends to a function on $F_{\mathbb{A}}^\times \times S_\kappa(U, \epsilon; A)$ with values in A. For $u \in \widehat{O}^\times$, we know from [HMI, Proposition 2.26]

$$c(yu, \mathbf{f}) = \epsilon_1(u) c(y, \mathbf{f}). \tag{19}$$

If $\mathbf{f}$ is a normalized Hecke eigenform, its eigenvalue $a(y, \mathbf{f})$ of $T(y)$ is given by $\epsilon_1(y)^{-1} c(y, \mathbf{f})$ which depends only on the ideal $\mathfrak{y} := y\widehat{O} \cap F$ by the above formula as claimed in the introduction. We define the q-expansion coefficients (at p) of $\mathbf{f} \in S_\kappa(U, \epsilon; A)$ by

$$\mathbf{c}_p(y, \mathbf{f}) = y_p^{-\kappa_1} c(y, \mathbf{f}). \tag{20}$$

The formal q-expansion of an A-rational $\mathbf{f}$ has values in the space of functions on $F_{\mathbb{A}^{(\infty)}}^\times$ with values in the formal monoid algebra $A[[q^\xi]]_{\xi \in F_+}$ of the multiplicative semi-group F_+ made up of totally positive elements, which is given by

$$\mathbf{f}(y) = \mathcal{N}(y)^{-1} \sum_{\xi \gg 0} \mathbf{c}_p(\xi y d, \mathbf{f}) q^\xi, \tag{21}$$

where $\mathcal{N} : F_{\mathbb{A}}^\times / F^\times \to \overline{\mathbb{Q}}_p^\times$ is the character given by $\mathcal{N}(y) = y_p^{-I} |y^{(\infty)}|_{\mathbb{A}}^{-1}$.

We now define for any p-adically complete $O[\kappa,\epsilon]$-algebra A in $\mathbb{C}_p$

$$S_\kappa(U,\epsilon;A) = \left\{\mathbf{f} \in S_\kappa(U,\epsilon;\mathbb{C}_p)\middle|\mathbf{c}_p(y,\mathbf{f}) \in A \text{ for integral } y\right\}. \tag{22}$$

As we have already seen, these spaces have geometric meaning as the space of A-integral global sections of a line bundle defined over A of the Hilbert modular variety of level U, and the q-expansion above for a fixed $y = y^{(\infty)}$ gives rise to the geometric q-expansion at the infinity cusp of the classical modular form $\mathbf{f}_x$ for $x = \left(\begin{smallmatrix} y & 0 \\ 0 & 1\end{smallmatrix}\right)$ (see [H91, (1.5)] and [PAF, (4.63)]).

We have chosen a complete representative set $\{c_i\}_{i=1,\dots,h}$ in finite ideles for the strict idele class group $F^\times\backslash F_{\mathbb{A}}^\times/\widehat{O}^\times F_{\infty+}^\times$, where h is the strict class number of F. Let $\mathfrak{c}_i = c_iO$. Write $t_i = \begin{pmatrix} c_id^{-1} & 0 \\ 0 & 1\end{pmatrix}$ and consider $\mathbf{f}_i = \mathbf{f}_{t_i}$ as defined in (S2). The collection $(\mathbf{f}_i)_{i=1,\dots,h}$ determines f, because of the approximation theorem. Then $\mathbf{f}(c_id^{-1})$ gives the q-expansion of $\mathbf{f}_i$ at the Tate abelian variety with $\mathfrak{c}_i$-polarization $\mathrm{Tate}_{\mathfrak{c}_i^{-1},O}(q)$ ($\mathfrak{c}_i = c_iO$). By (q-exp), the q-expansion $\mathbf{f}(y)$ determines $\mathbf{f}$ uniquely.

We write $T(y)$ for the Hecke operator acting on $S_\kappa(U,\epsilon;A)$ corresponding to the double coset $U\left(\begin{smallmatrix} y & 0 \\ 0 & 1\end{smallmatrix}\right)U$ for an integral idele y. We renormalize $T(y)$ to have a p-integral operator $\mathbb{T}(y)$: $\mathbb{T}(y) = y_p^{-\kappa_1}T(y)$. Since this only affects $T(y)$ with $y_p \neq 1$, $\mathbb{T}(\mathfrak{q}) = T(\varpi_{\mathfrak{q}}) = T(\mathfrak{q})$ if $\mathfrak{q} \nmid p$. However depending on weight, we can have $\mathbb{T}(\mathfrak{p}) \neq T(\mathfrak{p})$ for primes $\mathfrak{p}|p$. The renormalization is optimal to have the stability of the A-integral spaces under Hecke operators. We define $\langle\mathfrak{q}\rangle = N(\mathfrak{q})T(\mathfrak{q},\mathfrak{q})$ with $T(\mathfrak{q},\mathfrak{q}) = [U\varpi_{\mathfrak{q}}U]$ for $\mathfrak{q} \nmid \mathfrak{n}'p^\alpha$ ($\mathfrak{n}' = \mathfrak{n}_1 \cap \mathfrak{n}_2$), which is equal to the central action of a prime element $\varpi_{\mathfrak{q}}$ of $O_{\mathfrak{q}}$ times $N(\mathfrak{q}) = |\varpi_{\mathfrak{q}}|_{\mathbb{A}}^{-1}$. We have the following formula of the action of $\mathbb{T}(\mathfrak{q})$ (e.g., [HMI, (2.3.21)] or [PAF, Sect. 4.2.10]):

$$\mathbf{c}_p(y,\mathbf{f}|\mathbb{T}(\mathfrak{q})) = \begin{cases} \mathbf{c}_p(y\varpi_{\mathfrak{q}},\mathbf{f}) + \mathbf{c}_p(y\varpi_{\mathfrak{q}}^{-1},\mathbf{f}|\langle\mathfrak{q}\rangle) & \text{if } \mathfrak{q} \nmid \mathfrak{n}p \\ \mathbf{c}_p(y\varpi_{\mathfrak{q}},\mathbf{f}) & \text{otherwise,}\end{cases} \tag{23}$$

where the level $\mathfrak{n}$ of U is the ideal maximal under the condition: $U_1^1(\mathfrak{n}) \subset U \subset U_0(\mathfrak{n})$. Thus $\mathbb{T}(\varpi_{\mathfrak{q}}) = (\varpi_{\mathfrak{q}})_p^{-\kappa_1}U(\mathfrak{q})$ when $\mathfrak{q}$ is a factor of the level of U (even when $\mathfrak{q}|p$; see [PAF, (4.65–66)]). Writing the level of U as $\mathfrak{n}p^\alpha$, we assume

$$\text{either } p|\mathfrak{n}p^\alpha \text{ or } [\kappa] \geq 0, \tag{24}$$

since $\mathbb{T}(\mathfrak{q})$ and $\langle\mathfrak{q}\rangle$ preserve the space $S_\kappa(U,\epsilon;A)$ under this condition (see [PAF, Theorem 4.28]). We define the Hecke algebra $h_\kappa(U,\epsilon;A)$ [resp. $h_\kappa(\mathfrak{n},\epsilon_+;A)$] with coefficients in A by the A-subalgebra of the A-linear endomorphism algebra $\mathrm{End}_A(S_\kappa(U,\epsilon;A))$ [resp. $\mathrm{End}_A(S_\kappa(\mathfrak{n},\epsilon_+;A))$] generated by the action of the finite group $U_0(\mathfrak{n}p^\alpha)/U$, $\mathbb{T}(\mathfrak{q})$ and $\langle\mathfrak{q}\rangle$ for all $\mathfrak{q}$.

1.9 Hecke Algebras

We have canonical projections:

$$R(U_1^1(\mathfrak{n}p^\alpha, \Delta_0(\mathfrak{n}p^\alpha)) \twoheadrightarrow R(U, \Delta_0(\mathfrak{n}p^\alpha)) \twoheadrightarrow R(U_0(\mathfrak{n}p^\beta), \Delta_0(\mathfrak{n}p^\beta))$$

for all $\alpha \geq \beta$ taking canonical generators to the corresponding ones, which are compatible with inclusions

$$S_\kappa(U_0(\mathfrak{n}p^\beta), \epsilon; A) \hookrightarrow S_\kappa(U, \epsilon; A) \hookrightarrow S_\kappa(U_1^1(\mathfrak{n}p^\alpha), \epsilon; A).$$

We decompose $O_p^\times = \mathbf{\Gamma} \times \Delta$ as in the introduction and hence $\mathbf{G} = \mathbf{\Gamma} \times \Delta \times (O/\mathfrak{n}')^\times$. We fix κ and ϵ_+ and the initial $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_+)$. We suppose that ϵ_j ($j = 1, 2$) factors through $\mathbf{G}/\mathbf{\Gamma} = \Delta \times (O/\mathfrak{n}')^\times$ for $\mathfrak{n}'$ prime to p. We write $\mathfrak{n}$ for a factor of $\mathfrak{n}'$ such that ϵ^- is defined modulo $\mathfrak{n}p^{r_0+I_p}$ for $p^{r_0+I_p} = \prod_{\mathfrak{p}|p} \mathfrak{p}^{r_{0,\mathfrak{p}}+1}$ for a multi-index $r_0 = (r_{0,\mathfrak{p}})_\mathfrak{p}$ with $\mathfrak{p}$ running over prime factors of p. Then we get a projective system of Hecke algebras $\{h_\kappa(U, \epsilon; A)\}_U$ (U running through open subgroups of $U_0(\mathfrak{n}p^{r_0+1})$ containing $U_1^1(\mathfrak{n}p^\infty)$), whose projective limit (when $\kappa_2 - \kappa_1 \geq I$) gives rise to the universal Hecke algebra $\mathbf{h}(\mathfrak{n}, \epsilon; A)$ for a complete p-adic algebra A. We have a continuous character $T : \widehat{O}^\times \to \mathbf{h}(\mathfrak{n}, \epsilon; A)$ given by $u \mapsto T(u)$ where $\mathbf{f}|T(u)(x) = \epsilon_1(u)^{-1}\mathbf{f}\left(x\left(\begin{smallmatrix} u & 0 \\ 0 & 1 \end{smallmatrix}\right)\right)$ for $u \in \widehat{O}^\times$ (here $T(u)$ is the Hecke operator $T(y)$ taking $y = u$ as the double coset $U\left(\begin{smallmatrix} u & 0 \\ 0 & 1 \end{smallmatrix}\right)U$ is equal to the single coset $U\left(\begin{smallmatrix} u & 0 \\ 0 & 1 \end{smallmatrix}\right)$). This character T factors through $\mathbf{\Gamma} = \mathbf{G}/(\Delta \times (O/\mathfrak{n}')^\times)$ and induces a canonical algebra structure of $\mathbf{h}(\mathfrak{n}, \epsilon; A)$ over $A[[\mathbf{\Gamma}]]$.

Let W be a sufficiently large complete discrete valuation ring inside $\overline{\mathbb{Q}}_p$ (as before). Define $W[\epsilon] \subset \overline{\mathbb{Q}}_p$ by the W-subalgebra generated by the values of ϵ (over the finite adeles). It has canonical generators $\mathbb{T}(y)$ over $\mathbf{\Lambda} = W[[\mathbf{\Gamma}]]$. Here note that the operator $\langle \mathfrak{q} \rangle$ acts via multiplication by $N(\mathfrak{q})\epsilon_+(\mathfrak{q})$ for the fixed central character ϵ_+, where $N(\mathfrak{q}) = |O/\mathfrak{q}|$.

The (nearly) p-ordinary projector $e = \lim_n \mathbb{T}(p)^{n!}$ gives an idempotent of the Hecke algebras $h_\kappa(U, \epsilon; W)$, $h_\kappa(\mathfrak{n}p^\alpha, \epsilon_+; W)$ and $\mathbf{h}(\mathfrak{n}, \epsilon_+; W)$. By adding superscript "n.ord," we indicate the algebra direct summand of the corresponding Hecke algebra cut out by e; e.g., $h_\kappa^{\mathrm{n.ord}}(\mathfrak{n}p^\alpha, \epsilon_+; W) = e(h_\kappa(\mathfrak{n}p^\alpha, \epsilon_+; W))$. We simply write $\mathbf{h}$ for $\mathbf{h}^{\mathrm{n.ord}} = \mathbf{h}^{\mathrm{n.ord}}(\mathfrak{n}, \epsilon_+; W)$. The algebra $\mathbf{h}^{\mathrm{n.ord}}$ is by definition the universal nearly p-ordinary Hecke algebra over $\mathbf{\Lambda}$ of level $\mathfrak{n}p^\infty$ with "Neben character" ϵ. This algebra $\mathbf{h}^{\mathrm{n.ord}}(\mathfrak{n}, \epsilon; W)$ is exactly the one $\mathbf{h}(\psi^+, \psi')$ employed in [HT93, p. 240] (note that in [HT93] we assumed $\kappa_1 \geq \kappa_2$ reversing our normalization here).

The algebra $\mathbf{h}^{\mathrm{n.ord}}(\mathfrak{n}, \epsilon; W)$ is a torsion-free $\mathbf{\Lambda}$-algebra of finite rank. Take a point $P \in \mathrm{Spf}(\mathbf{\Lambda})(\overline{\mathbb{Q}}_p)$. If P is arithmetic, $\epsilon_P = P\kappa(P)^{-1}$ is a character of $\mathbf{\Gamma}$. By abusing a symbol, we write ϵ_P for the character $(\epsilon_{P,1}, \epsilon_{P,2}, \epsilon_+)$ given by $\epsilon_{P,j}$ on $\mathbf{\Gamma}$ and ϵ_j on $\Delta \times (O/\mathfrak{n}')^\times$. Writing the conductor of $\epsilon_P^-|_{O_p^\times}$ as $p^{f(P)}$, we define $r(P) \geq 0$ by $p^{r(P)+I_p} = p^{f(P)} \cap \mathfrak{p}$. Here $r(P)$ is an element of $\mathbb{Z}[I_p]$; so, $r(P) = \sum_{\mathfrak{p}|p} r(P)_\mathfrak{p}\mathfrak{p}$ indexed by prime factors $\mathfrak{p}|p$, and we write I_p for $\{1\}_{\mathfrak{p}|p}$. Therefore $r(P) + I_p = \sum_\mathfrak{p}(r(P)_\mathfrak{p} + 1)\mathfrak{p}$. As long as P is arithmetic, we have a canonical specialization morphism:

$$\mathbf{h}^{\text{n.ord}}(\mathfrak{n}, \epsilon_+; W) \otimes_{\mathbf{\Lambda}, P} W[\epsilon_P] \twoheadrightarrow h^{\text{n.ord}}_{\kappa(P)}(\mathfrak{n}p^{r(P)+I_p}, \epsilon_+; W[\epsilon_P]),$$

which is an isogeny and is an isomorphism if $\mathbf{h}^{\text{n.ord}}(\mathfrak{n}, \epsilon_+; W)$ is $\mathbf{\Lambda}$-free [PAF, Sect. 4.2.11] (note in [PAF] the order of κ_j is reversed so that $\kappa_1 > \kappa_2$). The specialization morphism takes the generators $\mathbb{T}(y)$ to $\mathbb{T}(y)$.

1.10 Analytic Families of Hecke Eigenforms

In summary, for a fixed κ and ϵ_+, we have the algebra $\mathbf{h} = \mathbf{h}^{\text{n.ord}}(\mathfrak{n}, \epsilon_+; W)$ characterized by the following two properties:

(C1) $\mathbf{h}$ is torsion-free of finite rank over $\mathbf{\Lambda}$ equipped with $\mathbb{T}(\mathfrak{l}) = \mathbb{T}(\varpi_{\mathfrak{l}}), \mathbb{T}(y) \in \mathbf{h}$ for all primes $\mathfrak{l}$ prime to p and $y \in O_p \cap F_p^{\times}$,

(C2) if $\kappa_2 - \kappa_1 \geq I$ and P is an arithmetic point of $\mathrm{Spec}(\mathbf{\Lambda})(\overline{\mathbb{Q}}_p)$, we have a surjective W-algebra homomorphism: $\mathbf{h} \otimes_{\mathbf{\Lambda}, P} W[\epsilon_P]) \to h^{\text{n.ord}}_{\kappa(P)}(\mathfrak{n}p^{r(P)+I_p}, \epsilon_+; W[\epsilon_P])$ with finite kernel, sending $\mathbb{T}(\mathfrak{l}) \otimes 1$ to $\mathbb{T}(\mathfrak{l})$ (and $\mathbb{T}(y) \otimes 1$ to $\mathbb{T}(y)$).

Actually, if $p \geq 5$ and $p \nmid |\Delta|$, in (C1), quite plausibly, $\mathbf{h}$ would be free over $\mathbf{\Lambda}$ (not just torsion-free), and we would have an isomorphism in (C2) (this fact holds true under unramifiedness of $p \geq 5$ in $F/\mathbb{Q}$; see [PAF, Corollary 4.31]), but we do not need this stronger fact.

By fixing an isomorphism $\mathbf{\Gamma} \cong \mathbb{Z}_p^m$ with $m = [F_p : \mathbb{Q}_p]$, we have identified $\mathbf{\Lambda} = \mathbf{\Lambda}_W$ with $W[[T_1, \dots, T_m]]$ for $\{t_i = 1 + T_i\}_{i=1,\dots,m}$ corresponding to a $\mathbb{Z}_p$-basis $\{\gamma_i\}_{i=1,\dots,m}$ of $\mathbf{\Gamma}$. Regard κ_2 as a character of $O_p^{\times}$ whose value at $\gamma \in O_p^{\times}$ is

$$\gamma^{\kappa_2} = \prod_{\sigma \in I} \sigma(\gamma)^{\kappa_{2,\sigma}}.$$

We may write an arithmetic prime P as a prime $\mathbf{\Lambda}$-ideal

$$P = (t_i - \epsilon_2(\gamma_i)^{-1}\gamma_i^{\kappa_2})\mathbf{\Lambda}_{W[\epsilon]} \cap \mathbf{\Lambda}_W.$$

When $\kappa_2 = kI$ for an integer k, $\gamma \mapsto \gamma^{\kappa_2}$ is given by $\gamma \mapsto N(\gamma)^k$ for the norm map $N = N_{F_p/\mathbb{Q}_p}$ on $O_p^{\times}$. For a point $P \in \mathrm{Spec}(\mathbf{\Lambda})(\overline{\mathbb{Q}}_p)$ killing $(t_i - \zeta_i^{-1}\gamma_i^{\kappa_2})$ for $\zeta_i \in \mu_{p^\infty}(W)$, we make explicit the character ϵ_P. First we define a character $\epsilon_{P,2,\mathbf{\Gamma}} : O_p^{\times} \to \mu_{p^\infty}(W)$ factoring through $\mathbf{\Gamma} = O_p^{\times}/\Delta$ by $\epsilon_{P,2,\mathbf{\Gamma}}(\gamma_i) = \zeta_i$ for all i. Then for the fixed ϵ_+, we put $\epsilon_{P,1,\mathbf{\Gamma}} = (\epsilon_+|_{\mathbf{\Gamma}})\epsilon_{P,2,\mathbf{\Gamma}}^{-1}$. With the fixed data $\epsilon_1^{(\mathbf{\Gamma})} := \epsilon_1|_{(O/\mathfrak{n}')^{\times} \times \Delta}$ and $\epsilon_2^{(\mathbf{\Gamma})} := \epsilon_2|_{(O/\mathfrak{n}')^{\times} \times \Delta}$, we put $\epsilon_{P,j} = \epsilon_{j,P,\mathbf{\Gamma}}\epsilon_j^{(\mathbf{\Gamma})}$. In this way, we form $\epsilon_P = (\epsilon_{P,1}, \epsilon_{P,2}, \epsilon^+)$.

Let $\mathrm{Spec}(\mathbb{I})$ be a reduced irreducible component $\mathrm{Spec}(\mathbb{I}) \subset \mathrm{Spec}(\mathbf{h})$. Since $\mathbf{h}$ is torsion-free of finite rank over $\mathbf{\Lambda}$, $\mathrm{Spec}(\mathbb{I})$ is a finite torsion-free covering of $\mathrm{Spec}(\mathbf{\Lambda})$. Write $a(y)$ and $a(\mathfrak{l})$ for the image of $T(y)$ and $T(\mathfrak{l})$ in $\mathbb{I}$ (so, $a(\varpi_{\mathfrak{p}})$ is the image of $T(\varpi_{\mathfrak{p}})$). We also write $\mathbf{a}(y)$ for the image of $\mathbb{T}(y)$; so, $\mathbf{a}(y) = y_p^{-\kappa_1}a(y)$. If $P \in \mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$ induces an arithmetic point P_0 of $\mathrm{Spec}(\boldsymbol{\Lambda})$, we call it again an *arithmetic* point of $\mathrm{Spec}(\mathbb{I})$, and put $\kappa_j(P) = \kappa_j(P_0)$. If P is arithmetic, by (C2), we have a Hecke eigenform $\mathbf{f}_P \in S_{\kappa(P)}(U_0(\mathfrak{n}p^{r(P)+I_p}), \epsilon_P; \overline{\mathbb{Q}}_p)$ such that its eigenvalue for $\mathbb{T}(\mathfrak{l})$ and $\mathbb{T}(y)$ is given by $a_P(\mathfrak{l}) := P(a(\mathfrak{l})), a_P(y) := P(a(y)) \in \overline{\mathbb{Q}}_p$ for all $\mathfrak{l}$ and $y \in F_p^\times$. Thus $\mathbb{I}$ gives rise to a family $\mathcal{F} = \mathcal{F}_{\mathbb{I}} = \{\mathbf{f}_P | \text{arithmetic } P \in \mathrm{Spec}(\mathbb{I})\}$ of classical Hecke eigenforms. We call this family a *p-adic analytic family of p-slope* 0 (with coefficients in $\mathbb{I}$) associated with an irreducible component $\mathrm{Spec}(\mathbb{I}) \subset \mathrm{Spec}(\mathbf{h})$. There is a sub-family corresponding to any closed integral subscheme $\mathrm{Spec}(\mathbb{J}) \subset \mathrm{Spec}(\mathbb{I})$ as long as $\mathrm{Spec}(\mathbb{J})$ has densely populated arithmetic points. Abusing our language slightly, for any covering $\pi : \mathrm{Spec}(\widetilde{\mathbb{I}}) \twoheadrightarrow \mathrm{Spec}(\mathbb{I})$, we will consider the pulled back family $\mathcal{F}_{\widetilde{\mathbb{I}}} = \{\mathbf{f}_P = \mathbf{f}_{\pi(P)} | \text{arithmetic } P \in \mathrm{Spec}(\widetilde{\mathbb{I}})\}$. The choice of $\widetilde{\mathbb{I}}$ is often the normalization of $\mathbb{I}$ or the integral closure of $\mathbb{I}$ in a finite extension of the quotient field of $\mathbb{I}$.

Identify $\mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$ with $\mathrm{Hom}_{W\text{-alg}}(\mathbb{I}, \overline{\mathbb{Q}}_p)$ so that each element $a \in \mathbb{I}$ gives rise to a "function" $a : \mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p) \to \overline{\mathbb{Q}}_p$ whose value at $(P : \mathbb{I} \to \overline{\mathbb{Q}}_p) \in \mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$ is $a_P := P(a) \in \overline{\mathbb{Q}}_p$. Then a is an analytic function of the rigid analytic space associated with $\mathrm{Spf}(\mathbb{I})$. We call such a family p-slope 0 because $|\mathbf{a}_P(\varpi_{\mathfrak{p}})|_p = 1$ for the p-adic absolute value $|\cdot|_p$ of $\overline{\mathbb{Q}}_p$ for all $\mathfrak{p}|p$ (it is also called a p-ordinary family).

1.11 Modular Galois Representations

Each (reduced) irreducible component $\mathrm{Spec}(\mathbb{I})$ of the Hecke spectrum $\mathrm{Spec}(\mathbf{h})$ has a 2-dimensional semi-simple (actually absolutely irreducible) continuous representation $\rho_{\mathbb{I}}$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ with coefficients in the quotient field of $\mathbb{I}$ (see [H86a] and [H89]). The representation $\rho_{\mathbb{I}}$ restricted to the $\mathfrak{p}$-decomposition group $D_{\mathfrak{p}}$ (for each prime factor $\mathfrak{p}|p$) is reducible (see [HMI, Sect. 2.3.8]). Define the p-adic avatar $\widehat{\epsilon}_+ : (F_{\mathbb{A}}^{(\infty)})^\times / F^\times \to \overline{\mathbb{Q}}_p^\times$ by $\widehat{\epsilon}_+(y) = \epsilon_+(y) y_p^{I-\kappa_1-\kappa_2}$ (note here $y_\infty = 1$ as $F_{\mathbb{A}}^{(\infty)}$ is made of finite adales in $F_{\mathbb{A}}$). We write $\rho_{\mathbb{I}}^{ss}$ for its semi-simplification over D_p. As is well known now (e.g., [HMI, Sect. 2.3.8]), $\rho_{\mathbb{I}}$ is unramified outside $\mathfrak{n}p$ and satisfies

$$\mathrm{Tr}(\rho_{\mathbb{I}}(Frob_{\mathfrak{l}})) = a(\mathfrak{l}) \ \text{ for all prime } \mathfrak{l} \nmid p\mathfrak{n}. \tag{Gal}$$

By (Gal) and Chebotarev density, $\mathrm{Tr}(\rho_{\mathbb{I}})$ has values in $\mathbb{I}$; so, for any integral closed subscheme $\mathrm{Spec}(\mathbb{J}) \subset \mathrm{Spec}(\mathbb{I})$ with projection $\pi : \mathbb{I} \to \mathbb{J}$, $\pi \circ \mathrm{Tr}(\rho_{\mathbb{I}}) : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathbb{J}$ gives rise to a pseudo-representation of Wiles (e.g., [MFG, Sect. 2.2]). Then by a theorem of Wiles, we can make a unique 2-dimensional semi-simple continuous representation $\rho_{\mathbb{J}} : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{GL}_2(Q(\mathbb{J}))$ unramified outside $\mathfrak{n}p$ with $\mathrm{Tr}(\rho_{\mathbb{J}}(Frob_{\mathfrak{l}})) = \pi(a(\mathfrak{l}))$ for all primes $\mathfrak{l} \nmid \mathfrak{n}p$, where $Q(\mathbb{J})$ is the quotient field of $\mathbb{J}$. If $\mathrm{Spec}(\mathbb{J})$ is one point $P \in \mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$, we write ρ_P for $\rho_{\mathbb{J}}$. This is the Galois representation associated with the Hecke eigenform $\mathbf{f}_P$ (given in [H89]). As for p-ramification, the restriction of $\rho_{\mathbb{I}}$ to the decomposition group at a prime

$\mathfrak{p}|p$ is reducible. Taking $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/F_{\mathfrak{p}})$ whose restriction to the maximal abelian extension of $F_{\mathfrak{p}}$ is the Artin symbol $[u, F_{\mathfrak{p}}]$, we have by Hida [H89]

$$\rho_P(\sigma) \sim \begin{pmatrix} \epsilon_{2,P}(u)u^{-\kappa_2} & * \\ 0 & \epsilon_{1,P}(u)u^{-\kappa_1} \end{pmatrix} \text{ for } u \in O_{\mathfrak{p}}^{\times} \text{ and } \rho_P(\sigma) \sim \begin{pmatrix} * & * \\ 0 & \mathbf{a}_P(u) \end{pmatrix} \text{ for } u \in O_{\mathfrak{p}} - \{0\}. \tag{Ram}$$

Thus $[u, F_{\mathfrak{p}}] \mapsto \epsilon_{1,P}(u)u^{-\kappa_1}$ is the quotient character at $\mathfrak{p}$ (and in this way, ϵ_j ($j = 1, 2$) are ordered).

1.12 CM Theta Series

Following the description in [H06, Sect. 6.2], we construct CM theta series with p-slope 0 and describe the CM component which gives rise to such theta series (the construction was first made in [HT93]). We first recall a cusp form $\mathbf{f}$ on $G(\mathbb{A})$ with complex multiplication by a CM field M top down without much proof. By computing its classical Fourier expansion, we can confirm that $\mathbf{f}$ is a cusp form. Let M/F be a CM field with integer ring O_M and choose a CM type Σ:

$$I_M = \mathrm{Hom}_{\mathrm{field}}(M, \overline{\mathbb{Q}}) = \Sigma \sqcup \Sigma c$$

for complex conjugation c. To assure the p-slope 0 condition, we need to assume that the CM type Σ is p-ordinary, that is, the set Σ_p of p-adic places induced by $\iota_p \circ \sigma$ for $\sigma \in \Sigma$ is disjoint from Σ_{pc} (its conjugate by the generator c of $\mathrm{Gal}(M/F)$). The existence of such a p-ordinary CM type implies that each prime factor $\mathfrak{p}|p$ of F split in M/F. Thus the set $I_{M,p}$ of p-adic places of M is given by $\Sigma_p \sqcup \Sigma_p^c$. Write $\mathfrak{p} = \mathfrak{P}\mathfrak{P}^c$ in O_M for two primes $\mathfrak{P} \neq \mathfrak{P}^c$ such that $\mathfrak{P} \in \Sigma_p$ is induced by $\iota_p \circ \sigma$ on M for $\sigma \in \Sigma$. For each $k \in \mathbb{Z}[I]$ and $X = \Sigma, I_M$, we write $kX = \sum_{\sigma \in X} k_{\sigma|F}\sigma$.

We choose $\kappa_2 - \kappa_1 \geq I$ with $\kappa_1 + \kappa_2 = [\kappa]I$ for an integer $[\kappa]$. We then choose a Hecke ideal character λ of conductor $\mathfrak{C}\mathfrak{P}^e$ ($\mathfrak{C}$ prime to $\mathfrak{p}$) such that

$$\lambda((\alpha)) = \alpha^{c\kappa_1\Sigma+\kappa_2\Sigma} \text{ for } \alpha \in M^{\times} \text{ with } \alpha \equiv 1 \mod \mathfrak{C}\mathfrak{P}^e O_{M,\mathfrak{C}\mathfrak{P}^e} \text{ in } \prod_{\mathfrak{l}|\mathfrak{C}\mathfrak{P}^e} M_{\mathfrak{l}},$$

where $\mathfrak{P}^e = \prod_{\mathfrak{P}\in\Sigma_p} \mathfrak{P}^{e(\mathfrak{P})}\mathfrak{P}^{ce(\mathfrak{P}^c)}$ for $e = \sum_{\mathfrak{P}\in\Sigma_p}(e(\mathfrak{P})\mathfrak{P}+e(\mathfrak{P}^c)\mathfrak{P}^c)$ and $O_{M,\mathfrak{a}} = \prod_{\mathfrak{l}|\mathfrak{a}} O_{M_{\mathfrak{l}}}$ for an integral ideal $\mathfrak{a}$ of O_M.

We now recall a very old idea of Weil (and history) to lift the ideal character λ to an "idele" Hecke character: $\tilde{\lambda} : M_{\mathbb{A}}^{\times}/M^{\times} \to \mathbb{C}^{\times}$ following to Weil (who invented this identification of two types of Hecke characters in [W55] as a part of the theory of complex multiplication of abelian varieties, established by himself together with Shimura and Taniyama in the Tokyo–Nikko symposium in 1955). For the moment, we write $\tilde{\lambda}$ for the lifted idele character following [W55], but once it is defined, we just write simply λ for the idele and the ideal characters removing the tilde "~", following the more recent tradition. We write $(M_{\mathbb{A}}^{(\mathfrak{C}\mathfrak{P}^e\infty)})^\times := \{x \in M_{\mathbb{A}}^\times | x_\infty = x_{\mathfrak{l}} = 1\}$ for all primes $\mathfrak{l}|\mathfrak{C}\mathfrak{P}^e$. For an idele $x \in (M_{\mathbb{A}}^{(\mathfrak{C}\mathfrak{P}^e\infty)})^\times$ whose $\mathfrak{C}\mathfrak{P}^e\infty$-component is trivial, we require $\tilde{\lambda}(x) := \lambda(xO_M)$, where $xO_M = M \cap x\widehat{O}_M$ inside $(M_{\mathbb{A}}^{(\infty)})^\times = \{x \in M_{\mathbb{A}}^\times | x_\infty = 1\}$ (which is a fractional ideal prime to $\mathfrak{C}\mathfrak{P}^e$). At the infinity component $M_\infty^\times = (M \otimes_{\mathbb{A}} \mathbb{R})^\times = \prod_{\sigma\in\Sigma} \mathbb{C}^\times$, for $x_\infty = (x_\sigma)_{\sigma\in\Sigma}$ requiring

$$\tilde{\lambda}(x_\infty) = x_\infty^{-\kappa_2\Sigma - c\kappa_1\Sigma} := \prod_{\sigma\in\Sigma} x_\sigma^{-\kappa_{2,\sigma} - c\kappa_{2,\sigma c}},$$

we get a continuous character $\tilde{\lambda} : (M_{\mathbb{A}}^{(\mathfrak{C}\mathfrak{P}^e\infty)})^\times \times M_\infty^\times \to \mathbb{C}^\times$. We consider $M^\times(M_{\mathbb{A}}^{(\mathfrak{C}\mathfrak{P}^e\infty)})^\times M_\infty^\times \subset M_{\mathbb{A}}^\times$ which is a dense subgroup of $M_{\mathbb{A}}^\times$, and in particular, we have $M_{\mathbb{A}}^\times = U(\mathfrak{C}\mathfrak{P}^e)(M_{\mathbb{A}}^{(\mathfrak{C}\mathfrak{P}^e\infty)})^\times M_\infty^\times$, where $U(\mathfrak{a}) = \widehat{O}_M^\times \cap 1 + \mathfrak{a}\widehat{O}_M$ for an O_M-ideal $\mathfrak{a}$. We can extend $\tilde{\lambda}$ to the entire idele group $M_{\mathbb{A}}^\times$ so that $\tilde{\lambda}(M^\times) = 1$. To verify this point, we only need to show $\tilde{\lambda}(\alpha) = 1$ for $\alpha \in M^\times \cap U(\mathfrak{C}\mathfrak{P}^e)M_\infty^\times$ inside $M_{\mathbb{A}}^\times$. Since the $\mathfrak{C}\mathfrak{P}^e$ component of $\alpha \in M_{\mathbb{A}}^\times$ is in $U(\mathfrak{C}\mathfrak{P}^e)$, we check $\alpha_{\mathfrak{C}\mathfrak{P}^e} \equiv 1 \mod \mathfrak{C}\mathfrak{P}^e$, and hence, writing $(\alpha) = xO_M$ for $x = \alpha^{(\mathfrak{C}\mathfrak{P}^e\infty)} \in (M_{\mathbb{A}}^{(\mathfrak{C}\mathfrak{P}^e\infty)})^\times$ (the projection of $\alpha \in M_{\mathbb{A}}^\times$ to $(M_{\mathbb{A}}^{(\mathfrak{C}\mathfrak{P}^e\infty)})^\times$), we have $\tilde{\lambda}(x\alpha_\infty) = \lambda((\alpha))\alpha^{-\kappa_2\Sigma - c\kappa_1\Sigma} = 1$. By continuity, this extension $\tilde{\lambda}$ of λ to the dense subgroup $M^\times(M_{\mathbb{A}}^{(\mathfrak{C}\mathfrak{P}^e\infty)})^\times M_\infty^\times$ extends uniquely to the entire idele group $M_{\mathbb{A}}^\times$ which is trivial on $M^\times U(\mathfrak{C}\mathfrak{P}^e)$. Hereafter, we just use the symbol λ for $\tilde{\lambda}$ (as identifying the ideal character λ with the corresponding idele character $\tilde{\lambda}$).

If we need to indicate that $\mathfrak{C}$ is the prime-to-$\mathfrak{p}$ conductor of λ, we write $\mathfrak{C}(\lambda)$ for $\mathfrak{C}$. We also decompose $\mathfrak{C} = \prod_{\mathfrak{L}} \mathfrak{L}^{e(\mathfrak{L})}$ for prime ideals $\mathfrak{L}$ of M. We extend λ to a p-adic idele character $\widehat{\lambda} : M_{\mathbb{A}}^\times/M^\times M_\infty^\times \to \overline{\mathbb{Q}}_p^\times$ so that $\widehat{\lambda}(a) = \lambda(aO)a_p^{-\kappa_2\Sigma - c\kappa_1\Sigma}$. By class field theory, for the topological closure $\overline{M^\times M_\infty^\times}$ in $M_{\mathbb{A}}^\times$, $M_{\mathbb{A}}^\times/\overline{M^\times M_\infty^\times}$ is canonically isomorphic to the Galois group of the maximal abelian extension of M; so, this is the first occurrence in the history (again due to Weil [W55]) of the correspondence between an automorphic representation $\lambda = \tilde{\lambda}$ of $GL_1(M_{\mathbb{A}})$ and the Galois representation $\widehat{\lambda}$. Pulling back to $\mathrm{Gal}(\overline{F}/M)$, we may regard $\widehat{\lambda}$ as a character of $\mathrm{Gal}(\overline{F}/M)$. Any character φ of $\mathrm{Gal}(\overline{F}/M)$ of the form $\widehat{\lambda}$ as above is called "of weight κ". For a prime ideal $\mathfrak{L}$ of M outside p, we write $\lambda_{\mathfrak{L}}$ for the restriction of $\widehat{\lambda}$ to $M_{\mathfrak{L}}^\times$; so, $\lambda_{\mathfrak{L}}(x) = \widehat{\lambda}(x) = \lambda(x)$ for $x \in M_{\mathfrak{L}}^\times$. For a prime ideal $\mathfrak{P}|p$ of M, we put $\lambda_{\mathfrak{P}}(x) = \widehat{\lambda}(x)x^{\kappa_2\Sigma + c\kappa_1\Sigma} = \lambda(x)$ for $x \in M_{\mathfrak{P}}^\times$. In particular, for the prime $\mathfrak{P}|\mathfrak{p}$ with $\mathfrak{P} \in \Sigma_p$, we have $\lambda_{\mathfrak{P}}(x) = \widehat{\lambda}(x)x^{\kappa_2\Sigma_{\mathfrak{p}}}$ for $x \in M_{\mathfrak{P}}^\times$, and $\lambda_{\mathfrak{P}^c}(x) = \widehat{\lambda}(x)x^{c\kappa_1\Sigma_{\mathfrak{p}}}$ for $x \in M_{\mathfrak{P}^c}^\times$. Then $\lambda_{\mathfrak{L}}$ for all prime ideals $\mathfrak{L}$ (including those above p) is a continuous character of $M_{\mathfrak{L}}^\times$ with values in $\overline{\mathbb{Q}}$ whose restriction to the $\mathfrak{L}$-adic completion $O_{M,\mathfrak{L}}^\times$ of O_M is of finite order. By the condition $\kappa_1 \neq \kappa_2$, $\widehat{\lambda}$ cannot be of the form $\widehat{\lambda} = \phi \circ N_{M/F}$ for an idele character $\phi : F_{\mathbb{A}}^\times/F^\times F_{\infty+}^\times \to \overline{\mathbb{Q}}_p^\times$.

We define a function $(F_{\mathbb{A}}^{(\infty)})^\times \ni y \mapsto c(y, \theta(\lambda))$ supported by integral ideles by

$$c(y, \theta(\lambda)) = \sum_{x \in (M_{\mathbb{A}}^{(\infty)})^\times, xx^c = y} \lambda(x) \quad \text{if } y \text{ is integral,} \tag{25}$$

where x runs over elements in $M_{\mathbb{A}}^{\times}/(\widehat{O}_M^{(\mathfrak{C}\mathfrak{P}^e)})^{\times}$ satisfying the following four conditions: (0) $x_\infty = 1$, (1) xO_M is an integral ideal of M, (2) $N_{M/F}(x) = y$ and (3) $x_{\mathfrak{Q}} = 1$ for prime factors $\mathfrak{Q}$ of the conductor $\mathfrak{C}\mathfrak{P}^e$. The q-expansion determined by the coefficients $c(y, \theta(\lambda))$ gives a unique element $\theta(\lambda) \in S_\kappa(\mathfrak{n}_\theta, \epsilon'_\lambda; \overline{\mathbb{Q}})$ ([HT93, Theorem 6.1] and [HMI, Theorem 2.72]), where $\mathfrak{n}_\theta = N_{M/F}(\mathfrak{C}\mathfrak{P}^e)d(M/F)$ for the discriminant $d(M/F)$ of M/F and ϵ'_λ is a suitable "Neben" character. We have

(C) The central character $\epsilon_{\lambda+}$ of the automorphic representation $\pi(\lambda)$ generated by $\theta(\lambda)$ is given by the product: $x \mapsto \lambda(x)|x|_{\mathbb{A}}\left(\frac{M/F}{x}\right)$ for $x \in F_{\mathbb{A}}^{\times}$ and the quadratic character $\left(\frac{M/F}{}\right)$ of the CM quadratic extension M/F.

Recall here that $\lambda : M_{\mathbb{A}}^{\times} \to \mathbb{C}^{\times}$ is trivial on $M^\times$ as $\lambda_\infty(x_\infty) = x_\infty^{-\kappa_2\Sigma - c\kappa_1\Sigma}$, and hence $\epsilon_{\lambda+}$ is a continuous character of the idele class group $F_{\mathbb{A}}^{\times}/F^{\times}$.

We describe the Neben character $\epsilon_\lambda = (\epsilon_{\lambda,1}, \epsilon_{\lambda,2}, \epsilon_{\lambda+})$ of the minimal form $\mathbf{f}(\lambda)$ in the automorphic representation $\pi(\lambda)$. For that, we choose a decomposition $\mathfrak{C} = \mathfrak{F}\mathfrak{F}_c\mathfrak{I}$ so that $\mathfrak{F}\mathfrak{F}_c$ is a product of split primes and $\mathfrak{I}$ for the product of inert or ramified primes, $\mathfrak{F} + \mathfrak{F}_c = O_M$ and $\mathfrak{F} \subset \mathfrak{F}_c^c$, where $\mathfrak{F}$ could be strictly smaller than $\mathfrak{F}_c^c$. If we need to make the dependence on λ of these symbols explicit, we write $\mathfrak{F}(\lambda) = \mathfrak{F}$, $\mathfrak{F}_c(\lambda) = \mathfrak{F}_c$ and $\mathfrak{I}(\lambda) = \mathfrak{I}$. We put $\mathfrak{f} = \mathfrak{F} \cap F$ and $\mathfrak{i} = \mathfrak{I} \cap F$. Define $\lambda^-(\mathfrak{a}) = \lambda(\mathfrak{a}^{c-1})$ (with $\mathfrak{a}^{c-1} = \mathfrak{a}^c\mathfrak{a}^{-1}$), and write its conductor as $\mathfrak{C}(\lambda^-)$. Decompose as above $\mathfrak{C}(\lambda^-) = \mathfrak{F}(\lambda^-)\mathfrak{F}^c(\lambda^-)\mathfrak{I}(\lambda^-)$ so that we have the following divisibility of radicals $\sqrt{\mathfrak{F}(\lambda^-)}|\sqrt{\mathfrak{F}(\lambda)}$ and $\sqrt{\mathfrak{F}_c(\lambda^-)}|\sqrt{\mathfrak{F}_c(\lambda)}$. Let $\mathcal{T}_M = \mathrm{Res}_{O_M/O}\mathbb{G}_m$. The $\mathfrak{l}$-component $\epsilon_{\lambda,j,\mathfrak{l}}$ ($j = 1, 2$) of the character $\epsilon_{\lambda,j}$ is given as follows:

(hk1) For $\mathfrak{l}|\mathfrak{f}$, we identify $\mathcal{T}_M(O_\mathfrak{l}) = O_{M,\mathfrak{L}}^{\times} \times O_{M,\mathfrak{L}^c}^{\times}$ with this order for the prime ideal $\mathfrak{L}|(\mathfrak{l}O_M \cap \mathfrak{F})$ and define $\epsilon_{\lambda,1,\mathfrak{l}} \times \epsilon_{\lambda,2,\mathfrak{l}}$ by the restriction of $\lambda_{\mathfrak{L}} \times \lambda_{\mathfrak{L}^c}$ to $\mathcal{T}_M(O_\mathfrak{l})$.

(hk2) For $\mathfrak{P} \in \Sigma_p$, we identify $\mathcal{T}_M(O_\mathfrak{p}) = \mathfrak{O}_{M_\mathfrak{P}}^{\times} \times \mathfrak{O}_{M_{\mathfrak{P}^c}}^{\times}$ and define $\epsilon_{\lambda,1,\mathfrak{p}} \times \epsilon_{\lambda,2,\mathfrak{p}}$ by the restriction of $\lambda_\mathfrak{P} \times \lambda_{\mathfrak{P}^c}$ to $\mathcal{T}_M(O_\mathfrak{p})$.

(hk3) For $\mathfrak{l}|(\mathfrak{I}(\lambda) \cap O)d(M/F)$ but $\mathfrak{l} \nmid (\mathfrak{I}(\lambda^-) \cap O)$, we can choose a character $\phi_\mathfrak{l} : F_\mathfrak{l}^{\times} \to \mathbb{C}^{\times}$ such that $\lambda_\mathfrak{L} = \phi_\mathfrak{l} \circ N_{M_\mathfrak{L}/F_\mathfrak{l}}$. Then we define $\epsilon_{\lambda,1,\mathfrak{l}}(a) = \left(\frac{M_\mathfrak{L}/F_\mathfrak{l}}{a}\right)\phi_\mathfrak{l}(a)$ and $\epsilon_{\lambda,2,\mathfrak{l}}(d) = \phi_\mathfrak{l}(d)$, where $\mathfrak{L}$ is the prime factor of $\mathfrak{l}$ in M and $\left(\frac{M_\mathfrak{L}/F_\mathfrak{l}}{d}\right)$ is the character of $M_\mathfrak{L}/F_\mathfrak{l}$.

(hk4) For $\mathfrak{l}|(\mathfrak{I}(\lambda^-) \cap O)$, $\epsilon_{\lambda,1,\mathfrak{l}} = \epsilon_{\lambda+,\mathfrak{l}}|_{O_\mathfrak{l}^{\times}}$ and $\epsilon_{\lambda,2,\mathfrak{l}} = 1$ for the central character $\epsilon_{\lambda+}$ given in (C).

We now give an explicit description of the automorphic representation $\pi(\lambda)$. In Cases (hk1–3), taking a prime $\mathfrak{L}|\mathfrak{l}$ in M, we have

$$\pi_\mathfrak{p}(\lambda) \cong \begin{cases} \pi(\lambda_\mathfrak{L}, \lambda_{\mathfrak{L}^c}) & \text{in Case (hk1)}, \\ \pi(\lambda_\mathfrak{P}, \lambda_{\mathfrak{P}^c}) & \text{in Case (hk2)}, \\ \pi(\left(\frac{M_\mathfrak{L}/F_\mathfrak{l}}{}\right)\phi_\mathfrak{l}, \phi_\mathfrak{l}) & \text{in Case (hk3)}. \end{cases} \tag{26}$$

In Case (hk4), $\pi_{\mathfrak{l}}(\lambda)$ is the super-cuspidal representation giving rise to $\mathrm{Ind}_{M_{\mathfrak{l}}}^{F_{\mathfrak{l}}} \widehat{\lambda}|_{\mathrm{Gal}(\overline{F}_{\mathfrak{l}}/M_{\mathfrak{l}})}$.

To describe of $\mathbf{f}(\lambda)$, we split $\mathfrak{n}_\theta$ into a product of co-prime ideals $\mathfrak{n}_{nc}$ and $\mathfrak{n}_{cusp}$ so that $\mathfrak{n}_{nc}$ is made up of primes in Cases (hk1–3). For $\mathfrak{l}|\mathfrak{n}_{nc}$, writing $\pi_{\mathfrak{l}}(\lambda) = \pi(\eta_{\mathfrak{l}}, \eta'_{\mathfrak{l}})$ for characters $\eta_{\mathfrak{l}}, \eta'_{\mathfrak{l}} : F_{\mathfrak{l}}^\times \to \mathbb{C}^\times$, we write $C_{\mathfrak{l}}$ for the conductor of $\eta_{\mathfrak{l}}^{-1}\eta'_{\mathfrak{l}}$. Define the minimal level of $\pi(\lambda)$ by

$$\mathfrak{n}(\lambda) = \mathfrak{n}_{cusp} \prod_{\mathfrak{l}|\mathfrak{n}_{nc}} C_{\mathfrak{l}},$$

where $\mathfrak{l}$ runs over primes satisfying one of the three conditions (hk1–3). Put

$$\Xi = \{\mathfrak{L}|\mathfrak{L} \supset \mathfrak{F} \prod_{\mathfrak{P}\in\Sigma_p} \mathfrak{P}, \mathfrak{L} \supset \mathfrak{n}(\lambda)\}$$

for primes $\mathfrak{L}$ of M. Then the minimal form $\mathbf{f}(\lambda)$ has the following q-expansion coefficient:

$$\mathbf{c}_p(y, \mathbf{f}(\lambda)) = \begin{cases} \sum_{xx^c=y, x_\Xi=1} \widehat{\lambda}(x) & \text{if } y \text{ is integral,} \\ 0 & \text{otherwise,} \end{cases} \tag{27}$$

where x runs over $(\widehat{O}_M \cap M^\times_{\mathbb{A}^{(\infty)}}/(O_M^{(\Xi)})^\times$ with $x_{\mathfrak{L}} = 1$ for $\mathfrak{L} \in \Xi$. See [H06, Sect. 6.2] for more details of this construction (though in [H06], the order of (κ_1, κ_2) is interchanged so that $\kappa_1 > \kappa_2$).

1.13 CM Components

We fix a Hecke character λ of type κ as in the previous subsection, and we continue to use the symbols defined above. We may regard the Galois character $\widehat{\lambda}$ as a character of $Cl_M(\mathfrak{C}p^\infty)$.

We consider the ray class group $Cl_M(\mathfrak{C}(\lambda^-)p^\infty)$ modulo $\mathfrak{C}(\lambda^-)p^\infty$. Since $\lambda^-(\mathfrak{a}^c) = (\lambda^-)^{-1}(\mathfrak{a})$, we have $\mathfrak{C}(\lambda^-) = \mathfrak{C}(\lambda^-)^c$. Thus $\mathrm{Gal}(M/F) = \langle c\rangle$ acts naturally on $Cl_M(\mathfrak{C}(\lambda^-)p^\infty)$. We define the anticyclotomic quotient of $Cl_M(\mathfrak{C}(\lambda^-)p^\infty)$ by

$$Cl_M^-(\mathfrak{C}(\lambda^-)p^\infty) := Cl_M(\mathfrak{C}(\lambda^-)p^\infty)/Cl_M(\mathfrak{C}(\lambda^-)p^\infty)^{1+c}.$$

We have canonical identities:

$$O_{M,\mathfrak{p}}^\times = O_{M,\mathfrak{P}}^\times \times O_{M,\mathfrak{P}^c}^\times = O_{\mathfrak{p}}^\times \times O_{\mathfrak{p}}^\times \text{ and } O_{M,p}^\times := (O_M \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times = O_{M,\Sigma_p}^\times \times O_{M,\Sigma_p^c}^\times = O_p^\times \times O_p^\times$$

on which c acts by interchanging the components. Here $O_{M,X} = \prod_{\mathfrak{P}\in X} O_{M,\mathfrak{P}}$ for $X = \Sigma$ and Σ^c. The natural inclusion $O_{M,p}^\times/\overline{O_M^\times} \hookrightarrow Cl(\mathfrak{C}(\lambda^-)p^\infty)$ induces an inclusion $\mathbf{\Gamma} \hookrightarrow Cl_M^-(\mathfrak{C}(\lambda^-)p^\infty)$. Decompose $Cl_M^-(\mathfrak{C}(\lambda^-)p^\infty) = \mathbf{\Gamma}_M \times \Delta_M$ with the maximal finite subgroup Δ_M so that $\mathbf{\Gamma}_M \supset \mathbf{\Gamma}$. Then $\mathbf{\Gamma}$ is an open subgroup in $\mathbf{\Gamma}_M$. In particular, $W[[\mathbf{\Gamma}_M]]$ is a regular domain finite flat over $\mathbf{\Lambda}_W$. Thus we call $P \in \mathrm{Spec}(W[[\mathbf{\Gamma}_M]])(\overline{\mathbb{Q}}_p)$ arithmetic if P is above an arithmetic point of $\mathrm{Spec}(\mathbf{\Lambda}_W)(\overline{\mathbb{Q}}_p)$. Regard the tautological character

$$\upsilon : Cl_M(\mathfrak{C}p^\infty) \xrightarrow{\text{projection}} \mathbf{\Gamma}_M \hookrightarrow W[\mathbf{\Gamma}_M]]^\times$$

as a Galois character $\upsilon : \mathrm{Gal}(\overline{M}/M) \to W[\mathbf{\Gamma}_M]]^\times$.

The composite $\upsilon_P = P \circ \upsilon$ for an arithmetic point $P \in \mathrm{Spec}(W[[\mathbf{\Gamma}_M]])$ is of the form $\widehat{\varphi}_P$ for a Hecke character φ_P with p-type $\kappa'_{P,2}\Sigma_p + \kappa'_{P,1}\Sigma_p^c$ for $\kappa'_P = (\kappa'_{P,1}, \kappa'_{P,2}) \in \mathbb{Z}[I_p]^2$ satisfying $\kappa_2 + \kappa'_{P,2} - (\kappa_1 + \kappa'_{P,1}) \geq I_p$. Assume that $\widehat{\lambda}$ has values in $W^\times$ (enlarging W if necessary). We then consider the product $\widehat{\lambda}\upsilon : \mathrm{Gal}(\overline{M}/M) \to W[[\mathbf{\Gamma}_M]]^\times$ and $\rho_{W[[\mathbf{\Gamma}_M]]} := \mathrm{Ind}_M^F \widehat{\lambda}\upsilon : \mathrm{Gal}(\overline{M}/M) \to \mathrm{GL}_2(W[[\mathbf{\Gamma}_M]])$. Define $\mathbb{I}_M \subset W[[\mathbf{\Gamma}_M]]$ by the $\mathbf{\Lambda}_W$-subalgebra generated by $\mathrm{Tr}(\rho_{W[[\Gamma_M]]})$. Then we have the localization identity $\mathbb{I}_{M,P} = W[[\mathbf{\Gamma}_M]]_P$ for any arithmetic point P (this follows from the irreducibility of $\rho_P = P \circ \rho_{W[[\mathbf{\Gamma}_M]]} = \mathrm{Ind}_M^F \widehat{\lambda}\upsilon_P$; e.g., [H86b, Theorem 4.3]).

Let $\mathbf{h} = \mathbf{h}^{\text{n.ord}}(\mathfrak{n}(\lambda), \epsilon_{\lambda+}; W)$, which is a torsion-free finite $\mathbf{\Lambda}_W$-algebra. We have a surjective projection $\pi_\lambda : \mathbf{h} \to \mathbb{I}_M$ sending $T(\mathfrak{l})$ to $\mathrm{Tr}(\rho_{W[[\mathbf{\Gamma}]]}(Frob_{\mathfrak{l}}))$ for primes $\mathfrak{l}$ outside $\mathfrak{n}(\lambda)$. Thus $\mathrm{Spec}(\mathbb{I}_M)$ is an irreducible component of $\mathrm{Spec}(\mathbf{h})$. In particular, $\rho_{\mathbb{I}_M} = \rho_{W[[\mathbf{\Gamma}_M]]}$. In the same manner as in [HMI, Proposition 3.78], we prove the following fact:

Proposition 1.1. *Let the notation be as above. Then for the reduced part* $\mathbf{h}^{red}$ *of* $\mathbf{h}$ *and each arithmetic point* $P \in \mathrm{Spec}(\mathbf{\Lambda}_W)(\overline{\mathbb{Q}}_p)$*,* $\mathrm{Spec}(\mathbf{h}_P^{red})$ *is finite étale over* $\mathrm{Spec}(\mathbf{\Lambda}_P)$*. In particular, no irreducible components cross each other at a point above an arithmetic point of* $\mathrm{Spec}(\mathbf{\Lambda}_W)$*.*

A component $\mathbb{I}$ is called a *CM component* if there exists a nontrivial character $\chi : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathbb{I}^\times$ such that $\rho_{\mathbb{I}} \cong \rho_{\mathbb{I}} \otimes \chi$. We also say that $\mathbb{I}$ has *complex multiplication* if $\mathbb{I}$ is a CM component. In this case, we call the corresponding family $\mathcal{F}$ a CM family (or we say $\mathcal{F}$ has complex multiplication). It is known essentially by deformation theory of Galois characters (cf. [H11, Sect. 4]) that any CM component is given by $\mathrm{Spec}(\mathbb{I}_M)$ as above for a specific choice of λ.

If $\mathcal{F}$ is a CM family associated with $\mathbb{I}$ with $\rho_{\mathbb{I}} \cong \rho_{\mathbb{I}} \otimes \chi$, then χ is a quadratic character of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ which cuts out a CM quadratic extension M/F, i.e., $\chi = \left(\frac{M/F}{}\right)$. Write $\widetilde{\mathbb{I}}$ for the integral closure of $\mathbf{\Lambda}_W$ inside the quotient field of $\mathbb{I}$. The following three conditions are known to be equivalent:

(CM1) *$\mathcal{F}$ has CM and $\rho_{\mathbb{I}} \cong \rho_{\mathbb{I}} \otimes \left(\frac{M/F}{}\right)$ ($\Leftrightarrow$ $\rho_{\mathbb{I}} \cong \mathrm{Ind}_M^F \Psi$ for a character $\Psi := \widehat{\lambda}\upsilon : \mathrm{Gal}(\overline{\mathbb{Q}}/M) \to Q(\mathbb{I})^\times$ for the quotient field $Q(\mathbb{I})$ of $\mathbb{I}$);*

(CM2) *For all arithmetic P of* $\mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$*,* $\mathbf{f}_P$ *is a binary theta series of the norm form of M/F;*

(CM3) *For some arithmetic P of* $\mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$*,* $\mathbf{f}_P$ *is a binary theta series of the norm form of M/F.*

Since the characteristic polynomial of $\underset{\sim}{\rho}_{\mathbb{I}}(\sigma)$ has coefficients in $\mathbb{I}$, its eigenvalues fall in $\widetilde{\mathbb{I}}$; so, the character Ψ has values in $\widetilde{\mathbb{I}}^{\times}$ (see [H86b, Corollary 4.2]). Then, (CM1) is equivalent to $\rho_{\mathbb{I}} \cong \mathrm{Ind}_M^F \Psi$ for a character $\Psi : \mathrm{Gal}(\overline{\mathbb{Q}}/M) \to \widetilde{\mathbb{I}}^{\times}$ unramified outside Np (e.g., [MFG, Lemma 2.15]). Then by (Gal) and (Ram), $\Psi_P = P \circ \Psi : \mathrm{Gal}(\overline{\mathbb{Q}}/M) \to \overline{\mathbb{Q}}_p^{\times}$ for an arithmetic $P \in \mathrm{Spec}(\widetilde{\mathbb{I}})(\overline{\mathbb{Q}}_p)$ is a locally algebraic p-adic character, which is the p-adic avatar of a Hecke character $\lambda_P : M_{\mathbb{A}}^{\times}/M^{\times} \to \mathbb{C}^{\times}$ of type A_0 of the quadratic extension $M_{/F}$. Then by the characterization (Gal) of $\rho_{\mathbb{I}}$, $\mathbf{f}_P$ is the theta series $\mathbf{f}(\lambda)$, where $\mathfrak{a}$ runs over all integral ideals of M. By $\kappa_2(P) - \kappa_1(P) \geq I$ (and (Gal)), M has to be a CM field in which $\mathfrak{p}$ is split (as the existence of Hecke characters of infinity type corresponding to such $\kappa(P)$ forces that M/F is a CM quadratic extension). This shows (CM1)$\Rightarrow$(CM2)$\Rightarrow$(CM3). If (CM2) is satisfied, we have an identity $\mathrm{Tr}(\rho_{\mathbb{I}}(Frob_{\mathfrak{l}})) = a(\mathfrak{l}) = \chi(\mathfrak{l})a(\mathfrak{l}) = \mathrm{Tr}(\rho_{\mathbb{I}} \otimes \chi(Frob_{\mathfrak{l}}))$ with $\chi = \left(\frac{M/F}{}\right)$ for all primes $\mathfrak{l}$ outside a finite set of primes (including prime factors of $\mathfrak{n}(\lambda)p$). By Chebotarev density, we have $\mathrm{Tr}(\rho_{\mathbb{I}}) = \mathrm{Tr}(\rho_{\mathbb{I}} \otimes \chi)$, and we get (CM1) from (CM2) as $\rho_{\mathbb{I}}$ is semi-simple. If a component $\mathrm{Spec}(\mathbb{I})$ contains an arithmetic point P with theta series $\mathbf{f}_P$ of M/F as above, either $\mathbb{I}$ is a CM component or otherwise P is in the intersection in $\mathrm{Spec}(\mathbf{h})$ of a component $\mathrm{Spec}(\mathbb{I})$ not having CM by M and another component having CM by M (as all families with CM by M are made up of theta series of M by the construction of CM components as above). The latter case cannot happen as two distinct components never cross at an arithmetic point in $\mathrm{Spec}(\mathbf{h})$ (i.e., the reduced part of the localization $\mathbf{h}_P$ is étale over $\mathbf{\Lambda}_P$ for any arithmetic point $P \in \mathrm{Spec}(\mathbf{\Lambda})(\overline{\mathbb{Q}}_p)$; see Proposition 1.1). Thus (CM3) implies (CM2). We call a binary theta series of the norm form of a CM quadratic extension of F a *CM theta series*.

Remark 1.2. If $\mathrm{Spec}(\mathbb{J})$ is an integral closed subscheme of $\mathrm{Spec}(\mathbb{I})$, we write the associated Galois representation as $\rho_{\mathbb{J}}$. By abuse of language, we say $\mathbb{J}$ has CM by M if $\rho_{\mathbb{J}} \cong \rho_{\mathbb{J}} \otimes \left(\frac{M/F}{}\right)$. Thus (CM3) is equivalent to having ρ_P with CM for some arithmetic point P. More generally, if we find some arithmetic point P in $\mathrm{Spec}(\mathbb{J})$ and ρ_P has CM, $\mathbb{J}$ and $\mathbb{I}$ have CM.

2 Weil Numbers

Since $\overline{\mathbb{Q}}$ sits inside $\mathbb{C}$, it has "the" complex conjugation c. For a prime l, a Weil l-number $\alpha \in \overline{\mathbb{Q}}$ of integer weight $k \geq 0$ is defined by the following two properties:

(1) α is an algebraic integer;

(2) $|\alpha^\sigma| = l^{k/2}$ for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$ for the complex archimedean absolute value $|\cdot|$.

Note that $\mathbb{Q}(\alpha)$ is in a CM field finite over $\mathbb{Q}$ (e.g., [Ho68, Proposition 4]), and the Weil l-number is realized as the Frobenius eigenvalue of a CM abelian variety over a finite field of characteristic l. We call two nonzero numbers $a, b \in \overline{\mathbb{Q}}$ equivalent (written as $a \sim b$) if a/b is a root of unity. We say that Weil numbers α and β are p-equivalent if $\alpha/\beta \in \mu_{p^\infty}(\overline{\mathbb{Q}})$. Here is an improvement of [H11, Corollary 2.5] proved as [H14, Corollary 2.2]:

Proposition 2.1. *Let d be a positive integer. Let $\mathcal{K}_d$ be the set of all finite extensions of $K = \mathbb{Q}[\mu_{p^\infty}]$ of degree d inside $\overline{\mathbb{Q}}$. If $l \neq p$, there are only finitely many Weil l-numbers of a given weight in the set-theoretic union $\bigcup_{L \in \mathcal{K}_d} L^\times$ (in $\overline{\mathbb{Q}}^\times$) up to p-equivalence.*

Let $L_{/F}$ be a finite field extension inside $\mathbb{C}_p$ with integer ring O_L as in the introduction. Recall $T_L = \mathrm{Res}_{O_L/\mathbb{Z}}\mathbb{G}_m$ (in the sense of [NMD, Sect. 7.6, Theorem 4]) and a morphism $\nu \in \mathrm{Hom}_{\text{gp scheme}}(T_L, T_F)$ in the introduction. Define an integral domain $R = R_\nu$ by the subalgebra of $\mathbf{\Lambda}$ generated over $\mathbb{Z}_{(p)}$ by the image G of $\nu(O_{L,(p)}^\times) \cap T_F(\mathbb{Z}_p)$ projected down to $\mathbf{\Gamma}$. If $\nu \neq 1$, $\nu(O_{L,(p)}^\times) \cap T_F(\mathbb{Z}_p)$ contains $G_0 := \{\xi^N | \xi \in \mathbb{Z}_{(p)}^\times\}$ for some $0 < N \in \mathbb{Z}$. Replacing N by its suitable multiple, G_0 is a free $\mathbb{Z}$-module of infinite rank. Since $R_\nu \cong \mathbb{Z}_{(p)}[G]$ (the group algebra of G), R_ν contains a polynomial ring over $\mathbb{Z}_{(p)}$ (isomorphic to $\mathbb{Z}_{(p)}[G_0]$) with infinitely many variables, and $Q(R_\nu)$ has infinite transcendental degree over $\mathbb{Q}$ (if $\nu \neq 1$). For any arithmetic point P and $\xi \in R_\nu$, the value $\xi_P \in \mathbb{C}_p$ falls in $L^{\mathrm{gal}}[\mu_N, \mu_{p^\infty}]$ for the Galois closure L^{gal} of $L/\mathbb{Q}$ and $N = |\Delta|$. For example, if $F = \mathbb{Q}$ and $L = \mathbb{Q}$ with the identity $\nu : \mathbb{G}_m \cong \mathbb{G}_m$, taking $\gamma_1 = 1 + \mathbf{p}$ for $\mathbf{p} = 4$ if $p = 2$ and $\mathbf{p} = p$ if $p > 2$, we have $G = \{t^{\log_p(\xi)/\log_p(\gamma_1)} | \xi \in \mathbb{Z}_{(p)}\}$; so, $P(t^{\log_p(\xi)/\log_p(\gamma_1)}) = \xi^{\kappa_2}\omega(\xi^{\kappa_2})^{-1}\zeta$ for $P = (t - \zeta\gamma_1^{\kappa_2})$, where ω is the Teichmüller character ($N = p - 1$ for $F = \mathbb{Q}$ and odd p). Note that ξ^{κ_2} has values in L^{gal} instead of L. Recall the algebraic closure $\overline{Q}$ (we fixed) of the quotient field Q of $\mathbf{\Lambda}$.

Proposition 2.2. *Let $\mathbb{I}$ be a finite normal extension of $\mathbf{\Lambda}$ inside $\overline{Q}$ and regard $R = R_\nu \subset \mathbf{\Lambda}$ as a subalgebra of $\mathbb{I}$. Let $A \subset \mathbb{I}$ be an R-subalgebra of finite type whose quotient field $Q(A)$ is a finite extension of the quotient field $Q(R)$ of R. Regarding an arithmetic point $P \in \mathrm{Spec}(\mathbb{I})$ as an algebra homomorphism $P : \mathbb{I} \to \overline{\mathbb{Q}}_p$, write A_P (resp. R_P) for the composite of the image $P(A)$ [resp. $P(R)$] with $\mathbb{Q}(\mu_{p^\infty})$ inside $\overline{\mathbb{Q}}_p$. Then there exists a closed subscheme E of codimension at least 1 of $\mathrm{Spec}(\mathbb{I})$ such that there are finitely many Weil l-numbers of a given weight in $\bigcup_{P \notin E} A_P \subset \overline{\mathbb{Q}}$ up to p-power roots of unity, where P runs over all arithmetic points of $\mathrm{Spec}(\mathbb{I})$ outside E.*

Proof. We may assume that $A = R[a]$ (i.e., A is generated over R by a single element a). The generator $a \in A$ satisfies an equation $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n \in R[x]$ with $a_0 \neq 0$. Then the zero locus E of a_0 is a closed formal subscheme of codimension at least 1. Since arithmetic points are Zariski dense in $\mathrm{Spec}(\mathbb{I})$, we have a plenty of arithmetic points outside E (i.e., the set arithmetic points outside

E is infinite). Thus as long as $P(a_0) \neq 0$, we have $[A_P : R_P] \leq n$. Since $R_P \subset L^{\mathrm{gal}}[\mu_N, \mu_{p^\infty}]$, we have $[R_P : \mathbb{Q}(\mu_{p^\infty})] \leq B$ for a constant B independent of arithmetic P outside E. Thus $[A_P : \mathbb{Q}(\mu_{p^\infty})]$ is bounded independently by $d := nB$ for all arithmetic $P \notin E$. Then we can apply Proposition 2.1 and get the desired result. □

3 Theorems and Conjectures

Hereafter,

(W) we fix $\kappa \in \mathbb{Z}[I]^2$ with $\kappa_2 - \kappa_1 \geq I$.

Though the weight κ is fixed, the character ϵ_P is a variable (so, we have densely populated arithmetic points $P \in \mathrm{Spec}(\mathbb{I})$ with $\kappa(P) = \kappa$). Let $\mathbf{f} \in S_\kappa(\mathfrak{n}p^{r+I_p}, \epsilon; W)$ be a Hecke eigenform with $\mathbf{f}|T(y) = a(y, \mathbf{f})\mathbf{f}$ for all y. We normalize $\mathbf{f}$ so that $c(1, \mathbf{f}) = 1$. For a prime $\mathfrak{l} \nmid p$, we write $\mathbf{f}|T(\mathfrak{l}) = (\alpha_\mathfrak{l} + \beta_\mathfrak{l})\mathbf{f}$ and $\alpha_\mathfrak{l}\beta_\mathfrak{l} = \epsilon(\mathfrak{l})l^{f_\mathfrak{l}}$ if $\mathfrak{l} \nmid \mathfrak{n}p^{r+1}$ ($\alpha_\mathfrak{l}, \beta_\mathfrak{l} \in \overline{\mathbb{Q}}$), where $f_\mathfrak{l}$ is the degree of the field $O/\mathfrak{l}$ over the prime field $\mathbb{F}_l$. If $l|\mathfrak{n}$, we put $\beta_\mathfrak{l} = 0$ and define $\alpha_\mathfrak{l} \in \overline{\mathbb{Q}}$ by $\mathbf{f}|U(\mathfrak{l}) = \alpha_\mathfrak{l}\mathbf{f}$. Then the Hecke polynomial $H_\mathfrak{l}(X) = (1 - \alpha_\mathfrak{l}X)(1 - \beta_\mathfrak{l}X)$ gives the Euler $\mathfrak{l}$-factor of $L(s, \mathbf{f}) = \sum_\mathfrak{n} a(\mathfrak{n}, \mathbf{f})N(\mathfrak{n})^{-s}$ after replacing X by $|O/\mathfrak{l}|^{-s} = N(\mathfrak{l})^{-s}$ and inverting the resulted factor. Here $\mathfrak{n}$ runs over all integral ideals of F.

Let $\mathcal{F} = \{\mathbf{f}_P\}_{P \in \mathrm{Spec}(\mathbb{I})(\mathbb{C}_p)}$ be a p-adic analytic family of p-ordinary Hecke eigen cusp forms of p-slope 0. The function $P \mapsto a(\mathfrak{y}, \mathbf{f}_P)$ is a function on $\mathrm{Spec}(\mathbb{I})$ in the structure sheaf $\mathbb{I}$; so, it is a formal (and analytic) function of P. We write $\alpha_{\mathfrak{l},P}, \beta_{\mathfrak{l},P}$ for $\alpha_\mathfrak{l}, \beta_\mathfrak{l}$ for $\mathbf{f}_P$. We write $\alpha_{\mathfrak{p},P}$ for $a(\mathfrak{p}, \mathbf{f}_P) = a(\varpi_\mathfrak{p}, \mathbf{f}_P)$. In particular, the field $F[\kappa][\mu_{Np^\infty}][\alpha_{\mathfrak{p},P}]$ (for the field $F[\kappa]$ of rationality of κ defined in Sect. 1.8) is independent of the choice of $\varpi_\mathfrak{p}$ (as long as $\varpi_\mathfrak{p}$ is chosen in F). By a result of Blasius [B02] (and by an earlier work of Brylinski–Labesse), writing $|\kappa_1| := \max_\sigma(|\kappa_{1,\sigma}|)$, $N(\mathfrak{l})^{|\kappa_1|}\alpha_{\mathfrak{l},P}$ is a Weil l-number of weight $([\kappa] + 2|\kappa_1|)f_\mathfrak{l}$ for $f_\mathfrak{l}$ given by $|O/\mathfrak{l}| = l^{f_\mathfrak{l}}$. Thus $\alpha_{\mathfrak{l},P}$ is a generalized Weil number in the sense of [H13, Sect. 2].

We state the horizontal theorem in a form different from the theorem in the introduction:

Theorem 3.1. *Let $K = \mathbb{Q}(\mu_{p^\infty})$. Suppose that there exist a subset Σ of primes of F with positive upper density outside $\mathfrak{n}p$ and an infinite set $\mathcal{A}_\mathfrak{l} \subset \mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$ of arithmetic points P of the fixed weight κ as in (W) such that $[K(\alpha_{\mathfrak{l},P}) : K] \leq B_\mathfrak{l}$ for all $P \in \mathcal{A}_\mathfrak{l}$ with a bound $B_\mathfrak{l}$ for each $\mathfrak{l} \in \Sigma$ (possibly dependent on $\mathfrak{l}$). If the Zariski closure $\overline{\mathcal{A}}_\mathfrak{l}$ in $\mathrm{Spec}(\mathbb{I})$ contains an irreducible subscheme $\mathrm{Spec}(\mathbb{J})$ of dimension $r \geq 1$ independent of $\mathfrak{l} \in \Sigma$ with Zariski-dense $\mathcal{A}_\mathfrak{l} \cap \mathrm{Spec}(\mathbb{J})$ in $\mathrm{Spec}(\mathbb{J})$, then $\mathbb{I}$ has complex multiplication.*

In the above theorem, κ is independent of $\mathfrak{l}$ but $B_\mathfrak{l}$ and $\mathcal{A}_\mathfrak{l}$ can be dependent on $\mathfrak{l}$. By replacing $\mathcal{A}_\mathfrak{l}$ by a suitable infinite subset of $\mathcal{A}_\mathfrak{l} \cap \mathrm{Spec}(\mathbb{J})$, we may assume that $\overline{\mathcal{A}}_\mathfrak{l}$ is irreducible with dimension r independent of $\mathfrak{l}$. By extending W if necessary, we may assume that $\mathrm{Spec}(\mathbb{J})$ is geometrically irreducible. From the proof of this theorem given in Sect. 6, it will be clear that we can ease the assumption of the theorem so that κ is also dependent on $\mathfrak{l}$.

Let R_ν be as in Proposition 2.2 for a number field L. Then we have the following result which implies Corollary I in the introduction:

Corollary 3.2. *Let the notation be as in* Proposition 2.2 *and in the above theorem. Let Σ be a set of primes of F with positive upper density. Let* $\mathrm{Spec}(\mathbb{I})$ *be a reduced irreducible component of* $\mathrm{Spec}(\mathbf{h})$, *and assume that $\mathbb{I}$ is a finite extension of $\mathbf{\Lambda}$ inside $\overline{Q}$. If there exists a pair (L, ν) of a finite extension $L_{/F}$ and a homomorphism $\nu \in$* Hom_{g}*pscheme(T_L, T_F) such that the ring $R_\nu[a(\mathfrak{l})]$ generated over R_ν by $a(\mathfrak{l})$ inside $\overline{Q}$ has quotient field $Q(R_\nu[a(\mathfrak{l})])$ finite over the quotient field $Q(R_\nu)$ for all $\mathfrak{l} \in \Sigma$, then $\mathbb{I}$ has complex multiplication.*

Proof. Applying Proposition 2.2 to $A_\mathfrak{l} = R_\nu[a(\mathfrak{l})]$, we take $\mathcal{A}_\mathfrak{l}$ to be the set of the arithmetic points outside the closed subscheme $E_\mathfrak{l}$ for $R_\nu[a(\mathfrak{l})]$ in Proposition 2.2. Then the Zariski closure of $\mathcal{A}_\mathfrak{l}$ is the entire $\mathrm{Spec}(\mathbb{I})$ as $E_\mathfrak{l}$ has codimension at least 1. Thus the assumption of the theorem is satisfied for $\mathcal{A}_\mathfrak{l}$ for all $\mathfrak{l} \in \Sigma$. Therefore, the above theorem tells us that $\mathbb{I}$ has CM. □

This corollary implies

Corollary 3.3. *Suppose that $\mathbb{I}$ is a non-CM component. Let (L, ν) be a pair of finite extension of F and $\nu \in$* Hom_{g}*pscheme(T_L, T_F). Then, for a density one set of primes Ξ of F outside $p\mathfrak{n}$, the ring $R_\nu[a(\mathfrak{l})] \subset \overline{Q}$ for each $\mathfrak{l} \in \Xi$ generated over $R_\nu \subset \overline{Q}$ by $a(\mathfrak{l})$ inside $\overline{Q}$ has quotient field of transcendental degree one over $Q(R_\nu)$ in $\overline{Q}$.*

Proof. Let Ξ be the set of primes $\mathfrak{l}$ of F made up of $\mathfrak{l}$ with $a(\mathfrak{l})$ transcendental over $Q(R_\nu)$ (as $a(\mathfrak{l}) \notin W$: non-constancy). Let Σ be the complement of Ξ outside $p\mathfrak{n}$. If Σ has positive upper density, by Corollary 3.2, $\mathbb{I}$ has complex multiplication by a subfield of L, a contradiction. Thus Σ has upper density 0, and hence Ξ has density 1. □

By Theorem 3.1, we get the following corollary:

Corollary 3.4. *Let $\mathcal{A}$ be an infinite set of arithmetic points of* $\mathrm{Spec}(\mathbb{I})$ *of fixed weight κ. Then there exists a subset Σ of primes of F with upper positive density such that $[K(a(\mathfrak{l}, \mathbf{f}_P)) : K]$ for $\mathfrak{l} \in \Sigma$ is bounded over $\mathcal{A}$ if and only if $\mathbf{f}_P$ is a CM theta series for an arithmetic P with $k(P) \geq I$.*

By the argument given after [H11, Conjecture 3.4], one can show $[K(a(\mathfrak{l}, \mathbf{f}_P)) : K]$ is bounded independently of arithmetic points $P \in \mathrm{Spec}(\mathbb{I})$ if $\mathbf{f}_{P_0}$ is square-integrable at a prime $\mathfrak{l} \nmid p$ (so, $\mathfrak{l}|\mathfrak{n}$) for one arithmetic P_0. Further, if a prime $\mathfrak{l}$ is a factor of $\mathfrak{n}$ (so $\mathfrak{l} \nmid p$) and $\mathbf{f}_P$ (or more precisely the automorphic representation generated by $\mathbf{f}_P$) is Steinberg (resp. super-cuspidal) at $\mathfrak{l}$ for an arithmetic point P, then all members of $\mathcal{F}$ are Steinberg (resp. super-cuspidal) at $\mathfrak{l}$ (see the remark after Conjecture 3.4 in [H11]). Take a prime $\mathfrak{l} \nmid \mathfrak{n}$ of O with $\alpha_{\mathfrak{l},P} \neq 0$ for some P (so, $\mathfrak{l}$ can be equal to $\mathfrak{p}$). If $\mathfrak{l} \nmid \mathfrak{n}p$, replacing $\mathbb{I}$ by a finite extension, we assume that $\det(T - \rho_\mathbb{I}(Frob_l)) = 0$ has roots in $\mathbb{I}$. Since $\alpha_{\mathfrak{l},P} \neq 0$ for some P (and hence $\alpha_{\mathfrak{l},P}$ is a p-adic unit), $\mathbf{f}_P$ is not super-cuspidal at $\mathfrak{l}$ for any arithmetic P.

Conjecture 3.5. *Let the notation be as in* Corollary 3.4. *Let $\mathcal{A}$ be an infinite subset of arithmetic points in* $\mathrm{Spec}(\mathbb{I})$ *of fixed weight κ. Then* $\lim_{P\in\mathcal{A}}[K(a(\mathfrak{l},\mathbf{f}_P)) : K] < \infty$ *for a single prime $\mathfrak{l}$ of F if and only if either $\mathbb{I}$ has complex multiplication or the automorphic representation generated by $\mathbf{f}_P$ is square integrable at $\mathfrak{l} \nmid p$ for a single $P \in \mathcal{A}$.*

4 Rigidity Lemmas

We study formal subschemes of $\widehat{G} := \widehat{\mathbb{G}}_m^n$ stable under the action of $t \mapsto t^z$ for all z in an open subgroup U of $\mathbb{Z}_p^\times$. The following lemma and its corollary were proven in [H13]. For the reader's convenience (and to make the paper self-contained), we recall the statements and their proof.

Lemma 4.1. *Let $X = \mathrm{Spf}(\mathcal{X})$ be a closed formal subscheme of $\widehat{G} = \widehat{\mathbb{G}}^n_{m/W}$ flat geometrically irreducible over W (i.e., $\mathcal{X} \cap \overline{\mathbb{Q}}_p = W$). Suppose there exists an open subgroup U of $\mathbb{Z}_p^\times$ such that X is stable under the action $\widehat{G} \ni t \mapsto t^u \in \widehat{G}$ for all $u \in U$. If there exists a subset $\Omega \subset X(\mathbb{C}_p) \cap \mu^n_{p^\infty}(\mathbb{C}_p)$ Zariski dense in X, then $\zeta^{-1}X$ is a formal subtorus for some $\zeta \in \Omega$.*

Proof. Let X^{sh} be the scheme associated with X given by $\mathrm{Spec}(\mathcal{X})$. Define X_s to be the singular locus of $X^{sh} = \mathrm{Spec}(\mathcal{X})$ over W, and put $X^\circ = X^{sh} \setminus X_s$. The scheme X_s is actually a closed formal subscheme of X. To see this, we note, by the structure theorem of complete noetherian rings, that $\mathcal{X}$ is finite over a power series ring $W[[X_1, \ldots, X_d]] \subset \mathcal{X}$ for $d := \dim_W X$ (cf. [CRT, Sect. 29]). The sheaf of continuous differentials $\Omega_{\mathcal{X}/\mathrm{Spf}(W[[X_1,\ldots,X_d]])}$ with respect to the formal Zariski topology of $\mathcal{X}$ is a torsion $\mathcal{X}$-module, and X_s is the support of the sheaf of $\Omega_{\mathcal{X}/\mathrm{Spf}(W[[X_1,\ldots,X_d]])}$ (which is a closed formal subscheme of X). The regular locus of X° is open dense in the generic fiber $X^{sh}_{/K} := X^{sh} \times_W K$ of X^{sh} (for the field K of fractions of W). Then $\Omega^\circ := X^\circ \cap \Omega$ is Zariski dense in $X^{sh}_{/K}$.

In this proof, by making scalar extension, we always assume that W is sufficiently large so that for $\zeta \in \Omega$ we focus on, we have $\zeta \in \widehat{G}(W)$ and that we have a plenty of elements of infinite order in $X(W)$ and in $X^\circ(K) \cap X(W)$, which we simply write as $X^\circ(W) := X^\circ(K) \cap X(W)$.

Note that the stabilizer U_ζ of $\zeta \in \Omega$ in U is an open subgroup of U. Indeed, if the order of ζ is equal to p^a, then $U_\zeta = U \cap (1 + p^a\mathbb{Z}_p)$. Thus making a variable change $t \mapsto t\zeta^{-1}$ (which commutes with the action of U_ζ), we may assume that the identity $\mathbf{1}$ of $\widehat{G}$ is in Ω°.

Let $\widehat{G}^{an}$, X_{an}, and X^s_{an} be the rigid analytic spaces associated with $\widehat{G}$, X, and X_s (in Berthelot's sense in [dJ95, Sect. 7]). We put $X^\circ_{an} = X_{an} \setminus X^s_{an}$, which is an open rigid analytic subspace of X_{an}. Then we apply the logarithm $\log : \widehat{G}^{an}(\mathbb{C}_p) \to \mathbb{C}_p^n = Lie(\widehat{G}^{an}_{/\mathbb{C}_p})$ sending $(t_j)_j \in \widehat{G}^{an}(\mathbb{C}_p)$ (the p-adic open unit ball centered at $\mathbf{1} = (1, 1, \ldots, 1)$) to $(\log_p(t_j))_j) \in \mathbb{C}_p^n$ for the p-adic Iwasawa logarithm map $\log_p$:

$\mathbb{C}_p^\times \to \mathbb{C}_p$. Then for each smooth point $x \in X^\circ(W)$, taking a small analytic open neighborhood V_x of x (isomorphic to an open ball in W^d for $d = \dim_W X$) in $X^\circ(W)$, we may assume that $V_x = G_x \cap X^\circ(W)$ for an n-dimensional open ball G_x in $\widehat{G}(W)$ centered at $x \in \widehat{G}(W)$. Since $\Omega^\circ \neq \emptyset$, $\log(X^\circ(W))$ contains the origin $0 \in \mathbb{C}_p^n$. Take $\zeta \in \Omega^\circ$. Write T_ζ for the Tangent space at ζ of X. Then $T_\zeta \cong W^d$ for $d = \dim_W X$. The space $T_\zeta \otimes_W \mathbb{C}_p$ is canonically isomorphic to the tangent space T_0 of $\log(V_\zeta)$ at 0.

If $\dim_W X = 1$, there exists an infinite order element $t_1 \in X(W)$. We may (and will) assume that $U = (1 + p^b\mathbb{Z}_p)$ for $0 < b \in \mathbb{Z}$. Then X is the (formal) Zariski closure $\overline{t_1^U}$ of

$$t_1^U = \{t_1^{1+p^bz} | z \in \mathbb{Z}_p\} = t_1\{t_1^{p^bz} | z \in \mathbb{Z}_p\},$$

which is a coset of a formal subgroup Z. The group Z is the Zariski closure of $\{t_1^{p^bz} | z \in \mathbb{Z}_p\}$; in other words, regarding t_1^u as a W-algebra homomorphism $t_1^u : \mathcal{X} \to \mathbb{C}_p$, we have $t_1Z = \mathrm{Spf}(\mathcal{Z})$ for $\mathcal{Z} = \mathcal{X}/\bigcap_{u\in U}\mathrm{Ker}(t_1^u)$. Since t_1^U is an infinite set, we have $\dim_W Z > 0$. From geometric irreducibility and $\dim_W X = 1$, we conclude $X = t_1Z$ and $Z \cong \widehat{\mathbb{G}}_m$. Since X contains roots of unity $\zeta \in \Omega \subset \mu_{p^\infty}^n(W)$, we confirm that $X = \zeta Z$ for $\zeta \in \Omega \cap \mu_{p^{b'}}^n$ for $b' \gg 0$. This finishes the proof in the case where $\dim_W X = 1$.

We prepare some result (still assuming $d = 1$) for an induction argument on d in the general case. Replacing t_1 by $t_1^{p^b}$ for b as above if necessary, we have the translation $\mathbb{Z}_p \ni s \mapsto \zeta t_1^s \in Z$ of the one parameter subgroup $\mathbb{Z}_p \ni s \mapsto t_1^s$. Thus we have $\log(t_1) = \frac{dt_1^s}{ds}|_{s=0} \in T_\zeta$, which is sent by "$\log : \widehat{G} \to \mathbb{C}_p^n$" to $\log(t_1) \in T_0$. This implies that $\log(t_1) \in T_0$ and hence $\log(t_1) \in T_\zeta$ for any $\zeta \in \Omega^\circ$ (under the identification of the tangent space at any $x \in \widehat{G}$ with $Lie(\widehat{G})$). Therefore T_ζ's over $\zeta \in \Omega^\circ$ can be identified canonically. This is natural as Z is a formal torus, and the tangent bundle on Z is constant, giving $Lie(Z)$.

Suppose now that $d = \dim_W X > 1$. Consider the Zariski closure Y of t^U for an infinite order element $t \in V_\zeta$ (for $\zeta \in \Omega^\circ$). Since U permutes finitely many geometrically irreducible components, each component of Y is stable under an open subgroup of U. Therefore $Y = \bigcup \zeta'\mathcal{T}_{\zeta'}$ is a union of formal subtori $\mathcal{T}_{\zeta'}$ of dimension ≤ 1, where ζ' runs over a finite set inside $\mu_{p^\infty}^n(\mathbb{C}_p) \cap X(\mathbb{C}_p)$. Since $\dim_W Y = 1$, we can pick $\mathcal{T}_{\zeta'}$ of dimension 1 which we denote simply by $\mathcal{T}$. Then $\mathcal{T}$ contains t^u for some $u \in U$. Applying the argument in the case of $\dim_W X = 1$ to $\mathcal{T}$, we find $u\log(t) = \log(t^u) \in T_\zeta$; so, $\log(t) \in T_\zeta$ for any $\zeta \in \Omega^\circ$ and $t \in V_\zeta$. Summarizing our argument, we have found

(T) The Zariski closure of t^U in X for an element $t \in V_\zeta$ of infinite order contains a coset $\xi\mathcal{T}$ of one dimensional subtorus $\mathcal{T}$, $\xi^{p^b} = 1$ and $t^{p^b} \in \mathcal{T}$ for some $b > 0$;
(D) Under the notation as above, we have $\log(t) \in T_\zeta$.

Moreover, the image $\overline{V}_\zeta$ of V_ζ in $\widehat{G}/\mathcal{T}$ is isomorphic to $(d-1)$-dimensional open ball. If $d > 1$, therefore, we can find $\overline{t}' \in \overline{V}_\zeta$ of infinite order. Pulling back $\overline{t}'$ to

$t' \in V_\zeta$, we find $\log(t), \log(t') \in T_\zeta$, and $\log(t)$ and $\log(t')$ are linearly independent in T_ζ. Inductively arguing this way, we find infinite order elements $t_1, \dots, t_d$ in V_ζ such that $\log(t_i)$ span over the quotient field $\mathbb{K} = Q(W)$ of W the tangent space $T_{\zeta/\mathbb{K}} = T_\zeta \otimes_W \mathbb{K} \hookrightarrow T_0$ (for any $\zeta \in \Omega^\circ$). We identify $T_{\mathbf{1}/\mathbb{K}} \subset T_0$ with $T_{\zeta/\mathbb{K}} \subset T_0$. Thus the tangent bundle over $X^\circ_{/\mathbb{K}}$ is constant as it is constant over the Zariski dense subset Ω°. Therefore X° is something close to an open dense subscheme of a coset of a formal subgroup. We pin-down this fact that X° is a coset of a formal scheme.

Take $t_j \in V_\zeta$ as above ($j = 1, 2, \dots, d$) which give rise to a basis $\{\partial_j = \log(t_j)\}_j$ of the tangent space of $T_{\zeta/\mathbb{K}} = T_{\mathbf{1}/\mathbb{K}}$. Note that $t_j^u \in X$ and $u\partial_j = \log(t_j^u) = u\log(t_j) \in T_{\mathbf{1}/\mathbb{K}}$ for $u \in U$. The embedding $\log : V_\zeta \hookrightarrow T_{\mathbf{1}} \subset Lie(\widehat{G}_{/W})$ is surjective onto a open neighborhood of $0 \in T_{\mathbf{1}}$ (by extending scalars if necessary). For $t \in V_\zeta$, as $t \to \zeta$, $\log(t) \to 0$. Thus by replacing $t_1, \dots, t_d$ inside V_ζ with elements in V_ζ closer to ζ, we may assume that $\log(t_i) \pm \log(t_j)$ for all $i \neq j$ belong to $\log(V_\zeta)$.

So, for each pair $i \neq j$, we can find $t_{i\pm j} \in V_\zeta$ such that $\log(t_i t_j^{\pm 1}) = \log(t_i) \pm \log(t_j) = \log(t_{i\pm j})$. The element $\log(t_{i\pm j})$ is uniquely determined in $\log(\widehat{G}_{an}(\mathbb{C}_p)) \cong \widehat{G}_{an}(\mathbb{C}_p)/\mu^n_{p^\infty}(\mathbb{C}_p)$. Thus we conclude $\zeta'_{i\pm j} t_i t_j^{\pm 1} = t_{i\pm j}$ for some $\zeta'_{i\pm j} \in \mu^n_{p^N}$ for sufficiently large N. Replacing X by its image under the p-power isogeny $\widehat{G} \ni t \mapsto t^{p^N} \in \widehat{G}$ and t_i by $t_i^{p^N}$, we may assume that $t_i t_j^{\pm 1} = t_{i\pm j}$ all in X. Since $t_i^U \subset X$, by (T), for a sufficiently large $b \in \mathbb{Z}$, we find a one dimensional subtorus $\widehat{H}_i$ containing $t_i^{p^b}$ such that $\zeta_i \widehat{H}_i \subset X$ with some $\zeta_i \in \mu^n_{p^b}$ for all i. Thus again replacing X by the image of the p-power isogeny $\widehat{G} \ni t \mapsto t^{p^b} \in \widehat{G}$, we may assume that the subgroup $\widehat{H}$ (Zariski) topologically generated by $t_1, \dots, t_d$ is contained in X. Since $\{\log(t_i)\}_i$ is linearly independent, we conclude $\dim_W \widehat{H} \geq d = \dim_W X$, and hence X must be the formal subgroup $\widehat{H}$ of $\widehat{G}$. Since X is geometrically irreducible, $\widehat{H} = X$ is a formal subtorus. Pulling it back by the p-power isogenies we have used, we conclude $X = \zeta\widehat{H}$ for the original X and $\zeta \in \mu^n_{p^{bN}}(W)$. Since Ω is Zariski dense in X, we may assume that $\zeta \in \Omega$. This finishes the proof. □

Corollary 4.2. *Let W be a complete discrete valuation ring in $\mathbb{C}_p$. Write $W[[T]] = W[[T_1, \dots, T_n]]$ for the tuple of variables $T = (T_1, \dots, T_n)$. Let*

$$\widehat{G} := \widehat{\mathbb{G}}^n_m = \mathrm{Spf}(\widehat{W[t_1, t_1^{-1} \dots, t_n, t_n^{-1}]}),$$

and identify $\widehat{W[t_1, t_1^{-1} \dots, t_n, t_n^{-1}]}$ with $W[[T]]$ for $t_j = 1 + T_j$. Let $\Phi(T_1, \dots, T_n) \in W[[T]]$. Suppose that there is Zariski dense subset $\Omega \subset \mu^n_{p^\infty}(\mathbb{C}_p)$ in $\widehat{G}(\mathbb{C}_p)$ such that $\Phi(\zeta - 1) \in \mu_{p^\infty}(\mathbb{C}_p)$ for all $\zeta \in \Omega$. Then there exists $\zeta_0 \in \mu_{p^\infty}(W)$ and $z = (z_j)_j \in \mathbb{Z}_p^n$ with $z_j \in \mathbb{Z}_p$ such that $\zeta_0^{-1}\Phi(t) = \prod_j (t_j)^{z_j}$, where $(1 + T)^x = \sum_{j=0}^\infty \binom{x}{j} T^j$ with $x \in \mathbb{Z}_p$.

Proof. Pick $\eta = (\eta_j) \in \Omega$. Making variable change $T \mapsto \eta^{-1}(T + 1) - 1$ (i.e., $T_j \mapsto \eta_j^{-1}(T_j + 1) - 1$ for each j) replacing W by its finite extension if necessary, we may replace Ω by $\eta^{-1}\Omega \ni 1$; so, rewriting $\eta^{-1}\Omega$ as Ω, we may assume that $\mathbf{1} \in \Omega$.

Then $\Phi(0) = \zeta_0 \in \mu_{p^\infty}$. Thus again replacing Φ by $\zeta_0^{-1}\Phi$, we may assume that $\Phi(0) = 1$.

For $\sigma \in \mathrm{Gal}(\mathbb{K}(\mu_{p^\infty})/\mathbb{K})$ with the quotient field $\mathbb{K}$ of W, $\Phi(\zeta^\sigma - 1) = \Phi(\zeta - 1)^\sigma$. Writing $\phi(\zeta) = \Phi(\zeta - 1)$, the above identity means $\phi(\zeta^\sigma) = \phi(\zeta)^\sigma$. Identify $\mathrm{Gal}(\mathbb{K}(\mu_{p^\infty})/\mathbb{K})$ with an open subgroup U of $\mathbb{Z}_p^\times$. This is possible as W is a discrete valuation ring, while $W[\mu_{p^\infty}]$ is not. Writing $\sigma_u \in \mathrm{Gal}(\mathbb{K}(\mu_{p^\infty})/\mathbb{K})$ for the element corresponding to $u \in U$, we find that

$$\Phi \circ u(\zeta - 1) = \Phi(\zeta^u - 1) = \Phi(\zeta^{\sigma_u} - 1) = \Phi(\zeta - 1)^{\sigma_u} = u \circ \Phi(\zeta - 1).$$

We find that $u \circ \phi = \phi \circ u$ is valid on the Zariski dense subset Ω of $\mathrm{Spec}(W[[T]])$; so, ϕ as a scheme morphism of $\widehat{G} = \widehat{\mathbb{G}}_m^n$ into $\widehat{\mathbb{G}}_m$ commutes with the action of $u \in U$.

Note that $u \in \mathbb{Z}_p^\times$ acts on $\widehat{\mathbb{G}}_m$ as a group automorphism induced by a W-bialgebra automorphism of $W[[T]]$ sending $t = (1 + T) \mapsto t^u = (1 + T)^u = \prod_j (1 + T_j)^u$. Take the morphism of formal schemes $\phi \in \mathrm{Hom}_{SCH/W}(\widehat{\mathbb{G}}_m^n, \widehat{\mathbb{G}}_m)$, which sends 1 to 1. Put $\widehat{\mathbf{G}} := \widehat{\mathbb{G}}_m^n \times \widehat{\mathbb{G}}_{m/W}$. We consider the graph Γ_ϕ of ϕ which is an irreducible formal subscheme $\Gamma_\phi \subset \widehat{\mathbb{G}}_m^n \times \widehat{\mathbb{G}}_m$ smooth over W. Writing the variable on $\widehat{\mathbf{G}}$ as (T, T'), Γ_ϕ is the geometrically irreducible closed formal subscheme containing the identity $\mathbf{1} \in \widehat{\mathbf{G}}$ defined by the principal ideal $(t' - \phi(t))$. Since $\phi \circ u = u \circ \phi$ for all u in an open subgroup U of $\mathbb{Z}_p^\times$ (where U acts on the source $\widehat{\mathbb{G}}_m^n$ and on the target $\widehat{\mathbb{G}}_m$ by $t \mapsto t^u$), Γ_ϕ is stable under the diagonal action of U on $\widehat{\mathbf{G}}$ and is finite flat over $\widehat{\mathbb{G}}_m^n$ (the left factor of $\widehat{\mathbf{G}}$). Then, applying Lemma 4.1 to Γ_ϕ, we find that Γ_ϕ is a subtorus of rank n surjecting down to the last factor $\widehat{\mathbb{G}}_m$. Since any subtorus of rank n in $\widehat{\mathbf{G}}$ whose projection to the last factor is defined by the equation $t' = (1 + T)^z$, $t' = \Phi(T)$, we have the power series identity $\Phi(T) = t' = (1 + T)^z$ in $W[[T]]$ identifying $\Gamma_\phi = \mathrm{Spf}(W[[T]])$. $\square$

5 Frobenius Eigenvalue Formula

Recall the fixed weight κ with $\kappa_2 - \kappa_1 \geq I$. We assume the following conditions and notations:

(J1) Let $\mathrm{Spec}(\mathbb{J})$ be a closed reduced geometrically irreducible subscheme of $\mathrm{Spec}(\mathbb{I})$ flat over $\mathrm{Spec}(W)$ of relative dimension r with Zariski dense set $\mathcal{A}$ of arithmetic points of the fixed weight κ.

(J2) We identify $\mathrm{Spf}(\mathbf{\Lambda})$ for $\mathbf{\Lambda} = W[[\mathbf{\Gamma}]]$ with $\widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \mathbf{\Gamma}^*$ for $\mathbf{\Gamma}^* := \mathrm{Hom}_{\mathbb{Z}_p}(\mathbf{\Gamma}, \mathbb{Z}_p)$ naturally.

Then for any direct $\mathbb{Z}_p$-summand $\Gamma \subset \mathbf{\Gamma}$, $\widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \Gamma^*$ is a closed formal torus of $\widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \mathbf{\Gamma}^*$. We insert here a lemma (essentially) proven in [H13, Lemma 5.1].

Lemma 5.1. *Let the notation and the assumption be as in (J1–2). Then, after making extension of scalars to a sufficiently large complete discrete valuation ring*

$W \subset \mathbb{C}_p$, we can find a $\mathbb{Z}_p$-direct summand Γ of $\mathbf{\Gamma}$ with rank $\dim_W \mathrm{Spf}(\mathbb{J})$ and an arithmetic point $P_0 \in \mathcal{A} \cap \mathrm{Spec}(\mathbb{J})(W)$ such that we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Spf}(\mathbb{J}) & \longrightarrow & P_0 \cdot (\widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \Gamma^*) \\ \cap\big\downarrow & & \big\downarrow\cap \\ \mathrm{Spf}(\mathbb{I}) & \longrightarrow & \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \mathbf{\Gamma}^* = \mathrm{Spf}(\mathbf{\Lambda}), \end{array}$$

which becomes Cartesian after localizing at each arithmetic point of $\mathrm{Spf}(\mathbb{J})$*, and* $\mathrm{Spf}(\mathbb{J})$ *gives a geometrically irreducible component of* $\mathrm{Spf}(\mathbb{I}) \times_{\mathrm{Spf}(\mathbf{\Lambda})} P_0 \cdot (\widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \Gamma^*)$*. Here $P_0 \cdot (\widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \Gamma^*)$ is the image of the multiplication by the point $P_0 \in \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \mathbf{\Gamma}^*$ inside $\widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \mathbf{\Gamma}^*$.*

In [H13, Lemma 5.1], it was claimed the diagram is Cartesian, which is wrong (as the fiber product could have several components). The correct statement is as above. This correction does not affect the results obtained in [H13].

Proof. Let $\pi : \mathrm{Spec}(\mathbb{J}) \to \mathrm{Spec}(\mathbf{\Lambda})$ be the projection. Then the smallest reduced closed subscheme $Z \subset \mathrm{Spec}(\mathbf{\Lambda})$ containing the topological image of π contains an infinitely many arithmetic points of weight κ. Since $\mathbb{J}$ is a domain with geometrically irreducible $\mathrm{Spec}(\mathbb{J})$, Z is geometrically irreducible. Take a basis $\{\gamma_1, \ldots, \gamma_m\}$ of $\mathbf{\Gamma}$, and write $\widehat{G} := \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \mathbf{\Gamma}^*$ as $\mathrm{Spf}(\widehat{W[t_j, t_j^{-1}]}_{j=1,\ldots,m})$ for the variable t_j corresponding to the dual basis $\{\gamma_j^*\}_j$ of $\mathbf{\Gamma}^*$. Let $P_1 \in Z$ be an arithmetic point of weight κ under $P \in \mathrm{Spec}(\mathbb{J})(W)$ (after replacing W by its finite extension, we can find a W-point P). Then by the variable change $t \mapsto P_1^{-1} \cdot t$ (which can be written as $t_j \mapsto \zeta_j \gamma_j^{-\kappa_2} t_j$ for suitable $\zeta_j \in \mu_{p^\infty}(W)$), the image of arithmetic points of $\mathrm{Spec}(\mathbb{J})$ of weight κ in Z are contained in $\mu_{p^\infty}^m(\overline{\mathbb{Q}}_p)$. Since Z is defined over W, $\Omega := Z(\mathbb{C}_p) \cap \mu_{p^\infty}^m(\mathbb{C}_p)$ is stable under $\mathrm{Gal}(\mathbb{K}[\mu_{p^\infty}]/\mathbb{K})$ for the quotient field $\mathbb{K}$ of W. Identify $\mathrm{Gal}(\mathbb{K}[\mu_{p^\infty}]/\mathbb{K})$ with a closed subgroup U of $\mathbb{Z}_p^\times$ by the p-adic cyclotomic character. Since W is a discrete valuation ring, U has to be also open in $\mathbb{Z}_p^\times$. Since $u \in U$ acts on Ω by $\zeta \mapsto \zeta^u$, Z is stable under the central action $\widehat{G} \ni t \mapsto t^u \in \widehat{G}$. Then by Lemma 4.1, we may assume, after making further variable change $t \mapsto \eta^{-1} t$ for $\eta \in \mu_{p^\infty}^m(W)$ (again replacing W by a finite extension if necessary), that Z is a formal subtorus; i.e., $Z = \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \Gamma^*$ for a direct summand Γ of $\mathbf{\Gamma}$. Since $\mathbb{J}$ is an integral extension of the normal domain $\Lambda := W[[\Gamma]]$, by Matsumura [CRT, Theorems 9.4 and 15.2–3], we conclude $\dim_W \mathbb{J} = \dim_W Z = \mathrm{rank}_{\mathbb{Z}_p} \Gamma$. Then putting $P_0 = P_1 \cdot \eta$, we get the commutative diagram. Thus we have a natural closed immersion $\mathrm{Spf}(\mathbb{J}) \hookrightarrow \mathrm{Spf}(\mathbb{I}) \times_{\mathrm{Spf}(\mathbf{\Lambda}_W)} P_0 \cdot (\widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \Gamma^*) \subset \mathrm{Spf}(\mathbb{I})$ by the universality of the fiber product. Since $\mathbb{I}$ is an integral extension of the normal domain $\mathbf{\Lambda}$, by Matsumura [CRT, Theorem 15.1], we have $\dim_W \mathrm{Spf}(\mathbb{I}) \times_{\mathrm{Spf}(\mathbf{\Lambda}_W)} P_0 \cdot (\widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \Gamma^*) = \mathrm{rank}_{\mathbb{Z}_p} \Gamma = \dim_W \mathbb{J}$. Thus $\mathrm{Spec}(\mathbb{J})$ is an irreducible component of $\mathrm{Spf}(\mathbb{I}) \times_{\mathrm{Spf}(\mathbf{\Lambda}_W)} P_0 \cdot (\widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \Gamma^*)$.

We can see that $\mathrm{Spec}(\mathbb{J})$ is an irreducible component of the fiber product in a more concrete way. At each arithmetic point $P \in \mathrm{Spf}(\mathbb{I})$, the localized ring extension $\mathbb{I}_P/\mathbf{\Lambda}_P$ is an étale extension (cf. [HMI, Proposition 3.78]). The morphism $\mathrm{Spec}(\mathbb{J}) \to Z$ is dominant of equal dimension; so, it is generically étale. Thus $\Omega_{\mathrm{Spf}(\mathbb{J})/Z}$ is a torsion $\mathbb{J}$-module. Hence the étale locus of $\mathrm{Spec}(\mathbb{J})^{\text{ét}}$ over Z is equal to the complement of the support of $\Omega_{\mathrm{Spf}(\mathbb{J})/Z}$. In particular, $\mathrm{Spec}(\mathbb{J})^{\text{ét}}$ is an open dense subscheme of $\mathrm{Spec}(\mathbb{J})$. Since arithmetic points are dense in $\mathrm{Spec}(\mathbb{J})$, we can find an arithmetic point $P \in \mathrm{Spec}(\mathbb{J})^{\text{ét}}$. Then we have the commutative diagram localized at P:

$$\begin{array}{ccc} \mathbb{J}_P & \xleftarrow{P_0^*} & \Lambda_P \\ \uparrow & & \uparrow \\ \mathbb{I}_P & \longleftarrow & \mathbf{\Lambda}_P \end{array}$$

By our choice of P, all horizontal morphisms in the above diagram are smooth (and all members of the diagram are integral domains). Thus the above diagram is Cartesian. In particular, $\mathrm{Spf}(\mathbb{J})$ is a geometrically irreducible component of the fiber of $\mathrm{Spf}(\mathbb{I})$ over $P_0 \cdot (\widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \Gamma^*)$. $\square$

Take Γ as in Lemma 5.1 given for $\mathbb{J}$, and write $\Lambda = W[[\Gamma]]$. Fix a basis $\gamma_1, \dots, \gamma_r \in \Gamma$ and identify Λ with $W[[T]](T = (T_i)_{i=1,\dots,r})$ by $\gamma_i \leftrightarrow t_i = 1 + T_i$. Let Q be the quotient field of Λ and fix its algebraic closure $\overline{Q}$. We embed $\mathbb{J}$ into $\overline{Q}$. We introduce one more notation:

(J3) If $\mathfrak{l}|p$, let $A_\mathfrak{l}$ be the image $a(\varpi_\mathfrak{l})$ in $\mathbb{J}$, and if $\mathfrak{l} \nmid \mathfrak{n}p$, fix a root $A_\mathfrak{l}$ in $\overline{Q}$ of $\det(T - \rho_\mathbb{J}(Frob_\mathfrak{l})) = 0$. Replacing $\mathbb{J}$ by a finite extension, we assume that $A_\mathfrak{l} \in \mathbb{J}$.

If the prime $\mathfrak{l}$ is clearly understood in the context, we simply write A for $A_\mathfrak{l}$. Recall the notation $A_P = P(A)$. Take and fix p^nth root t_i^{1/p^n} of t_i in $\overline{Q}$ $(i = 1, 2, \dots, r)$ and consider

$$W[\mu_{p^n}][[T]][t^{1/p^n}] := W[\mu_{p^n}][[T_1, \dots, T_r]][t_1^{1/p^n}, \dots, t_r^{1/p^n}] \subset \overline{Q}$$

which is independent of the choice of t^{1/p^n}. Take a basis $\{\gamma = \gamma_1, \dots, \gamma_m\}$ of $\mathbf{\Gamma}$ over $\mathbb{Z}_p$ (containing $\{\gamma_1, \dots, \gamma_r\}$). We write t_j for the variable of $\widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \mathbf{\Gamma}^*$ corresponding to the dual basis of $\{\gamma_j\}_j$ of $\mathbf{\Gamma}^*$. We recall another result from [H13, Proposition 5.2] and its proof (to make the paper self-contained and also by the request of one of the referees):

Proposition 5.2 (Frobenius Eigenvalue Formula). *Let the notation and the assumption be as in (J1–3), and fix a prime ideal $\mathfrak{l}$ prime to $\mathfrak{n}$ as in (J3). Write $K := \mathbb{Q}[\mu_{p^\infty}]$ and $L_P = K(A_P)$ for each arithmetic point P with $\kappa(P) = \kappa$. Suppose*

($\mathrm{BT}_\mathfrak{l}$) *L_P/K is a finite extension of degree bounded (independently of $P \in \mathcal{A}$) by a bound $B_\mathfrak{l} > 0$ dependent on $\mathfrak{l}$.*

Then, after making extension of scalars to a sufficiently large W, we have

$$A = A_{\mathfrak{l}} \in W[\mu_{p^n}][[T_1, \dots, T_r]][t_1^{1/p^n}, \dots, t_r^{1/p^n}] \cap \mathbb{J}$$

in $\overline{Q}$ for $0 \le n \in \mathbb{Z}$, and there exists $s = (s_i) \in \mathbb{Q}_p^r$ and a constant $c \in W^\times$ such that $A(T) = ct^s = c \prod_i t_i^{s_i}$ ($t_i = 1 + T_i$).

To simplify the notation, for $k = r$ or m, we often write $(\zeta\gamma^{-\kappa_2}t - 1)$ for the ideal in $W[[T_1, \dots, T_k]]$ generated by a tuple $(\zeta_j\gamma_j^{-\kappa_2}t_j - 1)$ for $j = 1, 2, \dots, k$ (where $\zeta = (\zeta_j)$ is also a tuple in $\mu_{p^\infty}^k(\overline{\mathbb{Q}}_p)$). The value of k should be clear in the context.

Proof. Since $\mathcal{A}$ is Zariski dense in $\mathrm{Spec}(\mathbb{J})$, for any $\mathrm{Gal}(\mathbb{K}[\mu_{p^\infty}]/\mathbb{K})$ for the field $\mathbb{K}$ of fractions of W, $\mathcal{A}_{st} := \bigcup_{\sigma \in \mathrm{Gal}(\mathbb{K}[\mu_{p^\infty}]/\mathbb{K})} \mathcal{A}^\sigma$ is Zariski dense in $\mathrm{Spec}(\mathbb{J})$ and stable under $\mathrm{Gal}(\mathbb{K}[\mu_{p^\infty}]/\mathbb{K})$. We replace $\mathcal{A}$ by $\mathcal{A}_{st}$. Let $Z = \mathrm{Spec}(\mathbf{\Lambda}/\mathfrak{a})$ for $\mathfrak{a} := \mathrm{Ker}(\mathbf{\Lambda} \to \mathbb{J})$ be the image of $\mathrm{Spec}(\mathbb{J})$ in $\mathrm{Spec}(\mathbf{\Lambda})$, and identify $\mathcal{A}$ with its image in Z. By Proposition 2.1 (and by a remark just above Theorem 3.1), we have only a finite number of generalized Weil l-numbers α of weight $[\kappa]f_{\mathfrak{l}}$ with bounded l-power denominator (i.e., $l^B\alpha$ is a Weil number of weight $([\kappa] + 2B)f_{\mathfrak{l}}$ for some $B > 0$) in $\bigcup_{P \in \mathcal{A}} L_P$ up to multiplication by p-power roots of unity. Here we can take $B = |\kappa_1|$. Hence, replacing $\mathcal{A}$ by a subset, we may assume that A_P for all $P \in \mathcal{A}$ hits one α of such generalized Weil l-numbers of weight $[\kappa]f_{\mathfrak{l}}$, up to p-power roots of unity, since the automorphic representation generated by $\mathbf{f}_P$ is not Steinberg because $\mathfrak{l} \nmid \mathfrak{n}$.

Let P_0 be as in Lemma 5.1 for this $\mathcal{A}$. By making a variable change $t \mapsto P_0 \cdot t$, we may assume that $P_0 = (t_j - 1)_{j=1,\dots,m}$, and $\mathcal{A}$ sits above $\mu_{p^\infty}^r(K)$, where we regard $\mu_{p^\infty}^r = \mu_{p^\infty} \otimes_{\mathbb{Z}_p} \Gamma^*$ as a subgroup of $\widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \Gamma^*$ (for $\Gamma \cong \mathbb{Z}_p^r$ as in Lemma 5.1) isomorphic to $\mathrm{Spf}(W[[\Gamma]]) = \mathrm{Spf}(\widehat{W[t_1, t_1^{-1}, \dots, t_r, t_r^{-1}]}) = \mathrm{Spf}(W[[T_1 \dots, T_r]])$ with $t_j = 1 + T_j$.

After the variable change $t \mapsto P_0 \cdot t$ ($\Leftrightarrow T_j \mapsto Y_j$) described above, suppose for the moment $\mathbb{J} \cong \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \Gamma^*$ (i.e., P_0 goes to the identity of $\widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \Gamma^*$ with $\mathbb{J} = W[[Y_1 \dots, Y_r]] = \Lambda$ (writing y_j for the variable corresponding to t_j and $y_j = 1 + Y_j$ and hence $A \in \Lambda$). Choosing $\gamma_1, \dots, \gamma_r$ to be a generator of Γ for $r = \mathrm{rank}_{\mathbb{Z}_p} \Gamma$, we may assume that the projection $\mathbf{\Lambda} \to \mathbb{J}$ has kernel $(t_{r+1} - 1, \dots, t_m - 1)$. In down to earth terms, for $A_{\mathfrak{l}} = A(T)$ in (J3), the variable change $t \mapsto P_0 \cdot t$ is the variable change $T_j \mapsto Y_j = \zeta_j\gamma_j^{-\kappa_2}(1 + T_j) - 1$ with $Y = (Y_1, \dots, Y_m)$, and we have $A(Y)|_{Y=0} = A(T)|_{T_j = \zeta_j\gamma_j^{\kappa_2} - 1}$. Let

$$\Phi_1(Y) := \alpha^{-1}A(Y) = \alpha^{-1}A(\gamma^{-\kappa_2}(1 + T) - 1) \in W[[Y]]$$

and $\mathbf{L}$ be the composite of L_P for P running through $\mathcal{A}$. By this variable change, $\mathcal{A}$ is brought into a Zariski dense subset Ω_1 of $\mu_{p^\infty}^r(\overline{\mathbb{Q}}_p) \subset \widehat{\mathbb{G}}_m^r = \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \Gamma^*$ made up of ζ such that $\Phi_1(\zeta - 1)$ is a root of unity in $\mathbf{L}$. It is easy to see (e.g., [H11, Lemma 2.6]) that the group of roots of unity of $\mathbf{L}$ contains $\mu_{p^\infty}(K)$ as a subgroup of finite index, and we find a subset $\Omega \subset \Omega_1$ Zariski dense in $\widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \Gamma^* = \mathrm{Spec}(\mathbb{J})$ and a root of unity ζ_1 such that $\{\Phi_1(\zeta - 1) | \zeta \in \Omega\} \subset \zeta_1\mu_{p^\infty}(K)$. Then $\Phi = \zeta_1^{-1}\Phi_1$ satisfies the assumption of Corollary 4.2, and for a root of unity ζ, we have $A(Y) = \zeta\alpha(1+Y)^s$ for $s \in \mathbb{Z}_p^r$, and $A(T) = \zeta\alpha(\gamma^{-\kappa_2}(1+T))^s$. Thus $A(T) = c(1+T)^s$ for a non-zero p-adic unit $c = \zeta\alpha\gamma^{-\kappa_2 s} \in W^\times$ as desired.

More generally, we now assume that $A \in W[[T]][t^{1/p^n}]$ (so, $\mathbb{J}$ is an extension of $W[[T_1 \dots, T_r]]$ and $A \in \mathbb{J} \cap W[[T_1 \dots, T_r]][t_1^{1/p^n}, \dots, t_r^{1/p^n}]$). Since

$$\mathrm{Spf}(W[[T]][t^{1/p^n}]]) \cong \widehat{\mathbb{G}}_m^r \xrightarrow{t \mapsto t^{p^n}} \widehat{\mathbb{G}}_m^r = \mathrm{Spf}(W[[T]]),$$

by applying the same argument as above to $W[[T]][t^{1/p^n}]]$, we get $A(T) = c(1+T)^{s/p^n}$ for $s \in \mathbb{Z}_p^r$ and a constant $c \neq 0$.

We thus need to show $A \in W[\mu_{p^n}][[T]][t^{1/p^n}]$ for sufficient large n, and then the result follows from the above argument. Again we make the variable change $T \mapsto Y$ we have already done. Replacing A by $\alpha^{-1}A$ for a suitable Weil l-number α of weight k (up to $\mu_{p^\infty}(\overline{\mathbb{Q}}_p)$), we may assume that there exists a Zariski dense set $\mathcal{A}_0 \subset \mathrm{Spec}(\mathbb{J})(\overline{\mathbb{Q}}_p)$ such that $P \cap \Lambda = (1 + Y - \zeta_P)$ for $\zeta_P \in \mu_{p^\infty}^r(\overline{\mathbb{Q}}_p)$ and $A_P \in \mu_{p^\infty}(\overline{\mathbb{Q}}_p)$ for all $P \in \mathcal{A}_0$. By another variable change $(1+Y) \mapsto \zeta(1+Y)$ for a suitable $\zeta \in \mu_{p^\infty}^r(\overline{\mathbb{Q}}_p)$, we may further assume that we have $P_0 \in \mathcal{A}_0$ with $\zeta_{P_0} = 1$ and $A_{P_0} = 1$ (i.e., choosing α well in $\alpha \cdot \mu_{p^\infty}(\overline{\mathbb{Q}}_p)$). We now write $\mathbb{J}'$ for the subalgebra of $\mathbb{J}$ topologically generated by A over $\Lambda = W[[Y]]$. Then we have $\mathbb{J}' := \Lambda[A] \subset \mathbb{J}$. Since $\mathbb{J}$ is geometrically irreducible, the base ring W is integrally closed in $\mathbb{J}'$. Since A is a unit in $\mathbb{J}$, we may embed the irreducible formal scheme $\mathrm{Spf}(\mathbb{J}')$ into $\widehat{\mathbb{G}}_m^r \times \widehat{\mathbb{G}}_m = \mathrm{Spf}(\widehat{W[y, y^{-1}, t', t'^{-1}]})$ by the surjective W-algebra homomorphism $\pi : \widehat{W[y, y^{-1}, t', t'^{-1}]} \twoheadrightarrow \mathbb{J}'$ sending (y, t') to $(1+Y, A)$. Write $Z \subset \widehat{\mathbb{G}}_m^r \times \widehat{\mathbb{G}}_m$ for the reduced image of $\mathrm{Spf}(\mathbb{J}')$. Thus we are identifying Λ with $\widehat{W[y, y^{-1}]}$ by $y \leftrightarrow 1+Y$. Then $P_0 \in Z$ is the identity element of $(\widehat{\mathbb{G}}_m^r \times \widehat{\mathbb{G}}_m)(\overline{\mathbb{Q}}_p)$. Since A is integral over Λ, it is a root of a monic polynomial $\Phi(t') = \Phi(y, t') = t'^d + a_1(y)t'^{d-1} + \cdots + a_d(y) \in \Lambda[t']$ irreducible over the quotient field Q of Λ, and we have $\mathbb{J}' \cong \Lambda[t']/(\Phi(y, t'))$. Thus $\mathbb{J}$ is free of rank, say d, over Λ; so, $\pi : Z \to \widehat{\mathbb{G}}_m^r = \mathrm{Spf}(\Lambda)$ is a finite flat morphism of degree d. We let $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ act on Λ by $\sum_{n=0}^\infty a_n Y^n \mapsto \sum_{n=0}^\infty a_n^\sigma Y^n$ and on $\Lambda[t']$ by $\sum_j A_j(Y)t'^j \mapsto \sum_j A_j^\sigma(Y)t'^j$ for $A_j(Y) \in \Lambda$. Note that $\Phi(\zeta_P, A_P) = 0$ for $P \in \mathcal{A}_0$. Since $A_P \in \mu_{p^\infty}(\overline{\mathbb{Q}}_p)$, $A_P^\sigma = A_P^{\nu(\sigma)}$ for the p-adic cyclotomic character $\nu : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathbb{Z}_p^\times$. Since W is a discrete valuation ring, for its quotient field F, the image of ν on $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$ is an open subgroup U of $\mathbb{Z}_p^\times$. Thus we have $\Phi^\sigma(\zeta_P^{\nu(\sigma)}, A_P^{\nu(\sigma)}) = \Phi(\zeta_P, A_P)^\sigma = 0$ for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and if $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$, $\Phi^\sigma = \Phi$. Thus we get

$$\Phi(\zeta_P^{\nu(\sigma)}, A_P^{\nu(\sigma)}) = \Phi(\zeta_P, A_P)^\sigma = 0 \text{ for all } P \in \mathcal{A}_0.$$

For $s \in \mathbb{Z}_p^\times$, consider the integral closed formal subscheme $Z_s \subset \widehat{\mathbb{G}}_m^r \times \widehat{\mathbb{G}}_m$ defined by $\Phi(y^s, t'^s) = 0$. If $s \in U$, we have $\mathcal{A}_0 \subset Z \cap Z_s$. Since Z and Z_s are finite flat over Λ and $\mathcal{A}_0$ is Zariski dense, we conclude $Z = Z_s$. Thus $Z \subset \widehat{\mathbb{G}}_m^r \times \widehat{\mathbb{G}}_m$ is stable under the diagonal action $(y, t') \mapsto (y^s, t'^s)$ for $s \in U$. By Lemma 4.1, Z is a formal multiplicative group and is a formal subtorus of $\widehat{\mathbb{G}}_m^r \times \widehat{\mathbb{G}}_m$, because $\mathbf{1} = P_0 \in Z$. The projection $\pi : Z \to \mathrm{Spf}(\Lambda) = \widehat{\mathbb{G}}_m^r$ is finite flat of degree d. So $\pi : Z \to \widehat{\mathbb{G}}_m^r$ is an isogeny. Thus we conclude $\mathrm{Ker}(\pi) \cong \prod_{j=1}^r \mu_{p^{m_j}}$ and hence $d = p^m$ for $m = \sum_j m_j \geq 0$. This implies $\mathbb{J}' = \Lambda[A] \subset W[\mu_{p^n}][[Y]][(1+Y)^{p^{-n}}] = W[\mu_{p^n}][[T]][t^{p^{-n}}]$ for $n = \max(m_j|j)$, as desired. □

6 Proof of Theorem 3.1

Let the notation be as in the previous section; so, $K := \mathbb{Q}[\mu_{p^\infty}]$. Put $L_{\mathfrak{l},P} = K(\alpha_{\mathfrak{l},P})$. Suppose that there exist a set Σ of primes of positive upper density as in Theorem 3.1. By the assumption of the theorem, we have an infinite set $\mathcal{A}_{\mathfrak{l}}$ of arithmetic points of a fixed weight κ with $\kappa_2 - \kappa_1 \geq I$ of $\mathrm{Spec}(\mathbb{I})$ (independent of $\mathfrak{l} \in \Sigma$) such that

(B) if $\mathfrak{l} \in \Sigma$, $L_{\mathfrak{l},P}/K$ is a finite extension of bounded degree independent of $P \in \mathcal{A}_{\mathfrak{l}}$.

Let $\overline{\mathcal{A}}_{\mathfrak{l}}$ be the Zariski closure of $\mathcal{A}_{\mathfrak{l}}$ in $\mathrm{Spec}(\mathbb{I})$. As remarked after stating Theorem 3.1, we may assume that $\overline{\mathcal{A}}_{\mathfrak{l}}$ is geometrically irreducible of dimension $r \geq 1$ independent of $\mathfrak{l}$. Thus (J1) is satisfied for $(\mathcal{A}_{\mathfrak{l}}, \mathrm{Spec}(\mathbb{J}) := \overline{\mathcal{A}}_{\mathfrak{l}})$ for all $\mathfrak{l} \in \Sigma$.

Since we want to find a CM quadratic extension M/F in which p splits such that the component $\mathbb{I}$ has complex multiplication by M, by absurdity, we assume that $\mathbb{I}$ is a non-CM component and try to get a contradiction.

By (B) and Proposition 5.2 applied to $\mathfrak{l} \in \Sigma$, for $A_{\mathfrak{l}}$ in (J3), we have

$$A_{\mathfrak{l}}(t) = c_{\mathfrak{l}} \prod_{i=1}^{r} t_i^{s_{i,\mathfrak{l}}} \quad \text{for } s_{\mathfrak{l}} = (s_{i,\mathfrak{l}}) \in \mathbb{Q}_p^r \text{ and } c_{\mathfrak{l}} \in W^\times. \tag{28}$$

As proved in Proposition 5.2, we have $A_{\mathfrak{l}} \in W[\mu_{p^n}][[T_1, \ldots, T_r]][t_1^{p^{-n}} - 1, \ldots, t_r^{p^{-n}} - 1]]$. Since $\mathrm{rank}_\Lambda \mathbb{J} \geq \mathrm{rank}_\Lambda \Lambda[A_{\mathfrak{l}}]$ with $A_{\mathfrak{l}} \in \mathbb{J} \cap W[\mu_{p^n}][[T_1, \ldots, T_r]][t_1^{p^{-n}} - 1, \ldots, t_r^{p^{-n}} - 1]]$, the integer n is also bounded independent of $\mathfrak{l}$. Thus by the variable change $t_i \mapsto t_i^{p^n}$, we may assume that $A_{\mathfrak{l}} \in W[[T_1, \ldots, T_r]]$ for all $\mathfrak{l} \in \Sigma$ (and hence $s_i \in \mathbb{Z}_p$). Up until this point, we only used the existence of $\mathcal{A}_{\mathfrak{l}}$ whose weight $\kappa_{\mathfrak{l}}$ depends on $\mathfrak{l}$ to conclude the above explicit form (28) of $A_{\mathfrak{l}}$. Since $A_{\mathfrak{l}}$ in (28) is independent of weight $\kappa_{\mathfrak{l}}$, we may now take any weight κ (with $\kappa_2 - \kappa_1 \geq I$) discarding the original choice $\kappa_{\mathfrak{l}}$ dependent on $\mathfrak{l}$ (as remarked after stating Theorem 3.1 that κ is allowed to be dependent on $\mathfrak{l}$). Once κ is chosen, we can take $\mathcal{A}$ to be all the arithmetic points of weight κ of $\mathrm{Spec}(\mathbb{J})$ (so, we may assume that $\mathcal{A} = \mathcal{A}_{\mathfrak{l}}$ is also independent of $\mathfrak{l}$). We use the symbols introduced in the proof of Proposition 5.2. We now vary $\mathfrak{l} \in \Sigma$.

Pick a p-power root of unity $\zeta \neq 1$ of order $1 < a = p^e$ and consider $\underline{\zeta} := (\zeta, \zeta, \dots, \zeta) \in \mu_{p^\infty}^r$, and write $\alpha_{f,\mathfrak{l}} = \alpha_{\mathfrak{l}} = A_{\mathfrak{l}}(\underline{\gamma}^{\kappa_2-1})$ for $\underline{\gamma}^{\kappa_2-1} := (\gamma_1^{\kappa_2-1}, \dots, \gamma_r^{\kappa_2-1})$ and $\alpha_{g,\mathfrak{l}} = \beta_{\mathfrak{l}} = A_{\mathfrak{l}}(\underline{\zeta\gamma}^{\kappa_2-1})$ for $\underline{\zeta\gamma}^{\kappa_2-1} := (\zeta\gamma_1^{\kappa_2-1}, \dots, \zeta\gamma_r^{\kappa_2-1})$. They are generalized Weil l-numbers of weight $[\kappa]f_{\mathfrak{l}}$. Write $f = \mathbf{f}_P$ for

$$P = (\underline{t} - \underline{\gamma}^{\kappa_2-1}) := (t_1 - \gamma_1^{\kappa_2-1}, \dots, t_r - \gamma_r^{\kappa_2-1})$$

and g for the cusp form $\mathbf{f}_{P'}$ for $P' = (\underline{t} - \underline{\zeta\gamma}^{\kappa_2-1})$. Consider the compatible system of Galois representation associated with f and g. Pick a prime $\mathfrak{Q}$ of $\mathbb{Q}(f, g) = \mathbb{Q}(f)(g)$ (with residual characteristic q sufficiently large) split over $\mathbb{Q}$. Write $\rho_{f,\mathfrak{Q}}$ (resp. $\rho_{g,\mathfrak{Q}}$) for the $\mathfrak{Q}$-adic member of the system associated with f (resp. g). Thus $\rho_{?,\mathfrak{Q}}$ has values in $\mathrm{GL}_2(\mathbb{Z}_q)$. Since proper compact subgroups of $SL_2(\mathbb{Z}_q)$ are either finite, open in a normalizer of a torus, open in a Borel subgroup or open in a unipotent subgroup, the non-CM property of f and g tells us that $\mathrm{Im}(\rho_{?,\mathfrak{Q}})$ contains an open subgroup of $SL_2(\mathbb{Z}_q)$ (e.g., [Di05, Sect. 0.1] or [CG14, Corollary 4.4]).

For a continuous representation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{GL}_2(R)$ (for $R = \overline{\mathbb{Q}}_q$ or any other topological ring), let $\rho^{sym\otimes j}$ denote the jth *symmetric* tensor representation into $\mathrm{GL}_{j+1}(R)$. Suppose that f [and hence g by the equivalence of (CM2–3)] does not have complex multiplication. Then by openness of $\mathrm{Im}(\rho_{?,\mathfrak{Q}})$ in $\mathrm{GL}_2(\mathbb{Z}_q)$, $\rho_{?,\mathfrak{Q}}^{sym\otimes j}$ is absolutely irreducible for all $j \geq 0$, and also the Zariski closure of $\mathrm{Im}(\rho_{?,\mathfrak{Q}}^{sym\otimes j})$ is connected isomorphic to a quotient of GL(2) by a finite subgroup in the center. Since $\beta_{\mathfrak{l}} = \zeta_{\mathfrak{l}}\alpha_{\mathfrak{l}}$ for a root of unity $\zeta_{\mathfrak{l}} = \prod_{i=1}^r \zeta^{s_{i,\mathfrak{l}}}$ (for $s_i \in \mathbb{Q}_p$ as in Proposition 5.2), we have $\beta_{\mathfrak{l}}^a = \alpha_{\mathfrak{l}}^a$ [for a p-power a with $\zeta^{s_{i,\mathfrak{l}}a} = 1$ $(j = 1, 2, \dots, r)$]. Thus $\mathrm{Tr}(\rho_{f,\mathfrak{Q}}^a(Frob_{\mathfrak{l}})) = \mathrm{Tr}(\rho_{g,\mathfrak{Q}}^a(Frob_{\mathfrak{l}}))$ for all prime $\mathfrak{l} \in \Sigma$ prime to $p\mathfrak{n}$, where $\mathrm{Tr}(\rho_{?,\mathfrak{Q}}^a)(g)$ is just the trace of ath matrix power $\rho_{?,\mathfrak{Q}}^a(g)$. Since the continuous functions $\mathrm{Tr}(\rho_{f,\mathfrak{Q}}^a)$ and $\mathrm{Tr}(\rho_{g,\mathfrak{Q}}^a)$ match on $\widetilde{\Sigma} := \{Frob_{\mathfrak{l}} | \mathfrak{l} \in \Sigma\}$, we find that $\mathrm{Tr}(\rho_{f,\mathfrak{Q}}^a) = \mathrm{Tr}(\rho_{g,\mathfrak{Q}}^a)$ on the closure of $\widetilde{\Sigma}$. Since we have

$$\mathrm{Tr}(\rho^a) = \mathrm{Tr}(\rho^{sym\otimes a}) - \mathrm{Tr}(\rho^{sym\otimes(a-2)} \otimes \det(\rho)),$$

we get over $\widetilde{\Sigma}$,

$$\mathrm{Tr}(\rho_{f,\mathfrak{Q}}^{sym\otimes a}) - \mathrm{Tr}(\rho_{f,\mathfrak{Q}}^{sym\otimes(a-2)} \otimes \det(\rho_{f,\mathfrak{Q}})) = \mathrm{Tr}(\rho_{g,\mathfrak{Q}}^{sym\otimes a}) - \mathrm{Tr}(\rho_{g,\mathfrak{Q}}^{sym\otimes(a-2)} \otimes \det(\rho_{g,\mathfrak{Q}})).$$

which implies

$$\mathrm{Tr}(\rho_{f,\mathfrak{Q}}^{sym\otimes a} \oplus (\rho_{g,\mathfrak{Q}}^{sym\otimes(a-2)} \otimes \det(\rho_{g,\mathfrak{Q}}))) = \mathrm{Tr}(\rho_{g,\mathfrak{Q}}^{sym\otimes a} \oplus (\rho_{f,\mathfrak{Q}}^{sym\otimes(a-2)} \otimes \det(\rho_{f,\mathfrak{Q}})))$$

over $\widetilde{\Sigma}$. Since Σ has positive upper Dirichlet density, by Rajan [Rj98, Theorem 2], there exists an open subgroup $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ such that as representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$

$$\rho_{f,\mathfrak{Q}}^{sym\otimes a} \oplus (\rho_{g,\mathfrak{Q}}^{sym\otimes(a-2)} \otimes \det(\rho_{g,\mathfrak{Q}})) \cong \rho_{g,\mathfrak{Q}}^{sym\otimes a} \oplus (\rho_{f,\mathfrak{Q}}^{sym\otimes(a-2)} \otimes \det(\rho_{f,\mathfrak{Q}})).$$

Since $\mathrm{Im}(\rho_{?,\mathfrak{Q}})$ contains open subgroup of $SL_2(\mathbb{Z}_q)$, $\rho_{?,\mathfrak{Q}}^{sym\otimes j}$ restricted to $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ is absolutely irreducible for all $j \geq 0$. Therefore, as representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$, we conclude $\rho_{f,\mathfrak{Q}}^{sym\otimes a} \cong \rho_{g,\mathfrak{Q}}^{sym\otimes a}$ from the difference of the dimensions of absolutely irreducible factors in the left and right-hand side. By Calegari and Gee [CG14, Corollary 4.4 and Theorem 7.1], each member of $\rho_f^{sym\otimes a}$ and $\rho_g^{sym\otimes a}$ is absolutely irreducible over $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$. Thus the ath symmetric tensor product of the two compatible systems ρ_f and ρ_g are isomorphic to each other over $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$. Again by Rajan [Rj98, Theorem 2], as compatible systems of Galois representations of the entire group $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$, we find $\rho_f^{sym\otimes a} \cong \rho_g^{sym\otimes a} \otimes \chi$ for a finite order character $\chi : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \overline{\mathbb{Q}}^{\times}$. In particular, we get the identity of their $\mathfrak{P}$-adic members

$$\rho_{f,\mathfrak{P}}^{sym\otimes a} \cong \rho_{g,\mathfrak{P}}^{sym\otimes a} \otimes \chi.$$

Note that $F_p := F \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong \prod_{\mathfrak{p}|p} F_{\mathfrak{p}}$ for the $\mathfrak{p}$-adic completion $F_{\mathfrak{p}}$ of F at prime factors $\mathfrak{p}$ of p. Pick a prime $\mathfrak{p}|p$ of F. Then $\mathfrak{p} = \{x \in O : |i_p(\sigma(x))|_p < 1\}$ for an embedding $\sigma : F \hookrightarrow \overline{\mathbb{Q}}$. Then $i_p \circ \sigma$ embeds $F_{\mathfrak{p}}$ into $\overline{\mathbb{Q}}_p$ continuously. Write $I_{\mathfrak{p}}$ for the set of all continuous embeddings of $F_{\mathfrak{p}}$ into $\overline{\mathbb{Q}}_p$ (including $i_p \circ \sigma$). By (Ram), we can write the restriction $\rho_{?,\mathfrak{P}}|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/F_{\mathfrak{p}})}$ in an upper triangular form $\begin{pmatrix} \epsilon_{?,\mathfrak{p}} & * \\ 0 & \delta_{?,\mathfrak{p}} \end{pmatrix}$ (up to isomorphisms) with

$$\delta_{?,\mathfrak{p}}([u, F_{\mathfrak{p}}]) = u^{-\kappa_1} \text{ and } \epsilon_{?,\mathfrak{p}}([u, F_{\mathfrak{p}}]) = u^{-\kappa_2} \text{ for } u \in O_{\mathfrak{p}}^{\times} \text{ sufficiently close to 1.} \tag{29}$$

Here $u^k = \prod_{i_p\circ\tau\in I_{\mathfrak{p}}} \tau(u)^{k_\tau}$ for $k = \sum_{\tau\in I} k_\tau$ (as the component of u in $F_{\mathfrak{p}'}^{\times}$ at $\mathfrak{p}' \neq \mathfrak{p}$ for other primes $\mathfrak{p}'|p$ is trivial in $F_p^{\times}$). This property distinguishes $\delta_{?,\mathfrak{p}}$ from $\epsilon_{?,\mathfrak{p}}$. Regard $\delta_{?,\mathfrak{p}}$ and $\epsilon_{?,\mathfrak{p}}$ as characters of $F_{\mathfrak{p}}^{\times}$ by local class field theory, and put $\delta_?((u_{\mathfrak{p}})_{\mathfrak{p}}) = \prod_{\mathfrak{p}} \delta_{?,\mathfrak{p}}(u_{\mathfrak{p}})$ and $\epsilon_?((u_{\mathfrak{p}})_{\mathfrak{p}}) = \prod_{\mathfrak{p}} \epsilon_{?,\mathfrak{p}}(u_{\mathfrak{p}})$ for $(u_{\mathfrak{p}})_{\mathfrak{p}} \in \prod_{\mathfrak{p}} F_{\mathfrak{p}}^{\times}$ as characters of $F_p^{\times} = \prod_{\mathfrak{p}} F_{\mathfrak{p}}^{\times}$ (in order to regard these characters as those of $F_p^{\times}$ not of the single $F_{\mathfrak{p}}^{\times}$). Then more precisely than (29), we have from our choice of f and g

$$\epsilon_f(\gamma_i) = \gamma_i^{-\kappa_2}, \epsilon_g(\gamma_i) = \zeta\gamma_i^{-\kappa_2}, \delta_f(\gamma_i) = \gamma_i^{-\kappa_1} \text{ and } \delta_g(\gamma_i) = \zeta^{-1}\gamma_i^{-\kappa_1} \tag{30}$$

as $\epsilon_{P'}(\gamma_i) = \zeta$ and $\epsilon_P(\gamma_i) = 1$ for all i. Since $\mathbf{\Gamma} \subset O_p^{\times} \subset F_p^{\times}$, and hence we may consider $\delta_?(\gamma_i)$ and $\epsilon_?(\gamma_i)$. Then we have from $\rho_{f,\mathfrak{P}}^{sym\otimes a} \cong \rho_{g,\mathfrak{P}}^{sym\otimes a} \otimes \chi$

$$\{\epsilon_f^j\delta_f^{a-j} | j = 0, \ldots, a\} = \{\epsilon_g^j\delta_g^{a-j}\chi | j = 0, \ldots, a\}.$$

Therefore we conclude from $\kappa_2 - \kappa_1 \geq I$ and (29) that $\epsilon_f^j\delta_f^{a-j} = \epsilon_g^j\delta_g^{a-j}\chi$. This means

$$\gamma_i^{-\kappa_2 j-\kappa_1(a-j)} = \epsilon_f^j\delta_f^{a-j}(\gamma_i) = \epsilon_g^j\delta_g^{a-j}\chi(\gamma_i) = \gamma_i^{-\kappa_2 j-\kappa_1(a-j)}\zeta^{2j-a}\chi(\gamma_i).$$

Therefore we get $\chi(\gamma_i) = \zeta^{a-2j}$ which has to be independent of j, a contradiction, as we can choose the p-power order of ζ as large as we want. Thus f and hence g must have complex multiplication by the same CM quadratic extension $M_{/F}$ by (CM1–3), and hence $\mathbb{I}$ is a CM component. □

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