L-functions and Galois Representations

Edited by
David Burns, Kevin Buzzard and Jan Nekovář

Contents

Non-vanishing modulo p of Hecke L–values and application

Haruzo Hida

Department of Mathematics
UCLA
Los Angeles, Ca 90095-1555
U.S.A.
hida@math.ucla.edu [a]

[a] *The author is partially supported by the NSF grant: DMS 0244401.*

Contents

1 Introduction

Let F be a totally real field and M/F be a totally imaginary quadratic extension (a CM field). We fix a prime $p > 2$ unramified in $M/\mathbb{Q}$ and suppose that all prime factors of p in F split in M (M is p–ordinary). Fixing two embeddings $i_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $i_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, we take a p–ordinary CM type Σ of M. Thus $\Sigma_p = \{i_p \circ \sigma\}$ is exactly a half of the p–adic places of M. Fix a Hecke character λ of infinity type $k\Sigma + \kappa(1-c)$ with $0 < k \in \mathbb{Z}$ and $\kappa = \sum_{\sigma\in\Sigma} \kappa_\sigma \sigma$ with $\kappa_\sigma \geq 0$. If the conductor $\mathfrak{C}$ of λ is a product of primes split in M/F, we call λ has *split* conductor. Throughout this paper, we assume that λ has split conductor. We fix a prime $\mathfrak{l} \nmid \mathfrak{C}p$ of F. As is well known ([K] and [Sh1]), for a finite order Hecke character χ and for a power Ω of the Néron period of an abelian scheme (over a p–adic valuation ring) of CM type Σ, the L–value $\frac{L^{(p)}(0,\lambda\chi)}{\Omega}$ is (p–adically) integral (where the superscript: "(p)" indicates removal of Euler factors at p). The purpose of this paper is three fold:

(1) To prove non-vanishing modulo p of Hecke L–values $\frac{L^{(p)}(0,\lambda\chi)}{\Omega}$ for "almost all" anticyclotomic characters χ of finite order with $\mathfrak{l}$–power conductor (under some mild assumptions; Theorem 4.3);

(2) To prove the divisibility: $L_p^-(\psi)|\mathcal{F}^-(\psi)$ in the anticyclotomic Iwasawa algebra Λ^- of M for an anticyclotomic character ψ of split conductor, where $L_p^-(\psi)$ is the anticyclotomic Katz p–adic L–function of the branch character ψ and $\mathcal{F}^-(\psi)$ is the corresponding Iwasawa power series (see Theorem 5.1).

(3) To prove the equality $L_p^-(\psi) = \mathcal{F}^-(\psi)$ up to units under some assumptions if $F/\mathbb{Q}$ is an abelian extension (Theorem 5.8) and $M = F[\sqrt{D}]$ for $0 > D \in \mathbb{Z}$.

Roughly speaking, $\mathcal{F}^-(\psi)$ is the characteristic power series of the ψ-branch of the Galois group of the only Σ_p–ramified p–abelian extension of the anticyclotomic tower over the class field of ψ.

The first topic is a generalization of the result of Washington [Wa] (see also [Si]) to Hecke L–values, and the case where λ has conductor 1 has been dealt with in [H04c] basically by the same technique. The phrase "almost all" is in the sense of [H04c] and means "Zariski densely populated characters". If $\mathfrak{l}$ has degree 1 over $\mathbb{Q}$, we can prove a stronger non-vanishing modulo p outside a (non-specified) finite set. In [HT1] and [HT2], we have shown the divisibility in item (2) in $\Lambda^- \otimes_{\mathbb{Z}} \mathbb{Q}$ and indicated that the full divisibility holds except for p outside an explicit finite set S of primes if one obtains the result claimed in (1). We will show that S is limited to ramified primes and even primes.

Though the result in (2) is a direct consequence of the vanishing of the μ–invariant of $L_p^-(\psi)$ proven in [H04c] by the divisibility in $\Lambda^- \otimes_{\mathbb{Z}} \mathbb{Q}$, we shall give another proof of this fact using the non-vanishing (1). We will actually show a stronger result (Corollary 5.6) asserting that the relative class number $h(M/F)$ times $L_p^-(\psi)$ divides the congruence power series of the CM component of the nearly ordinary Hecke algebra (which does not directly follow from the vanishing of μ). Our method to achieve (2) is a refinement of the work [HT1] and [HT2], and this subtle process explains the length of the paper.

Once the divisibility (2) is established, under the assumption of (3), if ψ descends to a character of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}[\sqrt{D}])$, we can restrict $L_p^-(\psi)$ and $\mathcal{F}^-(\psi)$ to a $\mathbb{Z}_p$-extension of an abelian extension of $\mathbb{Q}[\sqrt{D}]$, and applying Rubin's identity of the restricted power series ([R] and [R1]), we conclude the identity $L_p^-(\psi) = \mathcal{F}^-(\psi)$.

We should mention that the stronger divisibility of the congruence power series by $h(M/F)L_p^-(\psi)$ in this paper will be used to prove the equality of $L^-(\psi)$ and $\mathcal{F}^-(\psi)$ under some mild conditions on ψ for general base fields F in our forthcoming paper [H04d]. We shall keep the notation and the assumptions introduced in this introduction throughout the paper.

2 Hilbert Modular Forms

We shall recall algebro-geometric theory of Hilbert modular forms limiting ourselves to what we need later.

2.1 Abelian variety with real multiplication

Let O be the integer ring of F, and put $O^* = \{x \in F | \mathrm{Tr}(xO) \subset \mathbb{Z}\}$ (which is the inverse different $\mathfrak{d}^{-1}$). We fix an integral ideal $\mathfrak{N}$ and a fractional ideal $\mathfrak{c}$ of F prime to $\mathfrak{N}$. We write A for a fixed base algebra, in which $N(\mathfrak{N})$ and $N(\mathfrak{c})$ is invertible. The Hilbert modular variety $\mathfrak{M}(\mathfrak{c};\mathfrak{N})$ of level $\mathfrak{N}$ classifies triples $(X, \Lambda, i)_{/S}$ formed by

- An abelian scheme $\pi : X \to S$ for an A–scheme S with an embedding: $O \hookrightarrow \mathrm{End}(X_{/S})$ making $\pi_*(\Omega_{X/S})$ a locally free $O \otimes \mathcal{O}_S$–module of rank 1;
- An O–linear polarization $\Lambda : X^t = \mathrm{Pic}^0_{X/S} \cong X \otimes \mathfrak{c}$;
- A closed O–linear immersion $i = i_{\mathfrak{N}} : (\mathbb{G}_m \otimes O^*)[\mathfrak{N}] \hookrightarrow X$.

By Λ, we identify the O–module of symmetric O–linear homomorphisms with $\mathfrak{c}$. Then we require that the (multiplicative) monoid of symmetric O–linear isogenies induced locally by ample invertible sheaves be identified with the set of totally positive elements $\mathfrak{c}_+ \subset \mathfrak{c}$. Thus $\mathfrak{M}(\mathfrak{c};\mathfrak{N})_{/A}$ is the coarse moduli scheme of the following functor from the category of A–schemes into the category $SETS$:

$$\mathcal{P}(S) = \left[(X, \Lambda, i)_{/S}\right],$$

where $[\ ] = \{\ \}/\cong$ is the set of isomorphism classes of the objects inside the brackets, and we call $(X, \Lambda, i) \cong (X', \Lambda', i')$ if we have an O–linear isomorphism $\phi : X_{/S} \to X'_{/S}$ such that $\Lambda' = \phi \circ \Lambda \circ \phi^t$ and $i'^* \circ \phi = i^* (\Leftrightarrow \phi \circ i = i')$. The scheme $\mathfrak{M}$ is a fine moduli if $\mathfrak{N}$ is sufficiently deep. In [K] and [HT1], the moduli $\mathfrak{M}$ is described as an algebraic space, but it is actually a quasi-projective scheme (e.g. [C], [H04a] Lectures 5 and 6 and [PAF] Chapter 4).

2.2 Abelian varieties with complex multiplication

We write $|\cdot|_p$ for the p–adic absolute value of $\overline{\mathbb{Q}}_p$ and $\widehat{\mathbb{Q}}_p$ for the p–adic completion of $\overline{\mathbb{Q}}_p$ under $|\cdot|_p$. Recall the p–ordinary CM type (M, Σ), and let R be the integer ring of M. Thus $\Sigma \sqcup \Sigma c$ for the generator c of $\mathrm{Gal}(M/F)$ gives the set of all embeddings of M into $\overline{\mathbb{Q}}$. For each $\sigma \in (\Sigma \cup \Sigma c)$, $i_p \sigma$ induces a p–adic place $\mathfrak{p}_\sigma$ giving rise to the p–adic absolute value $|x|_{\mathfrak{p}_\sigma} = |i_p(\sigma(x))|_p$. We write $\Sigma_p = \{\mathfrak{p}_\sigma | \sigma \in \Sigma\}$ and $\Sigma_p c = \{\mathfrak{p}_{\sigma c} | \sigma \in \Sigma\}$. By ordinarity, we have $\Sigma_p \cap \Sigma_p c = \varnothing$.

For each O–lattice $\mathfrak{a} \subset M$ whose p–adic completion $\mathfrak{a}_p$ is identical to $R_p = R \otimes_{\mathbb{Z}} \mathbb{Z}_p$, we consider the complex torus $X(\mathfrak{a})(\mathbb{C}) = \mathbb{C}^\Sigma / \Sigma(\mathfrak{a})$, where $\Sigma(\mathfrak{a}) = \{(i_\infty(\sigma(a)))_{\sigma \in \Sigma} | a \in \mathfrak{a}\}$. By a theorem in [ACM] 12.4, this complex torus is algebraizable to an abelian variety $X(\mathfrak{a})$ of CM type (M, Σ) over a number field.

Let $\mathbb{F}$ be an algebraic closure of the finite field $\mathbb{F}_p$ of p–elements. We write $\widehat{W}$ for the p–adically closed discrete valuation ring inside $\widehat{\mathbb{Q}}_p$ unramified over $\mathbb{Z}_p$ with residue field $\mathbb{F}$. Thus $\widehat{W}$ is isomorphic to the ring of Witt vectors with coefficients in $\mathbb{F}$. Let $\mathcal{W} = i_p^{-1}(\widehat{W})$, which is a strict henselization of $\mathbb{Z}_{(p)} = \mathbb{Q} \cap \mathbb{Z}_p$. In general, we write W for a finite extension of $\widehat{W}$ in $\widehat{\mathbb{Q}}_p$, which is a complete discrete valuation ring. We suppose that p is unramified in $M/\mathbb{Q}$. Then the main theorem of complex multiplication ([ACM] 18.6) combined with the criterion of good reduction over $\widehat{W}$ [ST] tells us that $X(\mathfrak{a})$ is actually defined over the field of fractions $\mathcal{K}$ of $\mathcal{W}$ and extends to an abelian scheme over $\mathcal{W}$ (still written as $X(\mathfrak{a})_{/\mathcal{W}}$). All endomorphisms of $X(\mathfrak{a})_{/\mathcal{W}}$ are defined over $\mathcal{W}$. We write $\theta : M \hookrightarrow \mathrm{End}(X(\mathfrak{a})) \otimes_{\mathbb{Z}} \mathbb{Q}$ for the embedding of M taking $\alpha \in M$ to the complex multiplication by $\Sigma(\alpha)$ on $X(\mathfrak{a})(\mathbb{C}) = \mathbb{C}^\Sigma / \Sigma(\mathfrak{a})$.

Let $R(\mathfrak{a}) = \{\alpha \in R | \alpha\mathfrak{a} \subset \mathfrak{a}\}$. Then $R(\mathfrak{a})$ is an order of M over O. Recall the prime $\mathfrak{l} \nmid p$ of F in the introduction. The order $R(\mathfrak{a})$ is determined by its conductor ideal which we assume to be an $\mathfrak{l}$–power $\mathfrak{l}^e$. In other words, $R(\mathfrak{a}) = R_e := O + \mathfrak{l}^e R$. The following three conditions for a fractional

R_e–ideal $\mathfrak{a}$ are equivalent (cf. [IAT] Proposition 4.11 and (5.4.2) and [CRT] Theorem 11.3):

(I1) $\mathfrak{a}$ is R_e–projective;
(I2) $\mathfrak{a}$ is locally principal;
(I3) $\mathfrak{a}$ is a proper R_e–ideal (that is, $R_e = R(\mathfrak{a})$).

Thus $Cl_e := \mathrm{Pic}(R_e)$ is the group of R_e–projective fractional ideals modulo principal ideals. The group Cl_e is finite and called the ring class group modulo $\mathfrak{l}^e$.

We choose and fix a differential $\omega = \omega(R)$ on $X(R)_{/\mathcal{W}}$ so that

$$H^0(X(R), \Omega_{X(R)/\mathcal{W}}) = (\mathcal{W} \otimes_{\mathbb{Z}} O)\omega.$$

If $\mathfrak{a}_p = R_p$, $X(R \cap \mathfrak{a})$ is an étale covering of both $X(\mathfrak{a})$ and $X(R)$; so, $\omega(R)$ induces a differential $\omega(\mathfrak{a})$ first by pull-back to $X(R \cap \mathfrak{a})$ and then by pull-back inverse from $X(R \cap \mathfrak{a})$ to $X(\mathfrak{a})$. As long as the projection $\pi : X(R \cap \mathfrak{a}) \twoheadrightarrow X(\mathfrak{a})$ is étale, the pull-back inverse $(\pi^*)^{-1} : \Omega_{X(R\cap\mathfrak{a})/\mathcal{W}} \to \Omega_{X(\mathfrak{a})/\mathcal{W}}$ is a surjective isomorphism. We thus have

$$H^0(X(\mathfrak{a}), \Omega_{X(R)/\mathcal{W}}) = (\mathcal{W} \otimes_{\mathbb{Z}} O)\omega(\mathfrak{a}).$$

We choose a totally imaginary $\delta \in M$ with $\mathrm{Im}(i_\infty(\sigma(\delta))) > 0$ for all $\sigma \in \Sigma$ such that $(a, b) \mapsto (c(a)b - ac(b))/2\delta$ gives the identification $R \wedge R \cong \mathfrak{d}^{-1}\mathfrak{c}^{-1}$. We assume that $\mathfrak{c}$ is prime to $p\ell$ ($(\ell) = \mathfrak{l} \cap \mathbb{Z}$). This Riemann form: $R \wedge R \cong \mathfrak{c}^* = \mathfrak{d}^{-1}\mathfrak{c}^{-1}$ gives rise to a $\mathfrak{c}$–polarization $\Lambda = \Lambda(R) : X(R)^t \cong X(R) \otimes \mathfrak{c}$, which is again defined over $\mathcal{W}$. Here $\mathfrak{d}$ is the different of $F/\mathbb{Q}$, and $\mathfrak{c}^* = \{x \in F | \mathrm{Tr}_{F/\mathbb{Q}}(x\mathfrak{c}) \subset \mathbb{Z}\}$. Since we have $R_e \wedge R_e = \mathfrak{l}^e(O \wedge R) + \mathfrak{l}^{2e}(R \wedge R)$, the pairing induces $R_e \wedge R_e \cong (\mathfrak{c}\mathfrak{l}^{-e})^*$, and this pairing induces a $\mathfrak{c}\mathfrak{l}^{-e}N_{M/F}(\mathfrak{a})^{-1}$–polarization $\Lambda(\mathfrak{a})$ on $X(\mathfrak{a})$ for a proper R_e–ideal $\mathfrak{a}$.

We choose a local generator a of $\mathfrak{a}_\mathfrak{l}$. Multiplication by a induces an isomorphism $R_{e,\mathfrak{l}} \cong \mathfrak{a}_\mathfrak{l}$. Since $X(R_e)_{/\mathcal{W}}$ has a subgroup $C(R_e) = R/(O + \mathfrak{l}^e R) \subset X(R_e)$ isomorphic étale-locally to $O/\mathfrak{l}^e$. This subgroup $C(R_e)$ is sent by multiplication by a to $C(\mathfrak{a}) \subset X(\mathfrak{a})_{/\mathcal{W}}$, giving rise to a $\Gamma_0(\mathfrak{l}^e)$–level structure $C(\mathfrak{a})$ on $X(\mathfrak{a})$.

For our later use, we choose ideals $\mathfrak{F}$ and $\mathfrak{F}_c$ of R prime to $\mathfrak{c}$ so that $\mathfrak{F} \subset \mathfrak{F}_c^c$ and $\mathfrak{F} + \mathfrak{F}_c = R$. The product $\mathfrak{C} = \mathfrak{F}\mathfrak{F}_c$ shall be later the conductor of the Hecke character we study. We put $\mathfrak{f} = \mathfrak{F} \cap O$ and $\mathfrak{f}' = \mathfrak{F}_c \cap O$; so, $\mathfrak{f} \subset \mathfrak{f}'$. We shall define a level $\mathfrak{f}^2$–structure on $X(\mathfrak{a})$: supposing that $\mathfrak{a}$ is prime to $\mathfrak{f}$, we have $\mathfrak{a}_\mathfrak{f} \cong R_\mathfrak{f} = R_\mathfrak{F} \times R_{\mathfrak{F}^c}$, which induces a canonical identification

$$i(\mathfrak{a}) : \mathfrak{f}^*/O^* = \mathfrak{f}^{-1}/O \cong \mathfrak{F}^{-1}/R \cong \mathfrak{F}^{-1}\mathfrak{a}_\mathfrak{F}/\mathfrak{a}_\mathfrak{F} \subset X(\mathfrak{a})[\mathfrak{f}]. \tag{2.1}$$

This level structure induces $i'(\mathfrak{a}) : X(\mathfrak{a})[\mathfrak{f}] \twoheadrightarrow \mathfrak{f}^{-1}/O$ by the duality under Λ. In this way, we get many sextuples:

$$(X(\mathfrak{a}), \Lambda(\mathfrak{a}), i(\mathfrak{a}), i'(\mathfrak{a}), C(\mathfrak{a})[\mathfrak{l}], \omega(\mathfrak{a})) \in \mathcal{M}(\mathfrak{c}\mathfrak{l}^{-e}(\mathfrak{a}\mathfrak{a}^c)^{-1}; \mathfrak{f}^2, \Gamma_0(\mathfrak{l}))(\mathcal{W}) \tag{2.2}$$

as long as $\mathfrak{l}^e$ is prime to p, where $C(\mathfrak{a})[\mathfrak{l}] = \{x \in C(\mathfrak{a}) | \mathfrak{l}x = 0\}$. A precise definition of the moduli scheme of Γ_0–type: $\mathcal{M}(\mathfrak{c}\mathfrak{l}^{-e}(\mathfrak{a}\mathfrak{a}^c)^{-1}; \mathfrak{f}^2, \Gamma_0(\mathfrak{l}))$ classifying such sextuples will be given in 2.7. The point $x(\mathfrak{a}) = (X(\mathfrak{a}), \Lambda(\mathfrak{a}), i(\mathfrak{a}), i'(\mathfrak{a}))$ of the moduli scheme $\mathfrak{M}(\mathfrak{c}(\mathfrak{a}\mathfrak{a}^c)^{-1}; \mathfrak{f}^2)$ is called a *CM point* associated to $X(\mathfrak{a})$.

2.3 Geometric Hilbert modular forms

We return to the functor $\mathcal{P}$ in 2.1. We could insist on freeness of the differentials $\pi_*(\Omega_{X/S})$, and for ω with $\pi_*(\Omega_{X/S}) = (\mathcal{O}_S \otimes_{\mathbb{Z}} O)\omega$, we consider the functor classifying quadruples (X, Λ, i, ω):

$$\mathcal{Q}(S) = \left[(X, \Lambda, i, \omega)_{/S}\right].$$

Let $T = \mathrm{Res}_{O/\mathbb{Z}}\mathbb{G}_m$. We let $a \in T(S) = H^0(S, (\mathcal{O}_S \otimes_{\mathbb{Z}} O)^\times)$ act on $\mathcal{Q}(S)$ by $(X, \Lambda, i, \omega) \mapsto (X, \Lambda, i, a\omega)$. By this action, $\mathcal{Q}$ is a T–torsor over $\mathcal{P}$; so, $\mathcal{Q}$ is representable by an A–scheme $\mathcal{M} = \mathcal{M}(\mathfrak{c}; \mathfrak{N})$ affine over $\mathfrak{M} = \mathfrak{M}(\mathfrak{c}; \mathfrak{N})_{/A}$. For each character $k \in X^*(T) = \mathrm{Hom}_{gp-sch}(T, \mathbb{G}_m)$, if $F \neq \mathbb{Q}$, the k^{-1}–eigenspace of $H^0(\mathcal{M}_{/A}, \mathcal{O}_{\mathcal{M}/A})$ is by definition the space of modular forms of weight k integral over A. We write $G_k(\mathfrak{c}, \mathfrak{N}; A)$ for this space of A–integral modular forms, which is an A–module of finite type. When $F = \mathbb{Q}$, as is well known, we need to take the subsheaf of sections with logarithmic growth towards cusps (the condition (G0) below). Thus $f \in G_k(\mathfrak{c}, \mathfrak{N}; A)$ is a functorial rule assigning a value in B to each isomorphism class of $(X, \Lambda, i, \omega)_{/B}$ (defined over an A–algebra B) satisfying the following three conditions:

(G1) $f(X, \Lambda, i, \omega) \in B$ if (X, Λ, i, ω) is defined over B;
(G2) $f((X, \Lambda, i, \omega) \otimes_B B') = \rho(f(X, \Lambda, i, \omega))$ for each morphism $\rho : B_{/A} \to B'_{/A}$;
(G3) $f(X, \Lambda, i, a\omega) = k(a)^{-1} f(X, \Lambda, i, \omega)$ for $a \in T(B)$.

By abusing the language, we pretend f to be a function of isomorphism classes of test objects $(X, \Lambda, i, \omega)_{/B}$ hereafter. The sheaf of k^{-1}–eigenspace $\mathcal{O}_{\mathcal{M}}[k^{-1}]$ under the action of T is an invertible sheaf on $\mathfrak{M}_{/A}$. We write this sheaf as $\underline{\omega}^k$ (imposing (G0) when $F = \mathbb{Q}$). Then we have

$$G_k(\mathfrak{c}, \mathfrak{N}; A) = H^0(\mathfrak{M}(\mathfrak{c}; \mathfrak{N})_{/A}, \underline{\omega}^k_{/A})$$

as long as $\mathfrak{M}(\mathfrak{c};\mathfrak{N})$ is a fine moduli space. Writing $\underline{\mathbb{X}} = (\mathbb{X}, \boldsymbol{\lambda}, \mathbf{i}, \boldsymbol{\omega})$ for the universal abelian scheme over $\mathfrak{M}$, $s = f(\underline{\mathbb{X}})\boldsymbol{\omega}^k$ gives rise to the section of $\underline{\omega}^k$. Conversely, for any section $s \in H^0(\mathfrak{M}(\mathfrak{c};\mathfrak{N}), \underline{\omega}^k)$, taking a unique morphism $\phi : \mathrm{Spec}(B) \to \mathfrak{M}$ such that $\phi^*\underline{\mathbb{X}} = \underline{X}$ for $\underline{X} = (X, \Lambda, i, \omega)_{/B}$, we can define $f \in G_k$ by $\phi^* s = f(\underline{X})\omega^k$.

We suppose that the fractional ideal $\mathfrak{c}$ is prime to $\mathfrak{N}p$, and take two ideals $\mathfrak{a}$ and $\mathfrak{b}$ prime to $\mathfrak{N}p$ such that $\mathfrak{a}\mathfrak{b}^{-1} = \mathfrak{c}$. To this pair $(\mathfrak{a}, \mathfrak{b})$, we can attach the Tate AVRM $Tate_{\mathfrak{a},\mathfrak{b}}(q)$ defined over the completed group ring $\mathbb{Z}((\mathfrak{a}\mathfrak{b}))$ made of formal series $f(q) = \sum_{\xi \gg -\infty} a(\xi)q^\xi$ $(a(\xi) \in \mathbb{Z})$. Here ξ runs over all elements in $\mathfrak{a}\mathfrak{b}$, and there exists a positive integer n (dependent on f) such that $a(\xi) = 0$ if $\sigma(\xi) < -n$ for some $\sigma \in I$. We write $A[[(\mathfrak{a}\mathfrak{b})_{\geq 0}]]$ for the subring of $A[[\mathfrak{a}\mathfrak{b}]]$ made of formal series f with $a(\xi) = 0$ for all ξ with $\sigma(\xi) < 0$ for at least one embedding $\sigma : F \hookrightarrow \mathbb{R}$. Actually, we skipped a step of introducing the toroidal compactification of $\mathfrak{M}$ whose (completed) stalk at the cusp corresponding to $(\mathfrak{a}, \mathfrak{b})$ actually carries $Tate_{\mathfrak{a},\mathfrak{b}}(q)$. However to make exposition short, we ignore this technically important point, referring the reader to the treatment in [K] Chapter I, [C], [DiT], [Di], [HT1] Section 1 and [PAF] 4.1.4. The scheme $Tate(q)$ can be extended to a semi-abelian scheme over $\mathbb{Z}[[(\mathfrak{a}\mathfrak{b})_{\geq 0}]]$ adding the fiber $\mathbb{G}_m \otimes \mathfrak{a}^*$. Since $\mathfrak{a}$ is prime to p, $\mathfrak{a}_p = O_p$. Thus if A is a $\mathbb{Z}_p$–algebra, we have a canonical isomorphism:

$$Lie(Tate_{\mathfrak{a},\mathfrak{b}}(q) \mod \mathfrak{A}) = Lie(\mathbb{G}_m \otimes \mathfrak{a}^*) \cong A \otimes_{\mathbb{Z}} \mathfrak{a}^* \cong A \otimes_{\mathbb{Z}} O^*.$$

By Grothendieck-Serre duality, we have $\Omega_{Tate_{\mathfrak{a},\mathfrak{b}}(q)/A[[(\mathfrak{a}\mathfrak{b})_{\geq 0}]]} \cong A[[(\mathfrak{a}\mathfrak{b})_{\geq 0}]]$. Indeed we have a canonical generator ω_{can} of $\Omega_{Tate(q)}$ induced by $\frac{dt}{t} \otimes 1$ on $\mathbb{G}_m \otimes \mathfrak{a}^*$. We have a canonical inclusion $(\mathbb{G}_m \otimes O^*)[\mathfrak{N}] = (\mathbb{G}_m \otimes \mathfrak{a}^*)[\mathfrak{N}]$ into $\mathbb{G}_m \otimes \mathfrak{a}^*$, which induces a canonical closed immersion $i_{can} : (\mathbb{G}_m \otimes O^*)[\mathfrak{N}] \hookrightarrow Tate(q)$. As described in [K] (1.1.14) and [HT1] page 204, $Tate_{\mathfrak{a},\mathfrak{b}}(q)$ has a canonical $\mathfrak{c}$–polarization Λ_{can}. Thus we can evaluate $f \in G_k(\mathfrak{c}, \mathfrak{N}; A)$ at $(Tate_{\mathfrak{a},\mathfrak{b}}(q), \Lambda_{can}, i_{can}, \omega_{can})$. The value $f(q) = f_{\mathfrak{a},\mathfrak{b}}(q)$ actually falls in $A[[(\mathfrak{a}\mathfrak{b})_{\geq 0}]]$ (if $F \neq \mathbb{Q}$: Koecher principle) and is called the q–expansion at the cusp $(\mathfrak{a}, \mathfrak{b})$. When $F = \mathbb{Q}$, we impose f to have values in the power series ring $A[[(\mathfrak{a}\mathfrak{b})_{\geq 0}]]$ when we define modular forms:

(G0) $f_{\mathfrak{a},\mathfrak{b}}(q) \in A[[(\mathfrak{a}\mathfrak{b})_{\geq 0}]]$ for all $(\mathfrak{a}, \mathfrak{b})$.

2.4 p–Adic Hilbert modular forms

Suppose that $A = \varprojlim_n A/p^n A$ and that $\mathfrak{N}$ is prime to p. We can think of a functor

$$\widehat{\mathcal{P}}(A) = \left[(X, \Lambda, i_p, i_{\mathfrak{N}})_{/S}\right]$$

similar to $\mathcal{P}$ that is defined over the category of p–adic A–algebras $B = \varprojlim_n B/p^nB$. An important point is that we consider an isomorphism of ind-group schemes $i_p : \mu_{p^\infty} \otimes_{\mathbb{Z}} O^* \hookrightarrow X[p^\infty]$ (in place of a differential ω), which induces $\widehat{\mathbb{G}}_m \otimes O^* \cong \widehat{X}$ for the formal completion $\widehat{V}$ at the characteristic p–fiber of a scheme V over A.

It is a theorem (due to Deligne-Ribet and Katz) that this functor is representable by the formal Igusa tower over the formal completion $\widehat{\mathfrak{M}}(\mathfrak{c};\mathfrak{N})$ of $\mathfrak{M}(\mathfrak{c};\mathfrak{N})$ along the ordinary locus of the modulo p fiber (e.g., [PAF] 4.1.9). A p–adic modular form $f_{/A}$ for a p–adic ring A is a function (strictly speaking, a functorial rule) of isomorphism classes of $(X, \Lambda, i_p, i_{\mathfrak{N}})_{/B}$ satisfying the following three conditions:

(P1) $f(X, \Lambda, i_p, i_{\mathfrak{N}}) \in B$ if $(X, \Lambda, i_p, i_{\mathfrak{N}})$ is defined over B;
(P2) $f((X, \Lambda, i_p, i_{\mathfrak{N}}) \otimes_B B') = \rho(f(X, \Lambda, i_p, i_{\mathfrak{N}}))$ for each continuous A--algebra homomorphism $\rho : B \to B'$;
(P3) $f_{\mathfrak{a},\mathfrak{b}}(q) \in A[[(\mathfrak{a}\mathfrak{b})_{\geq 0}]]$ for all $(\mathfrak{a}, \mathfrak{b})$ prime to $\mathfrak{N}p$.

We write $V(\mathfrak{c}, \mathfrak{N}; A)$ for the space of p–adic modular forms satisfying (P1-3). This $V(\mathfrak{c}, \mathfrak{N}; A)$ is a p–adically complete A–algebra.

We have the q–expansion principle valid both for classical modular forms and p–adic modular forms f,

(q-exp) *f is uniquely determined by the q–expansion: $f \mapsto f_{\mathfrak{a},\mathfrak{b}}(q) \in A[[(\mathfrak{a}\mathfrak{b})_{\geq 0}]]$.*

This follows from the irreducibility of (the Hilbert modular version of) the Igusa tower proven in [DeR] (see also [PAF] 4.2.4).

Since $\widehat{\mathbb{G}}_m \otimes O^*$ has a canonical invariant differential $\frac{dt}{t}$, we have $\omega_p = i_{p,*}(\frac{dt}{t})$ on X. This allows us to regard $f \in G_k(\mathfrak{c}, \mathfrak{N}; A)$ a p–adic modular form by

$$f(X, \Lambda, i_p, i_{\mathfrak{N}}) := f(X, \Lambda, i_{\mathfrak{N}}, \omega_p).$$

By (q-exp), this gives an injection of $G_k(\mathfrak{c}, \mathfrak{N}; A)$ into the space of p–adic modular forms $V(\mathfrak{c}, \mathfrak{N}; A)$ (for a p–adic ring A) preserving q–expansions.

2.5 Complex analytic Hilbert modular forms

Over $\mathbb{C}$, the category of test objects (X, Λ, i, ω) is equivalent to the category of triples $(\mathcal{L}, \Lambda, i)$ made of the following data (by the theory of theta functions): $\mathcal{L}$ is an O–lattice in $O \otimes_{\mathbb{Z}} \mathbb{C} = \mathbb{C}^I$, an alternating pairing $\Lambda : \mathcal{L} \wedge_O \mathcal{L} \cong \mathfrak{c}^*$ and $i : \mathfrak{N}^*/O^* \hookrightarrow F\mathcal{L}/\mathcal{L}$. The alternating form Λ is supposed to be positive in the sense that $\Lambda(u, v)/\operatorname{Im}(uv^c)$ is totally positive definite. The differential ω can be recovered by $\iota : X(\mathbb{C}) = \mathbb{C}^I/\mathcal{L}$ so that $\omega = \iota^* du$ where $u = (u_\sigma)_{\sigma\in I}$ is the variable on $\mathbb{C}^I$. Conversely

$$\mathcal{L}_X = \left\{ \int_\gamma \omega \in O \otimes_{\mathbb{Z}} \mathbb{C} \middle| \gamma \in H_1(X(\mathbb{C}), \mathbb{Z}) \right\}$$

is a lattice in $\mathbb{C}^I$, and the polarization $\Lambda : X^t \cong X \otimes \mathfrak{c}$ induces $\mathcal{L} \wedge \mathcal{L} \cong \mathfrak{c}^*$.

Using this equivalence, we can relate our geometric definition of Hilbert modular forms with the classical analytic definition. Define $\mathfrak{Z}$ by the product of I copies of the upper half complex plane $\mathfrak{H}$. We regard $\mathfrak{Z} \subset F \otimes_{\mathbb{Q}} \mathbb{C} = \mathbb{C}^I$ made up of $z = (z_\sigma)_{\sigma\in I}$ with totally positive imaginary part. For each $z \in \mathfrak{Z}$, we define

$$\mathcal{L}_z = 2\pi\sqrt{-1}(\mathfrak{b}z + \mathfrak{a}^*),$$
$$\Lambda_z(2\pi\sqrt{-1}(az+b), 2\pi\sqrt{-1}(cz+d)) = -(ad-bc) \in \mathfrak{c}^*$$

with $i_z : \mathfrak{N}^*/O^* \to \mathbb{C}^I/\mathcal{L}_z$ given by $i_z(a \mod O^*) = (2\pi\sqrt{-1}a \mod \mathcal{L}_z)$.

Consider the following congruence subgroup $\Gamma_1^1(\mathfrak{N};\mathfrak{a},\mathfrak{b})$ given by

$$\left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(F) \middle| a, d \in O,\ b \in (\mathfrak{ab})^*,\ c \in \mathfrak{Nabd} \text{ and } d-1 \in \mathfrak{N} \right\}.$$

We let $g = (g_\sigma) \in SL_2(F \otimes_{\mathbb{Q}} \mathbb{R}) = SL_2(\mathbb{R})^I$ act on $\mathfrak{Z}$ by linear fractional transformation of g_σ on each component z_σ. It is easy to verify

$$(\mathcal{L}_z, \Lambda_z, i_z) \cong (\mathcal{L}_w, \Lambda_w, i_w) \iff w = \gamma(z) \text{ for } \gamma \in \Gamma_1^1(\mathfrak{N};\mathfrak{a},\mathfrak{b}).$$

The set of pairs $(\mathfrak{a},\mathfrak{b})$ with $\mathfrak{ab}^{-1} = \mathfrak{c}$ is in bijection with the set of cusps (unramified over ∞) of $\Gamma_1^1(\mathfrak{N};\mathfrak{a},\mathfrak{b})$. Two cusps are equivalent if they transform each other by an element in $\Gamma_1^1(\mathfrak{N};\mathfrak{a},\mathfrak{b})$. The standard choice of the cusp is $(O,\mathfrak{c}^{-1})$, which we call the infinity cusp of $\mathfrak{M}(\mathfrak{c};\mathfrak{N})$. Write $\Gamma_1^1(\mathfrak{c};\mathfrak{N}) = \Gamma_1^1(\mathfrak{N};O,\mathfrak{c}^{-1})$. For each ideal $\mathfrak{t}$, $(\mathfrak{t},\mathfrak{tc}^{-1})$ gives another cusp. The two cusps $(\mathfrak{t},\mathfrak{tc}^{-1})$ and $(\mathfrak{s},\mathfrak{sc}^{-1})$ are equivalent under $\Gamma_1^1(\mathfrak{c};\mathfrak{N})$ if $\mathfrak{t} = \alpha\mathfrak{s}$ for an element $\alpha \in F^\times$ with $\alpha \equiv 1 \mod \mathfrak{N}$ in $F_{\mathfrak{N}}^\times$. We have

$$\mathfrak{M}(\mathfrak{c};\mathfrak{N})(\mathbb{C}) \cong \Gamma_1^1(\mathfrak{c};\mathfrak{N})\backslash\mathfrak{Z}, \text{ canonically.}$$

Let $G = \mathrm{Res}_{O/\mathbb{Z}}GL(2)$. Take the following open compact subgroup of $G(\mathbb{A}^{(\infty)})$:

$$U_1^1(\mathfrak{N}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(\widehat{\mathbb{Z}}) \middle| c \in \mathfrak{N}\widehat{O} \text{ and } a \equiv d \equiv 1 \mod \mathfrak{N}\widehat{O} \right\},$$

and put $K = K_1^1(\mathfrak{N}) = \left(\begin{smallmatrix} d & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1} U_1^1(\mathfrak{N}) \left(\begin{smallmatrix} d & 0 \\ 0 & 1 \end{smallmatrix}\right)$ for an idele d with $dO = \mathfrak{d}$ and $d^{(\mathfrak{d})} = 1$. Then taking an idele c with $c\widehat{O} = \widehat{\mathfrak{c}}$ and $c^{(\mathfrak{c})} = 1$, we see that

$$\Gamma_1^1(\mathfrak{c};\mathfrak{N}) \subset \left(\left(\begin{smallmatrix} c & 0 \\ 0 & 1 \end{smallmatrix}\right) K \left(\begin{smallmatrix} c & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1} \cap G(\mathbb{Q})_+ \right) \subset O^\times \Gamma_1^1(\mathfrak{c};\mathfrak{N})$$

for $G(\mathbb{Q})_+$ made up of all elements in $G(\mathbb{Q})$ with totally positive determinant. Choosing a complete representative set $\{c\} \subset F_{\mathbb{A}}^{\times}$ for the strict ray class group $Cl_F^+(\mathfrak{N})$ modulo $\mathfrak{N}$, we find by the approximation theorem that

$$G(\mathbb{A}) = \bigsqcup_{c \in Cl_F^+(\mathfrak{N})} G(\mathbb{Q}) \left(\begin{smallmatrix} c & 0 \\ 0 & 1 \end{smallmatrix}\right) K \cdot G(\mathbb{R})^+$$

for the identity connected component $G(\mathbb{R})^+$ of the Lie group $G(\mathbb{R})$. This shows

$$G(\mathbb{Q})\backslash G(\mathbb{A})/KC_{\mathbf{i}} \cong G(\mathbb{Q})_+\backslash G(\mathbb{A})_+/KC_{\mathbf{i}} \cong \bigsqcup_{\mathfrak{c} \in Cl_F^+(\mathfrak{N})} \mathfrak{M}(\mathfrak{c};\mathfrak{N})(\mathbb{C}), \quad (2.3)$$

where $G(\mathbb{A})_+ = G(\mathbb{A}^{(\infty)})G(\mathbb{R})^+$ and $C_{\mathbf{i}}$ is the stabilizer in $G(\mathbb{R})^+$ of $\mathbf{i} = (\sqrt{-1}\ldots,\sqrt{-1}) \in \mathfrak{Z}$. By (2.3), a $Cl_F^+(\mathfrak{N})$–tuple $(f_{\mathfrak{c}})_{\mathfrak{c}}$ with $f_{\mathfrak{c}} \in G_k(\mathfrak{c},\mathfrak{N};\mathbb{C})$ can be viewed as a single automorphic form defined on $G(\mathbb{A})$.

Recall the identification of $X^*(T)$ with $\mathbb{Z}[I]$ so that $k(x) = \prod_\sigma \sigma(x)^{k_\sigma}$. Regarding $f \in G_k(\mathfrak{c},\mathfrak{N};\mathbb{C})$ as a holomorphic function of $z \in \mathfrak{Z}$ by $f(z) = f(\mathcal{L}_z,\Lambda_z,i_z)$, it satisfies the following automorphic property:

$$f(\gamma(z)) = f(z)\prod_\sigma (c^\sigma z_\sigma + d^\sigma)^{k_\sigma} \quad \text{for all } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1^1(\mathfrak{c};\mathfrak{N}). \quad (2.4)$$

The holomorphy of f is a consequence of the functoriality (G2). The function f has the Fourier expansion

$$f(z) = \sum_{\xi \in (\mathfrak{ab})_{\geq 0}} a(\xi)\mathbf{e}_F(\xi z)$$

at the cusp corresponding to $(\mathfrak{a},\mathfrak{b})$. Here $\mathbf{e}_F(\xi z) = \exp(2\pi\sqrt{-1}\sum_\sigma \xi^\sigma z_\sigma)$. This Fourier expansion gives the q–expansion $f_{\mathfrak{a},\mathfrak{b}}(q)$ substituting q^ξ for $\mathbf{e}_F(\xi z)$.

2.6 Differential operators

Shimura studied the effect on modular forms of the following differential operators on $\mathfrak{Z}$ indexed by $k \in \mathbb{Z}[I]$:

$$\delta_k^\sigma = \frac{1}{2\pi\sqrt{-1}}\left(\frac{\partial}{\partial z_\sigma} + \frac{k_\sigma}{2y_\sigma\sqrt{-1}}\right) \quad \text{and} \quad \delta_k^r = \prod_\sigma \left(\delta_{k_\sigma+2r_\sigma-2}^\sigma \cdots \delta_{k_\sigma}^\sigma\right), \quad (2.5)$$

where $r \in \mathbb{Z}[I]$ with $r_\sigma \geq 0$. An important point is that the differential operator preserves rationality property at CM points of (arithmetic) modular forms, although it does not preserve holomorphy (see [AAF] III and [Sh1]). We shall describe the rationality. The complex uniformization $\iota : X(\mathfrak{a})(\mathbb{C}) \cong \mathbb{C}^\Sigma/\Sigma(\mathfrak{a})$

induces a canonical base $\omega_\infty = \iota^* du$ of $\Omega_{X(\mathfrak{a})/\mathbb{C}}$ over $R \otimes_{\mathbb{Z}} \mathbb{R}$, where $u = (u_\sigma)_{\sigma\in\Sigma}$ is the standard variable on $\mathbb{C}^\Sigma$. Define a period $\Omega_\infty \in \mathbb{C}^\Sigma = O \otimes_{\mathbb{Z}} \mathbb{C}$ by $\omega(R) = \omega(\mathfrak{a}) = \Omega_\infty\omega_\infty$. Here the first identity follows from the fact that $\omega(\mathfrak{a})$ is induced by $\omega(R)$ on $X(R)$. We suppose that $\mathfrak{a}$ is prime to p. Here is the rationality result of Shimura for $f \in G_k(\mathfrak{c}, \mathfrak{f}^2; \mathcal{W})$:

$$\frac{(\delta_k^r f)(x(\mathfrak{a}), \omega_\infty)}{\Omega_\infty^{k+2r}} = (\delta_k^r f)(x(\mathfrak{a}), \omega(\mathfrak{a})) \in \overline{\mathbb{Q}}. \tag{S}$$

Katz interpreted the differential operator in terms of the Gauss-Manin connection of the universal AVRM over $\mathfrak{M}$ and gave a purely algebro-geometric definition of the operator (see [K] Chapter II and [HT1] Section 1). Using this algebraization of δ_k^r, he extended the operator to geometric modular forms and p–adic modular forms. We write his operator corresponding to δ_*^k as d^k : $V(\mathfrak{c}, \mathfrak{N}; A) \to V(\mathfrak{c}, \mathfrak{N}; A)$. The level p–structure $i_p(\mathfrak{a}) : (\mathbb{G}_m \otimes O^*)[p^\infty] \cong M_\Sigma/\mathfrak{a}_\Sigma \hookrightarrow X(\mathfrak{a})[p^\infty]$ ($\mathfrak{a}_\Sigma = \prod_{\mathfrak{P}\in\Sigma_p} \mathfrak{a}_\mathfrak{P} = R_\Sigma$) induces an isomorphism $\iota_p : \widehat{\mathbb{G}}_m \otimes O^* \cong \widehat{X}(\mathfrak{a})$ for the p–adic formal group $\widehat{X}(\mathfrak{a})_{/W}$ at the origin. Then $\omega(R) = \omega(\mathfrak{a}) = \Omega_p\omega_p$ ($\Omega_p \in O \otimes_{\mathbb{Z}} W = W^\Sigma$) for $\omega_p = \iota_{p,*}\frac{dt}{t}$. An important formula given in [K] (2.6.7) is: for $f \in G_k(\mathfrak{c}, \mathfrak{f}^2; \mathcal{W})$,

$$\frac{(d^r f)(x(\mathfrak{a}), \omega_p)}{\Omega_p^{k+2r}} = (d^r f)(x(\mathfrak{a}), \omega(\mathfrak{a})) = (\delta_k^r f)(x(\mathfrak{a}), \omega(\mathfrak{a})) \in \mathcal{W}. \tag{K}$$

The effect of d^r on q–expansion of a modular form is given by

$$d^r \sum_\xi a(\xi) q^\xi = \sum_\xi a(\xi)\xi^r q^\xi. \tag{2.6}$$

See [K] (2.6.27) for this formula.

2.7 Γ_0*–level structure and Hecke operators*

We now assume that the base algebra A is a $\mathcal{W}$–algebra. Choose a prime $\mathfrak{q}$ of F. We are going to define Hecke operators $U(\mathfrak{q}^n)$ and $T(1, \mathfrak{q}^n)$ assuming for simplicity that $\mathfrak{q} \nmid p\mathfrak{N}$, though we may extend the definition for arbitrary $\mathfrak{q}$ (see [PAF] 4.1.10). Then $X[\mathfrak{q}^r]$ is an étale group over B if X is an abelian scheme over an A–algebra B. We call a subgroup $C \subset X$ cyclic of order $\mathfrak{q}^r$ if $C \cong O/\mathfrak{q}^r$ over an étale faithfully flat extension of B.

We can think of quintuples $(X, \Lambda, i, C, \omega)_{/S}$ adding an additional information C of a cyclic subgroup scheme $C \subset X$ cyclic of order $\mathfrak{q}^r$. We define the space of classical modular forms $G_k(\mathfrak{c}, \mathfrak{N}, \Gamma_0(\mathfrak{q}^r); A)$ (resp. the space $V(\mathfrak{c}, \mathfrak{N}, \Gamma_0(\mathfrak{q}^r); A)$ of p–adic modular forms) of level $(\mathfrak{N}, \Gamma_0(\mathfrak{q}^r))$ by (G1-4) (resp. (P1-3)) replacing test objects (X, Λ, i, ω) (resp. $(X, \Lambda, i_\mathfrak{N}, i_p)$) by $(X, \Lambda, i, C, \omega)$ (resp. $(X, \Lambda, i_\mathfrak{N}, C, i_p)$).

Our Hecke operators are defined on the space of level $(\mathfrak{N}, \Gamma_0(\mathfrak{q}^r))$. The operator $U(\mathfrak{q}^n)$ is defined only when $r > 0$ and $T(1, \mathfrak{q}^n)$ is defined only when $r = 0$. For a cyclic subgroup C' of $X_{/B}$ of order $\mathfrak{q}^n$, we can define the quotient abelian scheme X/C' with projection $\pi : X \to X/C'$. The polarization Λ and the differential ω induce a polarization $\pi_*\Lambda$ and a differential $(\pi^*)^{-1}\omega$ on X/C'. If $C' \cap C = \{0\}$ (in this case, we say that C' and C are *disjoint*), $\pi(C)$ gives rise to the level $\Gamma_0(\mathfrak{q}^r)$–structure on X/C'. Then we define for $f \in G_k(\mathfrak{c}\mathfrak{q}^n; \mathfrak{N}, \Gamma_0(\mathfrak{q}^r); A)$,

$$f|U(\mathfrak{q}^n)(X, \Lambda, C, i, \omega) = \frac{1}{N(\mathfrak{q}^n)} \sum_{C'} f(X/C', \pi_*\Lambda, \pi \circ i, \pi(C), (\pi^*)^{-1}\omega), \tag{2.7}$$

where C' runs over all étale cyclic subgroups of order $\mathfrak{q}^n$ disjoint from C. Since $\pi_*\Lambda = \pi \circ \Lambda \circ \pi^t$ is a $\mathfrak{c}\mathfrak{q}^n$–polarization, the modular form f has to be defined for abelian varieties with $\mathfrak{c}\mathfrak{q}^n$–polarization. Since $\mathfrak{q} \nmid \mathfrak{N}$, forgetting the $\Gamma_0(\mathfrak{q}^n)$–structure, we define for $f \in G_k(\mathfrak{c}\mathfrak{q}^n; \mathfrak{N}; A)$

$$f|T(1, \mathfrak{q}^n)(X, \Lambda, i, \omega) = \frac{1}{N(\mathfrak{q}^n)} \sum_{C'} f(X/C', \pi_*\Lambda, \pi \circ i, (\pi^*)^{-1}\omega), \tag{2.8}$$

where C' runs over all étale cyclic subgroups of order $\mathfrak{q}^n$. We can check that $f|U(\mathfrak{q}^n)$ and $f|T(1, \mathfrak{q}^n)$ belong to $V(\mathfrak{c}, \mathfrak{N}, \Gamma_0(\mathfrak{q}^r); A)$ and also stay in $G_k(\mathfrak{c}, \mathfrak{N}, \Gamma_0(\mathfrak{q}^r); A)$ if $f \in G_k(\mathfrak{c}\mathfrak{q}, \mathfrak{N}, \Gamma_0(\mathfrak{q}^r); A)$. We have

$$U(\mathfrak{q}^n) = U(\mathfrak{q})^n.$$

2.8 Hilbert modular Shimura varieties

We extend the level structure i limited to $\mathfrak{N}$–torsion points to far bigger structure $\eta^{(p)}$ including all prime-to–p torsion points. Since the prime-to–p torsion on an abelian scheme $X_{/S}$ is unramified at p (see [ACM] 11.1 and [ST]), the extended level structure $\eta^{(p)}$ is still defined over S if S is a $\mathcal{W}$–scheme. Triples $(X, \overline{\Lambda}, \eta^{(p)})_{/S}$ for $\mathcal{W}$–schemes S are classified by an integral model $Sh^{(p)}_{/\mathcal{W}}$ (cf. [Ko]) of the Shimura variety $Sh_{/\mathbb{Q}}$ associated to the algebraic $\mathbb{Q}$–group $G = \mathrm{Res}_{F/\mathbb{Q}} GL(2)$ (in the sense of Deligne [De] 4.22 interpreting Shimura's original definition in [Sh] as a moduli of abelian schemes up to isogenies). Here the classification is up to prime-to–p isogenies, and $\overline{\Lambda}$ is an equivalence class of polarizations up to prime-to–p O–linear isogenies.

To give a description of the functor represented by $Sh^{(p)}$, we introduce some more notations. We consider the fiber category $\mathcal{A}^{(p)}_F$ over schemes defined by

(Object) abelian schemes X with real multiplication by O;

(Morphism) $\mathrm{Hom}_{\mathcal{A}_F^{(p)}}(X,Y) = \mathrm{Hom}(X,Y) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$,

where $\mathbb{Z}_{(p)}$ is the localization of $\mathbb{Z}$ at the prime ideal (p), that is,

$$\mathbb{Z}_{(p)} = \left\{ \frac{a}{b} \middle| b\mathbb{Z} + p\mathbb{Z} = \mathbb{Z},\ a, b \in \mathbb{Z} \right\}.$$

Isomorphisms in this category are isogenies with degree prime to p (called "prime-to–p isogenies"), and hence the degree of polarization Λ is supposed to be also prime to p. Two polarizations are equivalent if $\Lambda = c\Lambda' = \Lambda' \circ i(c)$ for a totally positive c prime to p. We fix an O–lattice $L \subset V = F^2$ with O–hermitian alternating pairing $\langle \cdot, \cdot \rangle$ inducing a self duality on $L_p = L \otimes_{\mathbb{Z}} \mathbb{Z}_p$. We consider the following condition on an AVRM $X_{/S}$ with $\theta : O \hookrightarrow \mathrm{End}(X_{/S})$:

(det) *the characteristic polynomial of $\theta(a)$ $(a \in O)$ on $Lie(X)$ over $\mathcal{O}_S$ is given by $\prod_{\sigma \in I}(T - \sigma(a))$, where I is the set of embeddings of F into $\overline{\mathbb{Q}}$.*

This condition is equivalent to the local freeness of $\pi_* \Omega_{X/S}$ over $\mathcal{O}_S \otimes_{\mathbb{Z}} O$ for $\pi : X \to S$.

For an open-compact subgroup K of $G(\mathbb{A}^{(\infty)})$ *maximal at p* (i.e. $K = GL_2(O_p) \times K^{(p)}$), we consider the following functor from $\mathbb{Z}_{(p)}$–schemes into $SETS$:

$$\mathcal{P}_K^{(p)}(S) = \left[(X, \overline{\Lambda}, \overline{\eta}^{(p)})_{/S} \text{ with (det)} \right]. \tag{2.9}$$

Here $\overline{\eta}^{(p)} : L \otimes_{\mathbb{Z}} \mathbb{A}^{(p\infty)} \cong V^{(p)}(X) = T(X) \otimes_{\mathbb{Z}} \mathbb{A}^{(p\infty)}$ is an equivalence class of $\eta^{(p)}$ modulo multiplication $\eta^{(p)} \mapsto \eta^{(p)} \circ k$ by $k \in K^{(p)}$ for the Tate module $T(X) = \varprojlim_{\mathfrak{N}} X[\mathfrak{N}]$ (in the sheafified sense that $\eta^{(p)} \equiv (\eta')^{(p)} \mod K$ étale-locally), and a $\Lambda \in \overline{\Lambda}$ induces the self-duality on L_p. As long as $K^{(p)}$ is sufficiently small (for K maximal at p), $\mathcal{P}_K^{(p)}$ is representable over any $\mathbb{Z}_{(p)}$–algebra A (e.g. [H04a], [H04b] Section 3.1 and [PAF] 4.2.1) by a scheme $Sh_{K/A} = Sh/K$, which is smooth by the unramifiedness of p in $F/\mathbb{Q}$. We let $g \in G(\mathbb{A}^{(p\infty)})$ act on $Sh^{(p)}_{/\mathbb{Z}_{(p)}}$ by

$$x = (X, \overline{\Lambda}, \eta) \mapsto g(x) = (X, \overline{\Lambda}, \eta \circ g),$$

which gives a right action of $G(\mathbb{A})$ on $Sh^{(p)}$ through the projection $G(\mathbb{A}) \twoheadrightarrow G(\mathbb{A}^{(p\infty)})$.

By the universality, we have a morphism $\mathfrak{M}(\mathfrak{c}; \mathfrak{N}) \to Sh^{(p)}/\widehat{\Gamma}_1^1(\mathfrak{c}; \mathfrak{N})$ for the open compact subgroup:

$$\widehat{\Gamma}_1^1(\mathfrak{c}; \mathfrak{N}) = \left(\begin{smallmatrix} c & 0 \\ 0 & 1 \end{smallmatrix}\right) K_1^1(\mathfrak{N}) \left(\begin{smallmatrix} c & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} cd^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right) U_1^1(\mathfrak{N}) \left(\begin{smallmatrix} cd^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1}$$

maximal at p. The image of $\mathfrak{M}(\mathfrak{c}; \mathfrak{N})$ gives a geometrically irreducible component of $Sh^{(p)}/\widehat{\Gamma}_1^1(\mathfrak{c}; \mathfrak{N})$. If $\mathfrak{N}$ is sufficiently deep, by the universality of

$\mathfrak{M}(\mathfrak{c};\mathfrak{N})$, we can identify $\mathfrak{M}(\mathfrak{c};\mathfrak{N})$ with its image in $Sh^{(p)}/\widehat{\Gamma}_1^1(\mathfrak{c};\mathfrak{N})$. By the action on the polarization $\Lambda \mapsto \alpha\Lambda$ for a suitable totally positive $\alpha \in F$, we can bring $\mathfrak{M}(\mathfrak{c};\mathfrak{N})$ into $\mathfrak{M}(\alpha\mathfrak{c};\mathfrak{N})$; so, the image of $\varprojlim_{\mathfrak{N}} \mathfrak{M}(\mathfrak{c};\mathfrak{N})$ in $Sh^{(p)}$ only depends on the strict ideal class of $\mathfrak{c}$.

For each $x = (X, \Lambda, i) \in \mathfrak{M}(\mathfrak{c};\mathfrak{N})(S)$ for a $\mathcal{W}$–scheme S, choosing $\eta^{(p)}$ so that $\eta^{(p)} \mod \widehat{\Gamma}_1^1(\mathfrak{c};\mathfrak{N}) = i$, we get a point $x = (X, \overline{\Lambda}, \eta^{(p)}) \in Sh^{(p)}(S)$ projecting down to $x = (X, \Lambda, i)$. Each element $g \in G(\mathbb{A})$ with totally positive determinant in $F^\times$ acts on $x = (X, \Lambda, \eta^{(p)}) \in Sh^{(p)}$ by $x \mapsto g(x) = (X, \det(g)\Lambda, \eta^{(p)} \circ g)$. This action is geometric preserving the base scheme $\mathrm{Spec}(\mathcal{W})$ and is compatible with the action of $G(\mathbb{A}^{(p\infty)})$ given as above (see [PAF] 4.2.2), because $\overline{\Lambda} = \overline{\det(g)\Lambda}$. Then we can think of the projection of $g(x)$ in $\mathfrak{M}(\mathfrak{c};\mathfrak{N})$. By abusing the notation slightly, if the lift $\eta^{(p)}$ of i is clear in the context, we write $g(x) \in \mathfrak{M}(\mathfrak{c};\mathfrak{N})$ for the image of $g(x) \in Sh^{(p)}$. If the action of g is induced by a prime-to–p isogeny $\alpha : X \to g(X)$, we write $g(x,\omega) = (g(x), \alpha_*\omega)$ for $(x,\omega) \in \mathcal{M}(\mathfrak{c};\mathfrak{N})$ if there is no ambiguity of α. When $\det(g)$ is not rational, the action of g is often non-trivial on $\mathrm{Spec}(\mathcal{W})$; see [Sh] II, [Sh1] and [PAF] 4.2.2.

2.9 Level structure with "Neben" character

In order to make a good link between classical modular forms and adelic automorphic forms (which we will describe in the following subsection), we would like to introduce "Neben" characters. We fix two integral ideals $\mathfrak{N} \subset \mathfrak{n} \subset O$. We think of the following level structure on an AVRM X:

$$i : (\mathbb{G}_m \otimes O^*)[\mathfrak{N}] \hookrightarrow X[\mathfrak{N}] \text{ and } i' : X[\mathfrak{n}] \twoheadrightarrow O/\mathfrak{n} \tag{2.10}$$

with $\mathrm{Im}(i) \times_{X[\mathfrak{N}]} X[\mathfrak{n}] = \mathrm{Ker}(i')$, where the sequence $(\mathbb{G}_m \otimes O^*)[\mathfrak{N}] \xrightarrow{i} X[\mathfrak{N}] \xrightarrow{i'} O/\mathfrak{n}$ is required to induce an isomorphism

$$(\mathbb{G}_m \otimes O^*)[\mathfrak{N}] \otimes_O O/\mathfrak{n} \cong (\mathbb{G}_m \otimes O^*)[\mathfrak{n}]$$

under the polarization Λ. When $\mathfrak{N} = \mathfrak{n}$, this is exactly a $\Gamma_1^1(\mathfrak{N})$–level structure. We fix two characters $\epsilon_1 : (O/\mathfrak{n})^\times \to A^\times$ and $\epsilon_2 : (O/\mathfrak{N})^\times \to A^\times$, and we insist for $f \in G_k(\mathfrak{c}, \mathfrak{N}; A)$ on the version of (G0-3) for quintuples $(X, \Lambda, i \cdot d, a \cdot i', \omega)$ and the equivariancy:

$$f(X, \overline{\Lambda}, i \cdot d, a \cdot i', \omega) = \epsilon_1(a)\epsilon_2(d) f(X, \overline{\Lambda}, i, i', \omega) \text{ for } a, d \in (O/\mathfrak{N})^\times. \tag{Neben}$$

Here $\overline{\Lambda}$ is the polarization class modulo multiple of totally positive numbers in F prime to $\mathfrak{n}$. We write $G_k(\mathfrak{c}, \Gamma_0(\mathfrak{N}), \epsilon; A)$ ($\epsilon = (\epsilon_1, \epsilon_2)$) for the A–module of geometric modular forms satisfying these conditions.

2.10 Adelic Hilbert modular forms and Hecke algebras

Let us interpret what we have said so far in automorphic language and give a definition of the adelic Hilbert modular forms and their Hecke algebra of level $\mathfrak{N}$ (cf. [H96] Sections 2.2-4 and [PAF] Sections 4.2.8–4.2.12). We first recall formal Hecke rings of double cosets. For that, we fix a prime element $\varpi_{\mathfrak{q}}$ of $O_{\mathfrak{q}}$ for every prime ideal $\mathfrak{q}$ of O.

We consider the following open compact subgroup of $G(\mathbb{A}^{(\infty)})$:

$$\begin{aligned} U_0(\mathfrak{N}) &= \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(\widehat{O}) \middle| c \equiv 0 \mod \mathfrak{N}\widehat{O} \right\}, \\ U_1^1(\mathfrak{N}) &= \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in U_0(\mathfrak{N}) \middle| a \equiv d \equiv 1 \mod \mathfrak{N}\widehat{O} \right\}, \end{aligned} \tag{2.11}$$

where $\widehat{O} = O \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ and $\widehat{\mathbb{Z}} = \prod_{\ell} \mathbb{Z}_{\ell}$. Then we introduce the following semigroup

$$\Delta_0(\mathfrak{N}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(\mathbb{A}^{(\infty)}) \cap M_2(\widehat{O}) \middle| c \equiv 0 \mod \mathfrak{N}\widehat{O}, d_{\mathfrak{N}} \in O_{\mathfrak{N}}^{\times} \right\}, \tag{2.12}$$

where $d_{\mathfrak{N}}$ is the projection of $d \in \widehat{O}$ to $\prod_{\mathfrak{q}|\mathfrak{N}} O_{\mathfrak{q}}$ for prime ideals $\mathfrak{q}$. Writing T_0 for the maximal diagonal torus of $GL(2)_{/O}$ and putting

$$D_0 = \left\{ \mathrm{diag}[a,d] = \left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) \in T_0(F_{\mathbb{A}^{(\infty)}}) \cap M_2(\widehat{O}) \middle| d_{\mathfrak{N}} = 1 \right\}, \tag{2.13}$$

we have (e.g. [MFG] 3.1.6 and [PAF] Section 5.1)

$$\Delta_0(\mathfrak{N}) = U_0(\mathfrak{N}) D_0 U_0(\mathfrak{N}). \tag{2.14}$$

In this section, writing $\mathfrak{p}^{\alpha} = \prod_{\mathfrak{p}|p} \mathfrak{p}^{\alpha(\mathfrak{p})}$ with $\alpha = (\alpha(\mathfrak{p}))$, the group U is assumed to be a subgroup of $U_0(\mathfrak{N}\mathfrak{p}^{\alpha})$ with $U \supset U_1^1(\mathfrak{N}\mathfrak{p}^{\alpha})$ for some multi-exponent α (though we do not assume that $\mathfrak{N}$ is prime to p). Formal finite linear combinations $\sum_{\delta} c_{\delta} U\delta U$ of double cosets of U in $\Delta_0(\mathfrak{N}\mathfrak{p}^{\alpha})$ form a ring $R(U, \Delta_0(\mathfrak{N}\mathfrak{p}^{\alpha}))$ under convolution product (see [IAT] Chapter 3 or [MFG] 3.1.6). The algebra is commutative and is isomorphic to the polynomial ring over the group algebra $\mathbb{Z}[U_0(\mathfrak{N}\mathfrak{p}^{\alpha})/U]$ with variables $\{T(\mathfrak{q}), T(\mathfrak{q},\mathfrak{q})\}_{\mathfrak{q}}$ for primes $\mathfrak{q}$, $T(\mathfrak{q})$ corresponding to the double coset $U \left(\begin{smallmatrix} \varpi_{\mathfrak{q}} & 0 \\ 0 & 1 \end{smallmatrix}\right) U$ and $T(\mathfrak{q},\mathfrak{q})$ (for primes $\mathfrak{q} \nmid \mathfrak{N}\mathfrak{p}^{\alpha}$) corresponding to $U_0 \varpi_{\mathfrak{q}} U$. Here we have chosen a prime element $\varpi_{\mathfrak{q}}$ in $O_{\mathfrak{q}}$. The group element $u \in U_0(\mathfrak{N}\mathfrak{p}^{\alpha})/U$ in $\mathbb{Z}[U_0(\mathfrak{N}\mathfrak{p}^{\alpha})/U]$ corresponds to the double coset UuU (cf. [H95] Section 2).

The double coset ring $R(U, \Delta_0(\mathfrak{N}\mathfrak{p}^{\alpha}))$ naturally acts on the space of modular forms on U whose definition we now recall. Recall that T_0 is the diagonal torus of $GL(2)_{/O}$; so, $T_0 = \mathbb{G}^2_{m/O}$. Since $T_0(O/\mathfrak{N}')$ is canonically a quotient of $U_0(\mathfrak{N}')$ for an ideal $\mathfrak{N}'$, a character $\epsilon : T_0(O/\mathfrak{N}') \to \mathbb{C}^{\times}$ can be considered as a character of $U_0(\mathfrak{N}')$. Writing $\epsilon\left(\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)\right) = \epsilon_1(a)\epsilon_2(d)$, if $\epsilon^- = \epsilon_1^{-1}\epsilon_2$

factors through $O/\mathfrak{N}$ for $\mathfrak{N}|\mathfrak{N}'$, then we can extend the character ϵ of $U_0(\mathfrak{N}')$ to $U_0(\mathfrak{N})$ by putting $\epsilon(u) = \epsilon_1(\det(u))\epsilon^-(d)$ for $u = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in U_0(\mathfrak{N})$. In this sense, we hereafter assume that ϵ is defined modulo $\mathfrak{N}$ and regard ϵ as a character of $U_0(\mathfrak{N})$. We choose a Hecke character $\epsilon_+ : F_{\mathbb{A}}^\times/F^\times \to \mathbb{C}^\times$ with infinity type $(1-[\kappa])I$ (for an integer $[\kappa]$) such that $\epsilon_+(z) = \epsilon_1(z)\epsilon_2(z)$ for $z \in \widehat{O}^\times$. We also write ϵ_+^t for the restriction of ϵ_+ to the maximal torsion subgroup $\Delta_F(\mathfrak{N})$ of $Cl_F^+(\mathfrak{N}p^\infty)$ (the strict ray class group modulo $\mathfrak{N}p^\infty$: $\varprojlim_n Cl_F^+(\mathfrak{N}p^n)$).

Writing I for the set of all embeddings of F into $\overline{\mathbb{Q}}$ and T^2 for $\mathrm{Res}_{O/\mathbb{Z}}T_0$ (the diagonal torus of G), the group of geometric characters $X^*(T^2)$ is isomorphic to $\mathbb{Z}[I]^2$ so that $(m,n) \in \mathbb{Z}[I]^2$ send $\mathrm{diag}[x,y] \in T^2$ to $x^m y^n = \prod_{\sigma\in I}(\sigma(x)^{m_\sigma}\sigma(y)^{n_\sigma})$. Taking $\kappa = (\kappa_1,\kappa_2) \in \mathbb{Z}[I]^2$, we assume $[\kappa]I = \kappa_1 + \kappa_2$, and we associate with κ a factor of automorphy:

$$J_\kappa(g,\tau) = \det(g_\infty)^{\kappa_2 - I} j(g_\infty,\tau)^{\kappa_1-\kappa_2+I} \quad \text{for } g \in G(\mathbb{A}) \text{ and } \tau \in \mathfrak{Z}. \tag{2.15}$$

We define $S_\kappa(U,\epsilon;\mathbb{C})$ by the space of functions $f : G(\mathbb{A}) \to \mathbb{C}$ satisfying the following three conditions (e.g. [H96] Section 2.2 and [PAF] Section 4.3.1):

(S1) $f(\alpha x u \delta) = \epsilon(u)\epsilon_+^t(z)f(x)J_\kappa(u,\mathbf{i})^{-1}$ for all $\alpha \in G(\mathbb{Q})$ and all $u \in U\cdot C_{\mathbf{i}}$ and $z \in \Delta_F(\mathfrak{N})$ ($\Delta_F(\mathfrak{N})$ is the maximal torsion subgroup of $Cl_F^+(\mathfrak{N}p^\infty)$);

(S2) Choose $u \in G(\mathbb{R})$ with $u(\mathbf{i}) = \tau$ for $\tau \in \mathfrak{Z}$, and put $f_x(\tau) = f(xu)J_\kappa(u,\mathbf{i})$ for each $x \in G(\mathbb{A}^{(\infty)})$ (which only depends on τ). Then f_x is a holomorphic function on $\mathfrak{Z}$ for all x;

(S3) $f_x(\tau)$ for each x is rapidly decreasing as $\eta_\sigma \to \infty$ ($\tau = \xi + \mathbf{i}\eta$) for all $\sigma \in I$ uniformly.

If we replace the word "rapidly decreasing" in (S3) by "slowly increasing", we get the definition of the space $G_\kappa(U,\epsilon;\mathbb{C})$. It is easy to check (e.g. [MFG] 3.1.5) that the function f_x in (S2) satisfies the classical automorphy condition:

$$f(\gamma(\tau)) = \epsilon(x^{-1}\gamma x)f(\tau)J_\kappa(\gamma,\tau) \quad \text{for all } \gamma \in \Gamma_x(U), \tag{2.16}$$

where $\Gamma_x(U) = xUx^{-1}G(\mathbb{R})^+ \cap G(\mathbb{Q})$. Also by (S3), f_x is rapidly decreasing towards all cusps of Γ_x (e.g. [MFG] (3.22)); so, it is a cusp form. Imposing that f have the central character ϵ_+ in place of the action of $\Delta_F(\mathfrak{N})$ in (S1), we define the subspace $S_\kappa(\mathfrak{N},\epsilon_+;\mathbb{C})$ of $S_\kappa(U_0(\mathfrak{N}),\epsilon;\mathbb{C})$. *The symbols* $\kappa = (\kappa_1,\kappa_2)$ *and* $(\varepsilon_1,\varepsilon_2)$ *here correspond to* (κ_2,κ_1) *and* $(\varepsilon_2,\varepsilon_1)$ *in [PAF] Section 4.2.6 (page 171) because of a different notational convention in [PAF].*

If we restrict f as above to $SL_2(F_{\mathbb{A}})$, the determinant factor $\det(g)^{\kappa_2}$ in the factor of automorphy disappears, and the automorphy factor becomes only dependent on $k = \kappa_1 - \kappa_2 + I \in \mathbb{Z}[I]$; so, the classical modular form in

G_k has single digit weight $k \in \mathbb{Z}[I]$. Via (2.3), we have an embedding of $S_\kappa(U_0(\mathfrak{N}'), \epsilon; \mathbb{C})$ into $G_k(\Gamma_0(\mathfrak{N}'), \epsilon; \mathbb{C}) = \bigoplus_{[\mathfrak{c}] \in Cl_F^+} G_k(\mathfrak{c}, \Gamma_0(\mathfrak{N}'), \epsilon; \mathbb{C})$ ($\mathfrak{c}$ running over a complete representative set for the strict ideal class group Cl_F^+) bringing f into $(f_\mathfrak{c})_{[\mathfrak{c}]}$ for $f_\mathfrak{c} = f_x$ (as in (S3)) with $x = \left(\begin{smallmatrix} cd^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)$ (for $d \in F_\mathbb{A}^\times$ with $d\widehat{O} = \widehat{\mathfrak{d}}$). The cusp form $f_\mathfrak{c}$ is determined by the restriction of f to $x \cdot SL_2(F_\mathbb{A})$. If we vary the weight κ keeping $k = \kappa_1 - \kappa_2 + I$, the image of S_κ in $G_k(\Gamma_0(\mathfrak{N}'), \epsilon; \mathbb{C})$ transforms accordingly. By this identification, the Hecke operator $T(\mathfrak{q})$ for non-principal $\mathfrak{q}$ makes sense as an operator acting on a single space $G_\kappa(U, \epsilon; \mathbb{C})$, and its action depends on the choice of κ. In other words, we have the double digit weight $\kappa = (\kappa_1, \kappa_2)$ for adelic modular forms in order to specify the central action of $G(\mathbb{A})$. For a given $f \in S_\kappa(U, \epsilon; \mathbb{C})$ and a Hecke character $\lambda : F_\mathbb{A}^\times / F^\times \to \mathbb{C}^\times$, the tensor product $(f \otimes \lambda)(x) = f(x)\lambda(\det(x))$ gives rise to a different modular form in $S_{\kappa_\lambda}(U, \epsilon_\lambda; \mathbb{C})$ for weight κ_λ and character ϵ_λ dependent on λ, although the two modular forms have the same restriction to $SL_2(F_\mathbb{A})$.

We identify I with $\sum_\sigma \sigma$ in $\mathbb{Z}[I]$. It is known that $G_\kappa = 0$ unless $\kappa_1 + \kappa_2 = [\kappa_1 + \kappa_2]I$ for $[\kappa_1 + \kappa_2] \in \mathbb{Z}$, because $I - (\kappa_1 + \kappa_2)$ is the infinity type of the central character of automorphic representations generated by G_κ. We write simply $[\kappa]$ for $[\kappa_1 + \kappa_2] \in \mathbb{Z}$ assuming $G_\kappa \neq 0$. The $SL(2)$–weight of the central character of an irreducible automorphic representation π generated by $f \in G_\kappa(U, \epsilon; \mathbb{C})$ is given by k (which specifies the infinity type of π_∞ as a discrete series representation of $SL_2(F_\mathbb{R})$). There is a geometric meaning of the weight κ: the Hodge weight of the motive attached to π (cf. [BR]) is given by $\{(\kappa_{1,\sigma}, \kappa_{2,\sigma}), (\kappa_{2,\sigma}, \kappa_{1,\sigma})\}_\sigma$, and thus, the requirement $\kappa_1 - \kappa_2 \geq I$ is the regularity assumption for the motive (and is equivalent to the classical weight $k \geq 2I$ condition).

Choose a prime element $\varpi_\mathfrak{q}$ of $O_\mathfrak{q}$ for each prime $\mathfrak{q}$ of F. We extend $\epsilon^- : \widehat{O}^\times \to \mathbb{C}^\times$ to $F_{\mathbb{A}(\infty)}^\times \to \mathbb{C}^\times$ just by putting $\epsilon^-(\varpi_\mathfrak{q}^m) = 1$ for $m \in \mathbb{Z}$. This is possible because $F_\mathfrak{q}^\times = O_\mathfrak{q}^\times \times \varpi_\mathfrak{q}^\mathbb{Z}$ for $\varpi_\mathfrak{q}^\mathbb{Z} = \{\varpi_\mathfrak{q}^m | m \in \mathbb{Z}\}$. Similarly, we extend ϵ_1 to $F_{\mathbb{A}(\infty)}^\times$. Then we define $\epsilon(u) = \epsilon_1(\det(u))\epsilon^-(a_\mathfrak{N})$ for $u = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Delta_0(\mathfrak{N})$. Let $\mathcal{U}$ be the unipotent algebraic subgroup of $GL(2)_{/O}$ defined by $\mathcal{U}(A) = \left\{\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right) \middle| a \in A\right\}$. For each $UyU \in R(U, \Delta_0(\mathfrak{N}\mathfrak{p}^\alpha))$, we decompose $UyU = \bigsqcup_{t \in D_0, u \in \mathcal{U}(\widehat{O})} utU$ for finitely many u and t (see [IAT] Chapter 3 or [MFG] 3.1.6) and define

$$f|[UyU](x) = \sum_{t,u} \epsilon(t)^{-1} f(xut). \tag{2.17}$$

We check that this operator preserves the spaces of automorphic forms: $G_\kappa(\mathfrak{N}, \epsilon; \mathbb{C})$ and $S_\kappa(\mathfrak{N}, \epsilon; \mathbb{C})$. This action for y with $y_\mathfrak{N} = 1$ is independent of the choice of the extension of ϵ to $T_0(F_{\mathbb{A}})$. When $y_{\mathfrak{N}} \neq 1$, we may assume that $y_{\mathfrak{N}} \in D_0 \subset T_0(F_{\mathbb{A}})$, and in this case, t can be chosen so that $t_{\mathfrak{N}} = y_{\mathfrak{N}}$ (so $t_{\mathfrak{N}}$ is independent of single right cosets in the double coset). If we extend ϵ to $T_0(F_{\mathbb{A}}^{(\infty)})$ by choosing another prime element $\varpi'_{\mathfrak{q}}$ and write the extension as ϵ', then we have

$$\epsilon(t_{\mathfrak{N}})[UyU] = \epsilon'(t_{\mathfrak{N}})[UyU]',$$

where the operator on the right-hand-side is defined with respect to ϵ'. Thus the sole difference is the root of unity $\epsilon(t_{\mathfrak{N}})/\epsilon'(t_{\mathfrak{N}}) \in \mathrm{Im}(\epsilon|_{T_0(O/\mathfrak{N})})$. Since it depends on the choice of $\varpi_{\mathfrak{q}}$, we make the choice once and for all, and write $T(\mathfrak{q})$ for $\left[U \left(\begin{smallmatrix} \varpi_{\mathfrak{q}} & 0 \\ 0 & 1 \end{smallmatrix}\right) U\right]$ (if $\mathfrak{q}|\mathfrak{N}$). By linearity, these action of double cosets extends to the ring action of the double coset ring $R(U, \Delta_0(\mathfrak{N}\mathfrak{p}^{\alpha}))$.

To introduce rationality of modular forms, we recall Fourier expansion of adelic modular forms (cf. [H96] Sections 2.3-4). Recall the embedding $i_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, and identify $\overline{\mathbb{Q}}$ with the image of i_∞. Recall also the differental idele $d \in F_{\mathbb{A}}^{\times}$ with $d^{(\mathfrak{d})} = 1$ and $d\widehat{O} = \mathfrak{d}\widehat{O}$. Each member f of $S_\kappa(U, \epsilon; \mathbb{C})$ has its Fourier expansion:

$$f\left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right) = |y|_{\mathbb{A}} \sum_{0 \ll \xi \in F} a(\xi y d, f)(\xi y_\infty)^{-\kappa_2} \mathbf{e}_F(i\xi y_\infty)\mathbf{e}_F(\xi x), \qquad (2.18)$$

where $\mathbf{e}_F : F_{\mathbb{A}}/F \to \mathbb{C}^{\times}$ is the additive character which has $\mathbf{e}_F(x_\infty) = \exp(2\pi i \sum_{\sigma \in I} x_\sigma)$ for $x_\infty = (x_\sigma)_\sigma \in \mathbb{R}^I = F \otimes_{\mathbb{Q}} \mathbb{R}$. Here $y \mapsto a(y, f)$ is a function defined on $y \in F_{\mathbb{A}}^{\times}$ only depending on its finite part $y^{(\infty)}$. The function $a(y, f)$ is supported by the set $(\widehat{O} \times F_\infty) \cap F_{\mathbb{A}}^{\times}$ of *integral* ideles.

Let $F[\kappa]$ be the field fixed by $\{\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F) | \kappa\sigma = \kappa\}$, over which the character $\kappa \in X^*(T^2)$ is rational. Write $O[\kappa]$ for the integer ring of $F[\kappa]$. We also define $O[\kappa, \epsilon]$ for the integer ring of the field $F[\kappa, \epsilon]$ generated by the values of ϵ over $F[\kappa]$. For any $F[\kappa, \epsilon]$–algebra A inside $\mathbb{C}$, we define

$$S_\kappa(U, \epsilon; A) = \left\{ f \in S_\kappa(U, \epsilon; \mathbb{C}) \middle| a(y, f) \in A \text{ as long as } y \text{ is integral} \right\}. \qquad (2.19)$$

As we have seen, we can interpret $S_\kappa(U, \epsilon; A)$ as the space of A–rational global sections of a line bundle of a variety defined over A; so, we have, by the flat base-change theorem (e.g. [GME] Lemma 1.10.2),

$$S_\kappa(\mathfrak{N}, \epsilon; A) \otimes_A \mathbb{C} = S_\kappa(\mathfrak{N}, \epsilon; \mathbb{C}). \qquad (2.20)$$

The Hecke operators preserve A–rational modular forms (e.g., [PAF] 4.2.9). We define the Hecke algebra $h_\kappa(U, \epsilon; A) \subset \mathrm{End}_A(S_\kappa(U, \epsilon; A))$ by the A–subalgebra generated by the Hecke operators of $R(U, \Delta_0(\mathfrak{N}\mathfrak{p}^{\alpha}))$.

For any $\overline{\mathbb{Q}}_p$–algebras A, we define

$$S_\kappa(U,\epsilon;A) = S_\kappa(U,\epsilon;\overline{\mathbb{Q}}) \otimes_{\overline{\mathbb{Q}},i_p} A. \tag{2.21}$$

By linearity, $y \mapsto a(y,f)$ extends to a function on $F_{\mathbb{A}}^\times \times S_\kappa(U,\epsilon;A)$ with values in A. We define the q–expansion coefficients (at p) of $f \in S_\kappa(U,\epsilon;A)$ by

$$\mathbf{a}_p(y,f) = y_p^{-\kappa_2} a(y,f) \text{ and } \mathbf{a}_{0,p}(y,f) = \mathcal{N}(yd^{-1})^{[\kappa_2]} a_0(y,f), \tag{2.22}$$

where $\mathcal{N} : F_{\mathbb{A}}^\times / F^\times \to \overline{\mathbb{Q}}_p^\times$ is the character given by $\mathcal{N}(y) = y_p^{-I}|y^{(\infty)}|_{\mathbb{A}}^{-1}$. Here we note that $a_0(y,f) = 0$ if $\kappa_2 \notin \mathbb{Z}I$. Thus, if $a_0(y,f) \neq 0$, $[\kappa_2] \in \mathbb{Z}$ is well defined. The formal q–expansion of an A–rational f has values in the space of functions on $F_{\mathbb{A}^{(\infty)}}^\times$ with values in the formal monoid algebra $A[[q^\xi]]_{\xi \in F_+}$ of the multiplicative semi-group F_+ made up of totally positive elements, which is given by

$$f(y) = \mathcal{N}(y)^{-1}\left\{\mathbf{a}_{0,p}(yd,f) + \sum_{\xi \gg 0} \mathbf{a}_p(\xi yd,f) q^\xi\right\}. \tag{2.23}$$

We now define for any p–adically complete $O[\kappa,\epsilon]$–algebra A in $\widehat{\mathbb{Q}}_p$

$$S_\kappa(U,\epsilon;A) = \left\{ f \in S_\kappa(U,\epsilon;\widehat{\mathbb{Q}}_p) \middle| \mathbf{a}_p(y,f) \in A \text{ for integral } y \right\}. \tag{2.24}$$

As we have already seen, these spaces have geometric meaning as the space of A–integral global sections of a line bundle defined over A of the Hilbert modular variety of level U (see [PAF] Section 4.2.6), and the q–expansion above for a fixed $y = y^{(\infty)}$ gives rise to the geometric q–expansion at the infinity cusp of the classical modular form f_x for $x = \left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right)$ (see [H91] (1.5) and [PAF] (4.63)).

We have chosen a complete representative set $\{c_i\}_{i=1,\dots,h}$ in finite ideles for the strict idele class group $F^\times \backslash F_{\mathbb{A}}^\times / \widehat{O}^\times F_{\infty+}^\times$, where h is the strict class number of F. Let $\mathfrak{c}_i = c_i O$. Write $t_i = \left(\begin{smallmatrix} c_i d^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and consider $f_i = f_{t_i}$ as defined in (S2). The collection $(f_i)_{i=1,\dots,h}$ determines f, because of the approximation theorem. Then $f(c_i d^{-1})$ gives the q–expansion of f_i at the Tate abelian variety with $\mathfrak{c}_i$–polarization $\mathrm{Tate}_{\mathfrak{c}_i^{-1},O}(q)$ ($\mathfrak{c}_i = c_i O$). By (q–exp), the q–expansion $f(y)$ determines f uniquely.

We write $T(y)$ for the Hecke operator acting on $S_\kappa(U,\epsilon;A)$ corresponding to the double coset $U \left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right) U$ for an integral idele y. We renormalize $T(y)$ to have a p–integral operator $\mathbb{T}(y)$: $\mathbb{T}(y) = y_p^{-\kappa_2} T(y)$. Since this only affects $T(y)$ with $y_p \neq 1$, $\mathbb{T}(\mathfrak{q}) = T(\varpi_{\mathfrak{q}}) = T(\mathfrak{q})$ if $\mathfrak{q} \nmid p$. However $\mathbb{T}(\mathfrak{p}) \neq T(\mathfrak{p})$

for primes $\mathfrak{p}|p$. The renormalization is optimal to have the stability of the A–integral spaces under Hecke operators. We define $\langle \mathfrak{q} \rangle = N(\mathfrak{q})T(\mathfrak{q},\mathfrak{q})$ for $\mathfrak{q} \nmid \mathfrak{N}\mathfrak{p}^\alpha$, which is equal to the central action of a prime element $\varpi_\mathfrak{q}$ of $O_\mathfrak{q}$ times $N(\mathfrak{q}) = |\varpi_\mathfrak{q}|_\mathbb{A}^{-1}$. We have the following formula of the action of $T(\mathfrak{q})$ and $T(\mathfrak{q},\mathfrak{q})$ (e.g., [PAF] Section 4.2.10):

$$\mathbf{a}_p(y, f|\mathbb{T}(\mathfrak{q})) = \begin{cases} \mathbf{a}_p(y\varpi_\mathfrak{q}, f) + \mathbf{a}_p(y\varpi_\mathfrak{q}^{-1}, f|\langle \mathfrak{q} \rangle) & \text{if } \mathfrak{q} \text{ is outside } \mathfrak{n} \\ \mathbf{a}_p(y\varpi_\mathfrak{q}, f) & \text{otherwise,} \end{cases} \tag{2.25}$$

where the level $\mathfrak{n}$ of U is the ideal maximal under the condition: $U_1^1(\mathfrak{n}) \subset U \subset U_0(\mathfrak{N})$. Thus $\mathbb{T}(\varpi_\mathfrak{q}) = U(\mathfrak{q})$ (up to p–adic units) when $\mathfrak{q}$ is a factor of the level of U (even when $\mathfrak{q}|p$; see [PAF] (4.65–66)). Writing the level of U as $\mathfrak{N}\mathfrak{p}^\alpha$, we assume

$$\text{either } p|\mathfrak{N}\mathfrak{p}^\alpha \text{ or } [\kappa] \geq 0, \tag{2.26}$$

since $\mathbb{T}(\mathfrak{q})$ and $\langle \mathfrak{q} \rangle$ preserve the space $S_\kappa(U,\epsilon;A)$ under this condition (see [PAF] Theorem 4.28). We then define the Hecke algebra $h_\kappa(U,\epsilon;A)$ (resp. $h_\kappa(\mathfrak{N},\epsilon_+;A)$) with coefficients in A by the A–subalgebra of the A–linear endomorphism algebra $\mathrm{End}_A(S_\kappa(U,\epsilon;A))$ (resp. $\mathrm{End}_A(S_\kappa(\mathfrak{N},\epsilon_+;A))$) generated by the action of the finite group $U_0(\mathfrak{N}\mathfrak{p}^\alpha)/U$, $\mathbb{T}(\mathfrak{q})$ and $\langle \mathfrak{q} \rangle$ for all $\mathfrak{q}$.

We have canonical projections:

$$R(U_1^1(\mathfrak{N}\mathfrak{p}^\alpha), \Delta_0(\mathfrak{N}\mathfrak{p}^\alpha)) \twoheadrightarrow R(U, \Delta_0(\mathfrak{N}\mathfrak{p}^\alpha)) \twoheadrightarrow R(U_0(\mathfrak{N}p^\beta), \Delta_0(\mathfrak{N}p^\beta))$$

for all $\alpha \geq \beta$ ($\Leftrightarrow \alpha(\mathfrak{p}) \geq \beta(\mathfrak{p})$ for all $\mathfrak{p}|p$) taking canonical generators to the corresponding ones, which are compatible with inclusions

$$S_\kappa(U_0(\mathfrak{N}p^\beta),\epsilon;A) \hookrightarrow S_\kappa(U,\epsilon;A) \hookrightarrow S_\kappa(U_1^1(\mathfrak{N}\mathfrak{p}^\alpha),\epsilon;A).$$

We get a projective system of Hecke algebras $\{h_\kappa(U,\epsilon;A)\}_U$ (U running through open subgroups of $U_0(\mathfrak{N}p)$ containing $U_1^1(\mathfrak{N}p^\infty)$), whose projective limit (when $\kappa_1 - \kappa_2 \geq I$) gives rise to the universal Hecke algebra $\mathbf{h}(\mathfrak{N},\epsilon;A)$ for a complete p–adic algebra A. This algebra is known to be independent of κ (as long as $\kappa_1 - \kappa_2 \geq I$) and has canonical generators $\mathbb{T}(y)$ over $A[[\mathbf{G}]]$ (for $\mathbf{G} = (O_p \times (O/\mathfrak{N}^{(p)}))^\times \times Cl_F^+(\mathfrak{N}p^\infty)$), where $\mathfrak{N}^{(p)}$ is the prime-to–p part of $\mathfrak{N}$. Here note that the operator $\langle \mathfrak{q} \rangle$ is included in the action of $\mathbf{G}$, because $\mathfrak{q} \in Cl_F^+(\mathfrak{N}p^\infty)$. We write $h_\kappa^{n.ord}(U,\epsilon;A)$, $h_\kappa^{n.ord}(\mathfrak{N}p^\alpha,\epsilon_+;A)$ and $\mathbf{h}^{n.ord} = \mathbf{h}^{n.ord}(\mathfrak{N},\epsilon;A)$ for the image of the (nearly) ordinary projector $e = \lim_n \mathbb{T}(p)^{n!}$. The algebra $\mathbf{h}^{n.ord}$ is by definition the universal nearly ordinary Hecke algebra over $A[[\mathbf{G}]]$ of level $\mathfrak{N}$ with "Neben character" ϵ. We also note here that this algebra $\mathbf{h}^{n.ord}(\mathfrak{N},\epsilon;A)$ is exactly the one $\mathbf{h}(\psi^+,\psi')$

employed in [HT1] page 240 (when specialized to the CM component there) if A is a complete p–adic valuation ring.

Let $\mathbf{\Lambda}_A = A[[\mathbf{\Gamma}]]$ for the maximal torsion-free quotient $\mathbf{\Gamma}$ of $\mathbf{G}$. We fix a splitting $\mathbf{G} = \mathbf{\Gamma} \times \mathbf{G}_{tor}$ for a finite group $\mathbf{G}_{tor}$. If A is a complete p–adic valuation ring, then $\mathbf{h}^{n.ord}(\mathfrak{N}, \epsilon; A)$ is a torsion-free $\mathbf{\Lambda}_A$–algebra of finite rank and is $\mathbf{\Lambda}_A$–free under some mild conditions on $\mathfrak{N}$ and ϵ ([PAF] 4.2.12). Take a point $P \in \mathrm{Spf}(\mathbf{\Lambda})(A) = \mathrm{Hom}_{cont}(\mathbf{\Gamma}, A^\times)$. Regarding P as a character of $\mathbf{G}$, we call P *arithmetic* if it is given locally by an algebraic character $\kappa(P) \in X^*(T^2)$ with $\kappa_1(P) - \kappa_2(P) \geq I$. Thus if P is arithmetic, $\epsilon_P = P\kappa(P)^{-1}$ is a character of $T^2(O/\mathfrak{p}^\alpha\mathfrak{N})$ for some multi-exponent $\alpha \geq 0$. Similarly, the restriction of P to $Cl_F^+(\mathfrak{N}p^\infty)$ is a p–adic Hecke character ϵ_{P+} induced by an arithmetic Hecke character of infinity type $(1 - [\kappa(P)])I$. As long as P is arithmetic, we have a canonical specialization morphism:

$$\mathbf{h}^{n.ord}(\mathfrak{N}, \epsilon; A) \otimes_{\mathbf{\Lambda}_A, P} A \twoheadrightarrow h^{n.ord}_{\kappa(P)}(\mathfrak{N}\mathfrak{p}^\alpha, \epsilon_{P+}; A),$$

which is an isogeny (surjective and of finite kernel) and is an isomorphism if $\mathbf{h}^{n.ord}$ is $\mathbf{\Lambda}_A$–free. The specialization morphism takes the generators $\mathbb{T}(y)$ to $\mathbb{T}(y)$.

3 Eisenstein series

We shall study the q–expansion, Hecke eigenvalues and special values at CM points of an Eisenstein series defined on $\mathfrak{M}(\mathfrak{c}; \mathfrak{N})$.

3.1 Arithmetic Hecke characters

Recall the CM type Σ ordinary at p and the prime ideal $\mathfrak{l}$ of O introduced in the introduction. We sometimes regard Σ as a character of $T_M = \mathrm{Res}_{M/\mathbb{Q}}\mathbb{G}_m$ sending $x \in M^\times$ to $x^\Sigma = \prod_{\sigma\in\Sigma} \sigma(x)$. More generally, each integral linear combination $\kappa = \sum_{\sigma\in\Sigma\sqcup\Sigma c} \kappa_\sigma \sigma$ is regarded as a character of T_M by $x \mapsto \prod_\sigma \sigma(x)^{\kappa_\sigma}$. We fix an arithmetic Hecke character λ of infinity type $k\Sigma + \kappa(1-c)$ for $\kappa = \sum_{\sigma\in\Sigma} \kappa_\sigma\sigma \in \mathbb{Z}[\Sigma]$ and an integer k. This implies, regarding λ as an idele character of $T_M(\mathbb{A})$, $\lambda(x_\infty) = x_\infty^{k\Sigma+\kappa(1-c)}$ for $x_\infty \in T_M(\mathbb{R})$. We assume the following three conditions:

(crt) $k > 0$ *and* $\kappa \geq 0$, *where we write* $\kappa \geq 0$ *if* $\kappa_\sigma \geq 0$ *for all* σ.

(opl) *The conductor* $\mathfrak{C}$ *of* λ *is prime to* p *and* $(\ell) = \mathfrak{l} \cap \mathbb{Z}$.

(spt) *The ideal* $\mathfrak{C}$ *is a product of primes split over* F.

3.2 Hilbert modular Eisenstein series

We shall define an Eisenstein series whose special values at CM points interpolate the values $L(0, \lambda\chi)$ for anticyclotomic characters χ of finite order.

We split the conductor $\mathfrak{C}$ in the following way: $\mathfrak{C} = \mathfrak{F}\mathfrak{F}_c$ with $\mathfrak{F} + \mathfrak{F}_c = R$ and $\mathfrak{F} \subset \mathfrak{F}_c^c$. This is possible by (spt). We then define $\mathfrak{f} = \mathfrak{F} \cap O$ and $\mathfrak{f}' = \mathfrak{F}_c \cap O$. Then $\mathfrak{f} \subset \mathfrak{f}'$. Here $X = O/\mathfrak{f} \cong R/\mathfrak{F}$ and $Y = O/\mathfrak{f}' \cong R/\mathfrak{F}_c$. Let $\phi : X \times Y \to \mathbb{C}$ be a function such that $\phi(\varepsilon^{-1}x, \varepsilon y) = N(\varepsilon)^k \phi(x, y)$ for all $\varepsilon \in O^\times$ with the integer k as above. We put $X^* = \mathfrak{f}^*/O^*$; so, X^* is naturally the Pontryagin dual module of X under the pairing $(x^*, x) = \mathbf{e}_F(x^*x) = \epsilon(\mathrm{Tr}(x^*x))$, where $\epsilon(x) = \exp(2\pi i x)$ for $x \in \mathbb{C}$. We define the partial Fourier transform $P\phi : X^* \times Y \to \mathbb{C}$ of ϕ by

$$P\phi(x, y) = N(\mathfrak{f})^{-1} \sum_{a \in X} \phi(a, y)\mathbf{e}_F(ax), \tag{3.1}$$

where $\mathbf{e}_F$ is the restriction of the standard additive character of the adele ring $F_{\mathbb{A}}$ to the local component $F_{\mathfrak{f}}$ at $\mathfrak{f}$.

A function ϕ as above can be interpreted as a function of $(\mathcal{L}, \Lambda, i, i')$ in 2.5. Here $i : X^* \hookrightarrow \mathfrak{f}^{-1}\mathcal{L}/\mathcal{L}$ is the level $\mathfrak{f}$–structure. We define an $O_{\mathfrak{f}}$–submodule $PV(\mathcal{L}) \subset \mathcal{L} \otimes_O F_{\mathfrak{f}}$ specified by the following conditions:

$$PV(\mathcal{L}) \supset \mathcal{L} \otimes_O O_{\mathfrak{f}},\ PV(\mathcal{L})/\mathcal{L}_{\mathfrak{f}} = \mathrm{Im}(i)\ (\mathcal{L}_{\mathfrak{f}} = \mathcal{L} \otimes_O O_{\mathfrak{f}}). \tag{PV}$$

By definition, we may regard

$$i^{-1} : PV(\mathcal{L}) \twoheadrightarrow PV(\mathcal{L})/\left(\mathcal{L} \otimes_O O_{\mathfrak{f}}\right) \cong \mathfrak{f}^*/O^*.$$

By Pontryagin duality under $\mathrm{Tr}\circ\lambda$, the dual map of i gives rise to $i' : PV(\mathcal{L}) \twoheadrightarrow O/\mathfrak{f}$. Taking a lift $\widetilde{i} : (\mathfrak{f}^2)^*/O^* \hookrightarrow PV(\mathcal{L})/\mathfrak{f}\mathcal{L}_{\mathfrak{f}}$ with $\widetilde{i} \bmod \mathcal{L}_{\mathfrak{f}} = i$, we have an exact sequence:

$$0 \to (\mathfrak{f}^2)^*/O^* \xrightarrow{\widetilde{i}} PV(\mathcal{L})/\mathfrak{f}\mathcal{L}_{\mathfrak{f}} \xrightarrow{i'} O/\mathfrak{f} \to 0.$$

This sequence is kept under $\alpha \in \mathrm{Aut}(\mathcal{L})$ with unipotent reduction modulo $\mathfrak{f}^2$, and hence, the pair (i, i') gives a level $\Gamma_1^1(\mathfrak{f}^2)$–structure: Once we have chosen a generator f of $\mathfrak{f}$ in $O_{\mathfrak{f}}$, by the commutativity of the following diagram:

$$\begin{array}{ccccc} (\mathfrak{f}^2)^*/O^* & \xrightarrow{\widetilde{i}} & PV(\mathcal{L})/\mathfrak{f}\mathcal{L}_{\mathfrak{f}} & \xrightarrow{i'} & O/\mathfrak{f} \\ \downarrow\wr & & f\downarrow\cap & & f\downarrow\cap \\ (\mathfrak{f}^2)^*/O^* & \longrightarrow & X[\mathfrak{f}^2] & \longrightarrow & O/\mathfrak{f}^2, \end{array} \tag{3.2}$$

giving (i, i') is equivalent to having the bottom sequence of maps in the above diagram. This explains why the pair (i, i') gives rise to a level $\Gamma_1^1(\mathfrak{f}^2)$–structure; strictly speaking, the exact level group is given by:

$$\Gamma_{1,0}^1(\mathfrak{f}^2) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(O_{\mathfrak{f}}) \middle| a \equiv d \equiv 1 \mod \mathfrak{f},\ c \equiv 0 \mod \mathfrak{f}^2 \right\}. \quad (3.3)$$

We regard $P\phi$ as a function of $\mathcal{L} \otimes_O F$ supported on $(\mathfrak{f}^{-2}\mathcal{L}) \cap PV(\mathcal{L})$ by

$$P\phi(w) = \begin{cases} P\phi(i^{-1}(w), i'(w)) & \text{if } (w \mod \mathcal{L}) \in \mathrm{Im}(i), \\ 0 & \text{otherwise.} \end{cases} \quad (3.4)$$

For each $w = (w_\sigma) \in F \otimes_{\mathbb{Q}} \mathbb{C} = \mathbb{C}^I$, the norm map $N(w) = \prod_{\sigma \in I} w_\sigma$ is well defined.

For any positive integer $k > 0$, we can now define the Eisenstein series E_k. Writing $\underline{\mathcal{L}} = (\mathcal{L}, \lambda, i)$ for simplicity, we define the value $E_k(\underline{\mathcal{L}}; \phi, \mathfrak{c})$ by

$$E_k(\underline{\mathcal{L}}; \phi, \mathfrak{c}) = \frac{\{(-1)^k \Gamma(k+s)\}^{[F:\mathbb{Q}]}}{\sqrt{|D_F|}} \sum_{w \in \mathfrak{f}^{-1}\mathcal{L}/O^\times}{}' \frac{P\phi(w)}{N(w)^k |N(w)|^{2s}}\Big|_{s=0}. \quad (3.5)$$

Here "$\sum'$" indicates that we are excluding $w = 0$ from the summation. As shown by Hecke, this type of series is convergent when the real part of s is sufficiently large and can be continued to a meromorphic function well defined at $s = 0$ (as long as either $k \geq 2$ or $\phi(a, 0) = 0$ for all a). The weight of the Eisenstein series is the parallel weight $kI = \sum_\sigma k\sigma$. If either $k \geq 2$ or $\phi(a, 0) = 0$ for all a, the function $E_k(\mathfrak{c}, \phi)$ gives an element in $G_{kI}(\mathfrak{c}, \mathfrak{f}^2; \mathbb{C})$, whose q–expansion at the cusp $(\mathfrak{a}, \mathfrak{b})$ computed in [HT1] Section 2 is given by

$$N(\mathfrak{a})^{-1} E_k(\phi, \mathfrak{c})_{\mathfrak{a},\mathfrak{b}}(q) = 2^{-[F:\mathbb{Q}]} L(1-k; \phi, \mathfrak{a}) + \sum_{0 \ll \xi \in \mathfrak{ab}} \sum_{\substack{(a,b) \in (\mathfrak{a} \times \mathfrak{b})/O^\times \\ ab = \xi}} \phi(a, b) \frac{N(a)}{|N(a)|} N(a)^{k-1} q^\xi, \quad (3.6)$$

where $L(s; \phi, \mathfrak{a})$ is the partial L–function given by the Dirichlet series:

$$\sum_{\xi \in (\mathfrak{a} - \{0\})/O^\times} \phi(\xi, 0) \left(\frac{N(\xi)}{|N(\xi)|} \right)^k |N(\xi)|^{-s}.$$

If $\phi(x, y) = \phi_X(x)\phi_Y(y)$ for two functions $\phi_X : X \to \mathbb{C}$ and $\phi_Y : Y \to \mathbb{C}$ with ϕ_Y factoring through $O/\mathfrak{f}'$, then we can check easily that $E_k(\phi) \in G_{kI}(\mathfrak{c}, \mathfrak{f}\mathfrak{f}'; \mathbb{C})$.

3.3 Hecke eigenvalues

We take a Hecke character λ as in 3.1. Then the restriction $\lambda_{\mathfrak{C}}^{-1} : R_{\mathfrak{F}}^{\times} \times R_{\mathfrak{F}_c}^{\times} \to \mathcal{W}^{\times}$ induces a locally constant function $\psi : (O/\mathfrak{f}) \times (O/\mathfrak{f}') \to \mathcal{W}$ supported on $(O/\mathfrak{f})^{\times} \times (O/\mathfrak{f}')^{\times}$, because $\lambda_{\mathfrak{C}}$ factor through $(R/\mathfrak{C})^{\times}$ which is canonically isomorphic to $(O/\mathfrak{f})^{\times} \times (O/\mathfrak{f}')^{\times}$. Since λ is trivial on $M^{\times}$, ψ satisfies

$$\psi(\varepsilon x, \varepsilon y) = \varepsilon^{k\Sigma+\kappa(1-c)}\psi(x,y) = N(\varepsilon)^k \psi(x,y)$$

for any unit $\varepsilon \in O^{\times}$.

We regard the local uniformizer $\varpi_{\mathfrak{q}} \in O_{\mathfrak{q}}$ as an idele. For each ideal $\mathfrak{A}$ of F, decomposing $\mathfrak{A} = \prod_{\mathfrak{q}} \mathfrak{q}^{e(\mathfrak{q})}$ for primes $\mathfrak{q}$, we define $\varpi^{e(\mathfrak{A})} = \prod_{\mathfrak{q}} \varpi_{\mathfrak{q}}^{e(\mathfrak{q})} \in F_{\mathbb{A}}^{\times}$. We then define a partial Fourier transform $\psi^{\circ} : X \times Y \to \mathcal{W}$ by

$$\psi^{\circ}(a,b) = \sum_{u \in O/\mathfrak{f}} \psi(u,b)\mathbf{e}_F(-ua\varpi^{-e(\mathfrak{f})}). \tag{3.7}$$

By the Fourier inversion formula, we have

$$P\psi^{\circ}(x,y) = \psi(\varpi^{e(\mathfrak{f})}x, y). \tag{3.8}$$

From this and the definition of $E_k(\underline{\mathcal{L}}) = E_k(\underline{\mathcal{L}}; \psi^{\circ}, \mathfrak{c})$, we find

$$E_k(X, \Lambda, i \circ x, i' \circ y, a\omega) = N(a)^{-k}\lambda_{\mathfrak{F}}(x)\lambda_{\mathfrak{F}_c}^{-1}(y)E_k(X, \Lambda, i, i', \omega) \tag{3.9}$$

for $x \in (O/\mathfrak{f})^{\times} = (R/\mathfrak{F})^{\times}$ and $y \in (O/\mathfrak{f}')^{\times} = (R/\mathfrak{F}_c)^{\times}$. Because of this, $E_k(\psi^{\circ}, \mathfrak{c})$ actually belongs to $G_{kI}(\mathfrak{c}, \Gamma_0(\mathfrak{f}\mathfrak{f}'), \epsilon_{\lambda}; \mathbb{C})$ for $\epsilon_{\lambda,1} = \lambda_{\mathfrak{F}_c}$ and $\epsilon_{\lambda,2} = \lambda_{\mathfrak{F}}$ identifying $O_{\mathfrak{F}} = R_{\mathfrak{F}}$ and $O_{\mathfrak{F}_c} = R_{\mathfrak{F}_c}$. Recall $G_{kI}(\Gamma_0(\mathfrak{N}), \epsilon; \mathbb{C}) = \bigoplus_{\mathfrak{c} \in Cl_F^+} G_{kI}(\mathfrak{c}, \Gamma_0(\mathfrak{N}), \epsilon; \mathbb{C})$. Via this decomposition we extend each Cl_F^+–tuple $(f_{\mathfrak{c}})_{\mathfrak{c}}$ in $G_{kI}(\Gamma_0(\mathfrak{f}\mathfrak{f}'), \epsilon_{\lambda}; \mathbb{C})$ to an automorphic form $f \in G_{kI}(\mathfrak{f}\mathfrak{f}', \epsilon_{\lambda+}; \mathbb{C})$ as follows:

(i) $f(zx) = \lambda(z)|z|_{\mathbb{A}} f(x)$ for z in the center $F_{\mathbb{A}}^{\times} \subset G(\mathbb{A})$ (so $\epsilon_{\lambda+}(z) = \lambda(z)|z|_{\mathbb{A}}$);

(ii) $f(xu) = \epsilon_{\lambda}(u)f(x)$ for $u \in U_0(\mathfrak{f}\mathfrak{f}')$;

(iii) $f_x = f_{\mathfrak{c}}$ if $x = \left(\begin{smallmatrix} c & 0 \\ 0 & 1 \end{smallmatrix}\right)$ for an idele c with $cO = \mathfrak{c}$ and $c^{(\mathfrak{c})} = 1$.

We now compute the effect of the operator $\langle \mathfrak{q} \rangle$ (defined above (2.25)) on E_k for a fractional ideal $\mathfrak{q}$ prime to the level $\mathfrak{f}$. A geometric interpretation of the operator $\langle \mathfrak{q} \rangle$ is discussed, for example, in [H04b] (5.3) (or [PAF] 4.1.9), and has the following effect on an AVRM X: $X \mapsto X \otimes_O \mathfrak{q}$. The level structure i is intact under this process. The $\mathfrak{c}$–polarization Λ induces a $\mathfrak{c}\mathfrak{q}^{-2}$–polarization on $X \otimes_O \mathfrak{q}$. On the lattice side, $\langle \mathfrak{q} \rangle$ brings $\mathcal{L}$ to $\mathfrak{q}\mathcal{L}$.

To simplify our notation, we write

$$t(w;s) = \frac{\lambda_{\mathfrak{F}}^{-1}(\varpi^{e(\mathfrak{f})} i^{-1}(w))\lambda_{\mathfrak{F}_c}^{-1}(i'(w))}{N(w)^k |N(w)|^{2s}}$$

for each term of the Eisenstein series and $c(s)$ for the Gamma factor in front of the summation, where $D = N(\mathfrak{d})$ is the discriminant of F. First we compute the effect of the operator $\langle \mathfrak{q} \rangle$ when $\mathfrak{q} = (\xi)$ for $\xi \in F$ naively as follows:

$$\begin{aligned} &E_k((\mathfrak{q}\mathcal{L}, \Lambda, i); \psi^\circ, \mathfrak{c}\mathfrak{q}^{-2}) = c(s) \sum_{w \in \mathfrak{f}^{-1}\mathfrak{q}\mathcal{L}/O^\times}{}' t(w;s)|_{s=0} \\ &\overset{\mathfrak{q}=(\xi)}{=} c(s) \sum_{w \in \mathfrak{f}^{-1}\mathcal{L}/O^\times}{}' t(\xi w;s)|_{s=0} = \lambda_{\mathfrak{F}}(\xi)^{-1}\lambda_{\mathfrak{F}_c}^{-1}(\xi) N(\xi)^{-k} E_k(\underline{\mathcal{L}}; \psi^\circ, \mathfrak{c}). \end{aligned} \tag{3.10}$$

Here we agree to put $\lambda_{\mathfrak{F}}^{-1}(x) = 0$ if $xR_{\mathfrak{F}} \neq R_{\mathfrak{F}}$ and $\lambda_{\mathfrak{F}_c}^{-1}(y) = 0$ if $yR_{\mathfrak{F}_c} \neq R_{\mathfrak{F}_c}$.

The result of the above naive calculation of the eigenvalue of $\langle \mathfrak{q} \rangle$ shows that our way of extending the Eisenstein series $(E_k(\psi^\circ; \mathfrak{c}))_{\mathfrak{c}}$ to an adelic automorphic form $G(\mathbb{A})$ is correct (and canonical): This claim follows from

$$\lambda_{\mathfrak{F}}(\xi)^{-1}\lambda_{\mathfrak{F}_c}^{-1}(\xi) N(\xi)^{-k} = \lambda(\mathfrak{q}) = \lambda(\varpi_{\mathfrak{q}})|\varpi_{\mathfrak{q}}|_{\mathbb{A}} N(\mathfrak{q}),$$

because the operator $\langle \mathfrak{q} \rangle$ on $G_{kI}(U, \epsilon; \mathbb{C})$ is defined (above (2.25)) to be the central action of $\varpi_{\mathfrak{q}} \in F_{\mathbb{A}}^\times$ (that is, multiplication by $\lambda(\varpi_{\mathfrak{q}})|\varpi_{\mathfrak{q}}|_{\mathbb{A}}$) times $N(\mathfrak{q})$. We obtain

$$\begin{aligned} &\lambda_{\mathfrak{F}_c}(\xi^2) E_k|\langle \mathfrak{q}^{-1} \rangle(\underline{\mathcal{L}}; \psi^\circ, \mathfrak{c}) \\ &\quad = E_k((\mathfrak{q}^{-1}\mathcal{L}, \Lambda, i); \psi^\circ, \mathfrak{c}\mathfrak{q}^2) = \lambda_{\mathfrak{F}}(\xi)\lambda_{\mathfrak{F}_c}(\xi) N(\xi)^k E_k(\underline{\mathcal{L}}; \psi^\circ, \mathfrak{c}). \end{aligned} \tag{3.11}$$

The factor $\lambda_{\mathfrak{F}_c}(\xi^2)$ in the left-hand-side comes from the fact that i' with respect to the $\mathfrak{c}$–polarization $\xi^2\Lambda$ of $\mathfrak{q}^{-1}\mathcal{L}$ is the multiple by ξ^2 of i' with respect to $\mathfrak{c}\mathfrak{q}^2$–polarization Λ of $\mathfrak{q}^{-1}\mathcal{L}$.

We now compute the effect of the Hecke operator $T(1, \mathfrak{q}) = T(\mathfrak{q})$ for a prime $\mathfrak{q} \nmid \mathfrak{f}$. Here we write $\mathcal{L}'$ for an O-lattice with $\mathcal{L}'/\mathcal{L} \cong O/\mathfrak{q}$. Then $\mathcal{L}' \wedge \mathcal{L}' = (\mathfrak{q}\mathfrak{c})^*$; so, Λ induces a $\mathfrak{q}\mathfrak{c}$–polarization on $\mathcal{L}'$, and similarly it induces $\mathfrak{q}^2\mathfrak{c}$–polarization on $\mathfrak{q}^{-1}\mathcal{L}$. By (2.8), $E_k|T(\mathfrak{q})$ is the sum of the terms $t(\ell; s)$ with multiplicity extended over $\mathfrak{q}^{-1}\mathfrak{f}^{-1}\mathcal{L}$. The multiplicity for each $\ell \in \mathfrak{f}^{-1}\mathcal{L}$ is $N(\mathfrak{q}) + 1$ and only once for $\ell \in \mathfrak{q}^{-1}\mathfrak{f}^{-1}\mathcal{L} - \mathfrak{f}^{-1}\mathcal{L}$ (thus, $N(\mathfrak{q})$ times for

$\ell \in \mathfrak{f}^{-1}\mathcal{L}$ and once for $\ell \in \mathfrak{q}^{-1}\mathfrak{f}^{-1}\mathcal{L}$). This shows

$$c(0)^{-1}N(\mathfrak{q})E_k|T(\mathfrak{q})(\underline{\mathcal{L}};\psi^\circ,\mathfrak{q}\mathfrak{c})$$
$$=\sum_{\mathcal{L}'}\left\{\sum_{w\in\mathfrak{f}^{-1}\mathcal{L}'/O^\times}{}' t(w;s)+\sum_{w\in\mathfrak{f}^{-1}\mathfrak{q}^{-1}\mathcal{L}'/O^\times}{}' t(w;s)\right\}\Big|_{s=0}$$
$$=c(0)^{-1}\left\{N(\mathfrak{q})E_k(\underline{\mathcal{L}};\psi^\circ,\mathfrak{c})+E_k|\langle\mathfrak{q}\rangle^{-1}(\underline{\mathcal{L}};\psi^\circ,\mathfrak{c}\mathfrak{q}^2)\right\}.$$

In short, we have

$$E_k(\psi^\circ,\mathfrak{q}\mathfrak{c})|T(\mathfrak{q})=E_k(\psi^\circ,\mathfrak{c})+N(\mathfrak{q})^{-1}E_k(\psi^\circ,\mathfrak{c}\mathfrak{q}^2)|\langle\mathfrak{q}\rangle^{-1}. \tag{3.12}$$

Suppose that $\mathfrak{q}$ is principal generated by a totally positive $\xi \in F$. Substituting $\xi^{-1}\Lambda$ for Λ, i' will be transformed into $\xi^{-1}i'$, and we have

$$\lambda_{\mathfrak{F}_c}(\xi)E_k(\psi^\circ,\mathfrak{c})|T(\mathfrak{q})=E_k(\psi^\circ,\mathfrak{q}\mathfrak{c})|T(\mathfrak{q})$$

We combine this with (3.11) assuming $\mathfrak{q}=(\xi)$ with $0 \ll \xi \in F$:

$$E_k(\psi^\circ,\mathfrak{c})|T(\mathfrak{q})(\underline{\mathcal{L}})=(\lambda_{\mathfrak{F}_c}^{-1}(\xi)+\lambda_{\mathfrak{F}}(\xi)N(\mathfrak{q})^{k-1})E_k(\psi^\circ,\mathfrak{c}), \tag{3.13}$$

which also follows from (3.6) noting that $\psi^\circ(a,b)=G(\lambda_{\mathfrak{F}}^{-1})\lambda_{\mathfrak{F}}(a)\lambda_{\mathfrak{F}_c}^{-1}(b)$ for the Gauss sum $G(\lambda_{\mathfrak{F}}^{-1})$.

We now look into the operator $[\mathfrak{q}]$ for a prime $\mathfrak{q}$ outside the level $\mathfrak{f}$. This operator brings a level $\Gamma_0(\mathfrak{q})$–test object (X,C,i) with level $\mathfrak{f}$ structure i outside $\mathfrak{q}$ to $(X/C,i)$, where the level $\mathfrak{f}$–structure i is intact under the quotient map: $X \to X/C$. On the lattice side, taking the lattice $\mathcal{L}_C$ with $\mathcal{L}_C/\mathcal{L}=C$, it is defined as follows:

$$f|[\mathfrak{q}](\mathcal{L},C,\Lambda,i)=N(\mathfrak{q})^{-1}f(\mathcal{L}_C,\Lambda,i). \tag{3.14}$$

The above operator is useful to relate $U(\mathfrak{q})$ and $T(\mathfrak{q})$. By definition,

$$f|U(\mathfrak{q})(\mathcal{L},\Lambda,C,i)=N(\mathfrak{q})^{-1}\sum_{\mathcal{L}',\mathcal{L}'\neq\mathcal{L}_C}f(\mathcal{L}',\Lambda,C',i)$$

for $C'=\mathcal{L}_C+\mathcal{L}'/\mathcal{L}'=\mathfrak{q}^{-1}\mathcal{L}/\mathcal{L}'$. Thus we have

$$U(\mathfrak{q})=T(\mathfrak{q})-[\mathfrak{q}]. \tag{3.15}$$

A similar computation yields:

$$[\mathfrak{q}]\circ U(\mathfrak{q})=N(\mathfrak{q})^{-1}\langle\mathfrak{q}\rangle^{-1}. \tag{3.16}$$

Lemma 3.1 *Let $\mathfrak{q}$ be a prime outside $\mathfrak{f}$. Suppose that $\mathfrak{q}^h=(\xi)$ for a totally positive $\xi \in F$. Let $\mathbb{E}'_k(\psi,\mathfrak{c})=E_k(\psi^\circ,\mathfrak{c})-E_k(\psi^\circ,\mathfrak{c}\mathfrak{q})|[\mathfrak{q}]$ and $\mathbb{E}_k(\psi,\mathfrak{c})=E_k(\psi^\circ,\mathfrak{c})-N(\mathfrak{q})E_k(\psi^\circ,\mathfrak{c}\mathfrak{q}^{-1})|\langle\mathfrak{q}\rangle|[\mathfrak{q}]$. Then we have*

(1) $\mathbb{E}'_k(\psi,\mathfrak{c})|U(\mathfrak{q}) = E_k(\psi^\circ,\mathfrak{q}^{-1}\mathfrak{c}) - E_k(\psi^\circ,\mathfrak{c})|[\mathfrak{q}]$,
(2) $\mathbb{E}'_k(\psi,\mathfrak{c})|U(\mathfrak{q}^h) = \lambda^{-1}_{\mathfrak{F}_c}(\xi)\mathbb{E}'_k(\psi,\mathfrak{c})$,
(3) $\mathbb{E}_k(\psi,\mathfrak{c})|U(\mathfrak{q}){=}(E_k(\psi^\circ,\mathfrak{q}\mathfrak{c}){-}N(\mathfrak{q})E_k(\psi^\circ,\mathfrak{c})|\langle\mathfrak{q}\rangle|[\mathfrak{q}])|(N(\mathfrak{q})^{-1}\langle\mathfrak{q}\rangle^{-1})$
(4) $\mathbb{E}_k(\psi,\mathfrak{c})|U(\mathfrak{q}^h) = \lambda_{\mathfrak{F}}(\xi)N(\mathfrak{q})^{h(k-1)}\mathbb{E}_k(\psi,\mathfrak{c})$.

Proof We prove (1) and (3), because (2) and (4) follow by iteration of these formulas combined with the fact: $\lambda_{\mathfrak{F}_c}(\xi)E_k(\psi^\circ,\mathfrak{c}) = E_k(\psi^\circ,\xi\mathfrak{c})$ for a totally positive $\xi\in F$. Since (3) can be proven similarly, we describe computation to get (1), writing $E_k(\mathfrak{c}) = E_k(\psi^\circ,\mathfrak{c})$:

$$\begin{aligned}
&\mathbb{E}'_k(\psi;\mathfrak{c})|U(\mathfrak{q}) = E_k(\mathfrak{c})|U(\mathfrak{q}) - E_k(\mathfrak{c}\mathfrak{q})|[\mathfrak{q}]|U(\mathfrak{q})\\
&\overset{(3.16)}{=}E_k(\mathfrak{c})|U(\mathfrak{q}) - N(\mathfrak{q})^{-1}E_k(\mathfrak{c}\mathfrak{q})|\langle\mathfrak{q}\rangle^{-1}\\
&\overset{(3.15)}{=}E_k(\mathfrak{c})|T(\mathfrak{q}) - E_k(\mathfrak{c}\mathfrak{q})|[\mathfrak{q}] - N(\mathfrak{q})^{-1}E_k(\mathfrak{c}\mathfrak{q})|\langle\mathfrak{q}\rangle^{-1}\\
&\overset{(3.12)}{=}E_k(\mathfrak{c}\mathfrak{q}^{-1}) + N(\mathfrak{q})^{-1}E_k(\mathfrak{c}\mathfrak{q})|\langle\mathfrak{q}\rangle^{-1} - E_k(\mathfrak{c})|[\mathfrak{q}] - N(\mathfrak{q})^{-1}E_k(\mathfrak{c}\mathfrak{q})|\langle\mathfrak{q}\rangle^{-1}\\
&= E_k(\mathfrak{c}\mathfrak{q}^{-1}) - E_k(\mathfrak{c})|[\mathfrak{q}].
\end{aligned}$$

□

Remark 3.2 As follows from the formulas in [H96] 2.4 (T1) and [H91] Section 7.G, the Hecke operator $T(\mathfrak{q})$ and $U(\mathfrak{q})$ commutes with the Katz differential operator as long as $\mathfrak{q}\nmid p$. Thus for $\mathbb{E}(\lambda,\mathfrak{c}) = d^\kappa\mathbb{E}_k(\psi,\mathfrak{c})$ and $\mathbb{E}'(\lambda,\mathfrak{c}) = d^\kappa\mathbb{E}'_k(\psi,\mathfrak{c})$, we have under the notation of Lemma 3.1

$$\begin{aligned}
\mathbb{E}'(\lambda,\mathfrak{c})|U(\mathfrak{q}^h) = &\quad \lambda^{-1}_{\mathfrak{F}_c}(\xi)\mathbb{E}'(\lambda,\mathfrak{c}),\\
\mathbb{E}(\lambda,\mathfrak{c})|U(\mathfrak{q}^h) = &\quad \lambda_{\mathfrak{F}}(\xi)N(\mathfrak{q})^{h(k-1)}\mathbb{E}(\lambda,\mathfrak{c}).
\end{aligned} \tag{3.17}$$

3.4 Values at CM points

We take a proper R_{n+1}–ideal $\mathfrak{a}$ for $n>0$, and regard it as a lattice in $\mathbb{C}^\Sigma$ by $a\mapsto(a^\sigma)_{\sigma\in\Sigma}$. Then $\Lambda(\mathfrak{a})$ induces a polarization of $\mathfrak{a}\subset\mathbb{C}^\Sigma$. We suppose that $\mathfrak{a}$ is prime to $\mathfrak{C}$ (the conductor of λ). For a p–adic modular form f of the form $d^\kappa g$ for classical $g\in G_{kI}(\mathfrak{c},\Gamma_{1,0}(\mathfrak{f}^2);\mathcal{W})$, we have by (K) in 2.6

$$\frac{f(x(\mathfrak{a}),\omega_p)}{\Omega_p^{k\Sigma+2\kappa}} = f(x(\mathfrak{a}),\omega(\mathfrak{a})) = \frac{f(x(\mathfrak{a}),\omega_\infty)}{\Omega_\infty^{k\Sigma+2\kappa}}.$$

Here $x(\mathfrak{a})$ is the test object: $x(\mathfrak{a}) = (X(\mathfrak{a}),\Lambda(\mathfrak{a}),i(\mathfrak{a}),i'(\mathfrak{a}))_{/\mathcal{W}}$.

We write $c_0 = (-1)^{k[F:\mathbb{Q}]}\frac{\pi^\kappa\Gamma_\Sigma(k\Sigma+\kappa)}{\mathrm{Im}(\delta)^\kappa\sqrt{D}\Omega_\infty^{k\Sigma+2\kappa}}$. Here $\Gamma_\Sigma(s) = \prod_{\sigma\in\Sigma}\Gamma(s_\sigma)$,

$\Omega_\infty^s = \prod_\sigma \Omega_\sigma^{s_\sigma}$, $\mathrm{Im}(\delta)^s = \prod_\sigma \mathrm{Im}(\delta^s)^{s_\sigma}$, and so on, for $s = \sum_\sigma s_\sigma \sigma$. By definition (see [H04c] 4.2), we find, for $e = [R^\times : O^\times]$,

$$\begin{aligned}
&(c_0 e)^{-1}\delta_{kI}^\kappa E_k(\mathfrak{c})(x(\mathfrak{a}),\omega(\mathfrak{a}))\\
&= \lambda_{\mathfrak{C}}^{-1}(\varpi^{e(\mathfrak{F})})\sum_{w\in\mathfrak{F}^{-1}\mathfrak{a}/R^\times}{}' \frac{\lambda_{\mathfrak{C}}^{-1}(w)\lambda(w^{(\infty)})}{N_{M/\mathbb{Q}}(w)^s}\Big|_{s=0}\\
&= \lambda_{\mathfrak{C}}^{-1}(\varpi^{e(\mathfrak{F})})\lambda(\mathfrak{a})N_{M/\mathbb{Q}}(\mathfrak{F}\mathfrak{a}^{-1})^s \sum_{w\mathfrak{F}\mathfrak{a}^{-1}\subset R_{n+1}}{}' \frac{\lambda(w\mathfrak{F}\mathfrak{a}^{-1})}{N_{M/\mathbb{Q}}(w\mathfrak{F}\mathfrak{a}^{-1})^s}\Big|_{s=0}\\
&= \lambda_{\mathfrak{C}}^{-1}(\varpi^{e(\mathfrak{F})})\lambda(\mathfrak{a})L_{[\mathfrak{F}\mathfrak{a}^{-1}]}^{n+1}(0,\lambda),
\end{aligned} \tag{3.18}$$

where for an ideal class $[\mathfrak{A}] \in Cl_{n+1}$ represented by a proper R_{n+1}–ideal $\mathfrak{A}$,

$$L_{[\mathfrak{A}]}^{n+1}(s,\lambda) = \sum_{\mathfrak{b}\in[\mathfrak{A}]}\lambda(\mathfrak{b})N_{M/\mathbb{Q}}(\mathfrak{b})^{-s}$$

is the partial L–function of the class $[\mathfrak{A}]$ for $\mathfrak{b}$ running over all R_{n+1}–proper integral ideals prime to $\mathfrak{C}$ in the class $[\mathfrak{A}]$. In the second line of (3.18), we regard λ as an idele character and in the other lines as an ideal character. For an idele a with $a\widehat{R} = \mathfrak{a}\widehat{R}$ and $a_{\mathfrak{C}} = 1$, we have $\lambda(a^{(\infty)}) = \lambda(\mathfrak{a})$.

We put $\mathbb{E}(\lambda,\mathfrak{c}) = d^\kappa\mathbb{E}_k(\psi,\mathfrak{c})$ and $\mathbb{E}'(\lambda,\mathfrak{c}) = d^\kappa\mathbb{E}'_k(\psi,\mathfrak{c})$ as in Remark 3.2. We want to evaluate $\mathbb{E}(\lambda,\mathfrak{c})$ and $\mathbb{E}'(\lambda,\mathfrak{c})$ at $x = (x(\mathfrak{a}),\omega(\mathfrak{a}))$. Here $\mathfrak{c}$ is the polarization ideal of $\Lambda(\mathfrak{a})$; so, if confusion is unlikely, we often omit the reference to $\mathfrak{c}$ (which is determined by $\mathfrak{a}$). Thus we write, for example, $\mathbb{E}(\lambda)$ and $\mathbb{E}'(\lambda)$ for $\mathbb{E}(\lambda,\mathfrak{c})$ and $\mathbb{E}'(\lambda,\mathfrak{c})$. Then by definition and (K) in 2.6, we have for $x = (x(\mathfrak{a}),\omega(\mathfrak{a}))$

$$\begin{aligned}
\mathbb{E}'(\lambda)(x) &= \delta_{kI}^\kappa E_k(\psi^\circ,\mathfrak{c})(x) - N(\mathfrak{q})^{-1}\delta_{kI}^\kappa E_k(\psi^\circ,\mathfrak{c}\mathfrak{q})(x(\mathfrak{a}R_n),\omega(\mathfrak{a}R_n))\\
\mathbb{E}(\lambda)(x) &= \delta_{kI}^\kappa E_k(\psi^\circ,\mathfrak{c})(x) - \delta_{kI}^\kappa E_k(\psi^\circ,\mathfrak{c}\mathfrak{q}^{-1})(x(\mathfrak{q}\mathfrak{a}R_n),\omega(\mathfrak{a}R_n))
\end{aligned} \tag{3.19}$$

because $C(\mathfrak{a}) = \mathfrak{a}R_n/\mathfrak{a}$ and hence $[\mathfrak{q}](x(\mathfrak{a})) = x(\mathfrak{a}R_n)$.

To simplify notation, write $\phi([\mathfrak{a}]) = \lambda(\mathfrak{a})^{-1}\phi(x(\mathfrak{a}),\omega(\mathfrak{a}))$. By (3.9), for $\phi = \mathbb{E}(\lambda)$ and $\mathbb{E}'(\lambda)$, the value $\phi([\mathfrak{a}])$ only depends on the ideal class $[\mathfrak{a}]$ but not the individual $\mathfrak{a}$. The formula (3.19) combined with (3.18) shows, for a proper R_{n+1}–ideal $\mathfrak{a}$,

$$\begin{aligned}
e^{-1}\lambda_{\mathfrak{C}}(\varpi^{e(\mathfrak{F})})\mathbb{E}'(\lambda)([\mathfrak{a}]) &= c_0\left(L_{[\mathfrak{F}\mathfrak{a}^{-1}]}^{n+1}(0,\lambda) - N(\mathfrak{q})^{-1}L_{[\mathfrak{F}\mathfrak{a}^{-1}R_n]}^{n}(0,\lambda)\right)\\
e^{-1}\lambda_{\mathfrak{C}}(\varpi^{e(\mathfrak{F})})\mathbb{E}(\lambda)([\mathfrak{a}]) &= c_0\left(L_{[\mathfrak{F}\mathfrak{a}^{-1}]}^{n+1}(0,\lambda) - \lambda(\mathfrak{q})L_{[\mathfrak{F}\mathfrak{q}^{-1}\mathfrak{a}^{-1}R_n]}^{n}(0,\lambda)\right)
\end{aligned} \tag{3.20}$$

where $e = [R^\times : O^\times]$. Now we define

$$L^n(s,\lambda) = \sum_{\mathfrak{a}} \lambda(\mathfrak{a}) N_{M/\mathbb{Q}}(\mathfrak{a})^{-s}, \tag{3.21}$$

where $\mathfrak{a}$ runs over all proper ideals in R_n prime to $\mathfrak{C}$ and $N_{M/\mathbb{Q}}(\mathfrak{a}) = [R_n : \mathfrak{a}]$. For each primitive character $\chi : Cl_f \to \overline{\mathbb{Q}}^\times$, we pick $n+1 = mh$ so that $(m-1)h \le f \le n+1$, where $\mathfrak{q}^h = (\xi)$ for a totally positive $\xi \in F$. Then we have

$$\begin{aligned} e^{-1}\lambda_{\mathfrak{C}}(\varpi^{e(\mathfrak{F})}) &\sum_{[\mathfrak{a}] \in Cl_{n+1}} \chi(\mathfrak{a}) \mathbb{E}'(\lambda)([\mathfrak{a}]) \\ &= c_0 \chi(\mathfrak{F}) \left(L^{n+1}(0, \lambda\chi^{-1}) - L^n(0,\lambda\chi^{-1}) \right) \\ e^{-1}\lambda_{\mathfrak{C}}(\varpi^{e(\mathfrak{F})}) &\sum_{[\mathfrak{a}] \in Cl_{n+1}} \chi(\mathfrak{a}) \mathbb{E}(\lambda)([\mathfrak{a}]) \\ &= c_0 \chi(\mathfrak{F}) \left(L^{n+1}(0, \lambda\chi^{-1}) - \lambda\chi^{-1}(\mathfrak{q}) N(\mathfrak{q}) L^n(0,\lambda\chi^{-1}) \right). \end{aligned} \tag{3.22}$$

As computed in [H04c] 4.1 and [LAP] V.3.2, if $k \ge f$ then the Euler $\mathfrak{q}$–factor of $L^k(s, \chi^{-1}\lambda)$ is given by

$$\begin{aligned} &\sum_{j=0}^{k-f} (\chi^{-1}\lambda(\mathfrak{q}))^j N(\mathfrak{q})^{j-2sj} \quad \text{if } f > 0, \\ &\sum_{j=0}^{k-1} (\chi^{-1}\lambda(\mathfrak{q}))^j N(\mathfrak{q})^{j-2sj} \\ &\quad + \left(N(\mathfrak{q}) - \left(\frac{M/F}{\mathfrak{q}}\right) \right) (\chi^{-1}\lambda(\mathfrak{q}))^k N(\mathfrak{q})^{k-1-2ks} L^0_{\mathfrak{q}}(s, \chi^{-1}\lambda) \quad \text{if } f = 0, \end{aligned} \tag{3.23}$$

where $\left(\frac{M/F}{\mathfrak{q}}\right)$ is 1, -1 or 0 according as $\mathfrak{q}$ splits, remains prime or ramifies in M/F, and $L^0_{\mathfrak{q}}(s,\chi^{-1}\lambda)$ is the $\mathfrak{q}$–Euler factor of the primitive L–function $L(s,\chi^{-1}\lambda)$. We define a possibly imprimitive L–function

$$L^{(\mathfrak{q})}(s,\chi^{-1}\lambda) = L_{\mathfrak{q}}(s,\chi^{-1}\lambda) L^0(s,\chi^{-1}\lambda)$$

removing the $\mathfrak{q}$–Euler factor.

Combining all these formulas, we find

$$e^{-1}\lambda\chi^{-1}(\varpi^{e(\mathfrak{F})}) \sum_{[\mathfrak{a}] \in Cl_{n+1}} \chi(\mathfrak{a}) \mathbb{E}(\lambda)([\mathfrak{a}]) = c_0 L^{(\mathfrak{q})}(0, \chi^{-1}\lambda), \tag{3.24}$$

$$e^{-1}\lambda\chi^{-1}(\varpi^{e(\mathfrak{F})})\sum_{[\mathfrak{a}]\in Cl_{n+1}}\chi(\mathfrak{a})\mathbb{E}'(\lambda)([\mathfrak{a}])$$
$$=\begin{cases} c_0 L^{(\mathfrak{q})}(0,\chi^{-1}\lambda) & \text{if } f>0,\\ c_0\left(\frac{M/F}{\mathfrak{q}}\right)L_{\mathfrak{q}}(1,\chi^{-1}\lambda)L(0,\chi^{-1}\lambda) & \text{if } f=0 \text{ and } \left(\frac{M/F}{\mathfrak{q}}\right)\neq 0,\\ -c_0\chi^{-1}\lambda(\mathfrak{Q})L_{\mathfrak{q}}(1,\chi^{-1}\lambda)L(0,\chi^{-1}\lambda) & \text{if } \mathfrak{q}=\mathfrak{Q}^2 \text{ in } R \text{ and } f=0.\end{cases} \tag{3.25}$$

All these values are algebraic in $\overline{\mathbb{Q}}$ and integral over $\mathcal{W}$.

4 Non-vanishing modulo p of L–values

We construct an $\mathbb{F}$–valued measure ($\mathbb{F}=\overline{\mathbb{F}}_p$ as in 2.1) over the anti-cyclotomic class group $Cl_\infty=\varprojlim_n Cl_n$ modulo $\mathfrak{l}^\infty$ whose integral against a character χ is the Hecke L–value $L(0,\chi^{-1}\lambda)$ (up to a period). The idea is to translate the Hecke relation of the Eisenstein series into a distribution relation on the profinite group Cl_∞. At the end, we relate the non-triviality of the measure to the q–expansion of the Eisenstein series by the density of $\{x(\mathfrak{a})\}_{\mathfrak{a}}$ (see [H04c]).

4.1 Construction of a modular measure

We choose a complete representative set $\{\mathfrak{c}\}_{[\mathfrak{c}]\in Cl_F^+}$ of the strict ideal class group Cl_F^+ made up of ideals $\mathfrak{c}$ prime to $p\mathfrak{f}\mathfrak{l}$. For each proper R_n–ideal $\mathfrak{a}$, the polarization ideal $\mathfrak{c}(\mathfrak{a})$ of $x(\mathfrak{a})$ is equivalent to one of the representatives $\mathfrak{c}$ (so $[\mathfrak{c}]=[\mathfrak{c}(\mathfrak{a})]$). Writing $\mathfrak{c}_0$ for $\mathfrak{c}(R)$, we have $\mathfrak{c}(\mathfrak{a})=\mathfrak{c}_0\mathfrak{l}^{-n}(\mathfrak{a}\mathfrak{a}^c)^{-1}$. Take a modular form g in $G_{kI}(\Gamma_0(\mathfrak{f}\mathfrak{f}'\mathfrak{l}),\epsilon_\lambda;W)$. Thus $g=(g_{[\mathfrak{c}]})$ is an h–tuple of modular forms for $h=|Cl_F^+|$. Put $f=(f_{[\mathfrak{c}]})_{\mathfrak{c}}$ for $f_{[\mathfrak{c}]}=d^\kappa g_{[\mathfrak{c}]}$ for the differential operator $d^\kappa=\prod_\sigma d_\sigma^{\kappa_\sigma}$ in 2.6. We write $f(x(\mathfrak{a}))$ for the value of $f_{[\mathfrak{c}(\mathfrak{a})]}(x(\mathfrak{a}))$. Similarly, we write $f(X,\Lambda,i,\omega)$ for $f_{[\mathfrak{c}]}(X,\Lambda,i,\omega)$ for the ideal class $\mathfrak{c}$ determined by $\overline{\Lambda}$. The Hecke operator $U(\mathfrak{l})$ takes the space $V(\mathfrak{c},\Gamma_0(\mathfrak{f}\mathfrak{f}'\mathfrak{l}),\epsilon_\lambda;W)$ into $V(\mathfrak{c}\mathfrak{l}^{-1},\Gamma_0(\mathfrak{f}\mathfrak{f}'\mathfrak{l}),\epsilon_\lambda;W)$. Choosing $\mathfrak{c}_{\mathfrak{l}}$ in the representative set equivalent to the ideal $\mathfrak{c}\mathfrak{l}^{-1}$, we have a canonical isomorphism $V(\mathfrak{c}\mathfrak{l},\Gamma_0(\mathfrak{f}\mathfrak{f}'\mathfrak{l}),\epsilon_\lambda;W)\cong V(\mathfrak{c}_{\mathfrak{l}},\Gamma_0(\mathfrak{f}\mathfrak{f}'\mathfrak{l}),\epsilon_\lambda;W)$ sending f to f' given by

$$f'(X,\xi\Lambda,i,i',\omega)=f(X,\Lambda,i,i',\omega)$$

for totally positive $\xi\in F$ with $\xi\mathfrak{c}_{\mathfrak{l}}=\mathfrak{l}^{-1}\mathfrak{c}$. This map is independent of the choice of ξ. Since the image of $\mathfrak{M}(\mathfrak{c};\mathfrak{N})$ in $Sh^{(p)}$ depends only on $\mathfrak{N}$ and the strict ideal class of $\mathfrak{c}$ as explained in 2.8, the Hecke operator $U(\mathfrak{l})$ is induced from the algebraic correspondence on the Shimura variety associated to the double coset $U\left(\begin{smallmatrix}\varpi_{\mathfrak{l}} & 0\\ 0 & 1\end{smallmatrix}\right)U$. So we regard $U(\mathfrak{l})$ as an operator acting on h–tuple of p–adic modular forms in $V(\Gamma_0(\mathfrak{f}\mathfrak{f}'\mathfrak{l}),\epsilon_\lambda;W)=\bigoplus_{\mathfrak{c}}V(\mathfrak{c},\Gamma_0(\mathfrak{f}\mathfrak{f}'\mathfrak{l}),\epsilon_\lambda;W)$ inducing permutation $\mathfrak{c}\mapsto\mathfrak{c}_{\mathfrak{l}}$ on the polarization ideals. Suppose that $g|U(\mathfrak{l})=ag$ with $a\in W^\times$; so, $f|U(\mathfrak{l})=af$ (see Remark 3.2). The Eisenstein series $(\mathbb{E}(\lambda,\mathfrak{c}))_{\mathfrak{c}}$ satisfies this condition by Lemma 3.1. The operator $U(\mathfrak{l}^h)$ ($h=|Cl_F^+|$) takes $V(\mathfrak{c},\Gamma_0(\mathfrak{f}\mathfrak{f}'\mathfrak{l}),\epsilon_\lambda;W)$ into itself. Thus $f_{\mathfrak{c}}|U(\mathfrak{l}^h)=a^hf_{\mathfrak{c}}$.

Choosing a base $w=(w_1,w_2)$ of $\widehat{R}=R\otimes_{\mathbb{Z}}\widehat{\mathbb{Z}}$, identify $T(X(R)_{/\overline{\mathbb{Q}}})=\widehat{R}$ with $\widehat{O}^2$ by $\widehat{O}\ni(a,b)\mapsto aw_1+bw_2\in T(X(R))$. This gives a level structure $\eta^{(p)}(R):F^2\otimes_{\mathbb{Q}}\mathbb{A}^{(p\infty)}\cong V^{(p)}(X(R))$ defined over $\mathcal{W}$. Choose the base w satisfying the following two conditions:

(B1) $w_{2,\mathfrak{l}}=1$ and $R_{\mathfrak{l}}=O_{\mathfrak{l}}[w_{1,\mathfrak{l}}]$;
(B2) By using the splitting: $R_{\mathfrak{f}}=R_{\mathfrak{F}}\times R_{\mathfrak{F}^c}$, $w_{1,\mathfrak{f}}=(1,0)$ and $w_{2,\mathfrak{f}}=(0,1)$.

Let $\mathfrak{a}$ be a proper R_n–ideal (for $R_n=O+\mathfrak{l}^nR$) prime to $\mathfrak{f}$. Recall the generator $\varpi=\varpi_{\mathfrak{l}}$ of $\mathfrak{l}O_{\mathfrak{l}}$. Regarding $\varpi\in F_{\mathbb{A}}^\times$, $w_n=(\varpi^nw_1,w_2)$ is a base of $\widehat{R}_n$ and gives a level structure $\eta^{(p)}(R_n):F^2\otimes_{\mathbb{Q}}\mathbb{A}^{(p\infty)}\cong V^{(p)}(X(R_n))$. We choose a complete representative set $A=\{a_1,\dots,a_H\}\subset M_{\mathbb{A}}^\times$ so that $M_{\mathbb{A}}^\times=\bigsqcup_{j=1}^H M^\times a_j\widehat{R}_n^\times M_\infty^\times$. Then $\mathfrak{a}\widehat{R}_n=\alpha a_j\widehat{R}_n$ for $\alpha\in M^\times$ for some index j. We then define $\eta^{(p)}(\mathfrak{a})=\alpha a_j\eta^{(p)}(R_n)$. The small ambiguity of the choice of α does not cause any trouble.

Write $x_0(\mathfrak{a})=(X(\mathfrak{a}),\Lambda(\mathfrak{a}),i(\mathfrak{a}),i'(\mathfrak{a}),C(\mathfrak{a}),\omega(\mathfrak{a}))$. This is a test object of level $\Gamma^1_{1,0}(\mathfrak{f}^2)\cap\Gamma_0(\mathfrak{l})$ (see (3.3) for $\Gamma^1_{1,0}(\mathfrak{f}^2)$). We pick a subgroup $C\subset X(R_n)$ such that $C\cong O/\mathfrak{l}^m$ ($m>0$) but $C\cap C(R_n)=\{0\}$. Then we define $x_0(R_n)/C$ by

$$\left(\frac{X(R_n)}{C},\pi_*\Lambda(R_n),\pi\circ i(R_n),\pi^{-1}\circ i'(R_n),\frac{C+C(R_n)[\mathfrak{l}]}{C},(\pi^*)^{-1}\omega(R_n)\right)$$

for the projection map $\pi:X(R_n)\twoheadrightarrow X(R_n)/C$. We can write

$$x_0(R_n)/C=x_0(\mathfrak{a})\in\mathcal{M}(\mathfrak{c}\mathfrak{l}^{-n-m},\mathfrak{f}^2,\Gamma_0(\mathfrak{l}))(\mathcal{W})$$

for a proper R_{n+m}–ideal $\mathfrak{a}\supset R_n$ with $(\mathfrak{a}\mathfrak{a}^c)=\mathfrak{l}^{-2m}$, and for $u\in O_{\mathfrak{l}}^\times$ we have

$$x_0(\mathfrak{a})=x_0(R_n)/C=\begin{pmatrix}1&\frac{u}{\varpi_{\mathfrak{l}}^m}\\0&1\end{pmatrix}(x_0(R_{m+n})). \tag{4.1}$$

See Section 2.8 in the text for the action of $g=\begin{pmatrix}1&\frac{u}{\varpi_{\mathfrak{l}}^m}\\0&1\end{pmatrix}$ on the point $x_0(R_{m+n})$, and see [H04c] Section 3.1 for details of the computation leading to (4.1).

Let $T_M=\mathrm{Res}_{M/\mathbb{Q}}\mathbb{G}_m$. For each proper R_n–ideal $\mathfrak{a}$, we have an embedding $\rho_{\mathfrak{a}}:T_M(\mathbb{A}^{(p\infty)})\to G(\mathbb{A}^{(p\infty)})$ given by $\alpha\eta^{(p)}(\mathfrak{a})=\eta^{(p)}(\mathfrak{a})\circ\rho_{\mathfrak{a}}(\alpha)$. Since

$\det(\rho_{\mathfrak{a}}(\alpha)) = \alpha\alpha^c \gg 0$, $\alpha \in T_M(\mathbb{Z}_{(p)})$ acts on $Sh^{(p)}$ through $\rho_{\mathfrak{a}}(\alpha) \in G(\mathbb{A})$. We have

$$\rho_{\mathfrak{a}}(\alpha)(x(\mathfrak{a})) = (X(\mathfrak{a}), (\alpha\alpha^c)\Lambda(\mathfrak{a}), \eta^{(p)}(\mathfrak{a})\rho_{\mathfrak{a}}(\alpha)) \\ = (X(\alpha\mathfrak{a}), \Lambda(\alpha\mathfrak{a}), \eta^{(p)}(\alpha\mathfrak{a}))$$

for the prime-to–p isogeny $\alpha \in \mathrm{End}_O(X(\mathfrak{a})) = R_{(p)}$. Thus $T_M(\mathbb{Z}_{(p)})$ acts on $Sh^{(p)}$ fixing the point $x(\mathfrak{a})$. We find $\rho(\alpha)^*\omega(\mathfrak{a}) = \alpha\omega(\mathfrak{a})$, and by (B2), we have

$$g(x(\alpha\mathfrak{a}), \alpha\omega(\mathfrak{a})) = g(\rho(\alpha)(x(\mathfrak{a}), \omega(\mathfrak{a}))) = \alpha^{-k\Sigma}\lambda_{\mathfrak{F}}(\alpha)\lambda_{\mathfrak{F}_c}(\alpha)g(x(\mathfrak{a}), \omega(\mathfrak{a})).$$

From this, we conclude

$$f(x(\alpha\mathfrak{a}), \alpha\omega(\mathfrak{a})) = f(\rho(\alpha)(x(\mathfrak{a}), \omega(\mathfrak{a}))) \\ = \alpha^{-k\Sigma-\kappa(1-c)}\lambda_{\mathfrak{F}}(\alpha)\lambda_{\mathfrak{F}_c}(\alpha)f(x(\mathfrak{a}), \omega(\mathfrak{a})),$$

because the effect of the differential operator d is identical with that of δ at the CM point $x(\mathfrak{a})$ by (K). By our choice of the Hecke character λ, we find

$$\lambda(\alpha\mathfrak{a}) = \alpha^{-k\Sigma-\kappa(1-c)}\lambda_{\mathfrak{F}}(\alpha)\lambda_{\mathfrak{F}_c}(\alpha)\lambda(\mathfrak{a}).$$

If $\mathfrak{a}$ and α is prime to $\mathfrak{C}p$, then the value $\alpha^{-k\Sigma-\kappa(1-c)}\lambda_{\mathfrak{C}}(\alpha)$ is determined independently of the choice of α for a given ideal $\alpha\mathfrak{a}$, and the value $\lambda(\mathfrak{a})^{-1}f(x(\mathfrak{a}), \omega(\mathfrak{a}))$ is independent of the representative set $A = \{a_j\}$ for Cl_n. Defining

$$f([\mathfrak{a}]) = \lambda(\mathfrak{a})^{-1}f(x(\mathfrak{a}), \omega(\mathfrak{a})) \text{ for a proper } R_n\text{–ideal } \mathfrak{a} \text{ prime to } \mathfrak{C}p, \quad (4.2)$$

we find that $f([\mathfrak{a}])$ only depends on the proper ideal class $[\mathfrak{a}] \in Cl_n$.

We write $x(\mathfrak{a}_u) = \begin{pmatrix} 1 & \frac{u}{\varpi\mathfrak{l}} \\ 0 & 1 \end{pmatrix}(x(\mathfrak{a}))$, where $\mathfrak{l}^h = (\varpi)$ for an element $\varpi \in F$. Then $\mathfrak{a}_u$ depends only on $u \bmod \mathfrak{l}^h$, and $\{\mathfrak{a}_u\}_{u \bmod \mathfrak{l}^h}$ gives a complete representative set for proper R_{n+h}–ideal classes which project down to the ideal class $[\mathfrak{a}] \in Cl_n$. Since $\mathfrak{a}_u R_n = \varpi^{-1}\mathfrak{a}$, we find $\lambda(\mathfrak{a}_u) = \lambda(\mathfrak{l})^{-h}\lambda(\mathfrak{a})$. Then we have

$$a^h f([\mathfrak{a}]) = \lambda(\mathfrak{a})^{-1}f|U(\mathfrak{l}^h)(x(\mathfrak{a})) = \frac{1}{\lambda(\mathfrak{l})^h N(\mathfrak{l})^h}\sum_{u \bmod \mathfrak{l}^h} f([\mathfrak{a}_u]),$$

and we may define a measure φ_f on Cl_∞ with values in $\mathbb{F}$ by

$$\int_{Cl_\infty} \phi d\varphi_f = b^{-m}\sum_{\mathfrak{a}\in Cl_{mh}} \phi(\mathfrak{a}^{-1})f([\mathfrak{a}]) \text{ (for } b = a^h\lambda(\mathfrak{l})^h N(\mathfrak{l})^h). \quad (4.3)$$

4.2 Non-triviality of the modular measure

The non-triviality of the measure φ_f can be proven in exactly the same manner as in [H04c] Theorems 3.2 and 3.3. To recall the result in [H04c], we need to describe some functorial action on p–adic modular forms, commuting with $U(\mathfrak{l}^h)$. Let $\mathfrak{q}$ be a prime ideal of F. For a test object $(X, \overline{\Lambda}, \eta)$ of level $\Gamma_0(\mathfrak{N}\mathfrak{q})$, η induces a subgroup $C \cong O/\mathfrak{q}$ in X. Then we can construct canonically $[\mathfrak{q}](X, \overline{\Lambda}, \eta) = (X', \overline{\Lambda}, \eta')$ with $X' = X/C$ (see [H04b] Subsection 5.3). If $\mathfrak{q}$ splits into $\mathfrak{Q}\overline{\mathfrak{Q}}$ in M/F, choosing $\eta_{\mathfrak{q}}$ induced by $X(\mathfrak{a})[\mathfrak{q}^\infty] \cong M_{\mathfrak{Q}}/R_{\mathfrak{Q}} \times M_{\overline{\mathfrak{Q}}}/R_{\overline{\mathfrak{Q}}} \cong F_{\mathfrak{q}}/O_{\mathfrak{q}} \times F_{\mathfrak{q}}/O_{\mathfrak{q}}$, we always have a canonical level $\mathfrak{q}$–structure on $X(\mathfrak{a})$ induced by the choice of the factor $\mathfrak{Q}$. Then $[\mathfrak{q}](X(\mathfrak{a})) = X(\mathfrak{a}\mathfrak{Q}_n^{-1})$ for $\mathfrak{Q}_n = \mathfrak{Q} \cap R_n$ for a proper R_n–ideal $\mathfrak{a}$. When $\mathfrak{q}$ ramifies in M/F as $\mathfrak{q} = \mathfrak{Q}^2$, $X(\mathfrak{a})$ has a subgroup $C = X(\mathfrak{a})[\mathfrak{Q}_n]$ isomorphic to $O/\mathfrak{q}$; so, we can still define $[\mathfrak{q}](X(\mathfrak{a})) = X(\mathfrak{a}\mathfrak{Q}_n^{-1})$. The effect of $[\mathfrak{q}]$ on the q–expansion at the infinity cusp $(O, \mathfrak{c}^{-1})$ is computed in [H04b] (5.12) and is given by the q–expansion of f at the cusp $(\mathfrak{q}, \mathfrak{c}^{-1})$. The operator $[\mathfrak{q}]$ corresponds to the action of $g = \begin{pmatrix} 1 & 0 \\ 0 & \varpi_{\mathfrak{q}}^{-1} \end{pmatrix} \in GL_2(F_{\mathfrak{q}})$. Although the action of $[\mathfrak{q}]$ changes the polarization ideal by $\mathfrak{c} \mapsto \mathfrak{c}\mathfrak{q}$, as in the case of Hecke operator, we regard it as a linear map well defined on $V(\Gamma_0(\mathfrak{f}\mathfrak{f}'\mathfrak{l}), \epsilon_\lambda; W)$ into $V(\Gamma_0(\mathfrak{f}\mathfrak{f}'\mathfrak{l}\mathfrak{q}), \epsilon_\lambda; W)$ (inducing the permutation $\mathfrak{c} \mapsto \mathfrak{c}_{\mathfrak{q}}$)

For ideals $\mathfrak{A}$ in F, we can think of the association $X \mapsto X \otimes_O \mathfrak{A}$ for each AVRM X. There are a natural polarization and a level structure on $X \otimes \mathfrak{A}$ induced by those of X. Writing $(X, \Lambda, \eta) \otimes \mathfrak{A}$ for the triple made out of (X, Λ, η) after tensoring $\mathfrak{A}$, we define $f|\langle\mathfrak{A}\rangle(X, \Lambda, \eta) = f((X, \Lambda, \eta) \otimes \mathfrak{A})$. For $X(\mathfrak{a})$, we have $\langle\mathfrak{A}\rangle(X(\mathfrak{a})) = X(\mathfrak{A}\mathfrak{a})$. The effect of the operator $\langle\mathfrak{A}\rangle$ on the Fourier expansion at $(O, \mathfrak{c}^{-1})$ is given by that at $(\mathfrak{A}^{-1}, \mathfrak{A}\mathfrak{c})$ (see [H04b] (5.11) or [PAF] (4.53)). The operator $\langle\mathfrak{A}\rangle$ induces an automorphism of $V(\Gamma_0(\mathfrak{f}\mathfrak{f}'\mathfrak{l}), \epsilon_\lambda; W)$. By q–expansion principle, $f \mapsto f|[\mathfrak{q}]$ and $f \mapsto f|\langle\mathfrak{A}\rangle$ are injective on the space of (p–adic) modular forms, since the effect on the q-expansion at one cusp of the operation is described by the q-expansion of the same form at another cusp.

We fix a decomposition $Cl_\infty = \Gamma_f \times \Delta$ for a finite group Δ and a torsion-free subgroup Γ_f. Since each fractional R–ideal $\mathfrak{A}$ prime to $\mathfrak{l}$ defines a class $[\mathfrak{A}]$ in Cl_∞, we can embed the ideal group of fractional ideals prime to $\mathfrak{l}$ into Cl_∞. We write Cl_∞^{alg} for its image. Then $\Delta^{alg} = \Delta \cap Cl_\infty^{alg}$ is represented by prime ideals of M non-split over F. We choose a complete representative set for Δ^{alg} as $\{\mathfrak{s}\mathfrak{R}^{-1} | \mathfrak{s} \in \mathcal{S},\ \mathfrak{r} \in \mathcal{R}\}$, where $\mathcal{S}$ contains O and ideals $\mathfrak{s}$ of F outside $p\ell\mathfrak{C}$, $\mathcal{R}$ is made of square-free product of primes in F ramifying in M/F, and $\mathfrak{R}$ is a unique ideal in M with $\mathfrak{R}^2 = \mathfrak{r}$. The set $\mathcal{S}$ is a complete representative set for the image Cl_F^0 of Cl_F in Cl_0 and $\{\mathfrak{R} | \mathfrak{r} \in \mathcal{R}\}$ is a complete representative set for 2–torsion elements in the quotient Cl_0/Cl_F^0. We fix a character $\nu : \Delta \to$

$\mathbb{F}^{\times}$, and define

$$f_{\nu} = \sum_{\mathfrak{r}\in\mathcal{R}} \lambda\nu^{-1}(\mathfrak{R}) \left(\sum_{\mathfrak{s}\in\mathcal{S}} \nu\lambda^{-1}(\mathfrak{s}) f|\langle\mathfrak{s}\rangle \right) |[\mathfrak{r}]. \tag{4.4}$$

Choose a complete representative set $\mathcal{Q}$ for $Cl_{\infty}/\Gamma_f \Delta^{alg}$ made of primes of M split over F outside $p\mathfrak{l}\mathfrak{C}$. We choose $\eta_n^{(p)}$ out of the base (w_1, w_2) of $\widehat{R}_n$ so that at $\mathfrak{q} = \mathfrak{Q} \cap F$, $w_1 = (1,0) \in R_{\mathfrak{Q}} \times R_{\mathfrak{Q}^c} = R_{\mathfrak{q}}$ and $w_2 = (0,1) \in R_{\mathfrak{Q}} \times R_{\mathfrak{Q}^c} = R_{\mathfrak{q}}$. Since all operators $\langle\mathfrak{s}\rangle$, $[\mathfrak{q}]$ and $[\mathfrak{r}]$ involved in this definition commutes with $U(\mathfrak{l})$, $f_{\nu}|[\mathfrak{q}]$ is still an eigenform of $U(\mathfrak{l})$ with the same eigenvalue as f. Thus in particular, we have a measure $\varphi_{f_{\nu}}$. We define another measure φ_f^{ν} on Γ_f by

$$\int_{\Gamma_f} \phi d\varphi_f^{\nu} = \sum_{\mathfrak{Q}\in\mathcal{Q}} \lambda\nu^{-1}(\mathfrak{Q}) \int_{\Gamma_f} \phi|\mathfrak{Q} d\varphi_{f_{\nu}|[\mathfrak{q}]},$$

where $\phi|\mathfrak{Q}(y) = \phi(y[\mathfrak{Q}]_f^{-1})$ for the projection $[\mathfrak{Q}]_f$ in Γ_f of the class $[\mathfrak{Q}] \in Cl_{\infty}$.

Lemma 4.1 *If $\chi : Cl_{\infty} \to \mathbb{F}^{\times}$ is a character inducing ν on Δ, we have*

$$\int_{\Gamma_f} \chi d\varphi_f^{\nu} = \int_{Cl_{\infty}} \chi d\varphi_f.$$

Proof Write $\Gamma_{f,n}$ for the image of Γ_f in Cl_n. For a proper R_n–ideal $\mathfrak{a}$, by the above definition of these operators,

$$f|\langle\mathfrak{s}\rangle|[\mathfrak{r}]|[\mathfrak{q}]([\mathfrak{a}]) = \lambda(\mathfrak{a})^{-1} f(x(\mathfrak{Q}^{-1}\mathfrak{R}^{-1}\mathfrak{a}), \omega(\mathfrak{Q}^{-1}\mathfrak{R}^{-1}\mathfrak{a})).$$

For sufficiently large n, χ factors through Cl_n. Since $\chi = \nu$ on Δ, we have

$$\begin{aligned}\int_{\Gamma_f} \chi d\varphi_f^{\nu} &= \sum_{\mathfrak{Q}\in\mathcal{Q}} \sum_{\mathfrak{s}\in\mathcal{S}} \sum_{\mathfrak{r}\in\mathcal{R}} \sum_{\mathfrak{a}\in\Gamma_{f,n}} \lambda\chi^{-1}(\mathfrak{Q}\mathfrak{R}\mathfrak{s}^{-1}\mathfrak{a}) f|\langle\mathfrak{s}\rangle|[\mathfrak{r}]|[\mathfrak{q}]([\mathfrak{a}]) \\ &= \sum_{\mathfrak{a},\mathfrak{Q},\mathfrak{s},\mathfrak{r}} \chi(\mathfrak{Q}\mathfrak{R}\mathfrak{s}^{-1}\mathfrak{a}) f([\mathfrak{Q}^{-1}\mathfrak{R}^{-1}\mathfrak{s}\mathfrak{a}]) = \int_{Cl_{\infty}} \chi d\varphi_f,\end{aligned}$$

because $Cl_{\infty} = \bigsqcup_{\mathfrak{Q},\mathfrak{s},\mathfrak{R}} [\mathfrak{Q}^{-1}\mathfrak{R}^{-1}\mathfrak{s}]\Gamma_f$. □

We identify $\mathrm{Hom}(\Gamma_f, \mathbb{F}^{\times}) \cong \mathrm{Hom}(\Gamma_f, \mu_{\ell^{\infty}})$ with $\mathrm{Hom}(\Gamma_f, \widehat{\mathbb{G}}_{m/\mathbb{Z}_{\ell}}) \cong \widehat{\mathbb{G}}_m^d$ for the formal multiplicative group $\widehat{\mathbb{G}}_m$ over $\mathbb{Z}_{\ell}$. Choosing a basis $\beta = \{\gamma_1, \ldots, \gamma_d\}$ of Γ_f over $\mathbb{Z}_{\ell}$ (so, $\mathbb{Z}^{\beta} = \sum_j \mathbb{Z}\gamma_j \subset \Gamma_f$) is to choose a multiplicative group $\mathbb{G}_m^{\beta} = \mathrm{Hom}(\mathbb{Z}^{\beta}, \mathbb{G}_m)$ over $\mathbb{Z}_{\ell}$ whose formal completion along the identity of $\mathbb{G}_m^{\beta}(\overline{\mathbb{F}}_{\ell})$ giving rise to $\mathrm{Hom}(\Gamma_f, \widehat{\mathbb{G}}_{m/\mathbb{Z}_{\ell}})$. Thus we may regard

$\mathrm{Hom}(\Gamma_f, \mu_{\ell^\infty})$ as a subset of $\mathrm{Hom}(\mathbb{Z}^\beta, \mathbb{G}_m) \cong \mathbb{G}_m^\beta$. We call a subset $\mathcal{X}$ of characters of Γ_f *Zariski-dense* if it is Zariski-dense as a subset of the algebraic group $\mathbb{G}^\beta_{m/\overline{\mathbb{Q}}_\ell}$ (for any choice of β). Then we quote the following result ([H04c] Theorems 3.2 and 3.3):

Theorem 4.2 *Suppose that p is unramified in $M/\mathbb{Q}$ and Σ is ordinary for p. Let $f \neq 0$ be an eigenform defined over $\mathbb{F}$ of $U(\mathfrak{l})$ of level $(\Gamma_0(\mathfrak{f}\mathfrak{f}'\mathfrak{l}), \epsilon_\lambda)$ with non-zero eigenvalue. Fix a character $\nu : \Delta \to \mathbb{F}^\times$, and define f_ν as in* (4.4)*. If f satisfies the following two conditions:*

- *(H1) There exists a strict ideal class $\mathfrak{c} \in Cl_F$ with the following two properties:*
  - (a) *the polarization ideal $\mathfrak{c}(\mathfrak{Q}^{-1}\mathfrak{R}^{-1}\mathfrak{s})$ is in $\mathfrak{c}$ for some $(\mathfrak{Q}, \mathfrak{R}, \mathfrak{s}) \in \mathcal{Q} \times \mathcal{S} \times \mathcal{R}$;*
  - (b) *for any given integer $r > 0$, the $N(\mathfrak{l})^r$ modular forms $f_{\psi,\mathfrak{c}}|\left(\begin{smallmatrix}1 & u\\ 0 & 1\end{smallmatrix}\right)$ for $u \in \mathfrak{l}^{-r}/O$ are linearly independent over $\mathbb{F}$,*
- *(H2) λ and f are rational over a finite field,*

then the set of characters $\chi : \Gamma_f \to \mathbb{F}^\times$ with non-vanishing $\int_{Cl_\infty} \nu\chi d\varphi_f \neq 0$ is Zariski dense. If $\mathrm{rank}_{\mathbb{Z}_\ell}\Gamma_f = 1$, under the same assumptions, the non-vanishing holds except for finitely many characters of Γ_f. Here $\nu\chi$ is the character of $Cl_\infty = \Gamma_f \times \Delta$ given by $\nu\chi(\gamma, \delta) = \nu(\delta)\chi(\gamma)$ for $\gamma \in \Gamma_f$ and $\delta \in \Delta$.

4.3 $\mathfrak{l}$–Adic Eisenstein measure modulo p

We apply Theorem 4.2 to the Eisenstein series $\mathbb{E}(\lambda)$ in (3.17) for the Hecke character λ fixed in 3.1. Choosing a generator π of $\mathfrak{m}_W$, the exact sequence $\underline{\omega}^{kI}_{/W} \xrightarrow{\varpi} \underline{\omega}^{kI}_{/W} \twoheadrightarrow \underline{\omega}^{kI}_{/\mathbb{F}}$ induces a reduction map: $H^0(\mathfrak{M}, \underline{\omega}^{kI}_{/W}) \to H^0(\mathfrak{M}, \underline{\omega}^{kI}_{/\mathbb{F}})$. We write $E_k(\psi^\circ, \mathfrak{c}) \bmod \Lambda$-mod$\mathfrak{m}_W$ for the image of the Eisenstein series $E_k(\psi^\circ, \mathfrak{c})$. Then we put

$$f = (d^\kappa(E_k(\psi^\circ, \mathfrak{c})) \bmod \mathfrak{m}_W)_\mathfrak{c} \in V(\Gamma_0(\mathfrak{f}\mathfrak{f}'\mathfrak{l}), \epsilon_\lambda; \mathbb{F}).$$

By definition, the q–expansion of $f_{[\mathfrak{c}]}$ is the reduction modulo $\mathfrak{m}_W$ of the q–expansion of $\mathbb{E}(\lambda, \mathfrak{c})$ of characteristic 0.

We fix a character $\nu : \Delta \to \mathbb{F}^\times$ as in the previous section and write $\varphi = \varphi_f$ and $\varphi^\nu = \varphi^\nu_f$. By (3.24) combined with Lemma 4.1, we have, for a character $\chi : CL_\infty \to \mathbb{F}^\times$ with $\chi|_\Delta = \nu$,

$$\int_{\Gamma_f} \chi d\varphi^\nu = \int_{Cl_\infty} \chi d\varphi = C\chi(\mathfrak{F}) \frac{\pi^\kappa \Gamma_\Sigma(k\Sigma + \kappa) L^{(\mathfrak{l})}(0, \chi^{-1}\lambda)}{\Omega_\infty^{k\Sigma + 2\kappa}} \bmod \mathfrak{m}_W, \tag{4.5}$$

where C is a non-zero constant given by the class modulo $\mathfrak{m}_W$ of

$$\frac{(-1)^{k[F:\mathbb{Q}]}(R^\times : O^\times)\lambda^{-1}(\varpi^{e(\mathfrak{F})})}{\mathrm{Im}(\delta)^\kappa\sqrt{D}}.$$

The non-vanishing of C follows from the unramifiedness of p in $M/\mathbb{Q}$ and that $\mathfrak{F}$ is prime to p.

Theorem 4.3 *Let p be an odd prime unramified in $M/\mathbb{Q}$. Let λ be a Hecke character of M of conductor $\mathfrak{C}$ and of infinity type $k\Sigma + \kappa(1-c)$ with $0 < k \in \mathbb{Z}$ and $0 \le \kappa \in \mathbb{Z}[\Sigma]$ for a CM type Σ that is ordinary with respect to p. Suppose* (spt) *and* (opl) *in* 3.1. *Fix a character $\nu : \Delta \to \overline{\mathbb{Q}}^\times$. Then $\frac{\pi^\kappa\Gamma_\Sigma(k\Sigma+\kappa)L^{(\mathfrak{l})}(0,\nu^{-1}\chi^{-1}\lambda)}{\Omega_\infty^{k\Sigma+2\kappa}} \in \mathcal{W}$ for all characters $\chi : Cl_\infty \to \mu_{\ell^\infty}(\overline{\mathbb{Q}})$ factoring through Γ_f. Moreover, for Zariski densely populated character χ in* $\mathrm{Hom}(\Gamma_f, \mu_{\ell^\infty})$*, we have*

$$\frac{\pi^\kappa\Gamma_\Sigma(k\Sigma+\kappa)L^{(\mathfrak{l})}(0,\nu^{-1}\chi^{-1}\lambda)}{\Omega_\infty^{k\Sigma+2\kappa}} \not\equiv 0 \bmod \mathfrak{m}_{\mathcal{W}},$$

unless the following three conditions are satisfied by ν and λ simultaneously:

- *(M1) M/F is unramified everywhere;*
- *(M2) The strict ideal class (in F) of the polarization ideal $\mathfrak{c}_0$ of $X(R)$ is not a norm class of an ideal class of M ($\Leftrightarrow \left(\frac{M/F}{\mathfrak{c}_0}\right) = -1$);*
- *(M3) The ideal character $\mathfrak{a} \mapsto (\lambda\nu^{-1}N(\mathfrak{a}) \bmod \mathfrak{m}_W) \in \mathbb{F}^\times$ of F is equal to the character $\left(\frac{M/F}{\cdot}\right)$ of M/F.*

If $\mathfrak{l}$ is a split prime of degree 1 over $\mathbb{Q}$, under the same assumptions, the non-vanishing holds except for finitely many characters of Γ_f. If (M1-3) *are satisfied, the L–value as above vanishes modulo $\mathfrak{m}$ for all anticyclotomic characters χ.*

See [H04b] 5.4 for an example of $(M, \mathfrak{c}_0, \Sigma)$ satisfying (M1-3).

Proof By Theorem 4.2, we need to verify the condition (H1-2) for $\mathbb{E}(\lambda)$. The rationality (H2) follows from the rationality of $E_k(\psi^\circ, \mathfrak{c})$ and the differential operator d described in 2.6. For a given q–expansion $h(q) = \sum_\xi a(\xi, h)q^\xi \in \mathbb{F}[[\mathfrak{c}^{-1}_{\ge 0}]]$ at the infinity cusp $(O, \mathfrak{c}^{-1})$, we know that, for $u \in O_{\mathfrak{l}} \subset F_{\mathbb{A}}$,

$$a(\xi, h|\alpha_u) = \mathbf{e}_F(u\xi)a(\xi, h) \ \text{ for } \alpha_u = \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right).$$

The condition (H1) for h concerns the linear independence of $h|\alpha_u$ for $u \in$

$\mathfrak{l}^{-r}O_{\mathfrak{l}}/O_{\mathfrak{l}}$. For any function $\phi : \mathfrak{c}^{-1}/\mathfrak{l}^r\mathfrak{c}^{-1} = O/\mathfrak{l}^r \to \mathbb{F}$, we write $h|\phi = \sum_\xi \phi(\xi)a(\xi,h)q^\xi$. By definition, we have

$$h|R_\phi = \sum_{u\in O/\mathfrak{l}^r} \phi(u)h|\alpha_u = h|\phi^*$$

for the Fourier transform $\phi^*(v) = \sum_u \phi(u)\mathbf{e}_F(uv)$. For the characteristic function χ_v of $v \in \mathfrak{c}^{-1}/\mathfrak{l}^r\mathfrak{c}$, we compute its Fourier transform

$$\chi_v^*(u) = \sum_{a\in O/\mathfrak{l}^r} \mathbf{e}_F(au)\chi_v(a) = \mathbf{e}_F(vu).$$

Since the Fourier transform of the finite group $O/\mathfrak{l}^r$ is an automorphism (by the inversion formula), the linear independence of $\{h|\alpha_u = h|\chi_u^*\}_u$ is equivalent to the linear independence of $\{h|\chi_u\}_u$.

We recall that f_ν is a tuple $(f_{\nu,[\mathfrak{c}]})_{\mathfrak{c}} \in V(\Gamma_{01}(\mathfrak{f}^2),\Gamma_0(\mathfrak{l});W)$. Thus we need to prove: there exists $\mathfrak{c}$ such that for a given congruence class $u \in \mathfrak{c}^{-1}/\mathfrak{l}^r\mathfrak{c}^{-1}$

$$a(\xi, f_{\nu,[\mathfrak{c}]}) \not\equiv 0 \bmod \mathfrak{m}_W \text{ for at least one } \xi \in u. \tag{4.6}$$

Since $a(\xi, d^\kappa h) = \xi^\kappa a(\xi,h)$ ((2.6)), (4.6) is achieved if

$$a(\xi, f'_{\nu,[\mathfrak{c}]}) \not\equiv 0 \bmod \mathfrak{m}_W \text{ for at least one } \xi \in u \text{ prime to } p \tag{4.7}$$

holds for

$$f' = (E_k(\psi^\circ,\mathfrak{c}) - N(\mathfrak{l})E_k(\psi^\circ,\mathfrak{c}\mathfrak{l}^{-1})|\langle\mathfrak{l}\rangle|[\mathfrak{l}])_{\mathfrak{c}},$$

because $\mathfrak{l} \nmid p$. Up to a non-zero constant, $\psi^\circ(a,b)$ in (3.7) is equal to $\phi(a,b) = \lambda_{\mathfrak{F}}(a)\lambda_{\mathfrak{F}_c}^{-1}(b)$ for $(a,b) \in (O/\mathfrak{f})^\times$. Thus we are going to prove, for a well chosen $\mathfrak{c}$,

$$a(\xi, f''_{\nu,[\mathfrak{c}]}) \not\equiv 0 \bmod \mathfrak{m}_W \text{ for at least one } \xi \in u \text{ prime to } p, \tag{4.8}$$

where $f''_{[\mathfrak{c}]} = E_k(\phi,\mathfrak{c}) - N(\mathfrak{l})E_k(\phi,\mathfrak{c}\mathfrak{l}^{-1})|\langle\mathfrak{l}\rangle|[\mathfrak{l}]$. Recall (4.4):

$$f''_\nu = \sum_{\mathfrak{r}\in\mathcal{R}} \lambda\nu^{-1}(\mathfrak{R})\left(\sum_{\mathfrak{s}\in\mathcal{S}} \nu\lambda^{-1}(\mathfrak{s})f''|\langle\mathfrak{s}\rangle\right)\big|[\mathfrak{r}]. \tag{4.9}$$

As computed in [H04b] (5.11) and (5.12), we have

$$\begin{aligned} N(\mathfrak{s}^{-1}\mathfrak{r})^{-1}E_k(\phi,\mathfrak{c})|\langle\mathfrak{s}\rangle|[\mathfrak{r}]_{O,\mathfrak{c}^{-1}}(q) &= 2^{-[F:\mathbb{Q}]}L(1-k;\phi,\mathfrak{s}^{-1}\mathfrak{r}) \\ &+ \sum_{0\ll\xi\in\mathfrak{c}^{-1}\mathfrak{r}} q^\xi \sum_{\substack{(a,b)\in(\mathfrak{s}^{-1}\mathfrak{r}\times\mathfrak{c}^{-1}\mathfrak{s})/O^\times \\ ab=\xi}} \phi(a,b)\frac{N(a)}{|N(a)|}N(a)^{k-1}. \end{aligned} \tag{4.10}$$

Thus we have, writing $t(a,b) = \phi(a,b)\frac{N(a)^k}{|N(a)|}$, $N(\mathfrak{a})^{-1}a(\xi, f''_{\mathfrak{a},\mathfrak{b}})$ is given by

$$\sum_{\substack{(a,b)\in(\mathfrak{a}\times\mathfrak{b})/O^\times \\ ab=\xi}} t(a,b) - \sum_{\substack{(a,b)\in(\mathfrak{a}\mathfrak{r}\times\mathfrak{l}\mathfrak{b})/O^\times \\ ab=\xi}} t(a,b)$$

$$= \sum_{\substack{(a,b)\in(\mathfrak{a}\times(\mathfrak{b}-\mathfrak{l}\mathfrak{b}))/O^\times \\ ab=\xi}} t(a,b). \quad (4.11)$$

We have the freedom of moving around the polarization ideal class $[\mathfrak{c}]$ in the coset $N_{M/F}(Cl_M)[\mathfrak{c}_0]$ for the polarization ideal class $[\mathfrak{c}_0]$ of $x(R)$. We first look into a single class $[\mathfrak{c}]$. We choose $\mathfrak{c}^{-1}$ to be a prime $\mathfrak{q}$ prime to $p\mathfrak{f}\mathfrak{l}$ (this is possible by changing $\mathfrak{c}^{-1}$ in its strict ideal class and choosing $\delta \in M$ suitably). We take a class $0 \ll \xi \in u$ for $u \in O/\mathfrak{l}^r$ so that $(\xi) = \mathfrak{q}\mathfrak{n}\mathfrak{l}^e$ for an integral ideal $\mathfrak{n} \nmid p\mathfrak{l}\mathfrak{C}$ prime to the relative discriminant $D(M/F)$ and $0 \le e \le r$. Since we have a freedom of choosing ξ modulo $\mathfrak{l}^r$, the ideal $\mathfrak{n}$ moves around freely in a given ray class modulo $\mathfrak{l}^{r-e}$.

We pick a pair $(a,b) \in F^2$ with $ab = \xi$ with $a \in \mathfrak{s}^{-1}$ and $b \in \mathfrak{q}\mathfrak{s}$. Then $(a) = \mathfrak{s}^{-1}\mathfrak{l}^\alpha\mathfrak{x}$ for an integral ideal $\mathfrak{x}$ prime to $\mathfrak{l}$ and $(b) = \mathfrak{s}\mathfrak{q}\mathfrak{l}^{e-\alpha}\mathfrak{x}'$ for an integral ideal $\mathfrak{x}'$ prime to $\mathfrak{l}$. Since $(ab) = \mathfrak{q}\mathfrak{n}\mathfrak{l}^e$, we find that $\mathfrak{x}\mathfrak{x}' = \mathfrak{n}$. By (4.11), b has to be prime to $\mathfrak{l}$; so, we find $\alpha = e$. Since $\mathfrak{x}\mathfrak{x}' = \mathfrak{n}$ and hence $\mathfrak{r} = O$ because $\mathfrak{n}$ is prime to $D(M/F)$. Thus for each factor $\mathfrak{x}$ of $\mathfrak{n}$, we could have two possible pairs $(a_\mathfrak{x}, b_\mathfrak{x})$ with $a_\mathfrak{x} b_\mathfrak{x} = \xi$ such that

$$((a_\mathfrak{x}) = \mathfrak{s}_\mathfrak{x}^{-1}\mathfrak{l}^e\mathfrak{x},\ (b_\mathfrak{x}) = (\xi a_\mathfrak{x}^{-1}) = \mathfrak{s}_\mathfrak{x}\mathfrak{q}\mathfrak{n}\mathfrak{x}^{-1})$$

for $\mathfrak{s}_\mathfrak{x} \in \mathcal{S}$ representing the ideal class of the ideal $\mathfrak{l}^e\mathfrak{x}$. We put $\psi = \nu^{-1}\lambda$. We then write down the q–expansion coefficient of q^ξ at the cusp $(O, \mathfrak{q})$ (see [H04c] (4.30)):

$$G(\psi_\mathfrak{f})^{-1}a(\xi, f''_\nu) = \psi_{\mathfrak{F}_c}^{-1}(\xi)\psi(\mathfrak{n}\mathfrak{l}^e)^{-1}N(\mathfrak{n}\mathfrak{l}^e)^{-1}\prod_{\mathfrak{y}|\mathfrak{n}} \frac{1-(\psi(\mathfrak{y})N(\mathfrak{y}))^{e(\mathfrak{y})+1}}{1-\psi(\mathfrak{y})N(\mathfrak{y})}, \quad (4.12)$$

where $\mathfrak{n} = \prod_{\mathfrak{y}|\mathfrak{n}} \mathfrak{y}^{e(\mathfrak{y})}$ is the prime factorization of $\mathfrak{n}$.

We define, for the valuation v of W (normalized so that $v(p) = 1$)

$$\mu_C(\psi) = \mathrm{Inf}_\mathfrak{n} v\left(\prod_{\mathfrak{y}|\mathfrak{n}} \frac{1-(\psi(\mathfrak{y})N(\mathfrak{y}))^{e(\mathfrak{y})+1}}{1-\psi(\mathfrak{y})N(\mathfrak{y})}\right), \quad (4.13)$$

where $\mathfrak{n}$ runs over a ray class C modulo $\mathfrak{l}^{r-e}$ made of all integral ideals prime to $D\mathfrak{l}$ of the form $\mathfrak{q}^{-1}\xi\mathfrak{l}^{-e}$, $0 \ll \xi \in u$. Thus if $\mu_C(\psi) = 0$, we get the desired non-vanishing. Since $\mu_C(\psi)$ only depends on the class C, we may assume (and will assume) that $e = 0$ without losing generality; thus ξ is prime to $\mathfrak{l}$, and C is the class of $u[\mathfrak{q}^{-1}]$.

Suppose that $\mathfrak{n}$ is a prime $\mathfrak{y}$. Then by (4.12), we have

$$G(\psi_{\mathfrak{f}})^{-1}a(\xi, f''_\nu) = \psi^{-1}_{\mathfrak{F}_c}(\xi)(1 + (\psi(\mathfrak{y})N(\mathfrak{y}))^{-1}).$$

If $\psi(\mathfrak{y})N(\mathfrak{y}) \equiv -1 \mod \mathfrak{m}_W$ for all prime ideals $\mathfrak{y}$ in the ray class C modulo $\mathfrak{l}^r$, the character $\mathfrak{a} \mapsto (\psi(\mathfrak{a})N(\mathfrak{a}) \mod \mathfrak{m}_W)$ is of conductor $\mathfrak{l}^r$. We write $\overline{\psi}$ for the character: $\mathfrak{a} \mapsto (\psi(\mathfrak{a})N(\mathfrak{a}) \mod \mathfrak{m}_W)$ of the ideal group of F with values in $\mathbb{F}^\times$. This character therefore has conductor $\widetilde{\mathfrak{C}}|\mathfrak{l}^r$. Since ν is anticyclotomic, its restriction to $F^\times_{\mathbb{A}}$ has conductor 1. Since λ has conductor $\mathfrak{C}$ prime to $\mathfrak{l}$, the conductor of $\overline{\psi}$ is a factor of the conductor of $\lambda \mod \mathfrak{m}_W$, which is a factor of $p\mathfrak{C}$. Thus $\widetilde{\mathfrak{C}}|p\mathfrak{C}$. Since $\mathfrak{l} \nmid p\mathfrak{C}$, we find that $\widetilde{\mathfrak{C}} = 1$.

We are going to show that if $\mu_C(\psi) > 0$, M/F is unramified and $\overline{\psi} \equiv \left(\frac{M/F}{}\right) \mod \mathfrak{m}_W$. We now choose two prime ideals $\mathfrak{y}$ and $\mathfrak{y}'$ so that $\mathfrak{q}\mathfrak{y}\mathfrak{y}' = (\xi)$ with $\xi \in u$. Then by (4.12), we have

$$G(\psi_{\mathfrak{f}})^{-1}a(\xi, f''_\nu) = \psi^{-1}_{\mathfrak{F}_c}(\xi)\left(1 + \frac{1}{\psi(\mathfrak{y})N(\mathfrak{y})}\right)\left(1 + \frac{1}{\psi(\mathfrak{y}')N(\mathfrak{y}')}\right). \tag{4.14}$$

Since $\overline{\psi}(\mathfrak{y}\mathfrak{y}') = \overline{\psi}(u[\mathfrak{q}^{-1}]) = \overline{\psi}(C) = -1$, we find that if $a(\xi, f''_\nu) \equiv 0 \mod \mathfrak{m}_W$,

$$-1 = \overline{\psi}(\mathfrak{y}/\mathfrak{y}') = \overline{\psi}(\mathfrak{l}^{-1})\overline{\psi}(\mathfrak{y}^2) = -\overline{\psi}(\mathfrak{y}^2).$$

Since we can choose $\mathfrak{y}$ arbitrary, we find that $\overline{\psi}$ is quadratic. Thus $\mu_C(\psi) > 0$ if and only if $\overline{\psi}(\mathfrak{c}) = -1$, which is independent of the choice of u. Since we only need to show the existence of $\mathfrak{c}$ with $\overline{\psi}(\mathfrak{c}) = 1$, we can vary the strict ideal class $[\mathfrak{c}]$ in $[\mathfrak{c}_0]N_{M/F}(Cl_M)$. By class field theory, assuming that $\overline{\psi}$ has conductor 1, we have

$$\overline{\psi}(\mathfrak{c}) = -1 \text{ for all } [\mathfrak{c}] \in [\mathfrak{c}_0]N_{M/F}(Cl_M)$$
$$\text{if and only if } \overline{\psi}(\mathfrak{c}_0) = -1 \text{ and } \overline{\psi}(\mathfrak{a}) = \left(\frac{M/F}{\mathfrak{a}}\right) \text{ for all } \mathfrak{a} \in Cl_F. \tag{4.15}$$

If M/F is unramified, by definition, $2\delta\mathfrak{c}^* = 2\delta\mathfrak{d}^{-1}\mathfrak{c}^{-1} = R$. Taking squares, we find that $(\mathfrak{d}\mathfrak{c})^2 = 4\delta^2 \ll 0$. Thus $1 = \overline{\psi}(\mathfrak{d}^{-2}\mathfrak{c}^{-2}) = (-1)^{[F:\mathbb{Q}]}$, and this never happens when $[F : \mathbb{Q}]$ is odd. Thus (4.15) is equivalent to the three conditions (M1-3). The conditions (M1) and (M3) combined is equivalent to $\psi^* \equiv \psi \mod \mathfrak{m}_W$, where the dual character ψ^* is defined by $\psi^*(x) = \psi(x^{-c})N(x)^{-1}$. Then the vanishing of $L(0, \chi^{-1}\nu^{-1}\lambda) \equiv 0$ for all anticyclotomic $\chi\nu$ follows from the functional equation of the p-adic Katz measure interpolating the p–adic Hecke L–values. This finishes the proof. □

5 Anticyclotomic Iwasawa series

We fix a conductor $\mathfrak{C}$ satisfying (spt) and (opl) in 3.1. We consider $Z = Z(\mathfrak{C}) = \varprojlim_n Cl_M(\mathfrak{C}p^n)$ for the ray class group $Cl_M(\mathfrak{x})$ of M modulo $\mathfrak{x}$. We split $Z(\mathfrak{C}) = \Delta_{\mathfrak{C}} \times \Gamma_{\mathfrak{C}}$ for a finite group $\Delta = \Delta_{\mathfrak{C}}$ and a torsion-free subgroup $\Gamma_{\mathfrak{C}}$. Since the projection: $Z(\mathfrak{C}) \twoheadrightarrow Z(1)$ induces an isomorphism $\Gamma_{\mathfrak{C}} = Z(\mathfrak{C})/\Delta_{\mathfrak{C}} \cong Z(1)/\Delta_1 = \Gamma_1$, we identify $\Gamma_{\mathfrak{C}}$ with Γ_1 and write it as Γ, which has a natural action of $\mathrm{Gal}(M/F)$. We define $\Gamma^+ = H^0(\mathrm{Gal}(M/F), \Gamma)$ and $\Gamma_- = \Gamma/\Gamma^+$. Write $\pi_- : Z \to \Gamma_-$ and $\pi_\Delta : Z \to \Delta$ for the two projections. Take a character $\varphi : \Delta \to \overline{\mathbb{Q}}^\times$, and regard it as a character of Z through the projection: $Z \twoheadrightarrow \Delta$. The Katz measure $\mu_{\mathfrak{C}}$ on $Z(\mathfrak{C})$ associated to the p–adic CM type Σ_p as in [HT1] Theorem II induces the anticyclotomic φ–branch μ_φ^- by

$$\int_{\Gamma_-} \phi d\mu_\varphi^- = \int_{Z(\mathfrak{C})} \phi(\pi_-(z))\varphi(\pi_\Delta(z))d\mu_{\mathfrak{C}}(z).$$

We write $L_p^-(\varphi)$ for this measure $d\mu_\varphi^-$ regarding it as an element of the algebra $\Lambda^- = W[[\Gamma_-]]$ made up of measures with values in W.

We look into the arithmetic of the unique $\mathbb{Z}_p^{[F:\mathbb{Q}]}$–extension M_∞^- of M on which we have $c\sigma c^{-1} = \sigma^{-1}$ for all $\sigma \in \mathrm{Gal}(M_\infty^-/M)$ for complex conjugation c. The extension M_∞^-/M is called the anticyclotomic tower over M. Writing $M(\mathfrak{C}p^\infty)$ for the ray class field over M modulo $\mathfrak{C}p^\infty$, we identify $Z(\mathfrak{C})$ with $\mathrm{Gal}(M(\mathfrak{C}p^\infty)/M)$ via the Artin reciprocity law. Then one has $\mathrm{Gal}(M(\mathfrak{C}p^\infty)/M_\infty^-) = \Gamma^+ \times \Delta_{\mathfrak{C}}$ and $\mathrm{Gal}(M_\infty^-/M) = \Gamma_-$. We then define M_Δ by the fixed field of $\Gamma_{\mathfrak{C}}$ in $M(\mathfrak{C}p^\infty)$; so, $\mathrm{Gal}(M_\Delta/M) = \Delta$. Since φ is a character of Δ, φ factors through $\mathrm{Gal}(M_\infty^- M_\Delta/M)$. Let $L_\infty/M_\infty^- M_\Delta$ be the maximal p–abelian extension unramified outside Σ_p. Each $\gamma \in \mathrm{Gal}(L_\infty/M)$ acts on the normal subgroup $X = \mathrm{Gal}(L_\infty/M_\infty^- M_\Delta)$ continuously by conjugation, and by the commutativity of X, this action factors through $\mathrm{Gal}(M_\Delta M_\infty^-/M)$. Then we look into the Γ_-–module: $X[\varphi] = X \otimes_{\Delta_{\mathfrak{C}},\varphi} W$.

As is well known, $X[\varphi]$ is a Λ^-–module of finite type, and in many cases, it is torsion by a result of Fujiwara (cf. [Fu], [H00] Corollary 5.4 and [HMI] Section 5.3) generalizing the fundamental work of Wiles [Wi] and Taylor-Wiles [TW]. If one assumes the Σ–Leopoldt conjecture for abelian extensions of M, we know that $X[\varphi]$ is a torsion module over Λ^- unconditionally (see [HT2] Theorem 1.2.2). If $X[\varphi]$ is a torsion Λ^-–module, we can think of the characteristic element $\mathcal{F}^-(\varphi) \in \Lambda^-$ of the module $X[\varphi]$. If $X[\varphi]$ is not of torsion over Λ^-, we simply put $\mathcal{F}^-(\varphi) = 0$. A character φ of Δ is called *anticyclotomic* if $\varphi(c\sigma c^{-1}) = \varphi^{-1}$.

We are going to prove in this section the following theorem:

Theorem 5.1 *Let ψ be an anticyclotomic character of Δ. Suppose* (spt) *and* (opl) *in* (3.1) *for the conductor $\mathfrak{C}(\psi)$ of ψ. If p is odd and unramified in $F/\mathbb{Q}$, then the anticyclotomic p–adic Hecke L–function $L_p^-(\psi)$ is a factor of $\mathcal{F}^-(\psi)$ in Λ^-.*

Regarding φ as a Galois character, we define $\varphi^-(\sigma) = \varphi(c\sigma c^{-1}\sigma^{-1})$ for $\sigma \in \mathrm{Gal}(\overline{M}/M)$. Then φ^- is anticyclotomic. By enlarging $\mathfrak{C}$ if necessary, we can find a character φ such that $\psi = \varphi^-$ for any given anticyclotomic ψ (e.g. [GME] page 339 or [HMI] Lemma 5.31). Thus we may always assume that $\psi = \varphi^-$.

It is proven in [HT1] and [HT2] that $L_p(\varphi^-)$ is a factor of $\mathcal{F}^-(\varphi^-)$ in $\Lambda^- \otimes_{\mathbb{Z}} \mathbb{Q}$. Thus the improvement concerns the p–factor of $L_p^-(\varphi^-)$, which has been shown to be trivial in [H04b]. The main point of this paper is to give another proof of this fact reducing it to Theorem 4.3. The new proof actually gives a stronger result: Corollary 5.6, which will be used in our forthcoming paper to prove the identity $L_p^-(\psi) = \mathcal{F}^-(\psi)$ under suitable assumptions on ψ. The proof is a refinement of the argument in [HT1] and [HT2]. We first deduce a refinement of the result in [HT1] Section 7 using a unique Hecke eigenform (in a given automorphic representation) of minimal level. The minimal level is possibly a proper factor of the conductor of the automorphic representation. Then we proceed in the same manner as in [HT1] and [HT2].

Here we describe how to reduce Theorem 5.1 to Corollary 5.6. Since the result is known for $F = \mathbb{Q}$ by the works of Rubin and Tilouine, we may assume that $F \neq \mathbb{Q}$. Put $\Lambda = W[[\Gamma]]$. By definition, for the universal Galois character $\widetilde{\psi} : \mathrm{Gal}(M(\mathfrak{C}p^\infty)/M) \to \Lambda^\times$ sending $\delta \in \Delta_{\mathfrak{C}}$ to $\psi(\delta)$ and $\gamma \in \Gamma$ to the group element $\gamma \in \Gamma \subset \Lambda$, the Pontryagin dual of the adjoint Selmer group $\mathrm{Sel}(Ad(\mathrm{Ind}_M^F \widetilde{\psi}))$ defined in [MFG] 5.2 is isomorphic to the direct sum of $X[\psi] \otimes_{\Lambda^-} \Lambda$ and $\frac{Cl_M \otimes_{\mathbb{Z}} \Lambda}{Cl_F \otimes_{\mathbb{Z}} \Lambda}$. Thus the characteristic power series of the Selmer group is given by $(h(M)/h(F))\mathcal{F}^-(\psi)$.

To relate this power series $(h(M)/h(F))\mathcal{F}^-(\psi)$ to congruence among automorphic forms, we identify $O_{\mathfrak{f}} \cong R_{\mathfrak{F}} \cong R_{\mathfrak{F}_c}$. Recall the maximal diagonal torus $T_0 \subset GL(2)_{/O}$. Thus ψ restricted to $(R_{\mathfrak{F}} \times R_{\mathfrak{F}_c})^\times$ gives rise to the character ψ of $T_0^2(O_{\mathfrak{f}})$. We then extend ψ to a character ψ_F of $T_0^2(O_{\mathfrak{f}} \times O_{D(M/F)})$ by $\psi_F(x_{\mathfrak{f}}, y_{\mathfrak{f}}, x', y') = \psi(x_{\mathfrak{f}}, y_{\mathfrak{f}}) \left(\frac{M/F}{y'}\right)$. Then we define the level ideal $\mathfrak{N}$ by $(\mathfrak{C}(\psi^-) \cap F)D(M/F)$ and consider the Hecke algebra $\mathbf{h}^{n.ord} = \mathbf{h}^{n.ord}(\mathfrak{N}, \psi_F; W)$. It is easy to see that there is a unique $W[[\mathbf{\Gamma}]]$–algebra homomorphism $\lambda : \mathbf{h}^{n.ord} \to \Lambda$ such that the associated Galois representation ρ_λ ([H96] 2.8) is $\mathrm{Ind}_M^F \widetilde{\psi}$. Here $\mathbf{\Gamma}$ is the maximal torsion-free quotient of $\mathbf{G}$ introduced in 2.10. Note that the restriction of ρ_λ to the decomposition group $D_{\mathfrak{q}}$ at a prime $\mathfrak{q}|\mathfrak{N}$ is the diagonal representation $\begin{pmatrix} \psi_{F,1} & 0 \\ 0 & \psi_{F,2} \end{pmatrix}$ with values in $GL_2(W)$, which we write $\rho_{\mathfrak{q}}$. We write $H(\psi)$ for the congruence power series $H(\lambda)$ of λ (see [H96] Section 2.9, where $H(\lambda)$ is written as $\eta(\lambda)$). Writing $\mathbb{T}$ for the local ring of $\mathbf{h}^{n.ord}$ through which λ factors, the divisibility: $H(\psi)|(h(M)/h(F))\mathcal{F}^-(\psi)$ follows from the surjectivity onto $\mathbb{T}$ of the natural morphism from the universal nearly ordinary deformation ring $R^{n.ord}$ of $\mathrm{Ind}_F^M \psi \bmod \mathfrak{m}_W$ (without deforming $\rho_{\mathfrak{q}}$ for each $\mathfrak{q}|\mathfrak{N}$ and the restriction of the determinant character to $\Delta(\mathfrak{N})$). See [HT2] Section 6.2 for details of this implication. The surjectivity is obvious from our construction of $\mathbf{h}^{n.ord}(\mathfrak{N}, \psi_F; W)$ because it is generated by $\mathrm{Tr}(\rho_\lambda(Frob_{\mathfrak{q}}))$ for primes $\mathfrak{q}$ outside $p\mathfrak{N}$ and by the diagonal entries of ρ_λ restricted to $D_{\mathfrak{q}}$ for $\mathfrak{q}|p\mathfrak{N}$. Thus we need to prove $(h(M)/h(F))L_p^-(\psi)|H(\psi)$, which is the statement of Corollary 5.6. This corollary will be proven in the rest of this section. As a final remark, if we write $\mathbb{T}^\chi$ for the quotient of $\mathbb{T}$ which parametrizes all p–adic modular Galois representations congruent to $\mathrm{Ind}_M^F \psi$ with a given determinant character χ, we have $\mathbb{T} \cong \mathbb{T}^\chi \widehat{\otimes}_W W[[\mathbf{\Gamma}^+]] = \mathbb{T}^\chi[[\mathbf{\Gamma}^+]]$ for the maximal torsion-free quotient $\mathbf{\Gamma}^+$ of $Cl_F^+(\mathfrak{N}p^\infty)$ (cf. [MFG] Theorem 5.44). This implies $H(\psi) \in W[[\Gamma^-]]$.

5.1 Adjoint square L–values as Petersson metric

We now set $G := \mathrm{Res}_{O/\mathbb{Z}} GL(2)$. Let π be a cuspidal automorphic representation of $G(\mathbb{A})$ which is everywhere principal at finite places and holomorphic discrete series at all archimedean places. Since π is associated to holomorphic automorphic forms on $G(\mathbb{A})$, π is rational over the Hecke field generated by eigenvalues of the primitive Hecke eigenform in π. We have $\pi = \pi^{(\infty)} \otimes \pi_\infty$ for representations $\pi^{(\infty)}$ of $G(\mathbb{A}^{(\infty)})$ and π_∞ of $G(\mathbb{R})$. We further decompose

$$\pi^{(\infty)} = \otimes_{\mathfrak{q}} \pi(\epsilon_{1,\mathfrak{q}}, \epsilon_{2,\mathfrak{q}})$$

for the principal series representation $\pi(\epsilon_{1,\mathfrak{q}}, \epsilon_{2,\mathfrak{q}})$ of $GL_2(F_{\mathfrak{q}})$ with two characters $\epsilon_{1,\mathfrak{q}}, \epsilon_{2,\mathfrak{q}} : F_{\mathfrak{q}}^\times \to \overline{\mathbb{Q}}^\times$. By the rationality of π, these characters have values in $\overline{\mathbb{Q}}$. The central character of $\pi^{(\infty)}$ is given by $\epsilon_+ = \prod_{\mathfrak{q}}(\epsilon_{1,\mathfrak{q}}\epsilon_{2,\mathfrak{q}})$, which is a Hecke character of F. However $\epsilon_1 = \prod_{\mathfrak{q}} \epsilon_{1,\mathfrak{q}}$ and $\epsilon_2 = \prod_{\mathfrak{q}} \epsilon_{2,\mathfrak{q}}$ are just characters of $F_{\mathbb{A}^{(\infty)}}^\times$ and may not be Hecke characters.

In the space of automorphic forms in π, there is a unique normalized Hecke eigenform $f = f_\pi$ of minimal level satisfying the following conditions (see [H89] Corollary 2.2):

(L1) The level $\mathfrak{N}$ is the conductor of $\epsilon^- = \epsilon_2\epsilon_1^{-1}$.

(L2) Note that $\epsilon_\pi : \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \epsilon_1(ad-bc)\epsilon^-(d)$ is a character of $U_0(\mathfrak{N})$ whose restriction to $U_0(C(\pi))$ for the conductor $C(\pi)$ of π induces the "Neben" character $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \epsilon_1(a)\epsilon_2(d)$. Then $f : G(\mathbb{Q})\backslash G(\mathbb{A}) \to \mathbb{C}$ satisfies

$$f(xu) = \epsilon_\pi(u)f(x).$$

(L3) The cusp form f corresponds to holomorphic cusp forms of weight $\kappa = (\kappa_1, \kappa_2) \in \mathbb{Z}[I]^2$.

In short, f_π is a cusp form in $S_\kappa(\mathfrak{N}, \epsilon_+; \mathbb{C})$. It is easy to see that $\Pi = \pi \otimes \epsilon_2^{-1}$ has conductor $\mathfrak{N}$ and that $v \otimes \epsilon_2$ is a constant multiple of f for the new vector v of Π (note here that Π may not be automorphic, but Π is an admissible irreducible representation of $G(\mathbb{A})$; so, the theory of new vectors still applies). Since the conductor $C(\pi)$ of π is given by the product of the conductors of ϵ_1 and ϵ_2, the minimal level $\mathfrak{N}$ is a factor of the conductor $C(\pi)$ and is often a proper divisor of $C(\pi)$.

By (L2), the Fourier coefficients $a(y, f)$ satisfy $a(uy, f) = \epsilon_1(u)a(y, f)$ for $u \in \widehat{O}^\times$ ($\widehat{O} = O \otimes_\mathbb{Z} \widehat{\mathbb{Z}}$). In particular, the function: $y \mapsto a(y, f)\overline{a(y, f)}$ only depends on the fractional ideal yO. Thus writing $a(\mathfrak{a}, f)\overline{a(\mathfrak{a}, f)}$ for the ideal $\mathfrak{a} = yO$, we defined in [H91] the self Rankin product by

$$D(s - [\kappa] - 1, f, f) = \sum_{\mathfrak{a} \subset O} a(\mathfrak{a}, f)\overline{a(\mathfrak{a}, f)} N(\mathfrak{a})^{-s},$$

where $N(\mathfrak{a}) = [O : \mathfrak{a}] = |O/\mathfrak{a}|$. We have a shift: $s \mapsto s - [\kappa] - 1$, because in order to normalize the L–function, we used in [H91] (4.6) the unitarization $\pi^u = \pi \otimes |\cdot|_\mathbb{A}^{([\kappa]-1)/2}$ in place of π to define the Rankin product. The weight κ^u of the unitarization satisfies $[\kappa^u] = 1$ and $\kappa^u \equiv \kappa \mod \mathbb{Q}I$. Note that (cf. [H91] (4.2b))

$$f_\pi^u(x) := f_{\pi^u}(x) = D^{-([\kappa]+1)/2} f_\pi(x)|\det(x)|_\mathbb{A}^{([\kappa]-1)/2}. \qquad (5.1)$$

We are going to define Petersson metric on the space of cusp forms satisfying (L1-3). For that, we write

$$X_0 = X_0(\mathfrak{N}) = G(\mathbb{Q})_+\backslash G(\mathbb{A})_+/U_0(\mathfrak{N})F_\mathbb{A}^\times SO_2(F_\mathbb{R}).$$

We define the inner product (f, g) by

$$(f, g)_\mathfrak{N} = \int_{X_0(\mathfrak{N})} \overline{f(x)}g(x)|\det(x)|_\mathbb{A}^{[\kappa]-1}dx \qquad (5.2)$$

with respect to the invariant measure dx on X_0 as in [H91] page 342. In exactly the same manner as in [H91] (4.9), we obtain

$$D^s(4\pi)^{-I(s+1)-(\kappa_1-\kappa_2)}\Gamma_F((s+1)I+(\kappa_1-\kappa_2))\zeta_F^{(\mathfrak{N})}(2s+2)D(s,f,f)$$
$$= N(\mathfrak{N})^{-1}D^{-[\kappa]-2}(f, f\mathbb{E}_{0,0}(x,\mathbf{1},\mathbf{1};s+1))_{\mathfrak{N}},$$

where D is the discriminant $N(\mathfrak{d})$ of F, $\zeta_F^{(\mathfrak{N})}(s)=\zeta_F(s)\prod_{\mathfrak{q}|\mathfrak{N}}(1-N(\mathfrak{q})^{-s})$ for the Dedekind zeta function $\zeta_F(s)$ of F and $\mathbb{E}_{k,w}(x,\mathbf{1},\mathbf{1};s)$ ($k=\kappa_1-\kappa_2+I$ and $w=I-\kappa_2$) is the Eisenstein series of level $\mathfrak{N}$ defined above (4.8e) of [H91] for the identity characters $(\mathbf{1},\mathbf{1})$ in place of $(\chi^{-1}\psi^{-1},\theta)$ there.

By the residue formula at $s=1$ of $\zeta_F^{(\mathfrak{N})}(2s)\mathbb{E}_{0,0}(x,\mathbf{1},\mathbf{1};s)$ (e.g. (RES2) in [H99] page 173), we find

$$(4\pi)^{-I-(\kappa_1-\kappa_2)}\Gamma_F(I+(\kappa_1-\kappa_2))\mathrm{Res}_{s=0}\zeta_F^{(\mathfrak{N})}(2s+2)D(s,f,f)$$
$$= D^{-[\kappa]-2}N(\mathfrak{N})^{-1}\prod_{\mathfrak{q}|\mathfrak{N}}(1-N(\mathfrak{q})^{-1})\frac{2^{[F:\mathbb{Q}]-1}\pi^{[F:\mathbb{Q}]}R_\infty h(F)}{w\sqrt{D}}(f,f)_{\mathfrak{N}}, \tag{5.3}$$

where $w=2$ is the number of roots of unity in F, $h(F)$ is the class number of F and R_∞ is the regulator of F.

Since f corresponds to $v\otimes\epsilon_2$ for the new vector $v\in\Pi$ of the principal series representation $\Pi^{(\infty)}$ of minimal level in its twist class $\{\Pi\otimes\eta\}$ (η running over all finite order characters of $F^\times_{\mathbb{A}^{(\infty)}}$), by making product $\overline{f}\cdot f$, the effect of tensoring ϵ_2 disappears. Thus we may compute the Euler factor of $D(s,f,f)$ as if f were a new vector of the minimal level representation (which has the "Neben" character with conductor exactly equal to that of Π). Then for each prime factor $\mathfrak{q}|\mathfrak{N}$, the Euler $\mathfrak{q}$–factor of $\zeta_F^{(\mathfrak{N})}(2s+2)D(s,f,f)$ is given by

$$\sum_{\nu=0}^{\infty} a(\mathfrak{q}^\nu,f)\overline{a(\mathfrak{q}^\nu,f)}N(\mathfrak{q})^{-\nu s}=\left(1-N(\mathfrak{q})^{[\kappa]-s}\right)^{-1},$$

because $a(\mathfrak{q},f)\overline{a(\mathfrak{q},f)}=N(\mathfrak{q})^{[\kappa]}$ by [H88] Lemma 12.2. Thus the zeta function $\zeta_F^{(\mathfrak{N})}(2s+2)D(s,f,f)$ has the single Euler factor $(1-N(\mathfrak{q})^{-s-1})^{-1}$ at $\mathfrak{q}|\mathfrak{N}$, and the zeta function $\zeta_F(s+1)L(s+1,Ad(f))$ has its square $(1-N(\mathfrak{q})^{-s-1})^{-2}$ at $\mathfrak{q}|\mathfrak{N}$, because $L(s+1,Ad(f))$ contributes one more factor $(1-N(\mathfrak{q})^{-s-1})^{-1}$. The Euler factors outside $\mathfrak{N}$ are the same by the standard computation. Therefore, the left-hand-side of (5.3) is given by

$$\zeta_F^{(\mathfrak{N})}(2s+2)D(s,f,f)$$
$$=\left(\prod_{\mathfrak{q}|\mathfrak{N}}(1-N(\mathfrak{q})^{-s-1})\right)\zeta_F(s+1)L(s+1,Ad(f)) \tag{5.4}$$

By comparing the residue at $s = 0$ of (5.4) with (5.3) (in view of (5.1)), we get

$$(f_\pi^u, f_\pi^u)_{\mathfrak{N}} = D^{-[\kappa]-1}(f_\pi, f_\pi)_{\mathfrak{N}} = \\ D\Gamma_F((\kappa_1-\kappa_2)+I)N(\mathfrak{N})2^{-2((\kappa_1-\kappa_2)+I)+1}\pi^{-((\kappa_1-\kappa_2)+2I)}L(1, Ad(f)) \tag{5.5}$$

for the primitive adjoint square L–function $L(s, Ad(f))$ (e.g. [H99] 2.3). Here we have written $x^s = \prod_\sigma x^{s_\sigma}$ for $s = \sum_\sigma s_\sigma \sigma \in \mathbb{C}[I]$, and $\Gamma_F(s) = \prod_\sigma \Gamma(s_\sigma)$ for the Γ–function $\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}dt$. This formula is consistent with the one given in [HT1] Theorem 7.1 (but is much simpler).

5.2 Primitive p–Adic Rankin product

Let $\mathfrak{N}$ and $\mathfrak{J}$ be integral ideals of F prime to p. We shall use the notation introduced in 2.10. Thus, for a p–adically complete valuation ring $W \subset \widehat{\mathbb{Q}}_p$, $\mathbf{h}^{n.ord}(\mathfrak{N}, \psi; W)$ and $\mathbf{h}^{n.ord}(\mathfrak{J}, \chi; W)$ are the universal nearly ordinary Hecke algebra with level $(\mathfrak{N}, \psi)$ and $(\mathfrak{J}, \chi)$ respectively. The character $\psi = (\psi_1, \psi_2, \psi_+^t)$ is made of the characters of ψ_j of $T_0(O_p \times (O/\mathfrak{N}'^{(p)}))$ (for an ideal $\mathfrak{N}' \subset \mathfrak{N}$) of finite order and for the restriction ψ_+^t to $\Delta_F(\mathfrak{N})$ (the torsion part of $Cl_F^+(\mathfrak{N}'p^\infty)$) of a Hecke character ψ_+ extending $\psi_1\psi_2$. Similarly we regard χ as a character of $\mathbf{G}(\mathfrak{J}')$ for an ideal $\mathfrak{J}' \subset \mathfrak{J}$); so, $\psi^- = \psi_1^{-1}\psi_2$ and χ^- are well defined (finite order) character of $T_0(O_p \times (O/\mathfrak{N}))$ and $T_0(O_p \times (O/\mathfrak{J}))$ respectively. In particular we have $\mathfrak{C}^{(p)}(\psi^-)|\mathfrak{N}$ and $\mathfrak{C}^{(p)}(\chi^-)|\mathfrak{J}$, where $\mathfrak{C}^{(p)}(\psi^-)$ is the prime-to–p part of the conductor $\mathfrak{C}(\psi^-)$ of ψ^-. We assume that

$$\mathfrak{C}^{(p)}(\psi^-) = \mathfrak{N}, \text{ and } \mathfrak{C}^{(p)}(\chi^-) = \mathfrak{J}. \tag{5.6}$$

For the moment, we also assume for simplicity that

$$\psi_{\mathfrak{q}}^- \neq \chi_{\mathfrak{q}}^- \text{ on } O_{\mathfrak{q}}^\times \text{ for } \mathfrak{q}|\mathfrak{J}\mathfrak{N} \text{ and } \psi_1 = \chi_1 \text{ on } \widehat{O}^\times. \tag{5.7}$$

Let $\lambda : \mathbf{h}^{n.ord}(\mathfrak{N}, \psi; W) \to \Lambda$ and $\varphi : \mathbf{h}^{n.ord}(\mathfrak{J}, \chi; W) \to \Lambda'$ be $\mathbf{\Lambda}$–algebra homomorphisms for integral domains Λ and Λ' finite torsion-free over $\mathbf{\Lambda}$. For each arithmetic point $P \in \mathrm{Spf}(\Lambda)(\overline{\mathbb{Q}}_p)$, we let $f_P \in S_{\kappa(P)}(U_0(\mathfrak{N}p^\alpha), \psi_P; \overline{\mathbb{Q}}_p)$ denote the normalized Hecke eigenform of minimal level belonging to λ. In other words, for $\lambda_P = P \circ \lambda : \mathbf{h}^{n.ord} \to \overline{\mathbb{Q}}_p$, we have $a(y, f_P) = \lambda_P(T(y))$ for all integral ideles y with $y_p = 1$. In the automorphic representation generated by f_P, we can find a unique automorphic form f_P^{ord} with $a(y, f_P^{ord}) = \lambda(T(y))$ for all y, which we call the (nearly) *ordinary projection* of f_P. Similarly, using φ, we define $g_Q \in S_{\kappa(Q)}(U_0(\mathfrak{J}p^\beta), \chi_Q; \overline{\mathbb{Q}}_p)$ for each arithmetic point $Q \in \mathrm{Spf}(\Lambda')(\overline{\mathbb{Q}}_p)$. Recall that we have two characters $(\psi_{P,1}, \psi_{P,2})$

of $T_0(\widehat{O})$ associated to ψ_P. Recall $\psi_P = (\psi_{P,1}, \psi_{P,2}, \psi_{P+}) : T_0(\widehat{O})^2 \times (F_{\mathbb{A}}^\times/F^\times) \to \mathbb{C}^\times$. The central character ψ_{P+} of f_P coincides with $\psi_{P,1}\psi_{P,2}$ on $\widehat{O}^\times$ and has infinity type $(1-[\kappa(P)])I$. We suppose

$$\text{The character } \psi_{P,1}\chi_{Q,1}^{-1} \text{ is induced by a global finite order character } \theta. \quad (5.8)$$

This condition combined with (5.6) implies that θ is unramified outside p. As seen in [H91] 7.F, we can find an automorphic form $g_Q|\theta^{-1}$ on $G(\mathbb{A})$ whose Fourier coefficients are given by $a(y, g_Q|\theta^{-1}) = a(y, g_Q)\theta^{-1}(yO)$, where $\theta(\mathfrak{a}) = 0$ if $\mathfrak{a}$ is not prime to $\mathfrak{C}(\theta)$. The above condition implies, as explained in the previous subsection,

$$y \mapsto a(y, f_P)\overline{a(y, g_Q|\theta^{-1})\theta(y)}$$

factors through the ideal group of F. Note that

$$a(y, f_P)\overline{a(y, g_Q|\theta^{-1})\theta(y)} = a(y, f_P)\overline{a(y, g_Q)}$$

as long as y_p is a unit. We thus write $a(\mathfrak{a}, f_P)\overline{a(\mathfrak{a}, g_Q|\theta^{-1})\theta(\mathfrak{a})}$ for the above product when $yO = \mathfrak{a}$ and define

$$D(s - \frac{[\kappa(P)] + [\kappa(Q)]}{2} - 1, f_P, g_Q|\theta^{-1}, \theta^{-1}) = \sum_{\mathfrak{a}} a(\mathfrak{a}, f_P)\overline{a(\mathfrak{a}, g_Q|\theta^{-1})\theta(\mathfrak{a})}N(\mathfrak{a})^{-s}.$$

Hereafter we write $\kappa = \kappa(P)$ and $\kappa' = \kappa(Q)$ if confusion is unlikely.

Note that for $g'_Q(x) = g_Q|\theta^{-1}(x)\theta(\det(x))$,

$$D(s, f_P, g'_Q) := D(s, f_P, g'_Q, \mathbf{1}) = D(s, f_P, g_Q|\theta^{-1}, \theta^{-1}).$$

Though the introduction of the character θ further complicates our notation, we can do away with it just replacing g_Q by g'_Q, since the local component $\pi(\chi'_{Q,1,\mathfrak{q}}, \chi'_{Q,2,\mathfrak{q}})$ of the automorphic representation generated by g'_Q satisfies $\chi'_{1,Q} = \psi_{1,P}$, and hence without losing much generality, we may assume a slightly stronger condition:

$$\psi_{P,1} = \chi_{Q,1} \text{ on } \widehat{O}^\times \quad (5.9)$$

in our computation.

For each holomorphic Hecke eigenform f, we write $M(f)$ for the rank 2 motive attached to f (see [BR]), $\widetilde{M}(f)$ for its dual, ρ_f for the $\mathfrak{p}$–adic Galois representation of $M(f)$ and $\widetilde{\rho}_f$ for the contragredient of ρ_f. Here $\mathfrak{p}$ is the p–adic place of the Hecke field of f induced by $i_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. Thus $L(s, M(f))$ coincides with the standard L–function of the automorphic representation generated by f, and the Hodge weight of $M(f_P)$ is given by

$\{(\kappa_{1,\sigma},\kappa_{2,\sigma}),(\kappa_{1,\sigma},\kappa_{2,\sigma})\}_\sigma$ for each embedding $\sigma : F \hookrightarrow \mathbb{C}$. We have $\det(\rho_{f_P}(Frob_{\mathfrak{q}})) = \psi_P^u(\mathfrak{q})N(\mathfrak{q})^{[\kappa]}$ ($\mathfrak{p} \neq \mathfrak{q}$; see [MFG] 5.6.1).

Lemma 5.2 *Suppose* (5.6) *and* (5.8). *For primes* $\mathfrak{q} \nmid p$, *the Euler* $\mathfrak{q}$*–factor of*

$$L^{(\mathfrak{N}\mathfrak{J})}(2s-[\kappa]-[\kappa'],\psi_P^u\chi_Q^{-u})D(s-\frac{[\kappa]+[\kappa']}{2}-1,f_P,g_Q')$$

is equal to the Euler $\mathfrak{q}$*–factor of* $L_{\mathfrak{q}}(s,M(f_P)\otimes\widetilde{M}(g_Q))$ *given by*

$$\det\left(1-(\rho_{f_P}\otimes\widetilde{\rho}_{g_Q})(Frob_{\mathfrak{q}})\big|_{V^I}N(\mathfrak{q})^{-s})\right)^{-1},$$

where V *is the space of the* $\mathfrak{p}$*–adic Galois representation of the tensor product:* $\rho_{f_P}\otimes\widetilde{\rho}_{g_Q}$ *and* $V^I = H^0(I,V)$ *for the inertia group* $I \subset \mathrm{Gal}(\overline{\mathbb{Q}}/F)$ *at* $\mathfrak{q}$.

Proof As already explained, we may assume (5.9) instead of (5.8). By abusing the notation, we write $\pi(\psi_{P,1,\mathfrak{q}},\psi_{P,2,\mathfrak{q}})$ (resp. $\pi(\chi_{Q,1,\mathfrak{q}},\chi_{Q,2,\mathfrak{q}})$) for the $\mathfrak{q}$–factor of the representation generated by f_P (resp. g_Q). By the work of Carayol, R. Taylor and Blasius-Rogawski combined with a recent work of Blasius [B], the restriction of ρ_{f_P} to the Decomposition group at $\mathfrak{q}$ is isomorphic to $\mathrm{diag}[\psi_{P,1,\mathfrak{q}},\psi_{P,2,\mathfrak{q}}]$ (regarding $\psi_{i,P,\mathfrak{q}}$ as Galois characters by local class field theory). The same fact is true for g_Q. If $\mathfrak{q}|\mathfrak{J}\mathfrak{N}$, then V^I is one dimensional on which $Frob_{\mathfrak{q}}$ acts by $\psi_{P,1,\mathfrak{q}}(\varpi_{\mathfrak{q}})\overline{\chi}_{Q,1,\mathfrak{q}}(\varpi_{\mathfrak{q}}) = a(\mathfrak{q},f_P)\overline{a(\mathfrak{q},g_Q)}$ because $\psi_{i,P,\mathfrak{q}}\overline{\chi}_{j,Q,\mathfrak{q}}$ is ramified unless $i=j=1$ ($\Leftrightarrow \psi_1=\chi_1$ on $\widehat{O}^\times$ and $\psi_{\mathfrak{q}}^- \neq \psi_{\mathfrak{q}}^-$). If $\mathfrak{q} \nmid \mathfrak{J}\mathfrak{N}$, both $\pi(\psi_{P,1,\mathfrak{q}},\psi_{P,2,\mathfrak{q}})\otimes\psi_{P,1,\mathfrak{q}}^{-1} = \pi(1,\psi_{P,\mathfrak{q}}^-)$ and $\pi(\chi_{Q,1,\mathfrak{q}},\chi_{Q,2,\mathfrak{q}})\otimes\chi_{Q,1,\mathfrak{q}}^{-1} = \pi(1,\chi_{Q,\mathfrak{q}}^-)$ are unramified principal series. By $\psi_{P,1,\mathfrak{q}} = \chi_{Q,1,\mathfrak{q}}$:(5.8), we have an identity:

$$\rho_{f_P}\otimes\rho_{g_Q^c} \cong (\rho_{f_P}\otimes\psi_{P,1,\mathfrak{q}}^{-1})\otimes(\rho_{g_Q^c}\otimes\chi_{Q,1,\mathfrak{q}})$$

on the inertia group, which is unramified. Therefore V is unramified at $\mathfrak{q}$. At the same time, the L–function has full Euler factor at $\mathfrak{q} \nmid \mathfrak{J}\mathfrak{N}$. □

We would like to compute $f_P|\tau(x) := \psi_P^u(\det(x))^{-1}f_P(x\tau)$ for $\tau(N) = \left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right) \in G(\mathbb{A}^{(\infty)})$ for an idele $N = N(P)$ with $N^{(p\mathfrak{N})} = 1$ and $NO = \mathfrak{C}(\psi_P^-)$ (whose prime-to–p factor is $\mathfrak{N}$). We continue to abuse notation and write $\pi(\psi_{P,1,\mathfrak{q}},\psi_{P,2,\mathfrak{q}})\otimes\psi_{P,1,\mathfrak{q}}^{-1}$ as $\pi(1,\psi_{P,\mathfrak{q}}^-)$ (thus $\psi_{P,\mathfrak{q}}^-$ is the character of $F_{\mathfrak{q}}^\times$ inducing the original $\psi_{P,\mathfrak{q}}^-$ on $O_{\mathfrak{q}}^\times$). We write $(\psi_{P,\mathfrak{q}}^-)^u = \psi_{P,\mathfrak{q}}^-/|\psi_{P,\mathfrak{q}}^-|$ (which is a unitary character). In the Whittaker model $V(1,\psi_{P,\mathfrak{q}}^-)$ of $\pi(1,\psi_{P,\mathfrak{q}}^-)$ (realized in the space of functions on $GL_2(F_{\mathfrak{q}})$), we have a unique function $\phi_{\mathfrak{q}}$ on $GL_2(F_{\mathfrak{q}})$ whose Mellin transform gives rise to the local L–function of $\pi(1,\psi_{\mathfrak{q}}^-)$. In particular, we have (cf. [H91] (4.10b))

$$\phi_{\mathfrak{q}}|\tau_{\mathfrak{q}}(x) := (\psi_P^-)^u(\det(x))^{-1}\phi_{\mathfrak{q}}(x\tau_{\mathfrak{q}}) = W(\phi_{\mathfrak{q}})|\psi_P^-(N_{\mathfrak{q}})|^{1/2}\overline{\phi}_{\mathfrak{q}}(x),$$

where $\overline{\phi}_{\mathfrak{q}}$ is the complex conjugate of $\phi_{\mathfrak{q}}$ belonging to the space representation $V(1, \overline{\psi}^{-}_{P,\mathfrak{q}})$ and $W(\phi_{\mathfrak{q}})$ is the epsilon factor of the representation $\pi(1, \psi^{-}_{P,\mathfrak{q}})$ as in [H91] (4.10c) (so, $|W(\phi_{\mathfrak{q}})| = 1$). Then $\phi_{\mathfrak{q}} \otimes \psi_{P,1,\mathfrak{q}}(x) := \psi_{P,1,\mathfrak{q}}(\det(x))\phi_{\mathfrak{q}}(x)$ is in $V(\psi_{P,1,\mathfrak{q}}, \psi_{P,2,\mathfrak{q}})$ and gives rise to the $\mathfrak{q}$–component of the global Whittaker model of the representation π generated by f_P. The above formula then implies

$$\begin{aligned}(\phi_{\mathfrak{q}} \otimes \psi_{P,1,\mathfrak{q}})|\tau_{\mathfrak{q}} &:= \psi^u_P(\det(x))^{-1}(\phi_{\mathfrak{q}} \otimes \psi_{P,1,\mathfrak{q}})(x\tau_{\mathfrak{q}}) \\ &= \psi^u_{P,1,\mathfrak{q}}(N_{\mathfrak{q}})|N_{\mathfrak{q}}|_{\mathfrak{q}}^{(1-[\kappa])/2} W(\phi_{\mathfrak{q}})(\overline{\phi}_{\mathfrak{q}} \otimes \overline{\psi}_{P,1,\mathfrak{q}})(x).\end{aligned}$$

Define the root number $W_{\mathfrak{q}}(f_P) = W(\phi_{\mathfrak{q}})$ and $W(f_P) = \prod_{\mathfrak{q}} W_{\mathfrak{q}}(f_P)$. Here note that $W_{\mathfrak{q}}(f_P) = 1$ if the prime $\mathfrak{q}$ is outside $\mathfrak{C}(\psi_{P,1})\mathfrak{C}(\psi_{P,2})D$. We conclude from the above computation the following formula:

$$f_P|\tau(x) := \psi^u_P(\det(x))^{-1} f_P(x\tau) = W(f_P)\psi^u_{P,1}(N)|N|_{\mathbb{A}}^{(1-[\kappa])/2} f^c_P(x), \tag{5.10}$$

where f^c_P is determined by $a(y, f^c_P) = \overline{a(y, f_P)}$ for all $y \in F^{\times}_{\mathbb{A}}$. This shows

$$W(f_P)W(f^c_P) = \psi^u_{P,\infty}(-1) = \psi^{+}_{P,\infty}(-1). \tag{5.11}$$

Using the formula (5.10) instead of [H91] (4.10b), we can prove in exactly the same manner as in [H91] Theorem 5.2 the following result:

Theorem 5.3 *Suppose* (5.6) *and* (5.7)*. There exists a unique element $\mathcal{D}$ in the field of fractions of $\Lambda\widehat{\otimes}_W\Lambda'$ satisfying the following interpolation property: Let $(P, Q) \in \mathrm{Spf}(\Lambda) \times \mathrm{Spf}(\Lambda')$ be an arithmetic point such that*

(W) $\kappa_1(P) - \kappa_1(Q) > 0 \geq \kappa_2(P) - \kappa_2(Q)$ *and* $\psi_{P,1} = \chi_{Q,1}$ *on* $\widehat{O}^{\times}$.

Then $\mathcal{D}$ is finite at (P, Q) and we have

$$\mathcal{D}(P, Q) = W(P, Q)C(P, Q)S(P)^{-1}E(P, Q)\frac{L^{(p)}(1, M(f_P) \otimes \widetilde{M}(g_Q))}{(f_P, f_P)},$$

where, writing $k(P) = \kappa_1(P) - \kappa_2(P) + I$,

$$W(P, Q) = \frac{(-1)^{k(Q)}}{(-1)^{k(P)}} \frac{N(\mathfrak{J})^{([\kappa(Q)]+1)/2}}{N(\mathfrak{N})^{([\kappa(P)]-1)/2}} \cdot \prod_{\mathfrak{p}|p} \frac{\chi^u_Q(d_{\mathfrak{p}})G(\chi^{-1}_{Q,1,\mathfrak{p}}\psi_{P,1,\mathfrak{p}})G(\chi^{-1}_{Q,2,\mathfrak{p}}\psi_{P,1,\mathfrak{p}})}{\psi^u_P(d_{\mathfrak{p}})G((\psi^{-}_{P,\mathfrak{p}})^{-1})}$$

$$\begin{aligned}C(P, Q) &= 2^{([\kappa(P)]-[\kappa(Q)])I-2k(P)}\pi^{2\kappa_2(P)-([\kappa(Q)]+1)I} \\ &\quad \times \Gamma_F(\kappa_1(Q) - \kappa_2(P) + I)\Gamma_F(\kappa_2(Q) - \kappa_2(P) + I),\end{aligned}$$

$$S(P) = \prod_{\mathfrak{p} \nmid \mathfrak{C}_p(\psi_P^-)} \left(\psi_P^-(\varpi_{\mathfrak{p}}) - 1\right) \left(1 - \psi_P^-(\varpi_{\mathfrak{p}})|\varpi_{\mathfrak{p}}|_{\mathfrak{p}}\right) \prod_{\mathfrak{p} | \mathfrak{C}_p(\psi_P^-)} (\psi_P^-(\varpi_{\mathfrak{p}})|\varpi_{\mathfrak{p}}|_{\mathfrak{p}})^{\delta(\mathfrak{p})},$$

$$E(P,Q) = \prod_{\mathfrak{p} \nmid \mathfrak{C}_p(\chi_Q^-)} \frac{(1 - \chi_{Q,1}\psi_{P,1}^{-1}(\varpi_{\mathfrak{p}}))(1 - \chi_{Q,2}\psi_{P,1}^{-1}(\varpi_{\mathfrak{p}}))}{(1 - \chi_{Q,1}^{-1}\psi_{P,1}(\varpi_{\mathfrak{p}})|\varpi_{\mathfrak{p}}|_{\mathfrak{p}})(1 - \chi_{Q,2}^{-1}\psi_{P,1}(\varpi_{\mathfrak{p}})|\varpi_{\mathfrak{p}}|_{\mathfrak{p}})} \times \prod_{\mathfrak{p} | \mathfrak{C}_p(\chi_Q^-)} \frac{\chi_{Q,2}\psi_{P,1}^{-1}(\varpi_{\mathfrak{p}}^{\gamma(\mathfrak{p})})(1 - \chi_{Q,1}\psi_{P,1}^{-1}(\varpi_{\mathfrak{p}}))}{(1 - \chi_{Q,1}^{-1}\psi_{P,1}(\varpi_{\mathfrak{p}})|\varpi_{\mathfrak{p}}|_{\mathfrak{p}})}.$$

Here $\mathfrak{C}_p(\psi_P^-) := \mathfrak{C}(\psi_P^-) + (p) = \prod_{\mathfrak{p}|p} \mathfrak{p}^{\delta(\mathfrak{p})}$ *and* $\mathfrak{C}_p(\chi_P^-) := \mathfrak{C}(\chi_Q^-) + (p) = \prod_{\mathfrak{p}|p} \mathfrak{p}^{\gamma(\mathfrak{p})}$. *Moreover for the congruence power series* $H(\lambda)$ *of* λ, $H(\lambda)\mathcal{D} \in \Lambda\widehat{\otimes}_W\Lambda'$.

The expression of p–Euler factors and root numbers is simpler than the one given in [H91] Theorem 5.1, because automorphic representations of f_P and g_Q are everywhere principal at finite places (by (5.6)). The shape of the constant $W(P,Q)$ appears to be slightly different from [H91] Theorem 5.2. Firstly the present factor $(-1)^{k(P)+k(Q)}$ is written as $(\chi_{Q+}\psi_{P+})_\infty(-1)$ in [H91]. Secondly, in [H91], it is assumed that $\chi_{Q,1}^{-1}$ and $\psi_{P,1}^{-1}$ are both induced by a global character ψ'_P and ψ'_P unramified outside p. Thus the factor $(\chi'_{Q,\infty}\psi'_{P,\infty})(-1)$ appears there. This factor is equal to $(\chi_{Q,1,p}\psi_{P,1,p})(-1) = \theta_p(-1)$, which is trivial because of the condition (W). We do not need to assume the individual extensibility of $\chi_{Q,1}$ and $\psi_{P,1}$. This extensibility is assumed in order to have a global Hecke eigenform $f_P^\circ = f_P^u \otimes \psi'_P$. However this assumption is redundant, because all computation we have done in [H91] can be done locally using the local Whittaker model. Also $C(P,Q)$ in the above theorem is slightly different from the one in [H91] Theorem 5.2, because $(f_P, f_P) = D^{[\kappa(P)]+1}(f_P^\circ, f_P^\circ)$ for f_P° appearing in the formula of [H91] Theorem 5.2.

Proof We start with a slightly more general situation. We shall use the symbol introduced in [H91]. Suppose $\mathfrak{C}(\psi^-)|\mathfrak{N}$ and $\mathfrak{C}(\chi^-)|\mathfrak{J}$, and take normalized Hecke eigenforms $f \in S_\kappa(\mathfrak{N}, \psi_+; \mathbb{C})$ and $g \in S_{\kappa'}(\mathfrak{J}, \chi_+; \mathbb{C})$. Suppose $\psi_1 = \chi_1$. We define $f^c \in S_\kappa(\mathfrak{N}, \overline{\psi}_+; \mathbb{C})$ by $a(y, f^c) = \overline{a(y,f)}$. Then $f^c(w) = \overline{f(\varepsilon^{-1}w\varepsilon)}$ for $\varepsilon = \left(\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right)$. We put $\Phi(w) = (\overline{f^c}g^c)(w)$. Then we see $\Phi(wu) = \psi(u)\chi^{-1}(u)\Phi(w)$ for $u \in U = U_0(\mathfrak{N}') \cap U_0(\mathfrak{J}')$. Since $\psi_1 = \chi_1$, we find that $\psi(u)\chi(u)^{-1} = \psi^-(\chi^-)^{-1}(d) = \psi^u(\chi^u)^{-1}(d)$ if

$u = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. We write simply ω for the central character of $\overline{f}^c g^c$, which is the Hecke character $\psi_+^u(\chi_+^u)^{-1}|\cdot|_{\mathbb{A}}^{-[\kappa']-[\kappa]}$. Then we have $\Phi(zw) = \omega(z)\Phi(w)$, and $\Phi(wu_\infty) = \overline{J_\kappa(u_\infty, \mathbf{i})}^{-1} J_{\kappa'}(u_\infty, \mathbf{i})^{-1}\Phi(w)$. We then define $\omega^*(w) = \omega(d_{\mathfrak{N}'\mathfrak{J}'})$ for $w = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in B(\mathbb{A})U \cdot G(\mathbb{R})^+$. Here B is the algebraic subgroup of G made of matrices of the form $\left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right)$. We extend $\omega^* : G(\mathbb{A}) \to \mathbb{C}$ outside $B(\mathbb{A})U \cdot G(\mathbb{R})^+$ just by 0. Similarly we define $\eta : G(\mathbb{A}) \to \mathbb{C}$ by

$$\eta(w) = \begin{cases} |y|_{\mathbb{A}} & \text{if } g = \left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right) zu \text{ with } z \in F_{\mathbb{A}}^\times \text{ and } u \in U \cdot SO_2(F_{\mathbb{R}}) \\ 0 & \text{otherwise.} \end{cases}$$

For each $\mathbb{Q}$–subalgebra $A \subset \mathbb{A}$, we write $B(A)_+ = B(A) \cap G(\mathbb{A}^{(\infty)}) \times G(\mathbb{R})^+$. Note that $\Phi(w)\overline{\omega}^*(w)\eta(w)^{s-1}$ for $s \in \mathbb{C}$ is left invariant under $B(\mathbb{Q})_+$. Then we compute

$$\mathcal{Z}(s, f, g) = \int_{B(\mathbb{Q})_+ \backslash B(\mathbb{A})_+} \Phi(w)\overline{\omega}^*(w)\eta(w)^{s-1} d\varphi_B(w)$$

for the measure $\varphi_B\left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right) = |y|_{\mathbb{A}}^{-1} dx \otimes d^\times y$ defined in [H91] page 340. We have

$$\begin{aligned} &\mathcal{Z}(s, f, g) \\ &= \int_{F_{\mathbb{A}+}^\times} \int_{F_{\mathbb{A}}/F} \Phi\left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right) dx |y|_{\mathbb{A}}^{s-1} d^\times y \\ &= D^{\frac{1}{2}} \int_{F_{\mathbb{A}}^\times} a(dy, f)\overline{a(dy, g)} \mathbf{e}_F(2\sqrt{-1}y_\infty) y_\infty^{-(\kappa_2+\kappa_2')} |y|_{\mathbb{A}}^s d^\times y \\ &\overset{dy \mapsto y}{=} D^{s+\frac{1}{2}} \int_{F_{\mathbb{A}}^\times} a(y, f)\overline{a(y, g)} \mathbf{e}_F(2\sqrt{-1}y_\infty) y_\infty^{-(\kappa_2+\kappa_2')} |y|_{\mathbb{A}}^s d^\times y \\ &= D^{s+\frac{1}{2}} (4\pi)^{-sI+\kappa_2+\kappa_2'} \Gamma_F(sI - \kappa_2 - \kappa_2') D(s - \frac{[\kappa]+[\kappa']}{2} - 1, f, g). \end{aligned}$$

Define $C_{\infty+} \subset G(\mathbb{R})^+$ by the stabilizer in $G(\mathbb{R})^+$ of $\mathbf{i} \in \mathfrak{Z}$. We now choose an invariant measure φ_U on $X(U) = G(\mathbb{Q})_+ \backslash G(\mathbb{A})_+ / UC_{\infty+}$ so that

$$\int_{X(U)} \sum_{\gamma \in O^\times B(\mathbb{Q})_+ \backslash G(\mathbb{Q})_+} \phi(\gamma w) d\varphi_U(w) = \int_{B(\mathbb{Q})_+ \backslash B(\mathbb{A})_+} \phi(b) d\varphi_B(b)$$

whenever ϕ is supported on $B(\mathbb{A}^{(\infty)})u \cdot G(\mathbb{R})^+$ and the two integrals are absolutely convergent. There exists a unique invariant measure φ_U as seen in [H91] page 342 (where the measure is written as μ_U). On $B(\mathbb{A})_+$, $\Phi(w) = \overline{f}^c g^c(w)\overline{J_\kappa(w, \mathbf{i})} J_{\kappa'}(w, \mathbf{i})$ and the right-hand-side is left $C_{\infty+}$ invariant (cf.

(S2) in 2.10). Then by the definition of φ_U, we have

$$\int_{B(\mathbb{Q})_+\backslash B(\mathbb{A})_+} \Phi\overline{\omega}^*\eta^{s-1}d\varphi_B = \int_{X(U)} \overline{f}^c(w)g^c(w)E(w,s-1)d\varphi_U(w), \tag{5.12}$$

where

$$E(w,s) = \sum_{\gamma\in O^\times B(\mathbb{Q})_+\backslash G(\mathbb{Q})_+} \overline{\omega}^*(\gamma w)\eta(\gamma w)^s\overline{J_\kappa(\gamma w,\mathbf{i})}J_{\kappa'}(\gamma w,\mathbf{i}).$$

Note that $E(zw,s) = (\psi_+{}^{-u}\chi_+^u)(z)E(w,s)$ for $z \in \widehat{O}^\times$. By definition, $E(\alpha x) = E(x)$ for $\alpha \in G(\mathbb{Q})_+$; in particular, it is invariant under $\alpha \in F^\times$. For $z \in F_\mathbb{R}^\times$, one has $E(zw,s) = N(z)^{[\kappa]+[\kappa']-2}E(w,s)$. It follows that $|\det(w)|_\mathbb{A}^{1-([\kappa]+[\kappa])/2}E(w,s)$ has eigenvalue $\psi_+^{-u}\chi_+^u(z)$ under the central action of $z \in F^\times\widehat{O}^\times F_\mathbb{R}^\times$. The averaged Eisenstein series:

$$\begin{aligned}\mathcal{E}(w,s) &= \sum_{a\in Cl_F} \psi_+^u\chi_+^{-u}(a)|\det(aw)|_\mathbb{A}^{1-([\kappa]+[\kappa'])/2}E(aw,s)\\ &= |\det(w)|_\mathbb{A}^{1-([\kappa]+[\kappa'])/2}\sum_{a\in Cl_F}\psi_+\overline{\chi}_+(a)E(aw,s)\end{aligned}$$

satisfies $\mathcal{E}(zw,s) = \psi_+^{-u}\chi_+^u(z)\mathcal{E}(w,s)$, where a runs over complete representative set for $F_\mathbb{A}^\times/F^\times\widehat{O}^\times F_\mathbb{R}^\times$ and ψ_+ is the central character of f and $\overline{\chi}_+$ is the central character of g^c. Defining the PGL_2 modular variety $\overline{X}(U) = X(U)/F_\mathbb{A}^\times$, by averaging (5.12), we find

$$\begin{aligned}&D^{s+\frac{1}{2}}(4\pi)^{-sI+\kappa_2+\kappa_2'}\Gamma_F(sI-\kappa_2-\kappa_2')D(s-\frac{[\kappa]+[\kappa']}{2}-1,f,g)\\ &\quad= \int_{\overline{X}(U)}\overline{f}^c(w)g^c(w)\mathcal{E}(w,s-1)|\det(w)|_\mathbb{A}^{([\kappa]+[\kappa'])/2-1}d\varphi_U(w).\end{aligned} \tag{5.13}$$

Writing $U = U_0(\mathfrak{L})$ and writing $r = \kappa_2'-\kappa_2$, we define an Eisenstein series $\mathbb{E}_{k-k',r}(\overline{\omega}^u,\mathbf{1};s)$ by

$$N(\mathfrak{L})^{-1}\sqrt{D}|\det(w)|_\mathbb{A}^{\frac{[\kappa']-[\kappa]}{2}}L^{(\mathfrak{L})}(2s,\omega^u)\mathcal{E}(w,s+\frac{[\kappa]+[\kappa']}{2}-1),$$

where $k = \kappa_1-\kappa_2$ and $k' = \kappa_1'-\kappa_2'$. The ideal $\mathfrak{L}$ is given by $\mathfrak{N}\cap\mathfrak{J}$. Then, changing variable $s - \frac{[\kappa]+[\kappa']}{2} - 1 \mapsto s$, we can rewrite (5.13) as

$$\begin{aligned} &D^{s+\frac{1}{2}}(4\pi)^{-sI-\frac{k+k'}{2}}\Gamma_F(sI + \frac{k+k'}{2})L^{(\mathfrak{L})}(2s+2, \psi^u\chi^{-u})D(s,f,g) \\ &= N(\mathfrak{L})^{-1}D^{-(3+[\kappa]+[\kappa'])/2}(f^c, g^c\mathbb{E}_{k-k',r}(\overline{\varpi}^u, \mathbf{1}; s+1))_{\mathfrak{L}} \\ &= N(\mathfrak{L})^{-s-\frac{[\kappa]+[\kappa']}{2}}D^{-(3+[\kappa]+[\kappa'])/2}(f^c|\tau, (g^c|\tau)\mathbf{G}_{k-k',r}(\omega^u, \mathbf{1}; s+1))_{\mathfrak{L}}, \end{aligned} \tag{5.14}$$

where $\mathbf{G}_{k-k',r}(\omega^u, \mathbf{1}; s) = N(\mathfrak{L})^{s-1+\frac{[\kappa']-[\kappa]}{2}}\mathbb{E}_{k-k',r}(\overline{\varpi}^u, \mathbf{1}; s)|\tau$ for τ of level $\mathfrak{L}$, and

$$(\phi, \varphi)_{\mathfrak{L}} = \int_{\overline{X}(U)} \overline{\phi}(w)\varphi(w)|\det(w)|_{\mathbb{A}}^{[\kappa]-1}d\varphi_U(w)$$

is the normalized Petersson inner product on $S_\kappa(U, \psi; \mathbb{C})$. This formula is equivalent to the formula in [H91] (4.9) (although we have more general forms f and g with character ψ and χ not considered in [H91]). In [H91] (4.9), k' is written as κ and r is written as $w - \omega$.

Let E be the Eisenstein measure of level $\mathfrak{L} = \mathfrak{N} \cap \mathfrak{J}$ defined in [H91] Section 8, where $\mathfrak{L}$ is written as L. We take an idele L with $LO = \mathfrak{L}$ and $L^{(\mathfrak{L})} = 1$. Similarly we take ideles J and N replacing in the above formula $\mathfrak{L}$ by $\mathfrak{J}$ and $\mathfrak{N}$, respective;y, and L by the corresponding J and N, respectively.

The algebra homomorphism $\varphi : \mathbf{h}^{n.ord}(\mathfrak{J}, \chi; W) \to \Lambda'$ induces, by the W–duality, $\varphi^* : \Lambda'^* \hookrightarrow \mathbf{S}^{n.ord}(\mathfrak{J}, \chi; W)$, where $\mathbf{S}^{n.ord}(\mathfrak{J}, \chi; W)$ is a subspace of p–adic modular forms of level $(\mathfrak{J}, \chi)$ (see [H96] 2.6). We then consider the convolution as in [H91] Section 9 (page 382):

$$\mathcal{D} = \lambda * \varphi = \frac{E *_\lambda ([L/J] \circ \varphi^*)}{H(\lambda) \otimes 1} \quad \text{for } E *_\lambda ([L/J] \circ \varphi^*) \in \Lambda\widehat{\otimes}\Lambda',$$

where $[L/J]$ is the operator defined in [H91] Section 7.B and all the ingredient of the above formula is as in [H91] page 383. An important point here is that we use the congruence power series $H(\lambda) \in \Lambda$ (so $H(\lambda) \otimes 1 \in \Lambda\widehat{\otimes}\Lambda'$) defined with respect to $\mathbf{h}^{n.ord}(\mathfrak{N}, \psi; W)$ instead of $\mathbf{h}(\psi^u, \psi_1)$ considered in [H91] page 379 (so, $H(\lambda)$ is actually a factor of H in [H91] page 379, which is an improvement).

We write the minimal level of f_P^{ord} as $\mathfrak{N}\mathfrak{p}^\alpha$ for $\mathfrak{p}^\alpha = \prod_{\mathfrak{p}|p} \mathfrak{p}^{\alpha(\mathfrak{p})}$. Then we define $\varpi^\alpha = \prod_{\mathfrak{p}|p} \varpi_{\mathfrak{p}}^{\alpha(\mathfrak{p})}$. The integer $\alpha(\mathfrak{p})$ is given by the exponent of $\mathfrak{p}$ in $\mathfrak{C}(\psi_P^-)$ or 1 whichever larger. We now compute $\mathcal{D}(P, Q)$. We shall give the argument only when $j = [\kappa(P)] - [\kappa(Q)] \geq 1$, since the other case can be treated in the same manner as in [H91] Case II (page 387). Put $\mathbf{G} = \mathbf{G}_{jI,0}\left(\chi_Q^{-u}\psi_P^u, \mathbf{1}; 1 - \frac{j}{2}\right)$. We write $(\cdot,\cdot)_{\mathfrak{N}\mathfrak{p}^\alpha} = (\cdot,\cdot)_\alpha$ and put $\mathfrak{m} = L\mathfrak{p}^\alpha$.

As before, $m = L\varpi^\alpha$ satisfies $mO = \mathfrak{m}$ and $m^{(\mathfrak{m})} = 1$. Put $r(P,Q) = \kappa_2(Q) - \kappa_2(P)$, which is non-negative by the weight condition (W) in the theorem. Then in exactly the same manner as in [H91] Section 10 (page 386), we find, for $c = (2\sqrt{-1})^{j[F:\mathbb{Q}]}\pi^{[F:\mathbb{Q}]}$,

$$\begin{aligned}(-1)^{k(Q)}N(\mathfrak{J}/\mathfrak{L})^{-1}N(\mathfrak{L}\mathfrak{p}^\alpha)^{[\kappa(Q)]-1}c((f_P^{ord})^c|\tau(N\varpi^\alpha), f_P^{ord})_\alpha\mathcal{D}(P,Q)\\ = ((f_P^{ord})^c|\tau(m), (g_Q^{ord}|\tau(J\varpi^\alpha)|\tau(m))\cdot(\delta_{jI}^{r(P,Q)}\mathbf{G}))_{\mathfrak{m}}. \quad (5.15)\end{aligned}$$

By [H91] Corollary 6.3, we have, for $r = r(P,Q)$,

$$\delta_{jI}^r\mathbf{G} = \Gamma_F(r+I)(-4\pi)^{-r}\mathbf{G}_{jI+2r,r}\left(\chi_Q^{-u}\psi_P^u, \mathbf{1}; 1-\frac{j}{2}\right).$$

Then by (5.14), we get

$$\begin{aligned}c(-1)^{k(Q)}N(\mathfrak{J}\mathfrak{p}^\alpha)^{-1}C(P,Q)^{-1}(f_P^{ord}|\tau(N\varpi^\alpha), f_P^{ord})_\alpha\mathcal{D}(P,Q)\\ = L^{(\mathfrak{m})}(2-[\kappa(P)]+[\kappa(Q)], \omega_{P,Q})\\ \times D(\frac{[\kappa(Q)]-[\kappa(P)]}{2}, f_P^{ord}, (g_Q^{ord}|\tau(J\varpi^\alpha))^c), \quad (5.16)\end{aligned}$$

where $\omega_{P,Q} = \chi_{Q+}^{-1}\psi_{P+}$ for the central characters χ_{Q+} of g_Q and ψ_{P+} of f_P.

Now we compute the Petersson inner product $(f_P^{ord}|\tau(N\varpi^\alpha), f_P^{ord})_\alpha$ in terms of (f_P, f_P). Note that for $f, g \in S_\kappa(U_0(N), \epsilon; \mathbb{C})$

$$(f^u)|\tau(N) = |N|_{\mathbb{A}}^{([\kappa]-1)/2}(f|\tau(N))^u \text{ and } (f,g)_{\mathfrak{N}} = D^{[\kappa]+1}(f^u, g^u). \quad (5.17)$$

The computation we have done in [H91] page 357 in the proof of Lemma 5.3 (vi) is valid without any change for each $\mathfrak{p}|p$, since at p–adic places, f_P in [H91] has the Neben type we introduced in this paper also for places outside p. The difference is that we compute the inner product in terms of (f_P, f_P) not (f_P°, f_P°) as in [H91] Lemma 5.3 (vi), where f_P° is the primitive form associated to $f_P^u \otimes \psi_{P,1}^u$ assuming that $\psi_{P,1}^u$ lifts to a global finite order character (the character $\psi_{P,1}^{-u}$ is written as ψ' in the proof of Lemma 5.3 (vi) of [H91]). Note here $f_P^\circ = f_P \otimes \psi_{P,1}^{-1}$ by definition and hence $(f_P^\circ, f_P^\circ) = (f_P^u, f_P^u)$, because tensoring a unitary character to a function does not alter the hermitian inner product. Thus we find

$$\frac{(f_P^{ord,u}|\tau(N\varpi^\alpha), f_P^{ord,u})}{(f_P^\circ, f_P^\circ)} = |N\varpi^\alpha|_{\mathbb{A}}^{([\kappa(P)]-1)/2}\frac{(f_P^{ord}|\tau(N\varpi^\alpha), f_P^{ord})}{(f_P, f_P)}. \quad (5.18)$$

A key point of the proof of Lemma 5.3 (vi) is the formula writing down $f_P^{ord,u} \otimes \psi_{P,1}^{-u}$ in terms of f_P°. Even without assuming the liftability of $\psi_{P,1}^u$

to a global character, the same formula is valid for f_P^{ord} and f_P before tensoring $\psi_{P,1}^{-1}$ (by computation using local Whittaker model). We thus have $f_P^{ord} = f_P|R$ for a product $R = \prod_{\mathfrak{p}|p} R_{\mathfrak{p}}$ of local operators $R_{\mathfrak{p}}$ given as follows: If the prime $\mathfrak{p}$ is a factor of $\mathfrak{C}(\psi_P^-)$, then $R_{\mathfrak{p}}$ is the identity operator. If $\mathfrak{p}$ is prime to $\mathfrak{C}(\psi_P^-)$ ($\Leftrightarrow \pi(1, \psi_{P,\mathfrak{p}}^-)$ is spherical), then $f|R_{\mathfrak{p}} = f - \psi_{P,2}(\varpi_{\mathfrak{p}})f|[\varpi_{\mathfrak{p}}]$, where $f|[\varpi_{\mathfrak{p}}](x) = |\varpi_{\mathfrak{p}}|_{\mathfrak{p}} f|g$ with $f|g(x) = f(xg)$ for $g = \begin{pmatrix} \varpi_{\mathfrak{p}}^{-1} & 0 \\ 0 & 1 \end{pmatrix}$. Writing U for the level group of f_P and $U' = U \cap U_0(\mathfrak{p})$, we note $f|T(\mathfrak{p}) = \mathrm{Tr}_{U/U'}(f|g^{-1})$. This shows

$$
\begin{aligned}
(f_P|[\varpi_{\mathfrak{p}}], f_P)_{U'} &= |\varpi_{\mathfrak{p}}|_{\mathfrak{p}}^{[\kappa(P)]}(f_P, f_P|g^{-1})_{U'} \\
&= |\varpi_{\mathfrak{p}}|_{\mathfrak{p}}^{[\kappa(P)]}(f_P, \mathrm{Tr}_{U/U'}(f_P|g^{-1}))_U = (a+b)(f_P, f_P)_U,
\end{aligned}
$$

where $a = |\varpi_{\mathfrak{p}}|_{\mathfrak{p}}^{[\kappa(P)]}\psi_{P,1}(\varpi_{\mathfrak{p}})$, $b = |\varpi_{\mathfrak{p}}|_{\mathfrak{p}}^{[\kappa(P)]}\psi_{P,2}(\varpi_{\mathfrak{p}})$, and $(\cdot,\cdot)_U$ is the Petersson metric on $\overline{X}(U)$. Similarly we have,

$$
\begin{aligned}
&(f_P, f_P|[\varpi_{\mathfrak{p}}])_{U'} = \overline{(a+b)}(f_P, f_P)_U \\
&\qquad \text{and } \ (f_P|[\varpi_{\mathfrak{p}}], f_P|[\varpi_{\mathfrak{p}}])_{U'} = |\varpi_{\mathfrak{p}}|_{\mathfrak{p}}^{[\kappa(P)]}(|\varpi_{\mathfrak{p}}|_{\mathfrak{p}} + 1)(f_P, f_P)_U.
\end{aligned}
$$

By (5.10) and by (5.18), we conclude from [H91] Lemma 5.3 (vi)

$$
\begin{aligned}
&\frac{((f_P^{ord})^c|\tau(N\varpi^{\alpha}), f_P^{ord})_{\alpha}}{(f_P, f_P)} \\
&\qquad = |N|_{\mathbb{A}}^{(1-[\kappa(P)])/2}(-1)^{k(P)}\psi_P^u(d_p)W'(f_P)S(P) \\
&\qquad\qquad \times \psi_{P,2}(\varpi^{\alpha}) \prod_{\mathfrak{p}|((p)+\mathfrak{C}(\psi_P^-))} G(\psi_{P,2,\mathfrak{p}}^{-1}\psi_{P,1,\mathfrak{p}}) \quad (5.19)
\end{aligned}
$$

for $\mathfrak{p}$ running over the prime factors of p.

We now compute the extra Euler factors: $E(P,Q)$ and $W(P,Q)$. Again the computation is the same as in [H91] Lemma 5.3 (iii)-(v), because the level structure and the Neben character at p–adic places are the same as in [H91] for f_P and g_Q and these factors only depend on p–adic places. Then we get the Euler p–factor $E(P,Q)$ and $W(P,Q)$ as in the theorem from [H91] lemma 5.3. □

Remark 5.4 We assumed the condition (5.7) to make the proof of the theorem simpler. We now remove this condition. Let $\mathcal{E}$ be the set of all prime factors $\mathfrak{q}$ of $\mathfrak{I}\mathfrak{N}$ such that $\chi_{\mathfrak{q}}^- = \psi_{\mathfrak{q}}^-$ on $O_{\mathfrak{q}}^{\times}$. Thus we assume that $\mathcal{E} \neq \varnothing$. Then in the proof of Lemma 5.2, the inertia group at $\mathfrak{q} \in \mathcal{E}$ fixes a two-dimensional subspace of $\rho_{f_P} \otimes \rho_{g_Q^c}$, one corresponding to $\psi_{1,\mathfrak{q}} \otimes \chi_{1,Q}^{-1}$ and the other coming from $\psi_{2,\mathfrak{q}} \otimes \chi_{2,Q}^{-1}$. The Euler factor corresponding to the latter does not appear in the Rankin product process; so, we get an imprimitive L–function, whose missing Euler factors are

$$E'(P,Q)^{-1} = \prod_{\mathfrak{q}\in\mathcal{E}}(1-\psi_{2,\mathfrak{q}}\chi_{2,Q}^{-1}(\varpi_{\mathfrak{q}}))^{-1}.$$

Thus the final result is identical to Theorem 5.3 if we multiply $E(P,Q)$ by $E'(P,Q)$ in the statement of the theorem. In our application, λ and φ will be (Λ–adic) automorphic inductions of Λ–adic characters $\widetilde{\lambda},\widetilde{\varphi}:\mathrm{Gal}(\overline{M}/M)\to\Lambda^{\times}$ for an ordinary CM field M/F. If the prime-to–p conductors of $\widetilde{\lambda}^{-}$ and $\widetilde{\varphi}^{-}$ are made of primes split in M/F, $\mathcal{E}$ is the set of primes ramifying in M/F. Then $E'(P,Q)$ is the specialization of $E' = \prod_{\mathfrak{q}\in\mathcal{E}}(1-\widetilde{\lambda}\otimes\widetilde{\varphi}^{-1}(Frob_{\mathfrak{q}}))$ at (P,Q), and $E'\in\Lambda\widehat{\otimes}\Lambda$ is not divisible by the prime element of W (that is, the μ–invariant of E' vanishes). Actually, we can choose $\widetilde{\lambda}$ and $\widetilde{\varphi}$ so that $\widetilde{\lambda}\widetilde{\varphi}^{-1}(Frob_{\mathfrak{q}})\not\equiv 1 \mod \mathfrak{m}_{\Lambda}$, and under this choice, we may assume that $E'\in(\Lambda\widehat{\otimes}\Lambda)^{\times}$.

5.3 Comparison of p–adic L–functions

For each character $\psi:\Delta_{\mathfrak{C}}\to W^{\times}$, we have the extension $\widetilde{\psi}:\mathrm{Gal}(\overline{\mathbb{Q}}/M)\to\Lambda$ sending $(\gamma,\delta)\in\Gamma\times\Delta_{\mathfrak{C}}$ to $\psi(\delta)\gamma$ for the group element $\gamma\in\Gamma$ inside the group algebra Λ. Regarding $\widetilde{\psi}$ as a character of $\mathrm{Gal}(\overline{\mathbb{Q}}/M)$, the induced representation $\mathrm{Ind}_M^F\widetilde{\psi}$ is modular nearly ordinary at p, and hence, by the universality of the nearly p–ordinary Hecke algebra $h_U^{n.ord}(W)$ defined in [H96] 2.5, we have a unique algebra homomorphism $\lambda: h_U^{n.ord}\to\Lambda$ such that $\mathrm{Ind}_M^F\widetilde{\psi}\cong\lambda\circ\rho^{Hecke}$ for the universal nearly ordinary modular Galois representation ρ^{Hecke} with coefficients in h, where $U=U_1^1(\mathfrak{N})$ with $\mathfrak{N}=D_{M/F}\mathfrak{c}(\psi)$ for the relative discriminant $D_{M/F}$ of M/F and $\mathfrak{c}=\mathfrak{C}(\psi)\cap F$ for the conductor $\mathfrak{C}(\psi)$ of ψ. Thus for each arithmetic point $P\in\mathrm{Spf}(\Lambda)(\overline{\mathbb{Q}}_p)$ (in the sense of [H96] 2.7), we have a classical Hecke eigenform $\theta(\widetilde{\psi}_P)$ of weight $\kappa(P)$.

We suppose that the conductor $\mathfrak{C}(\psi^-)$ of ψ^- consists of primes of M split over F. Then the automorphic representation $\pi(\widetilde{\psi}_P)$ of weight $\kappa(P)$ is everywhere principal at finite places. By [H96]7.1, the Hecke character $\widetilde{\psi}_P$ has infinity type

$$\infty(\widetilde{\psi}_P) = -\sum_{\sigma\in\Sigma}(\kappa_1(P)_{\sigma|_F}\sigma+\kappa_2(P)_{\sigma|_F}\sigma c).$$

In the automorphic representation $\pi(\widetilde{\psi}_P)$ generated by right translation of $\theta(\widetilde{\psi}_P)$, we have a unique normalized Hecke eigenform $f(\widetilde{\psi}_P)$ of minimal level.

The prime-to–p level of the cusp form $f(\widetilde{\psi}_P)$ is equal to

$$\mathfrak{N}(\psi) = N_{M/F}(\mathfrak{C}(\psi^-))D_{M/F},$$

and it satisfies (L1-3) in 5.1 for $\epsilon = \epsilon_P$ given by

$$\epsilon_{1,\mathfrak{q}} = \begin{cases} \widetilde{\psi}_P|_{\mathrm{Gal}(\overline{M}_{\mathfrak{L}}/M_{\mathfrak{L}})} & \text{if } \mathfrak{q} = \mathfrak{L}\overline{\mathfrak{L}}, \\ \text{an extension of } \widetilde{\psi}_P|_{\mathrm{Gal}(\overline{M}_{\mathfrak{L}}/M_{\mathfrak{L}})} \text{ to } \mathrm{Gal}(\overline{M}_{\mathfrak{L}}/F_{\mathfrak{q}}) & \text{otherwise,} \end{cases}$$

$$\epsilon_{2,\mathfrak{q}} = \begin{cases} \widetilde{\psi}_P|_{\mathrm{Gal}(\overline{M}_{\overline{\mathfrak{L}}}/M_{\overline{\mathfrak{L}}})} & \text{if } \mathfrak{q} = \mathfrak{L}\overline{\mathfrak{L}}, \\ \text{another extension of } \widetilde{\psi}_P|_{\mathrm{Gal}(\overline{M}_{\mathfrak{L}}/M_{\mathfrak{L}})} \text{ to } \mathrm{Gal}(\overline{M}_{\mathfrak{L}}/F_{\mathfrak{q}}) & \text{otherwise,} \end{cases} \tag{5.20}$$

where $\mathfrak{L}$ and $\overline{\mathfrak{L}}$ are distinct primes in M.

Write ϵ_+^t for the restriction of $\epsilon_+ = \epsilon_1\epsilon_2$ to $\Delta_F(\mathfrak{N}')$, which is independent of P (because it factors through the torsion part of $Cl_F^+(\mathfrak{N}'p^\infty)$). Since $\{f(\widetilde{\psi}_P)\}_P$ is again a p–adic analytic family of cusp forms, they are induced by a new algebra homomorphism $\lambda_\psi : \mathbf{h} = \mathbf{h}^{n.ord}(\mathfrak{N}, \epsilon; W) \to \Lambda$. Since λ_ψ is of minimal level, the congruence module $C_0(\lambda_\psi; \Lambda)$ is a well defined Λ–module of the form $\Lambda/H(\psi)\Lambda$ (see [H96] 2.9). Actually we can choose $H(\psi)$ in $\Lambda^- = W[[\Gamma_-]]$ (see [GME] Theorem 5.44). The element $H(\psi)$ is called the *congruence power series* of λ_ψ (identifying Λ^- with a power series ring over W of $[F : \mathbb{Q}]$ variables)

By Theorem 5.3 and Remark 5.4, we have the (imprimitive) p–adic Rankin product $\mathcal{D} = \lambda_\psi * \lambda_\varphi$ with missing Euler factor $E' \in (\Lambda\widehat{\otimes}\Lambda)^\times$ as in Remark 5.4 for two characters $\psi : \Delta_{\mathfrak{C}} \to W^\times$ and $\varphi : \Delta_{\mathfrak{C}'} \to W^\times$. Writing $\mathcal{R} = \mathcal{D} \cdot H(\psi) \in \Lambda\widehat{\otimes}_W\Lambda$, we have $\mathcal{D} = \frac{\mathcal{R}}{H(\psi)}$.

We define two p–adic L–functions $\mathcal{L}_p(\psi^{-1}\varphi)$ and $\mathcal{L}_p(\psi^{-1}\varphi_c)$ by

$$\mathcal{L}_p(\psi^{-1}\varphi)(P, Q) = E'(P, Q)L_p(\widetilde{\psi}_P^{-1}\widetilde{\varphi}_Q)$$

and

$$\mathcal{L}_p(\psi^{-1}\varphi_c)(P, Q) = L_p(\widetilde{\psi}_P^{-1}\widetilde{\varphi}_{Q,c})$$

for the Katz p–adic L–function L_p, where $\chi_c(\sigma) = \chi(c\sigma c^{-1})$.

We follow the argument in [H91], [HT1] and [H96] to show the following identity of p–adic L–functions, which is a more precise version of [HT1] Theorem 8.1 without the redundant factor written as $\Delta(M/F; \mathfrak{C})$ there:

Theorem 5.5 *Let ψ and φ be two characters of $\Delta_{\mathfrak{C}}$ with values in $W^\times$. Suppose the following three conditions:*

(i) $\mathfrak{C}(\psi^-)\mathfrak{C}(\varphi^-)$ *is prime to any inert or ramified prime of M;*

(ii) *At each inert or ramified prime factor* $\mathfrak{q}$ *of* $\mathfrak{C}(\psi)\mathfrak{C}(\varphi)$*,* $\psi_{\mathfrak{q}} = \varphi_{\mathfrak{q}}$ *on* $R_{\mathfrak{q}}^{\times}$*;*
(iii) *For each split prime* $\mathfrak{q}|\mathfrak{C}(\psi)\mathfrak{C}(\varphi)$*, we have a choice of one prime factor* $\mathfrak{Q}|\mathfrak{q}$ *so that* $\psi_{\mathfrak{Q}} = \varphi_{\mathfrak{Q}}$ *on* $R_{\mathfrak{Q}}^{\times}$*.*

Then we have, for a power series $\mathcal{L} \in \Lambda\widehat{\otimes}\Lambda$*,*

$$\frac{\mathcal{L}}{H(\psi)} = \frac{\mathcal{L}_p(\psi^{-1}\varphi)\mathcal{L}_p(\psi^{-1}\varphi_c)}{(h(M)/h(F))L_p^-(\psi^-)}.$$

The power series $\mathcal{L}$ *is equal to* $(\lambda_\psi * \lambda_\varphi) \cdot H(\psi)$ *up to units in* $\Lambda\widehat{\otimes}\Lambda$*.*

The improvement over [HT1] Theorem 8.1 is that our identity is exact without missing Euler factors, but we need to have the additional assumption (1-3) to assure the matching condition (5.6) for the automorphic induction of $\widetilde{\psi}_P$ and $\widetilde{\varphi}_Q$.

The proof of the theorem is identical to the one given in [HT1] Section 10, since the factors $C(P,Q)$, $W(P,Q)$, $S(P)$ and $E(P,Q)$ appearing in Theorem 5.3 are identical to those appearing in [H91] Theorem 5.2 except for the power of the discriminant D, which is compensated by the difference of (f_P°, f_P°) appearing in [H91] Theorem 5.2 from (f_P, f_P) in Theorem 5.3. Since we do not lose any Euler factors in (5.5) (thanks to our minimal level structure and the assumption (5.6) assuring principality everywhere), we are able to remove the missing Euler factor denoted $\Delta(1)$ in [HT1] (0.6b).

If M/F is ramified at some finite place, by Theorem 4.3 and Remark 5.4, we can always choose a pair $(\mathfrak{l}, \varphi)$ of a prime ideal $\mathfrak{l}$ and a character φ of $\mathfrak{l}$–power conductor so that

(i) $\mathfrak{l}$ is a split prime of F of degree 1;
(ii) $\mathcal{L}_p(\psi^{-1}\varphi)\mathcal{L}_p(\psi^{-1}\varphi_c)$ is a unit in $\Lambda\widehat{\otimes}\Lambda$.

Even if M/F is unramified, we shall show that such choice is possible: Writing Ξ for the set of primes $\mathfrak{q}$ as specified in the condition (3) in the theorem. Then we choose a prime $\mathfrak{L}$ split in $M/\mathbb{Q}$ so that $\mathfrak{L}\overline{\mathfrak{L}}$ is outside Ξ. We then take a finite order character $\varphi' : M_{\mathbb{A}}^{\times}/M^{\times} \to W^{\times}$ of conductor $\mathfrak{L}^m$ so that $\varphi'_{\overline{\mathfrak{L}}}$ is trivial but the reduction modulo $\mathfrak{m}_W$ of $\psi^{-1}\varphi\varphi'\mathcal{N}$ and $\psi^{-1}\varphi_c\varphi'_c\mathcal{N}$ is not equal to $\left(\frac{M/F}{\cdot}\right)$. Then again by Theorem 4.3, we find (infinitely many) φ' with unit power series $\mathcal{L}_p(\psi^{-1}\varphi\varphi')\mathcal{L}_p(\psi^{-1}(\varphi\varphi')_c)$. This implies the following corollary:

Corollary 5.6 *Suppose that* $\mathfrak{C}(\psi^-)$ *is prime to any inert or ramified prime of* M*. Then in* Λ^-*, we have* $(h(M)/h(F))L_p^-(\psi^-)\big|H(\psi)$*.*

As a byproduct of the proof of Theorem 5.5, we can express the p–adic

L–value $L_p(\widetilde{\psi}_P^-)$ (up to units in W and the period in $\mathbb{C}^\times$) by the Petersson metric of the normalized Hecke eigenform $f_P = f(\widetilde{\psi}_P)$ of minimal level (in the automorphic induction π_P of $\widetilde{\psi}_P$). We shall describe this fact.

We are going to express the value $L(1, Ad(f_P))$ in terms of the L–values of the Hecke character $\widetilde{\psi}_P$. The representation π_P is everywhere principal outside archimedean places if and only if ψ^- has split conductor. Then

$$L(s, Ad(f_P)) = L(1, \left(\frac{M/F}{\cdot}\right))L(0, (\widetilde{\psi}_P^-)^*).$$

The infinity type of $\widetilde{\psi}_P^-$ is $(\kappa_1(P) - \kappa_2(P)) + (\kappa_2(P) - \kappa_1(P))c = (k(P) - I)(1-c)$, where we identify $\mathbb{Z}[\Sigma]$ with $\mathbb{Z}[I]$ sending σ to $\sigma|_F$. Thus the infinity type of $(\widetilde{\psi}_P^-)^*$ is given by $k(P) - k(P)c + 2\Sigma c = 2\Sigma + (k(P) - 2I)(1 - c)$. Thus we have

$$\begin{aligned} L_p((\widetilde{\psi}_P^-)^*) &= N(\mathfrak{c})\widetilde{\psi}_P^-(\mathfrak{c})W'(\widetilde{\psi}_P^-)^{-1}L_p(\widetilde{\psi}_P^-) \\ &= CW_p(\widetilde{\psi}_P^-)E(P)\frac{\pi^{k(P)-I}\Gamma_\Sigma(k(P)-I)L(0,\psi_P^-)}{\Omega_\infty^{2(k(P)-I)}} \\ &= C'W_p((\widetilde{\psi}^-)_P^*)E(P)\frac{\pi^{k(P)-2I}\Gamma_\Sigma(k(P))L(0,(\psi^-)_P^*)}{\Omega_\infty^{2(k(P)-I)}}, \end{aligned} \tag{5.21}$$

where C and C' are constants in $W^\times$ and

$$E(P) = \prod_{\mathfrak{P}\in\Sigma_p} \left((1 - \widetilde{\psi}_P^-(\mathfrak{P}^c))(1 - (\widetilde{\psi}_P^-)^*(\mathfrak{P}^c))\right).$$

Since we have, for the conductor $\mathfrak{P}^{e(\mathfrak{P})}$ of $\psi_{P,\mathfrak{P}}^-$,

$$(\widetilde{\psi}_P^-)^*(\varpi_{\mathfrak{P}}^{-e(\mathfrak{P})}) = N(\mathfrak{P})^{e(\mathfrak{P})}\widetilde{\psi}_P^-(\varpi_{\mathfrak{P}}^{-e(\mathfrak{P})}),$$

by definition, $W((\widetilde{\psi}_P^-)^*)$ is the product over $\mathfrak{P} \in \Sigma$ of $G(2\delta_{\mathfrak{P}}, \psi_{P,\mathfrak{P}}^-)$, and hence we have

$$W_p((\widetilde{\psi}_P^-)^*) = N(\mathfrak{P}^{e(\Sigma)})W_p(\widetilde{\psi}_P^-),$$

where $\mathfrak{P}^{e(\Sigma)}$ is the Σ_p–part of the conductor of ψ_P^-. Thus we have, for $h(M/F) = h(M)/h(F)$,

$$\begin{aligned} L_p((\widetilde{\psi}_P^-)^*) &= C'W_p(\widetilde{\psi}_P^-)N(\mathfrak{P}^{e(\Sigma)})E(P)\frac{\pi^{k(P)-2I}\Gamma_\Sigma(k(P))L(0,(\widetilde{\psi}^-)_P^*)}{\Omega_\infty^{2(k(P)-I)}} \\ &= C''W_p(\widetilde{\psi}_P^-)E(P)\frac{\pi^{2(\kappa_1(P)-\kappa_2(P))}(f(\widetilde{\psi}_P),f(\widetilde{\psi}_P))}{h(M/F)\Omega_\infty^{2(\kappa_1(P)-\kappa_2(P))}}, \end{aligned} \tag{5.22}$$

where C'' and C' are constants in $W^\times$.

We suppose that the conductor $\mathfrak{C}(\widetilde{\psi}_P)$ is prime to p. Then $W_p(\widetilde{\psi}_P^-) = 1$ if p is unramified in $F/\mathbb{Q}$ (and even if p ramifies in $F/\mathbb{Q}$, it is a unit in W). Let $h = h(M)$, and choose a global generator ϖ of $\mathfrak{P}^h$. Thus $\varpi^\Sigma \equiv 0 \mod \mathfrak{m}_W$ and $\varpi^{\Sigma c} \not\equiv 0 \mod \mathfrak{m}_W$. By the arithmeticity of the point P, we have $k = k(P) \geq 2I$. Then we have, up to p–adic unit,

$$\widetilde{\psi}_P^-(\mathfrak{P}^c)^h = \widetilde{\psi}_P^-(\varpi) = \varpi^{(k-I)-(k-I)c}$$
$$(\widetilde{\psi}_P^-)^*(\mathfrak{P}^c)^h = (\widetilde{\psi}_P^-)^*(\varpi) = \varpi^{(k-2I)-kc}.$$

Thus $\widetilde{\psi}_P^-(\mathfrak{P}^c) \in \mathfrak{m}_W$ if $k \geq 2I$ and $(\widetilde{\psi}_P^-)^*(\mathfrak{P}^c) \in \mathfrak{m}_W$ if $k \geq 3I$. For each $\sigma \in \Sigma$, we write $\mathfrak{P}_\sigma$ for the place in Σ_p induced by $i_p \circ \sigma$. Thus we obtain

Proposition 5.7 *Suppose that either $k(P)_\sigma = \kappa_1(P)_\sigma - \kappa_2(P)_\sigma + 1 \geq 3$ or $(\widetilde{\psi}_P^-)^*(\mathfrak{P}_\sigma^c) \not\equiv 1 \mod \mathfrak{m}_W$ for all $\sigma \in I$ and that $\widetilde{\psi}_P$ has split conductor. Write $h(M/F)$ for $h(M)/h(F)$. Then up to units in W, we have*

$$h(M/F)L_p((\widetilde{\psi}_P^-)^*) = h(M/F)L_p(\widetilde{\psi}_P^-)$$
$$= \frac{\pi^{2(k(P)-I)}W_p(\widetilde{\psi}_P^-)(f(\widetilde{\psi}_P), f(\widetilde{\psi}_P))_{\mathfrak{N}}}{\Omega_\infty^{2(k(P)-I)}},$$

where $f(\widetilde{\psi}_P)$ is the normalized Hecke eigenform of minimal level $\mathfrak{N}$ (necessarily prime to p) of the automorphic induction of $\widetilde{\psi}_P$.

5.4 A case of the anticyclotomic main conjecture

Here we describe briefly an example of a case where the divisibility: $L_p^-(\psi)|\mathcal{F}^-(\psi)$ implies the equality $L_p^-(\psi) = \mathcal{F}^-(\psi)$ (up to units), relying on the proof by Rubin of the one variable main conjecture over an imaginary quadratic field in [R] and [R1]. Thus we need to suppose

(ab) $F/\mathbb{Q}$ is abelian, $p \nmid [F:\mathbb{Q}]$, and $M = F[\sqrt{D}]$ with $0 > D \in \mathbb{Z}$,

in order to reduce our case to the imaginary quadratic case treated by Rubin. Write $E = \mathbb{Q}[\sqrt{D}]$ and suppose that D is the discriminant of $E/\mathbb{Q}$. We have $p \nmid D$ since p is supposed to be unramified in $M/\mathbb{Q}$. We suppose also

(sp) The prime ideal (p) splits into $\mathfrak{p}\overline{\mathfrak{p}}$ in $E/\mathbb{Q}$.

Then we take $\Sigma = \{\sigma : M \hookrightarrow \overline{\mathbb{Q}} | \sigma(\sqrt{D}) = \sqrt{D}\}$. By (sp), Σ is an ordinary CM type.

We fix a conductor ideal $\mathfrak{c}$ (prime to p) of E satisfying (opl) and (spt) for $E/\mathbb{Q}$ (in place of M/F). We then put $\mathfrak{C} = \mathfrak{c}R$. We consider $Z_E = Z_E(\mathfrak{c}) =$

$\varprojlim_n Cl_E(\mathfrak{c}p^n)$. We split $Z_E(\mathfrak{c}) = \Delta_\mathfrak{c} \times \Gamma_\mathfrak{c}$ for a finite group $\Delta = \Delta_\mathfrak{c}$ and a torsion-free subgroup $\Gamma_\mathfrak{c}$. As before, we identify $\Gamma_\mathfrak{c}$ and Γ_1 for E and write it as Γ_E. We then write $\Gamma_{E,-} = \Gamma_E/\Gamma_E^+$ for $\Gamma_E^+ = H^0(\mathrm{Gal}(E/\mathbb{Q}), \Gamma_E)$. We consider the anticyclotomic $\mathbb{Z}_p$–extension E_∞^- of E on which we have $c\sigma c^{-1} = \sigma^{-1}$ for all $\sigma \in \mathrm{Gal}(E_\infty^-/E)$ for complex conjugation c. Writing $E(\mathfrak{c}p^\infty)$ (inside $M(\mathfrak{C}p^\infty)$) for the ray class field over E modulo $\mathfrak{c}p^\infty$, we identify $Z_E(\mathfrak{c})$ with $\mathrm{Gal}(E(\mathfrak{c}p^\infty)/E)$ via the Artin reciprocity law. Then $\mathrm{Gal}(E(\mathfrak{c}p^\infty)F/E_\infty^-) = \Gamma_E^+ \times \Delta_\mathfrak{c}$ and $\mathrm{Gal}(E_\infty^-/E) = \Gamma_{E,-}$. We then define E_Δ by the fixed field of $\Gamma_\mathfrak{c}$ in the composite $F \cdot E(\mathfrak{c}p^\infty)$; so, $\mathrm{Gal}(E_\Delta/E) = \Delta_\mathfrak{c}$ and $E_\Delta \supset F$. We have $M_\Delta \supset E_\Delta$. Thus we have the restriction maps $\mathrm{Res}_Z : Z(\mathfrak{C}) = \mathrm{Gal}(M(\mathfrak{C}p^\infty)/M) \to \mathrm{Gal}(E(\mathfrak{c}p^\infty)/M) = Z_E(\mathfrak{c})$, $\mathrm{Res}_\Gamma : \Gamma_- = \mathrm{Gal}(M_\infty^-/M) \to \mathrm{Gal}(E_\infty^-/E) = \Gamma_{E,-}$ and $\mathrm{Res}_\Delta : \Delta_\mathfrak{C} = \mathrm{Gal}(M_\Delta/M) \to \mathrm{Gal}(E_\Delta/E) = \Delta_\mathfrak{c}$. We suppose

(res) There exists an anticyclotomic character ψ_E of $\Delta_\mathfrak{c}$ such that $\psi = \psi_E \circ \mathrm{Res}_\Delta$.

Let $L_\infty^E/E_\infty^- E_\Delta$ be the maximal p–abelian extension unramified outside $\mathfrak{p}$. Each $\gamma \in \mathrm{Gal}(L_\infty^E/E)$ acts on the normal subgroup $X_E = \mathrm{Gal}(L_\infty^E/E_\infty^- E_\Delta)$ continuously by conjugation, and by the commutativity of X_E, this action factors through $\mathrm{Gal}(E_\Delta E_\infty^-/E)$. Then we look into the Λ_E^-–module: $X_E[\psi_E\chi] = X_E \otimes_{\Delta_\mathfrak{c},\psi\chi} W$ for a character χ of $\mathrm{Gal}(F/\mathbb{Q})$, where $\Lambda_E^- = W[[\Gamma_{E,-}]]$. The projection Res_Γ induces a W-algebra homomorphism $\Lambda^- \to \Lambda_E^-$ whose kernel we write as $\mathfrak{a}$.

Theorem 5.8 *Let the notation be as above. Suppose that ψ has order prime to p in addition to* (ab), (sp) *and* (res). *Then $L_p^-(\psi) = \mathcal{F}^-(\psi)$ up to units in Λ^-.*

Proof We shall use functoriality of the Fitting ideal $F_A(H)$ of an A-module H with finite presentation over a commutative ring A with identity (see [MW] Appendix for the definition and the functoriality Fitting ideals listed below):

(i) If $I \subset A$ is an ideal, we have $F_{A/I}(H/IH) = F_A(H) \otimes_A A/I$;

(ii) If A is a noetherian normal integral domain and H is a torsion A-module of finite type, the characteristic ideal $\mathrm{char}_A(H)$ is the reflexive closure of $F_A(H)$. In particular, we have $\mathrm{char}_A(H) \supset F_A(H)$.

By definition, we have

$$H_0(\mathrm{Ker}(\mathrm{Res}_\Gamma), X[\psi]) = X[\psi]/\mathfrak{a}X[\psi] \cong \bigoplus_\chi X_E[\psi_E\chi]$$

for χ running all characters of $\mathrm{Gal}(F/\mathbb{Q})$. By (i) above, we have

$$\prod_\chi F_{\Lambda_E^-}(X_E[\psi\chi]) = F_{\Lambda^-}(X[\psi]) \otimes_{\Lambda^-} \Lambda_E^-.$$

Since $\mathrm{char}_{\Lambda^-}(X[\psi]) = \mathcal{F}^-(\psi)$, by Theorem 5.1, $\mathrm{char}_{\Lambda^-}(X[\psi]) \subset L_p^-(\psi)$. Thus by (ii), we obtain

$$\prod_{\chi} \mathrm{char}_{\Lambda_E^-}(X_E[\psi\chi]) \subset L_p^-(\psi)\Lambda_E^-,$$

where $L_p^-(\psi)\Lambda_E^-$ is the ideal of Λ_E^- generated by the image of $L_p^-(\psi) \in \Lambda^-$ in Λ_E^-. Write R_E for the integer ring of E, and let $X(R_E)_{/\mathcal{W}}$ be the elliptic curve with complex multiplication whose complex points give the torus $\mathbb{C}/R_E$. Since $X(R) = X(R_E) \otimes_{R_E} R$ for our choice of CM type Σ, the complex and p-adic periods of $X(R)$ and $X(R_E)$ are identical. Thus by the factorization of Hecke L-functions, we have

$$L_p^-(\psi)\Lambda_E^- = \prod_{\chi} L_p^-(\psi_E\chi)\Lambda_E^-.$$

Then by Rubin [R] Theorem 4.1 (i) applied to the $\mathbb{Z}_p$-extension E_∞^-/E, we find that

$$\mathrm{char}_{\Lambda_E^-}(X_E[\psi_E\chi]) = L_p^-(\psi_E\chi)\Lambda_E^-.$$

Thus $(\mathcal{F}^-(\psi)/L_p^-(\psi))\Lambda_E^- = \Lambda_E^-$, and hence $\mathcal{F}^-(\psi)\Lambda^- = L_p^-(\psi)\Lambda^-$. □

Bibliography

Books

[AAF] G. Shimura, *Arithmeticity in the Theory of Automorphic Forms*, Mathematical Surveys and Monographs **82**, AMS, 2000

[ACM] G. Shimura, *Abelian Varieties with Complex Multiplication and Modular Functions*, Princeton University Press, 1998

[CRT] H. Matsumura, *Commutative Ring Theory*, Cambridge studies in advanced mathematics **8**, Cambridge Univ. Press, 1986

[GME] H. Hida, *Geometric Modular Forms and Elliptic Curves*, 2000, World Scientific Publishing Co., Singapore (a list of errata downloadable at `www.math.ucla.edu/˜hida`)

[HMI] H. Hida, *Hilbert Modular Forms and Iwasawa Theory*, Oxford University Press, In press

[IAT] G. Shimura, *Introduction to the Arithmetic Theory of Automorphic Functions*, Princeton University Press and Iwanami Shoten, 1971, Princeton-Tokyo

[LAP] H. Yoshida, *Lectures on Absolute CM period*, AMS mathematical surveys and monographs **106**, 2003, AMS

[LFE] H. Hida, *Elementary Theory of L–functions and Eisenstein Series*, LMSST **26**, Cambridge University Press, Cambridge, 1993

[MFG] H. Hida, *Modular Forms and Galois Cohomology*, Cambridge studies in advanced mathematics **69**, Cambridge University Press, Cambridge, 2000 (a list of errata downloadable at `www.math.ucla.edu/˜hida`)

[PAF] H. Hida, *p–Adic Automorphic Forms on Shimura Varieties*, Springer Monographs in Mathematics. Springer, New York, 2004 (a list of errata downloadable at www.math.ucla.edu/˜hida)

Articles

[B] D. Blasius, Ramanujan conjecture for Hilbert modular forms, preprint, 2002

[BR] D. Blasius and J. D. Rogawski, Motives for Hilbert modular forms, Inventiones Math. **114** (1993), 55–87

[C] C.-L. Chai, Arithmetic minimal compactification of the Hilbert-Blumenthal moduli spaces, Ann. of Math. **131** (1990), 541–554

[De] P. Deligne, Travaux de Shimura, Sem. Bourbaki, Exp. 389, Lecture notes in Math. **244** (1971), 123–165

[DeR] P. Deligne and K. A. Ribet, Values of abelian L–functions at negative integers over totally real fields, Inventiones Math. **59** (1980), 227–286

[Di] M. Dimitrov, Compactifications arithmétiques des variétés de Hilbert et formes modulaires de Hilbert pour $\Gamma_1(\mathfrak{c}, \mathfrak{n})$, in *Geometric aspects of Dwork theory*, Vol. I, II, 527–554, Walter de Gruyter GmbH & Co. KG, Berlin, 2004

[DiT] M. Dimitrov and J. Tilouine, Variété de Hilbert et arithmétique des formes modulaires de Hilbert pour $\Gamma_1(\mathfrak{c}, N)$, in *Geometric Aspects of Dwork's Theory*, Vol. I, II, 555–614, Walter de Gruyter GmbH & Co. KG, Berlin, 2004 Walter de Gruyter, Berlin, 2004.

[Fu] K. Fujiwara, Deformation rings and Hecke algebras in totally real case, preprint, 1999

[H88] H. Hida, On p-adic Hecke algebras for GL_2 over totally real fields, Ann. of Math. **128** (1988), 295–384

[H89] H. Hida, Nearly ordinary Hecke algebras and Galois representations of several variables, Proc. JAMI Inaugural Conference, Supplement to Amer. J. Math. (1989), 115–134

[H91] H. Hida, On p–adic L–functions of $GL(2) \times GL(2)$ over totally real fields, Ann. Inst. Fourier **41** (1991), 311–391

[H95] H. Hida, Control theorems of p–nearly ordinary cohomology groups for $SL(n)$, Bull. Soc. math. France, **123** (1995), 425–475

[H96] H. Hida, On the search of genuine p–adic modular L–functions for $GL(n)$, Mémoire SMF **67**, 1996

[H99] H. Hida, Non-critical values of adjoint L-functions for $SL(2)$, Proc. Symp. Pure Math., 1998

[H00] H. Hida, Adjoint Selmer group as Iwasawa modules, Israel J. Math. **120** (2000), 361–427

[H02] H. Hida, Control theorems of coherent sheaves on Shimura varieties of PEL–type, Journal of the Inst. of Math. Jussieu, 2002 **1**, 1–76

[H04a] H. Hida, p–Adic automorphic forms on reductive groups, Astérisque **298** (2005), 147–254 (preprint downloadable at www.math.ucla.edu/˜hida)

[H04b] H. Hida, The Iwasawa μ–invariant of p–adic Hecke L–functions, preprint, 2004 (preprint downloadable at www.math.ucla.edu/˜hida)

[H04c] H. Hida, Non-vanishing modulo p of Hecke L–values, in: "*Geometric Aspects of Dwork's Theory, II*" (edited by Alan Adolphson, Francesco Baldassarri, Pierre Berthelot, Nicholas Katz, and Francois Loeser), Walter de Gruyter, 2004, pp. 735–784 (preprint downloadable at www.math.ucla.edu/˜hida)

[H04d] H. Hida, Anticyclotomic main conjectures, preprint, 2004, (preprint downloadable at www.math.ucla.edu/˜hida)

[HT1] H. Hida and J. Tilouine, Anticyclotomic Katz p–adic L–functions and congruence modules, Ann. Sci. Ec. Norm. Sup. 4-th series **26** (1993), 189–259

[HT2] H. Hida and J. Tilouine, On the anticyclotomic main conjecture for CM fields, Inventiones Math. **117** (1994), 89–147

[K] N. M. Katz, p-adic L-functions for CM fields, Inventiones Math. **49** (1978), 199–297

[Ko] R. Kottwitz, Points on Shimura varieties over finite fields, J. Amer. Math. Soc. **5** (1992), 373–444

[MW] B. Mazur and A. Wiles, Class fields of abelian extensions of $\mathbb{Q}$, Inventiones Math. **76** (1984), 179–330

[R] K. Rubin, The "main conjectures" of Iwasawa theory for imaginary quadratic fields, Inventiones Math. **103** (1991), 25–68

[R1] K. Rubin, More"main conjectures" for imaginary quadratic fields. Elliptic curves and related topics, 23–28, CRM Proc. Lecture Notes, **4**, Amer. Math. Soc., Providence, RI, 1994.

[ST] J.-P. Serre and J. Tate, Good reduction of abelian varieties, Ann. of Math. **88** (1968), 452–517

[Sh] G. Shimura, On canonical models of arithmetic quotients of bounded symmetric domains, Ann. of Math. **91** (1970), 144-222; II, **92** (1970), 528-549

[Sh1] G. Shimura, On some arithmetic properties of modular forms of one and several variables, Ann. of Math. **102** (1975), 491–515

[Si] W. Sinnott, On a theorem of L. Washington, Astérisque **147-148** (1987), 209–224

[TW] R. Taylor and A. Wiles, Ring theoretic properties of certain Hecke modules, Ann. of Math. **141** (1995), 553–572

[Wa] L. Washington, The non-p–part of the class number in a cyclotomic $\mathbb{Z}_p$–extension, Inventiones Math. **49** (1978), 87–97

[Wi] A. Wiles, Modular elliptic curves and Fermat's last theorem, Ann. of Math. **141** (1995), 443–551