* André-Oort conjecture and non-vanishing of central L-values

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Talk in December, 2015 (Université de Paris-Nord).

See `www.math.ucla.edu/~hida/Hilbert.pdf`

*The author is partially supported by the NSF grant: DMS 1464106.

Abstract: This is a joint work with A. Burungale. Assume the André-Oort conjecture for products of Hilbert modular varieties for a totally real field F (whose proof has been announced by Tsimerman, Yuan-Zhang and Andreatta-Goren-Howard-Pera), and pick a Hilbert modular Hecke eigen-cusp form with root number 1. We study non-vanishing of central critical values of $L(s, f \otimes \chi)$ twisted by conductor 1 Hecke characters of CM quadratic extensions $K_{/F}$. As the discriminant of K tends to infinity, we prove that the number of characters χ with non-vanishing property also tends to infinity.

§0. Minimalist expectation:

For a self-dual family of L-functions with root number $\varepsilon(L) = 1$, the central-critical L-values are generically believed to be non-trivial. We can think of the following families

- $\{L(s, E) | E_{/\mathbb{Q}}$: elliptic curves$\}$ (Moduli theoretic family);
- $\{L(s, A_f) | f$: weight 2 Hecke eigenforms in an analytic family$\}$, where A_f is Shimura's abelian factor associated with f;
- $\{L(s, f \otimes \lambda_K) | C(\lambda_K) = 1, wt(f) = k\}$, where f is fixed and λ are Hecke characters of CM quadratic extension of K of conductor 1 with fixed infinity type (character twists family moving fields K).

To some extent, in the first two cases, non-triviality of the central critical values are known for infinitely family members.

Today, we study the third example.

§1. Tools to be used:

An L-value can often be expressed as a period of a modular form over an algebraic cycle in a Shimura variety. The non-triviality is thus typically related to an appropriate density of such cycles.

We study a self-dual Rankin-Selberg convolution of a cuspidal Hilbert modular new form and a Hecke character over CM quadratic extensions K/F with root number 1. Twists of the Hecke character by the class group characters of the CM extension keeps the root number unchanged and gives rise to a family alluded to.

The number of class group characters with the corresponding central-critical L-value being non-zero is expected to grow with the absolute value of the discriminant of the CM extension.

§2. Notation.

- K/F: a CM quadratic extension and $\Sigma = \Sigma_K$ a CM type (identified with infinite places of F by $\Sigma \ni \sigma \mapsto \sigma|_F$).
- c: complex conjugation on $\mathbb{C}$ which induces the unique non-trivial element of $\mathrm{Gal}(K/F)$.
- D_K: the discriminant of $K/\mathbb{Q}$.
- Cl_K: the ideal class group of K.
- $\widehat{Cl_K}$: the character group of Cl_K with values in $\mathbb{C}^\times$.
- f: a cuspidal Hilbert modular new form over F with unitary central character ω, weight $k = k\Sigma = k\sum_{\sigma\in\Sigma}\sigma$ $k > 0$ and on $\Gamma_0(\mathfrak{n})$ ($\mathfrak{n}$: conductor).
- For an ideal $\mathfrak{a}$ of O, we fix a decomposition $\mathfrak{a} = \mathfrak{a}^+\mathfrak{a}^-$ where $\mathfrak{a}^+$ (resp. $\mathfrak{a}^-$) is divisible only by split (resp. ramified or inert) primes in the extension K/F.
- Θ: any **infinite** set of CM quadratic extensions of F.

§3. Theorem.

Fix a unitary Hecke character $\lambda = \lambda_K$ of K of infinity type $m \in \mathbb{Z}[\Sigma \cup \Sigma c]$ such that

(C1) $\lambda_{|\mathbb{A}_F^\times} = \omega^{-1}$ and
(C2) $m = k \cdot \Sigma_K = k \cdot \sum_{\sigma \in \Sigma_K} \sigma$.
(RN) the Rankin convolution $L(s, f \otimes \lambda)$ has root number 1 ($\Leftrightarrow \epsilon_v(f \otimes \lambda) = 1$ for all $v|\mathfrak{n}^-$), where $\epsilon_v(f \otimes \lambda)$ is the local root number.

Main Theorem: We have

$$\liminf_{K \in \Theta} \left| \{\chi \in \widehat{Cl_K} : L(\frac{1}{2}, f \otimes \lambda_K \chi) \neq 0\} \right| = \infty.$$

§4. Strategy.

• the non-vanishing of the Rankin-Selberg L-values $\Leftrightarrow$ the non-vanishing of twisted **toric** periods of a holomorphic Hilbert modular form associated to f. Here the torus comes from the CM quadratic extension K/F (first observed by Waldspurger).

• An argument based on the Brauer-Siegel lower bound on the size of the class groups then reduces the theorem to a functional independence for functions induced by a holomorphic Hilbert modular form associated to f on Cl_K.

• Based on the geometric interpretation of holomorphic Hilbert modular forms, the independence in our setup is essentially equivalent to the Zariski density of well-chosen CM points on a self-product of the Hilbert modular Shimura variety. Here we need to consider arbitrary self-products of the Hilbert modular Shimura variety.

§5. Earlier analytic methods.

Here we only mention results of Michel–Venkatesh and Masri–Yang and refer to the references in their papers.

The approach in these articles is perhaps more analytic/ergodic. Typically, the first step is to obtain a mean value theorem for the Rankin-Selberg L-values $L(\frac{1}{2}, f \otimes \lambda\chi)$ over $\chi \in \widehat{Cl_K}$ via the Waldspurger formula. The equidistribution of CM points on the Hilbert modular Shimura variety then often implies that the average value is positive, for K with a large discriminant. Note that the average value result alone produces some non-vanishing twists but not a growing number of them.

The next step is to invoke a subconvex bound for the Rankin-Selberg L-values $L(\frac{1}{2}, f \otimes \lambda\chi)$. The previous step then allows to deduce a quantitative version of the non-vanishing i.e. a quantitative version of growing non-vanishing twists.

§6. Growing class number according to discriminant.

Since the proof is essentially the same for any totally real F, we assume $F = \mathbb{Q}$ for simplicity. Here is Brauer–Siegel lower bound for class numbers of imaginary quadratic field: For $\epsilon > 0$,

$$h_K \gg |D_K|^{1/2-\epsilon}.$$

This implies

Corollary: Let Θ be an infinite set consisting of imaginary quadratic fields. Then, for $h_K = |Cl_K|$,

$$\lim_{K \in \Theta} h_K = \infty.$$

Let n be a positive integer. Let d_n be a positive integer such that for imaginary quadratic field K with $|D_K| \geq d_n$, we have

$$h_K \geq n.$$

§7. CM points.

Assume $|D_K| > d_n$. We now choose n distinct ideal classes $[\mathfrak{a}_i]_{1 \leq i \leq n}$ such that $[\mathfrak{a}_1] = [O_K]$ so that

(N) The class $[\mathfrak{a}_i \mathfrak{a}_j^{-1}]$ cannot be represented by an integral ideal of bounded norm for all distinct i and j with $1 \leq i, j \leq n$ as $K \in \Theta$ varies.

Let

$$\Xi_n = \{(x(\mathfrak{a}), ..., x(\mathfrak{a}\mathfrak{a}_n)) \in V^n | [\mathfrak{a}] \in Cl_K, K \in \Theta\}$$

be the subset of CM points in the n-fold self-product V^n for $V := Sh_{\Gamma_1(\mathfrak{n})/\overline{\mathbb{Q}}}$. In particular, the CM points are well chosen skewed diagonal images of CM points arising from the ideal classes in the CM quadratic extensions. We recall that the CM points $x(\mathfrak{a})$ are of CM type (K, id) (so, $\Sigma = \{\mathrm{id}\}$).

§8. Zariski density in multiple modular curves.

The following is the consequence of André–Oort conjecture (proven by Pila for modular curves):

Theorem: *The subset Ξ_n of CM points is Zariski dense in the Shimura variety $Sh^n_{\Gamma_1(\mathfrak{n})}$.*

The case $n = 1$ follows because V is a curve and $|\Xi_1| = \infty$. When $F \neq \mathbb{Q}$, the case $n = 1$ follows from equidistribution result of Venkatesh (Annals of Math. **172** (2010)). We proceed by induction on n.

§9. **Proof when** $n \geq 2$. Let $X = \overline{\Xi_n} \subset V^n$ denote Zariski closure of the CM points Ξ_n. Let I be an irreducible component of X containing an infinite subset T_n of Ξ_n.

The André-Oort conjecture implies that I is a special subvariety of the self-product V^n. Write $\pi_{s,t} : V^n \to V^2$ to the s-th and t-th components. Suppose $I \subsetneq V^n$.

$$\pi_{s,t} T_n \subset \Delta_{g,h}$$

with $\Delta_{g,h} = \{(g(z), h(z)) \in V^2 | z \in Sh\}$ for $g, h \in \mathrm{GL}_2(\mathbb{A})$.

This description of $\Delta_{g,h}$ implies that the s^{th} and t^{th} components of I are isogenous by an isogeny of a fixed degree. Here isogeny refers to a finite morphism of bounded degree. For $[\mathfrak{a}] \in Cl_K$ with $K \in \Theta$, the CM points $x(\mathfrak{a}\mathfrak{a}_s)$ and $x(\mathfrak{a}\mathfrak{a}_t)$ are isogenous by construction. The corresponding isogeny degree is however not independent of K (cf. (N)). $\square$

§10. Single modular curve.

Let $\mathcal{F}$ be the $\overline{\mathbb{Q}}$-algebra of functions on Ξ_1 with values in $\mathbb{P}^1(\overline{\mathbb{Q}})$. Let

$$\phi : \mathcal{O}_V \to \mathcal{F}$$

be the morphism sending f to an element in $\mathcal{F}$ given by

$$x(\mathfrak{a}) \mapsto f(x(\mathfrak{a})).$$

Here $f \in \mathcal{O}_V$ and $\mathfrak{a} \in Cl_K$ for $K \in \Theta$.

Corollary: *Let the notation and assumptions be as above. Then, we have an embedding of the function field $\overline{\mathbb{Q}}(V)$ of the rational functions on V into $\mathcal{F}$.*

§11. Toric periods.

We first note that

$$[\mathfrak{a}] \mapsto f(x(\mathfrak{a}))\lambda_K(\mathfrak{a})$$

is a well defined function on Cl_K; i.e., the value $f(x(\mathfrak{a}))\lambda_K(\mathfrak{a})$ only depends on the class $[\mathfrak{a}]$.

For $\chi \in \widehat{Cl_K}$, let $P_{f,\lambda}(\chi)$ be the toric period given by

$$P_{f,\lambda}(\chi) = \frac{1}{|Cl_K|} \cdot \sum_{[\mathfrak{a}] \in Cl_K} \chi([\mathfrak{a}]) f(x(\mathfrak{a}))\lambda(\mathfrak{a}).$$

whose square is the central critical value $L(\frac{1}{2}, f \otimes \lambda_K \chi)$ by Waldspurger.

§12. Existence of non-vanishing $L(\frac{1}{2}, f \otimes \lambda_K \chi)$.

For a fixed holomorphic form f, we consider the non-vanishing of the toric periods as K varies over an infinite subset of CM quadratic extensions of the totally real field.

Lemma: *Let f be a holomorphic Hilbert modular form over a totally real field F of level $\Gamma_0(\mathfrak{n})$ as above and $Sh_{\Gamma_1(\mathfrak{n})}$ the Hilbert modular Shimura variety of level $\Gamma_1(\mathfrak{n})$. Let Θ be an infinite set of CM quadratic extensions of the totally real field. For $K \in \Theta$, let λ be a Hecke character over K as above. Then, for all but finitely many $K \in \Theta$, there exists $\chi \in \widehat{Cl_K}$ such that the toric period $P_{f,\lambda}(\chi)$ is non-zero.*

§13. Proof.

Apply Fourier analysis to the finite group Cl_K: In view of the Fourier inversion, it suffices to show that the above function

$$[\mathfrak{a}] \mapsto f(x(\mathfrak{a}))\lambda(\mathfrak{a})$$

is not identically zero on Cl_K for all but finitely many $K \in \mathbf{\Theta}$. This is precisely Corollary in §10.

In view of the sheaf theoretic definition of holomorphic modular forms under Zariski topology and division by another holomorphic form of the same weight, the assertion readily follows from the case of modular functions. $\square$

§14. Non-vanishing of many toric periods.

We in fact have the non-vanishing of many toric periods.

Theorem: *Let f be a non-constant holomorphic Hilbert modular form on $\Gamma_0(\mathfrak{n})$ as above. Then, we have*

$$\liminf_{K \in \Theta} \left| \{ \chi \in \widehat{Cl_K} : P_{f,\lambda}(\chi) \neq 0 \} \right| = \infty.$$

§15. Linear dependence, start of the proof.

Assume that there exists $l \in \mathbb{Z}$ such that exactly $l-1$ of $P_{f,\chi}$

$$\sum_{[\mathfrak{a}]\in Cl_K} \chi([\mathfrak{a}]) f(x(\mathfrak{a}))\lambda(\mathfrak{a})$$

are non-zero, for all $K \in \Theta$ with sufficiently large discriminant.

For $K \in \Theta$, let $[\mathfrak{b}_1], ..., [\mathfrak{b}_l] \in Cl_K$ be l-ideal classes. In view of the above assumption, it follows that the functions given by

$$[\mathfrak{a}] \mapsto f(x(\mathfrak{a}\mathfrak{b}_i))\lambda(\mathfrak{a}\mathfrak{b}_i)$$

viewed as elements in the vector space of maps $Cl_K \to \overline{\mathbb{Q}}$ are linearly dependent for $1 \le i \le l$; i.e., they vanishes at $|Cl_K| - l$ linear forms indexed by $\chi \in \widehat{Cl_K}$. Say,

$$\sum_{i=1}^{l} c_{K,i} f(x(\mathfrak{a}\mathfrak{b}_i))\lambda(\mathfrak{a}\mathfrak{b}_i) = 0 \quad \text{for } c_{K,i} \in \overline{\mathbb{Q}} \text{ and any } [\mathfrak{a}] \in Cl_K. \quad (*)$$

§16. Determinant.

We now choose l-ideal classes $[\mathfrak{c}_1], ..., [\mathfrak{c}_l] \in Cl_K$ and consider $(*)$ for $[\mathfrak{a}] = [\mathfrak{c}_1], ..., [\mathfrak{c}_l]$. It follows that

$$\det\Big(f(x(\mathfrak{c}_j\mathfrak{b}_i))\lambda(\mathfrak{c}_j\mathfrak{b}_i)\Big)_{1\le i,j\le l} = 0.$$

Note that

$$\Big(f(x(\mathfrak{c}_j\mathfrak{b}_i))\lambda(\mathfrak{c}_j\mathfrak{b}_i)\Big)_{1\le i,j\le l}$$
$$= diag\Big(\lambda(\mathfrak{c}_j)\Big)_{1\le j\le l}\Big(f(x(\mathfrak{c}_j\mathfrak{b}_i))\Big)_{1\le i,j\le l} diag\Big(\lambda(\mathfrak{b}_i)\Big)_{1\le i\le l}.$$

Here $diag\Big(a_k\Big)_{1\le k\le l}$ denotes the diagonal matrix with diagonal entries $\{a_1, ..., a_l\}$.

§17. Conclusion.

We conclude that

$$\det\left(f(x(\mathfrak{c}_j\mathfrak{b}_i))\right)_{1\le i,j\le l} = 0.$$

Consider the function h_l on the self-product Sh^{l^2} given by

$$\left(x_{i,j}\right)_{1\le i,j\le l} \mapsto \det\left(f(x_{i,j})\right)_{1\le i,j\le l}.$$

The lower triangular entries of the above matrix can be all arranged to be zero. Moreover, the product of the diagonal entries can be arranged to be non-constant simultaneously as f is non-constant. It follows that h_l is a non-constant function.

The function h_l vanishes on the collection Ξ_{l^2} of CM points. This contradicts the Zariski density of Ξ_{l^2} and finishes the proof.

§**18. End of the proof.** Recall the main theorem:

Theorem *We have*

$$\liminf_{K\in\Theta}\left|\{\chi\in\widehat{Cl_K} : L(\frac{1}{2}, f\otimes\lambda\chi)\neq 0\}\right| = \infty.$$

Proof. In view of our hypotheses and the Waldspurger formula, it follows that

$$L(\frac{1}{2}, f\otimes\lambda\chi)\neq 0 \iff P_{f,\lambda}(\chi)\neq 0.$$

Here f is a Gross-Prasad test vector/toric form associated to the pair (f,λ).

The theorem thus readily follows from Theorem we just finished its proof. $\square$

§19. Some consequences.

F. Castella proved many cases of Bloch–Kato conjecture for the motive M_f. Here are assumptions he made:
Let $p > 5$ be a prime, f a p-ordinary elliptic modular newform of weight $k \geq 2$ with $k \equiv 2 \mod p-1$ and level $\Gamma_0(N)$ with N prime to p. Let L be the p-adic Hecke field of f, $\rho_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(L)$ the p-adic Galois representation associated to f and $\overline{\rho}_f$ the mod p reduction of ρ_f. Let Θ be an infinite set of imaginary quadratic extensions K with integer ring $\mathcal{O}_K$ and odd discriminant D_K satisfying the following hypotheses.

(1). The prime p splits in K and $p \nmid h_K$,
(2). there exists an ideal $\mathfrak{N} \subset \mathcal{O}_K$ such that $\mathcal{O}_K/\mathfrak{N} \xrightarrow{\sim} \mathbb{Z}/N\mathbb{Z}$,
(3). $\overline{\rho}_f|_{G_K}$ is absolutely irreducible and
(4). $\overline{\rho}_f$ is ramified at all primes dividing the greatest common divisor (D_K, N).

§20. Main theorem + Castella's result.

Corollary: *Let Θ be the set of imaginary quadratic fields satisfying (1–4). Let λ_K be an unramified Hecke character over K as in Main Theorem and $\widehat{\lambda}_K$ the p-adic avatar. Then, we have*

$$\liminf_{K \in \Theta} \left| \{\chi \in \widehat{Cl_K} : \mathrm{rank}\,\mathrm{Sel}_K(\rho_f \otimes \widehat{\lambda}_K \chi) = 0\} \right| = \infty.$$

The existence of infinitely many imaginary quadratic fields satisfying the hypotheses (1)-(4) is shown by K. Prasanna.