Λ-ADIC p-DIVISIBLE GROUPS, II

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Contents

1. Mod p modular curves

We consider the following Drinfeld-style moduli problem classifying $(E, \phi'_p, \phi_N)_{/A}$ over $\mathbb{Z}_{(p)}$, where $\phi_N : \mu_N \hookrightarrow E[N]$ and ϕ'_p is made of isogenies $\pi : E \to E'$ and ${}^t\pi : E' \to E$ of degree p^r with two points $P \in E(A)$ and $P' \in E'(A)$ such that $\mathrm{Ker}(\pi)$ is equal to the relative Cartier divisor $\sum_{j=0}^{p^r-1}[jP] \subset E$ and $\mathrm{Ker}({}^t\pi)$ is equal to the relative Cartier divisor $\sum_{j=0}^{p^r-1}[jP'] \subset E'$. The canonical Cartier duality pairing $\mathrm{Ker}(\pi) \times \mathrm{Ker}({}^t\pi) \to \mu_{p^r}$ gives a point $\zeta_{p^r} = \langle P, P'\rangle$. Thus this moduli problem, we regard as defined over $\mathbb{Z}_{(p)}[\mu_{p^r}]$. As shown in [AME], this problem is represented by a regular affine scheme over $\mathbb{Z}_{(p)}[\mu_{p^r}]$ with regular projective compactification X'_r whose generic fiber is $X_{r/\mathbb{Q}[\mu_{p^r}]}$. The special fiber $X'_{r/\mathbb{F}_p}$ has the following description:

$$X'_{r/\mathbb{F}_p} = \bigcup_{a+b=r, a\geq 0, b\geq 0} X'_{(a,b)},$$

for smooth irreducible projective curves $X'_{(a,b)}$ intersecting only at super-singular points. The curve $X'_{(r,0)}$ removed super-singular points and cusps represents $(E, \mu_{p^r} \hookrightarrow E, \phi_N)$, and $X'_{(0,r)}$ removed super-singular points and cusps represents $(E, \mathbb{Z}/p^r\mathbb{Z} \hookrightarrow E, \phi_N)$. So they are both Igusa curves, and $X'_{(0,r)} \cong {X'}^{(p^r)}_{(r,0)}$ canonically. We put $Y'_r = X'_{(0,r)} \cup X'_{(r,0)}$. On the middle components $X'_{(a,b)}$ with $ab \neq 0$, $\pi : E \to E'$ factors $E \xrightarrow{F^a} E^{(p^a)} \cong {E'}^{(p^b)} \xrightarrow{V^b} E'$. As before, let J_r (resp. G_r and ${}^tG_r$) be the identity connected component of the Néron model of $J_{r/\mathbb{Q}}$ (resp. G_r and ${}^tG_r$) over $R_r := \mathbb{Z}_{(p)}[\mu_{p^r}]$. Mazur and Wiles in [MW] have shown the existence of a canonical

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isogeny $av(\mathrm{Pic}^0_{Y_r/\mathbb{F}_p}) \to av(G_{r/\mathbb{F}_p})$ for abelian variety part "av". By a theorem of Raynaud [NMD] Theorem 9.4.5, $J_r = \mathrm{Pic}^0_{X_r/R_r}$. Thus taking the special fiber, we have a surjection $J_{r/\mathbb{F}_p} = \mathrm{Pic}^0_{X_r/\mathbb{F}_p} \twoheadrightarrow \mathrm{Pic}^0_{Y_r/\mathbb{F}_p}$ corresponding to the inclusion $Y_r \hookrightarrow X_r$. Then by Theorem 3.1 in the first lecture combined with [MW] Proposition in page 267, we find

Corollary 1.1.
$$J_r^{ord}[p^\infty]_{/\mathbb{F}_p} \cong \mathrm{Pic}^0_{Y_r/\mathbb{F}_p}[p^\infty]^{ord} \cong G_r^{ord}[p^\infty]_{/\mathbb{F}_p}.$$

Adding the toric part to the isogeny in [MW], we have an isogeny $\mathrm{Pic}^0_{Y_r/\mathbb{F}_p}[p^\infty]^{ord} \to G_r^{ord}[p^\infty]_{/\mathbb{F}_p}$, but the projection: $J_r^{ord}[p^\infty]_{/\mathbb{F}_p} \cong \mathrm{Pic}^0_{Y_r/\mathbb{F}_p}[p^\infty]^{ord}$ composed with this isogeny is the special fiber of the isomorphism in Theorem 3.1. □

From this identification, we can give an alternating proof of the closed immersion of $G_r^{ord}[p^\infty]$ into $G_s^{ord}[p^\infty]$, similarly to the characteristic 0 case. Also they have shown that the $U(p)$ operator on the abelian quotient $\mathrm{Pic}^0_{X_{(r,0)}/\mathbb{F}_p} \times \mathrm{Pic}^0_{X_{(0,r)}/\mathbb{F}_p}$ of $\mathrm{Pic}^0_{Y_r/\mathbb{F}_p}$ has the following matrix shape

$$\begin{pmatrix} F & * \\ 0 & V\langle p^{(p)}\rangle \end{pmatrix} \quad \text{on} \quad \mathrm{Pic}^0_{X_{(0,r)}/\mathbb{F}_p} \times \mathrm{Pic}^0_{X_{(r,0)}/\mathbb{F}_p} \tag{1.1}$$

for the p-power relative Frobenius F and its dual V. If $N = 1$, $U(p) = \left(\begin{smallmatrix} F & 0 \\ 0 & V \end{smallmatrix}\right)$ is semi-simple on $\mathrm{Pic}^0_{X'_{(0,r)}/\mathbb{F}_p} \times \mathrm{Pic}^0_{X_{(r,0)}/\mathbb{F}_p}$. Here $\langle p^{(p)}\rangle$ is the diamond operator for $p \in (\mathbb{Z}/N\mathbb{Z})^\times$.

We can think of the following new moduli problem genuinely defined over $\mathbb{Z}_{(p)}$ classifying $(E, \phi_p, \phi_N)_{/A}$ over $\mathbb{Z}_{(p)}$, where $\phi_N : \mu_N \hookrightarrow E[N]$ and ϕ_p is made of an isogeny $\pi : E \to E'$ of degree p^r with a points $P \in E(A)$ such that $\mathrm{Ker}(\pi)$ is equal to the relative Cartier divisor $\sum_{j=0}^{p^r-1}[jP] \subset E$. Then the fine moduli scheme exists over $\mathbb{Z}$ which is a regular affine scheme of relative dimension 1 over $\mathbb{Z}$ (see [AME] Chapter 5). The compactification $X'_{r/\mathbb{Z}_{(p)}}$ of the moduli scheme is a projective regular curve whose generic fiber is canonically isomorphic to $X_1(Np^r)_{/\mathbb{Q}}$. The mod p curve $X'_{r/\mathbb{F}_p} = X'_{r/\mathbb{Z}_{(p)}} \otimes \mathbb{F}_p$ is a union of proper irreducible curves $X'_{(a,b)}$ as before intersecting at super-singular points. Also, $X_{r/\mathbb{Z}_{(p)}[\mu_{p^r}]}$ is the normalization of $X'_{r/\mathbb{Z}_{(p)}}$ in the function field of X_r. Thus we have a natural finite morphism $X_{r/\mathbb{Z}_{(p)}[\mu_{p^r}]} \to X'_{r/\mathbb{Z}_{(p)}} \otimes \mathbb{Z}_{(p)}[\mu_{p^r}]$. This morphism induces an isomorphism $X_{(0,r)} \cong X'_{(0,r)}$; so, we the identity connected component $J'_{r/\mathbb{Z}_{(p)}}$ of the Néron model of $J_{r/\mathbb{Q}}$ is canonically isomorphic to $\mathrm{Pic}^0_{X'_r/\mathbb{Z}_{(p)}}$ again by the theorem of Raynaud. In particular, $X'_{0/\mathbb{F}_p} \cong X_{0/\mathbb{F}_p}$ is a smooth projective curve. We write $G'_{r/\mathbb{Z}_{(p)}}$ (resp. ${}^tG'_{r/\mathbb{Z}_{(p)}}$) for the identity connected component of the Néron model over $\mathbb{Z}_{(p)}$ of $G_{r/\mathbb{Q}}$ (resp. ${}^tG_{r/\mathbb{Q}}$). We put $Y'_r = X'_{(0,r)} \cup X'_{(r,0)} \subset X'_{r/\mathbb{F}_p}$. Then we admit the following difficult fact:

Theorem 1.2.

$$ {}^tG'^{ord}_r[p^\infty]_{/\mathbb{F}_p} \cong J'^{ord}_r[p^\infty]_{/\mathbb{F}_p} \cong \mathrm{Pic}^0_{Y'_r/\mathbb{F}_p}[p^\infty]^{ord} \cong G'^{ord}_r[p^\infty]_{/\mathbb{F}_p}. $$

We now study the unipotent radical of $G'^{ord}_r[p^n]_{/\mathbb{F}_p}$. For simplicity, we assume that $N = 1$. As was shown in [AME] Theorem 13.5.4 and Corollary 13.5.3,

$$ X'_{(r,0)} \cong X_0 \otimes_{\mathbb{F}_p} \mathbb{F}_p[\widehat{x}]/(\widehat{x}^{\varphi(p^r)}), $$

where $m := \varphi(p^r) = p^{r-1}(p-1)$ and $\widehat{x}$ is a fixed parameter of the formal group of the universal elliptic curve $\mathbf{E}_{/X_0}$. Writing x for the function $Y'_r \ni P \mapsto \widehat{x}(P)$ and t for the parameter of the universal deformation space of a given super-singular elliptic curve over $S \in Y'_r(\overline{\mathbb{F}}_p)$. Then the completed local ring $\widehat{\mathcal{O}}_{Y'_r,S/\overline{\mathbb{F}}_p} \cong \overline{\mathbb{F}}_p[[t,x]]/(tx^m)$. Thus the nilradical $\mathfrak{n}$ of $\mathcal{O}_{Y'_r/\mathbb{F}_p}$ is Zariski locally generated by (tx); so, it has a filtration $\mathfrak{n} = \mathfrak{n}_1 \supset \mathfrak{n}_1 \supset \cdots \supset \mathfrak{n}_m = 0$ corresponding $(tx) \supset (tx^2) \supset \cdots \supset (tx^m) = 0$ at each super-singular point. Since $\widehat{x}$ on the ordinary locus of X_0 is the parameter of $\widehat{\mathbf{E}} \cong \widehat{\mathbb{G}}_m$, we may assume $\widehat{x} \circ \langle z\rangle = (1+\widehat{x})^z - 1 \equiv zx \mod \mathfrak{n}_2$. Thus on the sheaf $\mathfrak{n}_j/\mathfrak{n}_{j+1}$, $\mu_{p-1} \subset \mathbb{Z}_p^\times$ acts by $\zeta \mapsto \zeta^j = \omega^j$. Thus $\mathfrak{n}_{p-1}/\mathfrak{n}_p$ supported on $X'^{red}_{(r,0)} = X_0$ is a line bundle isomorphic to $\mathcal{O}(\Sigma)$ with $\mathcal{O}_{X_0}/\mathcal{O}(\Sigma) = \mathcal{O}_\Sigma$, where Σ is the disjoint union of super-singular points on X_0. Let $\pi : Ig_1 \to X_0$ be the first layer of the Igusa tower classifying $\mu_p \hookrightarrow E$ in addition to ϕ_N. The direct image

$$ \pi_*\mathcal{O}_{Ig}(\Sigma) = \mathcal{O}(\Sigma) \oplus \bigoplus_{a=1}^{p-2} \mathcal{O}(\omega^a) $$

for the ω^a-eigenspace $\mathcal{O}(\omega^a)$, which is a line bundle over X_0 of degree 0. Then we have for $j \not\equiv 0 \mod (p-1)$, $\mathfrak{n}_j/\mathfrak{n}_{j+1} \cong \mathcal{O}(\omega^a)$. Since $\widehat{x}$ is a parameter of formal group, it is unaffected dividing the elliptic curves by an étale subgroup. Thus $H^j(X_0, \mathfrak{n}_i/\mathfrak{n}_{i+1})$ the Hecke operator $T(n)$ for n prime to p and $U(q)$ for $q|Np$ acts naturally. Write $Y = Y'_r$ simply, and let $g : X_0 \to \mathrm{Spec}(\mathbb{F}_p)$ be the structural morphism. Note that

$$ 1 \to R^1g_*(1+\mathfrak{n}) \to \mathrm{Pic}_{Y/\mathbb{F}_p} \to \mathrm{Pic}_{Y^{red}/\mathbb{F}_p} \to 1 $$

is exact. The quotient $\mathrm{Pic}_{Y^{red}/\mathbb{F}_p}$ is a semi-abelian scheme with an exact sequence

$$ 0 \to \text{a torus} \to \mathrm{Pic}^0_{Y^{red}/\mathbb{F}_p} \to \mathrm{Pic}^0_{X'_{(0,r)}/\mathbb{F}_p} \times \mathrm{Pic}^0_{X_0/\mathbb{F}_p} \to 0. $$

Thus the unipotent radical $R_U(\mathrm{Pic}_{Y'_r/\mathbb{F}_p})$ has a filtration $\mathcal{U}_1 \supset \mathcal{U}_2 \supset \cdots \supset \mathcal{U}_m = 0$ such that

$$ \mathcal{U}_i/\mathcal{U}_{i+1} \cong H^1(X_0, (1+\mathfrak{n}_i)/(1+\mathfrak{n}_{i+1})) \cong \begin{cases} H^1(X_0, \mathcal{O}(\omega^i)) & \text{if } i \not\equiv 0 \mod (p-1) \\ H^1(X_0, \mathcal{O}(\Sigma)) & \text{if } (p-1)|i, \end{cases} $$

as Hecke modules. By the Serre duality, $H^0(X_0, \Omega_{X_0/\mathbb{F}_p}(-\Sigma))$ (resp. $H^0(X_0, \Omega_{X_0/\mathbb{F}_p}(\omega^{-i}))$ is the dual space of $H^1(X_0, \mathcal{O}(\Sigma))$ (resp. $H^1(X_0, \mathcal{O}(\omega^i))$). Since the Hasse invariant $H \in H^0(X_0, \underline{\omega}^{p-1})$ has simple zero at super-singular points ([AME] Theorem 12.4.3), we have $\Omega_{X_0/\mathbb{F}_p}(-\Sigma) \cong \Omega_{X_0/\mathbb{F}_p} \otimes \underline{\omega}^{(p-1)}$ as line bundles (by multiplying the Hasse invariant). By the Kodaira-Spencer isomorphism, as Hecke modules, we have

$$H^0(X_0, \Omega_{X_0/\mathbb{F}_p} \otimes \underline{\omega}^{(p-1)}) \cong S_{p+1}(SL_2(\mathbb{Z}), \mathbb{F}_p) \cong S_2(\Gamma_0(p), \mathbb{F}_p) \text{ if } (p-1)|i.$$

By dualizing back, the action of $U(p)$ on the left-hand side of the identity below

$$H^1(X_0, \mathfrak{n}_i/\mathfrak{n}_{i+1}) \cong S_{p+1}(SL_2(\mathbb{Z}), \mathbb{F}_p) \ ((p-1)|i.)$$

is by the natural contravariant action of $T(p) = F + V\langle p^{(p)}\rangle$ on the right-hand side. As for $H^1(X_0, \mathcal{O}(\omega^i))$ for i nontrivial modulo $(p-1)$, by a remark in [Ri] page 204, we have

$$H^1(X_0, \mathcal{O}(\omega^i)) \cong H^0(X_0, \Omega_{X_0/\mathbb{F}_p}(\omega^{-i})) \cong H^0(Ig, \Omega_{Ig/\mathbb{F}_p}(\omega^{-i})) \cong S_{[p+1-i]}(SL_2(\mathbb{Z}), \mathbb{F}_p),$$

where $[x] \equiv x \mod (p-1)$ and $3 \leq x \leq p+1$.

2. The α-eigen space in $G_\infty^{ord}[p^\infty]$

For an eigenvalue α of $U(p)$ on $S_2(\Gamma_1(p^r))$, under some assumptions, we try to show that $(G^{ord}_{\infty/R_\infty}[p^\infty] \otimes \mathbb{Z}[\alpha])[U(p)-\alpha]$ is contained in $G_r \otimes \mathbb{Z}[\alpha]$.

Look into the Hecke algebra

$$\mathbf{h} = \Lambda[T(n), U(p)]_{p\nmid n} \subset \mathrm{End}_\Lambda(G_\infty^{ord}[p^\infty]_{/R_\infty}).$$

Take a Gorenstein local ring $\mathbb{T}$ of $\mathbf{h}$ with maximal ideal $\mathfrak{m}$ on which $\langle\zeta\rangle$ ($\zeta \in \mu_{p-1} \subset \mathbb{Z}_p^\times$) acts by $\omega^a : \zeta \mapsto \zeta^a$ with $0 < a < p-1$. We give ourselves a Hecke eigenvalue α given by $f|U(p) = \alpha f$ for $f \in S_2(\Gamma_0(p^{r(\alpha)}), \varepsilon)$ with $\mathbb{T}\cdot f \neq 0$. Write $\mathfrak{G}_r$ for $G_r^{ord}[p^\infty] \otimes \mathbb{Z}[\alpha]$ over $\mathbb{Z}_{(p)}[\mu_{p^r}]$. Adding the subscript $\mathbb{T}$, we indicate the $\mathbb{T}$-eigenspace; so, for example, $\mathfrak{G}_{\mathbb{T},r} = \mathbb{T}(\mathfrak{G}_r)$. Then $\mathfrak{G}_{\mathbb{T},r}$ is a BT group over R_r. Identify Λ with $\mathbb{Z}_p[[\Gamma]]$, we write $t = [\gamma] - \varepsilon(\gamma)$, where γ is a fixed generator of Γ and $[\gamma] \in \Gamma \subset \Lambda$ for the group element corresponding to γ (which is $1+T$ if we write $\Lambda = \mathbb{Z}_p[[T]]$).

Then we expect

Conjecture 2.1. *Let $\mathfrak{G}[U(p)-\alpha, t^n] = \{x \in \mathfrak{G}(\overline{\mathbb{Q}}) | x|U(p) = \alpha x \text{ and } t^n x = 0\}$. Then there exists a positive integer $s \geq r$ independent of $n < \infty$ such that the divisible part $\mathfrak{G}_\infty[U(p)-\alpha, t^n]^{div}(\overline{\mathbb{Q}})$ is contained in G_s^α.*

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