CENTRAL CRITICAL VALUES OF MODULAR HECKE L-FUNCTION

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To the memory of Professor Masayoshi Nagata

Abstract. We give an explicit formula of the central critical value $L(\frac{1}{2},\widehat{\pi}\otimes\chi)$ of the base-change lift $\widehat{\pi}$ to an imaginary quadratic field K of an automorphic representation π as the square of a finite sum of the values of a nearly holomorphic cusp form in π at elliptic curves with complex multiplication by K. As long as the transcendental factor of the value is a CM period, χ is basically any unitary arithmetic Hecke character of K inducing the inverse of the central character of π.

Introduction

Let D be a quaternion algebra over a number field F, regarded as a quadratic space by its norm form $N : D \to F$. The orthogonal similitude group GO_D is isogenous to $D^\times \times D^\times$ by the action $(g,h)v = gvh^{-1}$ on $v \in D$. Pick a quadratic extension K/F with an embedding K into D; so, we have $K^\times\backslash K^\times_{\mathbb{A}} \hookrightarrow D^\times\backslash D^\times_{\mathbb{A}}$. Take a Hecke eigenform $\mathbf{f}$ on $D^\times\backslash D^\times_{\mathbb{A}}$ with central character $\boldsymbol{\psi}$, and pick a character χ of $K^\times\backslash K^\times_{\mathbb{A}}$ with $\chi|_{F^\times_{\mathbb{A}}} = \boldsymbol{\psi}^{-1}$. The unitarization $\mathbf{f}^u(g) := \mathbf{f}(g)|\boldsymbol{\psi}(\det(g))|^{-1/2}$ generates a unitary automorphic representation $\pi_{\mathbf{f}}$, which has a base-change lift $\widehat{\pi}_{\mathbf{f}}$ to $\mathrm{Res}_{K/F}D^\times$. Similarly we set $\chi^- = (\chi\circ c)/|\chi|$ for $\langle c\rangle = \mathrm{Gal}(K/F)$. Vigneras and Waldspurger [Wa] proved a striking (and ingenious) formula relating the square of $L_\chi(\mathbf{f}) := \int_{K^\times\backslash K^\times_{\mathbb{A}}} \mathbf{f}(t)\chi(t)d^\times t$ to the central critical value $L(\frac{1}{2},\widehat{\pi}_{\mathbf{f}}\otimes\chi^-)$ (up to sometimes undetermined local factors). When K/F is a totally imaginary quadratic extension of a totally real field F (a CM extension), $L_\chi(\mathbf{f})$ is basically a finite sum of the value of $\mathbf{f}$ at CM-abelian varieties and hence, it is essentially p-integral up to the Néron period of the abelian variety. If one wants to interpolate p-adically $L_\chi(\mathbf{f})$ over arithmetic χ's for a cusp form $\mathbf{f}$ as Katz did for Eisenstein series [K], we need an explicit formula without ambiguity. Such computation has been done by many people including Shou-Wu Zhang, Ben Howard, Kartik Prasanna and others (cf. [MSS], [YZZ] and [P]). However published computation seems limited to the case where the infinity type of χ is either the highest or the lowest determined by $\mathbf{f}$ and D and the conductor of χ could be limited to split primes of $K/\mathbb{Q}$ (the Heegner hypothesis). For simplicity, assuming $F = \mathbb{Q}$, K is imaginary quadratic and $D = M_2(\mathbb{Q})$, we present here such an explicit formula of $L_\chi(\mathbf{f})^2$ (Theorem 4.1) covering all arithmetic characters χ with $\chi|_{\mathbb{A}^\times} = \boldsymbol{\psi}^{-1}$ (producing "critical" central value). The formula involves an Euler-like factor (at primes dividing the level) which vanishes only in limited cases. A main point is to find a good Schwartz-Bruhat function on $D_{\mathbb{A}}$ making the theta correspondence optimal. This optimal choice is suggested by the explicit computation of the q-expansion of the theta lift of $\mathbf{f}$ to $GO(F_{\mathbb{A}})$ through "partial Fourier transform" of the Siegel–Weil theta series which was studied in [H06] in order to prove the anticyclotomic main conjecture for CM fields. Our method is elementary, classical and almost global without resorting much to Langlands theory, and we can extend it to general base fields. In this article, we restrict ourselves to $M_2(\mathbb{Q})$ for simplicity. Obviously one may use the same Schwarz-Bruhat function for division D fixing an isomorphism $D_\ell \cong M_2(F_\ell)$ (for almost all primes ℓ) or take a non-CM quadratic extension K/F. However we need a more careful analysis (e.g., [P]) of the rationality/transcendence of the theta correspondence in these slightly more general cases, which we hope to treat in future.

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Organization of the paper and a sketch of the proof. In Section 1, starting with a brief discussion of how to associate automorphic forms $\mathbf{f}$ on $GL_2(\mathbb{A})$ to classical holomorphic elliptic modular forms f, we recall the Siegel-Weil theta series Θ and its theta correspondence: $\mathbf{f} \mapsto \Theta(\mathbf{f})$ for the dual pairs $(SL(2), SO(2,2))$ and $(SL(2), SO(2))$ of the quadratic spaces $(M_2(\mathbb{Q}), \det)$ and $(K, \pm N_{K/\mathbb{Q}})$. For an explicitly given Schwartz–Bruhat function on $M_2(\mathbb{A})$, we make a computation of its partial Fourier transform, which later enables us to make explicit the image $\Theta(\mathbf{f})$ on the side of $SO(2,2) \sim SL(2) \times SL(2)$. In other words, starting with a normalized Hecke eigenform f of weight k, by our choice of a Schwartz–Bruhat function, we conclude the image $\Theta(\mathbf{f}) = \int_{Sh} \Theta \mathbf{f} dx$ is given $(2i)^k \mathbf{f} \otimes \mathbf{f}$ for a suitably chosen measure dx and an elliptic modular Shimura curve Sh. For this reason, we call the choice optimal. The precise choice of the Schwartz–Bruhat function is made in §1.4, and then we adjust the choice to make easier later computation of Rankin convolution in §1.7. In Section 2, we compute the restriction of the Siegel–Weil theta series to the orthogonal group $O(2) \times O(2)$ given by the quadratic space $(K, N_{K/\mathbb{Q}}) \oplus (K, -N_{K/\mathbb{Q}}) \cong (M_2(\mathbb{Q}), \det)$ and show that the restriction is a product $\theta_k \cdot \theta'$ of two binary theta series θ_k, θ' of K (of weight $1+k$ and 1, respectively). Via Siegel–Weil formula (and a more classical result of Hecke), we identify θ' with an explicitly given Eisenstein series E. In Section 3, we apply to Θ a (two variable) Maass–Shimura differential operator $\Delta = \delta_k^m \otimes \delta_k^m$ on $SO(2,2) \sim SL(2) \times SL(2)$ which is induced by (the m-th power of) a Lie invariant differential operator $X \otimes X$ on $SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$. The restriction of this derived $\Delta\Theta$ to $O(2) \times O(2)$ turns out to be $\theta_{k+2m} \delta_1^m E$ for a holomorphic binary theta series θ_{k+2m} with higher weight $1+k+2m$ than θ_k. In the final Section 4, we state our main theorem and compute $\int_{(K^\times \backslash K_{\mathbb{A}}^\times)^2} (\mathbf{f}\chi) \otimes (\mathbf{f}\chi) dt^\times \otimes dt^\times$ (with respect to a suitable Haar measure $d^\times t$ on $K_{\mathbb{A}}^\times$). On the one hand, this value is $L_\chi(\mathbf{f})^2$. Replacing $\mathbf{f} \otimes \mathbf{f}$ by $\Theta(\mathbf{f})$ transforms the integral into a double integral over $(K^\times \backslash K_{\mathbb{A}}^\times)^2 \times Sh$. Interchanging the order of integration, $L_\chi(\mathbf{f})^2$ is transformed into a Rankin convolution integral $\int_{Sh} \mathbf{f} \theta_{k+2m} \delta_1^m E dx$, which gives rise to the L-value. This proves the desired formula.

Contents

1. Quaternionic theta correspondence

1.1. **Classical modular forms and adelic ones.** Let S be the algebraic group $SL(2)_{/\mathbb{Z}}$. Let $f(\tau)$ be a cusp form in $S_k(\Gamma,\psi)$ ($\tau \in \mathfrak{H} = \{z \in \mathbb{C} | \mathrm{Im}(z) > 0\}$) for a congruence subgroup Γ of $S(\mathbb{Q})$. Here ψ is a finite order character whose kernel is a congruence subgroup Γ' of Γ. Write $\widehat{\Gamma}$ for the closure of Γ in $S(\mathbb{A}^{(\infty)})$. Then $\widehat{\Gamma}/\widehat{\Gamma}' \cong \Gamma/\Gamma'$, and hence we may regard ψ as a character of $\widehat{\Gamma}$. Then by the strong approximation theorem, we have $S(\mathbb{A}) = S(\mathbb{Q})\widehat{\Gamma}' S(\mathbb{R})$. Thus we can lift f to $\mathbf{f} : S(\mathbb{Q})\backslash S(\mathbb{A})/\widehat{\Gamma} \to \mathbb{C}$ by $\mathbf{f}(\alpha u) = f(u_\infty(i))\psi(u)j(u_\infty, i)^{-k}$ for $\alpha \in S(\mathbb{Q})$ and $u \in \widehat{\Gamma}$, where $j(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \tau) = (c\tau + d)$. For our later use, we put $J(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \tau) = (ad - bc)^{-1/2}(c\tau + d)$. We note that $j(r(\theta), i) = J(r(\theta), i) = e^{-i\theta}$ for $r(\theta) = \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right) \in SO_2(\mathbb{R})$. Similarly, writing Z for the center of $GL(2)$, we have $j(\zeta, \tau) = \zeta$ while $J(\zeta, \tau) = 1$ for $\zeta \in Z(\mathbb{R})$.

For an open compact subgroup $\widehat{\Gamma}$ of $GL_2(\mathbb{A}^{(\infty)})$ with $GL_2(\mathbb{A}) = GL_2(\mathbb{Q})\widehat{\Gamma} \cdot GL_2^+(\mathbb{R})$ ($GL_2^+(\mathbb{R}) = \{g \in GL_2(\mathbb{R}) | \det(g) > 0\}$), put $\Gamma = \widehat{\Gamma} \cdot GL_2^+(\mathbb{R}) \cap GL_2(\mathbb{Q})$. If $\psi : \widehat{\Gamma} \to \mathbb{C}^\times$ is a continuous character, we may regard ψ as a character of Γ. Write $S_k(\Gamma, \psi)$ for the space of holomorphic cusp forms with $f(\gamma(\tau)) = \psi^{-1}(\gamma)f(\tau)j(\gamma,\tau)^k$. Then we can define $\mathbf{f}(\alpha u) = f(u_\infty(i))\psi(u)j(u_\infty, i)^{-k}$, and $\mathbf{f}$ is a function on $GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A})/\mathrm{Ker}(\psi : \widehat{\Gamma} \to \mathbb{C}^\times)$ such that $\mathbf{f}(\alpha g u) = \psi(u)\mathbf{f}(g)$ for $u \in \widehat{\Gamma}$ and $\alpha \in G(\mathbb{Q})$. Write $\mathcal{S}_k(\widehat{\Gamma}, \psi)$ for the space of cusp forms $\mathbf{f}$ with holomorphic f satisfying $\mathbf{f}(\alpha u) = f(u_\infty(i))\psi(u)j(u_\infty, i)^{-k}$. Thus we have $S_k(\Gamma, \psi) \cong \mathcal{S}_k(\widehat{\Gamma}, \psi)$ by $\mathbf{f} \leftrightarrow f$. More generally, fixing $g \in GL_2(\mathbb{A}^{(\infty)})$, we may define $f_g(z) = \mathbf{f}(g g_\infty)j(g_\infty, i)^k$ with $g_\infty(i) = z$. Then $f_g \in S_k(\Gamma_g, \psi_g)$ for $\Gamma_g = (g\widehat{\Gamma}g^{-1}) \cdot GL_2^+(\mathbb{R}) \cap GL_2(\mathbb{Q})$ and $\psi_g(u) = \psi(g^{-1}ug)$; so, $S_k(\widehat{\Gamma}, \psi) \cong S_k(\Gamma_g, \psi_g)$ via $\mathbf{f} \leftrightarrow f_g$. For $\zeta \in Z(\mathbb{A})$ and $\mathbf{f} \in \mathcal{S}_k(\widehat{\Gamma}, \psi)$, we have $\mathbf{f}|\zeta(x) = \mathbf{f}(\zeta x)$ resides in $\mathcal{S}_k(\widehat{\Gamma}, \psi)$. Thus $Z(\mathbb{A})$ acts on $\mathcal{S}_k(\widehat{\Gamma}, \psi)$. Note that $\mathbf{f}|\zeta_\infty = \zeta_\infty^{-k}\mathbf{f}$. Thus $\mathcal{S}_k(\widehat{\Gamma}, \psi)$ can be decomposed into the direct sum of the eigenspaces of $Z(\mathbb{A})$. On each eigenspace, $Z(\mathbb{A})$ acts by a Hecke character $\boldsymbol{\psi} : \mathbb{A}^\times/\mathbb{Q}^\times \to \mathbb{C}^\times$ with $\boldsymbol{\psi}|_{\widehat{\Gamma} \cap Z(\mathbb{A})} = \psi$ and $\boldsymbol{\psi}(\zeta_\infty) = \zeta_\infty^{-k}$, and $\boldsymbol{\psi}| \cdot |_{\mathbb{A}}^k$ is of finite order. Write this eigenspace as $\mathcal{S}_k(\widehat{\Gamma}, \boldsymbol{\psi})$. Let $\widehat{\Gamma}_0(N) = \left\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(\widehat{\mathbb{Z}}) \middle| c \in N\widehat{\mathbb{Z}}\right\}$ and $\widehat{\Gamma}_1(N) = \left\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \widehat{\Gamma}_0(N) \middle| d - 1 \in N\widehat{\mathbb{Z}}\right\}$. If $\widehat{\Gamma} = \widehat{\Gamma}_0(N)$ for a positive integer N, a choice of ψ is given by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \psi(d)$ for a Dirichlet character ψ modulo N. Then $\boldsymbol{\psi}$ is a Hecke character whose restriction to $\mathbb{Z}_N^\times = \prod_{\ell | N} \mathbb{Z}_\ell^\times$ is given by ψ. Thus as usual, if we lift ψ to $\mathbb{A}^\times$ by $\psi^*(\ell_\ell) = \psi(\ell)$ for ℓ prime to N, we have $\boldsymbol{\psi} = \psi^* | \cdot |_{\mathbb{A}}^{-k}$. We write simply $\mathcal{S}_k(N, \boldsymbol{\psi})$ for $\mathcal{S}_k(\widehat{\Gamma}_0(N), \boldsymbol{\psi})$. Then we have $\mathcal{S}_k(N, \boldsymbol{\psi}) \cong S_k(\Gamma_0(N), \psi)$ via $\mathbf{f} \leftrightarrow f$. Note that $f \in S_k(\Gamma_0(N), \psi)$ satisfies $f(\gamma(z)) = \psi(a)^{-1}f(z)j(\gamma, z)^k$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ (note $\psi(a) = \psi^{-1}(d)$), which could be a common definition of $S_k(\Gamma_0(N), \psi)$).

If we start with an anti-holomorphic modular form $f(z) \in \overline{S}_k(\Gamma, \psi)$, we lift it to the adelic one $\mathbf{f}$ by $\mathbf{f}(\alpha u) = \mathbf{f}(u_\infty(i))\psi(u)j(u_\infty, -i)^{-k}$ for $\alpha \in S(\mathbb{Q})$ and $u \in \widehat{\Gamma}$. Again $\mathbf{f}(\alpha u) = \mathbf{f}(u_\infty(i))\psi(u)j(u_\infty, i)^{-k}$ for $\alpha \in S(\mathbb{Q})$ and $u \in \widehat{\Gamma}$. The corresponding spaces of antiholomorphic adelic modular forms are written as $\overline{\mathcal{S}}_k(\widehat{\Gamma}, \boldsymbol{\psi})$ and $\overline{\mathcal{S}}_k(N, \boldsymbol{\psi})$.

1.2. **Weil representation.** Let (V, Q) be a quadratic space over $\mathbb{Q}$ with dimension $2d$. The quadratic form $V \ni x \mapsto Q(x) \in \mathbb{Q}$ produces a $\mathbb{Q}$-bilinear symmetric pairing $s(x, y) = Q(x+y) - Q(x) - Q(y)$. If $V = D$ and $Q(x) = xx^\iota = N(x)$ (for the reduced norm $N : D \to \mathbb{Q}$ and the main involution ι), then $s(x, y) = \mathrm{Tr}(xy^\iota)$. If $V = K$ and $Q = N_{K/\mathbb{Q}}$, then $s(x, y) = \mathrm{Tr}_{K/\mathbb{Q}}(xy^c)$ ($\langle c \rangle = \mathrm{Gal}(K/\mathbb{Q})$). Write $\mathcal{S}(V_\mathbb{A})$ for the space of Schwartz-Bruhat functions on $V_\mathbb{A} = V \otimes_\mathbb{Q} \mathbb{A}$. The group $S(\mathbb{Q})$ is generated by $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and upper triangular matrices; so, by the density of $S(\mathbb{Q}) \subset S(\mathbb{A}^{(v)})$ diagonally embedded (removing one place v), $S(\mathbb{A}^{(v)})$ is topologically generated by these elements. The Weil representation $\mathbf{r}$ of $S(\mathbb{A})$ on $\mathcal{S}(V_\mathbb{A})$ is defined as follows:

$$(1.1)\quad \mathbf{r}\left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)\phi(v) = \mathbf{e}_\mathbb{A}(Q(v)u)\phi(v),\ \mathbf{r}\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)\phi(v) = \chi_V(a)|a|_\mathbb{A}^d\phi(av) \text{ and } \mathbf{r}\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\phi(v) = \gamma_V\widehat{\phi}(v),$$

where $\chi_V : \mathbb{A}^\times/\mathbb{Q}^\times \to \{\pm 1\}$ is a Hecke character, $\mathbf{e}_\mathbb{A} : \mathbb{A}/\mathbb{Q} \to \mathbb{C}^\times$ is an additive character with $\mathbf{e}_\mathbb{A}(x_\infty) = \exp(2\pi i x_\infty)$ for $x_\infty \in \mathbb{R}$, γ_V is an eighth root of unity both determined by (V, Q) (see [MSS] 8.5.3) and $\widehat{\phi}$ is the Fourier transform with respect to $\mathbf{e}_\mathbb{A}(s(x, y))$ normalized so that $\widehat{\widehat{\phi}}(x) = \phi(-x)$. We have (cf. [MSS] 8.5.3 and [HMI] Proposition 2.61)

- If $(V, Q) = (M_2(\mathbb{Q}), \pm\det)$ for the determinant $\det : M_2(\mathbb{Q}) \to \mathbb{Q}$, $\chi_D = \gamma_D = 1$,

- If $(V,Q)=(K,\pm N_{K/\mathbb{Q}})$ for an imaginary quadratic field K, $\chi_V=\left(\frac{K/\mathbb{Q}}{}\right)$ and $\gamma_V=\mp\sqrt{-1}$.

Let O_V be the orthogonal group and GO_V be its similitude group; so,

$$GO_V(A)=\left\{\alpha\in GL(V\otimes_{\mathbb{Q}}A)|Q(\alpha x)=\nu_V(\alpha)Q(x)\ \text{ with }\nu_V(\alpha)\in A^\times\right\}$$

and $O_V=\mathrm{Ker}(\nu_V:GO_V\to\mathbb{G}_m)$. We let $g\in GO_V(\mathbb{A})$ acts on $\mathcal{S}(V_{\mathbb{A}})$ by

$$L(g)\phi(v)=|\nu_V(g)|_{\mathbb{A}}^{-d/2}\phi(g^{-1}v).$$

Then by [We], the action $\mathbf{r}$ and L commutes on $S(\mathbb{A})\times O_V(\mathbb{A})$; so, we may regard $\mathbf{r}\otimes L$ as a representation of $S(\mathbb{A})\times O_V(\mathbb{A})$. The following result is a main theorem of [We].

Theorem 1.1. *The generalized theta series of Siegel–Weil*

$$\theta_S(\Phi)(x;g)=\sum_{v\in V}(\mathbf{r}(x)L(g))\Phi(v)\quad\textit{(for each }\Phi\in\mathcal{S}(V_{\mathbb{A}})\textit{)}$$

gives an automorphic form defined as a function on $(S(\mathbb{Q})\backslash S(\mathbb{A}))\times(O_V(\mathbb{Q})\backslash O_V(\mathbb{A}))$.

We define two projections $x\mapsto x_S$ and $x\mapsto {}_Sx$ of $GL(2)$ to S by $x_S=x\alpha^{-1}_{\det(x)}$ and ${}_Sx=\alpha^{-1}_{\det(x)}x$ for $\alpha_t=\left(\begin{smallmatrix}1&0\\0&t\end{smallmatrix}\right)$. Let $G_V=\{(x,g)\in GL(2)\times GO_V|\det(x)=\nu_V(g)\}$. Then we have the following skew commuting relation for $(x,g)\in G_V(\mathbb{A})$:

$$\mathbf{r}(x_S)\circ L(g)=L(g)\circ\mathbf{r}({}_Sx).\tag{1.2}$$

Thus we may extend the representation $\mathbf{r}\otimes L$ to a representation of $G_V(\mathbb{A})$ such that $\mathbf{r}(x_S)\otimes L(g)=L(g)\otimes\mathbf{r}({}_Sx)$. We can still think of

$$\theta_G(\phi)(x;g):=\sum_v\mathbf{r}(x_S)\circ L(g)\phi(v)=\sum_v L(g)\circ\mathbf{r}({}_Sx)\phi(v)=:\theta_G(\phi)(g;x).\tag{1.3}$$

In this definition, the variables x and g are not independent; so, we write $\theta_G(x;g)$ if we use the expression $\mathbf{r}(x_S)\circ L(g)$, and we write $\theta_G(g;x)$ if we use the expression $L(g)\circ\mathbf{r}({}_Sx)$ (though they produce the same function).

Lemma 1.2. *The above extended theta series $\theta_G(\phi)(x;g)$ on $G_V(\mathbb{A})$ is left $G_V(\mathbb{Q})$-invariant, that is, it factors through $G_V(\mathbb{Q})\backslash G_V(\mathbb{A})$.*

Proof. Take $\xi\in GO_V(\mathbb{Q})$. Since $GO_V(\mathbb{Q})$ leaves stable the vector space $V\subset V_{\mathbb{A}}$, noting ${}_S(\alpha_\xi x)={}_Sx$ and $|\nu_V(\xi)|_{\mathbb{A}}=1$ for $\xi\in GO_V(\mathbb{Q})$, we have

$$\begin{aligned}\theta_G(\phi)(\xi g;\alpha_{\nu(\xi)}x)&=\sum_{v\in V}L(\xi g)(\mathbf{r}({}_S(\alpha_{\nu(\xi)}x))\phi)(v)=\sum_v|\nu_V(\xi g)|_{\mathbb{A}}^{-d/2}(\mathbf{r}({}_S(\alpha_{\nu(\xi)}x))\phi)(g^{-1}\xi^{-1}v)\\&=\sum_v|\nu_V(g)|_{\mathbb{A}}^{-d/2}(\mathbf{r}({}_Sx)\phi)(g^{-1}\xi^{-1}v)=\sum_v|\nu_V(g)|_{\mathbb{A}}^{-d/2}(\mathbf{r}({}_Sx)\phi)(g^{-1}v)=\theta_G(\phi)(g;x).\end{aligned}$$

Thus $\theta_G(\phi)$ is left invariant under $(\alpha_{\nu(\xi)},\xi)\in G(\mathbb{Q})$. Since $(\alpha,\xi)\in G(\mathbb{Q})$ can be written as $(\alpha_{\nu(\xi)},\xi)({}_S\alpha,1)$, we now only need to prove left invariance of $\theta_G(\phi)$ under $S(\mathbb{Q})$. Since $(\alpha x)_S=\alpha(x_S)$ for $\alpha\in S(\mathbb{Q})$, we see

$$\theta_G(\phi)(\alpha x;g)=\sum_v\mathbf{r}((\alpha x)_S)(L(g)\phi)(v)=\theta_S(L(g)\phi)(\alpha(x_S);1)\overset{(*)}{=}\theta_S(L(g)\phi)(x_S;1)=\theta_G(\phi)(x;g),$$

where the identity at $(*)$ follows from $S(\mathbb{Q})$-invariance of θ_S (Theorem 1.1). □

1.3. **Partial Fourier transform.** Let $D=(M_2(\mathbb{Q}),\pm\det)$. Then $s(x,y)$ is the trace pairing $\langle x,y\rangle:=\mathrm{Tr}(xy^\iota)$ for the main involution ι. We define the partial Fourier transform $\phi\mapsto\phi^*$ for $\phi\in\mathcal{S}(D_{\mathbb{A}})$ as in [H06] Section 2.4:

$$\phi^*\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)=\int_{\mathbb{A}^2}\phi\left(\begin{smallmatrix}a'&b'\\c&d\end{smallmatrix}\right)\mathbf{e}_{\mathbb{A}}(ab'-ba')da'db',\tag{1.4}$$

where $\mathbf{e}_{\mathbb{A}}:\mathbb{A}/\mathbb{Q}\to\mathbb{C}^\times$ is the additive character with $\mathbf{e}_{\mathbb{A}}(x_\infty)=\exp(2\pi ix_\infty)$ for $x_\infty\in\mathbb{R}$ and $da'db'$ is the self-dual measure with respect to this Fourier transform.

Let ϕ be a Schwartz-Bruhat function on $D_{\mathbb{A}}$. Following [H06] (2.18), we choose ϕ so that $\phi = \phi^{(\infty)} \otimes \phi_\infty$ with $\phi^{(\infty)} : D_{\mathbb{A}}^{(\infty)} \to \mathbb{C}$ and $\phi_\infty : D_\infty \to \mathbb{C}$ given by, for $(\tau, z, w) \in \mathfrak{H}^3$,

$$\Psi_k(\tau; z, w)(v) = \mathrm{Im}(\tau)\left(\frac{\mathrm{Im}(\tau)[v; \overline{z}, \overline{w}]}{\mathrm{Im}(z)\,\mathrm{Im}(w)}\right)^k \mathbf{e}(-\det(v)\overline{\tau} + i\frac{\mathrm{Im}(\tau)}{2\,\mathrm{Im}(z)\,\mathrm{Im}(w)}|[v; z, w]|^2) \tag{1.5}$$

for $\mathbf{e}(x) = \exp(2\pi i x)$ and $[v; z, w] = -\mathrm{Tr}(v^\iota \cdot {}^t(z, 1)(w, 1)J) = -(w, 1)Jv^\iota \left(\begin{smallmatrix} z \\ 1 \end{smallmatrix}\right) = (z, 1)Jv \left(\begin{smallmatrix} w \\ 1 \end{smallmatrix}\right) = wcz - aw + dz - b$ with $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $v = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. We have

$$\begin{aligned} [g^{-1}vh; z, w] &= [v; g(z), h(w)] \det(g)^{-1} j(g, z) j(h, w), \\ \frac{[v; \overline{g(z)}, \overline{h(w)}]}{\mathrm{Im}(g(z))\,\mathrm{Im}(h(w))} &= \det(h)^{-1} j(g, z) j(h, w) \frac{[g^{-1}vh; \overline{z}, \overline{w}]}{\mathrm{Im}(z)\,\mathrm{Im}(w)}, \\ \frac{|[g^{-1}vh; z, w]|^2}{\mathrm{Im}(z)\,\mathrm{Im}(w)} &= \det(g^{-1}h) \frac{|[v; g(z), h(w)]|^2}{\mathrm{Im}(g(z))\,\mathrm{Im}(h(w))}, \end{aligned} \tag{1.6}$$

where $j(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \tau) = c\tau + d$. Consider Siegel's theta series $\theta_k(\phi^{(\infty)})(\tau; z, w) = \sum_{v \in D} \phi(v)$. As shown in [H06] Proposition 2.2, Poisson summation formula tells us

Lemma 1.3. *We have* $\theta_k(\tau; z, w; \phi^{(\infty)}) = \theta_k(z; \tau, w; \phi^{*(\infty)})$.

By Lemma 1.3, we get the following version of [Sh2] II Proposition 5.1 (see [H06] Theorem 3.2):

Theorem 1.4. *Suppose that f is a holomorphic cusp form of weight $k > 0$. Let Γ be a congruence subgroup of $SL_2(\mathbb{Q})$ fixing $f(\tau)\theta(\phi)(\tau)$. Then we have*

$$\int_{\Gamma\backslash\mathfrak{H}} \theta_k(\phi^{(\infty)})(\tau; z, w)\overline{f}_c(\tau) d\mu(\tau) = (2i)^k \sum_{\alpha \in \Gamma\backslash M_2(\mathbb{Q}); \det(\alpha) > 0} \phi^{*(\infty)}(\epsilon\alpha) \exp(2\pi i \det(\alpha) z) f|_k \alpha(w),$$

where $d\mu(\tau)$ is the invariant measure $\eta^{-2} d\xi d\eta$ on $\mathfrak{H}$ for $\tau = \xi + i\eta$, $\epsilon = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $f_c(z) = \overline{f(-\overline{z})}$ and $f|_k\alpha(w) = \det(\alpha)^{k-1} f(\alpha(w)) j(\alpha, w)^{-k}$ for $\alpha \in M_2(\mathbb{Q})$ with positive determinant.

1.4. **Optimal Schwartz–Bruhat function.** Let N be a positive integer and K be an imaginary quadratic field with discriminant $d(K)$. Define $d_0(K)$ to be $d(K)/4$ or $d(K)$ according as $4|d(K)$ or not. We split the set of prime factors in $N \cdot d_0(K)$ into two disjoint sets A and $C = C_0 \sqcup C_1$ (so, $A \sqcup C = \{\ell | N \cdot d_0(K)\}$). We put $C_1 = \{\ell | d_0(K)\}$. Decompose $N = \prod_{\ell \in A \cup C} \ell^{\nu(\ell)}$ and assume that $\ell \in A \Rightarrow \nu(\ell) > 0$ (but not necessarily the converse). Also $\nu(\ell)$ could be 0 for $\ell \in C$.

Definition 1.5. *Let*

$$\widehat{\Delta} = \widehat{\Delta}_0(A, C; N) = \widehat{\Delta}_0(A, C_0, C_1; N) \subset M_2(\widehat{\mathbb{Z}}) \cap GL_2(\mathbb{A}^{(\infty)})$$

be the semi-group made of elements $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(\widehat{\mathbb{Z}})$ satisfying the following conditions:

(a) $a - 1 \in N\widehat{\mathbb{Z}}$,
(c) $c_\ell \in N_\ell^2 \mathbb{Z}_\ell$ *for* $\ell \in A$, $c_\ell \in \ell^j N_\ell \mathbb{Z}_\ell$ *for* $\ell \in C_j$ *for* $j = 0, 1$,

where $N_\ell = \ell^{\nu(\ell)}$ is the ℓ-primary part of N.

We put $N_1 = N \prod_{\ell \in C_1} \ell$. Write δ_X for the characteristic function of a set X. Take $s, t \in \widehat{\mathbb{Z}}^\times$ with $t \equiv s \equiv 1 \mod N_C\widehat{\mathbb{Z}}$, where N_C is the C-part of N. Define $\boldsymbol{\phi}^* = \boldsymbol{\phi}^*_{t,s}$ to be a Schwartz-Bruhat function on $M_2(\mathbb{A}^{(\infty)})$ given by

$$\boldsymbol{\phi}^*_\ell \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \begin{cases} \delta_{\widehat{\Delta}_\ell} & \text{if } \ell \notin A \\ \delta_{(s_\ell + N_\ell\mathbb{Z}_\ell)}(a)\delta_{\mathbb{Z}_\ell}(b)\delta_{N_\ell(t_\ell + N_\ell\mathbb{Z}_\ell)}(c)\delta_{\mathbb{Z}_\ell}(d) & \text{if } \ell \in A. \end{cases} \tag{1.7}$$

Then $\boldsymbol{\phi}^*_{s,t}$ depends only on $(s, t) \mod N$ and is the characteristic function of $\gamma_{s,t}\widehat{\Delta}(A, C; N)$ for $\gamma_{s,t} = \left(\begin{smallmatrix} s & 0 \\ 0 & s^{-1} \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ stN & 1 \end{smallmatrix}\right) \in SL_2(\widehat{\mathbb{Z}})$. Let

$$\begin{aligned} \Gamma(A, C; N) &= SL_2(\mathbb{Z}) \cap \widehat{\Delta}(A, C; N) \ \text{ and } \ \widehat{\Gamma}(A, C; N) = SL_2(\widehat{\mathbb{Z}}) \cap \widehat{\Delta}(A, C; N), \\ U(A, C; N) &= \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \widehat{\Delta}(A, C; N) \cap GL_2(\widehat{\mathbb{Z}}) \middle| a_\ell \equiv d_\ell \mod N_\ell\mathbb{Z}_\ell \ \text{ for } \ell \in A \right\}. \end{aligned} \tag{1.8}$$

Note that $\gamma_{s,t}$ normalizes $U(A, C; N)$, $\widehat{\Gamma}(A, C; N)$ and $\Gamma(A, C; N)$.

Define $\boldsymbol{\phi}_{s,t}(x) := (\boldsymbol{\phi}^*_{s,t})^*(\epsilon x)$. Then, by [BNT] VII.7 Proposition 13, we have

$$\boldsymbol{\phi}_\ell\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \begin{cases} \delta_{M_2(\mathbb{Z}_\ell)} & \text{if } \ell \notin A \cup C, \\ \delta_{\mathbb{Z}_\ell}(a)\mathbf{e}_\ell(-sb)\delta_{N_\ell^{-1}\mathbb{Z}_\ell}(b)\delta_{N_\ell(t+N_\ell\mathbb{Z}_\ell)}(c)\delta_{\mathbb{Z}_\ell}(d) & \text{if } \ell \in A, \\ \delta_{\mathbb{Z}_\ell}(a)\mathbf{e}_\ell(-b)\delta_{N_\ell^{-1}\mathbb{Z}_\ell}(b)\delta_{\ell^j N_\ell\mathbb{Z}_\ell}(c)\delta_{\mathbb{Z}_\ell}(d) & \text{if } \ell \in C_j \ (j = 0, 1), \end{cases} \tag{1.9}$$

where $\mathbf{e}_\ell(x) = \exp(-2\pi i[x]_\ell)$ for the fractional part $[x]_\ell$ of $x \in \mathbb{Q}_\ell$. This shows

$$\boldsymbol{\phi}_{s,t}(v) = \boldsymbol{\phi}_{1,1}(\alpha_t^{-1}v\alpha_s)) \text{ and } \boldsymbol{\phi}^*_{s,t}(v) = \boldsymbol{\phi}^*_{1,1}(\alpha_t^{-1}\beta_s^{-1}v)) \tag{1.10}$$

for $\alpha_t = \left(\begin{smallmatrix} 1 & 0 \\ 0 & t \end{smallmatrix}\right)$ and $\beta_t = \left(\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right)$. By computation, we conclude $\boldsymbol{\phi}_{s,t}(\gamma x\delta^{-1}) = \boldsymbol{\phi}_{s,t}(x)$ for $\gamma, \delta \in \Gamma(A, C; N)$. Write, for Ψ_k in (1.5),

$$\Theta_{s,t}(\tau; z, w) = \Theta(\boldsymbol{\phi}_{s,t} \otimes \Psi_k)(\tau; z, w).$$

Then, by [H06] Proposition 2.3,

$$\Theta_{s,t}(\gamma(\tau); \alpha(z), \beta(z)) = j(\gamma, \overline{\tau})^{-k} j(\alpha, z)^k j(\beta, w)^k \Theta_{s,t}(\tau; z, w) \tag{1.11}$$

for $(\gamma, \alpha, \beta) \in \Gamma(A, C; N)^3$ and $j(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), z) = (cz+d)$ (recall also $J(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), z) = (ad-bc)^{-1/2}(cz+d)$).

Lemma 1.6. *Suppose $f \in S_k(\Gamma_0(N_1), \psi)$ for $N_1 = N \prod_{\ell \in C_1} \ell$. Then we have*

$$\sum_{\alpha \in \Gamma(A,C;N)\backslash M_2(\mathbb{Q}); \det(\alpha)>0} \boldsymbol{\phi}^*_{s,t}(\epsilon\alpha)\mathbf{e}(\det(\alpha)z)f|_k\alpha(w) = \psi(s)^{-1}\sum_{n=1}^\infty \mathbf{e}(nz)f|T(n)(w),$$

where $\mathbf{e}(z) = \exp(2\pi iz)$.

Proof. Abusing notation, we take an element γ in $SL_2(\mathbb{Z})$ with $\gamma \equiv \gamma_{s,t} \mod N^2$ (by the strong approximation theorem) and define $f|_k\gamma_{s,t}$ by $f|_k\gamma$. Also pick $\sigma_s \in SL_2(\mathbb{Z})$ with $\sigma_s \equiv \gamma_{s,t} \mod N$. By definition, we have

$$\begin{aligned} &\sum_{\alpha \in \Gamma(A,C;N)\backslash M_2(\mathbb{Q}); \det(\alpha)>0} \boldsymbol{\phi}^*_{s,t}(\epsilon\alpha)\mathbf{e}(\det(\alpha)z)f|_k\alpha(w) \\ &\quad = \sum_{\alpha \in \Gamma(A,C;N)\backslash M_2(\mathbb{Q})\cap\widehat{\Delta}(A,C;N); \det(\alpha)>0} \mathbf{e}(\det(\alpha)z)(f|_k\gamma_{s,t})\alpha(w) \\ &\quad = \psi(s)^{-1}\sum_{\alpha \in \Gamma(A,C;N)\backslash\Delta(A,C;N)} \mathbf{e}(\det(\alpha)z)f|_k\sigma_s\alpha(w) = \psi(s)^{-1}\sum_{n=1}^\infty \mathbf{e}(nz)f|T(n)(w). \end{aligned}$$

□

Thus if f is a normalized Hecke eigenform with $f|T(n) = a(n, f)f$, we have

$$\Theta_{\boldsymbol{\phi}_{1,1}}(f) = \int_{\Gamma(A,C;N)\backslash\mathfrak{H}} \Theta_{s,t}(\tau; z, w)\overline{f}_c(\tau)d\mu(\tau) = (2i)^k\psi(s)^{-1}f(z)f(w). \tag{1.12}$$

This is a version of a formula in [Sh2] II, Proposition 5.1 (see also [P] page 923).

1.5. **Adelic theta series.** Recall $S = SL(2)_{/\mathbb{Z}}$. Regard $(g, h) \in S(A)^2$ as a linear automorphism of $D \otimes_{\mathbb{Q}} A$ by $\ell(g, h) : v \mapsto gvh^{-1}$ in O_D. This gives rise an isogeny $S \times S \to O_D$. We pull back to $S(\mathbb{A})^3$ the theta series $\theta_S(\phi)(x; g, h)$ on $S(\mathbb{A}) \times O_D(\mathbb{A})$ by this isogeny, and we still write $\theta_S(\phi)$ for the resulting automorphic form on $S(\mathbb{A})^3$.

As for the classical Siegel's theta series, we first extend $\theta_k(\phi^{(\infty)})(\tau; z, w)$ to $S(\mathbb{A}) \times S(\mathbb{A}) \times S(\mathbb{A})$ as in 1.1 and write it as $\theta_k(\phi^{(\infty)})(x; g, h)$. Thus $\theta_k(\phi^{(\infty)})$ is a function on $(S(\mathbb{Q})\backslash S(\mathbb{A}))^3$. We have

Lemma 1.7. *Suppose $\phi(v) = \phi^{(\infty)}(v^{(\infty)})\Psi_k(i; i, i)(v_\infty)$. Then for $(x; g, h) \in S(\mathbb{A})^3$, we have $\theta_S(\phi)(x; g, h) = \theta_k(\phi^{(\infty)})(x; g, h)$.*

Proof. First suppose that $\theta_k(\phi^{(\infty)})(x_\infty; g_\infty, h_\infty) = \theta_S(\phi)(x_\infty; g_\infty, h_\infty)$ by definition. Thus they coincide on $(S(\mathbb{Q})S(\mathbb{R}))^3$. By the strong approximation theorem, $(S(\mathbb{Q})S(\mathbb{R}))^3$ is dense in $S(\mathbb{A})^3$; thus, they are equal on the entire $S(\mathbb{A})^3$. We need therefore to show $\theta_S(\phi)|_{S(\mathbb{R})^3} = \theta_k(\phi^{(\infty)})|_{S(\mathbb{R})^3}$. Note that $\phi_\infty(v) = \Psi_k(i; i, i) = [v; -i, -i]^k \mathbf{e}(\det(v)i + \frac{i}{2}\big|[v; i, i]\big|^2)$. Let $g_\tau = \sqrt{\mathrm{Im}(\tau)}^{-1}\left(\begin{smallmatrix}\mathrm{Im}(\tau) & \mathrm{Re}(\tau)\\ 0 & 1\end{smallmatrix}\right)$ for $\tau \in \mathfrak{H}$; so, $g_\tau(i) = \tau$. Note that $\theta_k(\phi^{(\infty)})$ is of weight $(-k, k, k)$ in $(\overline{\tau}, z, w)$ (cf. (1.11)), and hence

$$\theta_k(\phi^{(\infty)})(g_\tau; g_z, g_w) = \sum_v \Psi_k(\tau; z, w)(v) J(g_\tau, -i)^k J(g_z, i)^{-k} J(g_w, i)^{-k}.$$

We take the quadratic space $(D, -\det)$. We get from (1.6) and (1.1) (see also Section 3.1)

$$L(g_z, g_w)(\mathbf{r}(g_\tau)\Psi_k(i; i, i))(v) = \Psi_k(\tau; z, w) J(g_\tau, -i)^k J(g_z, i)^{-k} J(g_w, i)^{-k}.$$

This shows

$$\theta_S(\phi)|_{S(\mathbb{R})^3} = \sum_v L(g_z, g_w)(\mathbf{r}(g_\tau)\Psi_k(i; i, i))(v) = \theta_k(\phi^{(\infty)})|_{S(\mathbb{R})^3}$$

as desired. □

We further extend $\theta(\phi)(x; g, h)$ to $G(\mathbb{A}) = \{(x, g, h) \in GL_2(\mathbb{A})^3 | \det(x) = \det(g)/\det(h)\}$ by

$$(x; g, h) \mapsto \theta_S(\phi_{g,h})(x\left(\begin{smallmatrix}1 & 0\\ 0 & \det(x)\end{smallmatrix}\right)^{-1}; 1, 1) = \theta_S(\phi_{g,h})(1, 1; \left(\begin{smallmatrix}1 & 0\\ 0 & \det(x)\end{smallmatrix}\right)^{-1} x) \tag{1.13}$$

for $\phi_{g,h}(a) = |\det(h)/\det(g)|_{\mathbb{A}}\phi(g^{-1}ah)$. We write the above theta function on $G(\mathbb{A})$ as $\mathbf{\Theta}(\phi)(x; g, h)$.

Note that the action $(g, h)v = gvh^{-1}$ for $v \in D$ gives rise to an isogeny from G to G_D, and we regard $\theta_G(\phi)(x; g, h) = \theta_S(\phi)(x_S; g, h)$ as a function on $G(\mathbb{Q})\backslash G(\mathbb{A})$ by pull back. Note that $\theta_G(\phi)(g, h; x)$ can be defined using the left projection $GL(2) \ni x \mapsto {}_Sx \in S$. By (1.2), it turns out the two definitions produce the same function $\theta_G(\phi)$. In this sense, we write $\mathbf{\Theta}(\phi)(g, h; x) = \theta_G(g, h; x)$ if we adopt this left projection.

Lemma 1.8. *The function $\mathbf{\Theta}(\phi)(x; g, h)$ is an automorphic form on $G(\mathbb{Q})\backslash G(\mathbb{A})$ and is equal to $\theta_G(\phi)(x; g, h) = \theta_G(\phi)(g, h; x)$. Moreover, $\mathbf{\Theta}(\phi)(x; \zeta g, \zeta h) = \mathbf{\Theta}(\phi)(x; g, h)$ for $\zeta \in Z(\mathbb{A})$.*

Proof. For $\xi, \eta \in \mathbb{Q}$, we have

$$\mathbf{\Theta}(\phi_{\alpha_\xi g, \alpha_\eta h})(\alpha_\xi g, \alpha_\eta h; \alpha_{\xi\eta^{-1}}x) = \sum_v \phi_x(g^{-1}\alpha_\xi^{-1} v \alpha_\eta h)$$

for a Schwartz-Bruhat function ϕ_x only dependent on $x \in S(\mathbb{A})$ (given by the Weil representation $\mathbf{r}(x)\phi$). Since $v \mapsto \alpha_\xi^{-1} v \alpha_\eta$ is a linear automorphism of D, we get

$$\mathbf{\Theta}(\phi_{\alpha_\xi g, \alpha_\eta h})(\alpha_\xi g, \alpha_\eta h; \alpha_{\xi\eta^{-1}}x) = \mathbf{\Theta}(\phi_{\alpha_\xi g, \alpha_\eta h})(g, h; x).$$

Thus we only need to show $\mathbf{\Theta}(\phi)(\alpha x; \beta g, \gamma h) = \mathbf{\Theta}(x; g, h)$ for $\alpha, \beta, \gamma \in S(\mathbb{Q})$. This follows from Weil's generalized Poisson summation formula (Theorem 1.1). Thus $\theta_G(\phi) = \mathbf{\Theta}(\phi)$ on $S(\mathbb{A})^3$ by Lemma 1.7. Then the way of extending the two to $G(\mathbb{A})$ is the same; so, we get $\theta_G(\phi) = \mathbf{\Theta}(\phi)$. The last assertion follows from

$$\phi_{g,h}(v) = |\det(h)/\det(g)|_{\mathbb{A}}\phi(g^{-1}vh) = |\det(\zeta h)/\det(\zeta g)|_{\mathbb{A}}\phi(g^{-1}\zeta^{-1}v\zeta h) = \phi_{\zeta g, \zeta h}(v),$$

as ζ is in the center. □

1.6. **Adelic theta integral.** For a Dirichlet character ψ modulo N, we define $\psi^* : \mathbb{A}^\times/\mathbb{Q}^\times \to \mathbb{C}^\times$ by $\psi^*(s^{(N)}) = \psi(s)$ for positive integers s prime to N. Recall $\boldsymbol{\psi} = \psi^* |\cdot|_{\mathbb{A}}^{-k}$. Write $\theta_{A,C,N}$ for $\mathbf{\Theta}(\phi) = \theta_G(\phi)$ for $\phi = \boldsymbol{\phi}_{1,1} \otimes \Psi_k(i; i, i)$ given by $x \mapsto \boldsymbol{\phi}_{1,1}(x^{(\infty)})\Psi_k(i; i, i)(x_\infty)$. For $f \in S_k(\Gamma_0(N_1), \psi)$, we define $f_c(z) = \overline{f(-\overline{z})} \in S_k(\Gamma_0(N_1), \overline{\psi})$, and lift them to adelic modular forms on $GL_2(\mathbb{A})$

$$\mathbf{f}_c(g_\infty) = j(g_\infty, i)^{-k} f_c(g_\infty(i)) \in \overline{\mathcal{S}}_k(N_1, \overline{\boldsymbol{\psi}}), \text{ and } \mathbf{f}(g_\infty) = j(g_\infty, i)^{-k} f(g_\infty(i)) \in \mathcal{S}_k(N_1, \boldsymbol{\psi}).$$

We then have

$$\mathbf{f}(\zeta\gamma g_\infty u) = \boldsymbol{\psi}(\zeta)\boldsymbol{\psi}(d_N)\mathbf{f}(g_\infty) \text{ and } \mathbf{f}_c(\zeta\gamma g_\infty u) = \overline{\boldsymbol{\psi}}(\zeta)\overline{\boldsymbol{\psi}}(d_N)\mathbf{f}_c(g_\infty)$$

with $\zeta \in Z(\mathbb{A})$, $\gamma \in GL_2(\mathbb{Q})$ and $u = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) \in \widehat{\Gamma}_0(N_1)$ (see Proposition 3.5 of [MFG]).

The following result is the reason why we call our choice of the Schwartz-Bruhat function optimal.

Proposition 1.9. *Let $X := S(\mathbb{Q})\backslash S(\mathbb{A})/SO_2(\mathbb{R})$, and take the subgroup $U = \widehat{\Gamma}(A,C;N) \subset SL_2(\widehat{\mathbb{Z}})$ fixing the product $\theta_{A,C,N}(x\alpha_{\det(g^{-1}h)}; g, h)\overline{\mathbf{f}}_c(x)$. Write $d\mu(x)$ for the $SL_2(\mathbb{A})$-invariant measure on X inducing $X/U = \Gamma(A,C;N)\backslash\mathfrak{H}$ the measure $\frac{1}{2}\operatorname{Im}(\tau)^{-2}|d\tau \wedge d\overline{\tau}|$. Suppose that $f \in S_k(\Gamma_0(N_1),\psi)$ (for $N_1 = N\prod_{\ell\in C_1}\ell$) is a normalized Hecke eigenform. Then we have*

$$\int_X \theta_{A,C,N}(x\alpha_{\det(g^{-1}h)}; g, h)\overline{\mathbf{f}}_c(x)d\mu(x) = (2i)^k\boldsymbol{\psi}(\det(g))^{-1}\mathbf{f}(g)\mathbf{f}(h).$$

Proof. Since $U \cap SL_2(\mathbb{Q}) = \Gamma(A,C;N)$, we have from Lemma 1.6, for $g_1, h_1 \in S(\mathbb{A})$ and $s,t \in \widehat{\mathbb{Z}}^\times$,

$$\int_X \theta_S(\boldsymbol{\phi}_{s,t})(x; g_1, h_1)\overline{\mathbf{f}}_c(x)d\mu(x) = (2i)^k\psi(s)^{-1}\mathbf{f}(g_1)\mathbf{f}(h_1) = (2i)^k\psi^*(s_N)\mathbf{f}(g_1)\mathbf{f}(h_1),$$

as $d\mu(x)$ is the pullback of the measure $d\mu(\tau) = \frac{1}{2}\operatorname{Im}(\tau)^{-2}|d\tau\wedge d\overline{\tau}|$ on $\Gamma\backslash\mathfrak{H}$. Recall $\alpha_t = \left(\begin{smallmatrix}1&0\\0&t\end{smallmatrix}\right)$ for $t \in \mathbb{A}^\times$ with $t_\infty = 1$. Since $GL_2(\mathbb{A}) = GL_2(\mathbb{Q})GL_2(\widehat{\mathbb{Z}})GL_2^+(\mathbb{R})$, we may assume that $g, h \in GL_2(\widehat{\mathbb{Z}})$. Then, for $t = \det(g)$ and $s = \det(h)$, $|t|_\mathbb{A} = |s|_\mathbb{A} = 1$, and for $\phi = \boldsymbol{\phi}_{1,1}$, we have $\phi_{g,h}(v) = \boldsymbol{\phi}_{1,1}(g^{-1}vh) = \boldsymbol{\phi}_{1,1}(\alpha_t^{-1}g_S^{-1}vh_S\alpha_s) = \boldsymbol{\phi}_{s,t}(g_S^{-1}vh_S)$. Thus we have, for $g_S, h_S \in S(\mathbb{A})$,

$$\begin{aligned}\int_X \boldsymbol{\Theta}_G(\boldsymbol{\phi}_{1,1})(x\alpha_{t^{-1}s}; g, h)\overline{\mathbf{f}}_c(x)d\mu(x) &= \int_X \theta_S(\boldsymbol{\phi}_{s,t})((x\alpha_{t^{-1}s})_S; g_S, h_S)\overline{\mathbf{f}}_c(x)d\mu(x)\\ = \int_X \theta_S(\boldsymbol{\phi}_{s,t})(x; g_S, h_S)\overline{\mathbf{f}}_c|\alpha_t(x)d\mu(x) &= (2i)^k\psi^*(s_N)\mathbf{f}(g_S)\mathbf{f}(h_S) = (2i)^k\psi^*(s_N)\mathbf{f}(g\alpha_t^{-1})\mathbf{f}(h\alpha_s^{-1})\\ &= (2i)^k\psi^*(s_N)\psi^*(t_N)^{-1}\psi^*(s_N)^{-1}\mathbf{f}(g)\mathbf{f}(h) = (2i)^k\boldsymbol{\psi}(\det(g))^{-1}\mathbf{f}(g)\mathbf{f}(h),\end{aligned}$$

as $\psi^*(t) = \psi^*(t_N) = \boldsymbol{\psi}(t)$ since $t \in \widehat{\mathbb{Z}}^\times$. The left-hand-side and the right-hand-side are both functions on $GL_2(\mathbb{A}) \times GL_2(\mathbb{A})$ left invariant under $GL_2(\mathbb{Q})^2$, invariant under the diagonal action of $Z(\mathbb{A})$ and right invariant under $\widehat{\Gamma}_0(N_1)$ (by Lemma 1.8); so, they must coincide over $GL_2(\mathbb{A})^2$. □

1.7. **Adjustment of Schwartz-Bruhat function for convolution.** We now modify the theta series so that our computation of a Rankin convolution will be easier. Recall the fixed imaginary quadratic field K of discriminant $d = d(K)$. Let $d_0(K)$ be $d(K)/4$ or $d(K)$ according as $2|d(K)$ or not. Let $N_1 = N\prod_{\ell\in C_1}\ell$. Write $N_\ell = \ell^{\nu(\ell)}$, and we assume that $\ell|d_0(K) \Rightarrow \ell \in C$. Let $C'_+ = \{\ell \in C|\operatorname{ord}_l(N_1) > 0\}$. Note $C'_+ \supset C_0$. We decompose $C_0 = C_i \sqcup C_s \sqcup C_r$ so that C_i is made of primes inert in K and C_r is made of 2 if $4 \parallel d(K)$ and $\nu(2) > 0$ (so, C_s is made of split primes). Since $C_i \cup C_r \cup C_1$ is made of primes in C non-split in $K/\mathbb{Q}$, we often write C_{ns} for $C_i \cup C_r \cup C_1$. Define a new function $\varphi_{s,t}\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$ given by

$$\begin{cases}\delta_{M_2(\mathbb{Z}_\ell)} & \text{if } \ell \notin A \cup C'_+,\\ \delta_{\mathbb{Z}_\ell}(a)\mathbf{e}_\ell(-sb)\delta_{N_\ell^{-1}\mathbb{Z}_\ell}(b)\delta_{N_\ell(t+N_\ell\mathbb{Z}_\ell)}(c)\delta_{\mathbb{Z}_\ell}(d) & \text{if } \ell \in A,\\ \delta_{\mathbb{Z}_\ell}(a)\mathbf{e}_\ell(-b)\delta_{N_\ell^{-1}\mathbb{Z}_\ell}(b)\delta_{N_\ell\mathbb{Z}_\ell}(c)(\ell^{\nu(\ell)}\delta_{\ell^\nu\mathbb{Z}_\ell}(d) - \ell^{\nu(\ell)-1}\delta_{\ell^{\nu(\ell)-1}\mathbb{Z}_\ell}(d)) & \text{if } \ell \in C_s,\\ \delta_{\mathbb{Z}_\ell}(a)\mathbf{e}_\ell(-b)\delta_{N_\ell^{-1}\mathbb{Z}_\ell}(b)\delta_{N_{1,\ell}\mathbb{Z}_\ell}(c)\delta_{\mathbb{Z}_\ell}(d) & \text{if } \ell \in C_{ns},\end{cases} \tag{1.14}$$

where $\mathbf{e}_\ell(x) = \exp(-2\pi i[x]_\ell)$ for the fractional part $[x]_\ell$ of $x \in \mathbb{Q}_\ell$. Then $\varphi^*_{s,t}$ is given by

$$\begin{cases}\delta_{M_2(\mathbb{Z}_\ell)} & \text{if } \ell \notin A \cup C'_+,\\ \delta_{s+N_\ell\mathbb{Z}_\ell}(a)\delta_{\mathbb{Z}_\ell}(b)\delta_{N_\ell(t+N_\ell\mathbb{Z}_\ell)}(c)\delta_{\mathbb{Z}_\ell}(d) & \text{if } \ell \in A,\\ \delta_{1+N_\ell\mathbb{Z}_\ell}(a)\delta_{\mathbb{Z}_\ell}(b)\delta_{N_\ell\mathbb{Z}_\ell}(c)(\ell^{\nu(\ell)}\delta_{\ell^\nu\mathbb{Z}_\ell}(d) - \ell^{\nu(\ell)-1}\delta_{\ell^{\nu(\ell)-1}\mathbb{Z}_\ell}(d)) & \text{if } \ell \in C_s,\\ \delta_{1+N_\ell\mathbb{Z}_\ell}(a)\delta_{\mathbb{Z}_\ell}(b)\delta_{N_{1,\ell}\mathbb{Z}_\ell}(c)\delta_{\mathbb{Z}_\ell}(d) & \text{if } \ell \in C_{ns}.\end{cases} \tag{1.15}$$

Since for $\ell \in A \cup C$, we have $\widehat{\Delta}(A,C;N)_\ell = \bigsqcup_{j=0}^\infty \widehat{\Delta}(A,C;N)_\ell^\times \left(\begin{smallmatrix}1&0\\0&\ell^j\end{smallmatrix}\right) \widehat{\Delta}(A,C;N)_\ell^\times$, we get

$$\operatorname{Supp}(\boldsymbol{\phi}_{s,t,\ell}) = \gamma_{s,t}\widehat{\Delta}(A,C;N)_\ell = \bigsqcup_{j=0}^\infty \gamma_{s,t}\widehat{\Delta}(A,C;N)_\ell^\times \left(\begin{smallmatrix}1&0\\0&\ell^j\end{smallmatrix}\right) \widehat{\Delta}(A,C;N)_\ell^\times,$$

and $\operatorname{Supp}(\delta_{s+N_\ell\mathbb{Z}_\ell}(a)\delta_{\mathbb{Z}_\ell}(b)\delta_{N_\ell(t+N_\ell\mathbb{Z}_\ell)}(c)\delta_{\ell^\epsilon\mathbb{Z}_\ell}(d)) = \bigsqcup_{j=0}^\infty \gamma_{s,t}\widehat{\Delta}(A,C;N)_\ell^\times \left(\begin{smallmatrix}1&0\\0&\ell^{j\epsilon}\end{smallmatrix}\right) \widehat{\Delta}(A,C;N)_\ell^\times$. By Lemma 1.3 combined with [H06] Proposition 2.3, this shows

Lemma 1.10. *The theta series $\theta_G(\varphi_{1,1})(x;g,h)$ is an automorphic form on $U(A,C;N)$ with respect to the variable x and h.*

We write $\boldsymbol{\Theta}_{A,C,N}(x;g,h)$ for $\boldsymbol{\Theta}_G(\varphi_{1,1})$ and $\Theta^{(N)}(\mathbf{f})$ for $\int_X \boldsymbol{\Theta}_{A,C,N}(x\alpha_{\det(g^{-1}h)};g,h)\overline{\mathbf{f}}_c(x)d\mu(x)$. In basically the same way as in the proof of Proposition 1.9, we get

Lemma 1.11. *Let the notation be as in* Proposition 1.9. *Let $M=\prod_{\ell\in C_s}N_\ell$. Suppose that $f\in S_k(\Gamma_0(N_1),\psi)$ for $N_1=N\prod_{\ell\in C_1}\ell$ is a normalized Hecke eigenform. Then we have*

$$\Theta^{(N)}(\mathbf{f})=(2i)^k\boldsymbol{\psi}(\det(g))^{-1}\sum_{t|M}\mu(t)a(M/t,f)\,(M/t)\,\mathbf{f}|[\beta^{(\infty)}_{t/M}](g)\mathbf{f}(h)$$

for the Moebius function μ of $\mathbb{Q}$, where $\mathbf{f}|[\beta^{(\infty)}_{t/M}](g)=\mathbf{f}(g\beta^{(\infty)}_{t/M})$ for the finite part $\beta^{(\infty)}_{t/M}\in GL_2(\mathbb{A}^{(\infty)})$ of $\beta_{t/M}=\left(\begin{smallmatrix}t/M&0\\0&1\end{smallmatrix}\right)\in GL_2(\mathbb{Q})$.

Proof. Since $(f|[M/t])(g_\infty(i))j(g_\infty,i)^{-k}=\mathbf{f}(g\beta^{(\infty)}_{t/M})$, the proof is exactly the same as that of Proposition 1.9 if we get

$$\int_{\Gamma(A,C;N)\backslash\mathfrak{H}}\Theta_G(\varphi_{1,1})(\tau;z,w)\overline{f}_c(\tau)d\mu(\tau)=(2i)^k\sum_{t|M}\mu(t)(M/t)a(M/t,f)f|[M/t](z)f(w)\tag{1.16}$$

for a Hecke eigenform $f\in S_k(\Gamma_0(N_1),\psi)$. Note here

$$\sum_{t|M}\mu(t)(M/t)\delta_{(M/t)\mathbb{Z}_N}=\prod_{\ell\in C_s}(\ell^{\nu(\ell)}\delta_{\ell^\nu\mathbb{Z}_\ell}-\ell^{\nu(\ell)-1}\delta_{\ell^{\nu(\ell)-1}\mathbb{Z}_\ell})$$

for $\mathbb{Z}_N=\prod_{\ell\in A\cup C}\mathbb{Z}_\ell$. Define for a positive integer m,

$$\Delta_m(A,C;N)=\left\{\alpha\in\Delta(A,C;N)\middle|m|\det(\alpha)>0\right\}.$$

By Theorem 1.4, the left-hand-side of (1.16) is equal to

$$\begin{aligned}&\sum_{t|M}\mu(t)\sum_{\alpha\in\Gamma(A,C;N)\backslash(M_2(\mathbb{Q})\cap\gamma_{1,1}\widehat{\Delta}_{M/t}(A,C;N)}\varphi^*_{1,1}(\epsilon\alpha)\mathbf{e}(\det(\alpha)z)f|_k\alpha(w)\\&=\sum_{t|M}\mu(t)\frac{M}{t}\sum_{\alpha\in\Gamma(A,C;N)\backslash\Delta_{M/t}(A,C;N)}\mathbf{e}(\det(\alpha)z)f|_k\alpha(w)=\sum_{t|M}\mu(t)\frac{M}{t}\sum_{n=1,\,\frac{M}{t}|n}^{\infty}\mathbf{e}(nz)f|T(n)(w)\\&=\sum_{t|M}\mu(t)\frac{M}{t}\sum_{n=1}^{\infty}a(n(M/t),f)\mathbf{e}(n(M/t)z)f(w)=\sum_{t|M}\mu(t)\frac{M}{t}a(M/t,f)f|[M/t](z)f(w)\end{aligned}$$

as desired. □

2. Splitting of quaternionic theta series

Let K be an imaginary quadratic field with discriminant $d(K)$. Write O for the integer ring of K. We split the quadratic space $(D,\det)=(K,N)\oplus(K,-N)$ for the norm form $N=N_{K/\mathbb{Q}}$ and accordingly split the theta series into a product of theta series of K.

2.1. **Torus integral.** Choose $z_1\in O$ so that $O=\mathbb{Z}[z_1]$ with $z_1\in\mathfrak{H}$, and define $\rho:K\hookrightarrow M_2(\mathbb{Q})$ by a regular representation:

$$\rho(\xi)\left(\begin{smallmatrix}z_1\\1\end{smallmatrix}\right)=\begin{pmatrix}z_1\xi\\\xi\end{pmatrix}$$

and consider D as a right $K^\times\times K^\times$-module by $(\xi,\eta)x=\rho(\xi)^{-1}x\rho(\eta)$. Note that $\rho(\overline{b})=\rho(b)^\iota$. Let T be the algebraic torus defined over $\mathbb{Q}$ whose $\mathbb{Q}$-points are $K^\times\times K^\times$. We embed T into G by $(\xi,\eta)\mapsto(\alpha_{N(\xi\eta^{-1})};\rho(\xi),\rho(\eta))$. We then choose $g_1\in GL_2(\mathbb{A})$ with and $g_{1,\infty}(i)=z_1$.

Lemma 2.1. *Let $\chi:K^\times_{\mathbb{A}}/K^\times\to\mathbb{C}^\times$ be a Hecke character with $\chi^{-1}|_{\mathbb{A}^\times}=\boldsymbol{\psi}$ and $\chi(a_\infty)=a^k_\infty$. Then $a\mapsto\mathbf{f}(\rho(a)g_1)\chi(a)$ factors through $I^-_K:=K^\times_{\mathbb{A}}/K^\times\mathbb{A}^\times K^\times_\infty$ (the anticyclotomic idele class group).*

Proof. For $z \in Z(\mathbb{A})$, we have $\mathbf{f}(zx) = \boldsymbol{\psi}(z)\mathbf{f}(x)$; so, $a \mapsto \chi(a)\mathbf{f}(\rho(a)x)$ factors through $K^\times\backslash K_\mathbb{A}^\times/\mathbb{A}^\times$. Let K^1 be a torus over $\mathbb{Q}$ given by $K^1(A) = \{\xi \in K \otimes_\mathbb{Q} A | \xi\overline{\xi} = 1\}$, where the complex conjugation $\xi \mapsto \xi^c = \overline{\xi}$ is induced from K. We take $a_\infty \in K_\infty^\times$. Then $\rho(a_\infty)g_{1,\infty}(i) = \rho(a_\infty)(z_1) = z_1$, and we have, writing f' for $f_{g_1^{(\infty)}}$ as in Section 1.1,

$$\begin{aligned}\mathbf{f}(\rho(a_\infty)g_1) &= f'(\rho(a_\infty)g_{1,\infty}(i))j(\rho(a_\infty)g_{1,\infty}, i)^{-k}\\ &= f'(\rho(a_\infty)(z_1))j(\rho(a_\infty), z_1)^{-k}j(g_{1,\infty}, i)^{-k} = f'(z_1)j(\rho(a_\infty), z_1)^{-k}j(g_{1,\infty}, i)^{-k} = \mathbf{f}(g_1)a_\infty^{-k}.\end{aligned}$$

Since $\chi(a_\infty) = a_\infty^k$, we have $\mathbf{f}(\rho(a_\infty)g_1)\chi(a_\infty) = \mathbf{f}(g_1)$. Thus the function factors through I_K^-. $\square$

Let F be a number field with integer ring O_F. Normalize the Haar measure $d_F^\times a$ on $F_\mathbb{A}^\times/F_\infty^\times$ so that $\int_{\widehat{O}_F^\times} d^\times a = 1$. Then taking a fundamental domain $\Phi \subset F_\mathbb{A}^\times/F_\infty^\times$ of $I_F = F^\times\backslash F_\mathbb{A}^\times/F_\infty^\times$, we get the measure $d_F^\times a$ on I_F induced by $d_F^\times a$ on $\Phi \cong I_F$. Thus $\int_{\widehat{O}_F^\times/O_F^\times} d_F^\times a = |O_F^\times|^{-1}$ for $F = \mathbb{Q}$ and K. Write $d^\times a$ for $d_K^\times a$. We have an exact sequence: $1 \to I_\mathbb{Q} \to I_K \to I_K^- \to 1$. We define a measure d^-a on I_K^- by $\int_{I_K} \phi(a)d^\times a = \int_{I_K^-}\int_{I_\mathbb{Q}} \phi(ab)d_\mathbb{Q}^\times b d^-a$. Fix a Hecke character $\chi : K_\mathbb{A}^\times/K^\times \to \mathbb{C}^\times$ with $\chi^{-1}|_{\mathbb{A}^\times} = \boldsymbol{\psi}$. Taking χ as above such that $\chi^{-1}|_{\mathbb{A}^\times} = \boldsymbol{\psi}$ and $\chi(a_\infty) = a_\infty^k$. We put for $\mathbf{f} \in \mathcal{S}_k(N, \boldsymbol{\psi})$

$$L_\chi(\mathbf{f}) = \int_{I_K} \mathbf{f}(\rho(a)g_1)\chi(a)d^\times a; \quad \text{so,} \quad \int_{I_K^-} \mathbf{f}(\rho(a)g_1)\chi(a)d^-a = vol(I_\mathbb{Q})^{-1}L_\chi(\mathbf{f}) = 2L_\chi(\mathbf{f}),$$

where $vol(I_\mathbb{Q}) = \int_{I_\mathbb{Q}} d_\mathbb{Q}^\times a = \frac{1}{2}$. Then by Lemma 1.11, writing $\mathcal{T} = T(\mathbb{Q})\backslash T(\mathbb{A})/T(\mathbb{R}) = I_K \times I_K$ for simplicity, we get

$$\text{(2.1)} \quad (2i)^k \sum_{0<t|M} \mu(t)a(M/t, f)(M/t)L_\chi(\mathbf{f}|[\beta_{t/M}^{(\infty)}])L_\chi(\mathbf{f}) = \int_\mathcal{T} \boldsymbol{\psi}(N(a)\det(g_1))\Theta^{(N)}(\mathbf{f})(\rho(a)g_1, \rho(b)g_1)\chi(a)\chi(b)d^\times a d^\times b.$$

We have, for $t = N(a^{-1}b)$,

$$\begin{aligned}&\int_\mathcal{T} \boldsymbol{\psi}(N(a)\det(g_1))\Theta^{(N)}(\mathbf{f})(\rho(a)g_1, \rho(b)g_1)\chi(a)\chi(b)d^\times a d^\times b\\ &= \int_X\int_\mathcal{T} \boldsymbol{\psi}(N(a)\det(g_1))\boldsymbol{\Theta}_{A,C,N}(x\alpha_t; \rho(a)g_1, \rho(b)g_1)\chi(a)\chi(b)d^\times a d^\times b \cdot \overline{\mathbf{f}}_c(x)d\mu(x)\end{aligned}$$

By (1.13), we have, for $t = N(a^{-1}b)$,

$$\text{(2.2)} \qquad \boldsymbol{\Theta}_{A,C,N}(x\alpha_t; \rho(a)g_1, \rho(b)g_1) = |t|_\mathbb{A} \sum_{v\in D} \mathbf{r}(x)\left(\varphi_{1,1}(g_1^{-1}\rho(a)^{-1}v\rho(b)g_1)\right).$$

2.2. **Factoring the theta series.** We now study $\boldsymbol{\Theta}_{A,C,N}(x\alpha_t; \rho(a)g_1, \rho(b)g_1)$. Choose $\epsilon \in GL_2(\mathbb{Q})$ so that $(1, \epsilon)$ is a basis of D over K ($\Leftrightarrow D = \rho(K) + \rho(K)\epsilon$), $\epsilon^2 = 1$ and $\rho(K) \perp \rho(K)\epsilon$ under $s(x, y) = \mathrm{Tr}(xy^\iota)$ and $\epsilon\rho(\xi^c) = \rho(\xi)\epsilon$ for $\xi \in K$ and $\langle c\rangle = \mathrm{Gal}(K/\mathbb{Q})$. The norm form of D induces two quadratic forms on K: one Q_1 by pullback via $\rho : K \hookrightarrow D$ another Q_ϵ by pullback via $\rho \cdot \epsilon : K \hookrightarrow D$ ($\rho \cdot \epsilon(v) = \rho(v)\epsilon \in D$). Let $T_{j/\mathbb{Q}}$ ($j = 1, \epsilon$) be the orthogonal similitude group of (K, Q_j), which is a torus whose group of $\mathbb{Q}$-points is isomorphic to $K^\times$. We have $(a, b) \in (K^\times)^2$ acting on D by $x \mapsto x \cdot (a, b) = \rho(a)^{-1}x\rho(b)$. Thus we have

$$(\rho(x) + \rho(y)\epsilon) \cdot (a, b) = \rho(ab^{-1})^{-1}\rho(x) + \rho(a\overline{b}^{-1})^{-1}\rho(y)\epsilon.$$

The morphism $\pi : T \to T_1 \times T_\epsilon$ is given by $(a, b) \mapsto (ab^{-1}, a\overline{b}^{-1}) = (\alpha, \beta)$ identifying $T_1(\mathbb{Q}) = K^\times$ and $T_\epsilon(\mathbb{Q})$ with $K^\times$ by ρ. Note that $\mathrm{Ker}(\pi)$ is the diagonal image of $\mathbb{G}_{m/\mathbb{Q}}$ in T. Let $T' = T_1 \times T_\epsilon$. Assume the following two conditions:

(S1) $\mathrm{ord}_\ell(d(K)) = 1 \Leftrightarrow \ell \in C_1$, C_r is empty or a singleton $\{2\}$ according as $\nu(2) = 0$ or $4 \parallel d(K)$ and $\nu(2) > 0$, where $\mathrm{ord}_\ell : \mathbb{Q}_\ell \to \mathbb{Z}$ is the discrete valuation with $\mathrm{ord}_\ell(\ell) = 1$.

(S2) All $\ell \in A$ splits in K.

Proposition 2.2. *Assume* (S1–2). *Then, we have a decomposition*

$$\mathbf{\Theta}_{A,C,N}(x;\rho(a)g_1,\rho(b)g_1) = (-2i)^k\theta(\phi_1)(x,\alpha)\theta(\phi_\epsilon)(x,\beta)$$

for theta series $\theta(\phi_j)$ of Q_j. Here $\varphi^{(\infty)}_{1,1}(g_1^{-1}(\rho(v)+\rho(w)\epsilon)g_1) = \phi^{(\infty)}_1(v)\phi^{(\infty)}_\epsilon(w)$, and the explicit form of ϕ_j and the choice of ϵ and $g_1 \in GL_2(\mathbb{A})$ at each place will be given in the proof.

For the splitting in the proposition, the condition (S2) is an absolute requirement.

Proof. We now prove Proposition 2.2. We start with the infinity place. By Lemma 1.7, the infinity part of the Schwartz-Bruhat function defining $\Theta_{A,C,N}$ is given by $\Psi_k(i;i,i)$, and $L(g,h)\circ \mathbf{r}(x_\infty)\Psi_k(i;i,i)(v)$ is given roughly by $\Psi_k(\tau;i,i)(g^{-1}vh)$ if $x_\infty = g_\tau$ ($\tau \in \mathfrak{H}$) as in the proof of Lemma 1.7. More precisely, we have, by (1.6),

$$\begin{aligned}\frac{\Psi_k(\tau;i,i)(g^{-1}_{1,\infty}vg_{1,\infty})}{\mathrm{Im}(\tau)} &= \left(\mathrm{Im}(\tau)[g^{-1}_{1,\infty}vg_{1,\infty};-i,-i]\right)^k\mathbf{e}(\det(v)(-\overline{\tau})+i\frac{\mathrm{Im}(\tau)}{2}|[g^{-1}_{1,\infty}vg_{1,\infty};i,i]|^2)\\ &= J(g_{1,\infty},i)^{-2k}\left(\frac{\mathrm{Im}(\tau)[v;\overline{z}_1,\overline{z}_1]}{\mathrm{Im}(z_1)^2}\right)^k\mathbf{e}(\det(v)(-\overline{\tau})+i\frac{\mathrm{Im}(\tau)}{2}|[g^{-1}_{1,\infty}vg_{1,\infty};i,i]|^2)\\ &= \left(\frac{\mathrm{Im}(\tau)[v;\overline{z}_1,\overline{z}_1]}{\mathrm{Im}(z_1)}\right)^k\mathbf{e}(\det(v)(-\overline{\tau})+i\frac{\mathrm{Im}(\tau)}{2}|[g^{-1}_{1,\infty}vg_{1,\infty};i,i]|^2),\end{aligned}$$

where $g_{1,\infty} = \sqrt{\mathrm{Im}(z_1)}^{-1/2}\left(\begin{smallmatrix}\mathrm{Im}(z_1) & \mathrm{Re}(z_1)\\ 0 & 1\end{smallmatrix}\right)$. Write $v = \rho(\xi)+\rho(\eta)\epsilon$ for $\epsilon \in D$ with $\epsilon\rho(\xi)\epsilon^{-1} = \rho(\overline{\xi})$. If $K = \mathbb{Q}[\sqrt{d}]$, taking $z_0 = \sqrt{d}^{-1}$, we may realize $\rho_0(a+b\sqrt{d}) = \left(\begin{smallmatrix}a & b\\ db & a\end{smallmatrix}\right)$; so, $\rho_0(\eta)\left(\begin{smallmatrix}z_0\\ 1\end{smallmatrix}\right) = \left(\begin{smallmatrix}z_0\eta\\ \eta\end{smallmatrix}\right)$. We take ϵ for ρ_0 to be $\epsilon_0 = \left(\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right)$, and hence

$$\langle\rho_0(\xi),\rho_0(\eta)\epsilon_0\rangle = -\mathrm{Tr}(\rho_0(\xi)\epsilon_0\rho_0(\overline{\eta})) = -\mathrm{Tr}(\rho_0(\xi\eta)\epsilon_0) = -a+a = 0 \;(\Rightarrow \rho_0(K)\perp\rho_0(K)\epsilon_0)$$

if $\xi\eta = a+b\sqrt{d}$. Since any ρ is a conjugate of $\rho_0 : a+b\sqrt{d}\mapsto\left(\begin{smallmatrix}a & b\\ db & a\end{smallmatrix}\right)$, writing $\rho = \alpha\rho_0\alpha^{-1}$ for $\alpha \in GL_2(\mathbb{Q})$ with $z_1 = \alpha(z_0)$, we have $\langle\rho(\xi),\rho(\eta)\epsilon\rangle = 0$ with $\epsilon = \alpha\epsilon_0\alpha^{-1}$. We thus have

$$\langle\rho(\xi)+\rho(\eta)\epsilon,\rho(\xi')+\rho(\eta')\epsilon\rangle = \langle\rho(\xi),\rho(\xi')\rangle+\langle\rho(\eta)\epsilon,\rho(\eta')\epsilon\rangle = \mathrm{Tr}(\xi\overline{\xi}')-\mathrm{Tr}(\eta\overline{\eta}').$$

Thus the corresponding positive majorant is given by

$$\langle\rho(\xi)+\rho(\eta)\epsilon,\rho(\xi')+\rho(\eta')\epsilon\rangle_+ = \mathrm{Tr}_{K/\mathbb{Q}}(\xi\overline{\xi}')+\mathrm{Tr}_{K/\mathbb{Q}}(\eta\overline{\eta}'),$$

and defining $p(z,w) = -{}^t(z,1)(w,1)J$ (see [H06] (2.11)), $p(z_1,z_1)+\overline{p(z_1,z_1)}$ and $ip(z_1,z_1)-\overline{ip(z_1,z_1)}$ span $\rho(K_\infty)\epsilon$ (see [H06] 2.1 and 2.2). In other words,

$$\begin{aligned}(2.3)\quad [\rho(\xi)+\rho(\eta)\epsilon;\overline{z}_1,\overline{z}_1] &= \langle\rho(\xi)+\rho(\eta)\epsilon,p(\overline{z}_1,\overline{z}_1)\rangle = \langle\rho(\eta)\epsilon,p(\overline{z}_1,\overline{z}_1)\rangle\\ &= (\overline{z}_1,1)J\rho(\eta)\epsilon^t(\overline{z}_1,1) = (\overline{z}_1,1)J\epsilon\rho(\overline{\eta})^t(\overline{z}_1,1) = \eta[\epsilon;\overline{z}_1,\overline{z}_1],\end{aligned}$$

as $\rho(\eta)^t(z_1,1) = \eta^t(z_1,1)$, where we recall $J = \left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$. Similarly, we get

$$(2.4)\qquad [\rho(\xi)+\rho(\eta)\epsilon;\overline{z}_1,z_1] = \xi[1;\overline{z}_1,z_1] = -2i\xi\,\mathrm{Im}(z_1),$$

Note also

$$(2.5)\qquad [\epsilon;\overline{z}_1,\overline{z}_1]\,\mathrm{Im}(z_1)^{-1} = -2\sqrt{-1}.$$

Since for $v = \rho(\xi)+\rho(\eta)\epsilon$, we have

$$\begin{aligned}(2.6)\quad &\mathrm{Im}(\tau)^{k+1}\,\mathrm{Im}(z_1)^{-k}[v;\overline{z}_1,\overline{z}_1]^k\mathbf{e}(\det(v)(-\overline{\tau})+i\frac{\mathrm{Im}(\tau)}{2\,\mathrm{Im}(z_1)^2}|[v;z_1,z_1]|^2)\\ &= (-2i)^k\,\mathrm{Im}(\tau)^{k+1}\eta^k\mathbf{e}(\frac{1}{2}(-\langle v,v\rangle\,\mathrm{Re}(\tau)+i\,\mathrm{Im}(\tau)\langle v,v\rangle_+)) = (-2i)^k\,\mathrm{Im}(\tau)^{k+1}\eta^k\mathbf{e}(-\xi\overline{\xi}\overline{\tau}+\eta\overline{\eta}\tau),\end{aligned}$$

we now set

$$(2.7)\quad \phi_{1,\infty}(\xi) = \phi_{1,\infty}(\xi;\tau) = \mathrm{Im}(\tau)^{1/2}\mathbf{e}(-\xi\overline{\xi}\overline{\tau}),\quad \phi_{\epsilon,\infty}(\eta) = \phi_{\epsilon,\infty}(\eta;\tau) = \mathrm{Im}(\tau)^{k+(1/2)}\eta^k\mathbf{e}(\eta\overline{\eta}\tau).$$

For the quadratic space $(K,-N_{K/\mathbb{Q}})$, we have $\mathbf{r}(g_\tau)\phi_{1,\infty}(\xi;i)J(g_\tau,-i)^{-1} = \phi_{1,\infty}(\tau;\xi)$ and for the quadratic space $(K,N_{K/\mathbb{Q}})$, we have $\mathbf{r}(g_\tau)\phi_{\epsilon,\infty}(\eta;i)J(g_\tau,i)^{-k} = \phi_{\epsilon,\infty}(\tau;\eta)$.

Now suppose that ℓ is a prime split in K. Choose a prime factor $\mathfrak{l}|\ell$ in O, and identify $K_\ell = K_\mathfrak{l} \times K_{\overline{\mathfrak{l}}} = \mathbb{Q}_\ell \times \mathbb{Q}_\ell$. We write $\iota = \iota_\ell$ for the projection of K_ℓ to the left factor $K_\mathfrak{l}$ and $c \circ \iota_\ell$ for the other. We make explicit later the choice of $\mathfrak{l}$. Take $h_{1,\ell}$ so that $h_{1,\ell}^{-1}\rho(\alpha)h_{1,\ell} = \left(\begin{smallmatrix} \iota_\ell(\alpha) & 0 \\ 0 & c(\iota_\ell(\alpha)) \end{smallmatrix}\right)$. For example, $h_{1,\ell} = \left(\begin{smallmatrix} z_1 & \overline{z}_1 \\ 1 & 1 \end{smallmatrix}\right) \in GL_2(\mathbb{Z}_\ell)$ does the job. For one choice of ϵ with $\epsilon\rho(\xi)\epsilon^{-1} = \rho(\overline{\xi})$ for $\xi \in K$, all other choices fills the double coset $\rho(K_\ell^\times)\epsilon\rho(K_\ell^\times)$. Adjusting this way, we may choose $h_{1,\ell}$ so that $h_{1,\ell}^{-1}\epsilon h_{1,\ell} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\det(h_{1,\ell}) = 1$, as $\det(\rho(K_\ell)^\times) = \mathbb{Q}_\ell^\times$. Then we define $g_{1,\ell} = h_{1,\ell}\left(\begin{smallmatrix} \ell^{\nu(\ell)} & u \\ 0 & 1 \end{smallmatrix}\right)$ for $u = 1$ if $\ell \in A$ and $u = 0$ if $\ell \in C_s$; so, $\det(g_{1,\ell}) = \ell^\nu$. We simply write $\iota_\ell(\alpha) = \alpha$ and $c(\iota_\ell(\alpha)) = \overline{\alpha}$. Thus we have

$$g_{1,\ell}^{-1}(\rho(\xi) + \rho(\eta)\epsilon_0)g_{1,\ell} = \begin{cases} \left(\begin{smallmatrix} \xi - \overline{\eta} & \ell^{-\nu}(\eta - \overline{\eta}) + \ell^{-\nu}(\xi - \overline{\xi}) \\ \ell^\nu \overline{\eta} & \overline{\xi} + \overline{\eta} \end{smallmatrix}\right) & \text{if } \ell \in A, \\ \left(\begin{smallmatrix} \xi & \ell^{-\nu}\eta \\ \ell^\nu \overline{\eta} & \overline{\xi} \end{smallmatrix}\right) & \text{if } \ell \in C_s \text{ or } \ell \notin A \sqcup C \sqcup \{\infty\}. \end{cases}$$

We define $\mathcal{C}_s$ (resp. $\mathcal{A}$) for the set of this choice of split primes $\mathfrak{l}$ over $C_+ := \{\ell \in C | \nu(\ell) > 0\}$ (resp. over $\ell \in A$). For non-split primes over C, there is a unique choice of primes over ℓ in K. We write $\mathcal{C}_{ns}$ for the set of the non-split primes of K over C_+. Then $\mathcal{C} = \mathcal{C}_s \sqcup \mathcal{C}_{ns}$. Note that $g_{1,\ell} \in GL_2(\mathbb{Z}_\ell)$ if $\ell \notin A \sqcup C \sqcup \{\infty\}$. Then by definition, we get the following facts.

Lemma 2.3. *Suppose $\ell^\nu \parallel N$ and ℓ splits in K, and recall $\mathbf{e}_\ell(x) = \mathbf{e}(-[x]_\ell)$ for $x \in \mathbb{Q}_\ell$.*

(1) *If $\ell \in A$, we have*

$$\varphi_{1,1,\ell}(g_{1,\ell}^{-1}(\rho(\xi) + \rho(\eta)\epsilon)g_{1,\ell}) = \delta_{O_\mathfrak{l}}(\eta_\mathfrak{l})\delta_{(1+\ell^\nu O_{\overline{\mathfrak{l}}})}(\eta_{\overline{\mathfrak{l}}})\mathbf{e}_\ell(\ell^{-\nu}(1 - \eta_\mathfrak{l}))\delta_{O_\ell}(\xi_\ell)\mathbf{e}_\ell(\ell^{-\nu}(\xi_{\overline{\mathfrak{l}}} - \xi_\mathfrak{l}))$$

(2) *If $\ell \in C_s$, we have*

$$\varphi_{1,1,\ell}(g_{1,\ell}^{-1}(\rho(\xi) + \rho(\eta)\epsilon)g_{1,\ell}) = \delta_{O_\mathfrak{l}}(\xi_\mathfrak{l})\delta_{O_\ell}(\eta_\ell)(N(\overline{\mathfrak{l}})^\nu\delta_{\overline{\mathfrak{l}}^\nu}(\xi_{\overline{\mathfrak{l}}}) - N(\overline{\mathfrak{l}})^{\nu-1}\delta_{\overline{\mathfrak{l}}^{\nu-1}}(\xi_{\overline{\mathfrak{l}}}))\mathbf{e}_\ell(-\ell^{-\nu}\eta_\mathfrak{l}).$$

(3) *If $\ell \notin A \cup C$, we have $\varphi_{1,1,\ell}(g_{1,\ell}^{-1}(\rho(\xi) + \rho(\eta)\epsilon)g_{1,\ell}) = \delta_{O_\ell}(\eta_\ell)\delta_{O_\ell}(\xi_\ell)$.*

Proof. The assertion (2) and (3) are plain. We prove (1). Since $\varphi_{1,1,\ell} = \boldsymbol{\phi}_{1,1,\ell}$, we need to analyze

$$\delta_{\mathbb{Z}_\ell}(\xi_\mathfrak{l} - \eta_{\overline{\mathfrak{l}}})\delta_{\mathbb{Z}_\ell}(\eta_\mathfrak{l} - \eta_{\overline{\mathfrak{l}}} + \xi_\mathfrak{l} - \xi_{\overline{\mathfrak{l}}})\mathbf{e}_\ell(\ell^{-\nu}(\eta_{\overline{\mathfrak{l}}} - \eta_\mathfrak{l} + \xi_{\overline{\mathfrak{l}}} - \xi_\mathfrak{l}))\delta_{1+\ell^\nu\mathbb{Z}_\ell}(\eta_{\overline{\mathfrak{l}}})\delta_{\mathbb{Z}_\ell}(\xi_{\overline{\mathfrak{l}}} + \eta_{\overline{\mathfrak{l}}}).$$

If $\delta_{1+\ell^\nu\mathbb{Z}_\ell}(\eta_{\overline{\mathfrak{l}}}) \neq 0$, we get

$$\delta_{\mathbb{Z}_\ell}(\xi_\mathfrak{l} + \eta_{\overline{\mathfrak{l}}})\delta_{\mathbb{Z}_\ell}(\xi_{\overline{\mathfrak{l}}} - \eta_{\overline{\mathfrak{l}}}) \neq 0 \Leftrightarrow \delta_{O_\ell}(\xi_\ell) \neq 0.$$

Thus we get $\delta_{\mathbb{Z}_\ell}(\xi_\mathfrak{l} - \eta_{\overline{\mathfrak{l}}})\delta_{\mathbb{Z}_\ell}(\xi_{\overline{\mathfrak{l}}} + \eta_{\overline{\mathfrak{l}}})\delta_{1+\ell^\nu\mathbb{Z}_\ell}(\eta_{\overline{\mathfrak{l}}}) = \delta_{O_\ell}(\xi_\mathfrak{l})\delta_{1+\ell^\nu\mathbb{Z}_\ell}(\eta_{\overline{\mathfrak{l}}})$. Then we see

$$\delta_{\mathbb{Z}_\ell}(\xi_\mathfrak{l} - \eta_{\overline{\mathfrak{l}}})\delta_{\mathbb{Z}_\ell}(\eta_\mathfrak{l} - \eta_{\overline{\mathfrak{l}}} + \xi_\mathfrak{l} - \xi_{\overline{\mathfrak{l}}})\delta_{1+\ell^\nu\mathbb{Z}_\ell}(\eta_{\overline{\mathfrak{l}}})\delta_{\mathbb{Z}_\ell}(\xi_{\overline{\mathfrak{l}}} + \eta_{\overline{\mathfrak{l}}}) = \delta_{O_\ell}(\xi_\ell)\delta_{\mathbb{Z}_\ell}(\eta_\mathfrak{l})\delta_{1+\ell^\nu\mathbb{Z}_\ell}(\eta_{\overline{\mathfrak{l}}}).$$

□

We now deal with the case where ℓ is inert or ramified in K with $\ell^\nu \parallel N$. First we suppose $K = \mathbb{Q}_\ell[\sqrt{d_0}]$ with $O_\ell = \mathbb{Z}_\ell[\sqrt{d_0}]$ is the ℓ-adic integer ring of K_ℓ. Thus $d_0 = d(K)$ if ℓ is odd and $d_0 = \frac{d(K)}{4}$ if $\ell = 2$. For the moment, we suppose that 2 is not inert in $K/\mathbb{Q}$. Writing $\mathrm{ord}_\ell(d_0) = j$ and suppose that $\ell \in C_j$ if $j > 0$. We may take $g_{1,\ell}$ so that

$$g_{1,\ell}^{-1}\rho(a + b\sqrt{d_0})g_{1,\ell} = \left(\begin{smallmatrix} a & \ell^{-\nu}b \\ \ell^\nu d_0 b & a \end{smallmatrix}\right) \quad \text{and} \quad \det(g_{1,\ell}) = \ell^\nu.$$

Thus again $g_{1,\ell} \in GL_2(\mathbb{Z}_\ell)$ if $\ell \notin A \sqcup C \sqcup \{\infty\}$. Again by definition, we get

Lemma 2.4. *Suppose that $[K_\ell : \mathbb{Q}_\ell] = 2$ with $\ell^\nu \parallel N$ and $O_\ell = \mathbb{Z}_\ell[\sqrt{d_0}]$ for $d_0 = \frac{d(K)}{4} \in \mathbb{Z}_\ell$ (this implies that 2 ramifies if $\ell = 2$). Writing $\mathrm{ord}_\ell(d_0) = j$, suppose that $\ell \in C_j$ if $j > 0$ and $\ell = 2 \in C_0$ if $\mathrm{ord}_2(d(K)) > j = 0$ and $\nu(2) > 0$. For $v = \rho(\xi) + \rho(\eta)\epsilon$ with $\xi = a + b\sqrt{d_0}$, $\eta = a' + b'\sqrt{d_0}$ and $\epsilon = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, we have, for $\delta = \delta_{\mathbb{Z}_\ell}$*

$$\varphi_{1,1,\ell}(g_{1,\ell}^{-1}vg_{1,\ell}) = \begin{cases} \delta_{O_\ell}(\xi)\mathbf{e}_\ell(-\ell^{-\nu}\mathrm{Tr}(\frac{\xi}{\sqrt{d}}))\delta_{O_\ell}(\eta)\mathbf{e}_\ell(-\ell^{-\nu}\mathrm{Tr}(\frac{\eta}{\sqrt{d}})) & \textit{if } \ell \in C_{ns} \textit{ and } \nu = \nu(\ell) > 0, \\ \delta_{O_\ell}(\xi)\delta_{O_\ell}(\eta) & \textit{if } \nu(\ell) = 0 \end{cases}$$

Proof. We find $\rho(\xi)+\rho(\eta)\epsilon = \left(\begin{smallmatrix} a-a' & \ell^{-\nu}(b+b') \\ \ell^{\nu} d_0(b-b') & a+a' \end{smallmatrix}\right)$ for $\xi = a+b\sqrt{d}$ and $\eta = a'+b'\sqrt{d}$. Suppose $\ell \in C_{ns}$ or $\nu(\ell)=0$. Then

$$\varphi_{1,1,\ell}(\rho(\xi)+\rho(\eta)\epsilon) = \begin{cases} \delta(a-a')\delta(b+b')\mathbf{e}_\ell(-\ell^{-\nu}(b+b'))\delta_{\ell^\nu d_0\mathbb{Z}_\ell}(\ell^\nu d_0(b-b'))\delta_{\mathbb{Z}_\ell}(a+a') & \text{if } \nu>0, \\ \delta(a-a')\delta(b+b')\delta_{d_0\mathbb{Z}_\ell}(d_0(b-b'))\delta(a+a') & \text{otherwise.} \end{cases}$$

Since $a+a' \in \mathbb{Z}_\ell$ and $a-a' \in \mathbb{Z}_\ell \Leftrightarrow 2a, 2a' \in 2\mathbb{Z}_\ell \Leftrightarrow a, a' \in \mathbb{Z}_\ell$ (as $a+a' \equiv a-a' \mod 2$ if $\ell=2$), we find $\delta(a-a')\delta(a+a') = \delta(a)\delta(a')$. Similarly, $\delta(-b-b')\delta_{d_0\mathbb{Z}_\ell}(d_0(b-b')) = \delta(b)\delta(b')$; so, we have

$$\delta(a-a')\delta(a+a')\delta_{d_0\mathbb{Z}_\ell}(d_0(b-b')) = \delta(\xi)\delta(\eta).$$

This proves the formula when $\nu(\ell)=0$. Note that $b = \frac{1}{2}\mathrm{Tr}(\xi/\sqrt{d_0}) = \mathrm{Tr}(\xi/\sqrt{d})$ and $b' = \mathrm{Tr}(\eta/\sqrt{d})$. This proves the other case. $\square$

Lemma 2.5. *Assume that $[K_2:\mathbb{Q}_2]=2$ and $K_2/\mathbb{Q}_2$ is unramified. Then we can find $g_{1,\ell}$ for $\ell=2$ and units $u_1, u_\epsilon \in O_2^\times$ so that $\det(g_{1,\ell}) = \ell^\nu$, $\rho(O_\ell)+\rho(O_\ell)\epsilon = \alpha_{\ell^\nu} M_2(\mathbb{Z}_2)\alpha_{\ell^\nu}^{-1}$ and*

$$\rho(\xi)+\rho(\eta)\epsilon = \left(\begin{smallmatrix} * & \ell^{-\nu}(\mathrm{Tr}(u_1\xi\sqrt{d}^{-1})+\mathrm{Tr}(u_\epsilon\eta\sqrt{d}^{-1})) \\ * & * \end{smallmatrix}\right)$$

for all $(\xi,\eta) \in O_\ell \oplus O_\ell$.

Proof. First we assume that $\nu=0$. We pick a representation $\rho_1 : O_2 \hookrightarrow M_2(\mathbb{Z}_2)$ by choosing a basis of O_2 over $\mathbb{Z}_2$. Since 2 is unramified in K, we have $O_2 \otimes_{\mathbb{Z}_2} O_2 = O_2 \oplus O_2$ by $(a\otimes b) \mapsto (ab, a\overline{b})$. Since $M_2(\mathbb{Z}_2)$ is a module over $O_2 \otimes_{\mathbb{Z}_2} O_2$ by $(\xi\otimes\eta)x = \rho_1(\xi)x\rho_1(\eta)$, regarding $M_2(\mathbb{Z}_2)$ as an O_2-module by $\xi x = \rho_2(\xi)x$, $1 \in M_2(\mathbb{Z}_2)$ is an eigenvector under this action: $(\xi\otimes\eta)1 = \rho(\xi\eta)1$. Thus we have one more eigenvector ϵ_1 such that

$$(\xi\otimes\eta)\epsilon_1 = \rho_1(\xi)\epsilon\rho_1(\eta) = \rho(\xi\overline{\eta})\epsilon_1.$$

We may choose ϵ_1 so that $M_2(\mathbb{Z}_2) = \rho(O_2) \oplus \rho_1(O_2)\epsilon_1$. By reducing modulo 2, we get a representation $\overline{\rho}_1 = (\rho_1 \mod 2) : \mathbb{F}_4 \to M_2(\mathbb{F}_2)$ and the above decomposition indices $\overline{\rho}_1(\mathbb{F}_4) \oplus \overline{\rho}_1(\mathbb{F}_4)\overline{\epsilon}_1 = M_2(\mathbb{F}_2)$. Take any non-zero linear form $L : M_2(\mathbb{F}_2) \to \mathbb{F}_2$, $L \circ \overline{\rho}_1 \neq 0$, since otherwise $\overline{\rho}_1$ factors through $B = \{\alpha \in M_2(\mathbb{F}_2) | L\circ\alpha = 0\}$ making it reducible, a contradiction. Taking the linear from $b : M_2(A) \ni \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto b$, we find that $b|\overline{\rho}_1(\mathbb{F}_4) \neq 0$ because of this fact. So $b : \overline{\rho}_1(\mathbb{F}_4) \to \mathbb{F}_2$ is surjective. Similarly, $b : \overline{\rho}_1(\mathbb{F}_4)\overline{\epsilon}_1 \to \mathbb{F}_2$ is surjective. Then by Nakayama's lemma, we have $b : \rho_1(O_2) \to \mathbb{Z}_2$ and $b : \rho_1(O_2)\epsilon_1 \to \mathbb{Z}_2$ are surjective; so, we find $u_1, u_\epsilon \in O_2^\times$, as desired. For $\nu>0$, we just conjugate ρ and $g_{1,\ell}$ for $\nu=0$ by α_{ℓ^ν}. They do the job. $\square$

We choose $g_{1,\ell}$ as in the above lemmas. Then we have

$$\varphi_{1,1,\ell}(g_{1,\ell}^{-1}(\rho(\xi)+\rho(\eta)\epsilon)g_{1,\ell}) = \phi_1(\xi)\phi_\epsilon(\eta).$$

Indeed, for the discriminant $d = d(K)$ of $K/\mathbb{Q}$, we have

$$\phi_{1,\ell}(\xi) = \begin{cases} \delta_{O_\ell}(\xi) & \text{if } \nu(\ell)=0 \\ \delta_{O_\ell}(\xi_\ell)\mathbf{e}_\ell(\ell^{-\nu}(\xi_{\overline{\mathfrak{l}}}-\xi_{\mathfrak{l}})) & \text{if } \ell \in A, \\ \delta_{O_{\mathfrak{l}}}(\xi_{\mathfrak{l}})(N(\overline{\mathfrak{l}})^\nu\delta_{\overline{\mathfrak{l}}^\nu}(\xi_{\overline{\mathfrak{l}}}) - N(\overline{\mathfrak{l}})^{\nu-1}\delta_{\overline{\mathfrak{l}}^{\nu-1}}(\xi_{\overline{\mathfrak{l}}})) & \text{if } \ell \in C_s \text{ and } \nu=\nu(\ell)>0, \\ \delta_{O_\ell}(\xi)\mathbf{e}_\ell(-\ell^{-\nu}\mathrm{Tr}(\sqrt{d}^{-1}u_1\xi)) & \text{if } \ell \in C_{ns} \text{ and } \nu=\nu(\ell)>0, \end{cases}$$

$$\phi_{\epsilon,\ell}(\eta) = \begin{cases} \delta_{O_\ell}(\eta_\ell) & \text{if } \nu(\ell)=0 \\ \delta_{O_{\mathfrak{l}}}(\eta_{\mathfrak{l}})\delta_{(1+\ell^\nu O_{\overline{\mathfrak{l}}})}(\eta_{\overline{\mathfrak{l}}})\mathbf{e}_\ell(\ell^{-\nu}(1-\eta_{\mathfrak{l}})) & \text{if } \ell \in A, \\ \delta_{O_\ell}(\eta_\ell)\mathbf{e}_\ell(-\ell^{-\nu}\eta_{\mathfrak{l}}) & \text{if } \ell \in C_s \text{ and } \nu=\nu(\ell)>0, \\ \delta_{O_\ell}(\eta)\mathbf{e}_\ell(-\ell^{-\nu}\mathrm{Tr}(\sqrt{d}^{-1}u_\epsilon\eta)) & \text{if } \ell \in C_{ns} \text{ and } \nu=\nu(\ell)>0, \end{cases} \tag{2.8}$$

where u_1 and u_ϵ are units in O_ℓ and are equal to 1 except for the case where $\ell=2$ and 2 is inert in $K_\ell/\mathbb{Q}_\ell$.

Remark 2.6. *We note $\mathbf{e}_\ell(x) = \mathbf{e}(-[x]_\ell)$ for $x \in \mathbb{Q}_\ell$; so, if we replace $\mathbf{e}_\ell(x)$ by $\mathbf{e}([x]_\ell)$, we need to change the sign inside "$\mathbf{e}$".*

From the above consideration, for $\phi = \varphi_{1,1}$,

$$\phi_{\rho(a)g_1,\rho(b)g_1}(\rho(x)+\rho(y)\epsilon) = |N(\alpha)^{-1}|_{\mathbb{A}}^{1/2}\phi_1(\alpha^{-1}x)|N(\beta)^{-1}|_{\mathbb{A}}^{1/2}\phi_\epsilon(\beta^{-1}y),$$

and we conclude $\phi_{\rho(a)g_1,\rho(b)g_1}(\rho(x)+\rho(y)\epsilon) = \phi_{1,\alpha}(x)\phi_{\epsilon,\beta}(y)$, where $\phi_\alpha(x) = |N(\alpha)|_{\mathbb{A}}^{-1/2}\phi(\alpha^{-1}x)$ for $\phi = \phi_?$ with $? = 1, \epsilon$. Thus $\theta(\phi)(x;\alpha) = \sum_{v\in K}(\mathbf{r}({}_Sx)\phi)_\alpha(v)$ for $\phi = \phi_?$ with $? = 1, \epsilon$. This finishes the proof of Proposition 2.2. $\square$

2.3. **CM theta series.** Recall $T' = T_1 \times T_\epsilon$ and the character $\chi : K_{\mathbb{A}}^\times/K^\times \to \mathbb{C}^\times$ with $\chi^{-1}|_{\mathbb{A}^\times} = \boldsymbol{\psi}$ and $\chi(a_\infty) = a_\infty^k$. We have an exact sequence $1 \to \mathbb{G}_m \to T \xrightarrow{\pi} T' \xrightarrow{\nu} \mathbb{G}_m \to 1$ with $\nu(\alpha,\beta) = N(\alpha/\beta)$ and $\pi(a,b) = (ab^{-1}, a\overline{b}^{-1}) = (\alpha,\beta)$. Since $\boldsymbol{\psi}(N(a))\chi(ab) = \chi(a\overline{a})^{-1}\chi(ab) = \chi(\overline{a}^{-1}b) = \chi(\overline{\beta}^{-1})$, by Proposition 2.2, we have for $t = N(a^{-1}b) = N(\alpha)^{-1} = N(\beta)^{-1}$

$$\begin{aligned}(2.9)\quad &\int_{\mathcal{T}} \boldsymbol{\psi}(N(a)\det(g_1))\boldsymbol{\Theta}_{A,C,N}(x\alpha_t;\rho(a)g_1,\rho(b)g_1)\chi(ab)d^\times a d^\times b \\ &\overset{(*)}{=} \boldsymbol{\psi}(\det(g_1))\int_{T'(\mathbb{Q})\backslash T'(\mathbb{A})}\theta(\phi_1)(x\alpha_t;\alpha g_1)\theta(\phi_\epsilon)(x\alpha_t;\beta g_1)\chi(\overline{\beta}^{-1})d^\times\alpha d^\times\beta \\ &= \boldsymbol{\psi}(\det(g_1))\int_{T_1(\mathbb{Q})\backslash T_1(\mathbb{A})}\theta(\phi_1)(x\alpha_t;\alpha g_1)d^\times\alpha\int_{T_\epsilon(\mathbb{Q})\backslash T_\epsilon(\mathbb{A})}\theta(\phi_\epsilon)(x\alpha_t;\beta g_1)\chi(\overline{\beta}^{-1})d^\times\beta.\end{aligned}$$

Strictly speaking, the identity at $(*)$ has to be between the integrals over the image

$$\mathrm{Im}(T(\mathbb{Q})\backslash T(\mathbb{A})/T(\mathbb{R}) \xrightarrow{\pi} T'(\mathbb{Q})\backslash T'(\mathbb{A})/T'(\mathbb{R})).$$

However for the following reason, the identity $(*)$ is valid: By our way of extending the theta series to $S(\mathbb{A}) \times O_V(\mathbb{A})$ to $G_V(\mathbb{A})$ for $V = D$ and K, after the integral over $O_K \times O_K \subset T'$ is done, the result is just constant over the compact set $\mathrm{Coker}(T(\mathbb{Q})\backslash T(\mathbb{A})/T(\mathbb{R}) \xrightarrow{\pi} T'(\mathbb{Q})\backslash T'(\mathbb{A})/T'(\mathbb{R})) = I_{\mathbb{Q}}$ whose volume is canceled by the equal volume of

$$\mathrm{Ker}(T(\mathbb{Q})\backslash T(\mathbb{A})/T(\mathbb{R}) \xrightarrow{\pi} T'(\mathbb{Q})\backslash T'(\mathbb{A})/T'(\mathbb{R})) = I_{\mathbb{Q}}.$$

Write $\widetilde{\chi}(x) = \chi(\overline{x}^{-1})$ and ϕ for ϕ_ϵ. In this section, we write $\int_{T_\epsilon(\mathbb{Q})T_\epsilon(\mathbb{R})\backslash T_\epsilon(\mathbb{A})}\theta_G(\phi)(x;\beta)\widetilde{\chi}(\beta)d^\times\beta$ as a theta series of a Schwartz-Bruhat function Φ on $K_{\mathbb{A}}$. By (2.7), the infinity part of ϕ is given by $\phi_\infty(\eta) = \eta^k\mathbf{e}(\eta\overline{\eta}i)$. For g_τ as in the proof of Lemma 1.7, $\mathbf{r}(g_\tau)\phi_\infty(\eta)J(g_\tau,-i)^{-k} = \mathrm{Im}(\tau)^{k+(1/2)}\eta^k\mathbf{e}(\eta\overline{\eta}\tau)$. Then for $\beta \in K_\infty^\times$, $(x,\beta) \in G_V(\mathbb{R})$ for $V = K$ and $\tau = x(i) \in \mathfrak{H}$, we have ${}_Sx(i) = \alpha_{N(\beta)}^{-1}x(i) = N(\beta)\tau$. We may assume that $x_S = g_{N(\beta)\tau}$. Then

$$\begin{aligned}\mathbf{r}({}_Sx)\phi_\infty(\eta)J({}_Sx,-i)^{-k} &= \mathbf{r}(g_{N(\beta)\tau})\phi_\infty(\eta)J(g_{N(\beta)\tau},-i)^{-k} = \mathrm{Im}(N(\beta)\tau)^{k+(1/2)}\eta^k\mathbf{e}(\eta\overline{\eta}N(\beta)\tau),\\ L(\beta)\circ\mathbf{r}({}_Sx)\phi_\infty(\eta)J({}_Sx,-i)^{-k} &= \mathrm{Im}(\tau)^{k+(1/2)}N(\beta)^k\beta^{-k}\eta^k\mathbf{e}(\eta\overline{\eta}\tau) = \mathrm{Im}(\tau)^{k+(1/2)}\overline{\beta}^k\eta^k\mathbf{e}(\eta\overline{\eta}\tau).\end{aligned}$$

Thus the function $\beta \mapsto \theta(\phi)(x;\beta)\chi(\overline{\beta}^{-1})$ factors through $K_{\mathbb{A}}^\times/K_\infty^\times$ for $\phi_\infty(\eta) = c\eta^k\mathbf{e}(\eta\overline{\eta}i)$. Now regard $\theta(\phi)(\beta;\alpha_{N(\beta)}x)J(x_\infty,-i)^{-k}$ as a function of $x \in S(\mathbb{A})$ for which we integrate. Let $x_\infty = g_\tau$ ($\Rightarrow \tau = x_\infty(i)$), and we write

$$\begin{aligned}(2.10)\quad \theta(\phi)(\beta;\tau) &:= \theta(\phi)(\beta;\alpha_{N(\beta)}x_\infty)J(x_\infty,-i)^{-k} = \theta_S(\phi)(\beta;x_\infty)J(x_\infty,-i)^{-k}\\ &= \sum_{\eta\in K}(L(\beta)\circ\mathbf{r}(x_\infty)\phi)(\eta)J(x_\infty,-i)^{-k}\\ &= |N(\beta)|_{\mathbb{A}}^{-1/2}\,\mathrm{Im}(\tau)^{k+(1/2)}\sum_{\eta\in K}\phi^{(\infty)}(\beta^{-1}\eta)(\beta_\infty^{-1}\eta)^k\mathbf{e}(N(\beta_\infty)^{-1}\eta\overline{\eta}\tau).\end{aligned}$$

Decompose $T_\epsilon(\mathbb{Q})T_\epsilon(\mathbb{R})\backslash T_\epsilon(\mathbb{A}) = \bigsqcup_{i=1}^h a_iT_\epsilon(\widehat{\mathbb{Z}})/O^\times$ for $a_i \in K_{\mathbb{A}}^\times$ with $a_{i,N} = 1$ and $|N(a)|_{\mathbb{A}} = 1$. We can achieve $|N(a)|_{\mathbb{A}} = 1$ just taking $a_\infty = \sqrt{N(\mathfrak{a})} \in \mathbb{R}_+^\times$ for $\mathfrak{a} = a\widehat{O}\cap K$. Then we have

$$\int_{T_\epsilon(\mathbb{Q})T_\epsilon(\mathbb{R})\backslash T_\epsilon(\mathbb{A})}\theta(\phi)(\beta;\tau)\widetilde{\chi}(\beta)d^\times\beta = |O^\times|^{-1}\sum_{i=1}^h\int_{a_iT_\epsilon(\widehat{\mathbb{Z}})}\theta(\phi)(\beta;\tau)\widetilde{\chi}(\beta)d^\times\beta.$$

Pick $a \in K_{\mathbb{A}}^{\times}$ with $a_N = 1$, $|N(a)|_{\mathbb{A}} = 1$ and $a_\infty \in \mathbb{R}_+^{\times}$, and look at

$$\int_{aT_\epsilon(\widehat{\mathbb{Z}})} \theta(\phi)(\beta a_\infty;\tau)\widetilde{\chi}(\beta a_\infty)d^{\times}\beta = \widetilde{\chi}(a)\int_{T_\epsilon(\widehat{\mathbb{Z}})} \theta(\phi_a)(\beta;\tau)\widetilde{\chi}(\beta)d^{\times}\beta,$$

where $\phi_a(v) = |N(a)|_{\mathbb{A}}^{-1/2}\phi(a^{-1}v)$. Then $\theta(\phi_a)(\beta;\tau) = \sum_{\eta\in K}\phi_a(\beta^{-1}\eta;\tau)$ and hence

$$\int_{T_\epsilon(\widehat{\mathbb{Z}})} \theta(\phi_a)(\beta;\tau)\widetilde{\chi}(\beta)d^{\times}\beta = \sum_{\eta\in K}\int_{T(\widehat{\mathbb{Z}})} \phi_a(\beta^{-1}\eta;\tau)\widetilde{\chi}(\beta)d^{\times}\beta.$$

Write $\phi(\eta;\tau) = \phi_\infty(\eta_\infty;\tau)\prod_\ell \phi_\ell(\eta_\ell)$ for local function $\phi_\ell : K_\ell \to \mathbb{C}$ with

$$\phi_\infty(\eta;\tau) = \mathrm{Im}(\tau)^{k+(1/2)}\eta^k \mathbf{e}(\eta\overline{\eta}\tau).$$

Then we have, since a_∞ could be a nontrivial scalar with $N(a_\infty) = a_\infty^2 = N(\mathfrak{a})$ for $\mathfrak{a} = a\widehat{O}\cap K$,

$$\int_{T(\widehat{\mathbb{Z}})} (\mathbf{r}(_S(\alpha_{N(\beta_\infty)}x_\infty)\phi)_a(\beta^{-1}\eta)\widetilde{\chi}(\beta)d^{\times}\beta = \Phi_{a,\infty}(\eta_\infty;\tau)\prod_\ell\int_{T(\mathbb{Z}_\ell)} \phi_{\ell,a}(\beta_\ell^{-1}\eta_\ell)\widetilde{\chi}_\ell(\beta_\ell)d^{\times}\beta_\ell$$

for $x_\infty = \sqrt{\mathrm{Im}(\tau)}^{-1}\left(\begin{smallmatrix}\mathrm{Im}(\tau) & \mathrm{Re}(\tau)\\ 0 & 1\end{smallmatrix}\right)$ ($\Rightarrow \tau = x_\infty(i)$). We write as $\Phi_{a,\ell}(\eta_\ell)$ the individual factor $\int_{T(\mathbb{Z}_\ell)}\phi_{\ell,a}(\beta_\ell^{-1}\eta_\ell)\widetilde{\chi}_\ell(\beta_\ell)d^{\times}\beta_\ell$.

We have written the set of primes as $\mathcal{A}\cup\mathcal{C}$ for $\mathcal{A}$ made of prime factors one for each over $\ell\in A$ and $\mathcal{C}$ those over $\{\ell\in C|\nu(\ell)>0\}$. Recall $\nu=\nu(\ell)$ is the exponent of ℓ in N. The prime $\mathfrak{l}$ in $\mathcal{A}\cup\mathcal{C}_s$ was tentatively chosen (before stating Lemma 2.3) when we defined $g_{1,\ell}$. Here we make a specific choice depending on the conductor $\mathfrak{C}$ of the characters χ and χ_m later we introduce:

Definition 2.7. *Pick a conductor ideal $\mathfrak{C}$ of O and assume that $N(\mathfrak{C})|N^\mu$ for $\mu\gg 0$. We choose $\mathcal{A}$ and $\mathcal{C}$ so that $\mathfrak{C} = \prod_{\mathfrak{l}\in\mathcal{C}_s}\mathfrak{l}^{f_\mathfrak{l}}\overline{\mathfrak{l}}^{f_{\overline{\mathfrak{l}}}}\prod_{\mathfrak{l}\in\mathcal{A}}\mathfrak{l}^{f_\mathfrak{l}}\overline{\mathfrak{l}}^{f_{\overline{\mathfrak{l}}}}\prod_{\mathfrak{l}\in\mathcal{C}_{ns}}\mathfrak{l}^{f_\mathfrak{l}}$ with $\nu(\ell)\geq f_{\overline{\mathfrak{l}}}\geq f_\mathfrak{l}\geq 0$ for $\mathfrak{l}\in\mathcal{A}$ and $\nu(\ell)\geq f_{\overline{\mathfrak{l}}}\geq f_\mathfrak{l}=0$ for $\mathfrak{l}\in\mathcal{C}_s$. We also put $\mathcal{C}_0=\{\mathfrak{l}\in\mathcal{C}|\nu(\ell)>f_{\overline{\mathfrak{l}}}=0\}$, $\mathcal{A}_+=\{\mathfrak{l}\in\mathcal{A}|f_{\overline{\mathfrak{l}}}>0\}$ and $\mathcal{C}_+=\{\mathfrak{l}\in\mathcal{C}|f_{\overline{\mathfrak{l}}}>0\}$.*

We take $\mathfrak{C}$ to be the conductor of χ. Here is the explicit form of the function $\Phi_{a,\ell}$:

Lemma 2.8. *Assume* (S1–2), $|N(a)|_{\mathbb{A}}=1$ *and* $a_N=1$, *and write* $\chi^c(x)=\chi(\overline{x})$ *and* $\widetilde{\chi}(x)=\chi(\overline{x}^{-1})$.

1. *If ℓ is a prime with $\nu(\ell)=0$, $\Phi_{a,\ell}(\eta)=|N(a)|_\ell^{-1/2}\delta_{O_\ell}(a^{-1}\eta)$ for the characteristic function δ_{O_ℓ} of O_ℓ. At ∞, we have*

$$\widetilde{\chi}(a_\infty)\Phi_{a,\infty}(\eta;\tau) = N(\mathfrak{a})^{-k-(1/2)}\mathrm{Im}(\tau)^{k+(1/2)}\eta^k\mathbf{e}(N(\mathfrak{a})^{-1}\eta\overline{\eta}\tau),$$

where $\mathfrak{a}=a\widehat{O}\cap K$.

2. *If $\ell\in A$ (so, ℓ splits in $K/\mathbb{Q}$) and $\mathfrak{C}_\ell=\mathfrak{l}^{f_\mathfrak{l}}\overline{\mathfrak{l}}^{f_{\overline{\mathfrak{l}}}}$ with $0\leq f_\mathfrak{l}, f_{\overline{\mathfrak{l}}}\leq\nu$, we have,*

$$\Phi_{a,\ell}(\eta)=\begin{cases}|(O/\mathfrak{l}^\nu)^\times|^{-2}\mathbf{e}(-\ell^{-\nu})G(\chi_\mathfrak{l}^c)N(\mathfrak{l})^{\nu-f_{\overline{\mathfrak{l}}}}\delta_{\ell^{\nu-f_{\overline{\mathfrak{l}}}}O_\mathfrak{l}^\times}(\eta_\mathfrak{l})\delta_{O_{\overline{\mathfrak{l}}}^\times}(\eta_{\overline{\mathfrak{l}}})\widetilde{\chi}_\ell(\eta_\ell)\widetilde{\chi}_\mathfrak{l}(\ell^{f_{\overline{\mathfrak{l}}}-\nu}) & \text{if } f_{\overline{\mathfrak{l}}}>0,\\ |(O/\mathfrak{l}^\nu)^\times|^{-2}\mathbf{e}(-\ell^{-\nu})(N(\mathfrak{l})^\nu\delta_{\ell^\nu O_\mathfrak{l}}-N(\mathfrak{l})^{\nu-1}\delta_{\ell^{\nu-1}O_\mathfrak{l}})(\eta_\mathfrak{l})\delta_{O_{\overline{\mathfrak{l}}}^\times}(\eta_{\overline{\mathfrak{l}}}) & \text{if } f_{\overline{\mathfrak{l}}}=0,\end{cases}$$

where for a character ϕ of $O_\mathfrak{l}^\times$ of conductor $\mathfrak{l}^f$, $G(\phi)=\sum_{a\in O/\mathfrak{l}^f}\phi(a)\mathbf{e}\left(\frac{\mathrm{Tr}_{K_\ell/\mathbb{Q}_\ell}(a\sqrt{d}^{-1})}{\ell^f}\right)$ and δ_X is the characteristic function of $X\subset K_\ell$.

3. *If $\ell\in C_s$ and $\nu(\ell)>0$, writing $\mathfrak{C}_{\overline{\mathfrak{l}}}=\overline{\mathfrak{l}}^f$ with $0\leq f\leq\nu$,*

$$\Phi_{a,\ell}(\eta)=\begin{cases}|(O/\mathfrak{l}^\nu)^\times|^{-1}G(\chi_\mathfrak{l}^c)N(\mathfrak{l})^{\nu-f_{\overline{\mathfrak{l}}}}\delta_{\ell^{\nu-f}O_\mathfrak{l}^\times}(\eta_\mathfrak{l})\delta_{O_{\overline{\mathfrak{l}}}}(\eta_{\overline{\mathfrak{l}}})\widetilde{\chi}_\mathfrak{l}(\eta_\mathfrak{l})\widetilde{\chi}_\mathfrak{l}(\ell^{f-\nu}) & \text{if } f>0 \text{ and } \mathfrak{C}_\mathfrak{l}=1,\\ |(O/\mathfrak{l}^\nu)^\times|^{-1}(N(\mathfrak{l})^\nu\delta_{\ell^\nu O_\mathfrak{l}}-N(\mathfrak{l})^{\nu-1}\delta_{\ell^{\nu-1}O_\mathfrak{l}})(\eta_\mathfrak{l})\delta_{O_{\overline{\mathfrak{l}}}}(\eta_{\overline{\mathfrak{l}}}) & \text{if } \mathfrak{C}_\mathfrak{l}=\mathfrak{C}_{\overline{\mathfrak{l}}}=1,\\ 0 & \text{otherwise.}\end{cases}$$

4. *If $\ell\in C_{ns}$ with $\nu(\ell)>0$, writing $\mathfrak{C}_\ell=\ell^f$ with $0\leq f\leq\nu$, we have*

$$\Phi_{a,\ell}(\eta)=\begin{cases}|(O_\mathfrak{l}/\mathfrak{l}^\nu)^\times|^{-1}G(\chi_\mathfrak{l}^c)N(\mathfrak{l})^{\nu-f}\delta_{\ell^{\nu-f}O_\ell^\times}(\eta_\ell)\widetilde{\chi}_\mathfrak{l}(u_\epsilon\eta_\mathfrak{l})\widetilde{\chi}_\mathfrak{l}(\ell^{f-\nu}) & \text{if } f>0,\\ |(O_\mathfrak{l}/\mathfrak{l}^\nu)^\times|^{-1}(N(\mathfrak{l})^\nu\delta_{\mathfrak{l}^\nu O_\mathfrak{l}}-N(\mathfrak{l})^{\nu-1}\delta_{\mathfrak{l}^{\nu-1}O_\mathfrak{l}})(\eta_\mathfrak{l}) & \text{if } \mathfrak{C}_\ell=1,\end{cases}$$

where $u_\epsilon\in O_\ell^\times$ as in Lemma 2.5 *is equal to* 1 *except when $\ell=2$ is inert in $K_\ell/\mathbb{Q}_\ell$.*

Proof. The assertion (1) for finite place ℓ follows from the definition. As for the infinite place, note that $\beta \in T_\epsilon(\widehat{\mathbb{Z}})$; so, $\beta_\infty = 1$, and we get from (2.10) $\Phi_{a,\infty}(\eta) = |N(a_\infty)|^{-1/2} a_\infty^{-k}\eta^k \mathbf{e}(N(\mathfrak{a})^{-1}\eta\overline{\eta}\tau)$.

We now prove (2). As is well known (e.g. [LFE] page 259 (4b)), we have for $x \in O_{\mathfrak{l}}$,

$$\sum_{a\in O/\mathfrak{l}^\nu} \phi(a)\mathbf{e}\left(\left[\mathrm{Tr}_{K_\ell/\mathbb{Q}_\ell}(ax\sqrt{d}^{-1})/\ell^\nu\right]_\ell\right) = \begin{cases} N(\mathfrak{l})^{\nu-f}G(\phi)\delta_{\ell^{\nu-f}O_{\mathfrak{l}}^\times}(x)\phi^{-1}(x\ell^{f-\nu}) & \text{if } f > 0,\\ |(O/\mathfrak{l})^\times|^{-1}(N(\mathfrak{l})\delta_{\ell^\nu O_{\mathfrak{l}}} - \delta_{\ell^{\nu-1}O_{\mathfrak{l}}})(x) & \text{if } f = 0.\end{cases} \tag{2.11}$$

Since $\ell \in A$ splits in $K/\mathbb{Q}$, we may write $\beta = (a,b)$ for $a \in O_{\mathfrak{l}} = \mathbb{Z}_\ell$ and $b \in O_{\overline{\mathfrak{l}}} = \mathbb{Z}_\ell$. Then, for $\ell \in A$, we have, noting $\mathbf{e}_\ell(-\ell^{-\nu}\eta_{\mathfrak{l}}) = \mathbf{e}([\ell^{-\nu}\eta_{\mathfrak{l}}]_\ell)$ (see Remark 2.6),

$$\begin{aligned}
\Phi_{a,\ell}(x_\ell;\eta_\ell) &:= \int_{T_\epsilon(\mathbb{Z}_\ell)} \phi_{\epsilon,\ell}(x_\ell;\beta^{-1}\eta_\ell)\widetilde{\chi}(\beta)d^\times\beta\\
&= |(\mathbb{Z}/\ell^\nu\mathbb{Z})^\times|^{-2}\mathbf{e}_\ell(\ell^{-\nu}) \sum_{a,b\in(\mathbb{Z}/\ell^\nu\mathbb{Z})^\times} \widetilde{\chi}_{\mathfrak{l}}(a)\widetilde{\chi}_{\overline{\mathfrak{l}}}(b)\delta_{O_{\mathfrak{l}}}(a^{-1}\eta_{\mathfrak{l}})\mathbf{e}_\ell(-\ell^{-\nu}a^{-1}\eta_{\mathfrak{l}})\delta_{(1+\ell^\nu O_{\overline{\mathfrak{l}}})}(b^{-1}\eta_{\overline{\mathfrak{l}}})\\
&= |(\mathbb{Z}/\ell^\nu\mathbb{Z})^\times|^{-2}\mathbf{e}(-\ell^{-\nu}) \sum_{a\in(\mathbb{Z}/\ell^\nu\mathbb{Z})^\times} \widetilde{\chi}_{\mathfrak{l}}(a)\delta_{O_{\mathfrak{l}}}(a^{-1}\eta_{\mathfrak{l}})\mathbf{e}([\ell^{-\nu}a^{-1}\eta_{\mathfrak{l}}]_\ell) \times \sum_{b\in(\mathbb{Z}/\ell^\nu\mathbb{Z})^\times} \widetilde{\chi}_{\overline{\mathfrak{l}}}(b)\delta_{(1+\ell^\nu O_{\overline{\mathfrak{l}}})}(b^{-1}\eta_{\overline{\mathfrak{l}}})\\
&\overset{(2.11)}{=} \frac{\mathbf{e}(-\ell^{-\nu})\widetilde{\chi}_{\overline{\mathfrak{l}}}(\eta_{\overline{\mathfrak{l}}})\delta_{O_{\overline{\mathfrak{l}}}^\times}(\eta_{\overline{\mathfrak{l}}})}{|(\mathbb{Z}/\ell^\nu\mathbb{Z})^\times|} \times \begin{cases} |(\mathbb{Z}/\ell^{f_{\overline{\mathfrak{l}}}}\mathbb{Z})^\times|^{-1}\widetilde{\chi}_{\mathfrak{l}}(\eta_{\mathfrak{l}})\widetilde{\chi}_{\mathfrak{l}}(\ell^{f_{\overline{\mathfrak{l}}}-\nu})G(\chi_{\mathfrak{l}}^c)\delta_{\ell^{\nu-f_{\overline{\mathfrak{l}}}}O_{\mathfrak{l}}^\times}(\eta_{\mathfrak{l}}) & \text{if } f_{\overline{\mathfrak{l}}} > 0,\\ |(\mathbb{Z}/\ell\mathbb{Z})^\times|^{-1}(\ell\delta_{\ell^\nu O_{\mathfrak{l}}} - \delta_{\ell^{\nu-1}O_{\mathfrak{l}}})(\eta_{\mathfrak{l}}) & \text{if } f_{\overline{\mathfrak{l}}} = 0.\end{cases}
\end{aligned}$$

We prove (3). Write $\mathfrak{C}_{\overline{\mathfrak{l}}} = \overline{\mathfrak{l}}^f$. We have

$$\begin{aligned}
\Phi_{a,\ell}(x_\ell;\eta_\ell) &:= \int_{T_\epsilon(\mathbb{Z}_\ell)} \phi_{\epsilon,\ell}(x_\ell;\beta^{-1}\eta_\ell)\widetilde{\chi}(\beta)d^\times\beta\\
&= |(\mathbb{Z}/\ell^\nu\mathbb{Z})^\times|^{-2} \sum_{a,b\in(\mathbb{Z}/\ell^\nu\mathbb{Z})^\times} \widetilde{\chi}_{\mathfrak{l}}(a)\widetilde{\chi}_{\overline{\mathfrak{l}}}(b)\mathbf{e}_\ell(-\ell^{-\nu}a^{-1}\eta_{\mathfrak{l}})\delta_{O_{\mathfrak{l}}}(a^{-1}\eta_{\mathfrak{l}})\delta_{O_{\overline{\mathfrak{l}}}}(b^{-1}\eta_{\overline{\mathfrak{l}}})\\
&\overset{(2.11)}{=} \begin{cases} |(\mathbb{Z}/\ell^f\mathbb{Z})^\times|^{-1}\widetilde{\chi}_{\mathfrak{l}}(\ell^{f-\nu})\widetilde{\chi}_{\mathfrak{l}}(\eta_{\mathfrak{l}})G(\chi_{\mathfrak{l}}^c)\delta_{\ell^{\nu-f}O_{\mathfrak{l}}^\times}(\eta_{\mathfrak{l}})\delta_{O_{\overline{\mathfrak{l}}}}(\eta_{\overline{\mathfrak{l}}}) & \text{if } f > 0 \text{ and } \mathfrak{C}_{\mathfrak{l}} = 1,\\ |(\mathbb{Z}/\ell\mathbb{Z})^\times|^{-1}(\ell\delta_{\ell^\nu O_{\mathfrak{l}}} - \delta_{\ell^{\nu-1}O_{\mathfrak{l}}})(\eta_{\mathfrak{l}})\delta_{O_{\overline{\mathfrak{l}}}}(\eta_{\overline{\mathfrak{l}}}) & \text{if } \mathfrak{C}_{\mathfrak{l}} = \mathfrak{C}_{\overline{\mathfrak{l}}} = 1,\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

We prove (4). We have $\phi_\epsilon(\eta) = \delta_{O_\ell}(\eta)\mathbf{e}_\ell(-\ell^{-\nu}\mathrm{Tr}(u_\epsilon\eta/\sqrt{d})) = \delta_{O_\ell}(\eta)\mathbf{e}([\ell^{-\nu}\mathrm{Tr}(u_\epsilon\eta/\sqrt{d})]_\ell)$ and

$$\begin{aligned}
\Phi_{a,\ell}(x_\ell;\eta_\ell) &:= \int_{T_\epsilon(\mathbb{Z}_\ell)} \phi_{\epsilon,\ell}(x_\ell;\beta^{-1}\eta_\ell)\widetilde{\chi}(\beta)d^\times\beta\\
&= |(O_\ell/\ell^\nu O_\ell)^\times|^{-1} \sum_{a\in(O_\ell/\ell^\nu O_\ell)^\times} \widetilde{\chi}_\ell(a)\mathbf{e}([\ell^{-\nu}\mathrm{Tr}(a^{-1}u_\epsilon\eta/\sqrt{d})]_\ell).
\end{aligned}$$

Then the same computation as in (3) produces the result. □

We embed T into $G(\mathbb{A})$ by $(\xi,\eta) \mapsto (x\alpha_{N(\xi\eta^{-1})};\rho(\xi)g_1,\rho(\eta)g_1)$ for the choice of $g_1 \in GL_2(\mathbb{A})$ we made in Section 2.2, and computed the pull-back integral of $\theta_G(\varphi_{1,1})$. The corresponding embedding of the quadratic space $K_{\mathbb{A}}^2 \hookrightarrow D_{\mathbb{A}}$ is given by $(\xi,\eta) \mapsto g_1^{-1}(\rho(\xi)+\rho(\eta)\epsilon)g_1$.

To state the result, we fix some symbols: Write $\mathfrak{C}_{\mathfrak{l}} = \mathfrak{l}^{f_{\mathfrak{l}}}$. Let $\mathcal{C}_0 = \{\mathfrak{l} \in \mathcal{C} | f_{\overline{\mathfrak{l}}} = 0, \nu(\ell) > 0\}$, $\mathcal{A}_0 = \{\mathfrak{l} \in \mathcal{A} | f_{\overline{\mathfrak{l}}} = 0\}$, $\mathcal{C}_+ = \{\mathfrak{l} \in \mathcal{C} | f_{\overline{\mathfrak{l}}} > 0\}$, $\mathcal{A}_+ = \{\mathfrak{l} \in \mathcal{A} | f_{\overline{\mathfrak{l}}} > 0\}$, $\overline{\mathcal{A}}_+ = \{\overline{\mathfrak{l}} \in \mathcal{A} | f_{\overline{\mathfrak{l}}} > 0\}$. We then define

$$\mathfrak{t} = \prod_{\mathfrak{l}\in\mathcal{A}} \mathfrak{l}^{\nu(\ell)}\overline{\mathfrak{l}}^{\nu(\ell)} \prod_{\ell\in\mathcal{C}_s} \overline{\mathfrak{l}}^{\nu(\ell)} \prod_{\ell\in\mathcal{C}_{ns}} \mathfrak{l}^{\nu(\ell)},\ \mathfrak{s} = \prod_{\mathfrak{l}\in\mathcal{A}\cup\mathcal{C}} \mathfrak{l}^{\nu(\ell)-f_{\overline{\mathfrak{l}}}}, \text{ and } \mathfrak{s}_0 = \prod_{\mathfrak{l}\in\mathcal{A}_0\cup\mathcal{C}_0} \mathfrak{l}^{\nu(\ell)},$$

$L = \prod_{\ell\in A\cup C_r\cup C_i} \ell$, $\mathfrak{a}_{\overline{\mathcal{A}}} = \prod_{\mathfrak{l}\in\mathcal{A}} \overline{\mathfrak{l}}$, $\mathfrak{a}_J = \prod_{\mathfrak{l}\in J} \mathfrak{l}$, $\mathfrak{s}_J = \mathfrak{s}/\mathfrak{a}_J$ and $\mathfrak{s}_{0,J} = \mathfrak{s}_0/\mathfrak{a}_J$ for a subset $J \subset \mathcal{A}_0\cup\mathcal{C}_0$, where $\mathfrak{a}_\emptyset = O$. For each Hecke character λ with $\lambda(x_\infty) = x_\infty^{-k}$ and for each ideal $\mathfrak{a}$ prime to the conductor $\mathfrak{c}$ of λ, we have the corresponding ideal character given by $\lambda(\mathfrak{a}) = \lambda(a^{(\mathfrak{c})})$ where a is an idele a with $\mathfrak{a} = K \cap (a\widehat{O})$. We agree to put $\lambda(\mathfrak{a}) = 0$ if $\mathfrak{a}+\mathfrak{c} \subsetneq O$. Then we define

$$\mathbf{\Theta}_{\overline{\mathcal{A}}}(\lambda)(\tau) = \sum_{\mathfrak{a}\subset O} \lambda(\mathfrak{a})q^{N(\mathfrak{a})}$$

for $q = \mathbf{e}(\tau)$, where $\mathfrak{a}$ runs over O-ideals *prime to* all $\prod_{\mathfrak{l}\in\mathcal{A}}\overline{\mathfrak{l}}$. For any positive integer m and $f : \mathfrak{H} \to \mathbb{C}$, we define $f|[m](\tau) = f(m\tau)$. Then the result is

Lemma 2.9. *Let $\chi : K_{\mathbb{A}}^\times/K^\times \to \mathbb{C}^\times$ be a Hecke character of conductor $\mathfrak{C}$ with $\chi|_{\mathbb{A}^\times} = \boldsymbol{\psi}^{-1}$. Put $\lambda(x) = \widetilde{\chi}(x)^{-1}|N(x)|_{\mathbb{A}}^{-k} = \overline{\chi(\overline{x})}^{-1}$ (so, $\lambda(x_\infty) = x_\infty^{-k}$ and $\lambda^u := \lambda/|\lambda| = \chi^-$). Decompose $\mathfrak{C} = \prod_{\mathfrak{l}\in\mathcal{A}\cup\mathcal{C}_s}\mathfrak{l}^{f_\mathfrak{l}}\overline{\mathfrak{l}}^{f_{\overline{\mathfrak{l}}}}\prod_{\mathfrak{l}\in\mathcal{C}_{ns}}\mathfrak{l}^{f_\mathfrak{l}}$ with $0 \le f_\mathfrak{l}, f_{\overline{\mathfrak{l}}} \le \nu(\ell)$ as in* Definition 2.7*, and assume that $f_\mathfrak{l} = 0$ if either $\mathfrak{l} \in \mathcal{C}$ is split in K or $\ell \notin A \cup C$. Then the classical cusp form giving rise to the theta integral $\int_{T_\epsilon(\mathbb{Q})T_\epsilon(\mathbb{R})\backslash T_\epsilon(\mathbb{A})}\theta_G(\phi_\epsilon)(x;\beta)\chi(\overline{\beta}^{-1})d^\times\beta$ is a CM theta series given by*

$$C\,\mathrm{Im}(\tau)^{k+(1/2)}\sum_{\mathfrak{y}|\mathfrak{s}_0}\mu_K(\mathfrak{y})N(\mathfrak{s}/\mathfrak{y})\lambda(\mathfrak{s}_0/\mathfrak{y})\boldsymbol{\Theta}_{\overline{\mathcal{A}}}(\lambda)|[N(\mathfrak{s}/\mathfrak{y})]$$

for a constant $C = \mathbf{e}(-N_A^{-1})|(O/\mathfrak{t})^\times|^{-1}(\prod_{\mathfrak{l}\in\mathcal{A}_+\cup\mathcal{C}_+}N(\mathfrak{l})^{k(\nu(\ell)-f_{\overline{\mathfrak{l}}})}\chi_{\overline{\mathfrak{l}}}(\ell^{\nu(\ell)-f_{\overline{\mathfrak{l}}}}u_\epsilon^{-c})G(\chi_\mathfrak{l}\circ c))$, where $N_A = \prod_{\ell\in A}N_\ell$, u_ϵ is as in Lemma 2.5 *and is equal to* 1 *unless $\mathfrak{l}|2$, and the Gauss sum $G(\chi_\mathfrak{l}\circ c))$ is as in* Lemma 2.8 (2)*. Here μ_K is the Moebius function (for K) and $d^\times\beta$ is the Haar measure with $\int_{T_\epsilon(\widehat{\mathbb{Z}})}d^\times\beta = 1$.*

Proof. Each term of $\theta(\phi)(x;\beta)$ is given by $\Phi_a(x;\xi)$ which has been computed in Lemma 2.8. By our choice, $a_{i,N} = 1$ and $|N(a_i)|_{\mathbb{A}} = 1$ with scalar $a_{i,\infty} \in \mathbb{R}_+^\times$. Since $\theta(\phi)(a\beta;_S x) = \theta(\phi_{a\beta})(_S x)$ with $\phi_a(x) = |N(a)|_{\mathbb{A}}^{-1/2}\phi(a^{-1}x) = \phi(a^{-1}x)$, we may forget about the factor $|N(a)|_{\mathbb{A}}^{-1/2}$ (and we disregard $N(\mathfrak{a})^{-1/2}$ in $\Phi_{a,\infty}$ in Lemma 2.8 (1)). Note $N(\mathfrak{s}_{\overline{\mathfrak{C}}})^k = \prod_{\mathfrak{l}\in\mathcal{A}_+\cup\mathcal{C}_+}N(\mathfrak{l})^{k(\nu(\ell)-f_{\overline{\mathfrak{l}}})}$ and

$$\prod_{\mathfrak{l}\in\mathcal{A}_+\cup\mathcal{C}_+}N(\mathfrak{l})^{\nu(\ell)-f_{\overline{\mathfrak{l}}}}\delta_{\ell^{\nu(\ell)-f_{\overline{\mathfrak{l}}}}O_\mathfrak{l}}\prod_{\mathfrak{l}\in\mathcal{A}_0\cup\mathcal{C}_0}(N(\mathfrak{l})^{\nu(\ell)}\delta_{\ell^{\nu(\ell)}O_\mathfrak{l}} - N(\mathfrak{l})^{\nu(\ell)-1}\delta_{\ell^{\nu(\ell)}O_\mathfrak{l}})$$
$$= \sum_{J\subset\mathcal{A}_0\cup\mathcal{C}_0}(-1)^{|J|}N(\mathfrak{s}_J)\delta_{\mathfrak{s}_J O_\mathfrak{s}} = \sum_{\mathfrak{y}|\mathfrak{s}_0}\mu_K(\mathfrak{y})N(\mathfrak{s}/\mathfrak{y})\delta_{\mathfrak{s}/\mathfrak{y}O_\mathfrak{s}}.$$

Then we have

$$\mathrm{Im}(\tau)^{-k-(1/2)}\theta_i(\Phi_\epsilon) = \widetilde{\chi}(a_i)N(\mathfrak{a}_i)^{-k/2}\,\mathrm{Im}(\tau)^{-k-(1/2)}\int_{T_\epsilon(\widehat{\mathbb{Z}})/O^\times}\theta(\phi_{a_i})(\beta;\tau)\widetilde{\chi}(\beta)d^\times\beta$$
$$= CN(\mathfrak{s}_{\overline{\mathfrak{C}}})^{-k}|O^\times|^{-1}\widetilde{\chi}(a_i^{(\infty)})N(\mathfrak{a}_i)^{-k}\sum_{J\subset\mathcal{A}_0\cup\mathcal{C}_0}(-1)^{|J|}N(\mathfrak{s}_J)\sum_{\xi\in(\mathfrak{s}_J\mathfrak{a}_i)^\times}\widetilde{\chi}_{\overline{\mathfrak{C}}}(\xi)\xi^k\mathbf{e}(\xi\overline{\xi}N(\mathfrak{a}_i)^{-1}\tau),$$

where $(\mathfrak{s}_J\mathfrak{a}_i)^\times$ is the subset of $\mathfrak{s}_J\mathfrak{a}_i$ made of elements ξ with $\xi O_{\overline{\mathfrak{l}}} = O_{\overline{\mathfrak{l}}}$ for all $\mathfrak{l} \in \mathcal{A}$ and $\xi O_{\overline{\mathfrak{C}}} = \mathfrak{s}_{\overline{\mathfrak{C}}}$. If $\xi \in (\mathfrak{s}_J\mathfrak{a}_i)^\times$, $\xi_{\overline{\mathfrak{C}}}O_{\overline{\mathfrak{C}}} = \mathfrak{s}_{\overline{\mathfrak{C}}}$, and we have $\widetilde{\chi}_{\overline{\mathfrak{C}}}(\xi) = N(\mathfrak{s}_{\overline{\mathfrak{C}}})^k\lambda_{\overline{\mathfrak{C}}}^{-1}(\xi)$ from $\widetilde{\chi}\lambda = |N(\cdot)|_{\mathbb{A}}^{-k}$. Similarly, $\widetilde{\chi}(a_i^{(\infty)})N(\mathfrak{a}_i)^{-k} = \lambda^{-1}(a_i^{(\infty)})$. Thus, we have

$$\frac{\theta_i(\Phi_\epsilon)}{\mathrm{Im}(\tau)^{k+(1/2)}} = C|O^\times|^{-1}\sum_{J\subset\mathcal{A}_0\cup\mathcal{C}_0}(-1)^{|J|}\lambda^{-1}(a_i^{(\infty)})N(\mathfrak{s}_J)\sum_{\xi\in(\mathfrak{s}_J\mathfrak{a}_i)^\times}\lambda_{\overline{\mathfrak{C}}}^{-1}(\xi)\xi^k\mathbf{e}(\xi\overline{\xi}N(\mathfrak{a}_i)^{-1}\tau),$$

Since $\lambda_{\overline{\mathfrak{C}}}^{-1}(\xi)\lambda^{-1}(\xi^{(\overline{\mathfrak{C}}\infty)})\xi^k = 1$, $\mathfrak{s}_J\mathfrak{s}_{\overline{\mathfrak{C}}}^{-1} = \mathfrak{s}_{0,J}$ and $\lambda^{-1}(\xi^{(\overline{\mathfrak{C}}\infty)}) = \lambda^{-1}(\xi\mathfrak{s}_{\overline{\mathfrak{C}}}^{-1})$ for $\xi \in (\mathfrak{s}_J\mathfrak{a}_i)^\times$, we have

$$\frac{\theta_i(\Phi_\epsilon)}{\mathrm{Im}(\tau)^{k+(1/2)}} = C|O^\times|^{-1}\sum_{J\subset\mathcal{A}_0\cup\mathcal{C}_0}(-1)^{|J|}N(\mathfrak{s}_J)\lambda^{-1}(a_i^{(\infty)})\lambda(\mathfrak{s}_{0,J})\sum_{\xi\in(\mathfrak{s}_J\mathfrak{a}_i)^\times}\lambda(\xi\mathfrak{s}_J^{-1})\mathbf{e}(\xi\overline{\xi}N(\mathfrak{a}_i)^{-1}\tau),$$

Then by computation, we get

$$\lambda^{-1}(a_i^{(\infty)})\sum_{\xi\in(\mathfrak{s}_J\mathfrak{a}_i)^\times}\lambda(\xi\mathfrak{s}_J^{-1})\mathbf{e}(\xi\overline{\xi}N(\mathfrak{a}_i)^{-1}\tau) = \sum_{\xi\mathfrak{a}_i^{-1}\mathfrak{s}_J^{-1}\subset O,\xi\mathfrak{a}_i^{-1}+\mathfrak{a}_{\overline{\mathcal{A}}}=O}\lambda(\xi\mathfrak{a}_i^{-1}\mathfrak{s}_J^{-1})\mathbf{e}(\frac{N(\xi\mathfrak{a}_i^{-1}\mathfrak{s}_J^{-1})}{N(\mathfrak{s}_J^{-1})}\tau).$$

Changing variable $\xi\mathfrak{a}_i^{-1}\mathfrak{s}_J^{-1} \mapsto \mathfrak{a}$, this is equal to $\sum_\mathfrak{a}\lambda(\mathfrak{a})\mathbf{e}(N(\mathfrak{a}\mathfrak{s}_J)\tau) = \boldsymbol{\Theta}_{\overline{\mathcal{A}}}(\lambda)|[N(\mathfrak{s}_J)]$, where $\mathfrak{a}$ runs over all integral ideals prime to $\mathfrak{s}_J$ equivalent to $\mathfrak{a}_i^{-1}\mathfrak{s}_J^{-1}$. Summing up over ideal classes $\mathfrak{a}_i$, we get the desired formula. □

Corollary 2.10. *The cusp form $\int_{T_\epsilon(\mathbb{Q})T_\epsilon(\mathbb{R})\backslash T_\epsilon(\mathbb{A})}\theta_G(\phi_\epsilon)(x;\beta)\chi(\overline{\beta}^{-1})d^\times\beta$ is on $\Gamma_0(N_\epsilon)$ with Neben character $\psi^{-1}\chi_K$ for $\chi_K = \left(\frac{K/\mathbb{Q}}{}\right)$, where $N_\epsilon = |d(K)|\prod_{\mathfrak{l}\in\mathcal{A}\cup\mathcal{C}}N(\mathfrak{l})^{\nu(\ell)}\prod_{\mathfrak{l}\in\overline{\mathcal{A}}_+}N(\mathfrak{l}^{f_\mathfrak{l}})\prod_{\mathfrak{l}\in\mathcal{A}_0}N(\mathfrak{l})$.*

Proof. The primitive theta series $\boldsymbol{\Theta}(\lambda)$ associated to $\boldsymbol{\Theta}_{\overline{\mathcal{A}}}(\lambda)$ is on $\Gamma_0(|d|N(\mathfrak{C}))$ with character $\lambda|_{\mathbb{A}^\times}\chi_K = \boldsymbol{\psi}^{-1}\chi_K$ (e.g., [HMI] Theorem 2.71). Replacing $\boldsymbol{\Theta}(\lambda)$ by $\boldsymbol{\Theta}_{\overline{\mathcal{A}}}(\lambda)$, the level adds up only for a single power of $\mathfrak{l} \in \mathcal{A}_0$. Thus $\boldsymbol{\Theta}_{\overline{\mathcal{A}}}|[\mathfrak{s}]$ has the highest level: $d \cdot N(\mathfrak{C})N(\mathfrak{s})\prod_{\mathfrak{l}\in\mathcal{A}_0} N(\mathfrak{l})$ for $d = d(K)$. Since

$$|d|N(\mathfrak{C})N(\mathfrak{s}) = |d|N(\overline{\mathfrak{C}})N(\mathfrak{s}) = |d|\prod_{\mathfrak{l}\in\mathcal{A}\cup\mathcal{C}} N(\mathfrak{l}^{\nu(\ell)-f_{\overline{\mathfrak{l}}}+f_{\overline{\mathfrak{l}}}})N(\mathfrak{l}^{f_{\mathfrak{l}}}) = |d|\prod_{\mathfrak{l}\in\mathcal{A}\cup\mathcal{C}} N(\mathfrak{l})^{\nu(\ell)}\prod_{\overline{\mathfrak{l}}\in\overline{\mathcal{A}}_+} N(\mathfrak{l}^{f_{\mathfrak{l}}}),$$

we get the desired result. □

2.4. **The Siegel–Weil formula.** We now compute the first integral:

$$\int_{T_1(\mathbb{Q})T_1(\mathbb{R})\backslash T_1(\mathbb{A})} \theta_G(\phi_1)(x;\alpha)d^\times\alpha = \int_{K^\times K_\infty^\times\backslash K_{\mathbb{A}}^\times} \theta_G(\phi_1)(x;\alpha)d^\times\alpha.$$

In this section, we write ϕ for ϕ_1. By (2.7), we have $\phi_\infty(\xi) = \mathrm{Im}(\tau)^{1/2}\mathbf{e}(\xi\overline{\xi}\tau)$. By the same computation as in the previous subsection, we can verify that the function $\alpha \mapsto \theta_G(\phi)(x;\alpha)$ factors through $K_{\mathbb{A}}^\times/K_\infty^\times$, and the above integral is well defined.

Let $K_{\mathbb{A}}^{(1)} = \{x \in K_{\mathbb{A}}^\times \,|\, |N(x)|_{\mathbb{A}} = 1\}$. Then $K_{\mathbb{A}}^{(1)}/K_{\mathbb{A}}^1 \hookrightarrow \mathbb{Q}_+^\times$ by $\mathcal{N} : x \mapsto |N(x^{(\infty)})|_{\mathbb{A}}^{-1} = N(x_\infty)$. If $\xi \in \mathbb{Q}_+^\times$ is in the image of $\mathcal{N} : K_{\mathbb{A}}^{(1)}/K_{\mathbb{A}}^1 \hookrightarrow \mathbb{Q}_+^\times$, ξ is local norm at every finite place up to units, and $\mathrm{Im}(\mathcal{N}) = |N(K_{\mathbb{A}^{(\infty)}}^\times)|_{\mathbb{A}}$. Thus we have

$$\frac{N(K_{\mathbb{A}}^\times)}{N(K^\times)N(K_\infty^\times)} = \frac{N(K^\times K_\infty^\times K_{\mathbb{A}}^{(1)})}{N(K^\times K_\infty^\times)} = \frac{N(K_{\mathbb{A}}^{(1)})}{N(K^\times K_\infty^\times)\cap N(K_{\mathbb{A}}^{(1)})} =: \mathcal{T}_1.$$

In particular, $\mathcal{T}_1$ is a compact topological group. Indeed,

$$\mathcal{T}_1/N(\widehat{O}^\times) = \frac{N(K_{\mathbb{A}}^\times)}{N(K^\times)N(K_\infty^\times)N(\widehat{O}^\times)}$$

is a quotient of the class group Cl_K.

We have

$$\int_{K^\times K_\infty^\times\backslash K_{\mathbb{A}}^\times} \theta_G(\phi_1)(x;\alpha)d^\times\alpha = \int_{\mathcal{T}_1}\int_{K^1K_\infty^1\backslash K_{\mathbb{A}}^1} \theta(\phi)(\alpha_t x;\alpha\xi_t)d^\times\alpha d^\times t,$$

where $N(\xi_t) = t$ with $|N(t)|_{\mathbb{A}} = 1$ and $K^1 = \mathrm{Ker}(N_{K/\mathbb{Q}})$. By the Siegel–Weil formula [We],

$$\int_{K^1K_\infty^1\backslash K_{\mathbb{A}}^1} \theta(\phi)(\alpha_t x;\alpha\xi_t)d^\times\alpha = E(\phi_1)$$

where $E(\phi_1)(\alpha_t x) = \sum_{\gamma\in B(\mathbb{Q})\backslash SL_2(\mathbb{Q})}(\omega(\gamma\alpha_t x,\xi_t)\phi_1)(0)$ for $x \in S(\mathbb{A})$ and

$$L(\xi_t)(\phi)(v) = \phi_t(v) = |N(t)|_{\mathbb{A}}^{-1/2}\phi_1(\xi_t^{-1}v) = \phi_1(\xi_t^{-1}v)$$

as $|N(t)|_{\mathbb{A}} = 1$. Thus we get

$$\int_{K^\times K_\infty^\times\backslash K_{\mathbb{A}}^\times} \theta(\phi_1)(\alpha_{N(\xi)}x;\xi)d^\times\xi = \int_{\mathcal{T}_1} E(\phi_1)(\alpha_t x)d^\times t,$$

As explained in Section 1.2, we have

$$(2.12)\quad (\omega(\left(\begin{smallmatrix} a & b \\ 0 & a^{-1}\end{smallmatrix}\right)\alpha_t x,\xi_t)\phi_1)(0) = (\omega(\alpha_t\left(\begin{smallmatrix} a & bt \\ 0 & a^{-1}\end{smallmatrix}\right)x,\xi_t)\phi_1)(0) = \\ = (L(\xi_t)\mathbf{r}(\left(\begin{smallmatrix} a & bt \\ 0 & a^{-1}\end{smallmatrix}\right)x)\phi_1)(0) = |a|_{\mathbb{A}}(\mathbf{r}(x)\phi)(0),$$

since $\omega(x,\xi_{\det(x)}) = \mathbf{r}(x\alpha_{\det(x)}^{-1})L(\xi_{\det(x)}) = L(\xi_{\det(x)})\mathbf{r}(\alpha_{\det(x)}^{-1}x)$ (see (1.2)). This shows that $E(\phi_1)$ is well defined and is independent of $t \in \mathcal{T}_1$. We have proved

Lemma 2.11. *We have* $\int_{K^\times K_\infty^\times\backslash K_{\mathbb{A}}^\times} \theta_G(\phi_1)(x;\alpha)d^\times\alpha = \int_{\mathcal{T}_1} d^\times t \cdot E(\phi_1)(x)$.

2.5. **Explicit form of weight 1 theta series.** Strictly speaking, the Siegel–Weil formula we used is in the non-convergent range Weil [We1] did not cover (though it is briefly explained in [Wa] I.5). To show it actually works well and to exhibit the explicit form of the Eisenstein series we need, using a result of Hecke [H], we compute the theta series

$$\int_{T_1(\mathbb{Q})T_1(\mathbb{R})\backslash T_1(\mathbb{A})} \theta_G(\phi_1)(x;\alpha)d^\times\alpha$$

in the same way we did in Lemma 2.9. As before, in this section, we write ϕ for ϕ_1 for simplicity. By (2.7), the infinity part of ϕ is given by $\phi_\infty(\xi) = \mathrm{Im}(\tau)^{1/2}\mathbf{e}(-\xi\overline{\xi}\overline{\tau})$.

Decompose $T_1(\mathbb{Q})T_1(\mathbb{R})\backslash T_1(\mathbb{A}) = \bigsqcup_{i=1}^h a_iT_1(\widehat{\mathbb{Z}})/O^\times$ for $a_i \in K_\mathbb{A}^\times$ with $a_{i,N} = 1$ and $|N(a)|_\mathbb{A} = 1$. Then we have

$$\int_{T_1(\mathbb{Q})T_1(\mathbb{R})\backslash T_1(\mathbb{A})} \theta(\phi)(x;\alpha)d^\times\alpha = |O^\times|^{-1}\sum_{i=1}^h \int_{a_iT_1(\widehat{\mathbb{Z}})} \theta(\phi)(x;\alpha)d^\times\alpha.$$

Pick $a \in K_\mathbb{A}^\times$ with $a_N = 1$, $|N(a)|_\mathbb{A} = 1$ and $a_\infty \in \mathbb{R}_+^\times$, and look at

$$\int_{aT_1(\widehat{\mathbb{Z}})} \theta(\phi)(x;\alpha)d^\times\alpha = \int_{T_1(\widehat{\mathbb{Z}})} \theta(\phi_a)(x;\alpha)d^\times\alpha,$$

where $\phi_a(v) = |N(a)|_\mathbb{A}^{-1/2}\phi(a^{-1}v) = \phi(a^{-1}v)$. Suppose that $\phi = \prod_\ell \phi_\ell$ for local function $\phi_\ell : K_\ell \to \mathbb{C}$ and $x^{(\infty)} = 1$. Again we have, since a_∞ could be a nontrivial scalar with $N(a_\infty) = a_\infty^2 = N(\mathfrak{a})$ for $\mathfrak{a} = a\widehat{O} \cap K$,

$$\int_{T(\widehat{\mathbb{Z}})} (\mathbf{r}(_S(\alpha_{N(\alpha_\infty)}x_\infty)\phi)_a(\alpha^{-1}\xi)d^\times\alpha = \Psi_{a,\infty}(\xi_\infty;\tau)\prod_\ell \int_{T(\mathbb{Z}_\ell)} \phi_{\ell,a}(\alpha_\ell^{-1}\xi_\ell)d^\times\alpha_\ell$$

for $x_\infty = \sqrt{\mathrm{Im}(\tau)}^{-1}\left(\begin{smallmatrix}\mathrm{Im}(\tau) & \mathrm{Re}(\tau)\\ 0 & 1\end{smallmatrix}\right)$ ($\Rightarrow \tau = x_\infty(i)$). We write as $\Psi_{a,\ell}(\xi_\ell)$ the individual factor $\int_{T(\mathbb{Z}_\ell)} \phi_{\ell,a}(\alpha_\ell^{-1}\xi_\ell)d^\times\alpha_\ell$. Recall the prime factor $\mathfrak{l}$ of $\ell \in A \cup C$ we have chosen when we defined $g_{1,\ell}$. We write this set of primes as $\mathcal{A} \cup \mathcal{C}$ for $\mathcal{A}$ made of prime factors over A and $\mathcal{C}$ those over C. Write the conductor of χ as $\mathfrak{C}$. Recall $\nu = \nu(\ell)$ is the exponent of ℓ in N. Here is the explicit form of the function $\Psi_{a,\ell}$:

Lemma 2.12. *Assume* (S1–2)*, and $a_N = 1$ with $|N(a)|_\mathbb{A} = 1$. The we have*

(1) *If $\nu(\ell) = 0$, $\Psi_{a,\ell}(\xi) = |N(a)|_\ell^{-1/2}\delta_{O_\ell}(a^{-1}\xi)$ for the characteristic function δ_{O_ℓ} of O_ℓ. At ∞,*

$$\Psi_{a,\infty}(\xi;\tau) = N(\mathfrak{a})^{-1/2}\,\mathrm{Im}(\tau)^{1/2}\mathbf{e}(-N(\mathfrak{a})^{-1}\xi\overline{\xi}\overline{\tau}),$$

where $x_\infty(i) = \tau$ and $\mathfrak{a} = a\widehat{O} \cap K$.

(2) *If $\ell \in C_s$, $\Psi_{a,\ell}(\xi_\ell) = \phi_{1,\ell}(\xi_\ell) = \delta_{O_\mathfrak{l}}(\xi_\mathfrak{l})(N(\overline{\mathfrak{l}})^\nu\delta_{\overline{\mathfrak{l}}^\nu}(\xi_{\overline{\mathfrak{l}}}) - N(\overline{\mathfrak{l}})^{\nu-1}\delta_{\overline{\mathfrak{l}}^{\nu-1}}(\xi_{\overline{\mathfrak{l}}}))$.*

(3) *If $\ell \in C_{ns}$ with $\nu(\ell) > 0$, we have $\Psi_{a,\ell}(\xi) = |(O_\ell/\mathfrak{l}^\nu)^\times|^{-1}(N(\mathfrak{l}^\nu)\delta_{\mathfrak{l}^\nu O_\mathfrak{l}} - N(\mathfrak{l}^{\nu-1})\delta_{\mathfrak{l}^{\nu-1}O_\mathfrak{l}})(\xi_\mathfrak{l})$.*

(4) *If $\ell \in A$, we have $\Psi_{a,\ell}(\xi) = \Psi_{a,\mathfrak{l}}(\xi)\Psi_{a,\overline{\mathfrak{l}}}(\xi)$ for prime factors $\mathfrak{l}|\ell$, and*

$$\Psi_{a,\mathfrak{l}}(\xi) = |(O_\ell/\mathfrak{l}^\nu)^\times|^{-1}(N(\mathfrak{l}^\nu)\delta_{\mathfrak{l}^\nu O_\mathfrak{l}} - N(\mathfrak{l}^{\nu-1})\delta_{\mathfrak{l}^{\nu-1}O_\mathfrak{l}})(\xi_\mathfrak{l}),$$
$$\Psi_{a,\overline{\mathfrak{l}}}(\xi) = |(O_\ell/\overline{\mathfrak{l}}^\nu)^\times|^{-1}(N(\overline{\mathfrak{l}}^\nu)\delta_{\mathfrak{l}^\nu O_{\overline{\mathfrak{l}}}} - N(\overline{\mathfrak{l}}^{\nu-1})\delta_{\overline{\mathfrak{l}}^{\nu-1}O_{\overline{\mathfrak{l}}}})(\xi_{\overline{\mathfrak{l}}}).$$

Proof. The proof of assertion (1) is the same as the one for Lemma 2.8 (1). The assertion (2) follows from the fact that $\int_{O_\mathfrak{l}^\times} \delta_{\ell^m O_\mathfrak{l}}(a^{-1}x)d^\times a = \delta_{\ell^m O_\mathfrak{l}}(x)$.

We prove (3). Suppose that $\ell|N$ is non-split. We have $\phi_1(\xi) = \delta_{O_\ell}(\xi)\mathbf{e}([\ell^{-\nu}\mathrm{Tr}(u_1\xi/\sqrt{d})]_\ell)$ and

$$\int_{T_1(\mathbb{Z}_\ell)} \phi_{1,\ell}(x_\ell;\alpha^{-1}\xi_\ell)d^\times\alpha = |(O_\ell/\ell^\nu O_\ell)^\times|^{-1}\sum_{a\in(O_\ell/\ell^\nu O_\ell)^\times} \widetilde{\chi}_\ell(a)\mathbf{e}([\ell^{-\nu}\mathrm{Tr}(a^{-1}u_1\xi/\sqrt{d})]_\ell)$$
$$\overset{(2.11)}{=} |(O_\ell/\mathfrak{l}^\nu)^\times|^{-1}(N(\mathfrak{l}^\nu)\delta_{\mathfrak{l}^\nu O_\mathfrak{l}} - N(\mathfrak{l}^{\nu-1})\delta_{\mathfrak{l}^{\nu-1}O_\mathfrak{l}})(\xi_\mathfrak{l}).$$

As for (4), the computation is the same as in (3) replacing $\mathbf{e}([\ell^{-\nu}\mathrm{Tr}(a^{-1}u_1\xi/\sqrt{d})]_\ell)$ in the above formula by $\mathbf{e}([\ell^{-\nu}(a_\mathfrak{l}^{-1}\xi_\mathfrak{l} - a_{\overline{\mathfrak{l}}}^{-1}\xi_{\overline{\mathfrak{l}}})]_\ell)$. This finishes the proof. □

Recall $\mathfrak{t} = \prod_{\mathfrak{l}\in\mathcal{A}} \mathfrak{l}^{\nu(\ell)}\overline{\mathfrak{l}}^{\nu(\ell)} \prod_{\ell\in\mathcal{C}_s} \overline{\mathfrak{l}}^{\nu(\ell)} \prod_{\ell\in\mathcal{C}_{ns}} \mathfrak{l}^{\nu(\ell)}$, and define $\mathfrak{T} = \prod_{\mathfrak{l}\in\mathcal{A}} \mathfrak{l}^{\nu(\ell)}\overline{\mathfrak{l}}^{\nu(\ell)} \prod_{\mathfrak{l}\in\mathcal{C}_{ns}} \mathfrak{l}^{\nu(\ell)}$, $\mathfrak{a}_J = \prod_{\mathfrak{l}\in J} \mathfrak{l}$ for a subset $J \subset \mathcal{J} := \mathcal{A} \cup \overline{\mathcal{A}} \cup \mathcal{C}_s \cup \mathcal{C}_{ns}$ and $\mathfrak{t}_J = \mathfrak{t}/\mathfrak{a}_J$, where $\mathfrak{a}_\emptyset = O$. We define $\mathbf{\Theta}(\mathbf{1}) = \frac{h(K)}{|O^\times|} + \sum_{0\neq\mathfrak{a}\subset O} q^{N(\mathfrak{a})}$ for the class number $h(K)$ of K.

Lemma 2.13. *Let $\mathbf{1} : K_{\mathbb{A}}^\times/K^\times \to \{1\}$ be the identity Hecke character. Then the classical modular form giving rise to the theta integral $|(O/\mathfrak{T})^\times| \operatorname{Im}(\tau)^{-1/2} \int_{T_1(\mathbb{Q})T_1(\mathbb{R})\backslash T_1(\mathbb{A})} \theta_G(\phi_1)(x;\alpha)d^\times\alpha$ is an antiholomorphic CM theta series given by $\sum_{\mathfrak{x}|\mathfrak{t}} \mu_K(\mathfrak{x})N(\mathfrak{t}/\mathfrak{x})\overline{\mathbf{\Theta}}(\mathbf{1})|[N(\mathfrak{t}/\mathfrak{x})]$. Here $d^\times\alpha$ is the Haar measure with $\int_{T_1(\widehat{\mathbb{Z}})} d^\times\alpha = 1$.*

Proof. Each term of $\theta(\phi)(x;\alpha)$ is given by $\Psi_a(x;\xi)$ which has been computed in Lemma 2.12. By our choice, $a_{i,N} = 1$ and $|N(a_i)|_{\mathbb{A}} = 1$ with scalar $a_{i,\infty} \in \mathbb{R}_+^\times$. Thus, writing Ψ_i for Ψ_{a_i}, we have

$$\begin{aligned}|(O/\mathfrak{T})^\times| \operatorname{Im}(\tau)^{-1/2}\theta_i(\Psi_i) &= N(\mathfrak{a}_i)^{-1/2} \int_{T_1(\widehat{\mathbb{Z}})/O^\times} \theta(\phi_1)(a_i\alpha;_S x)d^\times\alpha \\ &= |O^\times|^{-1} \sum_{J\subset\mathcal{J}} (-1)^{|J|}N(\mathfrak{t}_J) \sum_{\xi\in(\mathfrak{t}_J\mathfrak{a}_i)} \mathbf{e}(-\xi\overline{\xi}N(\mathfrak{a}_i)^{-1}\overline{\tau}).\end{aligned}$$

Making variable change $\xi\mathfrak{a}_i^{-1}\mathfrak{t}_J^{-1} \mapsto \mathfrak{a}$ and summing up over ideals classes $\mathfrak{a}_i$, we get

$$\sum_i \sum_{\mathfrak{t}_J|\xi\mathfrak{a}_i^{-1}} \mathbf{e}(-N(\xi\mathfrak{a}_i^{-1})\overline{\tau}) = \sum_{\mathfrak{a}} \mathbf{e}(-N(\mathfrak{a}\mathfrak{t}_J)\overline{\tau}) = \overline{\mathbf{\Theta}}(\mathbf{1})|[N(\mathfrak{t}_J)],$$

where $\mathfrak{a}$ runs over all integral ideals. This shows

$$|(O/\mathfrak{T})^\times| \operatorname{Im}(\tau)^{-1/2} \sum_i \theta_i(\Psi_i) = \sum_{J\subset\mathcal{J}} (-1)^{|J|}\overline{\mathbf{\Theta}}(\mathbf{1})|[N(\mathfrak{t}_J)] = \sum_{\mathfrak{x}|\mathfrak{t}} \mu_K(\mathfrak{x})N(\mathfrak{t}/\mathfrak{x})\overline{\mathbf{\Theta}}(\mathbf{1})|[N(\mathfrak{t}/\mathfrak{x})]$$

as desired. □

Corollary 2.14. *The modular form $\int_{T_1(\mathbb{Q})T_1(\mathbb{R})\backslash T_1(\mathbb{A})} \theta_G(\phi_1)(x;\alpha)d^\times\alpha$ has character χ_K and level $N' := |d(K)| \prod_{\mathfrak{l}\in\mathcal{C}} N(\mathfrak{l})^{\nu(\ell)} \prod_{\ell\in A} \ell^{2\nu(\ell)}$, and hence $N_\epsilon|N'$ and $M|N'$ for M in* Lemma 1.11.

Proof. Since $\mathbf{\Theta}(\mathbf{1})|[\mathfrak{t}]$ has highest level in the summand over $J \subset \mathcal{J}$, the level of $\mathbf{\Theta}(\mathbf{1})$ is $|d|$, and the operation $[\mathfrak{t}]$ add the level $N(\mathfrak{t})$ as given in the lemma. Since $\mathbf{\Theta}(\mathbf{1})$ has Neben character χ_K, the character of the integral is the same. □

2.6. **Explicit form of Siegel Eisenstein series.** Recall $\chi_K = \left(\frac{K/\mathbb{Q}}{}\right) = \left(\frac{d(K)}{}\right)$. By definition, the Mellin transform of $\mathbf{\Theta}(\mathbf{1})$ is given by $\zeta_K(s) = \zeta(s)L(s,\chi_K)$. Then by Hecke [H], we can write $\mathbf{\Theta}(\mathbf{1})$ as an Eisenstein series:

$$\mathbf{\Theta}(\mathbf{1}) = \frac{\sqrt{d(K)}}{2\pi i}E_{1,1}(\tau;0). \tag{2.13}$$

Here for a positive integer L and $d = d(K)$,

$$E_{k,L}(\tau;s) = \frac{1}{2} \sum_{(m,n)\in\mathbb{Z}^2} \frac{\chi_{K,L}(n)}{(dLm\tau+n)^k|(dLm\tau+n)|^{2s}} = L^{(L)}(1+2s,\chi_K)E_{k,L}^*(\tau;s), \tag{2.14}$$

where $\chi_{K,L}(n) = \chi_K(n)$ if n is prime to Ld and otherwise $\chi_K(n) = 0$, and

$$E_{k,L}^*(\tau;s) = \sum_{\gamma\in\Gamma_0(Ld)/\Gamma_\infty} \chi_K(\gamma)j(\gamma,\tau)^{-k}|j(\gamma,\tau)|^{-2s}$$

with $\chi_K\left(\begin{smallmatrix} * & * \\ * & \delta \end{smallmatrix}\right) = \chi_K(\delta)$. Here $\Gamma_\infty = \{\pm\left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right) | m \in \mathbb{Z}\}$. We have a relation (e.g. [Sh1] (3.3))

$$E_{k,L} = \sum_{0<t|L} \mu(t)\chi_K(t)t^{-k}E_{k,1}|[L/t]. \tag{2.15}$$

We now write down the integral as a linear combination of $E_{1,L}$.

Lemma 2.15. *Let $\mathfrak{n}$ be an integral ideal of K. Decompose $\mathfrak{n} = \mathfrak{I}\mathfrak{F}\mathfrak{F}_c\mathfrak{R}$ so that $\mathfrak{I}$ is a product of primes ideal inert over $\mathbb{Q}$, $\mathfrak{R}$ is a product of primes ramified over $\mathbb{Q}$, and $\mathfrak{F} + \mathfrak{F}^c = O$ with $\mathfrak{F}_c \supset \mathfrak{F}^c$ for the complex conjugation c; so, $\mathfrak{F}\mathfrak{F}_c$ a product of prime ideals split over $\mathbb{Q}$. Write $I := N(\mathfrak{I})$, $R := N(\mathfrak{R})$ and $S := N(\mathfrak{F}_c)$. Then we have*

$$(\Theta 1) \qquad \sum_{\mathfrak{x}|\mathfrak{n}} \mu_K(\mathfrak{x}) N(\mathfrak{n}/\mathfrak{x}) \boldsymbol{\Theta}(\mathbf{1})|[N(\mathfrak{n}/\mathfrak{x})] = \frac{\sqrt{d(K)}N(\mathfrak{n})}{2\pi i} \sum_{a|IRS} \mu(a) a^{-1} E_{1,N(\mathfrak{n})/a},$$

where μ (resp. μ_K) is the Moebius function of $\mathbb{Q}$ (resp. K).

Proof. First suppose that $\mathfrak{n} + \mathfrak{n}^c = O$ ($\Leftrightarrow IRS = 1 \Leftrightarrow \mathfrak{I}\mathfrak{R}\mathfrak{F}_c = O$). Then ($\Theta$1) can be rewritten as

$$\frac{2\pi i}{\sqrt{d(K)}N(\mathfrak{n})} \sum_{t|N(\mathfrak{n})} \mu(t) \frac{N(\mathfrak{n})}{t} \Theta(\mathbf{1})| \left[\frac{N(\mathfrak{n})}{t}\right] \overset{(2.13)}{=} \sum_{t|N(\mathfrak{n})} \mu(t) \frac{N(\mathfrak{n})}{t} E_{1,1}| \left[\frac{N(\mathfrak{n})}{t}\right] \overset{(2.15)}{=} E_{1,N(\mathfrak{n})}.$$

Now we proceed on induction on the number (counting with multiplicity) of prime factors of $\mathfrak{I}\mathfrak{R}\mathfrak{F}_c$. Pick $\ell|IRS$ and the prime $\mathfrak{l}$ over ℓ. Let $\mathfrak{n}' = \mathfrak{n}/\mathfrak{l}$. Write R' (resp. I', S') for the corresponding factor of $N(\mathfrak{n}')$ for R (resp. I, S). We assume that

$$(2.16) \qquad \sum_{\mathfrak{x}|N(\mathfrak{n}')} \mu_K(\mathfrak{x}) N(\mathfrak{n}'/\mathfrak{x}) \Theta(\mathbf{1})|[N(\mathfrak{n}'/\mathfrak{x})] = \frac{\sqrt{d(K)}N(\mathfrak{n}')}{2\pi i} \sum_{s|S'} \sum_{i|I'} \sum_{r|R'} \mu(i)\mu(r)\mu(s)(irs)^{-1} E_{1,N(\mathfrak{n}')/irs}.$$

By applying $\frac{N(\mathfrak{n})}{N(\mathfrak{n}')}[\ell]$ if $\ell|I$ and $\frac{N(\mathfrak{n})}{N(\mathfrak{n}')}[\ell]$ otherwise to the above identity, we get

$$(2.17) \qquad \sum_{\mathfrak{x}|\mathfrak{n}'} \mu_K(\mathfrak{x}) N(\mathfrak{n}/\mathfrak{x}) \Theta(\mathbf{1})|[N(\mathfrak{n}/\mathfrak{x})] = \frac{\sqrt{d(K)}N(\mathfrak{n})}{2\pi i} \sum_{i|I'} \sum_{s|S'} \sum_{r|R'} \mu(i)\mu(r)\mu(s)(irs)^{-1} E_{1,N(\mathfrak{n}')/irs}|[N(\mathfrak{l})].$$

If $\ell|N(\mathfrak{n}')$, by (2.15), we have $E_{1,N(\mathfrak{n}')/ir}|[N(\mathfrak{l})] = E_{1,N(\mathfrak{n})/ir}$. Since

$$\left\{r|I'R'S'\middle|\mu(r) \neq 0\right\} = \left\{r|IRS\middle|\mu(r) \neq 0\right\},$$

we are done.

Suppose that $\mathfrak{l} \nmid \mathfrak{n}'$; so, $\mathfrak{n} = \mathfrak{n}'\mathfrak{l}$. We can rewrite ($\Theta$1) as

$$(\Theta 2) \qquad \sum_{\mathfrak{x}|\mathfrak{n}'} \mu_K(\mathfrak{x}) N(\mathfrak{n}/\mathfrak{x}) \Theta(\mathbf{1})|[N(\mathfrak{n}/\mathfrak{x})] + \sum_{\mathfrak{x}|\mathfrak{n}'} \mu_K(\mathfrak{x}\mathfrak{l}) N(\mathfrak{n}/\mathfrak{x}\mathfrak{l}) \Theta(\mathbf{1})|[N(\mathfrak{n}/\mathfrak{x}\mathfrak{l})]$$
$$= \left(\sum_{\mathfrak{x}|\mathfrak{n}'} \mu_K(\mathfrak{x}) N(\mathfrak{n}/\mathfrak{x}) \Theta(\mathbf{1})|[N(\mathfrak{n}'/\mathfrak{x})]\right)|[N(\mathfrak{l})] - \sum_{\mathfrak{x}|\mathfrak{n}'} \mu_K(\mathfrak{x}) N(\mathfrak{n}'/\mathfrak{x}) \Theta(\mathbf{1})|[N(\mathfrak{n}'/\mathfrak{x})],$$

which is, by induction assumption, equal to $\frac{\sqrt{d(K)}N(\mathfrak{n})}{2\pi i}$ times

$$\sum_{a|I'R'S'} a^{-1} N(\mathfrak{l}) \left(E_{1,N(\mathfrak{n}')/a}|[N(\mathfrak{l})] - \frac{1}{N(\mathfrak{l})} E_{1,N(\mathfrak{n}')/a}\right).$$

Then we need to show, for a prime $\ell|IRS$,

$$E_{1,N(\mathfrak{n}')}|[N(\mathfrak{l})] - \frac{1}{N(\mathfrak{l})} E_{1,N(\mathfrak{n}')} = E_{1,N(\mathfrak{n})} - \frac{1}{N(\mathfrak{l})} E_{1,N(\mathfrak{n})/\ell}.$$

When $\ell|RS$, by (2.15), we have $E_{1,N(\mathfrak{n}')}|[N(\mathfrak{l})] = E_{1,N(\mathfrak{n}'\mathfrak{l})} = E_{1,N(\mathfrak{n})}$ and $E_{1,N(\mathfrak{n}')} = E_{1,N(\mathfrak{n})/\ell}$, and hence the result follows. Assume that $\ell|I$. By (2.15), $E_{1,N(\mathfrak{n})} = E_{1,N(\mathfrak{n}')}|[\ell^2] + \frac{1}{\ell} E_{1,N(\mathfrak{n}')}|[\ell]$ and $E_{1,N(\mathfrak{n})/\ell} = E_{1,N(\mathfrak{n}')}|[\ell] + \frac{1}{\ell} E_{1,N(\mathfrak{n}')}$. From this, the desired identity clearly follows. $\square$

3. Derivative of theta series

3.1. **Lie derivatives of Schwartz functions.** Recall $J(g,z) = |\det(g)|^{-1/2}j(g,z)$ for $(g,z) \in GL_2(\mathbb{R}) \times \mathfrak{H}$. For any function $f : \mathfrak{H} \to \mathbb{C}$ such that $f(\gamma(z)) = \det(\gamma)^m J(\gamma,z)^k J(\gamma,\overline{z})^l f(z)$ for a discrete subgroup Γ of $PGL_2^+(\mathbb{R}) = GL_2^+(\mathbb{R})/Z(\mathbb{R})$ for the center Z of $GL(2)$, we define $\widetilde{f}(g) = f(g(i))J(g,i)^{-k}J(g,-i)^{-l}$ for $g \in SL_2(\mathbb{R})$. Similarly, for a function $f : GL_2^+(\mathbb{R}) \times \mathfrak{H} \to \mathbb{C}$ with $f(\gamma, g(z)) = \det(g)^m f(\gamma g, z)J(g,z)^k J(g,\overline{z})^l$, we define $\widetilde{f}(\gamma, g) = f(\gamma, g(i))J(g,i)^{-k}J(g,-i)^{-l}$. Then $\widetilde{f}$ factors through $\Gamma\backslash GL_2^+(\mathbb{R})$. Further, we define

$$\mathbf{f}(g) = f(\gamma, g(i))j(g,i)^{-k}j(g,-i)^{-l} = \det(g)^{-(k+l)/2}\widetilde{f}(g).$$

Recall $\left[\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right); z, w\right] = (z,1)J\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} w \\ 1 \end{smallmatrix}\right) = (cw+d)z - (aw+b) = (cz-a)w + dz - b$. Here is a table of corresponding functions on $\mathfrak{H}$ and on $PGL_2^+(\mathbb{R})$.

f	(m,k,l)	$\widetilde{f}$	$\mathbf{f}$
$\mathrm{Im}(z)$	$(0,-1,-1)$	1	$\det(g)$
$j(v,z)$	$(-\frac{1}{2},-1,0)$	$\det(g)^{-1/2}j(vg,i)$	$j(vg,i)$
$[v;z,w]$	$(\pm\frac{1}{2},-1,0)$	$\sqrt{\frac{\det(g)}{\det(h)}}[g^{-1}vh;i,i]$	$\sqrt{\frac{\det(g)^2}{\det(h)}}[g^{-1}vh;i,i]$
$[v;\overline{z},\overline{w}]$	$(\pm\frac{1}{2},0,-1)$	$\sqrt{\frac{\det(g)}{\det(h)}}[g^{-1}vh;-i,-i]$	$\sqrt{\det(g)}[g^{-1}vh;-i,-i]$
$\frac{\lvert[v;z,w]\rvert^2}{\mathrm{Im}(z)\,\mathrm{Im}(w)}$	$(\pm 1,0,0)$	$\frac{\det(g)}{\det(h)}\lvert[g^{-1}vh;i,i]\rvert^2$	$\frac{\det(g)}{\det(h)}\lvert[g^{-1}vh;i,i]\rvert^2$
$\mathbf{e}(i\frac{\mathrm{Im}(\tau)\lvert[v;z,w]\rvert^2}{\mathrm{Im}(z)\,\mathrm{Im}(w)})$	$(?,0,0)$	$\mathbf{e}(i\frac{\det(g)}{\det(h)}\mathrm{Im}(\tau)\lvert[g^{-1}vh;i,i]\rvert^2)$	$\mathbf{e}(i\frac{\det(g)}{\det(h)}\mathrm{Im}(\tau)\lvert[g^{-1}vh;i,i]\rvert^2)$

Let $Y \in \mathfrak{sl}(\mathbb{C})$ and regard it as a left invariant differential operator Y_g on $SL_2(\mathbb{R})$ for the variable matrix $g \in GL_2(\mathbb{R})$ (identifying $GL_2(\mathbb{R})$ with $SL_2(\mathbb{R}) \times \mathbb{R}^\times$ by the natural isogeny). Then we have

$$\begin{aligned} Y_g(g^{-1}vh) &= \frac{d}{dt}(\exp(-tY)g^{-1}vh)|_{t=0} = -Yg^{-1}vh \\ Y_h(g^{-1}vh) &= \frac{d}{ds}(g^{-1}vh\exp(sY))|_{s=0} = g^{-1}vhY \\ Y_gY_h(g^{-1}vh) &= \frac{d^2}{dtds}(\exp(-tY)g^{-1}vh\exp(sY))|_{t=s=0} = -Yg^{-1}vhY. \end{aligned} \tag{3.1}$$

Lemma 3.1. *Let $X = \frac{1}{2}\left(\begin{smallmatrix} 1 & i \\ i & -1 \end{smallmatrix}\right) \in \mathfrak{sl}(\mathbb{C})$ as an invariant differential operator. Then we have*

$$X\widetilde{f}(g) = -4\pi\widetilde{\mathrm{Im}(z)\delta_k f}\ (\Leftrightarrow X\widetilde{f} = -4\pi(\widetilde{\delta_k f}) \Leftrightarrow X\mathbf{f} = -4\pi\det(g)(\delta_k\mathbf{f})(g)), \tag{3.2}$$

where $2\pi i\delta_k = 2\pi i\delta_k(z) = \frac{k}{2i\,\mathrm{Im}(z)} + \frac{\partial}{\partial z}$ and $(\delta_k\mathbf{f})(g) := (\delta_k f)(g(i))j(g,i)^{-k-2}j(g,-i)^{-l}$ if f is of weight $(?,k,l)$.

Proof. We have $2X = A - iB + 2iC$ for $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $C = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$; so, $\exp(tA) = \left(\begin{smallmatrix} e^t & 0 \\ 0 & e^{-t} \end{smallmatrix}\right)$, $\exp(tB) = \left(\begin{smallmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{smallmatrix}\right)$ and $\exp(tC) = \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$. Let $g = \left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right)$; so, $z = x + iy = g(i)$. Suppose $f(\gamma(z)) = f(z)J(\gamma,z)^k$ for $\gamma \in \Gamma$. Write $F(x,y) = f(x+iy)$ as two variable function. Then

$$\begin{aligned} A\mathbf{f}(g) &= \frac{d}{dt}\left(f(g(e^{2t}i))e^{tk}\right)|_{t=0} = \frac{d}{dt}\left(f(x+ye^{2t}i)e^{tk}\right)\Big|_{t=0} = \frac{d}{dt}\left(F(x,ye^{2t})e^{tk}\right)\Big|_{t=0} \\ &= \left(2ye^{2t}\frac{\partial F}{\partial y}(x,ye^{2t})e^{tk} + ke^{tk}F(x,e^{2t}y)\right)|_{t=0} = 2y\frac{\partial F}{\partial y}(z) + kf(z) = 2y\frac{\partial f}{\partial y}(z) + kf(z), \end{aligned}$$

$$B\mathbf{f}(g) = \frac{d(f(z)e^{itk})}{dt}\Big|_{t=0} = kif(z),\ C\mathbf{f}(g) = \frac{df(x+yt+yi)}{dt}\Big|_{t=0} = y\frac{\partial f(x+yt+yi)}{\partial x}\Big|_{t=0} = y\frac{\partial f}{\partial x}(z).$$

These combined, we get the desired assertion. □

Let $X = \frac{1}{2}\left(\begin{smallmatrix} 1 & i \\ i & -1 \end{smallmatrix}\right) \in \mathfrak{sl}(\mathbb{C})$. To simplify notation, we write $[v]_{\pm,\pm} = [v;\pm i,\pm i]$. Then we have the following table of derivatives:

ϕ	$X_g\phi$	$X_h\phi$	$X_gX_h\phi$
$[g^{-1}vh]_{+,+}$	$-[g^{-1}vh]_{-,+}$	$-[g^{-1}vh]_{+,-}$	$[g^{-1}vh]_{-,-}$
$[g^{-1}vh]_{-,-}$	0	0	0
$[g^{-1}vh]_{+,-}$	$-[g^{-1}vh]_{-,-}$	0	0
$[g^{-1}vh]_{-,+}$	0	$-[g^{-1}vh]_{-,-}$	0
$\lvert[g^{-1}vh]_{+,+}\rvert^2$	$-[g^{-1}vh]_{-,+}[g^{-1}vh]_{-,-}$	$-[g^{-1}vh]_{+,-}[g^{-1}vh]_{-,-}$	$[g^{-1}vh]_{-,-}^2$

Using these, we compute Lie derivatives of the function $(g,h)\mapsto \Psi_k(\tau;i,i)(g^{-1}vh)$ considered in (1.5) roughly of the form: $v\mapsto [v]_{-,-}^k\mathbf{e}(-\det(v)\overline{\tau}+ia|[v]_{+,+}|^2)$ with a fixed $0<a\in\mathbb{R}$ (in our setting $a=\frac{\mathrm{Im}(\tau)}{2}$). Since $\det(g^{-1}vh)$ is a constant with respect to $g,h\in SL_2(\mathbb{R})$, we may forget about $\mathbf{e}(-\det(v)\overline{\tau})$. We compute Lie derivatives of $(g,h)\mapsto[v]_{-,-}^k\mathbf{e}(ia|[g^{-1}vh]_{+,+}|^2)$, and we get

$$\begin{aligned}
X_g\left(\mathbf{e}(ia|[g^{-1}vh]_{+,+}|^2)\right)&=-2\pi a[g^{-1}vh]_{-,+}[g^{-1}vh]_{-,-}\mathbf{e}(ia|[g^{-1}vh]_{+,+}|^2),\\
X_h\left(\mathbf{e}(ia|[g^{-1}vh]_{+,+}|^2)\right)&=-2\pi a[g^{-1}vh]_{+,-}[g^{-1}vh]_{-,-}\mathbf{e}(ia|[g^{-1}vh]_{+,+}|^2),\\
X_hX_g\left(\mathbf{e}(ia|[g^{-1}vh]_{+,+}|^2)\right)&=2\pi a[g^{-1}vh]_{-,-}^2\mathbf{e}(ia|[g^{-1}vh]_{+,+}|^2)\\
&\quad+(2\pi a)^2|[g^{-1}vh]_{+,-}|^2[g^{-1}vh]_{-,-}^2\mathbf{e}(ia|[g^{-1}vh]_{+,+}|^2).
\end{aligned}\tag{3.3}$$

In general, we get by induction on m,

Lemma 3.2. *For $m>0$, we have*

$$\begin{aligned}
&(X_hX_g)^m\left(\mathbf{e}(ia|[g^{-1}vh]_{+,+}|^2)\right)\\
&\qquad=(2\pi a)^{2m}[g^{-1}vh]_{-,-}^{2m}\mathbf{e}(ia|[g^{-1}vh]_{+,+}|^2)\sum_{j=0}^{m}c_j(m)(2\pi a)^{j-m}|[g^{-1}vh]_{+,-}|^{2j}
\end{aligned}\tag{3.4}$$

for constants $c_j(m)$. Moreover we have $c_m(m)=1$.

Definition 3.3. *Let $X=\frac{1}{2}\left(\begin{smallmatrix}1&i\\ i&-1\end{smallmatrix}\right)\in\mathfrak{sl}(\mathbb{C})$ as an invariant differential operator. We define, for a normalized Hecke eigenform $\mathbf{f}\in\mathcal{S}_k(N,\boldsymbol{\psi})$ and $0<m\in\mathbb{Z}$,*

$$\begin{aligned}
&\mathbf{f}_m(g)=(-4\pi)^{-m}|\det(g)|_{\mathbb{A}}^{-m}X_{g_\infty}^m\mathbf{f}(g),\ \ \boldsymbol{\psi}_m(z)=\boldsymbol{\psi}(z)|z|_{\mathbb{A}}^{-2m},\quad\text{and}\\
&\qquad\Theta_m^{(N)}(\mathbf{f})(x;g,h)=(4\pi)^{-2m}|\det(g^{-1}h)|_{\mathbb{A}}^{-m}(X_{g_\infty}X_{h_\infty})^m\Theta^{(N)}(\mathbf{f})(x;g,h).
\end{aligned}$$

By Lemma 3.1, $\delta_k^m\mathbf{f}(g_\infty)=\mathbf{f}_m(g_\infty)$ (and hence the value of $\mathbf{f}_m$ at g_1 has rationality after dividing a CM period; see [Sh]). By Lemma 1.11, we get

Lemma 3.4. *For a Hecke eigenform $f\in S_k(N,\psi)$, we have*

$$\Theta_m^{(N)}(\mathbf{f})(x;g,h)=(2i)^k\sum_{t|M}\mu(t)a(M/t,f)(M/t)^{1+m}\boldsymbol{\psi}_m(\det(g))^{-1}\mathbf{f}_m|[\beta_{t/M}^{(\infty)}](g)\mathbf{f}_m(h).$$

Proof. The proof is the same as the proof of Lemma 1.11, once we remark

$$\begin{aligned}
(-4\pi)^m\mathbf{f}_m|[\beta_{t/M}^{(\infty)}](g)&=|\det(g\beta_{t/M}^{(\infty)})|_{\mathbb{A}}^{-m}X_g^m\mathbf{f}(g\beta_{t/M}^{(\infty)})=(M/t)^{-m}|\det(g)|_{\mathbb{A}}^{-m}(X_g^m\mathbf{f})(g\beta_{t/M}^{(\infty)})\\
&=(M/t)^{-m}|\det(g)|_{\mathbb{A}}^{-m}X_g^m(\mathbf{f}(g\beta_{t/M}^{(\infty)}))=(M/t)^{-m}|\det(g)|_{\mathbb{A}}^{-m}X_g^m(\mathbf{f}|[\beta_{t/M}^{(\infty)}](g)).
\end{aligned}$$

□

3.2. **Lie derivative and derivative of Shimura–Maass.** We take $\rho:K\to D=M_2(\mathbb{Q})$ and $\epsilon,g_1\in GL_2(\mathbb{A})$ specified in the proof of Proposition 2.2. Write $(\xi,\eta)=g_1^{-1}(\rho(\xi)+\rho(\eta)\epsilon)g_1\in M_2(\mathbb{A}^{(\infty)})$ for $\xi,\eta\in K_{\mathbb{A}}^{(\infty)}$. We summarize a consequence of the proof of Proposition 2.2, in particular, from the computation in (2.6):

Lemma 3.5. *Write, for simplicity, $\boldsymbol{\Theta}(\phi)(\tau;g,h)$ for $\boldsymbol{\Theta}(\phi)(g_\tau;g,h)J(g_\tau,-i)^k$ for $g_\tau\in S(\mathbb{R})$ with $g_\tau(i)=\tau\in\mathfrak{H}$. Suppose that $\phi(v)=\phi^{(\infty)}(v^{(\infty)})\,\mathrm{Im}(\tau)^{k+1}[v_\infty]_{-,-}^k\mathbf{e}(-\det(v_\infty)\overline{\tau}+\frac{i}{2}\mathrm{Im}(\tau)|[v_\infty]_{+,+}|^2)$ for a Bruhat function $\phi^{(\infty)}$ on $D_{\mathbb{A}}^{(\infty)}$. Then we have*

$$\boldsymbol{\Theta}(\phi)(\tau;g,h)=\sum_{v\in V}\phi^{(\infty)}(g^{-1}vh)[g^{-1}vh]_{-,-}^k\mathbf{e}(-\det(v)\overline{\tau}+\frac{i}{2}\mathrm{Im}(\tau)|[g^{-1}vh]|^2).$$

Moreover, if $\phi^{(\infty)}(\xi,\eta)=\overline{\phi}_1(\xi^{(\infty)})\cdot\phi_\epsilon(\eta^{(\infty)})$ *for Bruhat functions* ϕ_1 *and* ϕ_ϵ *on* $K_{\mathbb{A}}^{(\infty)}$*, we have*

$$\mathbf{\Theta}(\phi)(\tau;g_1,g_1)=(-2i)^k\operatorname{Im}(\tau)^{k+1}\overline{\theta(\phi_1)}\cdot\theta_k(\phi_\epsilon) \tag{3.5}$$

for $\theta(\phi_1)=\sum_{\xi\in K}\phi_1(\xi^{(\infty)})\mathbf{e}(\xi\overline{\xi}\tau)$ *and* $\theta_k(\phi_\epsilon)=\sum_{\eta\in K}\phi_\epsilon(\eta^{(\infty)})\eta^k\mathbf{e}(\eta\overline{\eta}\tau)$.

Note here (3.5) follows from the computation in (2.3) and (2.6), noting (2.5): $\operatorname{Im}(z_1)^{-1}[\epsilon;\overline{z}_1,\overline{z}_1]=-2i$. Similarly, under the assumption of Lemma 3.5, we have

(3.6)

$$\begin{aligned}\frac{X_gX_h(\mathbf{\Theta}(\phi)(\tau;g,h))}{\operatorname{Im}(\tau)^{k+1}}&=\sum_{v\in V}\phi(g^{-1}vh)[g^{-1}vh]^k_{-,-}\mathbf{e}(-\det(v)\overline{\tau})(X_gX_h\mathbf{e}(\frac{i}{2}\operatorname{Im}(\tau)|[g^{-1}vh]_{+,+}|^2))\\&=\sum_{v\in V}\phi(g^{-1}vh)[g^{-1}vh]^k_{-,-}\mathbf{e}(-\det(v)\overline{\tau})\\&\quad\times(\pi\operatorname{Im}(\tau)[g^{-1}vh]^2_{-,-}+(\pi\operatorname{Im}(\tau))^2[g^{-1}vh]^2_{-,-}|[g^{-1}vh]_{-,+}|^2)\mathbf{e}(\frac{i}{2}\operatorname{Im}(\tau)|[g^{-1}vh]_{+,+}|^2)).\end{aligned}$$

Note that $g_{1,\infty}=\sqrt{\operatorname{Im}(z_1)}^{-1}\left(\begin{smallmatrix}\operatorname{Im}(z_1)&\operatorname{Re}(z_1)\\0&1\end{smallmatrix}\right)$. For $v=\rho(\xi)+\rho(\eta)\epsilon$, we have

$$[g_1^{-1}vg_1]_{-,-}\overset{(1.6)}{=}\frac{[v;\overline{z}_1,\overline{z}_1]}{\operatorname{Im}(z_1)}\overset{(2.3)}{=}\eta\frac{[\epsilon;\overline{z}_1,\overline{z}_1]}{\operatorname{Im}(z_1)},\ |[g_1^{-1}vg_1]_{-,+}|^2\overset{(1.6)}{=}\frac{|[v;\overline{z}_1,z_1]|^2}{\operatorname{Im}(z_1)^2}\overset{(2.4)}{=}4\xi\overline{\xi}. \tag{3.7}$$

If $\phi^{(\infty)}(\xi,\eta)=\overline{\phi}_1(\xi^{(\infty)})\cdot\phi_\epsilon(\eta^{(\infty)})$,

$$\begin{aligned}&\operatorname{Im}(\tau)^{-k-1}(X_gX_h\mathbf{\Theta})(\phi)(\tau;g_1,g_1) \quad (3.8)\\&=(2\pi\operatorname{Im}(\tau))^2\left(\frac{[\epsilon;\overline{z}_1,\overline{z}_1]}{\operatorname{Im}(z_1)}\right)^{k+2}\sum_{(\xi,\eta)\in V}\phi(\xi,\eta)(4\pi\operatorname{Im}(\tau))^{-1}\eta^{k+2}+\eta^{k+2}\xi\overline{\xi})\mathbf{e}(-\xi\overline{\xi}\overline{\tau}+\eta\overline{\eta}\tau)\\&\overset{(2.5)}{=}(2\pi\operatorname{Im}(\tau))^2(-2i)^{k+2}\theta_{k+2}(\phi_\epsilon)\overline{\delta_1\theta(\phi_1)(\tau)}.\end{aligned}$$

In general, for $m\ge 0$ and $\delta_k^m=\overbrace{\delta_{k+2m-2}\cdots\delta_{k+2}\delta_k}^{m}$, we get

Lemma 3.6. *Let the notation and the assumption be as in* Lemma 3.5. *Then we have*

$$\operatorname{Im}(\tau)^{-k-2m-1}X_g^mX_h^m\mathbf{\Theta}(\phi)(\tau;g_1,g_1)=(4\pi i)^{2m}(-2i)^k\theta_{k+2m}(\phi_\epsilon)(\tau)\overline{\delta_1^m\theta(\phi_1)(\tau)}. \tag{3.9}$$

if $\phi^{(\infty)}(\xi,\eta)=\overline{\phi}_1(\xi^{(\infty)})\cdot\phi_\epsilon(\eta^{(\infty)})$.

Proof. We can compute explicitly repeating the computation resulting (3.8) and get the result by induction on m. Here we prove this via a short-cut without much computation.

By Lemma 3.2, (2.3), (2.4) and (2.5), we can write the result as $\theta_{k+2m}(\phi_\epsilon)$ times a linear combination g of $(\pi\operatorname{Im}(\tau))^{j-m}\left(\frac{\partial}{\partial\overline{\tau}}\right)^j\overline{\theta(\phi_1)}$ for $j=0,\dots,m$. Thus g is in the (weight 1) limit of the discrete series representation of $SL_2(\mathbb{R})$ generated by $\overline{\theta(\phi_1)}$. In this representation, weight $1+2m$ vectors form 1-dimensional subspace spanned by $\overline{\delta_1^m\theta(\phi_1)}$ (cf. [AFG] Section I.5). Since g is an anti-holomorphic modular form of weight $1+2m$, g is a constant multiple of $\overline{\delta_1^m\theta(\phi_1)(\tau)}$. Then comparing the terms of $\left(\frac{\partial}{\partial\tau}\right)^m\theta(\phi_1)$ in g and $\overline{\delta_m^1\theta(\phi_1)}$, we get the result. □

3.3. **Torus integral again.** Let the notation be as in Lemma 2.1. Recall the central character of $\mathbf{f}_m$ is given by $\boldsymbol{\psi}_m(x)=\boldsymbol{\psi}(x)|x|_{\mathbb{A}}^{-2m}$ (see Definition 3.3).

Lemma 3.7. *Let* $\chi=\chi_m:K_{\mathbb{A}}^\times/K^\times\to\mathbb{C}^\times$ *be a Hecke character with* $\chi(zx)=\boldsymbol{\psi}_m^{-1}(z)\chi(x)$ *for* $z\in\mathbb{A}^\times$ *and* $\chi(a_\infty)=a_\infty^{k+2m}$. *Then* $a\mapsto\mathbf{f}_m(\rho(a)g_1)\chi_m(a)$ *factors through* $I_K^-:=K_{\mathbb{A}}^\times/K^\times\mathbb{A}^\times K_\infty^\times$ *(the anticyclotomic idele class group of* K*).*

Proof. For $z\in Z(\mathbb{A})$, we have $\mathbf{f}_m(zx)=\boldsymbol{\psi}_m(z)\mathbf{f}_m(x)$; so, $a\mapsto\chi_m(a)\mathbf{f}_m(\rho(a)x)$ factors through $K^\times\backslash K_{\mathbb{A}}^\times/\mathbb{A}^\times$. We take $a_\infty\in K_\infty^\times$. Then $\rho(a_\infty)g_1(i)=\rho(a_\infty)(z_1)=z_1$, and we have, writing f' for

$f_{m,g_1^{(\infty)}}$ as in Section 1.1,

$$\begin{aligned}\mathbf{f}_m(\rho(a_\infty)g_1) &= f'(\rho(a_\infty)g_{1,\infty}(i))j(\rho(a_\infty)g_{1,\infty},i)^{-k-2m}\\ &= f'(\rho(a_\infty)(z_1))j(\rho(a_\infty),z_1)^{-k-2m}j(g_{1,\infty},i)^{-k-2m}\\ &= f'(z_1)j(\rho(a_\infty),z_1)^{-k-2m}j(g_{1,\infty},i)^{-k-2m} = \mathbf{f}_m(g_1)a_\infty^{-k-2m}.\end{aligned}$$

Since $\chi(a_\infty) = a_\infty^{k+2m}$, we have $\mathbf{f}_m(\rho(a_\infty)g_1)\chi_m(a_\infty) = \mathbf{f}_m(g_1)$, and it factors through I_K^-. □

We again put for $\mathbf{f} \in \mathcal{S}_k(N,\boldsymbol{\psi})$

$$L_{\chi_m}(\mathbf{f}_m) = \int_{I_K}\mathbf{f}_m(\rho(a)g_1)\chi_m(a)d^\times a \ \text{ and } \ L_{\chi_m}(\mathbf{f}_m|[\beta_{t/M}^{(\infty)}]) = \int_{I_K}\mathbf{f}_m|[\beta_{t/M}^{(\infty)}](\rho(a)g_1)\chi_m(a)d^\times a.$$

Recall $M = \prod_{\ell\in C_s}N_\ell$. By Lemma 3.4, writing $\mathcal{T} := T(\mathbb{Q})T(\mathbb{R})\backslash T(\mathbb{A})$ and noting that $\mathbf{f}_m$ is of weight $k+2m$, we get

$$\begin{aligned}&\sum_{0<t|M}\mu(t)a(M/t,f)(M/t)^{1+m}L_{\chi_m}(\mathbf{f}_m|[\beta_{t/M}^{(\infty)}])L_{\chi_m}(\mathbf{f}_m)\\ &\qquad = (2i)^{-k}\int_{\mathcal{T}}\boldsymbol{\psi}_m(N(a)\det(g_1))\Theta_m^{(N)}(\mathbf{f})(\rho(a)g_1,\rho(b)g_1)\chi_m(a)\chi_m(b)d^\times a d^\times b.\end{aligned}$$

We note

Lemma 3.8. *There exists $\xi_{t/M} \in K_{\mathbb{A}^{(\infty)}}^\times$ such that $\mathbf{f}_m(\rho(a)g_1\beta_{t/M}^{(\infty)}) = \mathbf{f}_m(\rho(a\xi_{t/M})g_1)$. The projection $\xi_{t/M,M} \in \prod_{\ell\in C_s}K_\ell^\times$ of $\xi_{t/M}$ is uniquely determined by $\beta_{t/M}^{(\infty)}$ and satisfies $\xi_{J,M}\xi_{J',M} = \xi_{JJ',M}$ for fractions J and J' with $MJJ' \in \mathbb{Z}$. So, $L_{\chi_m}(\mathbf{f}_m|[\beta_{t/M}^{(\infty)}]) = \chi_m(\xi_{t/M,M}^{-1})L_{\chi_m}(\mathbf{f}_m)$.*

Proof. Since M is a product of primes split in K, at $\ell|M$, $g_{1,\ell}^{-1}\rho(a)g_{1,\ell} = \left(\begin{smallmatrix} a_\ell & 0\\ 0 & \overline{a}_\ell\end{smallmatrix}\right)$, we find $\beta_{t/M,\ell} \in g_{1,\ell}^{-1}\rho(K_\ell^\times)g_{1,\ell}$. We remark $\beta_{t/M}^{(M\infty)} \in \widehat{\Gamma}_1(N\cdot d(K))^{(M)}$. Hence we can find $\xi_{t/M} \in K_{\mathbb{A}}^\times$ such that $\rho(\xi_{t/M})g_1 = g_1\beta_{t/M}u$ with $u \in \widehat{\Gamma}_1(N\cdot d(K))$. By our construction, $\xi_{t/M,M}$ is uniquely determined, and indeed, $\xi_{t/M,\ell} = ((t/M)_\ell,1) \in K_{\mathfrak{l}}\times K_{\overline{\mathfrak{l}}}$ for $\mathfrak{l} \in \mathcal{A}\cup\mathcal{C}_s$ over ℓ. This $\xi_{t/M}$ does the job. The last assertion follows from the variable change: $a \mapsto \xi_{t/M,M}a$ of the integral defining L_{χ_m}. □

By this lemma, we have

$$\begin{aligned}(3.10)\quad &\left(\sum_{0<t|M}\mu(t)a(M/t,f)\chi_m(\xi_{t/M,M})^{-1}(M/t)^{1+m}\right)L_{\chi_m}(\mathbf{f}_m)^2\\ &\qquad = (2i)^{-k}\int_{\mathcal{T}}\boldsymbol{\psi}_m(N(a)\det(g_1))\Theta_m^{(N)}(\mathbf{f})(\rho(a)g_1,\rho(b)g_1)\chi(a)\chi(b)d^\times a d^\times b.\end{aligned}$$

Since $\xi_{t/M} = (t/M,1) \in K_{\mathcal{C}_s}\times K_{\overline{\mathcal{C}}_s}$ for $K_{\mathcal{C}_s} = \prod_{\mathfrak{l}\in\mathcal{C}_s}K_{\mathfrak{l}}$ and $K_{\overline{\mathcal{C}}_s} = \prod_{\mathfrak{l}\in\mathcal{C}_s}K_{\overline{\mathfrak{l}}}$, we have $\chi_m(\xi_{\ell/N_\ell})^{-1} = \chi_m(\mathfrak{l}^{\nu(\ell)-1})$, and assuming $a(\ell,f)\neq 0$

$$\begin{aligned}(3.11)\quad &\sum_{0<t|M}\mu(t)a(M/t,f)\chi_m(\xi_{t/M,M})^{-1}(M/t)^{1+m}\\ &\qquad = \prod_{\ell\in C_s}a(\ell^{\nu(\ell)},f)\chi_m(\mathfrak{l}^{\nu(\ell)})\ell^{\nu(\ell)(1+m)}\Big(1-\frac{1}{a(\ell,f)\chi_m(\mathfrak{l})\ell^{1+m}}\Big).\end{aligned}$$

If $a(\ell,f) = 0$ for one prime factor $\ell \in C_s$, the left-hand-side of (3.10) vanishes. Thus we hereafter assume that $a(\ell,f)\neq 0$ for all $\ell \in C_s$.

3.4. Factoring again the theta series.

We now study

$$(3.12)\qquad \boldsymbol{\Theta}_{A,C,N,m}(x;g,h) := (4\pi)^{-2m}|\det(g^{-1}h)|_{\mathbb{A}}^{-m}(X_{g_\infty}X_{h_\infty})^m\boldsymbol{\Theta}_{A,C,N}(x;g,h).$$

By the same computation as in Section 2.2 combined with Lemma 3.6 (for the infinite place), we get

Proposition 3.9. *Assume* (S1–2). *We have a decomposition*

$$|N(a^{-1}b)|_{\mathbb{A}}^{m}\mathbf{\Theta}_{A,C,N,m}(x;\rho(a)g_1,\rho(b)g_1)=(2i)^k(-1)^{k+m}\theta(\phi_{1,m})(x,\alpha)\theta(\phi_{\epsilon,m})(x,\beta).$$

Here

$$\begin{aligned}\phi_{1,m,\infty}(\xi)&=\phi_{1,m,\infty}(\xi;i)\ \textit{for}\ \phi_{1,m,\infty}(\xi;\tau)=\mathrm{Im}(\tau)^{1/2}\overline{\delta_1^m\mathbf{e}(\xi\overline{\xi}\tau)}\\ \phi_{\epsilon,m,\infty}(\eta)&=\phi_{\epsilon,m,\infty}(\eta;i)\ \textit{for}\ \phi_{\epsilon,m,\infty}(\eta;\tau)=\mathrm{Im}(\tau)^{k+2m+(1/2)}\eta^{k+2m}\mathbf{e}(\eta\overline{\eta}\tau).\end{aligned}\tag{3.13}$$

The finite part of $\phi_{j,m}$ for $j=1,\epsilon$ is independent of m as the differential operators only affect infinity type; so, its explicit form is given by (2.8)*.*

For the quadratic space $(K,-N_{K/\mathbb{Q}})$, we have $\mathbf{r}(g_\tau)\phi_{1,m,\infty}(\xi)J(g_\tau,-i)^{-1-2m}=\phi_{1,\infty}(\tau;\xi)$ and for the quadratic space $(K,N_{K/\mathbb{Q}})$, we have $\mathbf{r}(g_\tau)\phi_{\epsilon,\infty}(\eta)J(g_\tau,i)^{-k-2m}=\phi_{\epsilon,\infty}(\tau;\eta)$.

Since $\boldsymbol{\psi}_m(N(a))\chi_m(ab)=\chi_m(a\overline{a})^{-1}\chi_m(ab)=\chi_m(\overline{a}^{-1}b)=\chi_m(\overline{\beta}^{-1})$, by the same computation as in (2.9), we have again, for $t=N(a^{-1}b)=N(\alpha)^{-1}=N(\beta)^{-1}$,

$$\begin{aligned}&\int_{\mathcal{T}}\boldsymbol{\psi}_m(N(a))\mathbf{\Theta}_{A,C,N,m}(x\alpha_t;\rho(a)g_1,\rho(b)g_1)\chi_m(ab)d^\times a d^\times b\\ &\quad=\int_{T'(\mathbb{Q})\backslash T'(\mathbb{A})}|N(\alpha)|_{\mathbb{A}}^{m}\theta(\phi_{1,m})(x\alpha_t;\alpha g_1)\theta(\phi_{\epsilon,m}(x\alpha_t;\beta g_1)\chi_m(\overline{\beta}^{-1})d^\times\alpha d^\times\beta\\ &\quad=\int_{T_1(\mathbb{Q})\backslash T_1(\mathbb{A})}|N(\alpha)|_{\mathbb{A}}^{m}\theta(\phi_{1,m})(x\alpha_t;\alpha g_1)d^\times\alpha\int_{T_\epsilon(\mathbb{Q})\backslash T_\epsilon(\mathbb{A})}\theta(\phi_{\epsilon,m})(x\alpha_t;\beta g_1)\chi_m(\overline{\beta}^{-1})d^\times\beta.\end{aligned}\tag{3.14}$$

3.5. **CM theta series of higher weight.** In the same manner as in Section 2.3, we again compute

$$\int_{T_\epsilon(\mathbb{Q})T_\epsilon(\mathbb{R})\backslash T_\epsilon(\mathbb{A})}\theta_G(\phi_{\epsilon,m})(x;\beta)\chi_m(\overline{\beta}^{-1})d^\times\beta.$$

In this section, we write ϕ for $\phi_{\epsilon,m}$. By Proposition 3.13, the infinity part of ϕ is given by

$$\phi_\infty(\eta)=\mathrm{Im}(\tau)^{k+2m+(1/2)}\eta^{k+2m}\mathbf{e}(\eta\overline{\eta}\tau).$$

Let $x_\infty=\sqrt{\mathrm{Im}(\tau)}^{-1}\left(\begin{smallmatrix}\mathrm{Im}(\tau)&\mathrm{Re}(\tau)\\0&1\end{smallmatrix}\right)$ $(\Rightarrow\tau=x_\infty(i))$, and as in (2.10), we write

$$\begin{aligned}\theta(\phi)(\beta;\tau)&:=\sum_{\eta\in K}(L(\beta)\circ\mathbf{r}(x_\infty)\phi)(\eta)J(x_\infty,-i)^{-k-2m}\\ &=|N(\beta)|_{\mathbb{A}}^{-1/2}\mathrm{Im}(\tau)^{k+2m+(1/2)}\sum_{\eta\in K}\phi^{(\infty)}(\beta^{-1}\eta)(\beta_\infty^{-1}\eta)^{k+2m}\mathbf{e}(N(\beta_\infty)^{-1}\eta\overline{\eta}\tau).\end{aligned}\tag{3.15}$$

Write $\widetilde{\chi}_m(x)=\chi_m(\overline{x}^{-1})$. Then the computation resulting Lemma 2.9 using Lemma 2.8 is the same, because of $\phi_{\epsilon,m}^{(\infty)}=\phi_\epsilon^{(\infty)}$. We thus have

Lemma 3.10. *Let the assumption and notation be as in* Lemma 2.9*. Let $\chi_m:K_{\mathbb{A}}^\times/K^\times\to\mathbb{C}^\times$ be a Hecke character of conductor $\mathfrak{C}$ with $\chi|_{\mathbb{A}^\times}=\boldsymbol{\psi}_m^{-1}$. Put $\lambda_m(x)=\widetilde{\chi}_m(x)^{-1}|N(x)|_{\mathbb{A}}^{-k-2m}=\overline{\chi_m(\overline{x})}^{-1}$ (so, $\lambda_m(x_\infty)=x_\infty^{-k-2m}$ and $\lambda_m|_{\widehat{O}^\times}=\widetilde{\chi}_m|_{\widehat{O}^\times}$). Then the classical cusp form giving rise to the theta integral $\int_{T_\epsilon(\mathbb{Q})T_\epsilon(\mathbb{R})\backslash T_\epsilon(\mathbb{A})}\theta_G(\phi_{\epsilon,m})(x;\beta)\chi_m(\overline{\beta}^{-1})d^\times\beta$ is a CM theta series given by*

$$C_m\,\mathrm{Im}(\tau)^{k+2m+(1/2)}\sum_{\mathfrak{y}|\mathfrak{s}_0}\mu_K(\mathfrak{y})N(\mathfrak{s}/\mathfrak{y})\lambda_m(\mathfrak{s}_0/\mathfrak{y})\mathbf{\Theta}_{\overline{\mathcal{A}}}(\lambda_m)|[N(\mathfrak{s}/\mathfrak{y})]$$

for the constant C_m given by

$$\mathbf{e}(-N_A^{-1})|(O/\mathfrak{t})^\times|^{-1}\Big(\prod_{\mathfrak{l}\in\mathcal{A}_+\cup\mathcal{C}_+}N(\mathfrak{l})^{(k+2m)(\nu(\ell)-f_{\overline{\mathfrak{l}}})}\chi_{m,\overline{\mathfrak{l}}}(\ell^{\nu(\ell)-f_{\overline{\mathfrak{l}}}}u_\epsilon^{-c})G(\chi_{m,\mathfrak{l}}\circ c)\Big)$$

where u_ϵ is as in (2.8)*, and the standard Gauss sum $G(\chi_{\mathfrak{l}}\circ c)$ in* Lemma 2.8*).*

3.6. **The derived weight 1 theta series.** We look into

$$\int_{T_1(\mathbb{Q})T_1(\mathbb{R})\backslash T_1(\mathbb{A})} |N(\alpha)|_{\mathbb{A}}^m \theta_G(\phi_{1,m})(x;\alpha)d^\times\alpha = \int_{K^\times K_\infty^\times\backslash K_{\mathbb{A}}^\times} |N(\alpha)|_{\mathbb{A}}^m \theta_G(\phi_{1,m})(x;\alpha)d^\times\alpha.$$

In this section, we write ϕ for ϕ_1 for simplicity. By Proposition 3.13, the infinity part of ϕ is given by $\phi_\infty(\xi) = \mathrm{Im}(\tau)^{1/2}\delta_1^m \mathbf{e}(\xi\overline{\xi}\tau)\big|_{\tau=i}$. As before, we get

Lemma 3.11. *Let $\mathbf{1} : K_{\mathbb{A}}^\times/K^\times \to \{1\}$ be the identity Hecke character. Then the classical modular form giving rise to the integral $|(O/\mathfrak{T})^\times| \int_{T_1(\mathbb{Q})T_1(\mathbb{R})\backslash T_1(\mathbb{A})} |N(\alpha)|_{\mathbb{A}}^m \theta_G(\phi_{1,m})(x;\alpha)d^\times\alpha$ is an antiholomorphic derivative of a CM theta series given by $\mathrm{Im}(\tau)^{1/2}\sum_{\mathfrak{x}|\mathfrak{t}} \mu_K(\mathfrak{x})N(\mathfrak{t}/\mathfrak{x})\overline{\delta_1^m\Theta(\mathbf{1})}|[N(\mathfrak{t}/\mathfrak{x})]$.*

4. Main theorem

Let $f_0 \in S_k(\Gamma_0(N_0),\psi)$ be a normalized Hecke eigenform with corresponding adelic form $\mathbf{f}_0 \in \mathcal{S}_k(N_0,\boldsymbol{\psi})$. Assume that f_0 has conductor N_0. Recall $K = \mathbb{Q}[\sqrt{d(K)}]$ with the discriminant $0 > d(K) \in \mathbb{Z}$. Write $d = |d(K)|$, and recall $d_0(K) = d/4$ if $4|d(K)$ while $d_0(K) = d$ otherwise. Pick a Hecke character χ_m of K with conductor ideal $\mathfrak{C}$ of O and $\chi_m(a_\infty) = a_\infty^{k+2m}$. For a suitable normalized Hecke eigenform $\mathbf{f}$ in the automorphic representation generated by the unitarization $\mathbf{f}_0^u$ (depending on $\mathfrak{C}$), we compute the L-value which $L_{\chi_m}(\mathbf{f}_m)^2$ represents by a version of Rankin convolution method, where $\mathbf{f}_m$ is the m-th derivative defined in Definition 3.3. The form $\mathbf{f}$ is in $\mathcal{S}_k(N',\boldsymbol{\psi})$ for the least common multiple N' of N and $d_0(K)$ for a suitably chosen multiple N of N_0.

To specify N, recall the prime factorization $\mathfrak{C} = \prod_{\mathfrak{l}} \mathfrak{l}^{f_{\mathfrak{l}}}$. If ℓ is a prime factor in $N(\mathfrak{C})$ splitting in K, we choose a prime factor $\mathfrak{l}|\ell$ in K so that $0 \le f_{\mathfrak{l}} \le f_{\overline{\mathfrak{l}}}$ (we tacitly agree to write $f_{\mathfrak{l}} = f_{\overline{\mathfrak{l}}}$ if $\mathfrak{l} = \overline{\mathfrak{l}}$). Let $\mathcal{A} = \{\mathfrak{l}|f_{\mathfrak{l}} > 0, \mathfrak{l} \ne \overline{\mathfrak{l}}\}$, and define $A = \{N(\mathfrak{l})|\mathfrak{l} \in \mathcal{A}\}$. Let C be the set of rational prime factors in $N(\mathfrak{C})d_0(K)N_0$ outside A. Define $N = \prod_{\ell\in A\sqcup C} \ell^{\nu(\ell)}$ for the exponent $\nu(\ell)$ given by

$$\nu(\ell) = \max(f_{\overline{\mathfrak{l}}}, \mathrm{ord}_\ell(N_0)), \tag{4.1}$$

where for any non-zero integer n, its prime factorization is given by $n = \prod_\ell \ell^{\mathrm{ord}_\ell(n)}$.

The sets A and C are already given. Recall $N_A = \prod_{\ell\in A} \ell^{\nu(\ell)}$ and other finite subsets C_0 and C_1 of C defined in Definition 1.5 relative to N: C_1 is made up of all prime factors of $d_0(K)$ with $C = C_0 \sqcup C_1$. Then $C_+ = \{\ell \in C|\nu(\ell) > 0\}$ contains C_0. We decomposed $C_0 = C_i \sqcup C_s \sqcup C_r$ so that C_i is made of primes inert in K and $C_r = \{2\}$ if $\mathrm{ord}_2(d(K)) = 2$ with $\nu(2) > 0$ and otherwise $C_r = \emptyset$ (so, C_s is made of split primes). Since $C_i \cup C_r \cup C_1$ is made of primes in C non-split in $K/\mathbb{Q}$, we wrote C_{ns} for $C_i \cup C_r \cup C_1$. We chose a set $\mathcal{C}_s$ of prime ideals of K so that $\mathfrak{l} \in \mathcal{C}_s \Leftrightarrow f_{\overline{\mathfrak{l}}} \ge f_{\mathfrak{l}} = 0$ and $\mathfrak{l} \ne \overline{\mathfrak{l}}$. For non-split primes over C, there is a unique choice of primes over ℓ in K. We write $\mathcal{C}_{ns}$ for the set of the non-split primes of K over primes $\ell \in C_{ns} \cap C_+$. Then we put $\mathcal{C} = \mathcal{C}_s \sqcup \mathcal{C}_{ns}$. Decomposing $\mathfrak{C} = \prod_{\mathfrak{l}\in\mathcal{C}_s} \overline{\mathfrak{l}}^{f_{\overline{\mathfrak{l}}}} \prod_{\mathfrak{l}\in\mathcal{A}} \mathfrak{l}^{f_{\mathfrak{l}}}\overline{\mathfrak{l}}^{f_{\overline{\mathfrak{l}}}} \prod_{\mathfrak{l}\in\mathcal{C}_{ns}} \mathfrak{l}^{f_{\mathfrak{l}}}$, we put $\mathcal{C}_0 = \{\mathfrak{l} \in \mathcal{C}|f_{\overline{\mathfrak{l}}} = f_{\mathfrak{l}} = 0, \nu(\ell) > 0\}$, and $\mathcal{C}_+ = \{\mathfrak{l} \in \mathcal{C}|f_{\overline{\mathfrak{l}}} > 0\}$. We introduce $\mathcal{C}_s^0 = \mathcal{C}_s \cap \mathcal{C}_0$ and $\mathcal{C}_s^+ = \mathcal{C}_s \cap \mathcal{C}_+$ anew.

4.1. **Statement.** The L-value in question is $L(\frac{1}{2}, \widehat{\pi}_{\mathbf{f}} \otimes \chi_m^-)$ for χ_m in Section 3.3. For a positive integer S, we write $L^{(S)}(s, \widehat{\pi}_{\mathbf{f}} \otimes \chi_m^-)$ for the imprimitive L-function Euler factors at primes dividing S removed from the primitive one. For the starting normalized new Hecke eigenform $f_0 \in S_k(\Gamma_0(N_0),\psi)$ with $f_0|T(n) = a(n, f_0)f_0$, we define $\alpha_\ell, \beta_\ell \in \mathbb{C}$ for each prime $\ell \nmid N_0$ by $a(\ell, f_0)/\ell^{(k-1)/2} = \alpha_\ell + \beta_\ell$ and $\alpha_\ell\beta_\ell = \psi(\ell)$. If $\ell|N_0$, we simply put $\alpha_\ell = a(\ell, f_0)/\ell^{(k-1)/2}$ and $\beta_\ell = 0$. Write $\mathbf{f}_0$ for the adelic Hecke eigenform in $\mathcal{S}(N_0,\boldsymbol{\psi})$ corresponding to f_0. Let $\pi_{\mathbf{f}}$ be the unitary automorphic representation generated by the unitarization $\mathbf{f}_0^u$ whose base-change lift to K we write as $\widehat{\pi}_{\mathbf{f}}$. Write the primitive L-function $L(s, \widehat{\pi}_{\mathbf{f}} \otimes \chi_m^-)$ as a product $\prod_\ell E_\ell(s)$ for Euler ℓ-factors $E_\ell(s)$. Then for primes ℓ, the Euler factor $E_\ell(s)$ is given by

$$E_\ell(s) = \begin{cases} \left[(1 - \frac{\alpha_\ell\chi_m^-(\mathfrak{l})}{\ell^s})(1 - \frac{\alpha_\ell\chi_m^-(\overline{\mathfrak{l}})}{\ell^s})(1 - \frac{\beta_\ell\chi_m^-(\mathfrak{l})}{\ell^s})(1 - \frac{\beta_\ell\chi_m^-(\overline{\mathfrak{l}})}{\ell^s})\right]^{-1} & \text{if } \ell = \mathfrak{l}\overline{\mathfrak{l}}, \\ \left[(1 - \frac{\alpha_\ell^{2/e}\chi_m^-(\mathfrak{l})}{\ell^{2s/e}})(1 - \frac{\beta_\ell^{2/e}\chi_m^-(\mathfrak{l})}{\ell^{2s/e}})\right]^{-1} & \text{if } \mathfrak{l}^e = (\ell), \end{cases} \tag{4.2}$$

where $\chi_m^-(\mathfrak{l}) = 0$ if $\mathfrak{l}$ is a factor of the conductor $\overline{\mathfrak{C}}$ of $\chi_m^-(x) = \chi_m(\overline{x})/|\chi_m|$.

We now make f explicit out of f_0. Recall N' which is the least common multiple of N and $d_0(K)$. The form f is a normalized Hecke eigenform of level N' with $f|T(n) = a(n,f)f$ and $a(\ell, f) = a(\ell, f_0)$ for all primes ℓ outside N'. So, if $N' = N_0$, we put $f = f_0$. Otherwise, we choose f such that for primes $\ell | N'$, $a(\ell, f) = a(\ell, f_0)$ if $\ell | N_0$ and $a(\ell, f) = \alpha_l \ell^{(k-1)/2}$ if $\ell \nmid N_0$ (this is always possible and $a(\ell, f) \neq 0$ for $\ell \nmid N_0$). We write $\mathbf{f}$ for the adelic eigenform corresponding to f. Then $\mathbf{f}^u$ and $\mathbf{f}_0^u$ generate the same $\pi_{\mathbf{f}}$. Since $\mathbf{f}$ and f is also a Hecke eigenform of level $N_1 = N \cdot d_0(K)$ (as $\ell | N' \Leftrightarrow \ell | N_1$), all the result proven for Hecke eigenforms of level N_1 can be applied to $\mathbf{f}$.

Assume that $a(\ell, f) \neq 0$ for all $\ell \in C_s$. Recall (3.11) for $M = \prod_{\ell \in C_s} N_\ell$:

$$\sum_{0<t|M} \mu(t) a(\frac{M}{t}, f) \chi_m(\xi_{\frac{t}{M}})^{-1} \left(\frac{M}{t}\right)^{m+1} = \prod_{\ell \in C_s} a(\ell^{\nu(\ell)}, f) \chi_m(\mathfrak{l}^{\nu(\ell)}) \ell^{\nu(\ell)(m+1)} (1 - \frac{1}{a(\ell, f)\chi_m(\mathfrak{l})\ell^{m+1}})$$

in front of (3.10), which is equal to

$$E''(m) := \prod_{\ell \in C_s} \alpha_\ell^{\nu(\ell)} \ell^{\nu(\ell)/2} \chi_m^-(\bar{\mathfrak{l}}^{\nu(\ell)}) (1 - \frac{1}{\alpha_\ell \ell^{1/2} \chi_m^-(\bar{\mathfrak{l}})}). \tag{4.3}$$

The factor $E''(m)$ could vanish if $a(\ell, f)\chi_m(\mathfrak{l})\ell^{m+1} = 1$ for one prime $\ell \in C_s$. Since $|\chi_m(\mathfrak{l})| = \ell^{-(k+2m)/2}$, if this is the case, we have $|a(l, f)| = \ell^{(k/2)-1}$; so, $\pi_{\mathbf{f}}$ has to be a Steinberg representation at ℓ. If $\pi_{\mathbf{f}}$ is a Steinberg representation at ℓ, the primitive character ψ° associated to ψ has conductor prime to ℓ and $a(\ell, f) = \pm\sqrt{\psi^\circ(\ell)}\ell^{(k/2)-1}$. Thus we must have $\chi_m(\mathfrak{l}) = \pm\sqrt{\psi^\circ(\ell)}^{-1} \ell^{-m-(k/2)}$. Writing h for the class number of K and take a generator ϖ of $\mathfrak{l}^h$, we find that $\chi_m(\mathfrak{l}^h) = \varpi^{-(k+2m)h}$ up to roots of unity, and $\mathfrak{l} \neq \bar{\mathfrak{l}}$ prohibits $\chi_m(\mathfrak{l}) = \pm\sqrt{\psi^\circ(\ell)}^{-1} \ell^{-m-(k/2)}$ to happen.

Theorem 4.1. *Let f_0 and f be as above such that*

- *$f_0 | T(n) = a(n, f_0) f_0$ for all positive integer n;*
- *$a(\ell, f_0) \neq 0$ for all $\ell \in C_s$;*
- *the adelic form $\mathbf{f}_0$ (in 1.1) associated to f_0 has central character $\boldsymbol{\psi}$ with $\boldsymbol{\psi}_\infty(a_\infty) = a_\infty^{-k}$.*

For an integer $m \geq 0$, put $\boldsymbol{\psi}_m(x) = \boldsymbol{\psi}(x)|x|_{\mathbb{A}}^{-2m}$, and take a Hecke character $\chi_m : K_{\mathbb{A}}^\times / K^\times \to \mathbb{C}^\times$ with $\chi_m|_{\mathbb{A}^\times} = \boldsymbol{\psi}_m^{-1}$ and $\chi(a_\infty) = a_\infty^{k+2m}$. Suppose

(F) *χ_m has conductor $\mathfrak{C}$ such that $\ell^{\nu(\ell)} \parallel \mathfrak{C}$ for all $\mathfrak{l} \in \mathcal{A} \cup \mathcal{C}_{ns}$ (so, $f_{\mathfrak{l}} = f_{\bar{\mathfrak{l}}} = \nu(\ell) > 0$ for $\mathfrak{l} \in \mathcal{A} \cup \mathcal{C}_{ns}$), $\mathfrak{C}_{\bar{\mathfrak{l}}} | \bar{\mathfrak{l}}^{\nu(\ell)}$ for all $\mathfrak{l} \in \mathcal{C}_s$ and $\mathfrak{C}$ is prime to $\mathfrak{l}$ for all $\mathfrak{l} \in \mathcal{C}_s$.*

Let $\pi_{\mathbf{f}}$ be the unitary automorphic representation generated by the unitarization of $\mathbf{f}$, set $\chi_m^-(x) := \frac{\chi_m(x^c)}{|\chi_m(x)|}$ (the unitary projection), and write $\widehat{\pi}_{\mathbf{f}}$ be the base-change lift of $\pi_{\mathbf{f}}$ to K. Let $\mathbf{f}_m$ be the derivative of $\mathbf{f}$ as in Definition 3.3. *Write $L(s, \widehat{\pi}_{\mathbf{f}} \otimes \chi_m^-)$ for the primitive L-function. Then we have,*

$$L_{\chi_m}(\mathbf{f}_m)^2 = c \frac{\Gamma(k+m)\Gamma(m+1)}{(2\pi i)^{k+1+2m}} E(\frac{1}{2}) E'(m) L^{(Nd)}(\frac{1}{2}, \widehat{\pi}_{\mathbf{f}} \otimes \chi_m^-).$$

The constant $c = c_1 \cdot G \cdot v$ with $c_1 = \mathbf{e}(-N_A^{-1})\sqrt{d(K)}(2i)^{-(k+2m)} N^{k+2m}$ is given by

$$v = \frac{\prod_{\ell \in C_s} \ell^{\nu(\ell)}}{c_2 \prod_{\ell \in A} \ell^\nu (1 - \frac{1}{\ell})^3 \prod_{\ell \in C_i} \ell^{2\nu(\ell)} (1 + \frac{1}{\ell})^2 (1 - \frac{1}{\ell}) \prod_{\ell \in C_r \cup C_1, \nu(\ell) > 0} (1 - \frac{1}{\ell})}, \tag{4.4}$$

$$G = \left(\prod_{\ell \in A \cup C} \chi_{m,\ell}^-(\ell^{\nu(\ell)})^{-1} \prod_{\mathfrak{l} \in \mathcal{C}_s^+} \ell^{((k/2)+m)(\nu(\ell) - f_{\bar{\mathfrak{l}}})} \chi_{m,\mathfrak{l}}^-(\ell^{\nu(\ell) - f_{\bar{\mathfrak{l}}}}) G(\chi_{m,\mathfrak{l}}^-) \right) \left(\prod_{\mathfrak{l} \in \mathcal{A}} \chi_{m,\mathfrak{l}}^-(u_\epsilon)^{-1} G(\chi_{m,\mathfrak{l}}^-) \right),$$

where $\chi_{m,\ell}^- = \chi_m^-|_{\mathbb{Q}_\ell^\times}$, $\chi_{m,\mathfrak{l}}^- = \chi_m^-|_{K_{\mathfrak{l}}^\times}$, $u_\epsilon = 1$ unless $\mathfrak{l}|2$ and 2 is inert in K, and if $\mathfrak{l}|2$ is inert in $K/\mathbb{Q}$, u_ϵ is a dyadic unit in O_2 as in Lemma 2.5,

$$c_2 = \begin{cases} 1 & \text{if } 2 \nmid d(K) \text{ or } \nu(2) \geq 2, \\ 6 & \text{if } 4 \parallel d(K) \text{ and } \nu(2) = 0, \\ 4 & \text{if } 8 \parallel d(K) \text{ and } \nu(2) = 0, \\ 2 & \text{if } 2 | d(K) \text{ and } \nu(2) = 1, \end{cases}$$

and the modification Euler factors are

$$E(s) := \prod_{\mathfrak{l}|\ell\in C_s}\left((1-\frac{\chi_m^-(\mathfrak{l})\alpha_\ell}{N(\mathfrak{l})^s})\right)^{-1}\prod_{\mathfrak{l}|d(K)}\left((1-\frac{\chi_m^-(\mathfrak{l})\alpha_\ell}{N(\mathfrak{l})^s})(1-\frac{\chi_m^-(\mathfrak{l})\beta_\ell}{N(\mathfrak{l})^s})\right)^{-1},$$

$$E'(m) = \frac{\prod_{\mathfrak{l}\in\mathcal{C}_s^+}\frac{\alpha_\ell^{\nu(\ell)-f_{\bar{\mathfrak{l}}}}}{\ell^{(\nu(\ell)-f_{\bar{\mathfrak{l}}})(m+(k-1)/2)}}\prod_{\mathfrak{l}\in\mathcal{C}_s^0}\alpha_\ell^{\nu(\ell)}\ell^{\nu(\ell)/2}\chi_m^-(\mathfrak{l}^{\nu(\ell)})(1-\frac{1}{\alpha_\ell\ell^{1/2}\chi_m^-(\mathfrak{l})})}{\prod_{\ell\in C_s}\alpha_\ell^{\nu(\ell)}\ell^{\nu(\ell)/2}\chi_m^-(\bar{\mathfrak{l}}^{\nu(\ell)})(1-\frac{1}{\alpha_\ell\ell^{1/2)}\chi_m^-(\bar{\mathfrak{l}})})}.$$

Remark 4.2. (a) *Recall the conductor N_0 of $\pi_{\mathbf{f}}$. The theorem covers the value $L(\frac{1}{2},\widehat{\pi}_{\mathbf{f}}\otimes\chi^-)$ for all arithmetic characters χ^- with anticyclotomic $\widehat{\psi}\chi^-$ at least if the conductor of χ^- is prime to N_0 and the infinity type $\infty(\chi^-)=\kappa(c-1)$ for integers κ satisfies $|\kappa|\geq k/2$. We treated explicitly the case where $\kappa=(k/2)+m\geq(k/2)$. Replacing f by f_c and taking the complex conjugate of the value computed, we get the result for $\kappa\leq-(k/2)$. In order to treat the case where $|\kappa|<(k/2)$, we need to replace D by a definite quaternion algebra.*

(b) *For the conductor $\mathfrak{C}$ of χ_m, Suppose $(N_0)\supset\mathfrak{C}$. Then the condition (F) is satisfied automatically for $\ell\in C_{ns}$. For split prime factors ℓ, write $\mathfrak{C}_\ell=\mathfrak{l}^{f_\mathfrak{l}}\bar{\mathfrak{l}}^{f_{\bar{\mathfrak{l}}}}$ for $\ell|N_0$. Then we have $f_\mathfrak{l}=f_{\bar{\mathfrak{l}}}=\nu(\ell)$ by the condition $\boldsymbol{\psi}=\chi_m|_{\mathbb{A}^\times}^{-1}$ if $(N_0\ell)\supset\mathfrak{C}$. Thus* (F) *is satisfied if the conductor $\mathfrak{C}$ is deep enough with respect to N_0, and* Theorem 4.1 *covers such characters.*

(c) *The only cases which the theorem does not cover are the case (i) where $\mathrm{ord}_\ell(N_0)\geq f_{\bar{\mathfrak{l}}}>f_\mathfrak{l}>0$ (as we can place ℓ in C and take $\nu(\ell)\geq f_{\bar{\mathfrak{l}}}$ if $f_\mathfrak{l}=0$) for primes ℓ split in K and the case (ii) where $f_\mathfrak{l}<\mathrm{ord}_\ell(N_0)$ for $\ell\in C_{ns}$. We can actually compute $L_{\chi_m}(\mathbf{f}_m)^2$ explicitly in such an exceptional case, basically by the same argument we give in the following section, but the outcome turns out to be trivial (that is, $L_{\chi_m}(\mathbf{f}_m)=0$); so, we do not give more details.*

(d) *We have the identity*

$$L_{\chi_m}(\mathbf{f}_m)=L_{\chi_m\widehat{\lambda}^{-1}}(\mathbf{f}_m\otimes\lambda)\quad\textit{and}\quad L(s,\widehat{\pi}_{\mathbf{f}}\otimes\chi_m^-)=L(s,\widehat{\pi}_{\mathbf{f}\otimes\lambda}\otimes\widehat{\lambda}^{-1}\chi_m^-)$$

up to finitely many Euler factors for finite order character $\lambda:\mathbb{A}^\times/\mathbb{Q}^\times$ with $\widehat{\lambda}=\lambda\circ N_{K/\mathbb{Q}}:K_\mathbb{A}^\times/K^\times\to\mathbb{C}^\times$. Thus we may assume, after a twist, that $\mathbf{f}$ is a Hecke (possibly old) eigenform in the automorphic representation generated by a primitive new form with character χ_K if k is odd and with the identity character if k is even.

4.2. **Proof via Rankin convolution.** Actually our computation goes through under the following assumption milder than (F):

(F′) χ_m has conductor $\mathfrak{C}$ such that $\bar{\mathfrak{l}}^{\nu(\ell)}\parallel\mathfrak{C}$ for all $\mathfrak{l}\in\mathcal{A}\cup\mathcal{C}_{ns}$, $\mathfrak{C}_{\bar{\mathfrak{l}}}|\bar{\mathfrak{l}}^{\nu(\ell)}$ for all $\mathfrak{l}\in\mathcal{C}_s$ and $\mathfrak{C}$ is prime to $\mathfrak{l}$ for all $\mathfrak{l}\in\mathcal{C}_s$.

However as we will see, writing $\mathcal{A}'$ for the subset of $\mathcal{A}$ such that $\mathfrak{l}^{\nu(\ell)}\subsetneq\mathfrak{C}_\mathfrak{l}$ for $\mathfrak{l}\in\mathcal{A}$, if $\mathcal{A}'\neq\emptyset$, the integral vanishes, and this forces us to assume (F). Anyway for the moment, we assume only (F′).

By (3.10), (3.11) and (3.12), noting that $\det(g_{1,\ell})=\ell^{\nu(\ell)}\in\mathbb{Q}_\ell$, we get

$$(2i)^k\boldsymbol{\psi}_m(\det(g_1))^{-1}E''(m)L_{\chi_m}(\mathbf{f}_m)^2=\int_{\mathcal{T}}\boldsymbol{\psi}_m(N(a))\Theta_m^{(N)}(\mathbf{f})(\rho(a)g_1,\rho(b)g_1)\chi_m(ab)d^\times ad^\times b$$
$$=\int_X\left(\int_{\mathcal{T}}|N(a^{-1}b)|_\mathbb{A}^{-m}\boldsymbol{\Theta}_{A,C,N,m}(x\alpha_{N(a^{-1}b)};\rho(a)g_1,\rho(b)g_1)\chi_m(ab)d^\times ad^\times b\right)\overline{\mathbf{f}}_c(x)\mu(x).\tag{4.5}$$

Recall $\mathfrak{t}=\prod_{\mathfrak{l}\in\mathcal{A}}\mathfrak{l}^{\nu(\ell)}\bar{\mathfrak{l}}^{\nu(\ell)}\prod_{\ell\in\mathcal{C}_s}\bar{\mathfrak{l}}^{\nu(\ell)}\prod_{\ell\in\mathcal{C}_{ns}}\mathfrak{l}^{\nu(\ell)}$ and $\mathfrak{s}_0=\prod_{\mathfrak{l}\in\mathcal{C}_0}\mathfrak{l}^{\nu(\ell)}$. Since the integrand of (4.5) is invariant under $\widehat{\Gamma}_0(\mathbb{N})$ for $\mathbb{N}=N(\mathfrak{t})\cdot d(K)$ by Proposition 3.9, Lemmas 3.10 and 3.11 combined with Corollaries 2.10 and 2.14, we may integrate over $X':=X_0(\mathbb{N})$ in place of $X=\Gamma(A,C;N)\backslash\mathfrak{H}$, though, by our choice of the measure $d\mu(x)$ in Proposition 1.9, we need to divide the outcome by

$$[\Gamma(A,C;N):\Gamma_0(\mathbb{N})]:=\frac{[\Gamma(A,C;N):\Gamma(A,C;N)\cap\Gamma_0(\mathbb{N})]}{[\Gamma_0(\mathbb{N}):\Gamma(A,C;N)\cap\Gamma_0(\mathbb{N})]}=\frac{c_2\prod_{\ell\in C_i}N_\ell}{N\prod_{\ell|N}(1-\ell^{-1})}.\tag{4.6}$$

By Lemma 3.10 and Lemma 3.11, (4.5) is equal to, up to a non-zero explicit constant, the following classical convolution integral:

$$\int_{X'} \sum_{\mathfrak{y}|\mathfrak{s}_0} \mu_K(\mathfrak{y}) N(\mathfrak{s}/\mathfrak{y}) \lambda_m(\mathfrak{s}_0/\mathfrak{y}) \mathbf{\Theta}_{\overline{\mathcal{A}}}(\lambda_m)|[N(\mathfrak{s}/\mathfrak{y})] \times \sum_{\mathfrak{x}|\mathfrak{t}} \mu_K(\mathfrak{x}) N(\mathfrak{t}/\mathfrak{x}) \overline{\delta_1^m \Theta}(\mathbf{1})|[N(\mathfrak{t}/\mathfrak{x})] \cdot \overline{f}_c \operatorname{Im}(\tau)^{k+2m+1} d\mu(\tau). \tag{4.7}$$

This integral is absolutely convergent as $\mathbf{\Theta}_{\overline{\mathcal{A}}}(\lambda_m)$ is a cusp form and $\overline{\delta_1^m \Theta}(\mathbf{1})$ is slowly increasing towards cusps.

To transform this integral into a Rankin convolution integral, we recall the notation introduced in Lemma 2.15 for $\mathfrak{n} = \mathfrak{t}$: $R = \prod_{\ell|d(K)} \ell^{\nu(\ell)}$, $S = \prod_{\ell \in A} \ell^{\nu(\ell)}$ and $I = \prod_{\ell \in C_i} \ell^{2\nu(\ell)}$. Thus we have

$$\begin{aligned} &\sum_{\mathfrak{x}|\mathfrak{t}} \mu_K(\mathfrak{x}) N(\mathfrak{t}/\mathfrak{x}) \delta_1^m \Theta(\mathbf{1})|[N(\mathfrak{t}/\mathfrak{x})](\tau) \\ &\overset{\text{Lemma 2.15}}{=} \frac{\sqrt{d(K)}N(\mathfrak{t})}{2\pi i} \sum_{s|S} \sum_{i|I} \sum_{r|R} \mu(irs)(irs)^{-1} \delta_1^m E_{1,N(\mathfrak{t})/irs}(\tau;0) \\ &\overset{(2.14)}{=} \frac{\sqrt{d(K)}N(\mathfrak{t})}{2\pi i} L^{(N(\mathfrak{t}))}(1,\chi_K) \sum_{s|S} \sum_{i|I} \sum_{r|R} \mu(irs)(irs)^{-1} \delta_1^m E^*_{1,N(\mathfrak{t})/irs}(\tau;0). \end{aligned} \tag{4.8}$$

Note here $L^{(Nd)}(s,\chi_K) = L^{(N(\mathfrak{t})/irs)}(s,\chi_K)$ if $\mu(irs) \neq 0$. Thus we want to compute

$$\sum_{s|S} \sum_{i|I} \sum_{r|R} \mu(irs)(irs)^{-1} \int_{X'} \theta(\tau) \cdot \overline{\delta_1^m E^*_{1,N(\mathfrak{t})/irs}(\tau;0) \cdot f_c(\tau)} \operatorname{Im}(\tau)^{k+2m+1} d\mu \tag{4.9}$$

for $\theta = \sum_{\mathfrak{y}|\mathfrak{s}_0} \mu_K(\mathfrak{y}) N(\mathfrak{s}/\mathfrak{y}) \lambda_m(\mathfrak{s}_0/\mathfrak{y}) \mathbf{\Theta}_{\overline{\mathcal{A}}}(\lambda_m)|[N(\mathfrak{s}/\mathfrak{y})]$. Note (see [Sh1] (2.9) or [LFE] 10.2 (13)):

$$(-4\pi)^m \frac{\Gamma(s+k)}{\Gamma(s+k+m)} \delta_k^m E^*_{k,L}(\tau;s) = E^*_{k+2m,L}(\tau;s-m). \tag{4.10}$$

Lemma 4.3. *Let the notation be as above. We have* $\langle \theta, E \rangle = \int_{X_0(\mathbb{N})} \theta \overline{E} \operatorname{Im}(\tau)^{k+2m+1} d\mu = 0$ *for* $E := \delta_1^m E^*_{1,N(\mathfrak{t})/irs}(\tau;0) \cdot f_c(\tau)$ *if either a prime* $\ell|ir$ *under* (F′) *or* $\ell|irs$ *under* (F).

Proof. Let $N(\theta)$ (resp. $N(E)$, $N(\mathbf{\Theta})$, $N(f)$) for the exact level of θ (resp. E, $\mathbf{\Theta} := \mathbf{\Theta}(\lambda_m)$, f). We first show that $\operatorname{ord}_\ell(N(\mathbf{\Theta})) > \operatorname{ord}_\ell(N(E))$ if a prime $\ell|irs$ and $\mu(irs) \neq 0$. Note $N(\mathbf{\Theta}) = N(\mathfrak{C})d(K)$ and by definition $\operatorname{ord}_\ell(N(\theta)) \geq \operatorname{ord}_\ell(N(\mathbf{\Theta}))$, and under (F), $\operatorname{ord}_\ell(N(\theta)) = \operatorname{ord}_\ell(N(\mathbf{\Theta})) = (2/e)f_\mathfrak{l} + \operatorname{ord}_\ell(d(K))$ for $\ell|SIR$ and under (F′), $\operatorname{ord}_\ell(N(\theta)) = \operatorname{ord}_\ell(N(\mathbf{\Theta})) = (2/e)f_\mathfrak{l} + \operatorname{ord}_\ell(d(K))$ for $\ell|IR$ for the ramification index $e = e(\mathfrak{l}/\ell)$. The level of $E^*_{1,L}$ is $L \cdot d(K)$. Then E has level $N(E)$ at most the LCM of $N(\mathfrak{t})d(K)/irs$ and $N(f)$. By our choice, $N(f)$ is a factor of N' for the least common multiple N' of N and $d_0(K)$, and for $\ell|N'$, $\operatorname{ord}_\ell(N(f)) = 1$ if $\operatorname{ord}_\ell(N_0) = 0$ and otherwise, $\operatorname{ord}_\ell(N(f)) = \operatorname{ord}_\ell(N_0) > 0$. If $\mu(irs) \neq 0$, irs is square-free. Suppose $\ell|si$. Then $\operatorname{ord}_\ell(N(\mathfrak{t})) = 2\nu(\ell) \geq \nu(\ell)+1 = \operatorname{ord}_\ell(N')+1 \geq \operatorname{ord}_\ell(N(f))+1$ as $N(\mathfrak{l}) = \ell^2$ for $\ell|I$ (and $\ell \in A \Leftrightarrow \ell|S$). This if $\ell|si$ under (F) (resp. if $\ell|i$ under (F′)), $\operatorname{ord}_\ell(N(E)) < \operatorname{ord}_\ell(N(\mathbf{\Theta}))$. Suppose $\ell|r$. Then $\max(f_\mathfrak{l}, \operatorname{ord}_\ell(N_0)) = \nu(\ell) > 0$. If $\operatorname{ord}_\ell(N_0) = 0$, then $f_\mathfrak{l} > 0$ and $\operatorname{ord}_\ell(N(\mathfrak{t})d(K)) = \nu(\ell) + \operatorname{ord}_\ell(d(K)) = f_\mathfrak{l} + \operatorname{ord}_\ell(d(K)) \geq 2 > 1 = \operatorname{ord}_\ell(N(f))$ by (F′); so, $\operatorname{ord}_\ell(N(E)) < \operatorname{ord}_\ell(N(\mathbf{\Theta}))$. If $\operatorname{ord}_\ell(N_0) > 0$, then $\operatorname{ord}_\ell(N(f)) = \operatorname{ord}_\ell(N_0) \leq \operatorname{ord}_\ell(N) = \operatorname{ord}_\ell(N') = \nu(\ell)$ and $\operatorname{ord}_\ell(N(\mathfrak{t})d(K)) = f_\mathfrak{l} + \operatorname{ord}_\ell(d(K)) = \nu(\ell) + \operatorname{ord}_\ell(d(K)) > \operatorname{ord}_\ell(N')$ by (F′). Thus we get $\operatorname{ord}_\ell(N(\mathbf{\Theta})) > \operatorname{ord}_\ell(N(E))$ again. Since $\mathbf{\Theta}$ is a new form of conductor $N(\mathfrak{C})d(K)$, the Petersson inner product of θ with E having strictly lower level than $\mathbf{\Theta}$ at the prime ℓ vanishes if $irs > 1$ under the assumption (F) or (F′). □

Thus we only care the term with $ir = 1$. Let $A^{new} \subset A$ be such that at $\ell \in A$, θ is local new form. In other words, for $\ell \in A$, we have $\ell^{\nu(\ell)} \parallel \mathfrak{C} \Leftrightarrow \ell \in A^{new}$. Then by the same argument as above, θ can have nontrivial inner product only with $f_c \sum_{s'|S'} \mu(s') s'^{-1} \delta_1^m E_{1,N(\mathfrak{t})/s'}$ for $S' = \prod_{\ell \in A - A^{new}} \ell^{\nu(\ell)}$.

Note that the level of θ is a factor of $\mathbb{N}/\prod_{\ell|S'}\ell$. Then (4.9) is equal to

$$
\begin{aligned}
&(4.11)\quad \frac{\sqrt{d(K)}N(\mathfrak{t})}{2\pi i}L^{(N(\mathfrak{t}))}(1,\chi_K)\int_{X'}\theta(\tau)\overline{f}_c(\tau)\sum_{s'|S'}\frac{\mu(s')}{s'}\overline{\delta^m_{1+s}E^*_{1,N(\mathfrak{t})}(\tau;s)(\tau)}y^{s+k+2m-1}d\mu\Big|_{s=0}\\
&\overset{(4.10)}{=}\frac{\sqrt{d(K)}}{2\pi i}\frac{L^{(N(\mathfrak{t}))}(1,\chi_K)\Gamma(s+1+m)}{(-4\pi)^m\Gamma(s+1)}\int_{X'}\theta\overline{f}_c\sum_{s'|S'}\frac{\mu(s')}{s'}\overline{E^*_{1+2m,N(\mathfrak{t})}(\tau;s-m)}y^{s+k+m-1}d\mu\Big|_{s=0}\\
&\overset{[X':X_0(\mathbb{N}/s')]=s'}{=}\frac{\sqrt{d(K)}}{2\pi i}\frac{L^{(N(\mathfrak{t}))}(1,\chi_K)\Gamma(1+m)}{(-4\pi)^m}\sum_{s'|S'}\mu(s')\int_{X_0(\mathbb{N}/s')}\theta\overline{f}_c\overline{E^*_{1+2m,N(\mathfrak{t})}(\tau;-m)}y^{k+m-1}d\mu\\
&\overset{(*)}{=}\frac{\sqrt{|d|}N(\mathfrak{t})}{2\pi}\frac{L^{(N(\mathfrak{t}))}(1,\chi_K)\Gamma(1+m)}{(-4\pi)^m}(4\pi)^{-k-m}\Gamma(k+m)\Big(\sum_{s'|S'}\mu(s')\Big)\sum_n a(n,\theta)a(n,f)n^{-s}\Big|_{s=k+m}\\
&=(-1)^m(4\pi)^{-k-2m}\Gamma(k+m)\Gamma(m+1)\frac{\sqrt{|d|}N(\mathfrak{t})}{2\pi}\Big(\sum_{s'|S'}\mu(s')\Big)L^{(N(\mathfrak{t}))}(1,\chi_K)D(k+m,f\otimes\theta),
\end{aligned}
$$

where we have put $D(s,f\otimes g)=\sum_n a(n,f)a(n,g)n^{-s}$ (the Rankin product of f and g) and the equality $(*)$ follows from the Rankin convolution method (e.g., [LFE] §5.4). Thus if $\mathcal{A}'\neq\emptyset$, we have $\sum_{s'|S'}\mu(s')=0$, and we get nothing; so, we now assume (F). Note $L^{(N(\mathfrak{t}))}(s,\chi_K)=L^{(Nd)}(s,\chi_K)$ and $\mathfrak{l}|\mathfrak{s}_0\Rightarrow\mathfrak{l}\in\mathcal{C}_0\cap\mathcal{C}_s$ by our assumption (F). Then,we have

$$
\begin{aligned}
&(4.12)\quad L^{(Nd)}(1,\chi_K)D(k+m,f\otimes\theta)\\
&=L^{(Nd)}(1,\chi_K)\sum_{\mathfrak{y}|\mathfrak{s}_0}\mu_K(\mathfrak{y})N(\mathfrak{s}/\mathfrak{y})\lambda_m(\mathfrak{s}_0/\mathfrak{y})\sum_{\mathfrak{a}}\lambda_m(\mathfrak{a})a(N(\mathfrak{s}/\mathfrak{y})N(\mathfrak{a}),f)N(\mathfrak{s}/\mathfrak{y})^{-s}N(\mathfrak{a})^{-s}\Big|_{s=k+m}\\
&=\left(\sum_{\mathfrak{y}|\mathfrak{s}_0}\mu_K(\mathfrak{y})N(\mathfrak{s}/\mathfrak{y})^{1-k-m}\lambda_m(\mathfrak{s}_0/\mathfrak{y})a(N(\mathfrak{s}/\mathfrak{y}),f)\right)L^{(Nd)}(1,\chi_K)\sum_{\mathfrak{a}}\frac{\lambda_m(\mathfrak{a})a(N(\mathfrak{a}),f)}{N(\mathfrak{a})^s}\Big|_{s=k+m}\\
&\overset{(**)}{=}\left(\frac{a(N(\mathfrak{s}_{\overline{\mathfrak{C}}}),f)}{N(\mathfrak{s}_{\overline{\mathfrak{C}}})^{k+m-1}}\prod_{\mathfrak{l}\in\mathcal{C}_0\text{ and }\mathfrak{l}|\ell\in C_s}a(\ell^{\nu(\ell)},f)\ell^{\nu(\ell)(1-(k/2))}\chi_m^-(\mathfrak{l}^{\nu(\ell)})\Big(1-\frac{1}{a(\ell,f)\ell^{1-(k/2)}\chi_m^-(\mathfrak{l})}\Big)\right)\\
&\qquad\qquad\times E\Big(\frac{1}{2}\Big)L^{(Nd)}\Big(\frac{1}{2},\widehat{\pi}_{\mathbf{f}}\otimes\lambda_m^u\Big)\\
&=\prod_{\mathfrak{l}\in\mathcal{C}_s^+}\frac{\alpha_\ell^{\nu(\ell)-f_{\overline{\mathfrak{l}}}}}{\ell^{(\nu(\ell)-f_{\overline{\mathfrak{l}}})(m+(k-1)/2)}}\prod_{\mathfrak{l}\in\mathcal{C}_s^0}\alpha_\ell^{\nu(\ell)}\ell^{\nu(\ell)/2}\chi_m^-(\mathfrak{l}^{\nu(\ell)})\Big(1-\frac{1}{\alpha_\ell\ell^{1/2}\chi_m^-(\mathfrak{l})}\Big)E\Big(\frac{1}{2}\Big)L^{(Nd)}\Big(\frac{1}{2},\widehat{\pi}_{\mathbf{f}}\otimes\lambda_m^u\Big),
\end{aligned}
$$

where $\mathfrak{a}$ runs over integral ideals of K outside $\overline{\mathcal{A}}$, and the equality $(**)$ follows from, for example, [LFE] §5.4. This finishes the proof, noting $\lambda_m^u=\lambda_m/|\lambda_m|=\chi_m^-$.

We now compute the constant c. Here is a table of many constants of the right-hand-side we have computed along the way:

Source	Lemma 3.4	(4.12)	(4.3)	Proposition 3.9	Lemma 3.10	Lemma 3.11
Value	$(2i)^{-k}\boldsymbol{\psi}_m(\det(g_1))$	$E'(m)$	$E''(m)^{-1}$	$(2i)^k(-1)^{k+m}$	$\lvert(O/\mathfrak{t})^\times\rvert^{-1}$	$\lvert(O/\mathfrak{T})^\times\rvert^{-1}$

Lemma 3.10	(4.6)	(4.11)
$\prod_{\ell\in\mathcal{A}\cup\mathcal{C}_+}N(\mathfrak{l})^{(k+2m)(\nu(\ell)-f_{\overline{\mathfrak{l}}})}\chi_{m,\overline{\mathfrak{l}}}(\ell^{\nu(\ell)-f_{\overline{\mathfrak{l}}}}u_\epsilon^{-c})G(\chi_{m,\mathfrak{l}}\circ c)$	$[\Gamma(A,C;N):\Gamma_0(\mathbb{N})]^{-1}$	$\frac{\sqrt{\lvert d\rvert}N(\mathfrak{t})}{2\pi}$

(4.11)	(4.11)	Lemma 3.10
$(-1)^m(4\pi)^{-k-2m}$	$\Gamma(k+m)\Gamma(m+1)$	$\mathbf{e}(-N_A^{-1})$

We simply multiply out the constants appearing in the above table to get the constant $c=c_1Gv$. The volume factor v is the product of $|(O/\mathfrak{t})^\times|^{-1}$, $|(O/\mathfrak{T})^\times|^{-1}$ and $[\Gamma(A,C;N):\Gamma_0(\mathbb{N})]^{-1}$. Note $G(\chi_{m,\mathfrak{l}}\circ c)=G(\chi_{m,\mathfrak{l}}^-)$, $N(\mathfrak{l})^{(k+2m)(\nu(\ell)-f_{\overline{\mathfrak{l}}})}\chi_{m,\overline{\mathfrak{l}}}(\ell^{\nu(\ell)-f_{\overline{\mathfrak{l}}}}u_\epsilon^{-c})=\chi_{\mathfrak{l}}^-(\ell^{\nu(\ell)-f_{\overline{\mathfrak{l}}}})\ell^{(k/2)+m)(\nu(\ell)-f_{\overline{\mathfrak{l}}})}$ and

$$\boldsymbol{\psi}_m(\det(g_1))=\prod_{\ell|N}\boldsymbol{\psi}_{m,\ell}(\ell^{\nu(\ell)})=N^{k+2m}\prod_{\ell\in A\cup C}\chi_{m,\ell}^-(\ell^{\nu(\ell)})^{-1}$$

as $\boldsymbol{\psi}_m = \chi_m^{-1}$ on $\mathbb{A}^\times$ and $|\chi_{m,\ell}(\ell)| = \ell^{-k-2m}$. Thus the other constants aside from the volume factor v are in the Gauss sum factor G and c_1. This finishes the proof.

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