Ring theoretic properties of Hecke algebras and Cyclicity in Iwasawa theory

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A talk in June, 2016
at Banff conference center.

*The author is partially supported by the NSF grant: DMS 1464106.

We can formulate certain Gorenstein property of subrings of the universal deformation ring (i.e., the corresponding Hecke algebra) as a condition almost equivalent to the cyclicity of the Iwasawa module over $\mathbb{Z}_p$-extensions of an imaginary quadratic field if the starting residual representation is induced from the imaginary quadratic field. I will discuss this fact in some details.
§0. **Setting over an imaginary quadratic field.** Let $F$ be an imaginary quadratic field with discriminant $-D$ and integer ring $O$. Assume that the prime $(p)$ splits into $(p) = p\overline{p}$ in $O$ with $p \neq \overline{p}$. Let $L/F$ be a $\mathbb{Z}_p$-extension with group $\Gamma_L := \text{Gal}(L/F) \cong \mathbb{Z}_p$. Take a branch character $\phi : \text{Gal}(\overline{\mathbb{Q}}/F') \to F'^\times$ (for $F' = F_{p^f}$) with its Teichmüller lift $\bar{\phi}$ with values in $W = W(\mathbb{F})$. Regard it as an idele character $\phi : F_A^\times / F^\times \to \overline{\mathbb{Q}}_p^\times$ with

$$W = \mathbb{Z}_p[\phi] := \mathbb{Z}_p[\phi(x) | x \in F_A^\times] \subset \overline{\mathbb{Q}}_p.$$ 

Consider the Iwasawa algebra $W[[\Gamma_L]] = \varprojlim_n W[\Gamma_L / \Gamma_L^{p^n}]$.

Let $F(\phi)/F$ be the extension cut out by $\phi$ (i.e., $F(\phi) = \overline{\mathbb{Q}}^{\text{Ker}(\phi)}$). Let $Y_L$ be the Galois group of the maximal $p$-abelian extension over the composite $L(\phi) := L \cdot F(\phi)$ unramified outside $p$. By the splitting: $\text{Gal}(L(\phi)/F) = \text{Gal}(F(\phi)/F') \times \Gamma_L$, we have

$$Y_L(\phi) := Y_L \otimes_{W[\text{Gal}(F(\phi)/F)]} W$$ (the $\phi$-eigenspace).

This is a torsion module over $W[[\Gamma_L]]$ of finite type by Rubin.
§1. Cyclicity conjecture for an anti-cyclotomic branch. Let $c$ be complex conjugation in $\text{Gal}(F/\mathbb{Q})$. Suppose that $\phi(x) = \varphi(x)\varphi(x^{-c}) := \varphi^-(x)$ for a finite order character $\varphi : F_{\mathbb{A}}^\times / F_{\times}^\times \to \overline{\mathbb{Q}}_p^\times$.

Conjecture for $L$: Assume $\varphi^- \neq 1$ and that the conductor $\varphi$ is a product of split primes over $\mathbb{Q}$. If the class number $h_F$ of $F$ is prime to $p$, then $Y_L(\varphi^-)$ is pseudo isomorphic to $W[[\Gamma_L]]/(f_L)$ as $W[[\Gamma_L]]$-modules for an element $f_L \in W[[\Gamma_L]]$.

We know $f_L \neq 0$ by Rubin. For some specific $\mathbb{Z}_p$-extension (e.g., the anticyclotomic $\mathbb{Z}_p$-extension), we know that $(f_L)$ is prime to $pW[[\Gamma_L]]$ (vanishing of the $\mu$-invariant).

The anti-cyclotomic cyclicity conjecture is the one for the anticyclotomic $\mathbb{Z}_p$-extension $L = F_{\infty}^-$ such that on $\Gamma_- := \Gamma_{F_{\infty}^-}$, we have $c\sigma c^{-1} = \sigma^{-1}$. Write $Y^-(\varphi^-)$ for the $W[[\Gamma_-]]$-module $Y_{F_{\infty}^-}(\varphi^-)$. 
§2. **Anti-cyclotomic cyclicility ⇔ \( L \)-cyclicility.**

In this talk, we only deal with “pure” cyclicity. Hereafter, we suppose

(H1) We have \( \phi = \varphi^- \) for a character \( \varphi \) of conductor \( c \varphi \) with \( c + (p) = O \) and of order prime to \( p \),
(H2) \( N = DN_{F/Q}(c) \) for an \( O \)-ideal \( c \) prime to \( D \) with square-free \( N_{F/Q}(c) \) (so, \( N \) is cube-free),
(H3) \( p \) is prime to \( N \prod_{l \mid N}(l - 1) \) for prime factors \( l \) of \( N \),
(H4) the character \( \varphi^- \) has order at least 3,
(H5) the class number of \( F \) is prime to \( p \).

We first note:

**Theorem 1.** *The anticyclotomic pure cyclicility is equivalent to the pure cyclicility for \( Y_L(\varphi^-) \).* (Note that the branch character is anti-cyclotomic.)

This follows from a control theorem of Rubin.
§3. Anti-cyclotomic Cyclicity and Hecke algebra. Anti-cyclotomic cyclicity follows from a **ring theoretic assertion** on the big ordinary Hecke algebra $h$. We identify the Iwasawa algebra $\Lambda = W[[\Gamma]]$ with the one variable power series ring $W[[T]]$ by $\Gamma \ni \gamma = (1 + p) \mapsto t = 1 + T \in \Lambda$. Take a Dirichlet character $\psi : (\mathbb{Z}/Np\mathbb{Z})^\times \to W^\times$, and consider the big ordinary Hecke algebra $h$ (over $\Lambda$) of prime-to-$p$ level $N$ and the character $\psi$. We just mention here the following three facts about $h$:

- $h$ is an algebra flat over the Iwasawa (weight) algebra $\Lambda := W[[T]]$ **interpolating** $p$-ordinary Hecke algebras of level $Np^{r+1}$, of weight $k + 1 \geq 2$ and of character $\epsilon \psi \omega^{-k}$, where $\epsilon : \mathbb{Z}_p^\times \to \mu_p^r$ ($r \geq 0$) and $k \geq 1$ vary. If $N$ is cube-free, $h$ is a **reduced** algebra;
- Each prime $P \in \text{Spec}(h)$ has a unique Galois representation $\rho_P : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(\kappa(P))$, $\text{Tr}\rho_P(\text{Frob}_l) = T(l) \mod P(l \nmid Np)$ for the residue field $\kappa(P)$ of $P$;
- $\rho_P|_{\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)} \cong \begin{pmatrix} \epsilon_P^* & \delta_P^* \\ 0 & \delta_P \end{pmatrix}$ with unramified quotient character $\delta_P$. 
§ 4. **Ring theoretic setting.**

Let $\Spec(\mathbb{T})$ be the connected component of $\Spec(h)$ of $\overline{\rho} := \rho_\mathbb{T} \mod m_\mathbb{T} = \Ind_{F'}^Q \overline{\rho}$. Since $\mathbb{T}$ is universal among ordinary deformation of $\overline{\rho}$ with certain extra properties insensitive to the twist $\rho \mapsto \rho \otimes \chi$ for $\chi = \left( \frac{F}{Q} \right)$, $\mathbb{T}$ has an **algebra involution** $\sigma$ over $\Lambda$ coming from the twist. For any ring $A$ with an involution $\sigma$, we put $A_{\pm} = A^\pm := \{ x \in A | \sigma(x) = \pm x \}$. Then $A_+ \subset A$ is a subring and $A_-$ is an $A_+$-module.

It is easy to see

- For the ideal $I$ of $\mathbb{T}$ generated by $\mathbb{T}_-$ (the “−” eigenspace), we have a canonical isomorphism $\mathbb{T}/I \cong W[[\Gamma_-]]$ as $\Lambda$-algebras, where the $\Lambda$-algebra structure is given by sending $u \in \Gamma$ naturally into $u \in O_p^\times = \mathbb{Z}_p^\times$ and then projecting the local Artin symbol $\tau = [u, F_p] \in \Gamma$ to $\sqrt{\tau c \tau^{-1} c^{-1}} = \tau^{(1-c)/2} \in \Gamma_-$.  

§5. Non CM components.

- The fixed points $\text{Spec}(\mathbb{T})^{\sigma=1}$ is known to be canonically isomorphic to $\text{Spec}(W[[\Gamma_-]])$,
- $Y^-(\varphi^-) \neq 0$ if and only if $\sigma$ is non-trivial on $\mathbb{T}$ (and hence $\mathbb{T} \neq W[[\Gamma_-]]$).
- The ring $\mathbb{T}$ is reduced (as $N$ is cube-free), and for the kernel $I = \mathbb{T}(\sigma - 1)\mathbb{T} = \text{Ker}(\mathbb{T} \to W[[\Gamma_-]])$, $I$ span over $\text{Frac}(\Lambda)$ a ring direct summand $X$ complementary to $\text{Frac}(W[[\Gamma_-]])$.

We write $\mathbb{T}^{\text{ncm}}$ for the image of $\mathbb{T}$ in the ring direct summand $X$ (and call it the non-CM component of $\mathbb{T}$). Plainly $\mathbb{T}^{\text{ncm}}$ is stable under $\sigma$, but

$$\text{Spec}(\mathbb{T}^{\text{ncm}})^{\sigma=1}$$ has codimension 1 in $\text{Spec}(\mathbb{T}^{\text{ncm}})$,

which does not therefore contain an irreducible component.
§6. Galois deformation theory. By irreducibility of $\bar{\rho}$, we have a Galois representation

$$\rho_T : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(T) \quad \text{with} \quad \text{Tr}(\rho_T(Frob_l)) = T(l)$$

for all primes $l \nmid Np$. By the celebrated $R = \mathbb{T}$ theorem of Taylor–Wiles, the couple $(T, \rho_T)$ is universal among deformations $\rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(A)$ satisfying

$(D1)$ $\rho \mod m_A \cong \bar{\rho}$.

$(D2)$ $\rho|_{\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)} \cong (\epsilon^* \delta)$ with $\delta$ unramified.

$(D3)$ $\det(\rho)|_{I_l} = \psi_l$ for the $l$-part $\psi_l$ of $\psi$ for each prime $l|N$.

$(D4)$ $\det(\rho)|_{I_p} \equiv \psi|_{I_p} \mod m_A$ (⇔ $\epsilon|_{I_p} \equiv \psi|_{I_p} \mod m_A$).

By the $R = \mathbb{T}$ theorem and a theorem of Mazur, if $p \nmid h_F$,

$$I/I^2 = \Omega_{T/\Lambda} \otimes_T W[[\Gamma^-]] \cong Y^-(\varphi^-),$$

and principality of $I$ implies cyclicity.
§7. Theorem.

**Theorem A:** Suppose (H1–5). Then for the following statements

(1) The rings $T_{ncm}$ and $T_{ncm}^+$ are both local complete intersections free of finite rank over $\Lambda$.

(2) The $T_{ncm}$-ideal $I = T(\sigma - 1)T \subset T_{ncm}$ is principal and is generated by a non-zero-divisor $\theta \in T_- = T_{ncm}^-$ with $\theta^2 \in T_{ncm}^+$, and $T_{ncm} = T_{ncm}^+[\theta]$ is free of rank 2 over $T_{ncm}^+$.

(3) The Iwasawa module $Y^-(\varphi^-)$ is cyclic over $W[[\Gamma_-]]$.

(4) The Iwasawa module $Y^-(\varphi^- \omega)$ is cyclic over $W[[\Gamma_-]]$.

Under the condition (4), the ring $T_+$ is a local complete intersection.

(2) $\iff$ (3) follows from $I/I^2 \cong Y^-(\varphi^-)$, and we expect (3) $\iff$ (4) (a sort of modulo $p$ Tate duality).
§8. A key duality lemma from the theory of dualizing modules by Grothendieck, Hartshorne and Kleiman in a simplest case:

**Lemma 1** (Key lemma). Let \( S \) be a \( p \)-profinite Gorenstein integral domain and \( A \) be a reduced Gorenstein local \( S \)-algebra free of finite rank over \( S \). Suppose
- \( A \) has a ring involution \( \sigma \) with \( A_+ := \{ a \in A | \sigma(a) = a \} \),
- \( A_+ \) is Gorenstein,
- \( \text{Frac}(A)/\text{Frac}(A_+) \) is étale quadratic extension.
- \( \mathfrak{d}_{A/A_+}^{-1} := \{ x \in \text{Frac}(A) | \text{Tr}_{A/A_+}(xA) \subset A_+ \} \supseteq A \),

Then \( A \) is free of rank 2 over \( A_+ \) and \( A = A_+ \oplus A_+ \delta \) for an element \( \delta \in A \) with \( \sigma(\delta) = -\delta \).

**Lemma 2.** Let \( S \) be a Gorenstein local ring. Let \( A \) be a local Cohen–Macaulay ring and is an \( S \)-algebra with \( \dim A = \dim S \). If \( A \) is an \( S \)-module of finite type, the following conditions are equivalent:
- The local ring \( A \) is Gorenstein;
- \( A^\dagger := \text{Hom}_S(A, S) \cong A \) as \( A \)-modules.
§9. We can apply the key lemmas to $T^{\text{ncm}}$: (1) $\Leftrightarrow$ (2).

(1)$\Rightarrow$(2): $T^{\text{ncm}}$ and $T_+^{\text{ncm}}$ are local complete intersections by assumption; so, Gorenstein.

**Use of Main conjecture:** The proof of the anti-cyclotomic Main conjecture by Mazur–Tilouine (combined with a theorem of Tate on Gorenstein rings [MFG, Lemma 5.21]) shows

$$ T^{\text{ncm}}/\mathfrak{d}_{T^{\text{ncm}}/T_+^{\text{ncm}}} \cong W[[\Gamma_-]]/(L_p^-(\varphi^-)) $$ (Congruence module identity);

so, $\mathfrak{d}_{T^{\text{ncm}}/T_+^{\text{ncm}}} \subset \mathfrak{m}_{T^{\text{ncm}}}$ for the anti-cyclotomic Katz $p$-adic $L$-function $L_p^-(\varphi^-)$. The key lemma tells us (2).

(2)$\Rightarrow$(1): We have $I = (\theta) \subset T^{\text{ncm}}$ and $I_+ = (\theta^2) \subset T^{\text{ncm}}$.

Note that $T^{\text{ncm}}/(\theta) \cong W[[\Gamma_-]]/(L_p(\varphi^-)) \cong T_+^{\text{ncm}}/(\theta^2)$. Since $\theta$ is a non-zero divisor, the two rings $T^{\text{ncm}}$ and $T_+^{\text{ncm}}$ are local complete intersection.
§10. Presentation of $T$.

To see a possibility of applying the key lemma to $T/T_+$, we like to lift $T$ to a power series ring $\mathcal{R} = \Lambda[[T_1, \ldots, T_r]]$ with an involution $\sigma_\infty$ such that $\mathcal{R}^+ := \{ r \in \mathcal{R} | \sigma_\infty(r) = -r \}$ is Gorenstein and that $(\mathcal{R}/\mathfrak{A}, \sigma_\infty \bmod \mathfrak{A}) \cong (T, \sigma)$ for an ideal $\mathfrak{A}$ stable under $\sigma_\infty$.

Taylor and Wiles (with a later idea of Diamond and Fujiwara) found a pair $(\mathcal{R} := \Lambda[[T_1, \ldots, T_r]], (S_1, \ldots, S_r))$ with a regular sequence $S := (S_1, \ldots, S_r) \subset \Lambda[[T_1, \ldots, T_r]]$ such that

$$\Lambda[[T_1, \ldots, T_r]]/(S_1, \ldots, S_r) \cong T$$

by their Taylor–Wiles system argument.

We need to lift $\sigma$ somehow to an involution $\sigma_\infty \in \text{Aut}(\mathcal{R})$ and show also that $\mathcal{R}^+$ is Gorenstein. If further $d_{\mathcal{R}/\mathcal{R}^-} \subset \mathfrak{m}_\mathcal{R}$, $\mathcal{R} \cdot \mathcal{R}^- = (\delta_\infty)$ and the image $\delta \in T^-$ of $\delta_\infty$ in $T$ generates $I$ as desired.
§11. Taylor–Wiles method. Taylor–Wiles found an integer $r > 0$ and an infinite sequence of $r$-sets $Q := \{Q_m | m = 1, 2, \ldots\}$ of primes $q \equiv 1 \mod p^m$ such that for the local ring $\mathbb{T}^Q_m$ of $\overline{\rho}$ of the Hecke algebra $h^Q_m$ of tame-level $N_m = N \prod_{q \in Q_m} q$. The pair $(\mathbb{T}^Q_m, \rho_{\mathbb{T}^Q_m})$ is universal among deformation satisfying (D1–4) but ramification at $q \in Q_m$ is allowed. Then $\rho \mapsto \rho \otimes \chi$ induces an involution $\sigma_{Q_m}$ and $\mathbb{T}^Q_+ := \{x \in \mathbb{T}^Q_m | \sigma_{Q_m}(x) = x\}$ is Gorenstein.

Actually they work with $\mathbb{T}_{Q_m} = \mathbb{T}^Q_m/(t - \gamma^k)\mathbb{T}^Q_m$ ($t = 1 + T$, $\gamma = 1 + p \in \Gamma$; the weight $k$ Hecke algebra of weight $k \geq 2$ fixed). The product inertia group $I_{Q_m} = \prod_{q \in Q_m} I_q$ acts on $\mathbb{T}_{Q_m}$ by the $p$-abelian quotient $\Delta_{Q_m}$ of $\prod_{q \in Q_m} (\mathbb{Z}/q\mathbb{Z})^\times$. We choose an ordering of primes $Q_m = \{q_1, \ldots, q_r\}$ and a generator $\delta_{i,m(n)}$ of the $p$-Sylow group of $(\mathbb{Z}/q_i\mathbb{Z})^\times$. The sequence $Q$ is chosen so that for a given integer $n > 0$, we can find $m = m(n) > n$ so that we have ring projection maps $R_{n+1} \to R_n := \mathbb{T}_{Q_m(n)}/(p^n, \delta_{i,m(n)}^n - 1)_i$, and $R_\infty = \lim_{\leftarrow} R_n \cong W[[T_1, \ldots, T_r]]$ and $S_i = \lim_{\leftarrow} (\delta_{i,m(n)} - 1)$. 
§12. Lifting involution.

Write $\overline{S}_n$ for the image of $W[[S]]$ for $S = (S_1, \ldots, S_r)$ in $R_n$ ($\overline{S}_n$ is a Gorenstein local ring). We can add the involution to this projective system and an $R_n$-linear isomorphism $\phi_n : R_n^\dagger := \text{Hom}_{\overline{S}_n}(R_n, \overline{S}_n) \cong R_n$ commuting with the involution $\sigma_n$ of $R_n$ induced by $\sigma_{Q_{m(n)}}$ to the Taylor-Wiles system, and get the lifting

$$\sigma_\infty \in \text{Aut}(R_\infty)$$

with

$$\phi_\infty : R_\infty^\dagger := \text{Hom}_{W[[S]]}(R_\infty, W[[S]]) \cong R_\infty$$

compatible with $\sigma_\infty$; i.e., $\phi_\infty \circ \sigma_\infty = \sigma_\infty \circ \phi_\infty$. This shows

$$R_\infty^+,\dagger \cong R_\infty^+$$

as $R_\infty^+$-modules, as desired. Then we can further lift involution to $\mathcal{R} = \Lambda[[T_1, \ldots, T_r]]$ as $\mathcal{R}/(t - \gamma^k) = R_\infty$ for $t = 1 + T$.

The remaining point of the key lemma I have not done is to show

$$\mathfrak{d}_{\mathcal{R}/\mathcal{R}^+} \subset \mathfrak{m}_{\mathcal{R}}?$$
§13. Index set of $Q_m$ (towards $(4) \Leftrightarrow (2)$).

Write $D_q$ for the local version of the deformation functor associated to (D1–4) adding a fixed determinant condition

$$(\text{det}) \det(\rho) = \nu^k \psi$$

for the chosen $k \geq 2$ (the weight condition); so, the tangent spec of $T$ is given by a Selmer group $\text{Sel}(Ad)$ for $Ad = \mathfrak{s}\mathfrak{l}_2(\mathbb{F})$.

Then the index set of $Q_m$ is any choice of $\mathbb{F}$-basis of a “dual” Selmer group. Regard $D_q(\mathbb{F}[\epsilon])$ for the dual number $\epsilon$ as a subspace of $H^1(\mathbb{Q}_q, Ad)$ in the standard way: Thus we have the orthogonal complement $D_q(\mathbb{F}[\epsilon])^\perp \subset H^1(\mathbb{Q}_q, Ad^*(1))$ under Tate local duality. The dual Selmer group $\text{Sel}^\perp(Ad^*(1))$ is given by

$$\text{Sel}^\perp(Ad^*(1)) := \text{Ker}(H^1(\mathbb{Q}^{(Np)}/\mathbb{Q}, Ad^*(1)) \to \prod_{l|Np} \frac{H^1(\mathbb{Q}_l, Ad^*(1))}{D_l(\mathbb{F}[\epsilon])^\perp}).$$

Then $r = \dim_\mathbb{F} \text{Sel}^\perp(Ad^*(1))$. 
§14. Interpretation of the dual Selmer group.
Define $Q_m^\pm := \{ q \in Q_m | \chi(q) = \pm q \}$. Then if $S_q$ is the variable in $W[[S]]$ corresponding from $q \in Q_m^\pm$, then $\sigma(1 + S_q) = (1 + S_q)^{\pm 1}$.

We have splitting $Ad = \overline{\chi} \oplus \text{Ind}_{F}^{Q}(\varphi^-)$; so, $\text{Sel}^\perp(Ad^*(1)) = \text{Sel}^\perp(\overline{\chi}(1)) \oplus \text{Sel}^\perp(\text{Ind}_{F}^{Q}(\varphi^-(1)))$ and

$$\text{Sel}^\perp(\text{Ind}_{F}^{Q}(\varphi^-(1))) = \text{Hom}_{W[[\Gamma^-]]}(Y^-(\varphi^- \omega), F).$$

Thus the number $\mu_{W[[S]]_+}(W[[S]]_-)$ of generators of $W[[S]]_-$ over $W[[S]]_+$ is

$$\mu_{W[[S]]_+}(W[[S]]_-) = \dim_{F} Y^-(\varphi^- \omega) \otimes_{W[[\Gamma^-]]} F.$$

Writing a number of generators of an $A$-module $M$ over $A$ as $\mu_{A}(M)$, we thus have

$$\mu_{W[[S]]_+}(W[[S]]_-) = \text{codim}_{\text{Spec}(W[[S]])} \text{Spec}(W[[S]])^{\sigma=1}.$$
§15. Generator count $\mu_{R_\infty^+}(R_\infty^-)$ of $R_\infty^-$. 

**Lemma 3.** We have

$$\mu_{R_\infty^+}(R_\infty^-) = \text{codim}_{\text{Spec}(W[[S]])} \text{Spec}(W[[S]])^{\sigma=1}$$

$$= \dim_{\mathbb{F}} \text{Y}^- (\varphi^- \omega) \otimes W[[\Gamma_-]] \overline{\mathbb{F}}.$$

In the construction of Taylor–Wiles system, for each $q \in Q_m$, an eigenvalue of $\bar{\rho}(\text{Frob}_q)$ is chosen, which is equivalent to choose a factor $q|q$ if $q \in Q_m^+$. Then $\prod_{q \in Q_m^+} (O_F/q)^\times$ has $p$-Sylow subgroup $\Delta_{Q_m^+}$. The projective limit $\lim_{\leftarrow n} \Delta_{Q_m(n)} / \Delta_{Q_m(n)}^{p^n}$ gives rise to a group isomorphic to $\Delta_+ := \mathbb{Z}_{p^+}$ for $r_+ = |Q_m^+|$. 

§16. QED.

Let $I_\infty = R_\infty(\sigma - 1)R_\infty$, $I^Q = T^Q(\sigma - 1)T^Q$ and $H_Q = \Gamma_- \times \Delta_{Q^+}$.

By $T^Q/I^Q \cong W[[H_Q]] = T^Q_+/I^Q_+$, we get

$$R_\infty/I_\infty \cong W[[\Delta_+]] \cong R^+_\infty/I^+_\infty.$$ 

Note that

$$\text{Spec}(R_\infty)^{\sigma=1} = \text{Spec}(R_\infty/I_\infty) = \text{Spec}(W[[\Delta_+]]).$$

Thus we get

$$\mu_{R^+_\infty}(R^-_\infty) = \text{codim}_{\text{Spec}(W[[S]])} \text{Spec}(W[[\Delta_+]])$$

$$= r_- = \dim_{\mathbb{F}} Y^-(\varphi^- \omega) \otimes W[[\Gamma_-]] \mathbb{F}.$$

This shows the implication (4) $\Rightarrow$ (2) of Theorem A.