* Ring theoretic properties of Hecke algebras and Cyclicity in Iwasawa theory

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We can formulate certain Gorenstein property of subrings of the universal deformation ring (i.e., the corresponding Hecke algebra) as a condition almost equivalent to the cyclicity of the Iwasawa module over $\mathbb{Z}_p$-extensions of an imaginary quadratic field if the starting residual representation is induced from the imaginary quadratic field. I will discuss this fact in some details.

§0. **Setting over an imaginary quadratic field.** Let F be an imaginary quadratic field with discriminant $-D$ and integer ring O. Assume that the prime (p) **splits** into $(p) = \mathfrak{p}\overline{\mathfrak{p}}$ in O with $\mathfrak{p} \neq \overline{\mathfrak{p}}$. Let L/F be a $\mathbb{Z}_p$-extension with group $\mathbf{\Gamma}_L :=$ Gal$(L/F) \cong \mathbb{Z}_p$. Take a branch character $\overline{\phi}$: Gal$(\overline{\mathbb{Q}}/F) \to \mathbb{F}^\times$ (for $\mathbb{F} = \mathbb{F}_{p^f}$) with its Teichmüller lift ϕ with values in $W = W(\mathbb{F})$. Regard it as an idele character $\phi : F_{\mathbb{A}}^\times/F^\times \to \overline{\mathbb{Q}}_p^\times$ with

$$W = \mathbb{Z}_p[\phi] := \mathbb{Z}_p[\phi(x)|x \in F_{\mathbb{A}}^\times] \subset \overline{\mathbb{Q}}_p.$$

Consider the Iwasawa algebra $W[[\mathbf{\Gamma}_L]] = \varprojlim_n W[\mathbf{\Gamma}_L/\mathbf{\Gamma}_L^{p^n}]$.

Let $F(\phi)/F$ be the extension cut out by ϕ (i.e., $F(\phi) = \overline{\mathbb{Q}}^{\mathrm{Ker}(\phi)}$). Let Y_L be the Galois group of the maximal p-abelian extension over the composite $L(\phi) := L \cdot F(\phi)$ **unramified outside** $\mathfrak{p}$. By the splitting: Gal$(L(\phi)/F) =$ Gal$(F(\phi)/F) \times \mathbf{\Gamma}_L$, we have

$$Y_L(\phi) := Y_L \otimes_{W[\mathrm{Gal}(F(\phi)/F)],\phi} W \ \textbf{(the } \phi\textbf{-eigenspace)}.$$

This is a **torsion** module over $W[[\mathbf{\Gamma}_L]]$ of finite type by Rubin.

§**1. Cyclicity conjecture for an anti-cyclotomic branch.** Let c be complex conjugation in Gal$(F/\mathbb{Q})$. Suppose that $\phi(x) = \varphi(x)\varphi(x^{-c}) := \varphi^-(x)$ for a finite order character $\varphi : F_{\mathbb{A}}^\times/F^\times \to \overline{\mathbb{Q}}_p^\times$.

Conjecture for L: *Assume $\varphi^- \neq 1$ and that the conductor φ is a product of split primes over $\mathbb{Q}$. If the class number h_F of F is prime to p, then $Y_L(\varphi^-)$ is pseudo isomorphic to $W[[\mathbf{\Gamma}_L]]/(f_L)$ as $W[[\mathbf{\Gamma}_L]]$-modules for an element $f_L \in W[[\mathbf{\Gamma}_L]]$.*

We know $f_L \neq 0$ by Rubin. For some specific $\mathbb{Z}_p$-extension (e.g., the anticyclotomic $\mathbb{Z}_p$-extension), we know that (f_L) is prime to $pW[[\mathbf{\Gamma}_L]]$ (vanishing of the μ-invariant).

The anti-cyclotomic cyclicity conjecture is the one for the anticyclotomic $\mathbb{Z}_p$-extension $L = F_\infty^-$ such that on $\Gamma_- := \mathbf{\Gamma}_{F_\infty^-}$, we have $c\sigma c^{-1} = \sigma^{-1}$. Write $Y^-(\varphi^-)$ for the $W[[\Gamma_-]]$-module $Y_{F_\infty^-}(\varphi^-)$.

§2. Anti-cyclotomic cyclicity $\Leftrightarrow$ L-cyclicity.

In this talk, we only deal with "**pure**" cyclicity. Hereafter, we suppose

(H1) We have $\phi = \varphi^-$ for a character φ of conductor $\mathfrak{cp}$ with $\mathfrak{c} + (p) = O$ and of order prime to p,
(H2) $N = DN_{F/\mathbb{Q}}(\mathfrak{c})$ for an O-ideal $\mathfrak{c}$ prime to D with square-free $N_{F/\mathbb{Q}}(\mathfrak{c})$ (so, N is cube-free),
(H3) p is prime to $N\prod_{l|N}(l-1)$ for prime factors l of N,
(H4) the character φ^- has order at least 3,
(H5) the class number of F is prime to p.

We first note:

Theorem 1. *The anticyclotomic pure cyclicity is equivalent to the pure cyclicity for $Y_L(\varphi^-)$. (***Note that the branch character is anti-cyclotomic.***)*

This follows from a control theorem of Rubin.

§3. **Anti-cyclotomic Cyclicity and Hecke algebra.** Anti-cyclotomic cyclicity follows from a **ring theoretic assertion** on the big ordinary Hecke algebra $\mathbf{h}$. We identify the Iwasawa algebra $\Lambda = W[[\Gamma]]$ with the one variable power series ring $W[[T]]$ by $\Gamma \ni \gamma = (1+p) \mapsto t = 1+T \in \Lambda$. Take a Dirichlet character $\psi : (\mathbb{Z}/Np\mathbb{Z})^\times \to W^\times$, and consider the big ordinary Hecke algebra $\mathbf{h}$ (over Λ) of prime-to-p level N and the character ψ. We just mention here the following three facts about $\mathbf{h}$:

- $\mathbf{h}$ is an algebra flat over the Iwasawa (weight) algebra $\Lambda := W[[T]]$ **interpolating** p-ordinary Hecke algebras of level Np^{r+1}, of weight $k+1 \geq 2$ and of character $\epsilon\psi\omega^{-k}$, where $\epsilon : \mathbb{Z}_p^\times \to \mu_{p^r}$ $(r \geq 0)$ and $k \geq 1$ vary. If N is cube-free, $\mathbf{h}$ is a **reduced** algebra;
- Each prime $P \in \mathrm{Spec}(\mathbf{h})$ has a unique Galois representation

$$\rho_P : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\kappa(P)),\ \mathrm{Tr}\rho_P(\mathrm{Frob}_l) = T(l) \bmod P(l \nmid Np)$$

for the residue field $\kappa(P)$ of P;
- $\rho_P|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)} \cong \begin{pmatrix} \epsilon_P & * \\ 0 & \delta_P \end{pmatrix}$ with unramified quotient character δ_P.

§4. Ring theoretic setting.

Let Spec($\mathbb{T}$) be the connected component of Spec($\mathbf{h}$) of $\overline{\rho} := \rho_{\mathbb{T}}$ mod $\mathfrak{m}_{\mathbb{T}} = \mathrm{Ind}_F^{\mathbb{Q}}\overline{\varphi}$. Since $\mathbb{T}$ is universal among ordinary deformation of $\overline{\rho}$ with certain extra properties insensitive to the twist $\rho \mapsto \rho \otimes \chi$ for $\chi = \left(\frac{F/\mathbb{Q}}{}\right)$, $\mathbb{T}$ has an **algebra involution** σ over Λ coming from the twist. For any ring A with an involution σ, we put $A_{\pm} = A^{\pm} := \{x \in A | \sigma(x) = \pm x\}$. Then $A_{+} \subset A$ is a subring and A_{-} is an A_{+}-module.

It is easy to see

• For the ideal I of $\mathbb{T}$ generated by $\mathbb{T}_{-}$ (the "$-$" eigenspace), we have a canonical isomorphism $\mathbb{T}/I \cong W[[\Gamma_{-}]]$ as Λ-algebras, where the Λ-algebra structure is given by sending $u \in \Gamma$ naturally into $u \in O_{\mathfrak{p}}^{\times} = \mathbb{Z}_p^{\times}$ and then projecting the local Artin symbol $\tau = [u, F_{\mathfrak{p}}] \in \Gamma$ to $\sqrt{\tau c \tau^{-1} c^{-1}} = \tau^{(1-c)/2} \in \Gamma_{-}$.

§5. Non CM components.

- The fixed points $\mathrm{Spec}(\mathbb{T})^{\sigma=1}$ is known to be canonically isomorphic to $\mathrm{Spec}(W[[\Gamma_-]])$,
- $Y^-(\varphi^-) \neq 0$ if and only if σ is non-trivial on $\mathbb{T}$ (and hence $\mathbb{T} \neq W[[\Gamma_-]]$).
- The ring $\mathbb{T}$ is reduced (as N is cube-free), and for the kernel $I = \mathbb{T}(\sigma - 1)\mathbb{T} = \mathrm{Ker}(\mathbb{T} \twoheadrightarrow W[[\Gamma_-]])$, I span over $\mathrm{Frac}(\Lambda)$ a ring direct summand X complementary to $\mathrm{Frac}(W[[\Gamma_-]])$.

We write $\mathbb{T}^{\mathrm{ncm}}$ for the image of $\mathbb{T}$ in the ring direct summand X (and call it the non-CM component of $\mathbb{T}$). Plainly $\mathbb{T}^{\mathrm{ncm}}$ is stable under σ, but

$$\mathrm{Spec}(\mathbb{T}^{\mathrm{ncm}})^{\sigma=1} \text{ has codimension 1 in } \mathrm{Spec}(\mathbb{T}^{\mathrm{ncm}}),$$

which does not therefore contain an irreducible component.

§**6. Galois deformation theory.** By irreducibility of $\overline{\rho}$, we have a Galois representation

$$\rho_{\mathbb{T}} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{T}) \quad \text{with } \mathrm{Tr}(\rho_{\mathbb{T}}(\mathrm{Frob}_l)) = T(l)$$

for all primes $l \nmid Np$. By the celebrated $R = \mathbb{T}$ theorem of Taylor–Wiles, the couple $(\mathbb{T}, \rho_{\mathbb{T}})$ is universal among deformations $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(A)$ satisfying

(D1) $\rho \mod \mathfrak{m}_A \cong \overline{\rho}$.

(D2) $\rho|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)} \cong \left(\begin{smallmatrix} \epsilon & * \\ 0 & \delta \end{smallmatrix}\right)$ with δ unramified.

(D3) $\det(\rho)|_{I_l} = \psi_l$ for the l-part ψ_l of ψ for each prime $l|N$.

(D4) $\det(\rho)|_{I_p} \equiv \psi|_{I_p} \mod \mathfrak{m}_A$ ($\Leftrightarrow \epsilon|_{I_p} \equiv \psi|_{I_p} \mod \mathfrak{m}_A$).

By the $R = \mathbb{T}$ theorem and a theorem of Mazur, if $p \nmid h_F$,

$$I/I^2 = \Omega_{\mathbb{T}/\Lambda} \otimes_{\mathbb{T}} W[[\Gamma_-]] \cong Y^-(\varphi^-),$$

and principality of I implies cyclicity.

§7. Theorem.

Theorem A: *Suppose (H1–5). Then for the following statements* (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftarrow$ (4)*:*
(1) *The rings* $\mathbb{T}^{\mathsf{ncm}}$ *and* $\mathbb{T}_+^{\mathsf{ncm}}$ *are both local complete intersections free of finite rank over* Λ*.*
(2) *The* $\mathbb{T}^{\mathsf{ncm}}$*-ideal* $I = \mathbb{T}(\sigma - 1)\mathbb{T} \subset \mathbb{T}^{\mathsf{ncm}}$ *is principal and is generated by a non-zero-divisor* $\theta \in \mathbb{T}_- = \mathbb{T}_-^{\mathsf{ncm}}$ *with* $\theta^2 \in \mathbb{T}_+^{\mathsf{ncm}}$*, and* $\mathbb{T}^{\mathsf{ncm}} = \mathbb{T}_+^{\mathsf{ncm}}[\theta]$ *is free of rank 2 over* $\mathbb{T}_+^{\mathsf{ncm}}$*.*
(3) *The Iwasawa module* $Y^-(\varphi^-)$ *is cyclic over* $W[[\Gamma_-]]$*.*
(4) *The Iwasawa module* $Y^-(\varphi^-\omega)$ *is cyclic over* $W[[\Gamma_-]]$*.*
Under the condition (4)*, the ring* $\mathbb{T}_+$ *is a local complete intersection.*

(2) $\Leftrightarrow$ (3) **follows from** $I/I^2 \cong Y^-(\varphi^-)$, and we expect (3) $\Leftrightarrow$ (4) (a sort of modulo p Tate duality).

§**8. A key duality lemma** from the theory of dualizing modules by Grothendieck, Hartshorne and Kleiman in a simplest case:

Lemma 1 (Key lemma)**.** *Let S be a p-profinite* **Gorenstein integral domain** *and A be a* **reduced Gorenstein local S-algebra** *free of finite rank over S. Suppose*

- *A has a ring involution σ with $A_+ := \{a \in A | \sigma(a) = a\}$,*
- *A_+ is Gorenstein,*
- $\mathrm{Frac}(A)/\mathrm{Frac}(A_+)$ *is étale quadratic extension.*
- $\mathfrak{d}^{-1}_{A/A_+} := \{x \in \mathrm{Frac}(A) | \mathrm{Tr}_{A/A_+}(xA) \subset A_+\} \supsetneq A$,

Then A is free of rank 2 over A_+ and $A = A_+ \oplus A_+\delta$ for an element $\delta \in A$ with $\sigma(\delta) = -\delta$.

Lemma 2. *Let S be a* **Gorenstein local ring***. Let A be a local Cohen–Macaulay ring and is an S-algebra with $\dim A = \dim S$. If A is an S-module of finite type, the following conditions are equivalent:*

- *The local ring A is Gorenstein;*
- *$A^\dagger := \mathrm{Hom}_S(A, S) \cong A$ as A-modules.*

§9. We can apply the key lemmas to $\mathbb{T}^{\mathsf{ncm}}$: (1) $\Leftrightarrow$ (2).
(1)$\Rightarrow$(2): $\mathbb{T}^{\mathsf{ncm}}$ and $\mathbb{T}_{+}^{\mathsf{ncm}}$ are local complete intersections by assumption; so, Gorenstein.

Use of Main conjecture: The proof of the anti-cyclotomic Main conjecture by Mazur–Tilouine (combined with a theorem of Tate on Gorenstein rings [MFG, Lemma 5.21]) shows

$\mathbb{T}^{\mathsf{ncm}}/\mathfrak{d}_{\mathbb{T}^{\mathsf{ncm}}/\mathbb{T}_{+}^{\mathsf{ncm}}} \cong W[[\Gamma_{-}]]/(L_p^{-}(\varphi^{-}))$ (Congruence module identity);

so, $\mathfrak{d}_{\mathbb{T}^{\mathsf{ncm}}/\mathbb{T}_{+}^{\mathsf{ncm}}} \subset \mathfrak{m}_{\mathbb{T}^{\mathsf{ncm}}}$ for the anti-cyclotomic Katz p-adic L-function $L_p^{-}(\varphi^{-})$. The key lemma tells us (2).

(2)$\Rightarrow$(1): We have $I = (\theta) \subset \mathbb{T}^{\mathsf{ncm}}$ and $I_{+} = (\theta^2) \subset \mathbb{T}^{\mathsf{ncm}}$.

Note that $\mathbb{T}^{\mathsf{ncm}}/(\theta) \cong W[[\Gamma_{-}]]/(L_p(\varphi^{-})) \cong \mathbb{T}_{+}^{\mathsf{ncm}}/(\theta^2)$. Since θ is a non-zero divisor, the two rings $\mathbb{T}^{\mathsf{ncm}}$ and $\mathbb{T}_{+}^{\mathsf{ncm}}$ are local complete intersection.

§10. Presentation of $\mathbb{T}$.

To see a possibility of applying the key lemma to $\mathbb{T}/\mathbb{T}_+$, we like to lift $\mathbb{T}$ to a power series ring $\mathcal{R} = \Lambda[[T_1, \ldots, T_r]]$ with an involution σ_∞ such that $\mathcal{R}^+ := \{r \in \mathcal{R} | \sigma_\infty(r) = -r\}$ is Gorenstein and that $(\mathcal{R}/\mathfrak{A}, \sigma_\infty \bmod \mathfrak{A}) \cong (\mathbb{T}, \sigma)$ for an ideal $\mathfrak{A}$ stable under σ_∞.

Taylor and Wiles (with a later idea of Diamond and Fujiwara) found a pair $(\mathcal{R} := \Lambda[[T_1, \ldots, T_r]], (S_1, \ldots, S_r))$ with a regular sequence $S := (S_1, \ldots, S_r) \subset \Lambda[[T_1, \ldots, T_r]])$ such that

$$\Lambda[[T_1, \ldots, T_r]]/(S_1, \ldots, S_r) \cong \mathbb{T}$$

by their Taylor–Wiles system argument.

We need to lift σ somehow to an involution $\sigma_\infty \in \mathrm{Aut}(\mathcal{R})$ and show also that $\mathcal{R}^+$ is Gorenstein. If further $\mathfrak{d}_{\mathcal{R}/\mathcal{R}^-} \subset \mathfrak{m}_{\mathcal{R}}$, $\mathcal{R} \cdot \mathcal{R}^- = (\delta_\infty)$ and the image $\delta \in \mathbb{T}^-$ of δ_∞ in $\mathbb{T}$ generates I as desired.

§**11. Taylor–Wiles method.** Taylor–Wiles found an integer $r > 0$ and an infinite sequence of r-sets $\mathcal{Q} := \{Q_m | m = 1, 2, \dots\}$ of primes $q \equiv 1 \mod p^m$ such that for **the local ring** $\mathbb{T}^{Q_m}$ of $\overline{\rho}$ of the Hecke algebra $\mathbf{h}^{Q_m}$ of tame-level $N_m = N \prod_{q \in Q_m} q$. The pair $(\mathbb{T}^{Q_m}, \rho_{\mathbb{T}^{Q_m}})$ is universal among deformation satisfying (D1–4) but ramification at $q \in Q_m$ is allowed. Then $\rho \mapsto \rho \otimes \chi$ induces an involution σ_{Q_m} and $\mathbb{T}_+^{Q_m} := \{x \in \mathbb{T}^{Q_m} | \sigma_{Q_m}(x) = x\}$ is Gorenstein.

Actually they work with $\mathbb{T}_{Q_m} = \mathbb{T}^{Q_m}/(t - \gamma^k)\mathbb{T}^{Q_m}$ ($t = 1 + T$, $\gamma = 1 + p \in \Gamma$; the weight k Hecke algebra of weight $k \geq 2$ fixed). The product inertia group $I_{Q_m} = \prod_{q \in Q_m} I_q$ acts on $\mathbb{T}_{Q_m}$ by the p-abelian quotient Δ_{Q_m} of $\prod_{q \in Q_m} (\mathbb{Z}/q\mathbb{Z})^\times$. We choose an ordering of primes $Q_m = \{q_1, \dots, q_r\}$ and a generator $\delta_{i,m(n)}$ of the p-Sylow group of $(\mathbb{Z}/q_i\mathbb{Z})^\times$. The sequence $\mathcal{Q}$ is chosen so that for a given integer $n > 0$, we can find $m = m(n) > n$ so that we have ring projection maps $R_{n+1} \to R_n := \mathbb{T}_{Q_{m(n)}}/(p^n, \delta_{i,m(n)}^{p^n} - 1)_i$, and $R_\infty = \varprojlim_n R_n \cong W[[T_1, \dots, T_r]]$ and $S_i = \varprojlim_n (\delta_{i,m(n)} - 1)$.

§12. Lifting involution.

Write $\overline{\mathcal{S}}_n$ for the image of $W[[S]]$ for $S = (S_1, \ldots, S_r)$ in R_n ($\overline{\mathcal{S}}_n$ is a Gorenstein local ring). We can add the involution to this projective system and an R_n-**linear isomorphism** $\phi_n : R_n^\dagger := \mathrm{Hom}_{\overline{\mathcal{S}}_n}(R_n, \overline{\mathcal{S}}_n) \cong R_n$ commuting with the involution σ_n of R_n induced by $\sigma_{Q_{m(n)}}$ to the Taylor-Wiles system, and get the lifting

$$\sigma_\infty \in \mathrm{Aut}(R_\infty)$$

$$\text{with} \quad \phi_\infty : R_\infty^\dagger := \mathrm{Hom}_{W[[S]]}(R_\infty, W[[S]]) \cong R_\infty$$

compatible with σ_∞; i.e., $\phi_\infty \circ \sigma_\infty = \sigma_\infty \circ \phi_\infty$. This shows

$$R_\infty^{+,\dagger} \cong R_\infty^+$$

as R_∞^+-modules, as desired. Then we can further lift involution to $\mathcal{R} = \Lambda[[T_1, \ldots, T_r]]$ as $\mathcal{R}/(t - \gamma^k) = R_\infty$ for $t = 1 + T$.

The remaining point of the key lemma I have not done is to show

$$\mathfrak{d}_{\mathcal{R}/\mathcal{R}^+} \subset \mathfrak{m}_{\mathcal{R}}?$$

§13. Index set of Q_m (towards (4) $\Leftrightarrow$ (2)).

Write $\mathcal{D}_q$ for the local version of the deformation functor associated to (D1–4) adding a fixed determinant condition

(det) $\det(\rho) = \nu^k\psi$ for the chosen $k \geq 2$ (the weight condition); so, the tangent spec of $\mathbb{T}$ is given by a Selmer group $\mathrm{Sel}(Ad)$ for $Ad = \mathfrak{sl}_2(\mathbb{F})$.

Then the index set of Q_m is any choice of $\mathbb{F}$-basis of a "dual" Selmer group. Regard $\mathcal{D}_q(\mathbb{F}[\epsilon])$ for the dual number ϵ as a subspace of $H^1(\mathbb{Q}_q, Ad)$ in the standard way: Thus we have the orthogonal complement $\mathcal{D}_q(\mathbb{F}[\epsilon])^\perp \subset H^1(\mathbb{Q}_q, Ad^*(1))$ under Tate local duality. The dual Selmer group $\mathrm{Sel}^\perp(Ad^*(1))$ is given by

$$\mathrm{Sel}^\perp(Ad^*(1)) := \mathrm{Ker}(H^1(\mathbb{Q}^{(Np)}/\mathbb{Q}, Ad^*(1)) \to \prod_{l|Np} \frac{H^1(\mathbb{Q}_l, Ad^*(1))}{\mathcal{D}_l(\mathbb{F}[\epsilon])^\perp}).$$

Then $r = \dim_{\mathbb{F}} \mathrm{Sel}^\perp(Ad^*(1))$.

§14. Interpretation of the dual Selmer group.

Define $Q_m^{\pm} := \{q \in Q_m | \chi(q) = \pm q\}$. Then if S_q is the variable in $W[[S]]$ corresponding from $q \in Q_m^{\pm}$, then $\sigma(1+S_q) = (1+S_q)^{\pm 1}$.

We have splitting $Ad = \overline{\chi} \oplus \mathrm{Ind}_F^{\mathbb{Q}} \overline{\varphi}^-$; so, $\mathrm{Sel}^{\perp}(Ad^*(1)) = \mathrm{Sel}^{\perp}(\overline{\chi}(1)) \oplus \mathrm{Sel}^{\perp}(\mathrm{Ind}_F^{\mathbb{Q}}(\overline{\varphi}^-(1)))$ and

$$\mathrm{Sel}^{\perp}(\mathrm{Ind}_F^{\mathbb{Q}}(\overline{\varphi}^-(1))) = \mathrm{Hom}_{W[[\Gamma_-]]}(Y^-(\varphi^-\omega), \mathbb{F}).$$

Thus the number $\mu_{W[[S]]_+}(W[[S]]_-)$ of generators of $W[[S]]_-$ over $W[[S]]_+$ is

$$\mu_{W[[S]]_+}(W[[S]]_-) = \dim_{\mathbb{F}} Y^-(\varphi^-\omega) \otimes_{W[[\Gamma_-]]} \mathbb{F}.$$

Writing a number of generators of an A-module M over A as $\mu_A(M)$, we thus have

$$\mu_{W[[S]]_+}(W[[S]]_-) = \mathrm{codim}_{\mathrm{Spec}(W[[S]])} \mathrm{Spec}(W[[S]])^{\sigma=1}.$$

§15. Generator count $\mu_{R^+_\infty}(R^-_\infty)$ of R^-_∞.

Lemma 3. *We have*

$$\begin{aligned}\mu_{R^+_\infty}(R^-_\infty) &= \mathrm{codim}_{\mathrm{Spec}(W[[S]])}\,\mathrm{Spec}(W[[S]])^{\sigma=1} \\ &= \dim_{\mathbb{F}} Y^-(\varphi^-\omega)\otimes_{W[[\Gamma_-]]}\mathbb{F}.\end{aligned}$$

In the construction of Taylor–Wiles system, for each $q \in Q_m$, an eigenvalue of $\overline{\rho}(\mathrm{Frob}_q)$ is chosen, which is equivalent to choose a factor $\mathfrak{q}|q$ if $q \in Q^+_m$.

Then $\prod_{q\in Q^+_m}(O_F/\mathfrak{q})^\times$ has p-Sylow subgroup $\Delta_{Q^+_m}$. The projective limit $\varprojlim_n \Delta_{Q_{m(n)}}/\Delta^{p^n}_{Q_{m(n)}}$ gives rise to a group isomorphic to $\boldsymbol{\Delta}_+ := \mathbb{Z}_p^{r+}$ for $r_+ = |Q^+_m|$.

§16. QED.

Let $I_\infty = R_\infty(\sigma - 1)R_\infty$, $I^Q = \mathbb{T}^Q(\sigma - 1)\mathbb{T}^Q$ and $H_Q = \Gamma_- \times \Delta_{Q+}$. By $\mathbb{T}^Q/I^Q \cong W[[H_Q]] = \mathbb{T}^Q_+/I^Q_+$, we get

$$R_\infty/I_\infty \cong W[[\mathbf{\Delta}_+]] \cong R^+_\infty/I^+_\infty.$$

Note that

$$\mathrm{Spec}(R_\infty)^{\sigma=1} = \mathrm{Spec}(R_\infty/I_\infty) = \mathrm{Spec}(W[[\mathbf{\Delta}_+]]).$$

Thus we get

$$\begin{aligned}\mu_{R^+_\infty}(R^-_\infty) &= \mathrm{codim}_{\mathrm{Spec}(W[[S]])} \mathrm{Spec}(W[[\mathbf{\Delta}_+]]) \\ &= r_- = \dim_{\mathbb{F}} Y^-(\varphi^-\omega) \otimes_{W[[\Gamma_-]]} \mathbb{F}.\end{aligned}$$

This shows the implication (4) $\Rightarrow$ (2) of Theorem A.