ANTICYCLOTOMIC CYCLICITY CONJECTURE

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Abstract. Let $F$ be an imaginary quadratic field. We formulate certain Gorenstein/local complete intersection property of subrings of the universal deformation ring $\mathfrak{T}$ of a mod $p$ induced representation of a character of $\text{Gal}(\overline{\mathbb{Q}}/F)$. These conditions provide a base to prove pseudo-cyclicity of the Iwasawa module over $\mathbb{Z}_p$-extensions of $F$. Under mild conditions, we realize this scheme and prove anticyclicotonic pseudo-cyclicity.

Fix a prime $p > 3$. We have the following conjecture due to Iwasawa (cf. [CPI, No.62 and U3]):

**Cyclotomic cyclicity conjecture:** Let $Q_\infty/Q$ be the unique $\mathbb{Z}_p$-extension. Let $X_\pm$ be the Galois group of the maximal $p$-abelian extension everywhere unramified over $Q(\mu_{p\infty})$ on which complex conjugation acts by $\pm 1$. For an odd character $\psi : \text{Gal}(Q(\mu_{p\infty})/Q_\infty) \to \mu_{p-1}(\mathbb{Z}_p)$, define $X_-(\psi) := X_- \otimes_{\mathbb{Z}_p}[\text{Gal}(Q(\mu_{p\infty})/Q_\infty)], \psi \mathbb{Z}_p$ (the $\psi$-eigenspace of $X_-$). Then identifying $\text{Gal}(Q(\mu_{p\infty})/Q) = \mathbb{Z}_p^\times = \mu_{p-1} \times \Gamma$ and regarding $X_-(\psi)$ as $\mathbb{Z}_p[\Gamma]$-module naturally, if $X_-(\psi) \neq 0$, $X_-(\psi)$ is pseudo isomorphic to $\mathbb{Z}_p[\Gamma]/(f_\psi)$ for a power series $f_\psi$ prime to $p\mathbb{Z}_p[\Gamma]$.

This conjecture asserts the cyclicity (up to finite error) of $X_-(\psi)$ as an Iwasawa module (i.e., having a single generator over the Iwasawa algebra $\mathbb{Z}_p[\Gamma]$). Under the assumption that $X_+ = 0$ (the Kummer–Vandiver conjecture), in [CPI, No.48], Iwasawa proved (along with his main conjecture) pure cyclicity without finite pseudo-null error. The fact $p \nmid f_\psi$ is a combination of the vanishing of the $\mu$-invariant of the Kubota–Leopoldt $p$-adic L-function (proven by Ferrero–Washington) and the proof of Iwasawa’s main conjecture by Mazur–Wiles. There are some positive results towards this conjecture via Galois deformation theory (e.g. [Ku93], [O03], [Wa15] and [WE15]), relating it to Ribet’s proof of the converse of Herbrand’s theorem, Iwasawa main conjecture, Sharifi’s conjecture, a generalized version of the Kummer–Vandiver conjecture (which sometimes fails) and a conjecture of Greenberg.

Let $F$ be an imaginary quadratic field with discriminant $-D$ and integer ring $O$. Assume that the prime $(p)$ splits into $(p) = \mathfrak{p}\mathfrak{P}$ in $O$ with $\mathfrak{p} \neq \mathfrak{P}$. Let $F_\infty/F$ be the anticyclotomic $\mathbb{Z}_p$-extension with Galois group $\Gamma_- := \text{Gal}(F_\infty/F)$; so, $csc = \sigma^{-1}$ for complex conjugation $c$ and $\sigma \in \Gamma_- \cong \mathbb{Z}_p$. Take a branch character $\phi : \text{Gal}(\overline{\mathbb{Q}}/F) \to \mathbb{Q}_p^\times$, fixing an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}_p$. Regard it as a finite order idele character $\phi : F_\infty^\times/F^\times \to \mathbb{Q}_p^\times$. Most of the time, we suppose that $\phi$ is anticyclicotonic; so, $\phi(x^n) = \phi^{-1}(x)$. For an anticyclicotonic $\phi$, we always find a finite order character $\varphi$ of $F_\infty^\times/F^\times$ such that $\phi = \varphi$ for $\varphi$ given by $\varphi^{-}(x) = \varphi(x)\varphi(x^{-1})^{-1}$ (e.g., [HMI, Lemma 5.31]). However, controlling the conductor of $\varphi$ is a difficult task. We write $\mathbb{Z}_p[\phi]$ for the subring of $\mathbb{Q}_p$ generated by the values of $\phi$ over $\mathbb{Z}_p$. Consider the anticyclicotonic Iwasawa algebra $\mathbb{Z}_p[\phi][[\Gamma_-]] = \lim_{\leftarrow n} \mathbb{Z}_p[\phi][\Gamma_-/\Gamma_-^n]$. Let $F(\phi)/F$ be the abelian extension cut out by $\phi$ (i.e., $F(\phi) = \mathbb{Q}_{\ker(\phi)}$). Let $Y^-$ be the Galois group of the maximal $p$-ramified $p$-abelian extension over the composite $F_\infty^\phi := F_\infty F(\phi)$. The word: “$p$-ramified” means that it is unramified outside the prime $p$. Since $\text{Gal}(F(\phi)/F)$ acts on $Y$ naturally as a factor of $\text{Gal}(F_\infty(\phi)/F)$, we have the $\phi$-eigenspace $Y^-(\phi) = Y^- \otimes_{\mathbb{Z}_p[\text{Gal}(F(\phi)/F), \phi]} \mathbb{Z}_p[\phi]$, where $\mathbb{Z}_p[\phi]$ is the $\mathbb{Z}_p[\phi]$-free module of rank 1 on which $\text{Gal}(F(\phi)/F)$ acts via $\phi$.

**Anticyclicotonic cyclicity conjecture:** Assume $\phi \neq 1$ and that the conductor of $\phi$ is a product of split primes over $Q$. If the class number of $F$ is prime to $p$ and $Y^-(\phi) \neq 0$, then the $\mathbb{Z}_p[\phi][[\Gamma_-]]$-module $Y^-(\phi)$ is pseudo isomorphic to $\mathbb{Z}_p[\phi][[\Gamma_-]]/(f_\phi)$ as $\mathbb{Z}_p[\phi][[\Gamma_-]]$-modules for an element $f_\phi \in \mathbb{Z}_p[\phi][[\Gamma_-]]$ prime to $p\mathbb{Z}_p[\phi][[\Gamma_-]]$.

We prove in this paper:

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Theorem A: Let the notation be as above. Assume that \( \phi = \varphi^- \) for the Teichmüller lift \( \varphi \) of a modulo \( p \) Galois character \( \varphi \) of prime-to-\( p \) conductor \( c \), and let \( N = DN_{F/Q}(c) \). Suppose

- (h0) \( p > 3 \) is prime to \( N \prod l | N (l - 1) \) for prime factors \( l \) of \( N \),
- (h1) \( \varphi \) is prime to \( D \), and \( N_{F/Q}(c) \) is square-free (so, \( N \) is cube-free),
- (h2) \( \varphi : \text{Gal}(\overline{\mathbb{Q}}/F) \to \mathbb{F}^\times \) is unramified outside \( \mathfrak{p} \) with Teichmüller lift \( \varphi \),
- (h3) \( NP \) is the conductor of \( \det(\mathfrak{p}) \) for \( \mathfrak{p} = \text{Ind}_F^Q \varphi \),
- (h4) \( \varphi^- \) has order at least 3.

If the class number of \( F \) is prime to \( p \), the anticyclotomic cyclicity conjecture holds.

The proof of this theorem is technical ring theoretic tools applied to a local ring of the big Hecke algebra of tame level \( N \). In this introduction, to give a short outline of our argument without going into technicality, let us state a typical theorem which describe ring-theoretic properties of the Hecke algebra equivalent to pure anti-cyclotomic cyclicity (i.e., without pseudo-null error) of \( Y^-(\varphi^-) \). As a base ring, we take a (sufficiently large) complete discrete valuation ring \( W \subset \mathbb{C}_p \) flat over the \( p \)-adic integer ring \( \mathbb{Z}_p \). Here \( \mathbb{C}_p \) is the \( p \)-adic completion of a fixed algebraic closure \( \overline{\mathbb{Q}}_p \) of \( \mathbb{Q}_p \) under its norm \( | \cdot |_p \) normalized so that \( |p|_p = 1/p \). We identify the Iwasawa algebra \( \Lambda = W[\Gamma] \) with the one variable power series ring \( W[T] \), and \( \Gamma \geq \gamma = (1 + p) \mapsto 1 + T \in \Lambda \). Take a Dirichlet character \( \psi : (\mathbb{Z}/NP\mathbb{Z})^\times \to W^\times \), and consider the big ordinary Hecke algebra \( \mathfrak{h} \) (over \( \Lambda \)) of prime-to-\( p \) level \( N \) and the character \( \psi \) whose definition (including its CM components) will be recalled in the following section. We just mention here the following three facts

1. \( \mathfrak{h} \) is an algebra flat over \( \Lambda \) interpolating \( p \)-ordinary Hecke algebras of level \( NP^{r+1} \), of weight \( k + 1 \geq 2 \) and of character \( \epsilon \psi \omega^{-k} \) for the Teichmüller character \( \omega \), where \( \epsilon \in \mathbb{Z}_p^\times \to \mu_{p^r} \) \((r \geq 0)\) and \( k \geq 1 \) vary. If \( N \) is cube-free, \( \mathfrak{h} \) is a reduced algebra [H13, Corollary 1.3];
2. Each prime \( P \in \text{Spec}(\mathfrak{h}) \) has a unique (continuous) Galois representation \( \rho_P : \text{Gal}(\overline{\mathbb{Q}}/Q) \to \text{GL}_2(\kappa(P)) \) for the residue field \( \kappa(P) \) of \( P \);
3. \( \rho_P \) restricted to \( \text{Gal}(\overline{\mathbb{Q}}_p/Q_p) \) (the \( p \)-decomposition group) is isomorphic to an upper triangular representation whose quotient character is unramified.

By (2), each local ring \( \mathbb{T} \) has a mod \( p \) representation \( \overline{\mathfrak{p}} = \rho_{\text{red}} : \text{Gal}(\overline{\mathbb{Q}}/Q) \to \text{GL}_2(\mathbb{F}) \) for the residue field \( \mathbb{F} = \mathbb{F}/\mathfrak{m}^p \). If \( \overline{\mathfrak{p}} = \text{Ind}_F^Q \overline{\varphi} \) for the reduction \( \overline{\varphi} \) modulo \( \varphi \), we have an involution \( \sigma \in \text{Aut}(\mathbb{T}/\Lambda) \) such that \( \sigma \circ \rho_P \cong \rho_P \otimes \chi \) for \( \chi := (\mathbb{F}/\mathbb{Q}) \). For a subscheme \( \text{Spec}(A) \subset \text{Spec}(\mathbb{T}) \) stable under \( \sigma \), we put \( A_+ :=\{x \in A | \sigma(x) = x \} \). Then \( A_+ \subset A \) is a subring and \( A_+ \) is an \( A_+ \)-module.

Let \( Q \) be a finite set of rational primes in \( F/Q \) prime to \( Np \). Let \( Q^+ \) be the subset of primes in \( Q \) split in \( F \). Write \( K_Q \) for the ray class field over \( \mathbb{F} \). If \( N \geq 2 \), then \( K_Q = \text{Gal}(\overline{\mathbb{Q}}/Q) \). For a subscheme \( A \subset \text{Spec}(\mathbb{T}) \) stable under \( \sigma \), we put \( A_+ :=\{x \in A | \sigma(x) = x \} \). Then \( A_+ \subset A \) is a subring and \( A_+ \) is an \( A_+ \)-module.

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Theorem B: Let \( \text{Spec}(\mathbb{T}) \) be a connected component of \( \text{Spec}(\mathfrak{h}) \) associated to the induced Galois representation \( \overline{\varphi} = \text{Ind}_F^Q \overline{\varphi} \) for the reduction \( \overline{\varphi} \) modulo \( p \) of \( \varphi \). Suppose (h0–4) as in Theorem A. Then the class number of \( F \) is prime to \( p \), then, the following four statements are equivalent:

1. The rings \( \mathbb{T}_{\text{ncm}}^+ \) and \( \mathbb{T}_{\text{ncm}}^- \) are both local complete intersections free of finite rank over \( \Lambda \).
2. The \( \mathbb{T}_{\text{ncm}}^- \)-ideal \( \mathfrak{I} = \mathfrak{I} \sigma \mathbb{T}_{\text{ncm}}^+ \subset \mathbb{T}_{\text{ncm}}^- \) is principal and is generated by a non-zero-divisor \( \theta \in \mathbb{T}_{\text{ncm}}^- \) with \( \theta^2 \in \mathbb{T}_{\text{ncm}}^- \). The element \( \theta \) generates a free \( \mathbb{T}_{\text{ncm}}^- \)-module \( \mathbb{T}_{\text{ncm}}^- \), and \( \mathbb{T}_{\text{ncm}}^- = \mathbb{T}_{\text{ncm}}^- + \theta \) is free of rank 2 over \( \mathbb{T}_{\text{ncm}}^- \).
3. The Iwasawa module \( Y^-(-\varphi^-) \) is cyclic over \( W[\Gamma_-] \).
4. The Iwasawa module \( Y^-(\varphi^-) \) is cyclic over \( W[\Gamma_-] \).

Under these equivalent conditions, the ring \( \mathbb{T}_{\text{ncm}}^- \) is a local complete intersection.
The condition $p \nmid h_F$ could be an analogue of Iwasawa’s assumption $X_+ = 0$, and the cyclotomic cyclicity and anti-cyclotomic cyclicity could be closely related (as pointed out to the author by P. Wake). We actually do not assume $p \nmid h_F$ in the main text and prove a result equivalent (under $p \nmid h_F$) to the above theorem (see Theorem 5.4) replacing the assertions (3) and (4) by suitably modified statements, and if the statements for $T$ fail in this general setting, $Y^-(\phi)$ for $\phi = \varphi^-, \varphi^- \omega$ may not be cyclic over $W[[\Gamma_-]]$. The fact that $f_\phi$ in the conjecture is prime to $p\mathbb{Z}_p[\phi][[\Gamma_-]]$ follows from the vanishing of the $\mu$-invariant of the anti-cyclotomic Katz $p$-adic $L$-function [H10] (and [EAI, Theorem 3.37]) and the proof of the main conjecture by Rubin [Ru88], [Ru91], Tilouine [T89], Mazur [MT90] (and the author [H06]).

A slightly stronger and detailed version of Theorem B will be proven as Theorem 5.4 (and Corollary 2.5). The proof of equivalence of the assertion (4) and the rest of Theorem 5.4 relies on a new type of the Taylor–Wiles system argument proving Theorem 4.10 in Section 4 (and on the theory of relative dualizing modules of Grothendieck–Hartshorne–Kleiman recalled in Section 11). The Taylor–Wiles system is made of the deformation rings $R_Q$ of $F$ and the corresponding local rings $T_Q$ of the Hecke algebras (of level $N_Q := N \prod_{p \in Q} q$) allowing ramification at primes in $Q$ (for a suitably chosen infinite sequence of finite sets $Q$ of primes $q$ with $q \equiv 1 \mod p$; see Section 4 and [TW95]).

Here is a sketch of the proof of the equivalence of (2) $\iff$ (3) in Theorem B. For any commutative ring $A$, we write $\text{Frac}(A)$ for the total quotient ring of $A$ (i.e., $\text{Frac}(A)$ is the ring of fractions inverting all non-zero-divisors of $A$). We simply write $\mathbb{K}$ for $\text{Frac}(\Lambda)$. As is well known, under (h1), $\text{Frac}(T)$ can be decomposed as an algebra direct sum $\text{Frac}(W[[H]]) \oplus X$ in a unique way. Write $\mathbb{T}^{\text{ncm}}$ for the projected image of $T$ in $X$. Then we have $I \rightarrow \mathbb{T}^{\text{ncm}}$, and via the deformation theoretic technique of Mazur–Tilouine [MT90] (see also [H16, §6.3.6]), we show that $Y^-(\varphi^-) \otimes_{\mathbb{Z}_p} W$ is isomorphic to $I/IF^2$ (by an old formula in [H86c, Lemma 1.1]). Assume that the class number $h_F$ of $F$ is prime to $p$. Then the projection of $H$ to $\Gamma_-$ is an isomorphism. By the proof of the anticyclotomic main conjecture in [T89], [MT90] and [H06], for the Katz $p$-adic $L$-function $L_p^\phi(\varphi^-)$ with branch character $\varphi^-$ giving the characteristic ideal of $Y^-(\varphi^-)$, we have $W[[\Gamma_-]]/(L_p^\phi(\varphi^-)) \cong \mathbb{T}^{\text{ncm}}/I$ (which also shows that the generator of $I$ is a non-zero-divisor of $\mathbb{T}^{\text{ncm}}$). Since $I$ is principal generated by a non-zero divisor, we have $I/IF \cong \mathbb{T}^{\text{ncm}}/I \cong W[[\Gamma_-]]/(L_p^\phi(\varphi^-))$, getting the anticyclotomic cyclicity conjecture. If $H \rightarrow \Gamma_-$ has non-trivial kernel (which implies $p|h_F$), Theorem 5.4 tells us that $Y^-(\varphi^-) \otimes_{\mathbb{Z}_p} W$ is not cyclic over $W[[\Gamma_-]]$.

To reach (2) $\iff$ (4) in Theorem B, following the techniques of [H98] and [CV03], we construct an involution $\sigma$ of $T$ (Corollary 2.3). By Taylor–Wiles [TW95], $T$ is known to be a local complete intersection over $\Lambda$ (so, is Gorenstein over $\Lambda$). Adding to the data of the Taylor–Wiles system the involution $\sigma$ coming from the twist by $\chi = \left(\frac{\mathbb{L}/\mathbb{Q}}{p}\right)$, we argue in the same way as Taylor and Wiles did.

The limit ring $R$ (the system produced) is a power series ring over $\Lambda$ with the induced involution $\sigma$, and the ring $R_+$ fixed by involution is proven to be Gorenstein. By the theory of dualizing modules/sheaves for Gorenstein covering $X \rightarrow Y$ (studied by A. Grothendieck [SGA 2.VI–V], R. Hartshorne [RDD] and S. Kleiman [K80]), this is close to the cyclicity of $R_+ = \{x \in R | \sigma(x) = -x\}$ over $R_+$ (see Lemma 11.4), but we are bit short of proving it. Instead, we prove that the number of generator of $R_-$ over $R_+$ is actually given by the number of generators of $Y^-(\varphi^- \omega)$ over $W[[\Gamma_-]]$ via a refinement of the original Taylor–Wiles argument. Since $T_- = \{x \in T | \sigma(x) = -x\}$ is the surjective image of $R_-$, it is generated over $\mathbb{T}_+ = \{x \in T | \sigma(x) = x\}$ by a single element which is a generator of $I$, and essentially (4) $\iff$ (2).

The Gorenstein-ness of the rings $\mathbb{T}^{\text{ncm}}_+$ and $\mathbb{T}^{\text{ncm}}$ (i.e., (1)) implies (2) by Lemma 11.4 in the theory of dualizing modules. The identity $\mathbb{T}^{\text{ncm}}/(\theta) \cong \mathbb{T}^{\text{ncm}}_+/(\theta^2) \cong W[[H]]/(L_p^\phi(\varphi^-))$ tells us that $\mathbb{T}^{\text{ncm}}_+$ and $\mathbb{T}^{\text{ncm}}$ are actually local complete intersections; so, (2) $\Rightarrow$ (1).

The same ring theoretic analysis can be also done for a real quadratic field $F$, as the conditions (h0–4) do make sense for real $F$. We hope to come back to this problem for real quadratic fields in our future work. An example of $T \neq \Lambda$ given in [H85] is for $F = \mathbb{Q}[\sqrt{-3}]$, $p = 13$ and $N = 3$. This prime 13 is an irregular prime for $\mathbb{Q}[\sqrt{-3}]$ in the sense of [H82] and in the list [H81, (8.11)]. Of course, as easily checked (from the numerical values given in [H85]) the equivalent conditions of the theorem, and actually (the distinguished factor of) $L_p^\phi(\varphi^-)$ is a linear polynomial in this case.

The condition (h4) implies an assumption for “$R = T$” theorems of Wiles et al [Wi95] and [TW95].
\[ \text{(W) } \mathfrak{p} \text{ restricted to } \text{Gal}(\overline{\mathbb{Q}}/\mathbb{M}) \text{ for } M = \mathbb{Q}[\sqrt{(-1)^{(p-1)/2}p}] \text{ is absolutely irreducible, and the main reason for us to assume (h4) is the use of the “} R = T \text{” theorem for the minimal deformation ring } R \text{ of } \mathfrak{p} \text{ (see Theorem 2.1). The condition (W) is equivalent to the condition that the representation } \mathfrak{p} \text{ is not of the form } \text{Ind}_{\mathbb{Q}}^{\mathbb{H}} \xi \text{ for a character } \xi : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{M}) \to \mathbb{F}^\times \text{ from Frobenius reciprocity. The implication: (h4) } \Rightarrow \text{ (W) follows from [H15, Proposition 5.2]. Actually (W) also follows from the following condition: (h5) } \mathfrak{p}^{-1} \mathfrak{p} \text{ ramifies at a prime factor } l | N. \]

Indeed, if \( \mathfrak{p} = \text{Ind}_{\mathbb{Q}}^{\mathbb{H}} \xi \) for another quadratic field \( K \neq F \), by \cite[Proposition 5.2 (2)]{H15}, \( KF \) is uniquely determined degree 4 extension of \( \mathbb{Q} \) by \( \mathfrak{p} \), and the prime \( l \) in (h5) ramifies in \( KF/F \) as \( \mathfrak{p}|_{\mathfrak{I}_l} = \mathfrak{I}_l \oplus \mathfrak{I}_l \) for the inertia group \( \mathfrak{I}_l \subset \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}_l) \) with unramified \( \mathfrak{I}_l \). This is impossible if \( K = M \) as only \( p \) ramifies in \( M/\mathbb{Q} \). Because of this, in near future, we hope to prove Theorem A assuming (h5) (i.e., \( \epsilon \neq 0 \)) in place of assuming the order of \( \varphi^- \) has at least 3 (though this latter assumption is used for some other reason than (W) in this paper).

Since \( \mathfrak{p}|_{\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}_p)} \cong \tau \oplus \overline{\tau} \) with \( \overline{\tau} \) unramified, by (h2–3), \( \tau \) has to ramify at \( p \) (as \( F/\mathbb{Q} \) is unramified at \( p \)), and hence we conclude

\begin{align*}
(\text{Rg}) & \quad \overline{\tau} \neq \tau, \\
(\text{Rm}) & \quad \varphi^\sim \text{ in Theorems A and B ramifies at } p.
\end{align*}

Without (Rm), the choice of the \( p \)-unramified quotient \( \overline{\tau} \) of \( \mathfrak{p} \) is not uniquely determined: so, we have two different universal Hecke rings \( \mathbb{T} = \mathbb{T} \) for \( ? = \mathfrak{p} \) or \( \mathfrak{p}^\sim \) insisting \( \overline{\tau} = ? \) (the problem coming from the existence of companion forms giving rise to the same \( \mathfrak{p} \) but a different unramified quotient ? assigned). So, \( Y^-(\varphi^-) \otimes_{\mathbb{Z}_p[\varphi^-]} \mathbb{Q} \) is not exactly isomorphic to \( I_l/I_l^2 \) for the ideal \( I_l \subset \mathbb{T}_l \). Here \( I_l = I \) as above depends on the choice \( \mathbb{T} = \mathbb{T} \). We hope to study this more complicated case carefully in a future article. We freely therefore use (Rg) and (Rm) as we assume (h1–3) and (W) throughout the paper.

Here is a brief outline of the paper. In Section 1, we recall the theory of big ordinary Hecke algebras, paying particular attention to the Hecke algebra \( h^Q \) of auxiliary \( Q \)-level used to construct Taylor–Wiles systems and its CM components \( W[[H_Q]] \) as their residue rings. In Section 2, we recall the original \( R = \mathbb{T} \) theorem proven by Taylor–Wiles, and in Section 3, we recall some technical details of the Taylor–Wiles argument and discuss the relation of \( Y^-(\varphi^-) \) and \( \mathcal{R}_- \). In Section 4, we prove a sufficient condition for the local intersection property of the subring \( \mathbb{T}_+ \) of \( R = \mathbb{T} \) fixed by the involution \( \sigma \), employing the method of Taylor–Wiles adding the datum of the involution. In the following Section 5, we prove a finer version of Theorem B, applying the result of Section 4 to a residual representation induced from an imaginary quadratic field. In Section 6, we show that the minus part of the cotangent space of \( Q \)-ramified Hecke algebra \( \mathbb{T}_Q \) is generated by \( T(q) \) for primes \( q \) in \( F/\mathbb{Q} \) inert in \( F/\mathbb{Q} \). By a Selmer group computation, we show that the number \( r_\sim \) of such inert primes in \( Q \) is equal to the minimum number of generators of \( Y^-(\varphi^-) \) over the Iwasawa algebra. In Section 7, we prove Theorem A, introducing more \( r_\sim \)-local involutions acting on \( \mathbb{T} \). To have such involutions, the choice of generators given in Section 6 (Theorem 6.4) is crucial. These extra involutions fix subring smaller than \( \mathbb{T}_+ \) if \( r_\sim > 1 \), and studying intricate relations among eigenspaces of the involutions of \( \Omega_{\mathbb{T}/\mathcal{A}} \), we reach the proof of Theorem A. In Section 8, we show by a control theorem of Rubin that cyclicity (not pseudo cyclicity) of \( Y^-(\varphi^-) \) implies cyclicity of the corresponding Iwasawa module for any \( \mathbb{Z}_p \)-extension \( K/F \). In Section 9, we study CM irreducible components when the class number of the CM imaginary quadratic field is divisible by \( p \) and shows that the component is often far larger than the weight Iwasawa algebra \( \mathcal{A} \). In Section 10, we explore the close relation of a generator of the ideal \( I \) and the adjoint \( p \)-adic L-function. In the final section, we gather purely ring theoretic results on Gorenstein local rings and their duality theory used in the proofs of our main results.

Throughout this paper, we write \( \overline{\mathbb{Q}} \) (resp. \( \overline{\mathbb{Q}}_p \)) for an algebraic closure of \( \mathbb{Q} \) (resp. \( \mathbb{Q}_p \)) and fix embeddings \( \overline{\mathbb{Q}}_p \overset{i_p}{\rightarrow} \overline{\mathbb{Q}} \overset{i}{\rightarrow} \mathbb{C} \). We write \( \mathbb{C}_p \) for the \( p \)-adic completion of \( \overline{\mathbb{Q}}_p \). A number field is a subfield of \( \mathbb{C} \) by a fixed embedding. For each local ring \( A \), we write \( \mathfrak{m}_A \) for the maximal ideal of \( A \). For any profinite abelian group \( G \), we write \( W[G] \) for its group algebra, and put \( W[[G]] = \lim_{\leftarrow H} W[G/H] \) for \( H \) running over all open subgroups of \( G \); so, \( W[[G]] = W[G] \) is \( G \) is finite.
The author would like to thank R. Greenberg and R. Sharifi. Greenberg pointed out the missing hidden assumption (Rm) of Theorem A in the first draft of this paper by counter-examples presumably exist. Sharifi read the paper carefully and suggested many improvements. The author also appreciates the comments on the results of the paper made by P. Wake.

1. Big Hecke algebra

We recall the theory of $\mathfrak{h}$ to the extent we need. We assume that the starting prime-to-$p$ level $N$ is as in (h1); in particular, $N$ is cube-free and its odd part is square-free. We assume that the base discrete valuation ring $W$ flat over $\mathbb{Z}_p$ is sufficiently large so that its residue field $F_p$ is equal to $\mathbb{T}/m_{\mathbb{T}}$ for the maximal ideal of the connected component $\text{Spec}(\mathbb{T})$ (of our interest) in $\text{Spec}(\mathfrak{h})$.

The base ring $W$ may not be finite over $\mathbb{Z}_p$. For example, if we deal with Katz $p$-adic L-functions, the natural ring of definition is the Witt vector ring $W(F_p)$ of an algebraic closure $\mathbb{F}_p$ (realized in $\mathbb{C}_p$), though the principal ideal generated by a branch of the Katz $p$-adic L-function descends to an Iwasawa algebra over a finite extension $W$ of $\mathbb{Z}_p$ (and in this sense, the reader may assume finiteness over $\mathbb{Z}_p$ of $W$ just to understand our statement as it only depends on the ideal in the Iwasawa algebra over $W$).

We consider the following traditional congruence subgroups

$\Gamma_0(Np^r) := \{ (a \ b) \in SL_2(\mathbb{Z})|c \equiv 0 \mod Np^r \},$

$\Gamma_1(Np^r) := \{ (a \ b) \in \Gamma_0(Np^r)|d \equiv 1 \mod Np^r \}.$

A $p$-adic analytic family $\mathcal{F}$ of modular forms is defined with respect to the fixed embedding $i_p: \mathbb{T} \hookrightarrow \mathbb{C}_p$. We write $|\alpha|_p (\alpha \in \mathbb{T})$ for the $p$-adic absolute value (with $|p|_p = 1/p$) induced by $i_p$. Take a Dirichlet character $\psi: (\mathbb{Z}/Np^r\mathbb{Z})^\times \to W^\times$ with $(p \mid N, r \geq 0)$, and consider the space of elliptic cusp forms $S_{k+1}(\Gamma_0(Np^{r+1}), \psi)$ with character $\psi$ as defined in [IAT, (3.5.4)].

For our later use, we pick a finite set of primes $Q$ outside $Np$. We define

$\Gamma_0(Q) := \{ (a \ b) \in SL_2(\mathbb{Z})|c \equiv 0 \mod q \text{ for all } q \in Q \},$

$\Gamma_1(Q) := \{ (a \ b) \in \Gamma_0(Q)|d \equiv 1 \mod q \text{ for all } q \in Q \}.$

Let $\Gamma_Q^{(p)}$ be the subgroup of $\Gamma_0(Q)$ containing $\Gamma_1(Q)$ such that $\Gamma_0(Q)/\Gamma_Q^{(p)}$ is the maximal $p$-abelian quotient of $\Gamma_0(Q)/\Gamma_1(Q) \cong \prod_{q \in Q}(\mathbb{Z}/q\mathbb{Z})^\times$. We put

$\Gamma_{Q,r} := \Gamma_Q^{(p)} \cap \Gamma_0(Np^r),$

and we often write $\Gamma_Q$ for $\Gamma_{Q,r}$ when $r$ is well understood (mostly when $r = 0, 1$). Then we put

$\Delta_Q := (\Gamma_0(Np^r) \cap \Gamma_0(Q))/\Gamma_{Q,r},$

which is canonically isomorphic to the maximal $p$-abelian quotient of $\Gamma_0(Q)/\Gamma_1(Q)$ independent of the exponent $r$. If $Q = \emptyset$, we have $\Gamma_{Q,r} = \Gamma_0(Np^r)$, and if $q \neq 1 \mod p$ for all $q \in Q$, we have $\Gamma_1(NQp^r) \subset \Gamma_{Q,r} = \Gamma_0(NQp^r)$ for $NQ := N \prod_{q \in Q} q$. 

1. Contents

1. Big Hecke algebra
2. The $R = \mathbb{T}$ theorem and an involution of $R$
3. The Taylor–Wiles system and Taylor–Wiles primes
4. A sufficient condition for complete intersection property for $R_L$
5. Proof of Theorem B
6. A good choice of generators of $R_Q$ over $W$
7. Proof of Theorem A and local involutions
8. Cyclicity for a $Z_p$-extension $K/F$
9. Degree of CM components over the Iwasawa algebra
10. Divisibility of the adjoint $p$-adic L-function
11. Dualizing modules

References
Let the ring $\mathbb{Z}[\psi] \subset \mathbb{C}$ and $\mathbb{Z}_p[\psi] \subset \overline{\mathbb{Q}}_p$ be generated by the values $\psi$ over $\mathbb{Z}$ and $\mathbb{Z}_p$, respectively. The Hecke algebra over $\mathbb{Z}[\psi]$ is the subalgebra of the $\mathbb{C}$-linear endomorphism algebra of $S_{k+1}(\Gamma_{\infty}, \psi)$ generated over $\mathbb{Z}[\psi]$ by Hecke operators $T(n)$:

$$h = \mathbb{Z}[\psi][T(n)] | n = 1, 2, \ldots | \subset \text{End}_\mathbb{C}(S_{k+1}(\Gamma_{\infty}, \psi)),$$

where $T(n)$ is the Hecke operator as in [IAT, §3.5]. We put

$$h_{Q,k,\psi/W} = h_{(\Gamma_{\infty}, \psi; W)} := h \otimes_{\mathbb{Z}[\psi]} W.$$

Here $h_k(\Gamma_{\infty}, \psi; W)$ acts on $S_{k+1}(\Gamma_{\infty}, \psi; W)$ which is the space of cusp forms defined over $W$ (under the rational structure induced from the $q$-expansion at the infinity cusp; see, [MFG, §3.1.8]).

More generally for a congruence subgroup $\Gamma$ containing $\Gamma_1(Np^r)$, we write $h_k(\Gamma, \psi; W)$ for the Hecke algebra on $\Gamma$ with coefficients in $W$ acting on $S_{k+1}(\Gamma, \psi; W)$. The algebra $h_k(\Gamma, \psi; W)$ can be also realized as $W[T(n)] | n = 1, 2, \ldots | \subset \text{End}_W(S_{k+1}(\Gamma, \psi; W))$. When we need to indicate that our $T(l)$ is the Hecke operator of a prime factor $l$ of $Np^r$, we write it as $U(l)$, since $T(l)$ acting on a subspace $S_{k+1}(\Gamma_0(N^r), \psi) \subset S_{k+1}(\Gamma_0(Np^r), \psi)$ of level $N^r|Np^r$ prime to $l$ does not coincide with $T(l)$ on $S_{k+1}(\Gamma_0(Np^r), \psi)$. The ordinary part $h_{Q,k,\psi/W} \subset h_{Q,k,\psi/W}$ is the maximal ring direct summand on which $U(p)$ is invertible. If $Q = \emptyset$, we simply write $h_{k,\psi/W}$ for $h_{Q,k,\psi/W}$. We write $e$ for the idempotent of $h_{Q,k,\psi/W}$, and hence $e = \lim_{n \to \infty} U(p)^n$ under the $p$-adic topology of $h_{Q,k,\psi/W}$.

Fix a character $\psi_0$ modulo $Np$, and assume now $\psi_0(-1) = -1$. Let $\omega$ be the module $\psi$ Teichmüller character. Recall the multiplicative group $\Gamma := 1 + p\mathbb{Z}_p \subset \mathbb{Z}_p^\times$ and its topological generator $\gamma = 1 + p$. Then the Iwasawa algebra $\Lambda = W[[\Gamma]] = \lim_{n} W[T/\Gamma^n]$ is identified with the power series ring $W[[T]]$ by a $W$-algebra isomorphism sending $\gamma \in \Gamma$ to $t := 1 + T$. As constructed in [H86a], [H86b] and [GME], we have a unique ‘big’ ordinary Hecke algebra $h^Q$ (of level $\Gamma_\infty$). We write $h$ for $h^Q$.

Since $Np = DN_{\ell/\mathbb{Q}}(\ell)p \geq Dp > 4$, the algebra $h^Q$ is characterized by the following two properties (called Control theorems; see [H86a] Theorem 3.1, Corollary 3.2 and [H86b, Theorem 1.2] for $p > 5$ and [GME, Corollary 3.2.22] for general $p$):

1. $h^Q$ is free of finite rank over $\Lambda$ equipped with $T(n) \in h^Q$ for all $1 \leq n \in \mathbb{Z}$ prime to $Np$ and $U(l)$ for prime factors $l$ of $Np$.
2. If $k \geq 1$ and $\epsilon : \mathbb{Z}_p^\times \to \mu_{p^{\infty}}$ is a finite order character,

$$h^Q/\langle t - \epsilon(\gamma)\gamma^k \rangle h^Q \cong h_{Q,k,\epsilon \psi_0} (\gamma = 1 + p)$$

sending $T(n)$ to $T(n)$ (and $U(l)$ to $U(l)$) for $l|Np$.

Actually a slightly stronger fact than (C1) is known:

**Lemma 1.1.** The Hecke algebra $h^Q$ is flat over $\Lambda[\Delta_Q]$ with $h^Q/\mathfrak{A}_{\Delta_Q}h^Q \cong h^0$ for the augmentation ideal $\mathfrak{A}_{\Delta_Q} \subset \Lambda[\Delta_Q]$.

See [H89, Lemma 3.10] and [MFG, Corollary 3.20] for a proof. Hereafter, even if $k \leq 0$, abusing the notation, we put $h_{Q,k,\epsilon \psi_0} := h^Q/\langle t - \epsilon(\gamma)\gamma^k \rangle h^Q$ which acts on $p$-ordinary $p$-adic cusp forms of weight $k$ and of Neben character $\epsilon \psi_0$. By the above lemma, $h_{Q,k,\epsilon \psi_0}$ is free of finite rank $d$ over $W[\Delta_Q]$ whose rank over $W[\Delta_Q]$ is equal to rank$_W h_{\emptyset,k,\epsilon \psi_0}$ (independent of $Q$).

Since $NQ$ is cube-free, by [H13, Corollary 1.3], $h^Q$ is reduced. Let $\text{Spec}(\mathbb{I})$ be an irreducible component of $\text{Spec}(h^Q)$. Write $\text{al}(n)$ for the image of $T(n)$ in $\mathbb{I}$ (so, $\text{al}(p)$ is the image of $U(p)$). If a point $P$ of $\text{Spec}(\mathbb{I})(\overline{\mathbb{Q}})$ kills $(t - \epsilon(\gamma)\gamma^k)$ with $1 \leq k \in \mathbb{Z}$ (i.e., $P(t - \epsilon(\gamma)\gamma^k) = 0$), we call $P$ an arithmetic point, and we write $\epsilon_P := \epsilon, k_P := k \geq 1$ and $p^{\epsilon_P}$ for the order of $\epsilon_P$.

If $P$ is arithmetic, (by (C2)), we have a Hecke eigenform $f_P \in S_{k+1}(\Gamma_{\infty}(P), \epsilon \psi_0)$ such that its eigenvalue for $T(n)$ is given by $a_P(n) := P(a(n)) \in \overline{\mathbb{Q}}$ for all $n$. Thus $\mathbb{I}$ gives rise to a family $\mathcal{F} = \{f_P | \text{arithmetic } P \in \text{Spec}(\mathbb{I})\}$ of Hecke eigenforms. We define a $p$-adic analytic family of slope 0 (with coefficients in $\mathbb{I}$) to be the family as above of $\text{Hecke}$ eigenforms associated to an irreducible component $\text{Spec}(\mathbb{I}) \subset \text{Spec}(h^Q)$. We call this family slope 0 because $|a_P(p)|_p = 1$ for the $p$-adic absolute value $| \cdot |_p$ of $\overline{\mathbb{Q}}_p$ (it is also often called an ordinary family). This family is said to be analytic because the Hecke eigenvalue $a_P(n)$ for $T(n)$ is given by an analytic function $a(n)$ on (the rigid analytic space associated to) the $p$-profinite formal spectrum $\text{Spf}(\mathbb{I})$. Identify $\text{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$ with
Hom\textsubscript{\textit{W}-alg}(I, \overline{\mathbb{Q}}\textsubscript{p}) so that each element \( a \in I \) gives rise to a "function" \( a : \text{Spec}(I)(\overline{\mathbb{Q}}\textsubscript{p}) \rightarrow \overline{\mathbb{Q}}\textsubscript{p} \) whose value at \((P : I \rightarrow \overline{\mathbb{Q}}\textsubscript{p}) \in \text{Spec}(I)(\overline{\mathbb{Q}}\textsubscript{p})\) is \( a_P := P(a) \in \overline{\mathbb{Q}}\textsubscript{p} \). Then \( a \) is an analytic function of the rigid analytic space associated to Spf(I). Taking a finite covering \text{Spec}(\overline{I}) \text{ of Spec}(I) with surjection \text{Spec}(\overline{I})(\overline{\mathbb{Q}}\textsubscript{p}) \rightarrow \text{Spec}(I)(\overline{\mathbb{Q}}\textsubscript{p})\), abusing slightly the definition, we may regard the family \( \mathcal{F} \) as being indexed by arithmetic points of \text{Spec}(I)(\overline{\mathbb{Q}}\textsubscript{p})\), where arithmetic points of \text{Spec}(I)(\overline{\mathbb{Q}}\textsubscript{p})\) are made up of the points above arithmetic points of \text{Spec}(I)(\overline{\mathbb{Q}}\textsubscript{p})\). The choice of \( \overline{I} \) is often the normalization of \( I \) or the integral closure of \( I \) in a finite extension of the quotient field of \( I \).

Each irreducible component \( \text{Spec}(I) \subset \text{Spec}(h^Q) \) has a 2-dimensional semi-simple (actually absolutely irreducible) continuous representation \( \rho_I = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) with coefficients in the quotient field of \( I \) (see [H86b]). The representation \( \rho_I \) restricted to the \( p \)-decomposition group \( D_p \) is reducible with unramified quotient character (e.g., [GME, \S4.2]). As is well known now (e.g., [GME, \S4.2]), \( \rho_I \) is unramified outside \( N_{QP} \) and satisfies

\[
\text{(Gal)} \quad \text{Tr}(\rho_I(p\text{Frob}_l)) = a(l) \quad (l \nmid N_{QP}) \quad \rho_I((\gamma^s, Q_p)) \sim \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix},
\]

where \( \gamma^s = (1 + p)^s = \sum_{n=0}^{\infty} \binom{s}{n}p^n \in \mathbb{Z}_p^\times \) for \( s \in \mathbb{Z}_p \) and \([x, Q_p] \) is the local Artin symbol. As for primes in \( q \in \mathbb{Q} \), if \( q \equiv 1 \bmod p \) and \( \overline{p}(\text{Frob}_q) \) has two distinct eigenvalues, we have

\[
\text{(Gal)} \quad \rho_I([z, Q_p]) \sim \begin{pmatrix} \alpha(z) & 0 \\ 0 & \beta(z) \end{pmatrix}
\]

with characters \( \alpha_q \) and \( \beta_q \) of \( Q_p^\times \) for \( z \in Q_p^\times \), where one of \( \alpha_q \) and \( \beta_q \) is unramified (e.g., [MFG, Theorem 3.32 (2)] or [HMI, Theorem 3.75]). For each prime ideal \( P \) of \( \text{Spec}(I) \), writing \( \kappa(P) \) for the residue field of \( P \), we also have a semi-simple Galois representation \( \rho_p : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\kappa(P)) \) unramified outside \( N_{QP} \) such that \( \text{Tr}(\rho_p(p\text{Frob}_s)) \) is given by \( a(l)\rho \) for all primes \( l \nmid N_{QP} \). If \( P \) is the maximal ideal \( m_1 \), we write \( \overline{\mathcal{F}} \) for \( \rho_p \) which is called the residual representation of \( \rho_I \). The residual representation \( \overline{\mathcal{F}} \) is independent of \( I \) as long as \( \text{Spec}(I) \) belongs to a given connected component \( \text{Spec}(\mathcal{T}) \) of \( \text{Spec}(h^Q) \). Indeed, \( \text{Tr}(\rho_p) \bmod m_1 = \text{Tr}(\overline{\mathcal{F}}) \) for any \( P \in \text{Spec}(\mathcal{T}) \). If \( P \) is an arithmetic prime, we have \( \text{det}(\rho_p) = \epsilon_p \psi_p \kappa^k \) for the \( p \)-adic cyclotomic character \( \nu_P \) (regarding \( \epsilon_p \) and \( \psi_p \) as Galois characters by class field theory). This is the Galois representation associated to the Hecke eigenform \( f_P \) (constructed earlier by Shimura and Deligne) if \( P \) is arithmetic (e.g., [GME, \S4.2]).

A component \( I \) is called a CM component if there exists a nontrivial character \( \chi : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \mathbb{C}^\times \) such that \( \rho_I \cong \rho_I \otimes \chi \). We also say that \( I \) has complex multiplication if \( I \) is a CM component. In this case, we call the corresponding family \( \mathcal{F} \) a CM family (or we say that \( \mathcal{F} \) has complex multiplication). If \( \mathcal{F} \) is a CM family associated to \( I \) with \( \rho_I \cong \rho_I \otimes \chi \), then \( \chi \) is a quadratic character of \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) which cuts out an imaginary quadratic field \( F \), i.e., \( \chi = \begin{pmatrix} F/\mathbb{Q} \end{pmatrix} \). Write \( \overline{I} \) for the integral closure of \( \Lambda \) inside the quotient field of \( I \). The following three conditions are known to be equivalent:

\[
\text{(CM1)} \quad \mathcal{F} \text{ has CM with } \rho_I \cong \rho_I \otimes \begin{pmatrix} F/\mathbb{Q} \end{pmatrix} \quad (\Leftrightarrow \rho_I \cong \text{Ind}^\overline{I}_I \hat{\lambda} \text{ for a character } \hat{\lambda} : \text{Gal}(\overline{\mathbb{Q}}/F) \rightarrow \mathbb{C}^\times );
\]

\[
\text{(CM2)} \quad \text{For all arithmetic } P \text{ of } \text{Spec}(I)(\overline{\mathbb{Q}}_p), f_P \text{ is a binary theta series of the norm form of } F/\mathbb{Q};
\]

\[
\text{(CM3)} \quad \text{For some arithmetic } P \text{ of } \text{Spec}(I)(\overline{\mathbb{Q}}_p), f_P \text{ is a binary theta series of the norm form of } F/\mathbb{Q}.
\]

Indeed, \( \text{(CM1)} \) is equivalent to \( \rho_I \cong \text{Ind}^\overline{I}_I \hat{\lambda} \) for a character \( \hat{\lambda} : \text{Gal}(\overline{\mathbb{Q}}/F) \rightarrow \text{Frac}(I)^\times \) unramified outside \( N_F \) (e.g., [DHI98, Lemma 3.2] or [MFG, Lemma 2.15]). Since the characteristic polynomial of \( \rho_I(\sigma) \) has coefficients in \( I \), its eigenvalues fall in \( \overline{I} \), so, the character \( \hat{\lambda} \) has values in \( \overline{I}^\times \) (see, [H86c, Corollary 4.2]). Then by (Gal), \( \hat{\lambda}_P = P \circ \hat{\lambda} : \text{Gal}(\overline{\mathbb{Q}}/F) \rightarrow \overline{\mathbb{Q}}_p \) for an arithmetic \( P \in \text{Spec}(\overline{I})(\overline{\mathbb{Q}}_p) \) is a locally algebraic \( p \)-adic character, which is the \( p \)-adic avatar of a Hecke character \( \lambda_P : F/\mathbb{Q} \rightarrow \mathbb{C}^\times \) of type \( A_0 \) of the quadratic field \( F/\mathbb{Q} \). Then by the characterization (Gal) of \( \rho_I \), \( f_P \) is the theta series with \( q \)-expansion \( \sum_{\alpha} \lambda_P(\alpha)q^{N(\alpha)} \), where \( \alpha \) runs over all integral ideals of \( F \). For \( k \geq 1 \) (and \( (\overline{I}, \mathcal{F}) \), \( F \) has to be an imaginary quadratic field in which \( p \) is split (as holomorphic binary theta series of real quadratic field are limited to weight \( 1 \Leftrightarrow k = 0 \); cf., [MFM, \S4.8]). This shows \( \text{(CM1)} \Rightarrow \text{(CM2)} \Rightarrow \text{(CM3)} \). If \( \text{(CM2)} \) is satisfied, we have an identity \( \text{Tr}(\rho_l(p\text{Frob}_I)) = a(l) = \chi(l)a(l) = \text{Tr}(\rho_I \otimes \chi(p\text{Frob}_I)) \) with \( \chi = \begin{pmatrix} F/\mathbb{Q} \end{pmatrix} \) for all primes \( l \) outside \( N_P \). By Chebotarev density, we have \( \text{Tr}(\rho_l) = \text{Tr}(\rho_l \otimes \chi) \), and we get \( \text{(CM1)} \) from \( \text{(CM2)} \) as \( \rho_I \) is semi-simple. If a component \text{Spec}(\overline{I}) contains an arithmetic point \( P \) with theta series \( f_P \) as above of \( F/\mathbb{Q} \), either \( I \) is a CM component or otherwise \( P \) is in the intersection in \text{Spec}(h^Q) \) of a component \text{Spec}(\overline{I}) not
having CM by $F$ and another component having CM by $F$ (as all families with CM by $F$ are made up of theta series of $F$ by the construction of CM components in [H86, §7]). The latter case cannot happen as two distinct components never cross at an arithmetic point in $\text{Spec}(h^Q)$ (i.e., the reduced part of the localization $h^Q$ is étale over $\Lambda_p$ for any arithmetic point $P \in \text{Spec}(\Lambda(\overline{\mathbb{Q}}_p))$; see [HMI, Proposition 3.78]). Thus (CM3) implies (CM2). We can call a binary theta series of the norm form of an imaginary quadratic field a CM theta series.

We describe how to construct residue rings of $h^Q$ whose Galois representation is induced from a quadratic field $F$ (see [LFE, §7.6] and [HMI, §2.5.4]). Here $F$ is either real or imaginary. We write $c$ for the generator of $\text{Gal}(F/Q)$ (even if $F$ is real). Let $\varepsilon$ be the prime-to-$p$ conductor of a character $\varphi$ as in Theorem B in the introduction (allowing real $F$). Put $\varepsilon = c \cap c'$. By (h1), $c$ is a square free integral ideal of $F$ with $c + c' = O$ (for complex conjugation $c$). Since $Q$ is outside $N$, $Q$ is a finite set of rational primes unramified in $F/Q$ prime to $c_p$. Let $Q^+$ be the subset in $Q$ made up of primes split in $F$. We choose a prime factor $q$ of $q$ for each $q \in Q^+$ (once and for all), and put $\Omega^+ := \prod_{q \in Q^+} q$. We study some ray class groups isomorphic to $H_Q$. We put $\varepsilon = \varepsilon p$. We simply write $\varepsilon$ for $\varepsilon$. Consider the ray class group $Cl(a)$ (of $F$) modulo a for an integral ideal $a$ of $O$, and put

$$Cl(\varepsilon \Omega^+ p^\infty) = \lim_{r \to \infty} Cl(\varepsilon \Omega^+ p^r), \quad \text{and} \quad Cl(\varepsilon Q^+ p^\infty) = \lim_{r \to \infty} Cl(\varepsilon Q^+ p^r).$$

On $Cl(\varepsilon Q^+ p^\infty)$, complex conjugation $c$ acts as an involution.

Let $Z_{Q^+}$ (resp. $\varepsilon Q^+$) be the maximal $p$-profinite subgroup (and hence quotient) of $Cl(\varepsilon \Omega^+ p^\infty)$ (resp. $Cl(\varepsilon Q^+ p^\infty)$). Write $Z$ (resp. $\varepsilon$) for $Z_0$ (resp. $\varepsilon_0$). We have the finite level analogue $C_{Q^+}$ which is the $p$-profinite subgroup (and hence quotient) of $Cl(\varepsilon Q^+ p)$. We have a natural map of $(O_q^+ \times O_q^+)$ into $Cl(\varepsilon Q^+ p^\infty) = \lim_{r \to \infty} Cl(\varepsilon Q^+ p^r)$ (with finite kernel). Let $Z_{Q^+} = Z_{Q^+}/Z_{Q^+}^\infty$ (the maximal quotient on which $c$ acts by $-1$). We have the projections

$$\pi : \varepsilon Q^+ \to Z_{Q^+} \quad \text{and} \quad \pi^\perp : \varepsilon Q^+ \to Z_{Q^+}^\perp.$$

Recall $p > 3$; so, the projection $\pi^\perp$ induces an isomorphism $Z_{Q^+}^\perp = \{zz^\perp | z \in \varepsilon Q^+\} \to Z_{Q^+}^\perp$. Thus $\pi^\perp$ induces an isomorphism between the $p$-profinite groups $Z_{Q^+}^\perp$ and $Z_{Q^+}^\perp$. Similarly, $\pi$ induces $\pi : Z_{Q^+}^\perp \cong Z_{Q^+}$. Thus we have for the Galois group $H_Q$ as in the introduction

$$\iota : Z_{Q^+} \cong Z_{Q^+}^\perp \cong H_Q$$

by first lifting $z \in Z_{Q^+}$ to $\overline{z} \in \varepsilon Q^+$ and taking its square root and then project down to $\pi^\perp(\overline{z}^{1/2}) = \overline{z}^{1 - 1/2}$. Here the second isomorphism $Z_{Q^+}^\perp \cong H_Q$ is by Artin symbol of class field theory. The isomorphism $\iota$ identifies the maximal torsion free quotients of the two groups $Z_{Q^+}$ and $Z_{Q^+}^\perp$ which we have written as $\Gamma$. This $\iota$ also induces $W$-algebra isomorphism $W[[Z_{Q^+}]] \cong W[[Z_{Q^+}^\perp]]$ which is again written by $\iota$.

Let $\varphi$ be the Teichmüller lift of $\varphi$ as in Theorem B. Recall $N = N_{F/Q}(c)D$. Then we have a unique continuous character $\Phi : Gal(\overline{\mathbb{Q}}/F) \to W[[Z_{Q^+}]]$ characterized by the following two properties:

1. $\Phi$ is unramified outside $\varepsilon \Omega^+$.
2. $\Phi(Frob_l) = \varphi(Frob_l)[l]$ for each prime $l$ outside $Np$ and $\Omega^+$, where $[l]$ is the projection to $Z_{Q^+}$ of the class of $l$ in $Cl(\varepsilon \Omega^+ p^\infty)$.

When $F$ is real, all groups $Z_{Q^+}$, $Z_{Q^+}^\perp$, and $H_Q$ are finite groups; so, $W[[Z_{Q^+}]] \cong W[[Z_{Q^+}^\perp]]$ for example. The character $\Phi$ is uniquely determined by the above two properties because of Chebotarev density. We can prove the following result in the same manner as in [H86c, Corollary 4.2]:

**Theorem 1.2.** Suppose that $\varphi(Frob_q) \neq \varphi(Frob_q^\perp)$ for all $q | \Omega^+$. Then we have a surjective $\Lambda$-algebra homomorphism $h^{Q^+} \to W[[Z_{Q^+}]]$ such that

1. $T(l) \mapsto \Phi(l) + \Phi(l^c)$ if $l = W$ with $l$ not $c$ and $l \nmid N_{Q^+} p$;
2. $T(l) \mapsto 0$ if $l$ remains prime in $F$ and is prime to $N_{Q^+} p$;
3. $U(q) \mapsto \Phi(q^c)$ if $q | \Omega^+$;
4. $U(p) \mapsto \Phi(p^c)$.

If $F$ is real, the above homomorphism factors through the weight 1 Hecke algebra $h^{Q^+}/(p^n - 1)h^{Q^+}$ for a sufficiently large $m \geq 0$. 
The last point of the morphism factoring through the weight 1 Hecke algebra is because theta series of a real quadratic field are limited to weight 1.

Note that out of a Hecke eigenform \( f(z) \in S_{k+1}(\Gamma_0(N_{Q^+}p'), \phi) \) with \( f(T(q)) = a_q f \) for \( q \notin Q^+ \) and two roots \( \alpha, \beta \) of \( X^2 - a_qX + \phi(q)q^k = 0 \), we can create two Hecke eigenforms \( f_\alpha = f(z) - \beta f(qz) \) and \( f_\beta = f(z) - \alpha f(qz) \) of level \( N_{Q^+}q \) with \( f_\alpha U(q) = x f_x \) for \( x = \alpha, \beta \). This tells us that if we choose a set \( \Sigma^+ := \{ \xi_q | q \in Q^+ \} \) of mod \( p \) eigenvalues of \( (Frob_q) \) for \( q \in Q^+ := Q - Q^+ \), we have a unique local ring \( T^Q \) of \( h^Q \) and a surjective algebra homomorphism \( T^Q \to W[[Z_{Q^+}]] \) factorizing through \( h^Q \to W[[Z_{Q^+}]] \) such that \( U(q) \mod \mathfrak{m}_{T^Q} = \xi_q \) for all \( q \in Q^- \). For \( q \in Q^- \), if \( f \) is a theta series of \( F \), we have \( a_q = 0 \) so, the residual class (modulo \( \mathfrak{m}_{T^Q} \)) of \( \alpha, \beta \) in \( \mathbb{Z}_p \) are distinct (because of \( p > 2 \)). Therefore if we change \( \Sigma^- \), the local ring \( T^Q \) will be changed accordingly. We record this fact as

**Corollary 1.3.** Suppose that \( (Frob_q) \neq (Frob_{p'}) \) for all \( q | Q^+ \) and that \( W \) is sufficiently large so that we can choose a set \( \Sigma^- = \{ \xi_q \in F | q \in Q^- \} \) of mod \( p \) eigenvalues of \( (Frob_q) \) for \( q \in Q^- = Q - Q^+ \) in the residue field \( F \) of \( W \). Then we have a unique local ring \( T^Q \) of \( h^Q \) such that we have a surjective \( \Lambda \)-algebra homomorphism \( T^Q \to W[[Z_{Q^+}]] \) characterized by the following conditions:

1. \( T(l) \to \Phi(l) + \Phi(l') \) if \( l = l' \) and \( l \nmid N_{Q^+} \);\n2. \( T(l) \to 0 \) if \( l \) remains prime in \( F \) and is prime to \( N_{Q^+} \);\n3. \( U(q) \to \Phi(q') \) if \( q | Q^+ \);\n4. \( U(q) \to \pm \Phi(q) \) if \( q \in Q^- \), where the sign is determined by \( \pm \Phi(q) \mod \mathfrak{m}_{T^Q} = \xi_q \);\n5. \( U(p) \to \Phi(p') \).

If \( F \) is real, the above homomorphisms factors through the weight 1 Hecke algebra \( T^Q / (p^m - 1) T^Q \) for a sufficiently large \( m \geq 0 \).

We will later show that the quotient \( T^Q \to W[[Z_{Q^+}]] \) constructed above is the maximal quotient such that the corresponding Galois representation is induced from \( F \) under \( (h^0 - 4) \) (see the Proposition 2.6). Hereafter, more generally, fixing an integer \( k \geq 0 \) and the set \( \Sigma^- = \{ \xi_q \in F | q \in Q^- \} \), we put

\[
T_Q = T^Q / (t - \gamma^k) T^Q.
\]

The choice of \( q | Q^+ \) can be also considered to be the choice \( \Sigma^+ = \{ \xi_q \in F | q | Q^+ \} \) of the eigenvalue of \( U(q) \). Thus the local rings \( T^Q \) and \( T_Q \) are considered to be defined with respect to the choice \( \Sigma = \Sigma^+ \cup \Sigma^- \) of one of the mod \( p \) eigenvalues of \( U(q) \) for each \( q \in Q \). In other words, \( T_Q \) is a local factor of \( h_{Q, k, \psi^+} \) with the prescribed mod \( p \) eigenvalues \( \Sigma \) of \( U(q) \) for \( q \in Q \). Note that \( T_Q \) is classical if \( k \geq 1 \) but otherwise, it is defined purely \( p \)-adically. In the above corollary, we took \( k = 0 \) when \( F \) is real.

Assume that \( F \) is imaginary. In this case, we need later a rapid growth assertion of the group \( H_Q \) and the group ring \( W[|H_Q|] \) if we vary \( Q \) suitably. This growth result we describe now. We fix a positive integer \( r_+ \) and choose an infinite set \( Q^+ = \{ \Omega_m | m = 1, 2, \ldots \} \) of \( r_+ \)-sets \( \Omega_m \) of primes \( q \) of \( O \) such that \( N(q) \equiv 1 \) mod \( p^m \). We assume that \( \Omega_m \) is made of primes split in \( F/Q \) outside \( q p \) and that \( q \to q \cap \mathbb{Z} \) induces a bijection between \( \Omega_m^+ \) and \( Q_m^+ := \{ q \cap \mathbb{Z} | q \in \Omega_m^+ \} \). We regard \( Q_m^+ \) as a set of rational primes. We write \( \Omega_m^+ \) sometimes for the product \( \prod_{q \in Q_m^+} q \). Then the inclusion \( Z \to O \) induces a natural isomorphism \( \prod_{q \in Q_m^+} (\mathbb{Z} / q \mathbb{Z})^\times \cong (O/\Omega_m^+)^\times \). We identify the two groups by this isomorphism, and write \( \Delta_{Q_m^+} \) for the \( p \)-Sylow subgroup of this group. Then \( \Delta_{Q_m^+} \) is the product over \( q \in Q_m^+ \) of the \( p \)-Sylow subgroup \( \Delta_q \cong \Delta_q \) of \( O/\Omega_m^+ \cong (Z/q \mathbb{Z})^\times \). For the ray class group \( Cl(c \Omega_m^+ p^n) \), we have a natural exact sequence of abelian groups

\[
(O/\Omega_m^+)^\times \to Cl(c \Omega_m^+ p^n) \to Cl(p^n) \to 1,
\]

which induces the exact sequence of its maximal \( p \)-abelian quotients:

\[
1 \to \Delta_{Q_m^+} \to Cl(c \Omega_m^+ p^n) \to Cl(p^n) \to 1,
\]

since the order of the finite group \( \text{Ker}(i) \) is prime to \( p \) (as \( p > 3 \)). Passing to the projective limit with respect to \( n \), we have an exact sequence of compact modules

\[
1 \to \Delta_{Q_m^+} \to Z_{Q_m^+} \to Z_0 \to 1.
\]

We consider the group algebra \( W[[Z_{Q_m^+}]] \) which is an algebra over \( W[\Delta_{Q_m^+}] \). We choose a generator \( \delta_q \) of the cyclic group \( \Delta_q \) and put \( \Delta_{Q_m^+} \) to be the quotient of \( \Delta_{Q_m^+} \) by the subgroup generated by
\{q^p_{m_j}\}_{q \in Q^+_m}$ for $0 < n \leq m$; thus, $\Delta^+_n \cong (\mathbb{Z}/p^n\mathbb{Z})^+$. This include the ordering $Q^+_m = \{q_1, \ldots, q_{r_m}\}$ so that the above isomorphism sends $\Delta_{q_j}/(\delta^p_{q_j})$ to the $j$-th factor $\mathbb{Z}/p^n\mathbb{Z}$. In this way, we fix the identification of $\Delta^+_n$ with $(\mathbb{Z}/p^n\mathbb{Z})^+$ for all $n$ and $m$ once and for all. Thus, writing $W_n := W/p^nW$, we get a projective system

$$\{W_n[\Delta^+_n] \cong W_n[(\mathbb{Z}/p^n\mathbb{Z})^+]\}_{n>0}$$

sending $(\mathbb{Z}/p^n\mathbb{Z})^+ \ni x \mapsto (x \mod p^n) \in (\mathbb{Z}/p^n\mathbb{Z})^+$ for all $n$. We then have

$$W[[S_1, \ldots, S_{r_n}]] \cong \lim_{\longrightarrow_n} W_n[\Delta^+_n]$$

sending $s_j = 1 + S_j$ to the image of $\delta_{q_j}$ in $\Delta^+_n$ for all $j$, $q_j \in Q^+_m$ and $m \geq n$.

Assuming that $F$ has class number prime to $p$, the natural isomorphism $\mathbb{Z}^\times_p \cong O^\times_p$ induces a group morphism $\mathbb{Z}_p^\times \to \text{Cl}(\mathfrak{p}^\infty)$, which induces an isomorphism $\Gamma = 1 + p\mathbb{Z}_p \cong \mathbb{Z}_0$. Then we can canonically split exact sequence (1.8) so that $Z_{Q^+_m} = \Delta_{Q^+_m} \times \Gamma$, making the following diagram commutative for all $m' \geq n > n$: \[
\begin{array}{ccc}
W_n[\{\Gamma\}][\Delta^+_n] & \cong & W_n[[Z_{Q^+_m}]/\mathfrak{A}_n] \\
\pi_n' & \mapsto & W_n[[Z_\emptyset]] \\
\downarrow & & \downarrow \text{onto} \\
W_n[\{\Gamma\}][\Delta^+_n] & \cong & W_n[[Z_{Q^+_m}]/\mathfrak{A}_n] \longrightarrow W_n[[\mathbb{Z}_0]]
\end{array}
\]

where $\mathfrak{A}_n := (p^n, s^n_{j=1,\ldots,r_n})$ as an ideal of $W[[S_1, \ldots, S_{r_n}]]$. In this way, we get a (bit artificial) projective system

$$W_n[[Z_{Q^+_m}]/\mathfrak{A}_n] \mapsto \lim_{\longrightarrow_{n'}} W_n[[Z_{Q^+_m}]/\mathfrak{A}_n]_{n' > n}$$

By this map, $W_n[[Z_{Q^+_m}]/\mathfrak{A}_n]$ is naturally a $\Lambda$-algebra via the canonical splitting $Z_{Q^+_m} = \Delta_{Q^+_m} \times \mathbb{Z}_0$, and hence a $\Lambda[[S_1, \ldots, S_{r_n}]]$-algebra. Since $\mathbb{Z}_0 = \Gamma$, we get $\lim_{\longrightarrow_n} W_n[[Z_{Q^+_m}]/\mathfrak{A}_n] \cong \Lambda[[S_1, \ldots, S_{r_n}]]$.

We thus conclude

**Proposition 1.4.** Assume that $F$ is imaginary with class number prime to $p$. Identify $H_{Q^+_m}$ with $Z_{Q^+_m}$ by (1.6) (whence $\mathfrak{A}_n$ is the ideal of $W[[H_{Q^+_m}]]$). Then the limit ring $\lim_{\longrightarrow_n} W[[H_{Q^+_m}]]/\mathfrak{A}_n$ is isomorphic to $\Lambda[[S_1, \ldots, S_{r_n}]]$.

This follows from the above argument, after identifying $Z_{Q^+_m}$ with $H_{Q^+_m}$ and identifying $\Lambda$ with $W[[\Gamma_\emptyset]]$.

We now explore the case where the class number of $F$ is divisible by $p$. In this case, we again study the set $Q^+$ of $r_+$-sets $\Delta^+_m$ of split primes in $F$ outside $N$ such that $N(q) \equiv 1 \mod p^n$ with $Q^+_m := \{(q) = q \cap N \in Q^+_m\}$ with an ordering. We still have the following exact sequence (1.8):

$$1 \to \Delta_{Q^+_m} \to Z_{Q^+_m} \xrightarrow{\pi_{Q^+_m}} Z_\emptyset \to 1.$$ 

Write $Z_{\text{tor}}$ for the maximal torsion subgroup of $Z_\emptyset$, and fix a splitting $\mathbb{Z}_0 = \Gamma_F \times Z_{\text{tor}}$ with a torsion-free group $\Gamma_F$. The projection $\pi_{Q^+_m}$ identifies the maximal torsion-free quotient of $Z_{Q^+_m}$ with $\Gamma_F$. Write $Z_{Q^+_m, \text{tor}} : \text{Ker}(Z_{Q^+_m} \to \Gamma_F)$ (the maximal torsion subgroup of $Z_{Q^+}$). Note that $\Delta_{Q^+_m} \hookrightarrow Z_{Q^+_m, \text{tor}}$. For $m$ running over integers with $m \geq n$, the isomorphism classes of the set of cokernels $\{Z_{Q^+_m, \text{tor}}/\Delta_{Q^+_m}\}_{m \geq n}$ of pairs of abelian groups is finite. Here $Z_{Q^+_m, \text{tor}}/\Delta_{Q^+_m}$ and $Z_{Q^+_m, \text{tor}}/\Delta_{Q^+_m}$ are isomorphic if the following diagram for $m' > m$ is commutative:

$$\Delta_{Q^+_m}/\Delta_{Q^+_m} \longrightarrow Z_{Q^+_m, \text{tor}}/\Delta_{Q^+_m} \longrightarrow Z_{Q^+_m, \text{tor}}/\Delta_{Q^+_m}.$$ 

Here $\Delta_{Q^+_m}$ is induced by the generator $\delta_{q_j} \Delta_{Q^+_m}$ for $Q^+_m = \{q_1, \ldots, q_{r_m}\}$ to the generator $\delta_{q_j} \Delta_{Q^+_m}$ writing $Q^+_m = \{q_1', \ldots, q_{r_m}'\}$ according to our choice of ordering. Starting with $n = 1$, we have an isomorphism class $\mathcal{I}_1$ in $\{Z_{Q^+_m, \text{tor}}/\Delta_{Q^+_m}\}_{m \geq 1}$ with infinite elements. Suppose that we have
constructed a sequence \( \mathcal{I}_n \to \mathcal{I}_{n-1} \to \cdots \to \mathcal{I}_1 \) of isomorphism classes \( \mathcal{I}_j \) in \( \{ Z_{Q_m,tor}^+, \Delta_{Q_m}^+ \}_{m \geq j} \) such that \( Z_{Q_m,tor}^+/\Delta_{Q_m}^+ \in \mathcal{I}_j \) is sent onto \( Z_{Q_m,tor}^+/\Delta_{Q_m}^{p-1} \) in \( \mathcal{I}_{j-1} \) for all \( j = 2, 3, \ldots, n \). Since

\[
\mathcal{I}_n := \{ Z_{Q_m,tor}^+, (Z_{Q_m,tor}^+)/(\Delta_{Q_m}^+) \in \mathcal{I}_n \}_{m \geq n+1}
\]

is an infinite set, we can choose an isomorphism class \( \mathcal{I}_{n+1} \subset \mathcal{I}_{n+1} \) with \( |\mathcal{I}_{n+1}| = \infty \). Thus by induction on \( n \), we find an infinite sequence \( \cdots \to \mathcal{I}_n \to \mathcal{I}_{n-1} \to \cdots \to \mathcal{I}_1 \) as above. Then we define \( m(n) \) for each \( n \) to be the minimal \( m \) appearing \( \mathcal{I}_n \). Thus we have a projection \( \pi^n_{n+1,tor} : Z_{Q_m(n+1),tor}^+/\Delta_{Q_m(n+1)}^+ \to Z_{Q_m(n),tor}^+/\Delta_{Q_m(n)}^+ \) and a projective system of groups

\[
\begin{array}{cccc}
Z_{Q_m(n),tor}^+/\Delta_{Q_m(n)}^+ & \longrightarrow & Z_{Q_m(n+1),tor}^+/\Delta_{Q_m(n+1)}^+ & \longrightarrow & \Gamma_F \\
\pi^n_{n+1,tor} & & \pi^n_{n+1} & & \\
\end{array}
\]

Passing to the limit, we have an exact sequence:

\[
1 \longrightarrow \lim_n Z_{Q_m(n),tor}^+/\Delta_{Q_m(n)}^+ \longrightarrow \lim_n Z_{Q_m(n),tor}^+/\Delta_{Q_m(n)}^+ \longrightarrow \Gamma_F \longrightarrow 1.
\]

Note here the subgroup \( \Delta_{\infty} := \lim_n Q_{m(n),tor}^+ / \Delta_{m(n)}^+ \cong Z_p^+ \) with \( W[[\Delta_{\infty}]] = W[[S_1, \ldots, S_{r_1}]] \) for the variable chosen as in Proposition 1.4 and \( W[[Z_S]] \) for \( Z_S := \lim_n Z_{Q_m(n),tor}^+/\Delta_{Q_m(n)}^+ \) is an algebra free of finite rank over \( W[[\Delta_{\infty}]] \). We write \( \Gamma_S = Z_S/Z_S,tor \) for the maximal torsion subgroup \( Z_S,tor \) of \( Z_S \). Choose a splitting of the exact sequence \( Z_S,tor \hookrightarrow Z_S \twoheadrightarrow \Gamma_S \) so that \( \Gamma_S \) as a subgroup of \( Z_S \) contains \( \Delta_{\infty} \). Then \( W[[Z_S]] = W[[\Gamma_S]]/W[[Z_S,tor]] \cong W[[\Gamma_S]][Z_S/\Gamma_S] \). By splitting the projection \( Z_{\infty} := \lim_n Z_{Q_m(n),tor}^+/\Delta_{Q_m(n)}^+ \to \Gamma_F \), we have a \( W[\Gamma_F] \)-algebra structure of \( W[[\Delta_{\infty}]] \).

**Proposition 1.5.** Let the notation be as above. Assume that \( F \) is imaginary with class number divisible by \( p \). Identify \( H_{Q_m}^+ \) with \( Z_{Q_m}^+ \) by (1.6). Then there is a subsequence \( \{ Q_{m(n)} \}_{n=1,2,\ldots} \subset Q^+ \) such that \( \{ W[[H_{Q_m}^+]]/\mathfrak{A}_n \}_{n} \) forms a projective system of finite rings and that the limit ring \( \lim_n W[[H_{Q_m}^+]]/\mathfrak{A}_n \) is isomorphic to the profinite group algebra \( W[\Gamma_F \times \Gamma_S]/\Gamma_S/\Gamma_S \), and \( \Gamma_S \) (resp. \( \Gamma_F \)) contains \( \Delta_{\infty} \) (resp. \( \Gamma \)) as a subgroup of finite index. In particular, \( \lim_n W[[H_{Q_m}^+]]/\mathfrak{A}_n \) is free of finite rank over \( \Lambda[[S_1, \ldots, S_{r_1}]] \) and is a local complete intersection over \( \Lambda \).

2. THE \( R = T \) THEOREM AND AN INVOLUTION OF \( R \)

We place ourselves in the setting of Theorem B, but we allow any quadratic extension \( F/\mathbb{Q} \) (which can be real or imaginary). We assume that the residue field of \( W \) is given by \( F = \mathbb{Q}/m_T \). For the moment, we only assume (h0–3) for a fixed connected component \( \text{Spec}(T) \) of \( \text{Spec}(h) \) for \( h := h^0 \) and its residual representation \( \overline{\rho} \) of the form \( \text{Ind}_F^Q \overline{\tau} \) for a Galois character \( \overline{\tau} : \text{Gal}(\overline{Q}/F) \to F \).

We fix a weight \( k \geq 0 \) and pick a Hecke character \( \varphi_k : \text{Gal}(\overline{Q}/F) \to \mathbb{C}^\times \) of conductor at most \( cp \) with \( p \)-type \( -k_i \), \( i \) for the identity embedding \( i_p : F \to \mathbb{Q}_p \). Let \( \theta(\varphi_k) \in S_{k+1}(\Gamma_0(Np), \psi_k) \) for the corresponding theta series. Then \( \psi_k \) is determined by \( \varphi_k \) (i.e., \( \psi_k \cong \chi \varphi_k |_{A}^* \otimes \nu_p^k \) regarding \( \varphi_k \) and \( \psi_k \) as idele characters; see [HMI, Theorem 2.71]). When \( F \) is imaginary (that is usually the case), we assume that \( k \geq 1 \).

Recall the identity \( \psi_k \nu_p^k \equiv \text{det}(\overline{\theta}) \) for the \( p \)-adic cyclotomic character \( \nu_p^k \); so, \( \psi_0 \) is the Teichmüller lift of \( \text{det}(\overline{\theta}) \). Here, we simply write \( \psi \) for \( \psi_0 \equiv \psi_k \nu_p^k \). Writing \( c \) for the prime-to-\( p \) conductor of \( \overline{\rho} \), by (h3), \( N_{F/\mathbb{Q}}(c)D = N \) for the discriminant of \( F \) (cf. [GME, Theorem 5.1.9]). By (h1), the conductor \( c \) is square-free and only divisible by split primes in \( F/\mathbb{Q} \). Since \( \overline{\rho} = \text{Ind}_F^Q \overline{\tau} \), for \( l | N \), the prime \( l \) either splits in \( F \) or ramified in \( F \). Write \( l \) for the prime factor of \( (l) \) in \( F \). If \( (l) \) splits into \( \mathfrak{p} \mathfrak{l} \), we may assume that the character \( \overline{\tau} \) ramifies at \( \mathfrak{l} \) and is unramified at \( \mathfrak{p} \), and hence \( \overline{\rho}(\overline{\tau}) \cong \overline{\tau} \otimes \overline{\tau} \). If \( (l) = l \mathfrak{p} \) ramifies in \( F \), we have \( \overline{\rho}(I_l) \cong 1 \otimes \chi \) for the quadratic character \( \chi = \left( \frac{F/\mathbb{Q}}{l} \right) \). Here \( I_l \) is the inertia subgroup of \( \text{Gal}(\overline{Q}/\mathbb{Q}) \).
Write $CL_W$ for the category of $p$-profinite local $W$-algebras with residue field $\mathbb{F} := W/m_W$ whose morphisms are local $W$-algebra homomorphisms. Let $\mathbb{Q}^{(Np)} \subset \overline{\mathbb{Q}}$ be the maximal extension of $\mathbb{Q}$ unramified outside $Np\infty$. Consider the following deformation functor $D : CL_W \rightarrow SETS$ given by

$$D(A) = D^0(A) := \{\rho : \text{Gal}(\mathbb{Q}^{(Np)}/\mathbb{Q}) \rightarrow \text{GL}_2(A) : \text{a representation satisfying (D1–4)}\} \equiv .$$

Here are the conditions (D1–4):

(D1) $\rho \mod m_A \cong \overline{\rho}$ (i.e., there exists $a \in \text{GL}_2(\mathbb{F})$ such that $a\overline{\rho}(\sigma)a^{-1} = (\rho \mod m_A)$ for all $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$).

(D2) $\rho\text{}|_{\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} \cong (\rho \circ \sigma)$ with $\delta$ unramified.

(D3) $\det(\rho)|_{I_l}$ is equal to $\iota_A \circ \psi_I$ for the $l$-part $\psi_I$ of $\psi$ for each prime $l|N$, where $\iota_A : W \rightarrow A$ is the morphism giving $W$-algebra structure on $A$ and $\psi_I = \psi|_{I_l}$ regarding $\psi$ as a Galois character by class field theory.

(D4) $\det(\rho)|_{I_p} \equiv \psi|_{I_p} \mod m_A$ (which is equivalent to $\epsilon|_{I_p} \equiv \psi|_{I_p} \mod m_A$).

If we want to allow ramification at primes in a finite set $Q$ of primes outside $Np$, we write $\mathbb{Q}^{(QNp)}$ for the maximal extension of $\mathbb{Q}$ unramified outside $Q \cup \{l|Np\} \cup \{\infty\}$. Consider the following functor

$$D^Q(A) := \{\rho : \text{Gal}(\mathbb{Q}^{(Q(Np)}/\mathbb{Q}) \rightarrow \text{GL}_2(A) : \text{a representation satisfying (D1–4) and (UQ)}\} \equiv ,$$

where

(UQ) $\rho$ is unramified at all $q \in Q$.

We may also impose another condition if necessary:

(det) $\det(\rho) = \iota_A \circ \nu^k \psi_{k}$ for the $p$-adic cyclotomic character $\nu_p$,

and consider the functor

$$D_{Q,k,\psi_{k}}(A) := \{\rho : \text{Gal}(\mathbb{Q}^{(Q(Np)}/\mathbb{Q}) \rightarrow \text{GL}_2(A) : \text{a representation satisfying (D1–4) and (det)}\} \equiv .$$

The condition (det) implies that if deformation is modular and satisfies (D1–4), then it is associated to a weight $k + 1$ cusp form of Neben character $\psi_k$: strictly speaking, if $k = 0$ (i.e., $F$ is real), we allow non-classical $p$-ordinary $p$-adic cusp forms. We often write simply $D_{k,\psi_{k}}$ for $D^0_{k,\psi_{k}}$ when $Q$ is empty. For each prime $q$, we write $D^Q_{Q,k,\psi_{k}}$ for the deformation functor of $\overline{\rho}\text{|}_{\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}$ satisfying the local condition (D2–4) which applies to $q$.

By our choice of $\overline{\rho} = \text{Ind}_F^G \overline{\rho}$, we have $\overline{\rho}\text{|}_{\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} \cong \left(\begin{array}{cc} \tau_q & 0 \\ 0 & \overline{\tau}_q \end{array}\right)$ for two local characters $\tau_q$, $\overline{\tau}_q$ for all $q \in Q$. If $\overline{\delta} \neq \delta$ (i.e., $\overline{\text{Rg}}$) and $\tau_q(\text{Frob}_q) \neq \overline{\tau}_q(\text{Frob}_q)$ for all $q \in Q$, $D$, $D^Q$, $D_{k,\psi_{k}}$, and $D_{Q,k,\psi_{k}}$ are representable by universal objects $(R, \rho) = (W^\delta, \rho^\delta)$, $(R^Q, \rho^Q)$, $(R_k, \rho_k)$ and $(R_Q, \rho_Q)$, respectively (see [MFG, Proposition 3.30] or [HMI, Theorem 1.46 and page 186]).

Here is a brief outline of how to show the representability of $D$. It is easy to check the deformation functor $D^\text{ord}$ only imposing (D1–2) is representable by a $W$-algebra $R^\text{ord}$. The condition (D4) is actually redundant as it follows from the universality of the Teichmüller lift and the conditions (D1–2). Since $N$ is the prime-to-$p$ conductor of $\det(\overline{\rho})$ and $p$ is unramified in $F/\mathbb{Q}$ (h2–3), if $l$ is a prime factor of $N$, writing $\rho|_{I_l}$ for its semi-simplification of $\rho$ over $I_l$, we see from (h0) that $(\rho|_{I_l})^{ss} = \epsilon_l \oplus \delta_l$ for two characters $\epsilon_l$ and $\delta_l$ (of order prime to $p$) with $\delta_l$ unramified and $\epsilon_l \equiv \psi|_{I_l} \mod m_A$. Thus by the character $\epsilon_N := \prod_{l|N} \epsilon_l$ of $I_N = \prod_{l|N} I_l$, $A$ is canonically an algebra over the group algebra $W[I_N]$. Then $R$ is given by the maximal residue ring of $R^\text{ord}$ on which $I_N$ acts by $\psi_{1,N} = \prod_{l|N} \psi|_{I_l}$; so, $R = R^\text{ord} \otimes_{W[I_N],\psi_{1,N},W} W$, where the tensor product is taken over the algebra homomorphism $W[I_N] \rightarrow W$ induced by the character $\psi_{1,N}$. Since $\overline{\rho}$ is an induced representation, $\overline{\rho}|_{I_l}$ is semi-simple and $\overline{\rho}|_{I_l} = \tau_l \oplus \bar{\tau}_l$ with $\tau_l = \epsilon_l \mod m_A$. Similarly one can show the representability of $D^Q$ and $D_{Q,k,\psi_{k}}$.

Let $\mathbb{T}$ be the local ring of $h = h^\theta$ as in Theorem B whose residual representation is $\overline{\rho} = \text{Ind}_F^G \overline{\rho}$. The ring $\mathbb{T}$ is uniquely determined by (h1–3) as the unramified quotient of $\overline{\rho}$ at each $l|Np$ is unique. Without assuming (h1–3), to have a universal ring and to have uniquely defined $\mathbb{T}$, we need to specify in the deformation problem the unramified quotient character and for $\mathbb{T}$, the residue class of $U(l)$-eigenvalue (because of the existence of companion forms).

Since $\overline{\rho}$ is irreducible, by the technique of pseudo-representation, we have a unique representation

$$\rho_{\mathbb{T}} : \text{Gal}(\mathbb{Q}^{(Np)}/\mathbb{Q}) \rightarrow \text{GL}_2(\mathbb{T})$$
up to isomorphisms such that \( \text{Tr}(\rho_T(\text{Frob}_l)) = T(l) \in \mathbb{T} \) for all prime \( l \nmid Np \) (e.g., [HMI, Proposition 3.49]). This representation is a deformation of \( \mathbf{p} \) in \( \mathcal{D}^0(\mathbb{T}) \). Thus by universality, we have projections \( \pi: R = R^0 \to \mathbb{T} \), such that \( \pi \circ \rho \cong \rho_T \). Here is the “\( R = T \)” theorem of Taylor, Wiles et al:

**Theorem 2.1.** Assume (Rg) and (h0–3) with either (h4) or (h5). Then the morphism \( \pi: R \to \mathbb{T} \) is an isomorphism, and \( \mathbb{T} \) is a local complete intersection over \( \Lambda \).

See [Wi95, Theorem 3.3] and [DFG04] for a proof (see also [HMI, §3.2] or [MFG, Theorem 3.31] for details of how to lift the results in [Wi95] to the (bigger) ordinary deformation ring with varying determinant character). These references require the assumption (W) which is absolute irreducibility of \( \mathbf{p}_{\text{Gal}(\mathbb{Q}/M)} \) for \( M = \mathbb{Q}[\sqrt{p}] \) with \( p^* := (-1)^{(p-1)/2}p \). Note that (W) follows from either (h4) or (h5), as mentioned in the introduction. To eliminate the assumption (h0), we need to impose in addition to (D3) that \( H_0(I, \rho) \cong \Lambda \) for prime factors \( l \) of \( N \) with \( l \equiv 1 \pmod{p} \) to have the identity \( R = \mathbb{T} \) (or work with \( \Gamma_l(l) \)-level Hecke algebra), which not only complicates the setting but also the identification of \( T/I \cong W[[H]] \) (for \( I \) in Theorem B) could fail if (h0) fails (so, we always assume (h0); see Lemma 2.4). We will recall the proof of Theorem 2.1 in the following Section 4 to good extent in order to facilitate a base for a finer version we study there.

Perhaps the following fact is well known (e.g., [Ru91, Theorem 5.3]):

**Corollary 2.2.** Assume (h0–4) and that \( F \) is an imaginary quadratic field of class number prime to \( p \). Then \( Y^- (\varphi^-) \) has homological dimension 1 (so, it does not have any pseudo-null submodule non-null). Thus if \( Y^- (\varphi^-) \) is pseudo isomorphic to a cyclic \( \mathbb{Z}_p[\varphi^-][[\Gamma_-]]\)-module \( \mathbb{Z}_p[\varphi^-][[\Gamma_-]]/(f_{\varphi^-}) \) with \( f_{\varphi^-} \in \mathbb{Z}_p[\varphi^-][[\Gamma_-]] \), it has an injection into the cyclic module with finite cokernel.

**Proof.** Write the presentation of \( R \cong \mathbb{T} \) as \( R = \Lambda[[T_1, \ldots, T_r]]/(S_1, \ldots, S_r) \) for a regular sequence \( (S_1, \ldots, S_r) \) of \( \Lambda[[T_1, \ldots, T_r]] \). Then by the fundamental exact sequence of differentials (e.g., [CRT, Theorem 25.2] and [HMI, page 370]), we get the following exact sequence:

\[
0 \to \bigoplus_i \text{Ad} S_i = (S_1, \ldots, S_r)/(S_1, \ldots, S_r)^2 \to \bigoplus_i \text{Ad} T_i \to \Omega_{R/\Lambda} \to 0.
\]

Since the class number of \( F \) is prime to \( p \), the CM component \( W[[H]] \) of \( T = \mathbb{T} \) is isomorphic to \( \Lambda \); so, tensoring \( \Lambda \) over \( R \), we get another exact sequence:

\[
0 \to \bigoplus_i \text{Ad} S_i \to \bigoplus_i \text{Ad} T_i \to \Omega_{R/\Lambda} \otimes_R \Lambda \to 0.
\]

By a theorem of Mazur (cf. [MT90], [HT94, 3.3.7] and [H16, §6.3.6]), under (Rm) (which follows from (h2–3)) and (h0), we have \( \Omega_{R/\Lambda} \otimes_R \Lambda \cong Y^- (\varphi^-) \otimes_{\mathbb{Z}_p[\varphi^-]} W \). Thus we get a \( \Lambda \)-free resolution of length 2 of the Iwasawa module, and hence it has homological dimension 1.

Suppose that we have a pseudo-isomorphism \( i : Y^- (\varphi^-) \to \mathbb{Z}_p[\varphi^-][[\Gamma_-]]/(f_{\varphi^-}) \). Then \( i \) is an injection as \( Y^- (\varphi^-) \) does not have any pseudo-null submodule non-null, and \( \text{Coker}(i) \) is finite. □

Since \( \mathbf{p} = \text{Ind}_{\mathbb{F}}^\mathbb{Q} \mathbf{p} \), for \( \chi = \left( \frac{\mathbb{Q}}{\mathbb{Q}_l} \right) \), \( \mathbf{p} \otimes \chi \cong \mathbf{p} \). By ordinariness, \( p \) splits in \( F \); so, \( \chi \) is trivial on \( \text{Gal}(\overline{\mathbb{Q}}_l/\mathbb{Q}_l) \) for prime factors of \( pN_{F/\mathbb{Q}}(e) \) and ramified quadratic on \( \text{Gal}(\overline{\mathbb{Q}}_l/\mathbb{Q}_l) \) for \( l | D \). Thus \( \rho \mapsto \rho \otimes \chi \) is an automorphism of the functor \( \mathcal{D}^G \) and \( \mathcal{D}_{Q,k,\psi_\chi} \), and \( \rho \mapsto \rho \otimes \chi \) induces automorphisms \( \sigma_Q \) of \( R_Q \) and \( R^q \).

We identify \( R \) and \( \mathbb{T} \) now by Theorem 2.1; in particular, we have an automorphism \( \sigma = \sigma_Q \in \text{Aut}(\mathbb{T}) \) as above. We could think about \( \text{Ih} \mid W_0 \) defined over a smaller complete discrete valuation ring \( W_0 \subset W \) (the smallest possible ring is \( \mathbb{Z}_p[\psi] \)). After extending scalar from \( W_0 \) to \( W \), we get an involution. We may assume that \( W = W(\mathbb{F}) \) (the Witt vector ring of \( F = \mathbb{T}/m_F \)). Since \( \sigma \) fixes \( W \) as it is an identity on \( F \), we know that \( \sigma \) preserves \( \mathbb{T} \) before extending scalar to \( W \). Thus we get

**Corollary 2.3.** Assume (h0–4). Then for a complete discrete valuation ring \( W_0 \) flat over \( \mathbb{Z}_p[\psi] \), we have an involution \( \sigma \in \text{Aut}(\mathbb{T}/W_0) \) with \( \sigma \circ \rho_T \cong \rho_T \otimes \chi \).

We write \( T_+ \) for the subring of \( T \) fixed by the involution in Corollary 2.3. More generally, for any module \( X \) on which the involution \( \sigma \) acts, we put \( X_{\pm} = X^\pm = \{ x \in X | \sigma(x) = \pm x \} \). In particular, we have \( T_{\pm} := \{ x \in T | x^2 = \pm x \} \).
We now study the fixed subscheme \( \text{Spec}(\mathbb{T})^G \) of \( G := \langle \sigma \rangle \subset \text{Aut}(\mathbb{T}/\Lambda) \). Consider the functor \( D_F, D_F^\mathbb{C} : CL_W \to \text{SETS} \) defined by

\[
D_F(A) = \{ \lambda : \text{Gal}(\overline{\mathbb{C}}/F) \to A^\times | \lambda \equiv \overline{\rho} \mod m_\Lambda \text{ has conductor a factor of } \mathfrak{p} \},
\]
and

\[
D_F^\mathbb{C}(A) = \{ \lambda : \text{Gal}(\overline{\mathbb{C}}/F) \to A^\times | \lambda \equiv \overline{\rho} \mod m_\Lambda \text{ has conductor a factor of } \mathfrak{p}^\infty \}.
\]

Let \( F_{\mathfrak{p}} \) be the maximal abelian \( p \)-extension of \( F \) inside the ray class field of conductor \( \mathfrak{p} \). Put \( C = C_{\mathfrak{p}} := \text{Gal}(F_{\mathfrak{p}}/F) \). Similarly, write \( F_{\mathfrak{p}}^\infty \) for the maximal \( p \)-abelian extension inside the ray class field over \( F \) of conductor \( \mathfrak{p}^\infty \). Put \( H := \text{Gal}(F_{\mathfrak{p}}^\infty/F) \). Note that \( F_{\mathfrak{p}}^\infty/F \) is a finite extension if \( F \) is real. Then \( D_F \) is represented by \( (W[C], \Phi) \), where \( \Phi(x) = \varphi(x)x \) for \( x \in C \), where \( \varphi \) is the Teichmüller lift of \( \overline{\varphi} \) with values in \( W^\times \). Similarly \( D_F^\mathbb{C} \) is represented by \( W[[H]] = \varprojlim_{H < H, \text{open}} W[H/H'] \) for \( H := \text{Gal}(F_{\mathfrak{p}}^\infty/F) \). If \( F \) is real, \( H \) is a finite group, but it is an infinite \( p \)-profinite group if \( F \) is imaginary.

In the introduction, when \( F \) is imaginary, we defined \( H \) as the anticyclotomic \( p \)-primary part \( \text{Gal}(K^{-}/F) \) of the Galois group of the ray class field \( K \) of conductor \( (\mathfrak{c} \cap \mathfrak{c}^c)p^\infty \). The present definition is a bit different from the one given there. However, the present \( H \) is isomorphic to the earlier \( \text{Gal}(K^{-}/F) \) by sending \( \tau \) to \( \tau(1-c)/2 = \sqrt{\tau_{CT}}-1_{C^{-1}} \) by (1.6). Thus we identify the two groups by this isomorphism as the present definition makes the proof of the following results easier. We have the following simple lemma which can be proven in exactly the same way as \([CV03, \text{Lemma 2.1}] \) and \([H15, \text{Theorem 7.2}] \):

**Lemma 2.4.** Assume \((h0-4) \) and \( p > 3 \). Then the natural transformation \( \lambda \mapsto \text{Ind}_{\mathbb{C}}^\mathbb{C} \lambda \) induces an isomorphism \( D_F^\mathbb{C} \cong D_F^\mathbb{C} \) and \( D_F^\mathbb{C} \cong D_F^\mathbb{C} \), where

\[
D_F^\mathbb{C}(A) = \{ \rho \in D(A) | \rho \otimes \chi \equiv \rho \} \quad \text{and} \quad D_F^\mathbb{C}(A) = \{ \rho \in D^\mathbb{C}(A) | C(\det \rho) \supset (NP) \}
\]
for the conductor \( C(\det \rho) \) of \( \det(\rho) \).

**Proof.** Since the proof is essentially the same for the two cases, we only deal with \( D_F^\mathbb{C} \cong D_F^\mathbb{C} \). By \([DHI98, \text{Lemma 3.2}] \), we have \( \rho \otimes \chi \equiv \rho \) for \( \rho \in D(A) \) equivalent to having \( \lambda : \text{Gal}(\overline{\mathbb{C}}/F) \to A^\times \) such that \( \rho \equiv \text{Ind}_{\mathbb{C}}^\mathbb{C} \lambda \). We can choose \( \lambda \) so that \( \lambda \) has conductor a factor of \( \mathfrak{p}^\infty \) by (D4) and \( C(\det(\rho)) | NP^\infty \). Then \( \lambda \) is unique by (D2–3) and (h0). Thus we get the desired isomorphism. \( \Box \)

Since \( D_F^\mathbb{C} \) (resp. \( D_F^\mathbb{C} \)) is represented by \( T/(TT+1) = T/I \otimes_{\Lambda} (T) \) (resp. \( T/I \)) for \( I = T(\sigma-1)T \), this lemma shows

**Corollary 2.5.** Assume \((h0-4) \). Then we have \( T/I \otimes_{\Lambda} (T) \cong W[C] \) and \( T/I \cong W[[H]] \) canonically.

In the proof of Theorem 2.1, Taylor and Wile considered an infinite sets \( Q \) made up of finite sets \( Q \) of primes \( q \equiv 1 \mod p \) outside \( NP \) such that \( \overline{\mathfrak{p}}(\text{Frob}_q) \sim \left( \begin{array}{cc} \overline{\rho}_q & 0 \\ 0 & \overline{\delta}_q \end{array} \right) \) with \( \overline{\rho}_q \neq \overline{\delta}_q \in \mathbb{F} \). Over the inertia group \( I_q, \rho^Q \) has the following shape by a theorem of Faltings

\[
\rho^Q|_{I_q} = \left( \begin{array}{c} \delta_q \\ 0 \end{array} \right) = \left( \begin{array}{c} \overline{\rho}_q \\ 0 \end{array} \right)
\]

for characters \( \delta_q, \overline{\delta}_q : \text{Gal}(\overline{\mathbb{Q}}_q/Q_q) \to \langle R^Q \rangle^\times \) such that \( \delta_q|_{T_q} = \delta_q^{-1} \) and \( \delta_q([q, Q_q]) \equiv \overline{\rho}_q \mod m_\mathfrak{T} \) (e.g., \([MFH, \text{Theorem 3.32 (1)}] \) or \([HMI, \text{Theorem 3.75}] \)). Since \( \overline{\mathfrak{p}} \) is unramified at \( q \), \( \delta_q \) factors through the maximal \( p \)-abelian quotient \( \Delta_q \) of \( Z_q^\times \) by local class field theory, and in fact, it gives an injection \( \delta_q : \Delta_q \hookrightarrow \langle R^Q \rangle \) as we will see later. Note that \( \rho \mapsto \rho \otimes \chi \) is still an automorphism of \( D^Q \) and hence induces an involution \( \sigma = \sigma_Q \) of \( R^Q \).

We can choose infinitely many distinct \( Q \) with \( \overline{\mathfrak{p}}(\text{Frob}_q) \) for \( q \in Q \) having two distinct eigenvalues. We split \( Q = Q^+ \sqcup Q^- \) so that \( Q^\pm = \{ q \in Q | \chi(q) = \pm 1 \} \). By choosing an eigenvalue \( \overline{\rho}_q \) of \( \overline{\mathfrak{p}}(\text{Frob}_q) \) for each \( q \in Q \), we have a unique Hecke algebra local factor \( T_Q \) of the Hecke algebra \( h_{Q,k,\phi_a} \) whose residual representation is isomorphic to \( \overline{\mathfrak{p}} \) and \( U(q) \mod m_{T_Q} \) is the chosen eigenvalue \( \overline{\rho}_q \). This follows from Corollary 1.3 in the following way: We choose \( \overline{\rho}_q \) for \( q \in Q^- \) as in Corollary 1.3. For \( q \in Q^+ \), we choose a unique prime factor \( q \mid q \) so that \( \overline{\rho}(\text{Frob}_q^\prime) = \overline{\rho}_q \). In this way, we get a local factor \( T^Q \) of \( h^Q \) which covers \( W[[Z_Q]] \) as in Corollary 1.3. Recall (1.7):

\[
T_Q = T^Q/(t - \gamma^k)T^Q
\]
which is a local factor of $\mathfrak{h}_{Q,k,\psi}$, with the prescribed mod $p$ eigenvalues of $U(q)$ for $q \in Q$.

By absolute irreducibility of $\overline{\rho}$, the theory of pseudo representation tells us that the Galois representation $\rho_{Q,Q}$ in Section 1 can be arranged to have values in $\text{GL}_2(T^Q)$ (e.g., [MFG, Proposition 2.16]). The isomorphism class of $\rho_{Q,Q}$ as representation into $\text{GL}_2(T^Q)$ is unique by a theorem of Carayol–Serre [MFG, Proposition 2.13], as $\text{Tr}(\rho_{Q,Q} (\text{Frob}))$ is given by the image of $T(l)$ in $T^Q$ for all primes $l$ outside $N_{Q,Q}$ by (Gal) in Section 1 (and by Chebotarév density theorem). We need to twist $\rho_{Q,Q}$ slightly by a character $\delta$ to have $\rho_{Q,Q} \otimes \delta$ satisfy (UQ). This twisting is done in the following way:

By (Gal), write $\rho_{Q,Q} \sim (\phi^Q_0, 1)$ as a representation of the inertia group $I_q$ for $q \in Q$. Then $\epsilon_q \equiv 1 \mod m_{\mathfrak{m}_Q}$ as $\overline{\rho}$ is unramified at $q$. Thus $\epsilon_q$ has $p$-power order factoring through the maximal $p$-abelian quotient $\Delta_q$ of $Z_{\mathfrak{m}}^\times$; so, it has a unique square root $\sqrt{\epsilon_q}$ with $\sqrt{\epsilon_q} \equiv 1 \mod m_{\mathfrak{m}_Q}$. Since $\Delta_q$ is a unique quotient of $(\mathbb{Z}/q\mathbb{Z})^\times = \text{Gal} (\mathbb{Q}(\mu_q)/\mathbb{Q})$, we can lift $\sqrt{\epsilon_q}$ to a unique global character of $\text{Gal}(\mathbb{Q}(\mu_q)/\mathbb{Q})$. Write $\sqrt{\epsilon} := \prod_{q \in Q} \sqrt{\epsilon_q}$ as a character of $\text{Gal}(\mathbb{Q}(\mu_q)_{q \in Q}/\mathbb{Q}) \cong \prod_{q \in Q} (\mathbb{Z}/q\mathbb{Z})^\times$. Then we define

$$\rho^Q := \rho_{Q,Q} \otimes \sqrt{\epsilon}^{-1}. \tag{2.2}$$

Then $\rho^Q$ satisfies (UQ) and $\rho^Q \in D^Q(T^Q)$. In the same manner, we can define a unique global character $\delta : \text{Gal}(\mathbb{Q}(\mu_q)_{q \in Q}/\mathbb{Q}) \rightarrow (R^Q)^\times$ such that $\delta|_{I_q} = \delta_q$ for all $q \in Q$.

By local class field theory, we identify $\Delta_q$ with the $p$-Sylow subgroup of $Z_{\mathfrak{m}}^\times$. Then the $p$-abelian group $\Delta_q$ defined above Theorem 1.2 has a canonical factorization: $\Delta_q := \prod_{q \in Q} \Delta_q$. By Lemma 1.1, the inertia action $I_q \rightarrow R^Q \rightarrow T^Q$ makes $T^Q$ free (of finite rank) over $W[\Delta_q]$, and hence $\Delta_q \rightarrow R^Q$ and $\Delta_q \rightarrow T^Q$. The character $\delta_q : I_q \rightarrow (R^Q)^\times$ (resp. $\delta^{-1}_q : I_q \rightarrow (R^Q)^\times$) extends uniquely to $\delta_q : \text{Gal}(\mathbb{Q}_q/Q)/q) \rightarrow R^Q$ (resp. $\delta^{-1}_q : \text{Gal}(\mathbb{Q}_q/Q)/q) \rightarrow R^Q$) so that

$$\rho^Q|_{\text{Gal}(\mathbb{Q}_q/Q)} = \begin{pmatrix} \delta_q & 0 \\ 0 & \delta^{-1}_q \end{pmatrix} \tag{2.3}$$

with $\delta_q(\phi_q)$ mod $m_{R^Q} = \overline{\rho}_q$ (resp. $\delta^{-1}_q(\phi_q)$ mod $m_{R^Q} = \overline{\rho}_q$) for any $\phi_q \in \text{Gal}(\mathbb{Q}_q/Q)$ with $\phi_q \mod I_q = \text{Frob}_q$ (e.g., [MFG, Theorem 3.32] or [HMI, Theorem 3.75]).

We choose $q|q$ for $q \in Q^+$ so that $\overline{\rho}(\text{Frob}_q) = \overline{\rho}_q$, and define $\Omega_+$ by the product over $q \in Q^+$ of $q$ thus chosen. Define the functor $\mathcal{D}^\Sigma_{F,Q} : CL_W \rightarrow SETS$ by

$$\mathcal{D}^\Sigma_{F,Q}(A) = \{ \lambda : \text{Gal}(\overline{\mathbb{Q}}/F) \rightarrow A^\times | \lambda \equiv \overline{\rho} \mod m_A \text{ has conductor a factor of } \Omega_+ \sqrt{\varphi} \}. \tag{1.6}$$

Hereafter we simply write $Z_Q$ for $Z_{Q^+}$. Then plainly $\mathcal{D}^\Sigma_{F,Q}$ is representable by $W[[Z_Q]] \cong W[[H_Q]]$ in (1.6). Here is a generalization of Corollary 2.5:

**Proposition 2.6.** Assume (h0–4). Let $I^Q = R^Q(\sigma_Q - 1)R^Q$. Then $R^Q/I^Q \cong W[[H_Q]]$ and $R^Q/I^Q \otimes_{\Lambda} A/(T) \cong W[\mathcal{C}_Q]$ for $C_Q$ defined above Theorem B.

**Proof.** Since the proof is basically the same for $H_Q$ and $C_Q$, we shall give a proof for $H_Q$. If a finite group $G$ acts on an affine scheme $\text{Spec}(A)$ over a base ring $B$, the functor $\text{Spec}(A)^G : C \mapsto \text{Spec}(A(C))^G = \text{Hom}_{B-alg}(A,C)^G$ sending $B$-algebras $C$ to the set of fixed points is a closed subscheme of $\text{Spec}(A)$ represented by $A_G := A/\sum_{g \in G} (g - 1)A$; i.e., $\text{Spec}(A)^G = \text{Spec}(A_G)$. Thus we need to prove that the natural transformation $\lambda \mapsto \text{Ind}_B^A \lambda$ induces an isomorphism $\mathcal{D}^\Sigma_{F,Q} \cong (D^Q)^G$, where $(D^Q)^G(A) = \{ \rho \in D^Q(A)|\rho \otimes \chi \cong \rho \}$. If $\rho \in D^Q(A)$, we have a unique algebra homomorphism $\phi : R^Q \rightarrow A$ such that $\rho \cong \phi \circ \rho^Q$ and $\rho|_{I_q} \cong \left( \begin{smallmatrix} \phi|_{I_q} & 0 \\ 0 & \phi|_{I_q} \end{smallmatrix} \right)$. This implies $\rho \otimes (\phi \circ \delta)|_{I_q} \sim (\phi \circ \delta)|_{I_q}$ for the global character $\delta : \text{Gal}(\mathbb{Q}(\mu_q)_{q \in Q}/\mathbb{Q}) \rightarrow (R^Q)^\times$, and hence its prime-to-p conductor is a factor of $N_Q$. On the other hand, for $\rho = \text{Ind}_B^A \lambda$ in $D^Q(A)$, if $\rho$ ramifies at $q \in Q^+$, the $q$-conductor of $\rho \otimes (\phi \circ \delta)$ is $N_{F/Q}(q) = q^2$, a contradiction as $q^2 \nmid N_Q$. Thus $\lambda$ is unramified at $q \in Q^+$, and we may assume $\lambda \in \mathcal{D}^\Sigma_{F,Q}(A)$. Indeed, among $\lambda, \lambda_1, \lambda_2 = \lambda(e \sigma e^{-1})$, we can characterize $\lambda$ uniquely (by (h0)) so that $\lambda \mod m_A = \overline{\rho}$. Thus $\mathcal{D}^\Sigma_{F,Q}(A) \rightarrow (D^Q)^G(A)$ is an injection. Surjectivity follows from [DH98, Lemma 3.2].

3. The Taylor–Wiles system and Taylor–Wiles primes

In their proof of Theorem 2.1, Taylor and Wiles used an infinite family $Q$ of finite sets $Q$ made of primes $q \equiv 1 \mod p$ outside $N$. We can choose infinitely many distinct $Q$s with $\overline{\rho}(\text{Frob}_q)$ for
Recall the projection $\pi_q$ of $\mathcal{F}(\text{Frob}_q)$ for each $q \in Q$, we have a unique local factor $\mathbb{T}_Q$ (resp. $\mathbb{T}_Q$) of the Hecke algebra $\mathfrak{h}^Q$ (resp. $\mathfrak{h}_{Q,k,v}$) as in (1.7), whose residual representation is isomorphic to $\mathfrak{p}$ and $U(q)$ mod $\mathfrak{m}_q$. This affects on the inertia action of $\Delta$ (with $\mathfrak{p}$). In [HMI, page 198], though in the original work of Taylor–Wiles, the choice is the $\mathbb{T}_Q$-factor $H_1(X(\mathcal{G}_Q), W) \otimes_{\mathfrak{h}_Q} \mathbb{T}_Q$ of the homology group $H_1(X(\mathcal{G}_Q), W)$ for the modular curve $X(\mathcal{G}_Q)$ associated to $\mathcal{G}_Q := \mathcal{G}_{\mathfrak{p},1}$ defined in (1.3).

The Hecke algebra $h_Q(\mathcal{G}_Q, \psi; W)$ has an involution coming from the action of the normalizer of $\Gamma_Q$. Taking $\gamma \in SL_2(\mathbb{Z})$ such that $\gamma \equiv \begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix}$ mod $2^3$ and $\gamma \equiv 1 \mod (N_{Q}/D)^2$, put $\eta := \gamma \begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix}$. Then $\eta$ normalizes $\Gamma_Q$, and the action of $\eta$ satisfies $\eta = 1, qU(l)^{-1} = \chi(l)(U(l))$ for each prime $l|N_{Q}/D$ and $\eta T(l)^{-1} = \chi(l)(T(l))$ for each prime $l|N_{Q}$ (see [FMF, (4.6.22), page 168]). Thus the conjugation of $\eta$ induces on $\mathfrak{T}_Q$ an involution compatible with $\sigma_Q$ under the canonical surjection $R_Q \rightarrow \mathbb{T}_Q$. Note that $\sigma_Q(U(q)) = -U(q)$ for $q \in \mathbb{Z}^+$, so, the role of $\overline{\pi}_q$ will be played by $-\overline{\pi}_q = \overline{\pi}_q$. This affects on the inertia action of $\Delta_q$ at $q$ by $\delta_q \rightarrow \delta_q^{-1}$ for $q \in \mathbb{Z}^+$, because the action is normalized by the choice of $\overline{\pi}_q$ with $\overline{\pi}_q \equiv U(q) \mod \mathfrak{m}_q$ (see Lemma 3.1 and [HMI, Theorem 3.74]). Since $\mathbb{T}_Q$ is the local component of the big Hecke algebra of tame level $\Gamma_Q$ whose reduction modulo $t - \gamma^k$ is $\mathbb{T}_Q$, again $\mathbb{T}_Q$ has involution $\sigma_Q$ induced from $\eta$. We write $\mathbb{T}_Q^+$ (resp. $\mathbb{T}_Q^+$) for the fixed subring of $\mathbb{T}_Q$ (resp. $\mathbb{T}_Q$) under the involution.

Since we follow the method of Taylor–Wiles for studying the local complete intersection property of $R_+ \cong \mathbb{T}_+$, we recall the Taylor–Wiles system argument (which proves Theorem 2.1) formulated by Fujiwara [Fu06] (see also [HMI, §3.2]). Identify the image of the inertia group $I_q$ for $q \in \mathbb{Q}$ in the Galois group of the maximal abelian extension over $\mathcal{O}_Q$ with $\mathbb{Z}_q^\times$ by the $q$-adic cyclotomic character. Recall the $p$-Sylow subgroup $\Delta_q$ of $\mathbb{Z}_q^\times$ and $\Delta_q := \bigcap_{q \in \mathbb{Q}} \Delta_q$ in (1.4). If $q \equiv 1 \mod p^m$ for $m > 0$ for all $q \in \mathbb{Q}$, $\Delta_q / \Delta_q^p$ for $0 < n \leq m$ is a cyclic group of order $p^n$. We put $\Delta_n := \bigcap_{q \in \mathbb{Q}} \Delta_q / \Delta_q^p$. By Lemma 1.1, the inertia action $I_q \rightarrow \mathbb{Z}_q^\times \rightarrow R_Q \rightarrow \mathbb{T}_Q$ makes $\mathbb{T}_Q$ free of finite rank over $W[\Delta_q]$. Then they found an infinite sequence $Q = \{Q_m|m = 1, 2, \ldots\}$ of ordered finite sets $Q = Q_m$ of primes $q$ (with $q \equiv 1 \mod p^m$) which produces a projective system:

$$\{(R_{n,m(n)}, \alpha = \alpha_n, \tilde{R}_{n,m(n)}, (f_1 = f_1^{(n)}, \ldots, f_r = f_r^{(n)}))\}_n$$

made of the following objects:

$$(1)\ R_{n,m} := \mathbb{T}_{Q_m}/(p^n, \delta_q^m - 1)_{q \in \mathbb{Q} \cap \mathbb{Q}_m} \mathbb{T}_{Q_m} \text{ for each } 0 < n \leq m. \text{ Since the integer } m \text{ in the system (3.1) is determined by } n, \text{ we have written it as } m. \text{ In [HMI, page 191], } R_{n,m} \text{ is defined to be the image of } \mathbb{T}_{Q_m} \text{ in } \text{End}_{W[\Delta_n]}(M_{n,m}) \text{ for } M_{n,m} := M_{Q_m}/(p^n, \delta_q^m - 1)_{q \in \mathbb{Q} \cap \mathbb{Q}_m} M_{Q_m}, \text{ but by our choice } M_Q = \mathbb{T}_Q, \text{ the image is identical to } \mathbb{T}_{Q_m}/(p^n, \delta_q^m - 1)_{q \in \mathbb{Q} \cap \mathbb{Q}_m} \mathbb{T}_{Q_m}. \text{ An important point is that } R_{n,m} \text{ is a finite ring whose order is bounded independent of } m \text{ (by (Q0) below).}$$

$$(2)\ \tilde{R}_{n,m} := R_{n,m}/(\delta_q - 1)_{q \in \mathbb{Q} \cap \mathbb{Q}_m} \mathbb{T}_{Q_m}. \text{ Notice that } \tilde{R}_{n,m} \cong R_{n,m}/(\delta_q - 1)_{q \in \mathbb{Q} \cap \mathbb{Q}_m} \mathbb{T}_{Q_m}. \text{ The order of } \tilde{R}_{n,m} \text{ is } \Delta_n / \Delta_n^p. \text{ Thus we have chosen } M_Q \text{ to be } \mathbb{T}_Q, \text{ and define } R_{n,m} \text{ by definition } R_{n,m}; \text{ so, } \beta \text{ is just the identity map (and hence we forget about it).}$$

The infinite set $Q$ satisfies the following conditions (Q0–8):
(Q0) \( M_{Q_m} = T_{Q_m} \) is free of finite rank \( d \) over \( W[\Delta_{Q_m}] \) with \( d \) independent of \( m \) (see Lemma 1.1 and the remark after the lemma and [HMI, 190 and 199] taking \( M_{Q_m} := T_{Q_m} \)).

(Q1) \( |Q_m| = r \geq \dim D_{Q_m,k,\psi}(\mathbb{F}[e]) \) for \( r \) independent of \( m \) [HMI, Propositions 3.29 and 3.33], where \( \epsilon \) is the dual number with \( \epsilon^2 = 0 \). (Note that \( \dim D_{Q_m,k,\psi}(\mathbb{F}[e]) \) is the minimal number of generators of \( R_{Q_m} \) over \( W_{sp} \).

(Q2) \( q \equiv 1 \mod p^n \) and \( \overline{p}(\text{Frob}_q) \sim (\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}) \) with \( \overline{\pi}_q \neq \overline{\pi}_q \in \mathbb{F} \) if \( q \in Q_m \) (so, \( |\Delta_q| = p^e \geq p^n \)). Actually as we will see later in Lemma 3.2, we can impose a slightly stronger condition: \( q \equiv 1 \mod C \) for \( C = N_{E/F}(\mathbb{F}) \).

(Q3) the set \( Q_m = \{ q_1, \ldots, q_r \} \) is ordered so that
- \( \Delta_{q_j} \subset \Delta_{Q_m} \) is identified with \( \mathbb{Z}/p^n \mathbb{Z} \) by \( \delta_{q_j} \mapsto 1 \); so, \( \Delta_n = \Delta_{n,Q_m(n)} = (\mathbb{Z}/p^n \mathbb{Z})^{Q_m(n)} \),
- \( \Delta_n = (\mathbb{Z}/p^n \mathbb{Z})^{Q_m(n)} \) is identified with \( \Delta_{n+1}/\Delta_{n+1} = ((\mathbb{Z}/p^{n+1} \mathbb{Z})/p^n(\mathbb{Z}/p^{n+1} \mathbb{Z}))^{Q_m(n)} \),
- the diagram

\[
\begin{array}{ccc}
W_{n+1}[\Delta_{n+1}] & \xrightarrow{\alpha_{n+1}} & R_{n+1,m(n+1)} \\
\downarrow & & \downarrow \\
W_n[\Delta_n] & \xrightarrow{\alpha_n} & R_{n,m(n)}
\end{array}
\]

is commutative for all \( n > 0 \) (and by (Q0), \( \alpha_n \) is injective for all \( n \)).

(Q4) There exists an ordered set of generators \( \{ f_1^{(n)}, \ldots, f_r^{(n)} \} \subset m_{R,m(n)}(n) \) of \( R_{n,m(n)}(n) \) over \( W \) for the integer \( r \) in (Q1) such that \( \pi_{n+1}(f_j^{(n+1)}) = f_j^{(n)} \) for each \( j = 1, 2, \ldots, r \). In our case of \( \mathfrak{p} = \text{Ind}_{\mathfrak{p}}(\mathfrak{p}) \), we can make the choice of generators canonical to good extent dependent on \( Q_{m(n)}(n) \) (see Theorem 6.4).

(Q5) \( R_{\infty} := \lim_{\longrightarrow} R_{n,m(n)}(n) \) is isomorphic to \( W[[T_1, \ldots, T_r]] \) by sending \( T_j \) to \( f_j^{(\infty)} := \lim_{\longleftarrow} f_j^{(n)} \) for each \( j \) (e.g., [HMI, page 193]).

(Q6) Inside \( R_{\infty} \), \( \lim_{\longrightarrow} W_n[\Delta_n] \) is isomorphic to \( W[[S_1, \ldots, S_r]] \) so that \( s_j := (1 + S_j) \) is sent to the generator \( \delta_{q_j}(\Delta_n) / \Delta_n^{(n)} \) for the ordering \( q_1, \ldots, q_r \) of primes in \( Q_m \) in (Q3).

(Q7) \( R_{\infty}/(S_1, \ldots, S_r) \cong \lim_{\longrightarrow} \tilde{R}_{n,m(n)}(n) \cong R_{\Phi} \cong T_{\mathcal{H}} \), where \( R_{\Phi} \) is the universal deformation ring for the deformation functor \( D_{\mathfrak{p},k,\psi} \) and \( T_{\mathcal{H}} \) is the local factor of the Hecke algebra \( h_{\mathfrak{p},k,\psi} \) whose residual representation is isomorphic to \( \mathfrak{p} \).

(Q8) We have \( R_{Q_m} \cong T_{Q_m} \) by the canonical morphism, and \( R_{Q_m} \cong R_{\infty}/(S_{Q_m}) \) for the ideal \( \mathfrak{I}_{Q_m} := (1 + S_j)^{(1)} \) of \( W[[S_1, \ldots, S_r]] \) is a local complete intersection.

All the above facts (Q0–Q8) follows, for example, from [HMI, Theorem 3.23] and its proof. Since \( m(n) \) is determined by \( n \), if confusion is unlikely, we simply drop “\( m(n) \)” from the notation (so, we often write \( R_n \) for \( R_{n,m(n)} \)). For \( q \in Q = Q_m \), we write \( S_q \) for the one of the variables in \( \{ S_1, \ldots, S_r \} \) in (Q6) corresponding to \( q \).

Lemma 3.1. Let \( \chi := \left( \begin{array}{c} F/Q \\ \mathcal{H} \end{array} \right) \) as before. Then the involution \( \sigma_{Q_m} \) on \( T_{Q_m} \) acts on \( \delta_{q}(I_q) \) (the image of \( s_q = 1 + S_q \)) for \( q \in Q_m \) by \( \sigma_{Q_m}(\delta_{q}(I_q)) = (\delta_{q}(I_q))^{(n)} \). In particular, the ideal \( (p^n, \delta_q^{(n)} - 1)_{Q_m} \) of \( T_{Q_m} \) is stable under \( \sigma_{Q_m} \), and the involution \( \sigma_{Q_m} \) induces an involution \( \sigma = \sigma_n \) of \( R_n = R_{n,m} \).

Proof. For each \( q \in Q \), by (2.1), the restriction of \( \rho^Q \) to the inertia group \( I_q \subset \text{Gal}(\overline{\mathbb{Q}}_{\mathfrak{p}}/\mathbb{Q}) \) has the form \( \begin{pmatrix} 0 & 1 \\ \delta_q & 0 \end{pmatrix} \) and the choice of the eigenvalue \( \overline{\pi}_q \) determines the character \( \delta_q \) (i.e., \( \overline{\pi}_q \)-eigenspace of \( \overline{\rho}(\text{Frob}_q) \) is the image of \( \delta_q^{-1} \)-eigenspace in \( \overline{\mathfrak{p}} \) by (2.3); see also [MFG, Theorem 3.32 and its proof] or [HMI, Theorem 3.75]). By tensoring \( \chi \), \( \overline{\pi}_q \) is transformed to \( \chi(q) \overline{\pi}_q = \overline{\pi}_q \), and hence \( \delta_q \) will be transformed to \( \delta_q^{(n)} \) under \( \sigma_{Q_m} \). Thus, we get the desired result as the canonical morphism \( R_{Q_m} \to T_{Q_m} \) is \( W[\Delta_{Q_m}] \)-linear.

Since \( \delta_q^{p^n} - 1 = -\delta_q^{p^n}(\delta_q^{p^n} - 1) \), the ideal \( (p^n, \delta_q^{p^n} - 1)_{Q_m} \) of \( T_{Q_m} \) is stable under \( \sigma_{Q_m} \). Therefore \( \sigma_{Q_m} \in \text{Aut}(T_{Q_m}) \) induces an involution \( \sigma_n \) on \( R_n = R_{n,m} = T_{Q_m}/(p^n, \delta_q^{p^n} - 1)_{Q_m} \). □

We recall the way Wiles chose the sets \( Q \) as we make a finer choice building on his way relating \( q \in Q^* \) with generator choice \( f_j \). Write \( Ad \) for the adjoint representation of \( \mathfrak{p} \) acting on \( \mathfrak{sl}_2(\mathbb{F}) \) by conjugation, and put \( Ad^* \) for the \( \mathbb{F} \)-contradegent. Then \( Ad^*(1) \) is one time Tate twist of \( Ad^* \). Note
that $Ad^* \cong Ad$ by the trace pairing as $p$ is odd. Let $Q$ be a finite set of primes, and consider
\[
\beta_Q : H^1(Q^{(Q^{Np})}/Q, Ad) \to \prod_{q \in Q} H^1(Q_q, Ad),
\]
\[
\beta'_Q : H^1(Q^{(Q^{Np})}/Q, Ad^*(1)) \to \prod_{q \in Q} H^1(Q_q, Ad^*(1)).
\]

Here is a lemma due to A. Wiles [Wi95, Lemma 1.12] which shows the existence of the sets $Q_m$. We state the lemma slightly different from [Wi95, Lemma 1.12], and for that, we write $K_1 = \mathbb{Q}^{(\ker Ad)}$ (the splitting field of $Ad = Ad(\overline{n})$). Since $Ad \cong \mathbb{Q} \oplus Ind^{\overline{p}}_{p} \mathbb{Q}$, we have $K_1 = F(p^-)$.

**Lemma 3.2.** Assume (W). Pick $0 \neq x \in \ker(\beta'_Q)$ and $0 \neq y \in \ker(\beta_Q)$, and write
\[
f_x : \text{Gal}(Q^{(Q^{Np})}/K_1(\mu_p)) \to Ad^*(1) \in \text{Hom}_{\text{Gal}(K_1(\mu_p)/Q)}(\text{Gal}(Q^{(Q^{Np})}/K_1(\mu_p), Ad^*(1))
\]
\[
f_y : \text{Gal}(Q^{(Q^{Np})}/K_1) \to Ad \in \text{Hom}_{\text{Gal}(K_1/Q)}(\text{Gal}(Q^{(Q^{Np})}/K_1), Ad)
\]
for the restriction of the cocycle representing $x$ and $y$ to $\text{Gal}(Q^{(Q^{Np})}/K_1(\mu_p))$ and $\text{Gal}(Q^{(Q^{Np})}/K_1)$, respectively. Let $\overline{\rho}$ be the composite of $\mathfrak{p}$ with the projection $GL_2(F) \to PGL_2(F)$, and pick a positive integer $C$ which is a product of primes $l \neq p$ split in $F/Q$. Then, $f_x$ (resp. $f_y$) factors through $\text{Gal}(Q^{(Np)}/K_1(\mu_p))$ (resp. $\text{Gal}(Q^{(Np)}/K_1)$), and there exists $\sigma \in \text{Gal}(Q^{(Np)}/Q)$ for $? = x, y$ such that

1. $\overline{\rho}(\sigma) \neq 1$ (so, $Ad(\sigma) \neq 1$),
2. $\sigma$ fixes $Q(\mu_{C^p_m})$ for an integer $m > 0$,
3. $f_\sigma(\sigma^a) \neq 0$ for $a := \text{ord}(\overline{\rho}(\sigma)) = \text{ord}(Ad(\sigma))$.

The argument is the same for $x$ and $y$, we give Wiles’ proof in details for $x$ and indicate how to modify the argument for $y$ at the end of the proof. Strictly speaking, [Wi95, Lemma 1.12] gives the above statement replacing $K_1$ by the splitting field $K_0$ of $\mathfrak{p}$. Since the statement is about the cohomology group of $Ad$ and $Ad^*(1))$, we can replace $K_0$ in his argument by $K_1$. We note also $\ker(Ad(\overline{n})) = \ker(\overline{\rho})$ as the kernel of the adjoint representation: $GL(2) \to GL_3$ is the center of $GL_2$ (so it factors through $PGL_2$).

**Proof.** Since $x \in \ker(\beta'_Q)$, $f_x$ is unramified at $q \in Q$; so, $f_x$ factors through $\text{Gal}(Q^{(Np)}/K_1(\mu_p))$.

We have two possibilities of $F' := K_1 \cap \mathbb{Q}(\mu_{C^p_m})$; i.e., $F' = \mathbb{Q}$ or a quadratic extension of $\mathbb{Q}$ disjoint from $F$. Indeed, the maximal abelian extension of $\mathbb{Q}$ inside $K_1$ is either $F$ (when $\text{ord}(\mathfrak{p})$ is odd > 1) or a composite $FF'$ of the quadratic extensions $F$ and $F'$ over $\mathbb{Q}$ (if $\text{ord}(\mathfrak{p})$ is even $2n > 2$). If $\mathfrak{p}$ has odd order, $F' = \mathbb{Q}(\mu_{C^p_m}) \cap K_1 = \mathbb{Q}$ as it is a subfield of $F$ and $\mathbb{Q}(\mu_{C^p_m})$ (because $(C, D) = 1$ and $F \cap \mathbb{Q}(\mu_p) = \mathbb{Q}$).

Assume that $\text{ord}(\mathfrak{p}) = 2n > 3$. Let $D := \text{Gal}(K_1/Q)$ and $C := \text{Gal}(K_1/F)$. Then $C$ is a cyclic group of order $2n$. Pick a generator $g \in C$. Then $D = C \cup Cc$ for complex conjugation $c$, and we have a characterization $Cc = \{r \in D | \tau gr^{-1} = g^{-1}, \tau^2 = 1\}$. For the derived group $D'$ of $D$, we have $D'^{ab} := D/D' \cong (\mathbb{Z}/2\mathbb{Z})^2$. We have $K_1^P = FF'$, and $\text{Gal}(K_1/F')$ is equal to $C^2 \rtimes \langle \alpha \rangle$ (a dihedral group of order $2n$). If $n > 2$ (so, $2n > 4$), $\text{Ind}^{\overline{\mathfrak{p}}}_{\mathfrak{p}}$ restricted to $\text{Gal}(K_1/F')$ is still irreducible isomorphic to $\text{Ind}^{\overline{\mathfrak{p}}}_{\mathfrak{p}}$. If $n = 2$, $F'$ is a unique quadratic extension in $K_1^P$ unramified at $D$. In any case, $F' \neq F$ which is quadratic over $\mathbb{Q}$. Since $F' = \mathbb{Q}(\mu_{C^p_m}) \cap K_1$ is at most quadratic disjoint from $F$, we can achieve (1)-(2) by picking up suitable $\sigma$ in $C^2 \rtimes \langle \alpha \rangle$ because $Ad = \mathbb{Q} \oplus \text{Ind}^{\overline{\mathfrak{p}}}_{\mathfrak{p}}$.

Let $M_x := \mathbb{Q}^{\ker(f_x)}$. Then $Y := \mathbb{Q}(M_x/K_1(\mu_p))$ is embedded into $Ad^*(1)$ by $f_x$ and $f_x$ is equivariant under the action of $\text{Gal}(K_1(\mu_p)/Q)$ which acts on $Y$ by conjugation. Since $Ad = \mathbb{Q} \oplus \text{Ind}^{\overline{\mathfrak{p}}}_{\mathfrak{p}}$, we have two irreducible invariant subspaces $X \subset Ad^*(1): X = \mathbb{Q}^\sigma$ and $\text{Ind}^{\overline{\mathfrak{p}}}_{\mathfrak{p}}(\mathfrak{p}^\sigma)$. Thus $f_x(Y)$ contains one of $X$ as above. By (1), $\mathfrak{p}(\sigma) \sim \langle 0 \; \alpha \rangle$ with $\alpha \neq \beta$. By (2), $\alpha \beta = 1$ and hence $\alpha$ is a primitive $\alpha$-th root of unity with $\alpha > 1$, and $\alpha \not\in \{ \pm 1 \}$. The eigenvalue of $Ad^*(1)(\sigma)$ is therefore $\alpha^2, 1, \alpha^2$, which are distinct.

If $f_x(Y) \supset X$, we claim to find $\sigma$ satisfying (1), (2) and $\sigma$ has eigenvalue 1 in $X$. If $X = \mathbb{Q}^\sigma$, the splitting field of $X$ is $F(\mu_p)$. Note that $F(\mu_{C^p_m})$ is abelian over $\mathbb{Q}$. Thus choosing $\sigma$ fixing $F(\mu_{C^p_m})$ with $\sigma \in C^2 \rtimes K_1$ and having $\text{ord}(\mathfrak{p}^\sigma) \geq \text{ord}((\mathfrak{p}^\sigma)^2) = |C^2| \geq 2$, we have $\sigma$ having eigenvalue 1 on $X = \mathbb{Q}^\sigma$. 
If $X = \text{Ind}^Q_F \mathfrak{g}$, we just choose $\sigma \in \text{Gal}(K(1(\mu_{C_p^m})/Q(\mu_{C_p^m})))$ inducing the non-trivial automorphism on $F$ (i.e., the projection to the factor $\langle c \rangle$ of $C^2 \times \langle c \rangle$ is non-trivial). Since $\sigma$ fixes $Q(\mu_{C_p^m})$, we have $\omega(\sigma) = 1$; so, we forget about $\omega$-twist. Then on $\chi$, $\text{Ad}(\sigma)$ has eigenvalue $-1$, and hence $\text{Ad}(\sigma)$ has to have the eigenvalue $1$ on $\text{Ind}^Q_F \mathfrak{g}$.

Since $f_x(Y) \supset X[1] = \{v \in X|\text{Ad}(\sigma)(v) = v\}$, we can find $1 \neq \tau \in Y$ such that $f_x(\tau) \in X[1]$; so, $f_x(\tau) \neq 0$. Thus $\tau$ commutes with $\sigma \in \text{Gal}(M_x/Q)$. This shows $(\sigma \tau)^a = \sigma \tau \sigma, f_x(\sigma \tau)^a = f(\sigma \tau)^a = f(x \tau)^a$. Since $f_x(\tau) \neq 0$, at least one of $f(\sigma \tau)^a$ and $f(\sigma)^a$ is non-zero. Then $\sigma^a_\tau = \sigma_\tau$ satisfies the condition (3) in addition to (1–2).

Now we describe the case for $f_y$. In this case, we write $M_x$ for the splitting field of $f_y$ over $K_1$. We put $Y := \text{Gal}(M_y/K_1)$. Since $\text{Ad} = \mathfrak{g} \oplus \text{Ind}^Q_F \mathfrak{g}$, for $X = \chi$ or $\text{Ind}^Q_F \mathfrak{g}$, we have $f_y(Y) \supset X$. Then we argue in exactly the same way as above and find $\sigma_y$ with the required property.

Let $Q = \emptyset$ and choose a basis $\{x\}_x$ over $F$ of the “dual” Selmer group $\text{Sel}_Q^* (\text{Ad}^*(1))$ inside $H^1(Q(\mathcal{N}_p)/Q, \text{Ad}^*(1))$ (see (3.2) below for the definition of the Selmer group). Then Wiles’ choice of $Q_m$ is a set of primes $p$ so that $\text{Frob}_p = \sigma_x$ on $M_x$ as in the above lemma. By Chebotarev density, we have infinitely many sets $Q_m$ with this property.

**Corollary 3.3.** Let the assumptions and the notation be as in Lemma 3.2 and its proof. Assume that $0 \neq f_x(\text{Gal}(M_x/K_1)) \subset \text{Ind}^Q_F \mathfrak{g}$ and $0 \neq f_y(\text{Gal}(M_y/K_1)) \subset \text{Ind}^Q_F \mathfrak{g}$. If $p > 3$, then there exists $\sigma \in \text{Gal}(Q(\mathcal{N}_p)/Q)$ for $x = y$ such that

1. $\rho(\sigma) \neq 1$,
2. $\sigma$ fixes $Q(\mu_{C_p^m})$,
3. $f_x(\sigma^a) \neq 0$ and $f_y(\sigma^a) \neq 0$ for $a := \text{ord}(\mathfrak{g}(\sigma))$.

**Proof.** We first show that $\mathfrak{g}$ on $\text{Gal}(K_1(K_1(\mu_p)/K_1)$ is non-trivial. Let $F' = K_1 \cap F(\mu_p)$. Thus $F'$ is an abelian extension of $Q$. Since $\text{Gal}(K_1/Q)$ is a dihedral group, its maximal abelian quotient has either order 2 or order 4 according as $\varphi^-$ has odd order or even order. If it has order 4, we have $\text{Gal}(F'/Q) \cong (\mathbb{Z}/2\mathbb{Z})^2$ (as any element in the Galois group has order 2). Thus if $\varphi^-$ has an odd order, we have $F' = F$.

Suppose $\text{Gal}(F'/Q) \cong (\mathbb{Z}/2\mathbb{Z})^2$. Then $\varphi^-$ must have even order $2n > 3$. Since we have only three quadratic subfields in $F(\mu_p)$, i.e., $F, \sqrt{p}$ and $\sqrt{-D}$ for $p^* = (-1)^{(p-1)/2}$. We have $F' = F(\sqrt{p})$ which is a splitting field of $\mathfrak{g}$. Thus if $F(\mu_p) = F'$, we have $p = 3$. This is not the case, as assumed that $p > 3$. Thus $K_1(\mu_p) \neq K_1$ and hence $\mathfrak{g}$ on $\text{Gal}(K_1(K_1(\mu_p)/K_1)$ is non-trivial.

Note that $f_x(\text{Gal}(M_x/K_1(\mu_p)) \subset \text{Ind}^Q_F \mathfrak{g}$ and $f_y(\text{Gal}(M_y/K_1)) \subset \text{Ind}^Q_F \mathfrak{g}$ and that $f_x$ and $f_y$ are $\mathbb{F}[\text{Gal}(K_1(\mu_p)/Q)]$-linear. Here the Galois group $\tau \in \text{Gal}(K_1(\mu_p)/Q)$ acts on $\text{Gal}(M_x/K_1(\mu_p))$ and $\text{Gal}(M_y(K_1(\mu_p))$ by $\tau g = \tau g \tilde{\tau}^{-1}$ taking a lifting $\tilde{\tau} \in \text{Gal}(Q(\mathcal{N}_p)/K_1(\mu_p))$ of $\tau$ (i.e., $\tilde{\tau}|_{K_1(\mu_p)} = \tau$). We may assume that $\mathfrak{g}$ is generated over $F_p$ by the values of $\mathfrak{g}$ (and hence $\mathfrak{g}$ is generated over $F_p$ by the values of $\mathfrak{g}$ as $\mathfrak{g}$ has values in $F_p$). Thus $f_y$ restricted to $\text{Gal}(Q(\mathcal{N}_p)/K_1(\mu_p))$ has image isomorphic to $\text{Ind}^Q_F \mathfrak{g}$ as $\text{Gal}(Q(\mathcal{N}_p)/Q)$-modules; i.e.,

$$X_y := f_y(\text{Gal}(M_y(\mu_p)/K_1)) = f_x(\text{Gal}(M_x(K_1(\mu_p))) := X_x$$

as $\text{Gal}(Q(\mathcal{N}_p)/Q(\mu_p))$-modules, since $\text{Gal}(K_1\mu_p)/Q)$ acts on $X_y$ via the quotient $\text{Gal}(F(\mathfrak{g})/Q)$ for the splitting field $F(\mathfrak{g}) = \overline{\text{Ker}(\mathfrak{g})}$. Thus we have two possibilities inside $\text{Gal}(Q(\mathcal{N}_p))$: (i) $M_x(\mu_p) = M_x$ or (ii) $M_y(\mu_p) \neq M_x$. If $\text{Ind}^Q_F \mathfrak{g} \cong \text{Ind}^Q_F \mathfrak{g}$ as $\text{Gal}(Q(\mathcal{N}_p)/Q)$-modules, $\mathfrak{g}$ has to be quadratic and $p = 3$ and $p$ ramifies in $F'$. This is impossible by our assumption $p \nmid D$. Thus we may assume that $\text{Ind}^Q_F \mathfrak{g} \neq \text{Ind}^Q_F \mathfrak{g} \cong \text{Gal}(K_1(\mu_p)/Q)$-modules.

Since $f_x$ and $f_y$ are $F[\text{Gal}(K_1(\mu_p)/Q)]$-linear and the values of $\mathfrak{g}$ generate $F$ over $F_p$, $X_x \cong \text{Ind}^Q_F \mathfrak{g}$ as $\text{Gal}(K_1(\mu_p)/Q)$-modules and $X_y \cong \text{Ind}^Q_F \mathfrak{g}$ as $\text{Gal}(K_1(\mu_p)/Q)$-modules. In particular, elements $\in \text{Gal}(K_1(K_1(\mu_p)/K_1)$ acts on $X_x$ by the scalar multiplication by $\mathfrak{g}(\tau)$ and on $X_y$ trivially. As a $\text{Gal}(K_1(K_1(\mu_p)/K_1)$-module, the Galois group $Z := \text{Gal}(M_y(K_1)/K_1(\mu_p))$ is the quotient of $X_x$ and $X_y$, which implies $Z$ is trivial; so, $M_y(\mu_p) \cap M_x = K_1(\mu_p)$. Thus $M_y(\mu_p)$ and $M_x$ are linearly disjoint over $K_1(\mu_p)$. Then starting with the same $\sigma_0$ as in the proof of Lemma 3.2 to find $\sigma_x$ and $\sigma_y$, we find $\sigma_x \in \text{Gal}(M_x(\mu_p)/Q)$ and $\sigma_y \in \text{Gal}(M_y(\mu_p)/Q)$ (satisfying the requirements of Lemma 3.2 respectively for $\text{Ind}^Q_F \mathfrak{g}$ and $\text{Ind}^Q_F \mathfrak{g}$) which coincides with $\sigma_0|_{K_1(\mu_p)}$ over $K_1(\mu_p)$. Note that $\sigma_0^{-1} \sigma_x|_{M_x} \neq \sigma_0^{-1} \sigma_y|_{M_y(\mu_p)}$ is in $\text{Gal}(M_x/K_1(\mu_p)) \times \text{Gal}(M_y(K_1)/K_1(\mu_p))$ which is isomorphic (by the)
restriction maps) to $\text{Gal}(M_z M_p/K_1(\mu_p))$ by linear disjointness of $M_z$ and $M_p(\mu_p)$ over $K_1(\mu_p)$ for the composite $M_z M_p$. Thus we can find $\sigma' \in \text{Gal}(\mathbb{Q}(N^p)/K_1(\mu_p))$ such that $\sigma'|_{M_z} = \sigma_0^{-1}\sigma_x|_{M_z}$ and $\sigma'|_{M_p(\mu_p)} = \sigma_0^{-1}\sigma_y|_{M_p(\mu_p)}$. Then $\sigma := \sigma_0\sigma'$ does the job. \hfill \Box

**Corollary 3.4.** Let the notation be as in Lemma 3.2 and its proof. If $0 \neq f_2(Y) \subset \text{Ind}_F^Q \varphi \varpi$, the field automorphism $\sigma$ in Lemma 3.2 satisfies $\left( \frac{F/Q}{\sigma} \right) = -1$. Otherwise, we can choose $\sigma$ so that $\left( \frac{F/Q}{\sigma} \right) = 1$.

**Proof.** In this case, we can have $X[1] \subset \text{Ind}_F^Q \varphi \varpi \neq 0$; so, $\text{Ad}(\sigma)(1) = \text{Ad}(\sigma)$ (as $\omega(\sigma) = 1$) must have two distinct eigenvalues $\{1, -1\}$ on $\text{Ind}_F^Q \varphi$, which implies $\left( \frac{F/Q}{\sigma} \right) = -1$ as $\sigma$ has to have eigenvalues $-1$ with multiplicity 2. \hfill \Box

**Remark 3.5.** The space $X[1] \subset \text{Ind}_F^Q \varphi \varpi$ in the above proof of the corollary is given by the subspace of “anti-scalars” $\{\frac{1}{a} \varphi|_a \} \subset \text{Ad}^*(1)$, and therefore the anti-diagonal trace map $T : \text{Ad}^*(1) \rightarrow F$ induces an isomorphism $X[1] \cong F$ and $\text{Ker}(T) = X[-1]$ (as $p > 2$). Here the anti-diagonal trace means $T(a \varphi|_a) = b + c$. This fact becomes important later in the proof of Corollary 6.3.

**Definition 3.6.** Let $\mathcal{Y}^-$ be the Galois group over $K_0 F(\phi)$ of the maximal $p$-abelian extension $L_\phi$ of $K_0 F(\phi)$ unramified outside $p$. Define $\mathcal{Y}^-(\phi) := \mathcal{Y}^- \otimes_{\mathbb{Z}_p[\phi]} \text{Gal}(F(\phi)/F), \phi \mathbb{Z}_p[\phi]$ similarly to $Y^-(\phi)$ in the introduction. More generally write $\mathcal{Y}_Q^-$ for the Galois group over $K_\phi F(\phi)$ of the maximal $p$-abelian extension $L_Q$ of $K_\phi F(\phi)$ unramified outside $p$ and $Q$. Define $\mathcal{Y}_Q^-(\phi) := \mathcal{Y}_Q^- \otimes_{\mathbb{Z}_p[\phi]} \text{Gal}(F(\phi)/F), \phi \mathbb{Z}_p[\phi]$.

Thus we have a natural restriction map $\mathcal{Y}^- \rightarrow \mathcal{Y}^-$ which is an isomorphism if $p \nmid h_F$. In particular $\mathcal{Y}^-(\phi) = \mathcal{Y}^-(\phi)$ if $p \nmid h_F$.

Let $D_Q := D_{Q,k,\psi_k}$ and $D_Q^p$ for the corresponding local functor at a prime $l|\mathcal{N}_Q$ defined below (det) in Section 2. Regard $D_Q^p(F[e])$ for the dual number $e$ as a subspace of $H^1(Q, Ad)$ in the standard way: For $r \in D_Q^p(F[e])$, we write $r \mathbb{Z}_p^{-1} = 1 + eu_p$. Then $u_p$ is the cocycle with values in $s\phi_Q(F) = Ad$. Thus we have the orthogonal complement $D_Q^p(F[e])^\perp \subset H^1(Q, A_d(1))$ under Tate local duality. We recall the definition of the Selmer group giving the global tangent space $D_Q(F[e])$ and its dual from the work of Wiles–Taylor–Wiles (e.g., [HMI, §3.2.4]):

$$\text{Sel}_Q(Ad) := \text{Ker}(H^1(Q^{QN^p}/Q, Ad) \rightarrow \prod_{l|\mathcal{N}_p} H^1(Q_l, Ad)/D_Q^p(F[e])) \cong D_Q(F[e]),$$

$$\text{Sel}_Q(Ad^*(1)) := \text{Ker}(H^1(Q^{QN^p}/Q, Ad^*(1)) \rightarrow \prod_{l|\mathcal{N}_p} \frac{H^1(Q_l, Ad^*(1))}{D_Q^p(F[e])^\perp} \times \prod_{q \in Q} H^1(Q_q, Ad^*(1)).$$

**Remark 3.7.** As noticed in [CV03, Theorem 3.1], the decomposition $Ad = \chi \oplus \text{Ind}_F^Q \varphi \varpi$ for $\chi := (\chi \mod p)$, $\text{Sel}_Q(Ad)$ (resp. $\text{Sel}_Q(Ad^*(1))$) induces the direct sum of the Selmer groups $\text{Sel}_Q(\chi)$ (resp. $\text{Sel}_Q(\text{Ind}_F^Q \varphi \varpi)$) and $\text{Sel}_Q(\text{Ind}_F^Q \varphi \varpi)$ (resp. $\text{Sel}_Q(\text{Ind}_F^Q \varphi \varpi)$).

To prove the direct sum decomposition in Remark 3.7, we need to decompose $D_Q^p(F[e])^\perp$ as in (3.3) (which is equivalent to the decomposition of the original $D_Q^p(F[e])$). We consider $\text{Sel}_Q(Ad^*(1))$ (whose decomposition as above is equivalent to (3.3) below). Then $D_Q^p(F[e])$ is made of classes of cocycles such that $u_p$ is upper nilpotent and $u_p|\text{Gal}(\mathbb{Q}_p/Q)$ is upper triangular. Thus we confirm for $l = p$ that

$$D_Q^p(F[e])^\perp = (D_Q^p(F[e])^\perp \cap H^1(Q_1, \chi \mathbb{Z}_p)) \oplus (D_Q^p(F[e])^\perp \cap H^1(Q_1, \text{Ind}_F^Q \varphi \varpi)),$$

and $D_Q^p(F[e])^\perp \cap H^1(Q_p, \text{Ind}_F^Q \varphi \varpi)$ is made of upper nilpotent matrices in $Ad^*(1)$ (since $\text{Ind}_F^Q \varphi \varpi$ (1) is the direct sum of the upper nilpotent Lie algebra and the lower nilpotent Lie algebra). Therefore $D_Q^p(F[e])^\perp \cap H^1(Q_p, \text{Ind}_F^Q \varphi \varpi)$ is the direct factor $H^1(F_p, \varphi \varpi)$ of

$$H^1(F_p, \text{Ind}_F^Q \varphi \varpi) = H^1(F_p, \varphi \varpi) \oplus H^1(F_p, \varphi^{-1} \varpi).$$
This implies all non-zero classes in $H^1(\mathcal{Q}_p, \overline{\omega})$ are ramified. Similarly, since $\overline{\chi}$ is unramified and $\overline{Z}$ has cohomological dimension 1, we have a commutative diagram with exact rows:

$$
\begin{array}{c}
H^1(\text{Frob}^2_{\mathcal{Q}_p}, \overline{\chi}) \\ \downarrow \\
\text{Hom}(\text{Frob}^2_{\mathcal{Q}_p}, \mathbb{F}) \\
\end{array}
\begin{array}{c}
H^1(\mathcal{Q}_p, \overline{\chi}) \\ \downarrow \\
\text{Hom}(\mathcal{Q}_p, \mathbb{F}) \\
\end{array}
\begin{array}{c}
H^1(\mathcal{Q}_p, \overline{\chi})_{\text{Frob}^2_{\mathcal{Q}_p}}=1 \\
\end{array}
$$

By the requirement of the cocycle in $D^*_p([\mathbb{F}]_1)$ is upper nilpotent over $I_p$ and is upper triangular over $D^*_p := \text{Gal}(\overline{\mathcal{Q}_p}/\mathcal{Q}_p)$, we have $D^*_p([\mathbb{F}]_1) \cap H^1(\mathcal{Q}_p, \overline{\chi}) = \text{Hom}(\text{Frob}^2_{\mathcal{Q}_p}, \mathbb{F})$ whose $p$-local Tate dual is $(p^2/p^p)^\perp \otimes \mathbb{F} \subset (\mathcal{Q}_p^\times/(\mathcal{Q}_p^\times)^p) \otimes \mathbb{F} = H^1(\mathcal{Q}_p, \overline{\omega})$ by Kummer theory. Thus we have

$$
D^*_p([\mathbb{F}]_1) \cap H^1(\mathcal{Q}_p, \overline{\omega}) = H^1(\mathcal{Q}_p, \overline{\omega})_{\text{Frob}^2_{\mathcal{Q}_p}}=1 = (\mathcal{Z}_p^\times/(\mathcal{Z}_p^\times)^p) \otimes \mathbb{F}.
$$

So, it is ramified, and hence

(Km) the Selmer cocycle $u$ in $\text{Sel}_1^p(\overline{\omega})$ for $\overline{\omega}$ can ramify at $p$ and is a Kummer cocycle in $(\mathcal{Q}_p^\times/(\mathcal{Q}_p^\times)^p) \otimes \mathbb{F} \subset (\mathcal{Q}_p^\times/(\mathcal{Q}_p^\times)^p) \otimes \mathbb{F}$ projecting down trivially to $\mathbb{F}$ by sending $z \in \mathcal{Q}_p^\times$ to its $p$-adic valuation modulo $p$.

For a prime $l | N_{\mathcal{Q}_p/Q}(c), Ad \cong \overline{\sigma} \otimes (\overline{\tau})^{-1}$ and $Ad^*(1) \cong \overline{\tau} \otimes (\overline{\sigma})^{-1} \overline{\omega}$ over $\text{Gal}(\overline{\mathcal{Q}_l}/\mathcal{Q}_l)$ (as $F_l = Q_l \oplus Q_l$). Write $\overline{\sigma}$ (resp. $\overline{\tau}$) for $\overline{\tau}$ and $\overline{\sigma}$ (resp. for $\overline{\chi}$ and $\overline{\omega}$) in order to treat the two cases at the same time. We normalize $Ad$ so that the character $\overline{\tau}$ is realized on $\mathbb{F} \left( \frac{1}{0}, \frac{0}{1} \right)$ and $\overline{\sigma}$ appears on the upper nilpotent matrices and $(\overline{\tau})^{-1}$ acts on lower nilpotent matrices, and we also normalize $Ad^*(1)$ accordingly. Since $H^0(I_1, \overline{\tau}) = 0$, we have an isomorphism $H^1(\mathcal{Q}_l, \overline{\sigma}) \cong H^1(\mathcal{Q}_l, \overline{\tau})_{\text{Frob}^2_{\mathcal{Q}_l}}=1$ by the restriction map. Since $\overline{\omega}$ is unramified at $l$, we have $\overline{\tau}/ \overline{\omega}[l] = \overline{\sigma}/ \overline{\omega}$. We have the following exact sequence

$$
0 \to H^1(\overline{\tau}, I_l) \to H^1(I_l) \to \text{Hom}(\overline{\omega}, I_l) \to H^2(\overline{\sigma}, I_l).
$$

Since $\overline{\sigma}/I_l$ has order prime to $p$, we have $H^j(\overline{\sigma}/I_l, \overline{\sigma}) = 0$ for all $j > 0$. Thus $H^1(I_l, \overline{\sigma}) \cong \text{Hom}(\overline{\omega}, I_l)$ factors through the tame quotient of $I_l$ which is abelian, the conjugation action of $\overline{\sigma}$ on $\text{Ker}(\overline{\omega}/I_l)$ is trivial, while $\overline{\sigma}$ is non-trivial; so, we conclude $H^1(I_l, \overline{\sigma}) \cong \text{Hom}(\overline{\omega}, I_l)$. Thus, we get

$$
H^1(\mathcal{Q}_l, Ad) = \text{Hom}(\text{Gal}(\overline{\mathcal{Q}_l}/\mathcal{Q}_l), \mathbb{F} \left( \frac{1}{0}, \frac{0}{1} \right)) \cong \mathbb{F} \text{ and } H^1(\mathcal{Q}_l, Ad^*(1)) = H^1(\mathcal{Q}_l, \mathbb{F} \left( \frac{1}{0}, \frac{0}{1} \right) \otimes \overline{\omega}) = H^1(\text{Frob}^2_{\mathcal{Q}_l}, \mathbb{F} \left( \frac{1}{0}, \frac{0}{1} \right)) \cong \mathbb{F},
$$

which is the Tate dual of $H^1(\mathcal{Q}_l, Ad)$. This tells us that the Selmer cocycle $u_p$ giving a class in $D^*_p([\mathbb{F}]_1)$ for $Ad$ has values in $\mathbb{F} \left( \frac{1}{0}, \frac{0}{1} \right)$ over $\text{Gal}(\overline{\mathcal{Q}_l}/\mathcal{Q}_l)$ and is unramified. In other words, we have $D^*_p([\mathbb{F}]_1) = H^1(\mathcal{Q}_l, Ad)$; so, again the direct sum decomposition (3.3) holds, and we find $D^*_p([\mathbb{F}]_1) = H^1(\mathcal{Q}_l, Ad)$.
decomposition (3.3) holds, and \( \mathcal{D}^2(F[\epsilon]) = H^4(Q_1, Ad) \). Thus, for the dual Selmer groups of \( \text{Ind}_F^Q(\varphi - \omega) \) and \( \varphi - \omega \), triviality at \( l \mid N \) is imposed (under (h0)). In particular, for the splitting field \( K \) of \( \chi \omega \), writing \( \text{Cl}_{\chi \omega}(p^\infty) := \lim_{n \to \infty} \text{Cl}_{\chi \omega}(p^n) \) for the ray class group modulo \( p^n \) (\( n = 0, \ldots, \infty \)) of \( K \), we have

\[
\text{Sel}_{\varphi}(\text{Ind}_F^Q(\varphi - \omega), F) = \text{Hom}(\text{Cl}_{\chi \omega}(p^\infty), F[\varphi - \omega], \varphi - \omega)
\]

where \( \text{Hom}(\text{Cl}_{\chi \omega}(p^\infty), F[\varphi - \omega], \varphi - \omega) \) is the \( \varphi - \omega \)-eigen subspace of \( \text{Hom}(\text{Cl}_{\chi \omega}(p^\infty), F[\varphi - \omega], \varphi - \omega) \) under the action of \( \text{Gal}(K/Q) \). Note that \( \varphi - \omega \) ramifies both at two primes \( l \) and \( \mathfrak{p} \) over \( l \mid N_F/Q(c) \). Since \( \varphi - \omega \) is anticyclotomic, any prime \( l \mid D \) is fully split in \( F(\varphi - \omega)/F \).

Let \( F^Q \) be the maximal extension of \( K_0 \) unramified outside \( Q \). By (h0), all deformations of \( \varphi - \omega \) are fully \( p^\infty \)-ramified, while \( F(\varphi - \omega) \) is at most tamely \( p^\infty \)-ramified. Therefore the inertia subgroup of \( \varphi - \omega \) for the extension \( K_0 \mid F(\varphi - \omega) \) is the entire Galois group \( \text{Gal}(K_0/F(\varphi - \omega))/F(\varphi - \omega) \). This tells us that \( M_0 \cap K_0 \mid F(\varphi - \omega) = F(\varphi - \omega) \). Thus we have the vanishing of the \( \varphi - \omega \)-eigenspace.

\[
\text{Coker}(\text{Res} \xrightarrow{\text{Res}} \text{Gal}(M_0/F(\varphi - \omega)))[\varphi - \omega] = \text{Coker}(\text{Res} \xrightarrow{\text{Res}} \text{Gal}(M_0/F(\varphi - \omega))) \otimes_{\mathbb{Z}[\text{Gal}(F(\varphi - \omega)/F), \varphi - \omega]} W = 0,
\]

and we find \( \text{Gal}(M_0/F(\varphi - \omega))[\varphi - \omega] = Y^-(\varphi - \omega)_H = H_0(H, Y^-(\varphi - \omega)) \) and

\[
\text{Hom}_{\text{Gal}(F(\varphi - \omega)/F)}(\text{Gal}(\text{M}_0/F(\varphi - \omega)), \varphi - \omega) = \text{Hom}(Y^-(\varphi - \omega)_H, F) = \text{Hom}_{W[H]}(Y^-(\varphi - \omega), F).
\]

**Proposition 3.8.** Let \( \text{Cl}_{Q \ast} = \{ x \in \text{Cl}_Q \mid c(x) = x^{-1} \} \), and write \( \text{Cl}_{\chi \omega}(p^\infty) \) for the class group of the splitting field of \( \chi \omega \). Then, under (h0–4), we have \( Y^Q_\infty(\varphi - \omega) = Y^-(\varphi - \omega) \) for \( Q \in \mathcal{Q} \),

\[
\text{Sel}_{Q}(\chi) \cong \text{Hom}(\text{Cl}_{Q \ast}^+, F) \text{ including } Q = 0,
\]

\[
\text{Sel}_{Q}(\text{Ind}_F^Q(\varphi - \omega)) \cong \text{Hom}_{W[H]}(Y^-(\varphi - \omega), F) \text{ including } Q = 0,
\]

\[
\text{Sel}_{\varphi}(\chi \omega) \cong \text{Hom}(\text{Cl}_{\chi \omega}(p^\infty), F[\varphi - \omega], \varphi - \omega)
\]

and

\[
\text{Sel}_{Q}(Ad) \cong \text{Hom}(\text{Cl}_{Q \ast}^+, F) \oplus \text{Hom}_{W[H]}(Y^-(\varphi - \omega), F).
\]

**Proof.** We have already proven the last two identities of (3.5) and the second identity of (3.6). Thus we deal the rest. The subspace \( \mathcal{D}^2(F[\epsilon]) \) is made of classes of cocycles with values in \( Ad = \text{sl}_2(F) \) such that \( u_\rho = u_\rho[D_v] \) is upper nilpotent and \( u_\rho[D_p] = \text{Gal}(\overline{Q}_p/Q_p) \) is upper triangular. Similarly \( \mathcal{D}^1(F[\epsilon]) \) is made of classes of ramified cocycles \( u_\rho \) with values in diagonal matrices over \( D_1 \). Then by the same argument proving (3.3) (or by the dual statement of (3.3)), we note that

\[
\text{Sel}_{Q}(Ad) = \text{Sel}_{Q}(\chi) \oplus \text{Sel}_{Q}(\text{Ind}_F^Q(\varphi - \omega)),
\]

where \( \text{Sel}_{Q}(\chi) = \ker(H^1(Q^{(QNP)}/Q, \chi) \rightarrow \prod_{l \mid N} H^1(I_l, \chi)) \) and

\[
\text{Sel}_{Q}(\text{Ind}_F^Q(\varphi - \omega)) = \ker(H^1(Q^{(QNP)}/Q, \text{Ind}_F^Q(\varphi - \omega)) \rightarrow \prod_{l \mid N} H^1(Q_l, \text{Ind}_F^Q(\varphi - \omega)).
\]

By the inflation restriction sequence,

\[
\text{Sel}_{Q}(\chi) \cong \ker(\text{Hom}_{\text{Gal}(F/Q)}(\text{Gal}(F^Q/F), \chi) \rightarrow \prod_{l \mid N} H^1(I_l, \chi)) \cong \text{Hom}(\text{Cl}_{Q \ast}^+, F).
\]
However the order of $\text{Ker}(C_l^Q, C_{l+1}^Q)$ is $\prod_{q \in Q^{-}} (q + 1)$, which is prime to $p$; so, we conclude

$$\text{Sel}_Q(\chi) \cong \text{Hom}(C_l^Q, \mathbb{F}) \cong \text{Hom}(C_{l+1}^Q, \mathbb{F}).$$

Again by the inflation restriction sequence, identifying $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ with the decomposition group at $\mathbb{F}$, we have an exact sequence

$$0 = H^1(\text{Frob}_p^\mathbb{F}, H^0(I_p, \mathbb{F}^{-})) \to H^1(I_p^\mathbb{F}, \mathbb{F}^{-}) \to H^1(I_p^\mathbb{F}, \mathbb{F}(\mathbb{F}^{-}))(\text{Frob}_p \to 0),$$

since $H^0(I_p, \mathbb{F}^{-}) = 0$ (i.e., $\varphi^{-}$ ramified at $p$ and also at $\mathbb{F}$). Thus we conclude

$$\text{Ker}(H^1(I_p, \mathbb{F}^{-}) \to H^1(I_p, \mathbb{F}^{-})) = 0,$$

and $\text{Sel}_Q(\text{Ind}_p^Q \varphi^{-})$ is actually given (by replacing $H^1(I_p, \mathbb{F}^{-})$ by $H^1(I_p, \mathbb{F}^{-})$ in (3.7))

$$\text{Ker}(H^1(Q(QN_p)/Q, \text{Ind}_p^Q \varphi^{-}) \to H^1(I_p, \mathbb{F}^{-}) \times \prod_{l|N} H^1(I_l, \text{Ind}_p^Q \varphi^{-}).$$

By the inflation-restriction sequence, we have an exact sequence $H^1(\text{Frob}_p^\mathbb{F}, (\varphi^{-})^{I_l}) \to H^1(D_l, \mathbb{F}^{-}) \to H^1(I_l, \mathbb{F}^{-})$ with $(\mathbb{F}^{-})^{I_l} = 0$ for $l|N$, and hence by Shapiro’s lemma (and (h0), we can rewrite

$$\text{Sel}_Q(\text{Ind}_p^Q \varphi^{-}) = \text{Ker}(H^1(Q(QN_p)/Q, \varphi^{-}) \to H^1(I_p, \mathbb{F}^{-}) \times \prod_{l|N} H^1(I_l, \mathbb{F}^{-})), $$

where I running over all prime factors of $N$ in $F$. Thus, restricting to the Galois group over $F(\varphi^{-})$, by the restriction-inflation sequence, we have

$$\text{Sel}_Q(\text{Ind}_p^Q \varphi^{-}) \cong \text{Hom}_{W[[H_q]]}(\mathbb{Y}_Q^{-}(\varphi^{-}), \mathbb{F}).$$

Similarly, $\text{Sel}_Q(\mathbb{Y}) \cong \text{Hom}_{\text{Gal}(F/Q)}(\text{Gal}(Q(QN_p)/Q), \mathbb{Y}) = \text{Hom}(C_l^Q, \mathbb{F})$. Therefore the first identity of (3.6) follows if we prove $\mathbb{Y}_Q^{-}(\varphi^{-}) \otimes_{W[[H_q]]} \mathbb{F} = \mathbb{Y}^{-}(\varphi^{-}) \otimes_{W[[H]]} \mathbb{F}$. To prove $\mathbb{Y}_Q^{-}(\varphi^{-}) \otimes_{W[[H_q]]} \mathbb{F} = Y^{-}(\varphi^{-}) \otimes_{W[[H]]} \mathbb{F}$, writing $I_{Q, p}^{\text{ab}}$ for the maximal $p$-abelian quotient of the inertia group $I_Q \subset \text{Gal}(\overline{\mathbb{Q}}_p/K_Q^F(\varphi^{-}))$ of a prime $\Omega|q$ in $K_Q^F(\varphi^{-})$, we have an exact sequence

$$\prod_{\Omega|q \neq Q} I_{Q, p}^{\text{ab}} \to Y_q^{-} \to Y^{-} \to 0$$

as $\text{Ker}(Y_q^{-} \to Y^{-})$ is generated by the image $I_{Q, p}^{\text{ab}} \cong \mathbb{Z}_p$. The surjectivity of the restriction map: $Y_q^{-} \to Y^{-}$ follows from linear-disjointness of $I_q$ and $K_Q^F(\varphi^{-})$ over $K^{-}F(\varphi^{-})$ as at least one of $q \in Q$ ramifies in any intermediate field of $K_Q^F(\varphi^{-})/K^{-}F(\varphi^{-})$. Note that $q \in Q^{-}$ totally splits in $K_Q^F(\varphi^{-})/F$. Thus $I_q^{-} := \prod_{\Omega|q} I_{Q, p}^{\text{ab}}$ for $q \in Q^{-}$ is isomorphic to

$$Z_p^{\text{Gal}(K_Q^F(\varphi^{-})/F)} = Z_p[[\text{Gal}(K_Q^F(\varphi^{-})/F)]] = Z_p[[H_q]][\text{Im}(\varphi^{-})]$$

as $Z_p[[\text{Gal}(K_Q^F(\varphi^{-})/F)]]$-modules. Since $I_{Q, p}^{\text{ab}} \cong Z_p$ is the quotient of the maximal $q$-tame quotient of $I_Q$, $\text{Frob}_q$ (for the prime $q|q \in Q^{-}$ in $F$) acts on it via multiplication by $q^2$. Since $\varphi^{-}(\text{Frob}_q) = 1$, the map $I_q^{-} \otimes_{Z_p[\text{Im}(\varphi^{-})]} \varphi^{-} W \to Y_q^{-}(\varphi^{-})$ factors through

$$I_q^{-} \varphi^{-} = I_q^{-} \otimes_{Z_p[\text{Im}(\varphi^{-})]} \varphi^{-} W \cong W[[H_q]]/(q^2 - 1).$$

Thus we have $I_q^{-} \varphi^{-} \otimes_{W[[H_q]]} \mathbb{F} = \mathbb{F}(\varphi^{-})$ (one dimensional space over $\mathbb{F}$ on which $\text{Gal}(F(\varphi^{-})/F)$ acts by $\varphi^{-}$). Note that $\text{Frob}_q$ acts on $I_q^{-} \varphi^{-} \otimes_{W[[H_q]]} \mathbb{F}$ via multiplication by $q$, which is trivial as $q \equiv 1 \mod p$. Thus the image of $I_q^{-} \varphi^{-} \otimes_{W[[H_q]]} \mathbb{F}$ in $Y_q^{-}$ is stable under $\text{Frob}_q = c$, and hence stable under $\text{Gal}(F(\varphi^{-})/Q)$. Since $I_q^{-} \varphi^{-} \otimes_{W[[H_q]]} \mathbb{F} = \mathbb{F}(\varphi^{-})$, if the image is non-trivial, it must contain the irreducible $\text{Gal}(F(\varphi^{-})/Q)$-module $I_q^{-} \varphi^{-}$, which is impossible as the image has dimension $\leq 1$. Thus the image of $I_q^{-} \varphi^{-} \otimes_{W[[H_q]]} \mathbb{F}$ in $Y_q^{-} \varphi^{-}$ is trivial.

The set $\Omega_q^+$ of primes $\Omega$ in $K_Q^F(\varphi^{-})$ above $q|q \in Q^+$ is a finite set on which the Galois group $\text{Gal}(K_Q^F(\varphi^{-})/F)$ acts by permutation. Then, writing $D(\Omega/q) \subset \text{Gal}(K_Q^F(\varphi^{-})/F)$ for the decomposition group of $\Omega$, we have

$$I_q^+ := \prod_{\Omega \in \Omega_q^+} I_{\Omega}^{\text{ab}} \cong Z_p^{\Omega_q^+} \cong Z_p[\text{Gal}(K_Q^F(\varphi^{-})/F)/D(\Omega/q)]$$
on which $\text{Frob}_q$ acts by $\sigma D(\Omega/q) \mapsto q\sigma \text{Frob}_q D(\Omega/q) = q\sigma D(\Omega/q)$ for $\sigma \in \text{Gal}(K^{-}_Q F(\varphi^-)/F)$ and $\Delta_q \subset H_{Q}$ act trivially. Thus putting $I_{F}^{+}(\varphi^-) := I_{F}^{+} \otimes_{\mathbb{Z}_p[\varphi^-]} W$, we conclude from $q \equiv 1 \mod p$

$$I_{F}^{+}(\varphi^-) \otimes W[H_{Q}] \mathbb{F} = \begin{cases} 0 & \text{if } \varphi^-(\text{Frob}_q) \neq 1, \\ \mathbb{F} & \text{if } \varphi^-(\text{Frob}_q) = 1, \end{cases}$$

since $q \equiv 1 \mod p$ (i.e., after tensoring $\mathbb{F}$, $\text{Frob}_q$ acts on $\mathbb{F}[\text{Gal}(K^{-}_Q F(\varphi^-)/F)/D(\Omega/q)]$ by multiplication by $q \equiv 1 \mod p$). By our choice of $Q \in Q$, $\pi(q)$ has two distinct eigenvalues, and hence $\varphi^-(\text{Frob}_q) \neq 1$. Thus we get the following isomorphism: $Y_{Q}^{-}(\varphi^-) \otimes W[H_{Q}] \mathbb{F} \cong Y^-(\varphi^-) \otimes W[H] \mathbb{F}$ which implies

$$Y_{Q}^{-}(\varphi^-) \otimes W[H_{Q}] \mathbb{F} = Y^-(\varphi^-) \otimes W[H] \mathbb{F}$$

as desired. \hfill \Box

The primes $q_x \in Q_m$ is indexed by a basis $\{x\}_x$ of the Selmer group $\text{Sel}^\dagger_{\mathbb{Q}}(Ad^+(1))$ so that $f_x$ as in Lemma 3.2 has non-trivial value at $\text{Frob}_{q_x}$. Thus writing $Q_m := \{q \in Q_m | \chi(q) = \pm 1\}$, we get from our choice in Corollary 3.4

$$|Q_m^-| = \dim_{\mathbb{F}} \text{Hom}_{W[H]}(Y^-(\varphi^- \omega), \mathbb{F}) \text{ and } |Q_m^+| = \dim_{\mathbb{F}} \text{Sel}^\dagger_{\mathbb{Q}}(\overline{\chi} \omega).$$

4. A SUFFICIENT CONDITION FOR COMPLETE INTERSECTION PROPERTY FOR $R_+$

We now claim to be able to add the compatibility (Q9) to the above list of the conditions (Q0–8):

(Q9) $\pi_{n}^{n+1} \circ \sigma_{n+1} = \sigma_{n} \circ \pi_{n}^{n+1}$, and the set $\{f_{1}^{(n)}, \ldots, f_{t}^{(n)}\}$ is made of eigenvectors of $\sigma_{n}$ for all $n$ (i.e., $\sigma_{n}(f_{j}^{(n)}) = \pm f_{j}^{(n)}$).

**Lemma 4.1.** We can find an infinite family $Q = \{Q_m\}_m$ of $r$-sets of primes outside $Np$ satisfying (Q0–9).

**Proof.** Pick an infinite family $Q$ satisfying (Q0–8). We modify $Q$ to have it satisfy (Q9). Since $p > 2$, plainly, $R_n$ is generated over $W$ by $\sigma_n$-eigenvectors $\{\sigma_n(f_{j}^{(n)}) \pm f_{j}^{(n)}\}_{j=1, \ldots, r}$. Since $r$ is larger than or equal to the minimal number of generators $\dim_{\mathbb{F}} T_{R_n}^{*} \leq \dim_{\mathbb{F}} D_{Q_m,k,\psi_{r}}(\mathbb{F}[\varepsilon])$ for the co-tangent space $T_{R_n}^{*} := m_{R_n}/(m_{R_n}^{2} + m_{W})$, we can choose $r$ generators among $\{\sigma_n(f_{j}^{(n)}) \pm f_{j}^{(n)}\}$.

Once compatibility $\pi_{n}^{n+1} \circ \sigma_{n+1} = \sigma_{n} \circ \pi_{n}^{n+1}$ is shown, we get

$$\pi_{n}^{n+1}(\sigma_{n}(f_{j}^{(n+1)}) \pm f_{j}^{(n+1)}) = \sigma_{n}(f_{j}^{(n)}) \pm f_{j}^{(n)}$$

for each $j$ from $\pi_{n}^{n+1}(f_{j}^{(n+1)}) = f_{j}^{(n)}$; so, we may assume that the set of generators is made of eigenvectors of the involution (and is compatible with the projection $\pi_{n}^{n+1}$).

We now therefore show that we can make the system compatible with the involution. The triple with $0 < n \leq m(n)$:

$$((R_{n,m(n)}(\alpha), \tilde{R}_{n,m(n)}(f_1, \ldots, f_r))$$

in the system (3.1) actually represents an isomorphism class $I_{T}^{TW}$ made of infinite triples

$$\{(R_{n,m} \alpha), \tilde{R}_{n,m}((f_1, \ldots, f_r))\}_{m \geq n}$$

satisfying (Q0–8) with $m$ varying in the choosing process of $Q$ (of Taylor–Wiles; see [HMI, page 191] or [MFG, §3.2.6]). Then $m(n)$ is chosen to be minimal choice of $m$ in the class $T_{T}^{TW}$; so, we can replace $m(n)$ by a bigger one if we want (as $T_{T}^{TW}$ is an infinite set). In other words, choosing $m$ appearing in $T_{T}^{TW}$ possibly bigger than $m(n)$, we would like to show that we are able to add the datum of the involution $\sigma$ induced by $\sigma_{Q_m}$. Therefore, we look into isomorphism classes in the infinite set of $(\sigma$-added) quadruples (varying $m$)

$$\{(R_{n,m} \alpha), \tilde{R}_{n,m}((f_1, \ldots, f_r)), \sigma_{n,m}\}_{m \geq n+1}$$

of level $n$ in place of triples $\{(R_{n,m} \alpha), \tilde{R}_{n,m}((f_1, \ldots, f_r))\}_{m \geq n}$, where $\sigma_{n,m}$ indicates the involution of $R_{n,m}$ induced by $\sigma_{Q_m}$ (which is compatible with the projection $R_{n,m} \rightarrow \tilde{R}_{n,m}$).

We start an induction on $n$ to find the projective system satisfying $\pi_{n+1} \circ \sigma_{n+1} = \sigma_{n} \circ \pi_{n+1}$.

The projection $\pi_{Q_m} : R_{Q_m} \rightarrow R_{\theta}$ (for any $m \geq 1$) of forgetting ramification at $Q_m$ is $\sigma$-compatible (by definition) for the involution $\sigma_{Q_m}$ and $\sigma_{\theta}$ coming from the $\chi$-twist, which induces a surjective $W$-algebra homomorphism $\pi_{0} : R_{1,m} \rightarrow R_{1,0}$ for $R_{1,0} = T_{\theta}/pT_{\theta}$ satisfying $\pi_{0} \circ \sigma_{1} = \sigma_{0} \circ \pi_{1}$. Thus the
initial step of the induction is verified. In the same way, the projection $R_{n,m} \to \tilde{R}_{n,m}$ is compatible with the involution.

Now suppose that we find an isomorphism class $\mathcal{I}_n$ of the $(\sigma$-added) quadruples (indexed by $r$-sets $Q_m \in \mathbb{Q}$ satisfying (Q0–9) varying $m$ with $m \geq n$) containing infinitely many quadruples of level $n$ whose reduction modulo $(p^{n+1}, \delta^q_{n+1} - 1)_{q \in \mathbb{Q}}$ is in the unique isomorphism class $\mathcal{I}_{n-1}$ (already specified in the induction process). Since the subset of such $Q \in \mathbb{Q}$ of level $m \geq n + 1$ (so $q \equiv 1 \pmod{p^{n+1}}$ for all $q \in \mathbb{Q}$) whose reduction modulo $(p^n, \delta^q_{n} - 1)_{q \in \mathbb{Q}}$ falls in the isomorphism class $\mathcal{I}_n$ is infinite, we may replace $\mathcal{I}_n$ by an infinite subset $\mathcal{I}_n \subset \mathcal{I}_n$ coming with this property (i.e., $m > n$), and we find an infinite set $\mathcal{I}_{n+1}$ of $\{(R_{n,m+1}, \alpha), \overline{R}_{n,m+1}, (f_1, \ldots, f_r, \sigma_{n,m+1})\}_{m \geq n+1}$ which surjects down modulo $(p^n, \delta^q_{n} - 1)_{q \in \mathbb{Q}}$ isomorphically to a choice

$\{((R_{n,m}, \alpha), \overline{R}_{n,m}, (f_1, \ldots, f_r, \sigma_{n,m})) \in \mathcal{I}_n\}$

at the level $n$. Indeed if all $q \in \mathbb{Q}$ satisfies $q \equiv 1 \pmod{p^{n+1}}$, as we now vary $m$ so that $m > n$ (rather than $m \geq n$), we can use the same $Q = Q_m$ to choose the isomorphism class of level $n + 1$. Therefore, for $R_{Q,j} \to Q/(p^j, \delta^q_j - 1)_{q \in \mathbb{Q}}$, the projections

$R_{Q,n+1} \to R_{Q,n}$ and $\overline{R}_{Q,n+1} = R_{Q}/(p^{n+1}, \delta^q_{n+1} - 1)_{q \in \mathbb{Q}} \to \overline{R}_{Q,n} = R_{Q}/(p^n, \delta^q_n - 1)_{q \in \mathbb{Q}}$

are compatible with the involutions induced by $\sigma_Q$, and hence for the same set of generators $\{f_j\}_j$, the two quadruples

$\{(R_{Q,j}, \alpha), \overline{R}_{Q,j}, (f_1, \ldots, f_r, \sigma_j)\}_j$

of level $j = n + 1$, $n$ are automatically $\sigma_j$-compatible.

Since the number of isomorphism classes of level $n + 1$ in $\mathcal{I}_n$ is finite, we can choose an isomorphism class $\mathcal{I}_{n+1}$ of level $n + 1$ with $|\mathcal{I}_{n+1}| = \infty$ inside $\mathcal{I}_n$ whose members are isomorphic each other (this is the pigeon-hole principle argument of Taylor–Wiles). Thus by induction on $n$, we get the desired compatibility $\pi_n \circ \sigma_{n+1} = \sigma_n \circ \pi_{n+1}$ for $\mathcal{I}_{n+1}$; i.e., $\mathcal{I}_{n+1} \xrightarrow{\text{reduction}} \mathcal{I}_n \to \mathcal{I}_{n-1} \to \cdots \to \mathcal{I}_1$ with $|\mathcal{I}_j| = \infty$ for all $j = 1, 2, \ldots, n + 1$. We hereafter write $m(n)$ for the minimal of $m$ with $((R_{n,m}, \alpha), \overline{R}_{n,m}, (f_1, \ldots, f_r, \sigma_{n,m})$ appearing in $\mathcal{I}_n$.

\[|Q_m| = \dim \text{Hom}_{\mathbb{W}}[\mathbb{A}^{|\mathbb{A}|}] \langle \gamma^-, \phi^- \rangle, \mathbb{F} \rangle \text{ by Proposition 3.8, it is independent of } m.\]

\[|Q_m| \text{ (and hence } |Q_m|) \text{ is independent of } m. \]
We can impose that these $f_j$ and $g_j$ are made of eigenvectors of the involution. By sending $T_i = f_i$ to $g_i$, we have $R/\mathfrak{q}R \cong T, R^+ / \mathfrak{q}R = T^+, R/ (t - \gamma^k) = R_\infty$ and $R^+ / (t - \gamma^k) = R_\infty^+$. 

We reformulate the ring $W[[S_1, \ldots, S_r]]$ in terms of group algebras. Let $\Delta_{Q_m^\pm} := \prod_{q \in Q_m^\pm} \Delta_q$ and $\Delta_n^\pm := \prod_{q \in Q_n^\pm} \Delta_q / \Delta_q^\pm$; so, $\Delta_n = \Delta_n^\pm \times \Delta_n^\mp$. Define $p$-profinite groups $\Delta$ and $\Delta_+$ by $\Delta = \lim_n \Delta_n \cong \mathbb{Z}_p^r$ and $\Delta_+ = \lim_n \Delta_n^\pm \cong \mathbb{Z}_p^{r_+}$ for $r_+ := |Q_m^+|$. Here the limits are taken with respect to $\pi_n^{n+1}$ restricted to $\Delta_{n+1}$.

Set

\[(4.3) \quad S := W[[\Delta]] = \lim_n W[\Delta / \Delta^p] = \lim_n W[\Delta_n]
\]

for the $p$-profinite group $\Delta = \lim_n \Delta_n \cong \mathbb{Z}_p^r$ with $\Delta = \Delta_+ \times \Delta_-$. and $A$ be a local $S$-algebra. Thus by identifying $\Delta / \Delta^p$ with $\Delta_n$, we have the identification $S = W[[S_1, \ldots, S_r]]$. The image $S_n := W_n[\Delta_n]$ ($W_n = W/p^nW$) of $S$ in $R_n$ is a local complete intersection and hence Gorenstein. We assume that the ordering of primes in $Q \in Q$ preserves $Q_m^+$ and $Q_m^-$. In other words, the ordering of $(Q)$ induces $Q_m^+ := \{q_1, \ldots, q_{r_+}\}$ and $Q_m^- := \{q_{r_+ + 1} := q_1', \ldots, q_r = q_{r+1}\}$. We now write $s_j^\pm$ for the generator of $\Delta$ corresponding to $\delta_{q_j}^\pm$.

**Definition 4.3.** Write $s_j^\pm$ for the generator of $\Delta_\pm$ corresponding to $\delta_{q_j}^\pm$. Then define $S_j^+ = s_j^+ - 1$ and $S_j^- = s_j^- - (s_j^-)^{-1}$. Thus $\sigma_\infty(S_j^+) = \pm S_j^+$. Write $G$ for the subgroup of involutions in $\text{Aut}(W[[\Delta]])$ generated by the involutions $b_i (i = 1, \ldots, r_+)$ such that $b_i(s_j^-) = (-1)^{\delta_i j} s_j^-$. For the Kronecker's delta $\delta_{ij}$ and $b_j(s_j) = s_j^+$ for all $j = 1, 2, \ldots, r_+$. Put $S := S^G = W[[\Delta]]^G$.

Since $\sigma_\infty$ acts as $\sigma_\infty(s_j^-) = -s_j^-$ for all $j = 1, 2, \ldots, r_-$, the group $G = \langle \sigma \rangle$ embeds into $G$ so that $G = \prod_i b_i$ on $W[[\Delta]]$.

For the ideal $A := \text{Ker}(W[[\Delta_+]] \to W_n[\Delta_n])$ for $W_n := W/p^nW$, we put $A_n = a_n + ((s_1^+)p^n - 1, \ldots, (s_{r_+}^-)p^n - 1) \subset S$ as an $S$-ideal. Then $A_n$ is stable under $\sigma$, and $A := \text{Ker}(S \to W_n[\Delta / \Delta^p])$. Put

\[(4.4) \quad S_n := A_n \cap W[\Delta]^G = \text{Ker}(W[\Delta]^G \to W_n[\Delta / \Delta^p]^G] = a_n + (((s_1^+)p^n - 1) + \sigma((s_1^+)p^n - 1), \ldots, ((s_{r_+}^-)p^n - 1) + \sigma((s_{r_+}^-)p^n - 1)).
\]

By this expression, we confirm the following fact:

**Lemma 4.4.** The ring $S_n := S / S_n = W_n[\Delta / \Delta^p]^G$ is a local complete intersection over $W_n := W/p^nW$ and is a Gorenstein ring free of finite rank over $W_n$.

Using the natural projection $\Delta \to \Delta_{Q_m}$ sending $s_j^\pm$ to $\delta_{q_j}^\pm$, we get $\mathfrak{g}_{Q_m} = \text{Ker}(S \to W[\Delta_{Q_m}])$. We define $\mathfrak{g}_{Q_m} := \text{Ker}(S \to W[\Delta_{Q_m}]^G)$. Let $A$ be a local $S_n$-algebra for $S_n = S / \mathfrak{g}_{Q_n} = W_n[\Delta / \Delta^p]$ (and hence $A$ is an $S_n$-algebra for $S_n = S / \mathfrak{g}_{Q_n} \subset S_n$). We suppose that $\sigma$ acts on $A$ as an involution extending its action on $S_n$. Then $\sigma$ acts on $A^! = \text{Hom}_S(A, S_n)$ (resp. $A^\# = \text{Hom}_S(A, S_n)$) by $f^\#(x) = \sigma(f(\sigma(x)))$. Indeed, $f^\#(s_n x) = \sigma(f(\sigma(s_n x))) = \sigma(\sigma(s_n)(\sigma(x))) = s^\# f(x)$, and hence $f^\#$ is $S$-linear. We put $S_\infty = S$ and $S_\infty = S$ and allow $n = \infty$.

**Remark 4.5.** Let $C \subset A$ be $B$-algebras. Suppose that

1. $B$ and $C$ are Gorenstein,
2. $A$ and $C$ are $B$-modules of finite type,
3. $C$ is $B$-free of finite rank.

Then we have $\text{Hom}_B(C, B) \cong C$ as $B$-modules (cf., Lemma 11.1). Thus by [BAL, Proposition II.4.1.1],

$$\text{Hom}_C(A, C) \cong \text{Hom}_C(A, \text{Hom}_B(C, B)) \cong \text{Hom}_B(A \otimes_C B, B) = \text{Hom}_B(A, B).$$

This isomorphism is sending $g \in \text{Hom}_C(A, \text{Hom}_B(C, B))$ to $\bar{g} \in \text{Hom}_B(A \otimes_C B, B)$ given by $\bar{g}(a \otimes c) = g(a)(c)$. Applying this to $(A, B, C) := (R_n, S_n, S_n)$ and then to $(A, B, C) := (R_n, W_n, S_n)$, we get $A^\# \cong A^! \cong A^*$ as $A$-modules for $A^* = \text{Hom}_{W_n}(A, W_n)$. The identity $A^\# \cong A^! \cong A^*$ is valid for $n = \infty$. 

**ANTICYCLOMATIC CYCLICITY CONJECTURE 26**
also. Since the isomorphism $S_n \cong S'_n$ can be chosen to be compatible with the action of $G$ (including $\sigma$), the isomorphisms

\begin{equation}
A^\# \cong A^\dagger \cong A^*
\end{equation}

can be chosen to be $\sigma$-compatible. Note that $W_n$-duality is equivalent to Pontryagin duality for profinite $W$-modules as long as $W$ is finite over $\mathbb{Z}_p$.

By the above remark, noting $R_\infty$ is free of finite rank over $S$, we get the following $\sigma$-compatible identity:

\begin{equation}
\lim_{n \to \infty} R_n^{\dagger} \cong \lim_{n \to \infty} R_n^\# = \lim_{n \to \infty} \text{Hom}_{S_n}(R_n, S_n)
\end{equation}

and

\begin{equation}
\lim_{n \to \infty} \text{Hom}_{S}(R_\infty/\mathfrak{A}_n R_\infty, S/\mathfrak{A}_n) \cong \text{Hom}_{S}(R_\infty, S) \cong R_\infty^\# \cong R_\infty^\dagger.
\end{equation}

Here the identities (1) are from Remark 4.5 and the identity (2) is by the fact: $R_n = R_\infty/\mathfrak{A}_n R_\infty$ and by the definition $S_n := S/\mathfrak{A}_n$.

Define

$$
\text{Hom}_B(A, B)^\pm := \{ \phi \in \text{Hom}_B(A, B) \mid \phi \circ \sigma = \pm \sigma \circ \phi \}
$$

for $A = T_{Q_m}^\dagger := \text{Hom}_{W[\Delta_m]}(T_{Q_m}, W[\Delta_m]^G)$ or $R_n^\dagger$ and $B = T_{Q_m}$ or $R_n$ accordingly. Write $\text{Isom}_B(A, B)^\pm \subset \text{Hom}_B(A, B)^\pm$ for the subset made of isomorphisms. Using the Gorenstein-ness of $T_Q$ for $Q = Q_m$ or $Q = \emptyset$ (which follows from the presentation (4.1) and for $Q = 0$ from Theorem 2.1), by Lemma 11.2 (1) applied to the involution $\sigma_{Q_m}$ of $T_{Q_m}$, we have

\begin{equation}
\text{Isom}_{T_{Q_m}}(T_{Q_m}^\dagger, T_{Q_m})^\varepsilon \neq \emptyset
\end{equation}

for at least a sign $\varepsilon \in \{ \pm \}$.

**Lemma 4.6.** We have

\begin{equation}
\text{Isom}_{T_{Q_m}}(T_{Q_m}^\dagger, T_{Q_m})^\varepsilon \neq \emptyset \iff \text{Isom}_{R_m}(R_m^\dagger, R_m)^\varepsilon \neq \emptyset
\end{equation}

for each $0 < n \leq m$.

**Proof.** The direction ($\Rightarrow$) is just reduction modulo $(p^n, \delta p^n - 1)_{q \in Q_m}$. We prove the converse. If we have $\phi \in \text{Isom}_{R_m}(R_m^\dagger, R_m)^\varepsilon$, then $\sigma(\phi^{-1}(1)) = \varepsilon \phi^{-1}(1)$. We can lift $\phi^{-1}(1)$ to $v \in T_{Q_m}$ with $v \equiv \varepsilon v \mod (p^n, \delta p^n - 1)_{q \in Q_m} = \phi^{-1}(1)$. Define $\Phi : T_{Q_m} \to T_{Q_m}$ by $\Phi(t) = tv$. Then $\Phi$ is a $T_{Q_m}$-linear map. By definition, $\Phi \mod (p^n, \delta p^n - 1)_{q \in Q_m} = \phi^{-1}$; so, by Nakayama's lemma, $\Phi$ is onto. Since $T_{Q_m}$ and $T_{Q_m}^\dagger$ are $W$-free of equal rank, $\Phi$ must be an isomorphism. Thus $\Phi^{-1} \in \text{Isom}_{T_{Q_m}}(T_{Q_m}^\dagger, T_{Q_m})^\varepsilon$. \hfill \Box

We want to add one more datum $\phi_n \in \text{Isom}_{R_m}(R_m^\dagger, R_m)^\varepsilon$ to the data $((R_n, \alpha), \bar{R}_n, (f_1, \ldots, f_r), \sigma_n)$ which is required to satisfy the following compatibility condition:

\begin{enumerate}
\item[(Q10)] We have $\phi_n \in \text{Isom}_{R_m}(R_m^\dagger, R_m)^\varepsilon$ with $\varepsilon \in \{ \pm \}$ independent of $n$ for all $n > 0$.
\end{enumerate}

**Remark 4.7.** Let $A$ and $B$ be a finite Gorenstein local rings of residual characteristic $p$. We suppose to have a surjective ring homomorphism $\pi : A \to B$. By adding $\ast$, we denote the Pontryagin dual module. Since $A$ and $B$ are Gorenstein, we have isomorphisms $A^* \cong A$ as $A$-modules and $B^* \cong B$ as $B$-modules. Thus we have a diagram

$$
A \xrightarrow{\pi} B
$$

$$
\downarrow \phi_A \quad \downarrow \phi_B
$$

$$
A^* \xrightarrow{\omega} B^*.
$$

By defining $\omega := \phi_B^{-1} \circ \pi \circ \phi_A$, the above diagram is commutative. Thus we can always adjust $A^* \to B^*$ making the above diagram commutative. Suppose that $A$ and $B$ have involutions $\sigma_X \subset X$ for $X = A, B$. By duality, the involution $\sigma_X$ acts on the dual $X^*$, which we denote by $\sigma_X^*$.

If $\phi_X \circ \sigma_X^* = \sigma_X \circ \phi_X$ for $\varepsilon = \pm 1$ independent of $X = A, B$ and $\sigma_B \circ \pi = \pi \circ \sigma_A$, we have

$$
\omega \circ \sigma_A^* = \phi_B^{-1} \circ \pi \circ \phi_A \circ \sigma_A^* = \phi_B^{-1} \circ \pi \circ \varepsilon \sigma_A \circ \phi_A = \phi_B^{-1} \circ \varepsilon \sigma_A \circ \phi_A = \varepsilon^2 \sigma_B \circ \phi_B^{-1} \circ \pi \circ \phi_A = \sigma_B \circ \omega.
$$
Thus the adjusted \( \varpi \) commutes with the involution.

This remark shows that if we have a projective system \( \{(R_n, \alpha), \tilde{R}_n, (f_1, \ldots, f_r), \sigma_n, \phi_n\}_n \) satisfying (Q0–9), we can add the datum of an \( R_n \)-linear isomorphism \( \phi_n : R_n^1 \xrightarrow{(4.5)} R_n^\ast \cong R_n \) compatible with \( \sigma \); i.e., (Q10) is automatically satisfied for \( \phi_n \) induced by \( \phi_{Q_{m(n)}} \), as long as we can take \( \phi_{Q_{m(n)}} \in \text{Isom}_{\tilde{R}_n}(\tilde{T}_{Q_{m(n)}}, \tilde{T}_{Q_{m(n)}}) \) with \( \varepsilon \) independent of \( m(n) \). Explicitly, the compatibility of \( \phi_n \) means the following:

1. the datum \( \phi_n \) satisfies \( \phi_n \circ \sigma_n^\ast = \varepsilon \sigma_n \circ \phi_n \) for all \( n \) and for \( \varepsilon \) as in (Q10) independent of \( m = m(n) \), and
2. the projections \( \pi_{n', n} : R_{n'} \to R_n \) and \( \varpi_{n', n} : R_{n'}^\dagger \to R_n^\dagger \) commute with the involution in addition to the commutativity of the diagram:

\[
\begin{array}{ccc}
R_{n'} & \xrightarrow{\pi_{n', n}} & R_n \\
| & \phi_{n'} | & | \phi_n | \\
| R_{n'}^\dagger & \xrightarrow{\varpi_{n', n}} & R_n^\dagger \\
\end{array}
\]

Now we again go through the Taylor–Wiles system argument made of the augmented tuples

\( \{(R_n, \alpha), \tilde{R}_n, (f_1, \ldots, f_r), \sigma_n, \phi_n\}_n \)

with \( \phi_n = (\phi_{Q_{m(n)}} \mod (p^n, \delta_{Q_{m(n)}} - 1))_{q \in \mathbb{Q}_{m(n)}} \in \text{Isom}_{\tilde{R}_n}(R_n^1, R_n^\ast) \) for \( m = m(n) \); then, we obtain \( R_\infty \)

with the limit involution \( \sigma_\infty \) and the limit isomorphism \( \phi_\infty \in \text{Isom}_{R_\infty}(R_\infty', R_\infty) \). Here \( R_\infty^1 = \text{Hom}_{S}(R_n, S_n) \). Thus we get

**Corollary 4.8.** Suppose (h0–4). Then we can choose the Taylor–Wiles projective system

\( \{(R_n, \alpha), \tilde{R}_n, (f_1, \ldots, f_r), \sigma_n, \phi_n\}_n \)

satisfying (Q0–10). If \( \varepsilon = + \) in (Q10), then we conclude that \( R_\infty^+ \) is a Gorenstein ring over \( S = S^G \), \( R_\infty/\mathfrak{Q}_{Q_\infty} R_\infty \cong R_\infty \cong \mathbb{T}_{Q_\infty} \).

Here is a prototypical example of the rings of type \( R_\infty, R_\infty^+ \) corresponding to the choice \( r_+ = r' = 0 \) and \( r_- = r'' = 1 \):

**Example 4.9.** Consider \( 0 \neq \delta \in m W \) and put

\[
W[\sqrt{\delta}] = \begin{cases} 
W + W\sqrt{\delta} & \text{if } \delta \notin W^2, \\
\{(x, y) \in W + W((x \mod \sqrt{\delta}) = y \mod \sqrt{\delta})\} & \text{if } \sqrt{\delta} \in W.
\end{cases}
\]

Define

\[
A = \{(x, y) \in W + W[\sqrt{\delta}] | (x \mod \delta) = (y \mod \sqrt{\delta}) \}
\]

and

\[
B = \{(x, y) \in W + W | x \equiv y \mod (\delta)\}.
\]

Note that \( A = W[[T_-]]/(S_-) \) with \( S_- = T_- (T_-^2 - \delta) \) by sending \( T_- \) to \( (0, \sqrt{\delta}) \in A \) and \( B = W[[T_-]]/(S_-) \) by sending \( T_-^2 \) to \( (0, \delta) \in B \). Then \( W[[T_-]] \supset W[[T_-^2]] \) and \( W[[T_-^2]] \supset W[[S_-]] \). We have an involution \( \sigma \) of \( W[[T_-^2]] \) over \( W[[T_-]] \) with \( \sigma(T_-) = -T_- \) and \( \sigma(S_-) = -S_- \).

For \( Q \in \mathbb{Q} \), recall \( r_- = |Q^-| \) with

\[
Q^- := \{q \in Q | q \text{ is inert in } F/Q\} \quad \text{and} \quad Q^+ := \{q \in Q | q \text{ is split in } F/Q\}.
\]

Now we would like to prove

**Theorem 4.10.** Suppose (h0–4) and that the family \( Q \) satisfies (Q0–10). Let \( Q \in \mathbb{Q} \) or \( Q = \emptyset \). Suppose that \( \sigma \) is non-trivial on \( T_0 \) (so, nontrivial on \( T^0 \)). Then we have \( \varepsilon = + \) in (Q10), and the following three assertions holds.

1. We have \( 0 < r_- = \text{dim}_F \text{Hom}_{W[[H]]}((\varphi^- \omega), F) = r'' \).
2. If \( r_+ = 1 \), the \( T_+^Q \)-module \( T_+^Q \) is generated by a single element over \( T_+^Q \).
3. If \( r_- = 1 \), the ring \( T_+ = T_+^Q \) is a local complete intersection over \( \Lambda \). More generally, for \( Q \in \mathbb{Q} \), the rings \( T_+^Q \) and \( T^Q_+ \) are local complete intersection.
Proof. By (Q9), σ is compatible with the projective system of tuples
\[(R_n, (f_1, \ldots, f_r), \sigma_n) \in \mathcal{I}_n,\]
and by constancy of ε, we can find an isomorphism class \(T'_n \) with \(|T'_n| = \infty \) of the tuples
\[(R_n, (f_1, \ldots, f_r), \sigma_n, \phi_n)\]
with an extra datum \(\phi_n\) compatible with projections. Indeed, we will see in Lemma 5.3 that if \(\sigma\) is non-trivial on \(T^0\), we have \(\text{Isom}_{\mathcal{I}_Q} (T^1_{Q_m}, T_{Q_m}^-) = \emptyset \) for all \(m\), where we recall \(T^1_{Q_m} = \text{Hom}_{W[\Delta_{Q_m}]}(T_{Q_m}, W[\Delta_{Q_m}])\), and hence \(\text{Isom}_{\mathcal{I}_Q} (T^1_{Q_m}, T_{Q_m}^+) \neq \emptyset\) by Lemma 11.2 (1), proving \(\varepsilon = +\) for \(\varepsilon\) in (Q10). As explained after Remark 4.7, by Proposition 4.6, we can add the datum \(\phi_n\) to our tuples without changing the isomorphism class \(\mathcal{I}_n\) as long as \(\varepsilon\) is constant for all \(Q_m\). In other words,
\[(R_n, (f_1, \ldots, f_r), \sigma_n, \phi_n) \mapsto (R_n, (f_1, \ldots, f_r), \sigma_n)\]
induces a bijection between \(\mathcal{I}_n\) and \(\mathcal{I}_n\). Then by the finiteness of isomorphism classes of the tuples
\[(R_n, (f_1, \ldots, f_r), \sigma_n)\]
of level \(n + 1\) in \(\mathcal{I}_n\) combined with infiniteness of \(\mathcal{I}_n\), the projection maps \(R_{n+1} \rightarrow R_n\) and its dual are compatible with \(\phi_j \in \text{Isom}_{R_j} (R^1_j, R_j)^+ \) for \(j = n + 1, n\) and \(R^1_j = \text{Hom}_{S_j} (R_j, S_j)\) with \(S_j\) as in Remark 4.5). Since \(T'_n\) and \(\mathcal{I}_n\) are in bijection, hereafter we use the symbol \(\mathcal{I}_n\) also for \(T'_n\) (identifying the two index sets).

We have the limit involution \(\sigma_\infty\) acting on \(R_\infty\) which is uniquely lifted to an involution \(\sigma = \sigma_\infty\) acting on \(R := \mathcal{R}_\infty\) for \(\mathcal{R}_\infty\) defined just below (4.2). Put
\[R_{\pm} := \{x \in R | \sigma(x) = \pm x\}.\]
Let \(I_\infty = R(\sigma - 1)R = \mathcal{R}_{\mathcal{R}_-}\). Note that \(r_{\pm} := |Q_\pm|\) is independent of \(Q\) by Corollary 4.2.

We now claim that \(r_- > 0\) if \(\sigma\) acts non-trivially on \(T^0 = R^0\). Here is a proof of this claim. First assume that the class number of \(F\) is prime to \(p\) (so, \(\mathcal{C} = C_0\) in the introduction is trivial). Note that \(T^0/I^0 \cong W[[H_Q]]\) for \(I^0 := T^0(\sigma - 1)T^0\) by Proposition 2.6 and \(H_Q = H_Q^+\) by definition. By our choice of \(Q\), if \(r_0 = 0\) (i.e., \(r = r_+\) and hence \(Q = Q^+\)), by Proposition 1.4, for \(I_\infty = R(\sigma_\infty - 1)R\), we have \(R/I_\infty = \lim_n W[|H_Q]|/\mathfrak{A}_n \cong W[[S_1, \ldots, S_\infty]]\); so, \(\dim R = \dim R/I_\infty\).

If the class number of \(F\) is divisible by \(p\), by Proposition 2.6, we have a canonical isomorphism
\[R^Q/I^Q \otimes_{\Lambda}(T) \cong W[C_{Q_m}]\]
for \(C_{Q_m}\) defined above Theorem B in the introduction. By [H16, Section 3.1], the ring \(W[C_{Q_m}]\) determines functorially the group \(C_{Q_m}\); so, the projection \(R^Q/I^Q \otimes_{\Lambda}(T) \rightarrow R^Q/I^Q \otimes_{\Lambda}(T)\) induces a surjective group homomorphism
\[C_{Q_m}/\Delta_{Q_m}^n \rightarrow C_{Q_m}/\Delta_{Q_m}^{n+1}\]
Here \(C_{Q_m}\) is as in the introduction. This tells us that we have a surjective group homomorphism
\[Z_{Q_m}/\Delta_{Q_m}^{n+1} \rightarrow Z_{Q_m}/\Delta_{Q_m}^n\]. Thus the sequence \(\{Q_m(n)\}_n\) satisfies the requirement of the sequence in Proposition 1.5, and by Proposition 1.5, we have \(R/I_\infty = \lim_n W[|H_Q]|/\mathfrak{A}_n\), which is free of finite rank over \(\Lambda[\Delta]\).\(\Lambda[\Delta]\); so, \(\dim R = \dim R/I_\infty\) without assuming that the class number is prime to \(p\). Thus, if \(|Q^-| = 0\), then \(\text{Spec}(R/I_\infty)\) contains an irreducible component of the integral scheme \(\text{Spec}(\mathcal{R})\). This implies \(\text{Spec}(R/I_\infty)\) is a torsion \(S_\Lambda\)-module of finite type for \(S_\Lambda = \Lambda[\Delta]\), \(S_\Lambda = \Lambda[I^0][I^02, \ldots, I^02]\) and \(S_\Lambda S_\Lambda\) in \(S\) as in Definition 4.3.

Since \(R/I_\infty \cong R_{\infty}/I_{\infty}\) has finite rank over \(W[|\Delta|]\), which is the ramification locus (fixed by \(\sigma_\infty\)), we find that \(r' = \dim_W \text{Spec}(R/I_\infty) = \dim_W \text{Spec}(W[|\Delta|]) = r_+\), which implies \(0 < r_- = r'' = r'_+ + r_-\). The identity \(r_- = \dim_W \text{Spec}(W[|\Delta|]) = r_+\) follows from (3.9). Later we see directly that \(r' = \dim_W \text{Spec}(W[|\Delta|]) = r_+\). Since \(R\) is free of finite rank over \(S_\Lambda\) by the Auslander-Buchsbaum formula (e.g. [CMA, Theorem 19.9]), regularity of \(R\) implies that \(R\) is a Gorenstein ring over \(S_\Lambda\); in particular, \(R^1 := \text{Hom}_{S_\Lambda}(R, S_\Lambda) \cong R\) as \(R\)-modules. By Corollary 4.8 (and (4.6)), \(\phi_\infty\) commutes with \(\sigma_\infty\), and we conclude that \(\phi_\infty : \text{Hom}_{S}(R_{\infty}, S) \cong R_\infty\) induces \(\phi_\infty : \text{Hom}_{S}(R^+_\infty, S) \cong R^+_\infty\) as \(R^+_\infty\)-modules.
Since $R_+/(t - \gamma k)R_+ \cong R_+[[t]]$, $R_+$ is Gorenstein by [CRT, Exercise 18.1], and by Lemma 11.1, $R_+ := \text{Hom}_{\mathbb{T}}(R_+, S_{\Lambda}) \cong R_+$ as $R_+$-modules.

Suppose $r_- = r' = 1$. Let $S_{\Lambda} = S \otimes W \Lambda = \Lambda[[\Delta]]$. Then plainly $S_{\Lambda}$ is flat over $S_{\Lambda}^{\text{f}} := S_{\Lambda}^Q$. By Lemma 11.4, $R_-$ is generated over $R_+$ by a single element $\delta$ with $\sigma(\delta) = -\delta$. If a power series $\Phi(T_{1,+}, \ldots, T_{r,-+}, T_{r,-})$ is fixed by $\sigma_{\infty}$, by equating the coefficients of the identity:

$$
\Phi(T_{1,+}, \ldots, T_{r,-+}, T_{r,-}) = \sigma(\Phi(T_{1,+}, \ldots, T_{r,-+}, T_{r,-})) = \Phi(T_{1,+}, \ldots, T_{r,-+}, -T_{r,-})
$$

we find that $\Phi$ is actually a power series of $(T_{1,+}, \ldots, T_{r,-+}, T_{r,-})$. Thus the fixed part $R_+ := R^G$ for $G = \mathbb{G} = \text{id}, \sigma_{\infty} \}$ is still a power series ring, and we have $R_+ = \Lambda[[T_{1,+}, \ldots, T_{r,-+}, T_{r,-}]]$. Since $T_{r} = \lim_{m \to \infty} \bar{T}_{r}$ by the original Taylor–Wiles argument (e.g., [HMI, page 194]), lifting it to $\Lambda$, we get $T_\infty = \mathbb{T}_0 = \mathbb{Q}/\mathbb{Q}_\theta \mathbb{R}$ and $\mathbb{T}_n$ is the surjective image of $R_-$. Since $R_-$ is generated by one element $\delta$ over $R_+$ (which can be given by T_{r,-}), its image $\mathbb{T}_n$ in $\mathbb{T}$ is generated by one element $\theta$ over $\mathbb{T}_n$. This proves the assertion (2) for $Q = 0$.

For a given $Q = Q_{n_0} \neq 0$, we take $n_0$ such that $p^{n_0} = \max_{e \in \mathbb{Q}} |\Delta_{\mathbb{Q}}(e)|$. Then we restart the Taylor-Wiles argument from $\mathbb{T}_Q$ in place of $\mathbb{T}_0$. In other words, we consider the projective system for $n \geq n_0$:

$$(4.7) \quad ((R_n, \alpha, \tilde{R}_{Q,n}, (f_1, \ldots, f_r), \sigma_n, \phi_n) \in \mathcal{I}_n)$$

for $\tilde{R}_{Q,n} = R_n/((p^{n}) + \mathfrak{A}_Q) \mathfrak{R}$. Then by the same argument, we get

$$
\mathbb{T}_Q \cong \lim_{n \geq n_0} \tilde{R}_{Q,n} = \mathbb{R}_{\infty}/\mathfrak{A}_Q.
$$

Thus again lifting over $\Lambda$, we get $\mathbb{T}_Q^Q = \mathbb{R}/\mathfrak{A}_Q \mathbb{R}$. Since $R_-$ is generated by one element $\delta$ over $R_+$, $\mathbb{T}_Q^Q$ (which is a surjective image of $R_-)$ is generated by a single element $\theta_Q$ over $\mathbb{T}_Q$. We may assume that the projection maps send $T_{r,-} \mapsto \theta_Q \mapsto \theta$ in $\mathbb{T}_-$. This finishes the proof of the assertion (2).

We now prove (3). Since $r'' = r = 1$, we can write $Q^+ = Q_0^+ = \{q_1, \ldots, q_r\}$ and $Q^+ = Q_m = \{q_r\}$. Recall $S_{\Lambda} = S \otimes W \Lambda = \Lambda[[\Delta]]$, and write $\{s_j = 1 + S_j\}_{j = 1, \ldots, r}$ for the basis of $\Delta$ corresponding to $\lim_{m \to \infty} \delta_j$. Since $r'' = r = 1$, $R_+ = \Lambda[[T_{1,+}, \ldots, T_{r,-+}, T_{r,-}]]$ and $S_{\Lambda} = \mathfrak{A}_Q \cap S_{\Lambda}$ is generated by an $S$-sequence

$$
\{s^{|\Delta_{n_1}|} - 1, \ldots, s_r^{\Delta_{n_r}} - 1, s_r^{\Delta_{n_r}} - s_r^{-|\Delta_{n_r}|} - 2\}
$$

(which is hence an $R_+$-sequence), $R_+/\mathfrak{A}_Q \mathbb{R}$ is a local complete intersection and hence is a Gorenstein ring (e.g., [CRT, Exercise 18.1]). We have a surjection $\mathbb{R}_+ \rightarrow \mathbb{T}_Q$ and hence a surjection $\mathbb{R}_+/\mathfrak{A}_Q \mathbb{R} \rightarrow \mathbb{T}_Q \subset \mathbb{T}_Q$. Then we have

$$
b_Q := \text{Ker}(\mathbb{R}_+/\mathfrak{A}_Q \mathbb{R} \rightarrow \mathbb{T}_Q \subset \mathbb{T}_Q) = \text{Ker}(\mathbb{R} \rightarrow \mathbb{T}_Q) \cap \mathbb{R}_+
$$

$$
= \mathfrak{A}_Q \mathbb{R} \cap \mathbb{R}_+ = H^0(G, \mathfrak{A}_Q \mathbb{R}) = s_{Q} + (T_{r,-}(s_r^{\Delta_{n_r}} - s_r^{-|\Delta_{n_r}|})),
$$

since $\mathfrak{A}_Q \mathbb{R} \cap \mathbb{R}_+$ is generated by $T_{r,-} \mathfrak{A}_Q \mathbb{R} = T_{r,-} \mathfrak{A}_Q \mathbb{R} + (T_{r,-}(s_r^{\Delta_{n_r}} - s_r^{-|\Delta_{n_r}|}))$. Thus $b_Q$ is generated by the regular sequence

$$
\{s_{j}^{\Delta_{n_j}} - 1, \ldots, s_r^{\Delta_{n_r}} - 1, T_{r,-}(s_r^{\Delta_{n_r}} - s_r^{-|\Delta_{n_r}|})\}.
$$

Since $S_{j}$ ($j \leq r - 1$) is fixed by $\sigma$, we find that $\mathbb{T}_Q = \Lambda[[T_{1,+}, \ldots, T_{r,-+}, T_{r,-}]]/b_Q$ is a local complete intersection. \qed

5. Proof of Theorem B

In Sections 5–10, unless otherwise mentioned, we assume that $\mathfrak{T} = \text{Ind}_{\mathbb{F}_p}^\mathbb{F} \mathfrak{T}$ for the imaginary quadratic field $F$. Let $Q$ be either $Q = \mathbb{Q}$ as in Theorem 4.10 or $Q = \emptyset$. Thus $\mathbb{T}_Q^0 = \mathbb{T}$ by our convention. So, when $Q = \emptyset$, we omit the superscript or subscript “$Q$” from the notation. Recall the fixed integer $k \geq 1$ and the local direct summand $\mathbb{T}_Q = \mathbb{T}_Q^Q/(t - \gamma k)^Q \mathbb{T}_Q^Q$ of $h_{Q,k,\psi_k}$. Since we use the anticyclotomic Katz $p$-adic L-function $L_p$ defined as an element of $W[[H]]$, the base ring $W$ is a finite extension of $W(\mathbb{F}_p)$ (see [Ka78]), though, replacing $L_p$ by a generator of the ideal $(L_p)$ defined in $W_0[[H]]$ for a finite extension $W_0$ of $Z_p$ (see Theorem 5.2), we do not need to take $W$ bigger than $W_0$. By Corollary 2.5 and Proposition 2.6, we have $\mathbb{T}_Q^Q/IQ \cong W[[H_Q]]$. Write $K := \text{Frac}(\Lambda)$ for the weight Iwasawa algebra $\Lambda$. Since $\mathbb{T}_Q^Q$ is a reduced algebra finite flat over $\Lambda$ (cf. [H13, Corollary 1.3]),
we have $\text{Frac}(\mathbb{T}^Q) = \mathbb{T}^Q \otimes_{\Lambda} \mathbb{K} = X \oplus \text{Frac}(W[[H_Q]])$ for a ring direct summand $X$. Put $\mathbb{T}_{Q,\text{ncm}}$ for the image of $\mathbb{T}^Q$ in $X$. Then we have $I^Q = (\mathbb{T}_{Q,\text{ncm}} \oplus 0) \cap \mathbb{T}^Q$ in $\text{Frac}(\mathbb{T}^Q)$. In particular, the involution $\sigma_Q$ preserves the quotient ring $\mathbb{T}_{Q,\text{ncm}}$ as an automorphism of $\text{Frac}(\mathbb{T}^Q)$.

Since $W[[H_Q]]$ is $\Lambda$-free of finite rank, the exact sequence of Proposition 2.6

$$0 \to I^Q \to \mathbb{T}^Q \to W[[H_Q]] \to 0$$

is split exact, and hence $I^Q$ is $\Lambda$-free of finite rank. Recall $M^\vee = \text{Hom}_\Lambda(M, \Lambda)$ for $\Lambda$-modules $M$. Since $(\mathbb{T}^Q)^\vee \cong \mathbb{T}^Q$ by Theorem 2.1 and Theorem 4.10 and $W[[H_Q]] \cong W[[H_Q]]$ as $\mathbb{T}^Q$-modules, from the above exact sequence, we get the dual diagram with exact rows:

$$\begin{array}{ccc}
W[[H_Q]] & \xrightarrow{\sigma} & (\mathbb{T}^Q)^\vee \\
| & | & | \\
W[[H_Q]] & \xrightarrow{\sigma} & \mathbb{T}^Q \\
& & \downarrow \\
& & T_Q \\
& & T_{Q,\text{ncm}}.
\end{array}$$

Thus we get

**Lemma 5.1.** Suppose $(h0\text{-}4)$. Let $a_Q := \mathbb{T}^Q \cap (0 \oplus \text{Frac}(W[[H_Q]])) = \text{Ker}(\mathbb{T}^Q \to \mathbb{T}^Q_{\text{ncm}})$. Then $a_Q$ is a principal ideal generated by $a_Q \in \mathbb{T}_{Q,\text{ncm}}$ in $\mathbb{T}_{Q,\text{ncm}}^\vee$ isomorphic to $W[[H_Q]]$ as $\mathbb{T}_{Q,\text{ncm}}$-modules (since $W[[H_Q]]$ is fixed by $\sigma_Q$).

If $Q = \emptyset$, we have the anticyclotomic Katz measure $L_p^- \in \text{Frac}((\mathbb{Z}/p[-\epsilon])$ with branch character given by the anticyclotomic projection $\varphi$ of the Teichmüller lift $\varphi$ of $\mathbb{F}$ (see [H15, §6]). Identifying $H$ with $\mathbb{Z}^-$ when $Q = \emptyset$, we regard $L_p^- \in \text{Frac}([H])$. Then from [H15, Theorem 7.2], we get

**Theorem 5.2.** Suppose $(h0\text{-}4)$ and $p > 3$. The ideal $a = a_Q$ is generated by $L_p^- \in \text{Frac}([H])$.

Let $T_{Q}^Q = \{ x \in \mathbb{T}^Q | x^2 = \pm x \}$, $T_{Q,\text{ncm}}^Q = \{ x \in \mathbb{T}_{Q,\text{ncm}}^Q | x^\sigma = \pm x \}$ and $T_{Q}^\pm = \{ x \in \mathbb{T}^Q | x^\sigma = \pm x \}$.

Since no irreducible components of $\text{Spec}(\mathbb{T}_{Q,\text{ncm}})$ is fixed by $\sigma_Q$ and $I^Q = \mathbb{T}^Q(\sigma^-_{Q} - 1)\mathbb{T}^Q = \mathbb{T}_{Q,\text{ncm}}^Q$, we have $T_{Q,\text{ncm}}^Q = T_{Q,\text{ncm}}^Q$. Also $I^Q \subset \mathbb{T}_{Q,\text{ncm}}^Q$, as $I^Q$ is generated by $T_{Q,\text{ncm}}^Q$. Taking $\sigma_Q$-invariant, from $\mathbb{T}^Q/I^Q = W[[H_Q]]$, we conclude $\mathbb{T}^Q/I^Q = W[[H_Q]]$.

We now prove the following key lemma.

**Lemma 5.3.** Assume $(h0\text{-}4)$ and that $F$ is imaginary. Let $Q = Q_m \in Q$ or $Q = \emptyset$ as in Theorem 4.10. If $\sigma$ acts non-trivially on $\mathbb{T} = \mathbb{T}^Q$, then the condition (Q10) is satisfied with $\epsilon = +$ and the ring $T_{Q,\text{ncm}}^Q$ and $T_{Q,\text{ncm}}^Q$ are both Gorenstein. Indeed, we have $\text{Isom}_{\mathbb{T}^Q}(\mathbb{T}^Q)^\vee, \mathbb{T}^Q)^+ \neq \emptyset$ and

$$\text{Isom}_{\mathbb{T}^Q}(\mathbb{T}^Q)^\vee, \mathbb{T}^Q)^- = \text{Isom}_{\mathbb{T}^Q}(\mathbb{T}^Q, \mathbb{T}^Q)^- = \emptyset,$$

where $M^\ast = \text{Hom}_{\mathbb{W}}(M, \mathbb{W})$ and $M^\vee = \text{Hom}_\Lambda(M, \Lambda)$.

In the lemma, we can replace $T_{Q,\text{ncm}}^Q$ (resp. $(\mathbb{T}^Q)^\vee$) by $\text{Hom}_{\mathbb{S}^Q}(\mathbb{T}^Q, \mathbb{S}^Q)$ (resp. $\text{Hom}_{\mathbb{S}^Q}(\mathbb{T}^Q, \mathbb{S}^Q)$) for the image $S^Q$ (resp. $S_{\Lambda, Q}$) of $S$ (resp. $S_{\Lambda, Q}$) in $\mathbb{T}^Q$ (resp. in $\mathbb{T}^Q$) (e.g., Remark 4.5).

**Proof.** Since the proof is the same for any $Q$ including $Q = \emptyset$ and also for $T_{Q}$ and $T_{Q}^Q$, we prove the lemma for $T_{Q,\text{ncm}}^Q$.

Recall $C := \text{Gal}(F_{\varphi}/F)$ for the maximal $p$-abelian extension $F_{\varphi}/F$ of conductor dividing $\varphi$. Since $T/I = W[[H]]$ by Corollary 2.5 and $W[[H]]$ is $\Lambda$-free of rank $|C|$, $I$ is an $\Lambda$-direct summand of $T$, and hence $I$ is $\Lambda$-free. Taking the $\Lambda$-dual sequence of $0 \to I \to T \to W[[H]] \to 0$ (with all $\Lambda$-free terms), we have another exact sequence: $0 \to I^\vee \to T^\vee \to W[[H]]^\vee \to 0$ of $\mathbb{T}$-modules. By Theorem 2.1, $T$ is a local complete intersection. Since $W[[H]]$ is a group algebra, it is a local complete intersection, and hence they are Gorenstein. Then we have $T^\vee \cong T$ and $W[[H]]^\vee \cong W[[H]]$ as $\mathbb{T}$-modules. From this, we conclude $\mathbb{T}_{\text{ncm}} \cong I^\vee$. Thus $\mathbb{T}_{\text{ncm}}$ is $\Lambda$-free and is non-trivial as $\sigma$ acts on $\mathbb{T}$ non-trivially. Since $\mathbb{T}_{\text{ncm}}$ is reduced (by cube-freeness of $N$; see [H13, Corollary 1.3]) and there is no irreducible component of $\text{Spec}(\mathbb{T}_{\text{ncm}})$ on which $\sigma$-acts trivially, $\text{Frac}(\mathbb{T}_{\text{ncm}})$ is equal to $\text{Frac}(\mathbb{T}_{\text{ncm}}) \oplus \text{Frac}(\mathbb{T}_{\text{ncm}})\delta$ for a non-zero divisor $\delta$ with $\delta^2 \in \text{Frac}(\mathbb{T}_{\text{ncm}})$. In other words, $\text{Frac}(\mathbb{T}_{\text{ncm}})$ is a $\text{Frac}(\mathbb{T}_{\text{ncm}})$-free module of rank 2, and $\mathbb{T}_{\text{ncm}} \otimes \mathbb{K} = \mathbb{T}_- \otimes \mathbb{K}$ is a $\text{Frac}(\mathbb{T}_{\text{ncm}})$-free module of rank 1. In particular, we have

$$\begin{align*}
\text{rank}_\Lambda \mathbb{T}_{\text{ncm}}^+ = \text{rank}_\Lambda \mathbb{T}_{\text{ncm}}^- = \text{rank}_\Lambda \mathbb{T}_- > 0.
\end{align*}$$
Under these equivalent conditions, the ring $\Lambda = A$ by Lemma 11.2 (2) applied to $\mathcal{H}$. Thus, by Theorem 5.2, we have (cf. [MFG, §5.3.3])

$$\text{T}^{\text{ncm}}/I = \text{Isom}_T W[[H]] \cong W[[H]]/(L_p^+),$$

which is a torsion $\Lambda$-module. Thus we get

$$\text{rank}_A I_\pm = \dim_K I_\pm \otimes_A K = \dim_K \text{Frac}(\mathcal{T}^{\text{ncm}}) = \text{rank}_A \mathcal{T}^{\text{ncm}} = \dim_K \mathcal{T}^{\text{ncm}}.$$

Taking the $\sigma$-invariant of the two sides of the identity $T/I = W[[H]]$, we have $\text{T}_+ / I_+ \cong W[[H]]$. Thus we get

$$\text{rank}_A T_+ = \text{rank}_A I_+ + \text{rank}_A W[[H]] = \text{rank}_A T_- + \text{rank}_A W[[H]] > \text{rank}_A T_- > 0.$$ 

By Lemma 11.2 (2) applied to $A = \mathcal{T}$ and $S = \Lambda$, $\phi \in \text{Isom}_T(\mathcal{T}_v, \mathcal{T})$ must commute with the involution; so, we get $\text{Isom}_T(\mathcal{T}_v, \mathcal{T})^+ \neq \emptyset$ and $\mathcal{T}_+ \cong \mathcal{T}_v$. Thus $\mathcal{T}_+$ is a Gorenstein ring (by Lemma 11.1) as well as $\text{Isom}_T(\mathcal{T}_v, \mathcal{T}) = \emptyset$. □

We want to prove the following slightly stronger version of Theorem B in the introduction allowing the case when $p|\mathcal{h}_F$:

**Theorem 5.4.** Assume (h0–4). Suppose that $\sigma$ acts non-trivially on $\mathcal{T}$. Then the following four statements are equivalent:

1. The rings $\mathcal{T}^{\text{ncm}}$ and $\mathcal{T}_+^{\text{ncm}}$ are both local complete intersections.
2. The $\mathcal{T}^{\text{ncm}}$-ideal $I = \mathcal{T}(n-1)\mathcal{T} \subset \mathcal{T}^{\text{ncm}}$ is principal and is generated by a non-zero-divisor $\theta \in \mathcal{T}_- = \mathcal{T}^{\text{ncm}}$ with $\theta^2 \in \mathcal{T}_+^{\text{ncm}}$. The element $\theta$ generates a free $\mathcal{T}^{\text{ncm}}$-module $\mathcal{T}_-$, and $\mathcal{T}^{\text{ncm}}/\mathcal{T}_+^{\text{ncm}}[\theta]$ is free of rank 2 over $\mathcal{T}_+^{\text{ncm}}$.
3. The Iwasawa module $\mathcal{Y}^{-}(\varphi^{-})$ is cyclic over $W[[H]]$.
4. The Iwasawa module $\mathcal{Y}^{-}(\varphi^{-}\omega)$ is cyclic over $W[[H]]$.

Under these equivalent conditions, the ring $\mathcal{T}_+$ is a local complete intersection (not just a Gorenstein ring).

Note here that $H = \Gamma_- \cong \Gamma$ if $p \nmid \mathcal{h}_F$ and that $\text{rank}_W W[[H]]/(L_p^-)$ is the sum of the Iwasawa $\lambda$-invariant of the branches of the $p$-adic $L$-function $L_p^-$ since the $\mu$-invariant of $L_p^-$ vanishes by [H10, Theorem 1]. Thus if $p \nmid \mathcal{h}_F$, we have $\mathcal{Y}^{-}(\xi) = \mathcal{Y}^{-}(\xi)$.

**Proof.** For simplicity, we write $A := \mathcal{T}^{\text{ncm}}$ and $A_+ := \mathcal{T}_+^{\text{ncm}}$ and $S = W$. Suppose (1). Then $A, A_+$ are local complete intersections; so, Gorenstein. Thus the different inverse $\mathfrak{d}_{A/A_+}$ and $\mathfrak{d}_{A/W}^{-1}$ are $A$-free modules of rank 1 and $\mathfrak{d}_{A_+/W}^{-1}$ is an $A_+$-module of rank 1. (See Section 11 for the definition of the different inverse. Since

$$\text{Spec}(A)^{\mathcal{h}_F = 1} = \text{Spec}(A/I) \cong \text{Spec}(W[[H]]/(L_p^-)) = \text{Spec}(A_+/I_+),$$

the ramified locus of $\text{Spec}(\mathcal{T}^{\text{ncm}})$ is a non-trivial divisor given by the zero set of $L_p^-$ which is a non-zero divisor of $W[[H]]$. Thus $\mathfrak{d}_{A/A_+}$ is the characteristic ideal ($L_p^-$) (e.g., [MFG, Lemma 5.21]), which is contained in $\mathfrak{m}_A$. Thus by Lemma 11.4, we get the assertion (2).

Suppose (2). By the proof of the anticyclotomic main conjecture in [H06] (see also [H15, Section 7]), we have an identity $\mathcal{T}^{\text{ncm}}/I = W[[H]]/(L_p^-)$ and by the technique of [MT90] (see also [H16, §6.3.6]), we have an isomorphism $\mathcal{Y}^{-}(\varphi^{-}) \otimes_{\mathcal{O}_p[\varphi^{-}]} W \cong \Omega_{\mathcal{T}/\Lambda} \otimes_T W[[H]]$ as $\Lambda$-modules, where $\varphi$ is the unique cyclic satisfying the assumption of the anticyclotomic cyclicity conjecture such that $\chi_{\varphi^{-} | A_+}$ is the Teichmüller lift of $\det(\mathcal{T})$ (i.e., the Neben character of $\mathcal{T}$). Thus we conclude $L_p = L_p^+(\varphi^{-})$ for the Katz measure $L_p$ in Theorem 5.2. Then by [H86c, Lemma 1.1], we have a canonical isomorphism of $W[[H]]$-modules:

$$\Omega_{\mathcal{T}/\Lambda} \otimes_T W[[H]] \cong I/I^2 = (\theta)/(\theta)^2$$

whose left-hand-side is isomorphic to $\mathcal{Y}^{-}(\varphi^{-}) \otimes_{\mathcal{O}_p[\varphi^{-}]} W$. Here $\theta$ is the generator of $I$ as in Theorem 5.4 (2). Since $\theta$ is a non-zero divisor, multiplication by $\theta$ induces an isomorphism of $W[[H]]$-modules

$$\Lambda/(L_p(\varphi^{-})) \cong \mathcal{T}^{\text{ncm}}/I \otimes_{\mathcal{T}} (\theta)/(\theta)^2 \cong \mathcal{Y}^{-}(\varphi^{-}) \otimes_{\mathcal{O}_p[\varphi^{-}]} W.$$
This shows the cyclicity of $Y^-(\varphi^-)$ over $W[[H]]$, which proves (3).

Assume (3). Then by the above identity, $I/I^2 \cong Y^-(\varphi^-)$ is cyclic over $W[[H]]$; so, $I$ is generated by one element by Nakayama’s lemma. Let $I/I^2$ be the tangent space of $\mathbb{T}_Q$ over $W[\Delta_Q]$. Then $I_Q/I^2_Q \cong \text{Sel}(A^n)$ and its minus-eigenspace for $\sigma_Q$ is isomorphic to $\text{Hom}_{W[[H]]}(Y^-(\varphi^-), \mathbb{F})$ by (3.6). Thus $I_Q/I^2_Q$ is generated by one element over $W[\Delta_Q]$. Consider the Taylor Wiles system $(R_n, \ldots)_n$ as in (3.1). Writing $I_n = R_n(\sigma_n - 1)R_n$ for the involution $\sigma_n$ of $R_n$. Since $I_n/I^2_n$ is the image of $I_{Q(m)}/I^2_{Q(m)}$, it is generated by one element over $R_n$. Since $I_{\infty}/I^2_{\infty} = \lim I_n/I^2_n$, $I_{\infty}/I^2_{\infty} \otimes_{R_{\infty}} \mathbb{F}$ factor through $I_n/I^2_n \otimes R_n \mathbb{F}$ for some $n$; so, $I_{\infty}/I^2_{\infty}$ is generated by one element over $R_{\infty}^+$. Since $R_{\infty} = W[[T^1_+ \ldots, T^r_+, T^1_-, \ldots, T^r_-]]$, $I_{\infty}/I^2_{\infty}$ is generated by $r''$ elements over $R_{\infty}$, we conclude $r'' = 1$. Since $r'' = r_-$ by Theorem 4.10 (1) and $r_- = \text{dim} \text{Hom}_{W[[H]]}(Y^-(\varphi^-), \mathbb{F})$ for all $Q$ including $Q = \emptyset$ by Proposition 3.8, we conclude $r_- = 1$ and $Y^-(\varphi^-)$ is cyclic over $W[[H]]$, proving (4).

Assume (4). By Lemma 5.3 combined with Lemma 4.6, the assumption of Theorem 4.10 is satisfied. Then $r_- = 1 = r''$ by Theorem 4.10 (1). Then $\mathbb{T}_-$ is generated by a non-zero divisor $\theta$ by Theorem 4.10 (2), and $I_+$ is generated by $\theta^2$. This implies $\mathbb{T}_{mcm}/(\theta) \cong W[[H]]/(L^-_p) \cong \mathbb{T}_{mcm}/(\theta^2)$. Since $W[[H]]/(\theta)$ is a local complete intersection over $W$, by Lemma 5.5, the assertion (1) holds. Moreover, by Theorem 4.10 (3), $\mathbb{T}_+$ is a local complete intersection. \hfill \Box

Here is the ring theoretic lemma we used:

**Lemma 5.5.** Let $A$ be a complete local noetherian ring finite flat over $\Lambda$. Then $A$ is a local complete intersection if and only if for a non-zero divisor $\delta \in \mathfrak{m}_A$, $A/(\delta)$ is a local complete intersection.

**Proof.** We first prove the “if”-part. Take a presentation $\Lambda[[x_1, \ldots, x_m]] \to A$ for the $m$-variable power series ring $\Lambda[[x_1, \ldots, x_m]]$ over $\Lambda$. Write the kernel of this map as $\mathfrak{a}$. Lifting $\delta$ to $\tilde{\delta} \in \Lambda[[x_1, \ldots, x_m]]$ so that $\tilde{\delta}$ has image $\delta$ in $A$, we have $\Lambda[[x_1, \ldots, x_m]]/(\mathfrak{a} + (\tilde{\delta})) = A/(\delta)$. Write $\mu(\mathfrak{b})$ for the minimal number of generators of an ideal $\mathfrak{b}$ of a ring. Since $A/(\delta)$ is a local complete intersection of dimension 1, $\mathfrak{a} + (\tilde{\delta})$ is generated by a regular sequence of length $m + 1$ as $\mu(\mathfrak{a} + (\tilde{\delta}))$ is equal to $m + 1 = \text{dim} \Lambda[[x_1, \ldots, x_m]] - \text{dim} A/(\delta)$ for the complete intersection ring $A/(\delta)$ (cf. Theorems 17.1 and 21.2 of [CRT]). Since the height of $\mathfrak{a} + (\tilde{\delta})$ is $m + 1$ and the height of $\mathfrak{a}$ is $m$ (by $\text{dim} A = 1 + \text{dim} A/(\delta)$ as $\delta$ is a non-zero divisor; see [CRT, Theorem 17.4]), we conclude $\mu(\mathfrak{a} + (\tilde{\delta})) = \mu(\mathfrak{a} + (\tilde{\delta}))$ is a local complete intersection by [CRT, Theorem 21.2 (iii)].

We now prove the “only if”-part. Let $(a_1, \ldots, a_m)$ be a sequence generating $A$. Pick a non-zero divisor $\delta \in \mathfrak{m}_A$ and lift it to $\tilde{\delta} \in \Lambda[[x_1, \ldots, x_m]]$. Then plainly $(a_1, \ldots, a_m, \delta)$ is a regular sequence $A \cong \Lambda[[x_1, \ldots, x_m]]/(a_1, \ldots, a_m)$ is a local complete intersection by [CRT, Theorem 21.2 (ii)]. \hfill \Box

**Conjecture 5.6** (Semi-simplicity). Suppose $p > 3$. If $c$ is a square-free product of primes split in $F/\mathbb{Q}$, then the projection of $L_p^{-}$ to each irreducible component of $\text{Spec}(W[[H]])$ is square-free.

Note that each irreducible component of $\text{Spec}(W[[H]])$ is the spectrum of a regular local ring $\Lambda := W[[\Gamma_-]]$, which is a unique factorization domain; so, square-freeness of elements of $\Lambda$ is well defined. If $c$ is divisible by non-split primes, there are some cases where $L_p^{-}$ is divisible by $p^3$ (e.g., [H10, §5.5]). It is a well known conjecture that the Kubota-Leopoldt $p$-adic $L$ function is square-free in the Iwasawa algebra (the semi-simplicity conjecture of Iwasawa; see [CPI, (P3–4), No.62 and see also U3]). Thus the above conjecture is an anti-cyclotomic version of Iwasawa’s semi-simplicity conjecture.

6. A Good Choice of Generators of $R_Q$ over $W$

To prove Theorem A, we need to refine further the Taylor–Wiles system choosing a good system of generators over $F$ of the minus part of the cotangent space of $R_Q$ (resp. $R^2$) $(Q \in \mathfrak{Q})$ at $\mathfrak{m}_W$ (resp. at $\mathfrak{m}_Q$). For the system $\mathfrak{Q} = \{Q_m|m = 1, 2, \ldots\}$ of set of primes as in Sections 3 and 4, we will show, choosing $\mathfrak{Q}$ further well, that the image of $T(q)$ for $q \in Q^-$ gives rise to a canonical set of generators at the end. As shown in Lemma 3.2, we hereafter choose $\mathfrak{Q}$ so that $q \equiv 1 \pmod{Cp^m}$ for every $q \in Q_m \in \mathfrak{Q}$ $(m = 1, 2, \ldots)$. The main result in this section is Theorem 6.4. Throughout this section, we assume (h0–4) and $Q \in \mathfrak{Q}$. 


Let $F^Q$ denote the maximal extension of $K_0$ (the splitting field of $\mathfrak{p} = \text{Ind}^Q_{\mathbb{F}} \mathfrak{p}$) unramified outside $Q$ and $p$. Any deformation satisfying (D1–4) of $\mathfrak{p}$ factors through $\text{Gal}(F^Q/\mathbb{Q})$ by (h0).

**Lemma 6.1.** Let $F$ be real or imaginary. The image of the cohomology group $H^1(F^Q/\mathbb{Q}, \text{Ind}^Q_{\mathbb{F}} \mathfrak{p}^-)$ (resp. $H^1(F^Q/\mathbb{Q}, \text{Ind}^Q_{\mathbb{F}} \mathfrak{p}^-)$) to the local cohomology group of $\text{Gal}(\overline{\mathbb{Q}}_q/F_q)$ falls in the one-dimensional subspace $\text{Hom}(\text{Frob}_q^\mathbb{F}, \mathbb{F})$ for $q \in Q^+$ after applying the restriction map and then Shapiro’s lemma. For $q \in Q^+$, we have $H^1(Q_q, \text{Ind}^Q_{\mathbb{F}} \mathfrak{p}^-) = H^1(Q_q, \text{Ind}^Q_{\mathbb{F}} \mathfrak{p}^-) = 0$.

Since the proof for $\text{Ind}^Q_{\mathbb{F}} \mathfrak{p}^-$ and $\text{Ind}^Q_{\mathbb{F}} \mathfrak{p}^+$ is the same, we give a proof for $\text{Ind}^Q_{\mathbb{F}} \mathfrak{p}^-$. 

**Proof.** Note that $\mathfrak{p}|_{\text{Gal}(\overline{\mathbb{Q}}_q/Q_q)} = 1$ as $q \equiv 1$ mod $p$. Thus we ignore $\mathfrak{p}$ when we compute local cohomology groups at $q$. We compute $H^1(Q_q, \text{Ind}^Q_{\mathbb{F}} \mathfrak{p}^-)$. First suppose that $q$ is inert in $F/\mathbb{Q}$ (i.e., $q \in Q^-$). Then by Shapiro’s lemma, we have

$$H^1(Q_q, \text{Ind}^Q_{\mathbb{F}} \mathfrak{p}^-) = H^1(F_q, \mathbb{F}) = \text{Hom}(\mu_{q^2-1} \times q^2, \mathbb{F}) = \text{Hom}(\text{Frob}_q^\mathbb{F}, \mathbb{F}) \equiv \mathbb{F}^2,$$

since $\mathfrak{p}^-$ is trivial on $\text{Gal}(\overline{\mathbb{Q}}_q/F_q)$ as $\mathfrak{p}^- (\text{Frob}_q) = 1$ and $\mathfrak{p} = q \equiv 1$ mod $p$. By the inflation restriction sequence applies to $\text{Gal}(\overline{\mathbb{Q}}_q/Q_q)/I_q = \text{Frob}_q^\mathbb{F}$, we have an exact sequence

$$0 \to H^1(\text{Frob}_q^\mathbb{F}, \text{Ind}^Q_{\mathbb{F}} \mathfrak{p}^-) \to H^1(Q_q, \text{Ind}^Q_{\mathbb{F}} \mathfrak{p}^-) \to \text{Hom}_{\text{Frob}_q^\mathbb{F}}(I_q, \mathfrak{p}^- + \mathfrak{p}^-^{-1}) \to 0,$$

since $H^2(\text{Frob}_q^\mathbb{F}, M) = 0$ for all discrete finite modules $M$. Since $\text{Frob}_q$ acts by multiplication by $q \equiv 1$ mod $p$ on the tame inertia group $I_q^t$ and $\text{Frob}_q$ interchange $\mathfrak{p}^-$ and its inverse, we have $\text{Hom}_{\text{Frob}_q^\mathbb{F}}(I_q, \mathfrak{p}^- + \mathfrak{p}^-^{-1}) \equiv \mathbb{F}$. By Shapiro’s lemma, we have

$$H^1(\text{Frob}_q^\mathbb{F}, \text{Ind}^Q_{\mathbb{F}} \mathfrak{p}^-) = H^1(\text{Frob}_q^\mathbb{F}, \mathfrak{p}^-) = H^1(\text{Frob}_q^\mathbb{F}, \mathbb{F}) \equiv \mathbb{F}$$
as $\mathfrak{p}^-$ is trivial on $\text{Frob}_q^\mathbb{F}$. We have the following commutative diagram:

$$\begin{array}{ccc}
H^1(F^Q/\mathbb{Q}, \text{Ind}^Q_{\mathbb{F}} \mathfrak{p}^-) & \xrightarrow{\text{Res}_q} & H^1(Q_q, \text{Ind}^Q_{\mathbb{F}} \mathfrak{p}^-) \\
\downarrow & & \downarrow \\
H^1(F^Q/F, \mathfrak{p}^-) & \xrightarrow{\text{Res}_q} & \text{Hom}(\text{Gal}(\overline{\mathbb{Q}}_q/F_q), \mathbb{F}) \equiv \mathbb{F}^2.
\end{array}$$

The image of $\text{Res}_F$ further restricted to $I_q$ has trivial intersection with $\text{Hom}_{\text{Frob}_q^\mathbb{F}}(I_q, \mathfrak{p}^- + \mathfrak{p}^-^{-1})$ (while the image of the top $\text{Res}_q$ lands in single $\text{Hom}_{\text{Frob}_q^\mathbb{F}}(I_q, \mathfrak{p}^-)$). Thus $\text{Im}(H^1(F^Q/\mathbb{Q}, \text{Ind}^Q_{\mathbb{F}} \mathfrak{p}^-) \xrightarrow{\text{Res}_q} H^1(Q_q, \text{Ind}^Q_{\mathbb{F}} \mathfrak{p}^-))$ is unramified and is canonically isomorphic to a subgroup of $\text{Hom}(\text{Frob}_q^\mathbb{F}, \mathbb{F}) \equiv \mathbb{F}$.

Now assume that $q$ splits in $F/\mathbb{Q}$. Then we see $\text{Ind}^Q_{\mathbb{F}} \mathfrak{p}^-|_{\text{Gal}(\overline{\mathbb{Q}}_q/Q_q)} = \mathfrak{p}^- + \mathfrak{p}^-^{-1}$. We first compute $H^1(Q_q, \mathfrak{p}^-)$. By the inflation restriction sequence applies to $\text{Gal}(\overline{\mathbb{Q}}_q/Q_q)/I_q = \text{Frob}_q^\mathbb{F}$, we have an exact sequence

$$0 \to H^1(\text{Frob}_q^\mathbb{F}, \mathfrak{p}^-) \to H^1(Q_q, \mathfrak{p}^-) \to \text{Hom}_{\text{Frob}_q^\mathbb{F}}(I_q, \mathfrak{p}^-).$$

Since $\mathfrak{p}^- (\text{Frob}_q) \neq 1$ by our choice of $q \in Q^+$, we have $H^1(\text{Frob}_q^\mathbb{F}, \mathfrak{p}^-) \equiv \mathbb{F}/(\mathfrak{p}^- (\text{Frob}_q) - 1) \mathbb{F} = 0$. Since $\text{Hom}_{\text{Frob}_q^\mathbb{F}}(I_q, \mathfrak{p}^-) \equiv \mathbb{F}$ for the maximal tame quotient $I_q^t$ of $I_q$ and on $I_q^t$, $\text{Frob}_q$ acts by multiplication by $q \equiv 1$ mod $p$. Since $\mathfrak{p}^- (\text{Frob}_q) \neq 1 = (q \text{ mod } p)$, we find $\text{Hom}_{\text{Frob}_q^\mathbb{F}}(I_q^t, \mathfrak{p}^-) = 0$. Thus $H^1(Q_q, \mathfrak{p}^-) = 0$; so, $H^1(Q_q, \text{Ind}^Q_{\mathbb{F}} \mathfrak{p}^-) = H^1(Q_q, \text{Ind}^Q_{\mathbb{F}} \mathfrak{p}^-) = 0$ as $\mathfrak{p}^- = 1$ on $\text{Gal}(\overline{\mathbb{Q}}_q/Q_q)$.

**Lemma 6.2.** For the minus part $t_Q$ of the tangent space $t_Q$ of $\mathbb{T}_Q$, we have

$$t_Q \cong \text{Sel}_Q(\text{Ind}^Q_{\mathbb{F}} \mathfrak{p}^-) \cong \text{Sel}_q(\text{Ind}^Q_{\mathbb{F}} \mathfrak{p}^-) \cong \text{Sel}_q(\text{Ind}^Q_{\mathbb{F}} \mathfrak{p}^-)$$
as vector spaces over $\mathbb{F}$ for all $Q \in Q$ contributing to the Taylor–Wiles system in (3.1) including $Q = \emptyset$.

The isomorphism at (*) is not canonical, and we just choose one.
Proof. By Proposition 3.8 and the construction of the system, we have \( r_- = \dim_{\mathbb{F}} \text{Sel}_0^1(\text{Ind}_F^Q \varphi \mathbb{F}) \) which is independent of \( Q \). For \( n > 1 \), we have defined \( R_n = T_{Q_m} \otimes SS_n \) with \( S_n = S/\mathfrak{A}_n \) for the ideal \( \mathfrak{A}_n := ((s^+_j)^n - 1, (s^+_j)^n - 1, ...)^m \) for some \( m \geq n \). Therefore \( t_{R_m} \equiv t_{R_n} \equiv t_Q \), as \( R_n = T_{R_m} \mathfrak{A}_n \) and taking modulo the ideal \( \mathfrak{A}_n \) for \( n > 1 \) does not change the tangent space. Thus \( \dim_{\mathbb{F}} t_Q \equiv r_- = \dim_{\mathbb{F}} \text{Sel}_0^1(\text{Ind}_F^Q \varphi \mathbb{F}) \). On the other hand, by Proposition 3.8, we have \( \text{Sel}_Q(\text{Ind}_F^Q \varphi \mathbb{F}) \cong t_Q \cong \text{Sel}_0(\text{Ind}_F^Q \varphi \mathbb{F}) \). Thus \( \dim_{\mathbb{F}} \text{Sel}_0(\text{Ind}_F^Q \varphi \mathbb{F}) = \dim_{\mathbb{F}} \text{Sel}_0^1(\text{Ind}_F^Q \varphi \mathbb{F}) \), and we can choose an \( \mathbb{F} \)-linear isomorphism \( (\ast) \). This finishes the proof. \( \square \)

We pick an \( \mathbb{F} \)-linear isomorphism \( \iota : \text{Sel}_0^1(\text{Ind}_F^Q \varphi \mathbb{F}) \cong \text{Sel}_0(\text{Ind}_F^Q \varphi \mathbb{F}) \). Choose a basis \( \{ x \}_x \) of \( \text{Sel}_0^1(\text{Ind}_F^Q \varphi \mathbb{F}) \) over \( \mathbb{F} \) and write \( y = y(x) = \iota(x) \) which gives rise to a basis of \( \text{Sel}_0(\text{Ind}_F^Q \varphi \mathbb{F}) \). Then we define Galois homomorphisms \( f_x \in \text{Hom}_{\text{Gal}(K_1(\mu_p)/Q)}(\text{Gal}(Q(N_p)/K_1), \text{Ind}_F^Q(\varphi \mathbb{F})) \) and \( f_y \in \text{Hom}_{\text{Gal}(K_1/Q)}(\text{Gal}(Q(N_p)/K_1), \text{Ind}_F^Q(\varphi \mathbb{F})) \) by restricting the classes \( x \) and \( y \) (as in Lemma 3.2). Since \( p \nmid |K_1(\mu_p) : Q| \), the two homomorphisms \( f_x \) and \( f_y \) are non-zero. Choose \( \sigma \in \text{Gal}(Q(N_p)/Q) \) as in Corollary 3.3 for \( f_x \) and \( f_y(x) \). Thus we have from Lemma 6.1

\[
(\ast) \quad \text{Sel}_0^1(\text{Ind}_F^Q \varphi \mathbb{F}) \cong \prod_{q \in Q^-} \text{Hom}(\text{Frob}_q^2, \mathbb{F}) \cong \text{Sel}_Q(\text{Ind}_F^Q \varphi \mathbb{F}) \cong \text{Sel}_0(\text{Ind}_F^Q \varphi \mathbb{F}).
\]

We then choose primes \( q \in Q^- = Q_m^- \) so that \( \text{Frob}_q = \text{Frob}_q = \sigma|_{M, M_q} \). By Chebotarev density theorem, we thus have infinite sets \( \{ Q_m^- | m = 1, 2, \ldots \} \). We choose as in Wiles’ work \( Q_m^+ \) as explained in Lemma 3.2 for split primes and put \( Q_m = Q_m^- \cup Q_m^+ \), which makes \( Q \).

Corollary 6.3. We make the choice of \( Q \) as above. The image \( \text{diag}(\sigma) \) of Hecke operator \( T(q) \) in \( T_{\varphi} \) for \( q \in Q^- \) gives rise to a basis of the minus eigenspace \( t_Q^- \) of the cotangent space \( t_Q^- \) under the involution \( \sigma \) induced by \( \rho \mapsto \rho \otimes \chi \) for the deformation \( \rho \). More generally, writing \( b(q) \) for \( \text{image of } U(q) - q^k U(q)^{-1} \) in \( T_{\varphi} \) which projects down to \( o(q) \) in \( T_{\varphi} \), the set \( \{ b(q) \}_{q \in Q^-} \) gives rise to a basis of the minus eigenspace \( t_Q^- \).

Since \( \text{det}(X - \rho_Q([q, q]) = (X - U(q))(X - q^k \chi(q)U(q)^{-1}) \in T_{Q}[X] \) and \( \text{det}(X - \rho_Q(\text{Frob}_q)) = X - T(q)X - q^k \chi(q) \) in \( T_{\varphi}[X] \), the image of \( b(q) \) in \( T_{\varphi} \) is \( a(q) \).

Proof. Write simply \( R = T_{\varphi} \) and \( \sigma \) for the involution. Similarly we write \( R_Q \) for \( T_{Q} \). Since \( \text{diag}(\sigma(T(q))) = \chi(q) T(q) = - T(q) \), we have \( a(q) \in I = R(\sigma - 1) \). Since \( t_Q^- = I/I \cdot m_R \), we have the natural image \( \phi(q) \in t_Q^- \). As Mazur noticed, the tangent space is identified with \( \text{Sel}_Q(\text{Ad}(\varphi)) \) (e.g., [H16, §6.3.6]). The minus part is then identified with \( \text{Sel}_Q(\text{Ind}_F^Q \varphi \mathbb{F}) \subset \text{Sel}_0(\text{Ad}(\varphi)) \) in the following way. Consider \( \rho_Q := \rho_{Q} \text{ mod } m_R \oplus m_W \) for the universal representation \( \rho_Q : \text{Gal}(F(Q)/Q) \to \text{GL}_2(R_Q) \). Then \( \rho_Q \in \mathcal{D}_Q[F[t_Q]] \) as \( R/(m_R^2 + m_W) \cong F[t_Q] \) as rings. Writing \( \rho_Q = \varphi \otimes \ell \) for \( \ell : \text{Gal}(F(Q)/Q) \to t_Q \otimes_F M_2(\mathbb{F}) = M_2(t_Q) \), the map \( c_\varphi : g \mapsto (\Phi \circ \ell(g)) |_{\text{Frob}_q} \) for each \( \Phi \in t_Q^- \), gives rise to a cocycle representing an element of \( \text{Sel}_Q(\text{Ad}(\varphi)) \). Here for the matrix \( \ell(g) = (\ell_{a, b}) \in M_2(t_Q^-) \), we have \( \Phi \circ \ell(g) \) and \( \varphi \) acts on \( M_2(\mathbb{F}) \) from the right. Note that \( \text{Ad}(\varphi) = \text{Ind}_F^Q \varphi \mathbb{F} \otimes \varphi \) and the subspace \( V := \text{Ind}_F^Q \varphi \mathbb{F} \subset \text{Ad}(\varphi) \) is made of anti-diagonal matrices. The anti-diagonal trace map \( T : \text{Ad}(\varphi) \to \text{F} \) induces a linear form on \( V \) and gives rise to an isomorphism of \( X[1] \) in Remark 3.5 (in which the value \( c_\varphi(q) \) lands) onto \( \text{F} \). The isomorphism \( \text{Sel}_Q(\text{Ind}_F^Q \varphi \mathbb{F}) \cong \text{Hom}(\prod_{q \in Q^-} \text{Frob}_q^2, \text{F}) \) by our choice of \( Q^- \) is given by evaluating each Selmer 1-cocycle at \( \phi_q := [q, Q_q] \) (the local Artin symbol) and then projecting the value in \( \text{F} \) via \( T \). Thus we have

\[
T((\Phi \circ \ell(q)) |_{\text{Frob}_q}) = T(\Phi |_{\text{Frob}_q}) - \left( \begin{array}{cc} 0 & \varphi(q)^{-1} \end{array} \right) \left( \begin{array}{cc} \varphi(q)^{-1} & 0 \end{array} \right)^{-1}
\]

\[
= \Phi(\text{Tr}(\rho_Q(\phi_q)) |_{\text{Frob}_q}) = \Phi(\bar{b}(q) \varphi(q)^{-1}) \in \text{F}.
\]

Since \( \Phi \) is arbitrary, this shows the evaluation at \( \phi_q \) corresponds to \( \Phi(\bar{b}(q) \varphi(q)^{-1}) \in t_Q^- \). Thus \( \{ \bar{b}(q) \}_{q \in Q^-} \) (resp. \( \{ \varphi(q) \}_{q \in Q^-} \)) generates \( I_Q/I_Q \cdot m_{R_Q} \) (resp. \( I_{\varphi}/I_{\varphi} \cdot m_R \) over \( \text{F} \)). \( \square \)

Combining all these lemmas, we record what we have proven:
Theorem 6.4. Assume (h0–4). We can choose a set of generators \{f_q\}_{q \in Q} \subset \mathbb{Z}_Q of \mathbb{T}_Q indexed by \(q \in Q\) for \(Q \in \mathbb{Q}\) over \(W\) such that (i) for \(q \in Q^+\), \(f_q\) is the image of \(U(q)^{-1}\) in \(\mathbb{T}_Q\) and (ii) \(\{f_q\}_{q \in Q^-}\) gives rise to a basis of the minus part of the co-tangent space \(t_Q^* = m_{\mathbb{Z}_Q}m_W\).

For \(q \in Q^+\), to form our Taylor–Wiles system, we just choose generators \(\{f_q\}_{q \in Q^+}\) in \(m_{\mathbb{Z}_Q}^+\) whose image generate over \(F\) the plus part of the co-tangent space \(t_Q^*\).

Remark 6.5. Out of the above choices of generators, we create the Taylor–Wiles system for each \(Q \in \mathbb{Q}\). Since \(\mathbb{Q}\) is infinite, we still make the system work and produces \(R_\infty\) (and \(R\)) as before.

We note the following fact, although we have proven finer results in the above theorem.

Lemma 6.6. Suppose that the order of \(\phi^-\) is either odd > 2 or larger than 4. For any \(m \geq 0\), the ideal \(T^Q\) (resp. \(I_Q\)) is generated by the image \(a(l)\) of \(T(l)\) in \(\mathbb{T}_Q\) (resp. \(\mathbb{T}_Q\)) for primes \(l \equiv 1\) mod \(Cp^m\) in \(F/\mathbb{Q}\) outside \(Q\) and \(N\), where \(C = N_{F/\mathbb{Q}}(c)\).

Proof. Let \(\Sigma\) be the set of totally split primes of \(\mathbb{Q}(\mu_{Cp^m})\) outside \(Np\) and \(Q\), and define \(S \subset \Sigma\) to be the set of all primes in \(\Sigma\) split in \(F(\mu_{Cp^m})/\mathbb{Q}(\mu_{Cp^m})\). The set of primes of \(\mathbb{Q}(\mu_{Cp^m})\) above \(\Sigma\) has Dirichlet density 1 in the set of primes of \(\mathbb{Q}(\mu_{Cp^m})\). For a deformation \(\rho_A : \text{Gal}(\mathbb{T}/\mathbb{Q}(\mu_{Cp^m})) \rightarrow \text{GL}_2(A)\) in \(\mathbb{D}^Q(A)\) for a local \(p\)-profinite \(W\)-algebra \(A\), if \(\text{Tr}(\rho_A(Frob_l)) = 0\) for a set \(S_l\) of primes \(l \in \Sigma\) in \(F(\mu_{Cp^m})/\mathbb{Q}(\mu_{Cp^m})\) such that the set of primes of \(\mathbb{Q}(\mu_{Cp^m})\) above \(S_uS_l\) has Dirichlet density 1 with respect to \(\mathbb{Q}(\mu_{Cp^m})\), plainly \(\text{Tr}((\rho_A \otimes \chi(Frob_l)) = \text{Tr}(\rho_A(Frob_l))\) for all \(l \in S_uS_l\). By Chebotarev density, we conclude the trace identity: \(\text{Tr}(\rho_A \otimes \chi) = \text{Tr}(\rho_A)\) over \(\text{Gal}(\mathbb{T}/\mathbb{Q}(\mu_{Cp^m}))\).

As shown in the proof of Lemma 3.2, under the assumption of the order of \(\phi^-\), \(T\) is irreducible over \(\text{Gal}(\mathbb{T}/\mathbb{Q}(\mu_{Cp^m}))\). Then by a theorem of Carayol–Serre (e.g., [MFG, Proposition 2.13]), we have \(\rho_A \otimes \chi \cong \rho_A\) over \(\text{Gal}(\mathbb{T}/\mathbb{Q}(\mu_{Cp^m}))\). Then by [DHI98, Lemma 3.2], \(\rho_A|_{\text{Gal}(\mathbb{T}/\mathbb{Q}(\mu_{Cp^m}))} \cong \text{Ind}_{F(\mu_{Cp^m})}^\mathbb{Q}(\mu_{Cp^m}) \phi\) for a Galois character \(\phi\). This character \(\phi\) extends to \(\text{Gal}(\mathbb{T}/\mathbb{Q})\) as \(\rho_A\) extends to \(\text{Gal}(\mathbb{T}/\mathbb{Q})\) and \(F\) and \(\mathbb{Q}(\mu_{Cp^m})\) are linearly disjoint over \(\mathbb{Q}\), and we conclude that \(\rho_A\) is an induced representation from \(F\) over \(\text{Gal}(\mathbb{T}/\mathbb{Q})\). Thus \(\rho_A \otimes \chi \cong \rho_A\) over \(\text{Gal}(\mathbb{T}/\mathbb{Q})\).

Since the proof is the same for \(I^Q\) and \(I_Q\), we deal with \(I^Q\). Let \(I_0\) be the ideal generated by \(a(l)\) for all rational primes \(l \in S \cup S_l\) as above. Then \(\rho : \rho_{\tau_Q}\) mod \(I_0\) satisfies \(\text{Tr}(\rho_A(Frob_l)) = 0\) for the set \(S \cup S_l\). Therefore by the above argument, we conclude \(\rho \otimes \chi \cong \rho\). Then by Lemma 2.4, we have \(I_0 \supset I^Q\).

Since \(a(l)\) mod \(I^Q\) \(\equiv 0\) if a rational prime \(l\) is inert in \(F\) (as \(\rho_{\tau_Q}\) mod \(I^Q\) is an induced representation from \(F\)), we conclude the reverse inclusion \(I^Q \supset I_0\).

7. PROOF OF THEOREM A AND LOCAL INVOLUTIONS

Throughout this section, we assume (h0–4) and that \(\mathbb{T}_{nm} \neq 0\) (\(\Leftrightarrow I \neq 0\)); i.e., \(\Omega_{T/A} \otimes \mathbb{T}_{nm} \cong I/I^2 \cong Y^-(\varphi^-) \neq 0\) (as otherwise, we have nothing to prove).

Before starting the proof, we ask if we can extend the involutive action of \(G\) on \(W[[\Delta]]\) to \(R_\infty\). The answer is yes if we make a good choice of the generators \(T_j\) depending on \(q_j \in Q\). For that, we tentatively write \(*_j\) for an involution of \(R_\infty\) \((j = 1, 2, \ldots, r_*)\) such that \(*_j(T_j^-) = (-1)^{\delta_{ij}} T_j^-\) for Kronecker’s delta \(\delta_{ij}\). Plainly this involution depends on the choice of generators \(f_j\) (and hence on the variable \(T_j^-\)) in the Taylor–Wiles system. Therefore, to answer the question, we need to choose generators of the Taylor-Wiles system carefully as was done in Theorem 6.4.

Our idea of extending the involution \(*_j\) (acting on \(W[[\Delta]]\)) associated to \(q = q_j \in Q^-\) is to use the local involution coming from the normalizer of \(\Gamma_0(q)\). Thus we give here a description of the local component at \(q \in Q^-\) of the automorphic representation arising from \(\mathbb{T}_Q\). For simplicity, we choose \(k := 1\) (otherwise, we need to replace \(q\) in the following argument by \(q^k\) which does not cause any harm), and suppose that \(q \equiv 1\) mod \(Cp^m\) for \(C = N_{F/\mathbb{Q}}(c)\). If \(f\) is a Hecke eigenform in \(S_2(\Gamma_0(q), \psi_k)\) whose Hecke eigenvalues gives rise to an algebra homomorphism \(\lambda : \mathbb{T}_Q \rightarrow \mathbb{T}_p\); so, \(f(T(n) = \lambda(a(n))f\) for the image \(a(n)\) of \(T(n)\) in \(\mathbb{T}_Q\). After lifting \(f\) to an adelic automorphic form on \(\text{GL}_2(\mathbb{A})\) as in [MFG], the right translation of the adelic automorphic form provides an automorphic representation \(\pi\). Since we have twisted in (2.2) the original \(\rho_{\tau_Q}\) by a global Galois character \(\sqrt{\varphi^-}\) to get the universal representation \(\rho\), we need to study the twisted automorphic representation \(\pi := \pi \otimes \sqrt{\varphi^-}\) of \(\text{GL}_2(\mathbb{A})\). The corresponding automorphic form \(f : \text{GL}_2(\mathbb{Q}) \backslash \text{GL}_2(\mathbb{A}) \rightarrow C\) is the minimal form of \(\pi\) as described in [HMI, after Lemma 2.41 in page 122]. Write \(\pi = \otimes_i \pi_i\) for the
local representation $\pi_q$ of $GL_2(\mathbb{Q}_p)$. Since $\pi_q$ having conductor a factor of $q$, its twist $\pi_\rho$ belongs to a principal series, and hence we have two characters $\alpha, \beta : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ with $\pi_q \cong \pi(\alpha, \beta)$. The cusp form $f$ being minimal at $q$ means that $f(xu) = \alpha(a)\beta(b)f(x)$ for $u = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in the Iwahori subgroup of $GL_2(\mathbb{Q}_q)$. Here the Iwahori subgroup $I$ at $q$ is given by

$$I := \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}_q) | c \equiv 0 \mod q\mathbb{Z}_q \}.$$ 

We write $U(q)$ for the Hecke operator parameterized on $\pi$ by the double coset $I \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} I$ which corresponds to the classical operator $\Gamma Q \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix}$ of $\Gamma_Q$ acting on $f$ (see [MFG, (3.33)] for this fact). Writing $\pi_{\alpha} \in F$ for the chosen eigenvalue of $T(q)$ to define $T_Q$, we require $f(U(q) = \alpha(q)f$ with $\alpha(q) \mod \mathfrak{m}_{T_Q} = \pi_q$. As is well known, the normalized $N(I)$ of $I$ is generated by the class of $w_q := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in GL_2(\mathbb{Q}_q)$. We then define the involution $i = i_q$ by $f(x) \mapsto \chi(\det(x))f(xw_qw_D)$ for $w_D := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in GL_2(\mathbb{Q}_D)$ with $Q_D = \prod_{\mathfrak{p} \mid D} Q_{\mathfrak{p}}$, and

we write $q = q_0$ for the involution induced on $h_Q = h^Q/(t - \gamma^k)h^Q$ (i.e., $h \mapsto q_0(h) = i_q \circ h_{i_q}$).

Suppose that the conductor of $\pi$ at $q$ is $q^2$ (i.e., the conductor of $\alpha$ and $\beta$ are both equal to $q$). Then $\pi_q$ has conductor $q$ and hence $f$ (and $f$) cannot be a theta series induced from $F$ (as $\pi_q$ has to be super-cuspidal if $f$ were a theta series). Recall $\rho_{Q_0}|_{\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)} = \left( \begin{smallmatrix} \delta_q & 0 \\ 0 & \delta_q' \end{smallmatrix} \right)$ with two characters $\delta_q := (\delta_q \mod (t - \gamma^k))$ and $\delta_q' = (\delta_q' \mod (t - \gamma^k))$ distinct even modulo $\mathfrak{m}_{T_{\mathbb{Q}_q}}$ for $\delta_q$ and $\delta_q'$ in (2.3). Since $\alpha$ and $\beta$ are specialization of $\delta_q$ and $\delta_q'$ as Galois characters, we have $\alpha \not\equiv \beta$. In any case, $w_qf(x) = f(xw_q)$ belongs to $\pi$ and satisfies $(w_qf)(xu) = \beta(a)\alpha(d)(w_qf)(x)$ for $u = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in I$ and $U(q)(w_qf) = \beta(q)w_qf$. Since $\pi$ is not a theta series, $\iota_q(f)$ belongs to another automorphic representation $\pi' = \pi \otimes \chi \not\equiv \pi$. Define $\lambda : T_Q \to \overline{\mathbb{Q}_p}$ by $h(i_qf) = \lambda(h)e_q$. Notating $\pi_{\alpha} = \pi(\alpha, \beta)$. Then $U(q)i_qf = \lambda(U(q)\iota_qf) = -\beta(q)i_qf$ (i.e., $\lambda = \lambda \circ g_0$). Then $h_Q/(\ker(\lambda) \cap \ker(\lambda'))$ (for $h_Q = h^Q/(t - \gamma^k)h^Q$) is stable under the involution $q_0$. Thus the involution $q_0$ on $h_Q/(\ker(\lambda) \cap \ker(\lambda'))$ flips $\alpha$ and $\beta$. Since $\alpha\beta(q) = \chi(q)q$ (as we have chosen $q \equiv 1 \mod \mathfrak{p}^m$), the operator $U(q) + \chi(q)qU(q)^{-1}$ has eigenvalue $\alpha(q) + \beta(q)$ for $f$. Since we have an arithmetic point $P \in \text{Spec}(\overline{\mathbb{Q}})$ such that $f = f_P$ is a theta series, we have the $q$-local component of $f_P$ is of the form $\pi(\alpha_0, \alpha_0\chi)$. Therefore $\alpha \equiv \alpha_0 \equiv \beta_0 \mod \mathfrak{m}_Q$ modulo the ideal maximal, and hence $\lambda$ and $\lambda'$ both belong to $\text{Spec}(\overline{\mathbb{Q}})$ for the local ring $T_Q$. This implies that the involution $q_0$ of $h_Q$ preserves the local ring $T_Q$; so, hereafter we regard $q_0$ as an involution of $T_Q$.

Now we deal with the $q$-old case; so, $\pi_q$ has conductor 1. We need to show the involution $q_0$ is well defined on $T_0 = T_0/\mathbb{Q}_Q$ (i.e., $q_0$ preserves $\mathfrak{a}_Q$). In any case, locally at $p$, we have $\pi_q = \pi(\phi, \phi\chi)$ and associated to the induced local Galois representation $\text{Ind}_{F_p}^Q \phi \cong \phi \oplus \phi\chi$ for a character $\phi : \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \to \mathbb{C}^\times$. Therefore $\pi$ is spherical at $q$ and hence has two linearly independent $I$-fixed vectors. Write $H$ (resp. $H_I$) for the spherical (resp. Iwahori) Hecke algebra over $\mathbb{Z}_p$ with respect to $GL_2(\mathbb{Z}_p) \subset GL_2(\mathbb{Q}_p)$ (resp. with respect to the Iwahori subgroup $I \subset GL_2(\mathbb{Q}_p)$). Let $J$ be the Iwahori subgroup of $PGL_2(\mathbb{Z}_q)$ (i.e., the image of $I$ in $PGL_2(\mathbb{Q}_q)$) and $H_J$ be the Iwahori Hecke algebra with respect to $J \subset PGL_2(\mathbb{Q}_q)$ over $\mathbb{Z}_p$. Let $Z \subset GL_2(\mathbb{Q}_q)$ be the center. Write the image of $x \in GL_2(\mathbb{Q}_q)$ in $PGL_2(\mathbb{Q}_q)$ as $x^\gamma$, and take a character $\gamma : Z \to \mathbb{C}^\times$ of conductor at most $q$ (so, $\xi$ is $q$-tamely ramified).

For the moment, assume that $\xi$ is unramified. Consider the ideal $\mathfrak{a} = \mathfrak{a}_Q$ of $H_I$ generated by $Z_q - \xi(q)$ for $Z_q := I \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} I$. Since $Z_q$ commutes with every element in $H_I$, the ideal $\mathfrak{a}$ is a two-sided ideal, and $H_I/\mathfrak{a} \cong H_I$ as $\mathbb{Z}_p$-modules by sending $ixI \mod \mathfrak{a}$ to $ixJ$.

Plainly $H_I$ contains the Hecke algebra $H_0$ of $J$ with respect to the maximal compact subgroup $PGL_2(\mathbb{Z}_q)_\mathfrak{a}$. Since $H_0$ is isomorphic to the Hecke algebra of $I$ with respect to $GL_2(\mathbb{Z}_q)$, we may also regard $H_0 \subset H_I$. Then for the cocharacter group $X^*$ of the maximal split torus $PT$ of $PGL_2$, we have $H_J \cong H_0 \otimes_{\mathbb{Z}_p} Z_p[X_*]$ as $\mathbb{Z}_p$-modules (see [HKP10, Lemma 1.7.1]). Though [HKP10] treats Hecke algebras over $C$, as long as $q^{-1}$ can be found in the base integral domain, this type of result remains true. The group algebra $Z_p[X_\ast]$ is actually a subalgebra of $H_J$ and the generating co-character $q \mapsto \left( \begin{smallmatrix} q & 0 \\ 0 & 1 \end{smallmatrix} \right)$ is sent to $q^{-1}U(q) \in H_J$ (and its inverse $q \mapsto \left( \begin{smallmatrix} 0 & q \\ 1 & 0 \end{smallmatrix} \right)$ is sent to $U(q^{-1})$). Here we have written $U(x)$ for the double coset $J \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} J$. Since $\xi$ is unramified, $\xi(\det(J)) = 1$, and hence $H_I$ acts on functions $\phi : GL_1(\mathbb{Q}_q)$ by $[ixI]\phi(g) := \sum_i \phi(gx_i)$ decomposing $ixI = \bigsqcup_i x_i I$. 


For the center $Z$, we have a splitting of the diagonal torus $T$ of $GL_2(\mathbb{Q}_p)$: $T = Z \times PT$ given by $\begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix} \mapsto (b, \begin{pmatrix} a^{-1} & 0 \\ 0 & b^{-1} \end{pmatrix})$. Thus $X_r$ can be considered a subgroup of $X_s(T)$. In this way, we have a section $H_f \hookrightarrow H_1$ identifying $\mathbb{Z}_p[X_r]$ with subalgebra of $\mathbb{Z}_p[X_1(T)] \subset H_f$. Therefore the isomorphism $i$ induces an identification of $\mathbb{Z}_p[X_r] \cong \mathbb{Z}_p[q^{-1}U(q)]$ as $\mathbb{Z}_p$-sub-algebras of $H_f/\mathbb{Z}_p[X_r]$.

We now allow $\xi$ and $\phi$ to be defined on $1 + q \mathbb{Z}_p$. We let $Z_{\mathbb{Z}_p[X_1(T)]} = \mathbb{Z}_p[q^{-1}U(q)]$ acts on functions $\phi : GL_2(\mathbb{Q}_p) \to A$ for a $\mathbb{Z}[\xi]$-algebra $A$ with $\phi(g(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix})) = \xi(a)\xi(d)^{-1}\chi(d)\phi(g)$ in the following way:

First the central operator $Z_{\mathbb{Z}_p[X_1(T)]}$ acts by $\xi$, and

$$U(q)\phi(g) := \sum_i \xi(\det(x_i))^{-1}\xi(d_i)^2\chi(d_i)\phi(gx_i)$$

decomposing $U(q) = I(\begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix})I = \bigcup_i x_iI$ with $x_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$. Note that for $u = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in I$, we have $\xi(\det(u))^{-1}\xi(d)^2 = \xi(ad)^{-1}\xi(d)^2 = \xi(a)^{-1}\xi(d); \text{ so, } U(q)\phi$ satisfies $U(q)\phi(g(\begin{pmatrix} a & b \\ c & d \end{pmatrix})) = \xi(a)\xi(d)^{-1}\chi(d)U(q)\phi(g)$.

**Remark 7.1.** Keep assuming $k = 1$ for simplicity. We take $\xi = \chi|_{\mathbb{Q}^\times_\ell}$, since the central character of $\pi_q$ is $\chi|_{\mathbb{Q}^\times_\ell}$, as long as $q \equiv 1 \mod C$ for $C = N_{\ell/\mathbb{Q}}(\ell)$. Thus $Z_{\mathbb{Z}_p[X_1(T)]}$ acts by the character $\chi$ and hence the action of $H_f$ on $f$ factors through $H_f/\mathbb{Z}_p$. Since $U(q) \in H_f$ satisfies $X^2 - T(q)X + y(q)q = 0$, $T(q) \in H_f$ corresponds to $U(q) + \chi(q)U(q)^{-1} \in H_f$. The involution $i_q : x \mapsto w_q^{-1}xw_q$ flips the diagonal entry of $x \in T$. In $H_f/\mathbb{Z}_p$, we have for $U = U(q)$,

$$i_q(U) = I_i((\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix})I = I(q(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix})I = I_q(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix})I = I_q(U(q)^{-1} = \chi(q)qU(q)^{-1} = \chi(q)U^{-1} \mod p^m).$$

We have another involution $q_q$ coming from $x \mapsto x^{-1}$ for $x \in GL_2(\mathbb{Q}_p)$ (e.g., [HKP10, 1.8]). The two involutions coincide if $q = 1$ and $\chi(q) = 1$ which is not in our case. In our case of $q \equiv 1 \mod p^m$ and $\chi(q) = 1$, the involution $\xi_q$ modulo $p^m$ coincides with $i_q$ modulo $p^m$ up to the twist by $\chi$ (i.e., $I_x^{-1} = \xi_q((I_xI)) = \chi(\det(x))I_i(q(x)I) \equiv \mod p^m$).

(7.1) We have $\xi_q = i_q \circ \chi$ of $H_f/\mathbb{Z}_p$ and $\xi_q \equiv q_q \mod p^m$ on the image of $H_f/\mathbb{Z}_p$ in $\mathbb{T}_Q$.

where $[\chi]$ sends $T(q)$ to $\chi(q)T(q)$ modulo $p^m$. Thus "modulo $p^m$" the involution $\xi_q$ coincides with the involution of $H_f/\mathbb{Z}_p$ with the property $U(q) - U(q)^{-1} \mapsto -(U(q) - U(q)^{-1})$, and hence $\xi_q \equiv \ast_q \mod p^m$ locally at $q$. Also by (7.1), $q_q$ is well defined on $\mathbb{T}_Q$ and $\mathbb{T}_q$.

If we replace $qU(q)$ by $q^kU(q)$ and take $\xi = \chi|_{\mathbb{Q}^\times_\ell}$, the above remark remains true for general $k \geq 1$. We can see the content of Remark 7.1 (i.e., $i(U(q)) = \chi(q)U(q)^{-1} \mod p^m$) in an elementary matrix computation without using the theory of Iwahori Hecke algebra. To see this (still assuming $k = 1$), consider the two dimensional space $V$ spanned by $f$ and $(qf)(\eta) = f(q\eta)$ for $\eta = \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix}$. We prepare two copies of $V_0 = CF \subset \pi$. Define $I : V^2_0 \to V$ by $(f_1, f_2) \mapsto \frac{f_1}{\eta_2}$. As seen in [HMI, Lemma 3.13], $U(q)$ as a matrix form on $V_0$ is given by $U(q) = \begin{pmatrix} T(q)^{-1} \\ \chi(q) \end{pmatrix}$ (i.e., $U(q)((f_1, f_2)) = I((f_1, f_2)U)$). Defining $w_qf(x) = f(xw_qq)$, it is easy to see $w_qf = f|I$ and $w_q(q\eta) = Z_q^t = \chi(q)(q\eta)$ if $w_qGL_2(\mathbb{Z}_p) = q\eta GL_2(\mathbb{Z}_p)$ and $w_qq \in qGL_2(\mathbb{Z}_p)$; so, $I((f_1, f_2)W_qq) = w_qf(I(f_1, f_2))$ with $W_q = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

Thus we have $U^* := i(U) = W_qUq^{-1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and hence, by $q \equiv 1 \mod p^m$,

$$UU^* = \begin{pmatrix} x(q) & 0 \\ 0 & q^{-1}\chi(q) \end{pmatrix} \equiv \chi(q)G_1(T(q)) \mod p^m$$

as desired. Note that when we lift the classical modular forms to the adele one $f$, the standard “global” action of $w_q$ on the space of adele forms is given by $w_qf(x) = \chi(\det(x))w_qf(x) = \chi(\det(x))f(xw_q)$. If we use this construction of $w_q$, defining $U^*(q) := [w_q] \ast U(q) \ast [w_q]^{-1}$, on new forms, we have $U(q)U^*(q) = q^k$; so, it is consistent with the standard fact $\lambda(T^*(n)) = \lambda(T(n))$ for classical modular forms, e.g., [MFM, Theorem 4.5.4 and (4.6.17)].

Recall the involution $s_j$ of $R_{\infty}$ given by $s_j(T_q^j) = (-1)^{j+1}T_q^{-j}$.

**Lemma 7.2.** For the variables $(T_q^j)_{j=1,...,r}$ of $R_{\infty}$ and $R$, we can choose them so that $s_j$ leaves $W[[\Delta]]$ stable and $s_j(s^+_j) = (s_j)^{-1}$ and fix $s^-_j$ for $i \neq j$ and $s^+_j$ for all $l$ for $s^+_l$ in Definition 4.3.

**Proof.** By Theorem 6.4, we can choose the Taylor–Wiles prime $q := q_j \in \mathbb{Q}_p$ so that $T_q^j$ is the image of $U(q) - q^kU(q)^{-1}$ (as in Remark 7.1) in $R_n, m = \mathbb{T}_Q/(p^n, \delta^n - 1)_{\eta \in \mathbb{Q}} (n \leq m)$. Even imposing this condition, as remarked in Remark 6.5, we have infinitely many Taylor–Wiles prime
sets in \( Q \). Recall the involution \( \varsigma_q \) on \( \mathbb{Z}_p[\Delta_q]|X_s(T) \) in Remark 7.1. Then we have the following diagram becoming commutative after taking modulo \( p^m \) by (7.1):

\[
\begin{array}{ccc}
\mathbb{Z}_p[\Delta_q]|X_s(T) & \longrightarrow & T_{Q_m} \\
\varsigma_q \downarrow & & \downarrow e_v \\
\mathbb{Z}_p[\Delta_q]|X_s(T) & \longrightarrow & T_{Q_m}.
\end{array}
\]

By the argument preceding the lemma, the horizontal map is given by the action of \( \Delta_q \) (via \( \overline{\delta}_q \)) and the unramified cocharacter: “\( q \mapsto \binom{0}{q} \)” is sent to \( q_j^{-1}U(q_j) \) in \( T_{Q_m} \). Let \( S^I_{m,n} \) be the image of \( \mathbb{Z}_p[\Delta_q]|X_s(T) \) in \( R_{m,n} \). The commutativity modulo \( p^m \) of the above diagram shows that the involution \( *_j \) is well defined at least on the \( \mathbb{Z}/p^n\mathbb{Z}[\Delta_q] \)-subalgebra \( S^I_{m,n} \subset R_{m,n} \text{ containing } U(q) \) and is equal to \( \varrho_q \) modulo \( p^m \). Since \( *_j \) is well defined on the entire \( R_{\infty} \), this shows that \( *_j \) preserves \( W[(S^I_j)] \subset R_{\infty} \) after passing to the limit with respect to \( n \). Note that the involution \( \varsigma_q \) on the Iwahori Hecke algebra induced for \( h \in H_I \) by \( h(x) \mapsto h(x^{-1}) \) is not well defined on modular forms \( f \) in characteristic 0 as \( x \mapsto f(x^{-1}) \) is not at all left invariant under \( \text{GL}_2(\mathbb{Q}) \), and \( *_j \) only makes sense on \( R_{m,n} \) killed by \( p^m \) and their limit \( R_{\infty} \). Thus implies \( *_j(\overline{\delta}_{q_j}|I_{q_j}) = \overline{\delta}_{q_j}|I_{q_j} = \overline{\delta}_{q_j}|I_{q_j} \) in \( R_{m,n} \) and hence \( *_{q_j}(\varsigma_q) = (\varsigma_q)^{-1} \) in \( R_{m,n} \).

Note that the involution \( i_q \) induced via conjugation by \( w_q \) sends the Hecke algebra \( T_Q \) for \( Q = Q_m \) in which \( U(q) = \delta_q(q, Q_q) \) to another local component \( T_Q \) of \( h_Q \) in which \( U(q) = \delta_q(q, Q_q) \).

The two Hecke algebras are a twin pair both acting on the same automorphic representations \( \pi \) as \( \pi_q \) (\( \cong \pi(\alpha, \beta) \)) has two independent minimal forms \( f \) and \( f' \) such that \( f|a, b, c, d = \alpha(a)\beta(b)f \) and \( f'|a, b, c, d = \alpha(a)\beta(b)f' \) for all \((a, b, c, d) \) in the Iwahori subgroup \( I \) in \( \text{GL}_2(\mathbb{Q}). \)

The algebra \( T_Q \) corresponds to \( f \) since the image \( b(q) \) of \( U(q) \) in \( T_Q \) satisfies \( b(q) = \pi_{q_0} \mod \pi_{q_0} \) (by our choice of \( U(q) \)-eigenvalue \( \pi_{q_0} = \pi(q) \mod m_{\pi_{q_0}} \), and similarly the image \( b'(q) \in T_Q \) of \( U(q) \) satisfies \( b'(q) = \pi_{q_0} \mod m_{\pi_{q_0}} \). Therefore identifying \( T_Q \) and \( T_{Q'} \) by this isomorphism, we have

\[
(b(q) \mod m_{\pi_{q_0}}) = -(b(q) \mod m_{\pi_{q_0}}).
\]

This identification is a \( \mathbb{Z}_p[\Delta_q]-\)algebra isomorphism because \( \overline{\delta}_q|I_{q_j} = \overline{\delta}_q|I_{q_j} \).

Strictly speaking, the above argument produces a twin pair of two local rings of \( h_Q \) for each choice \( \pi_q \) out of the two eigenvalues of \( U(q) \) for each \( q \in Q \). Thus in total we have \( 2^n \) local rings indexed by the choice \( \Sigma := (\pi_q \in \{\pi_q| q \in Q\}: q \in Q) \); so, we write \( \Sigma \) for the local ring corresponding to \( \Sigma \), and hence \( T_Q = T_{\Sigma_0} \) for \( \Sigma_0 = (\pi_q | q \in Q) \). All these \( 2^n \) local rings are isomorphic each other as \( \mathbb{Z}_p[\Delta_q]-\)algebras, because they are generated by \( \text{Tr}(\rho(\text{Frob}_b)) \) with primes \( l \mid N \) and \( l \not\in Q \) for the Galois representation \( \rho \) attached to the starting local ring \( T_Q = T_{\Sigma_0} \) by Chebotarev density (e.g., \cite[Lemma 5.4]{H15}). However the isomorphism: \( \Sigma_0 \cong T_{\Sigma_0} \) changes the image of \( U(q) \) modulo the maximal ideal by a sign depending on \( \Sigma_0 \). We write this isomorphism \( \chi|\Sigma : T_Q = T_{\Sigma_0} \cong T_{\Sigma_0} \) as it is a sort of local twist by \( \chi \) (at least, the global twist gives rise to \( T_{\Sigma_0} \cong T_{\Sigma_0} \) for \( \Sigma_0 := (\overline{\delta}_q | q \in Q) \)).

We identify all these \( 2^n \) local rings by \( \chi|\Sigma \). Under this identification, \( w_q \) induces an involution on the identified \( T_Q \), and we find \( *_{q_j} \) induces an involution by \( w_{q_j} \) (as it is \( \chi \circ i_q \) as in (7.1)).

The involution \( g_j = g_{q_j} \) induces an involution on the image \( S_{m,n} \) of \( W[\Delta]|T_{Y_j}^{-1} \) in \( R_{m,n} \) identical to \( *_j \) if \( Q = Q_m \). Indeed, the purely local involution \( i = i_q \) (induced from conjugation by \( w_q \)) in Remark 7.1 interchanges \( \overline{\delta}_q \) and \( \overline{\delta}_{q_j} \) by our identification via \( T_{\Sigma_0} \) for all \( \Sigma_0 \). By the pure local nature of \( w_q \) and \( i_q \) and \( \overline{\delta}_q \) (and unramifiedness of \( \varsigma_q \) at all \( q \in Q_m \)), we find \( g_j \circ \overline{\delta}_q = \overline{\delta}_{q_j} \) on \( R_{m,n} \) for all \( i \neq j \) in \( Q_m \) as in (7.1). This implies \( g_j(s_i) = s_{i} \) on \( R_{m,n} \) for all \( i \neq j \), and hence \( g_j \) preserves the image of the entire \( W[\Delta] \) in \( R_{m,n} \). Then we obtain the desired result as \( g_j = s_{j} \) on \( S_{m,n} \), and passing to the limit, \( g_j = *_{j} \) on \( W[\Delta]|T_{Y_j} | \) by (7.1) (as \( g_j \) for \( H_I = \varsigma_q \) for \( H_I \) at \( q_j \)). Since \( *_j \) is well defined on \( R_{\infty} \), it leaves \( W[\Delta] \) stable as desired (although \( *_j \) acts non-trivially on \( W[\Delta] \)).

As remarked in Remark 7.1 and in the proof of Lemma 7.2, the involution \( *_{j} \) with the property of Lemma 7.2 makes sense only modulo \( p^m \) (and hence only on \( R_{m,n} \) with \( 0 < n \leq m \)) for \( q = q_j \) (in other words, \( *_{j} \equiv *_{j} \) modulo \( p^m \) on \( W[\Delta_{Q_{q_j}}] \)), and hence taking the coherent limit \( R_{\infty} = \lim_n R_n \) as in Remark 6.5, we have well defined \( *_{j} \) on \( R_{\infty} \). Making this choice of the variable \( T_{Y_j} \) (as in Theorem 6.4), we write \( b_j \) for \( *_{j} \) in Lemma 7.2 now acting on \( R_{\infty} \) made via Remark 6.5 (not just on \( W[\Delta] \)). Thus the group \( G := \prod_j b_j \) (of order \( 2^n \)) acts on \( R_{\infty} \) and also on \( R \) by patching.
Lemma 7.3. If $p \nmid h_F$, we have $\text{Sel}_\chi^1(\chi\omega) = 0$; so, $r_+ = 0$.

Proof. By Proposition 3.8, $\text{Sel}_\chi^1(\chi\omega)$ is isomorphic to the $F$-dual of the $\chi\omega$-eigenspace of the ray class group modulo $p\infty$ of the splitting field of $\chi\omega$ (which is the maximal totally real subfield of $F(\mu_p)$). This eigenspace is trivial. Indeed the Iwasawa power series for $\chi\omega$ is a unit since the corresponding Kubota–Leopoldt $p$-adic $L$ evaluated at $s = 0$ is a $p$-adic unit by $p \nmid h_F$ (see [LPL, §7]).

We now start the proof of Theorem A. In this proof, we give an argument which applies to $T$ and $T/(t - \gamma^k)$ at the same time. So we write for simplicity $A$ for either $T$ or $T/(t - \gamma^k)$, and put

$$B = \begin{cases} W & \text{if } A = T/(t - \gamma^k), \\ A & \text{if } A = T, \end{cases}$$

which is the base subalgebra of $A$. Similarly, we write $A$ for $R$ or $R_\infty$ according as $A = T$ or $A = T/(t - \gamma^k)$. Plainly by Lemma 7.3, $A^G = B[[T_1^+, \ldots, T_r^+]]^G = B[[T_1^-, \ldots, (T_r^-)^2]]$. To make notation simple, we just write $Y_j$ for $(T_j^-)^2$; so, $A^G = B[[T_1^-, \ldots, T_r^-]]^G = B[[Y_1, \ldots, Y_r]]$. Though, we do not have the “$+$” variables $T_j^+$ by Lemma 7.3, our argument goes through taking the quotient of $R_\infty$ and $R$ by the ideal $(T_1^+, \ldots, T_r^+)$. This remark possibly applies to the case of a totally real base field where $r_+$ might be positive; so, we argue in the following proof as if $r_+$ were positive (forgetting the above lemma).

Let $\Theta_i$ be the image of $T_i^-$ in $m_A$. By Proposition 3.8, the image of $\Theta_i$ ($i = 1, \ldots, r$) in the co-tangent space of $A$ over $\Lambda$ spans the minus part of dimension $r_-$. As $G$ acts on $R$ (and $R_\infty$) by $\hat{b}_i(T_j^-) = (-1)^{i,j} T_j^-$, $G$ acts non-trivially on $A$. For the cotangent space $t_{A/B} = m_A/(m_A^2 + m_B) = \Omega_{A/B} \otimes A F$, we have $G \hookrightarrow \text{Aut}(t_{A/B})$, and hence $G \hookrightarrow \text{Aut}_{B\text{-alg}}(A)$. Let $\tilde{G} := \text{Hom}_{\text{group}}(G, \mu_2(W))$ with the identity character $1 \in \tilde{G}$, and define $\varepsilon_i \in \tilde{G}$ by $\hat{b}_i(T_j^-) = \varepsilon_i(b) T_j^-$ for $b \in G$; so, $G = \bigoplus_i \langle \varepsilon_i \rangle$. Since $G$ acts on $A$ and $A$, we can decompose $A = \bigoplus_{\varepsilon \in \tilde{G}} A[\varepsilon]$ and $A = \bigoplus_{\varepsilon \in \tilde{G}} A[\varepsilon]$ as direct sum of $\varepsilon$-eigenspaces $A[\varepsilon]$ and $A[\varepsilon]$. Note that $A[\varepsilon]$ is free of rank $1$ over $A^G = A[1]$. More generally, we write $X[\varepsilon]$ for the $\varepsilon$-eigenspace of any $p$-profinite module $X$ on which $G$ acts.

Since $A$ is reduced, $\text{Frac}(A)$ is a product of fields finite over $\text{Frac}(B)$. Therefore $b_j \in G$ either permutes some simple components of $\text{Frac}(A)$ or induces a non-trivial automorphism of some other simple components of $\text{Frac}(A)$; so, we have $\langle b_j \rangle \subset \text{Aut}(A/B)$, and $b_j$ are all distinct in $\text{Aut}(A/B)$. Thus $b_1, \ldots, b_r$ generate a subgroup of $\text{Aut}(A/B)$ isomorphic to $G$. Therefore, the $\varepsilon$-eigenspace $A[\varepsilon]$ is $0$ for all $\varepsilon \in \tilde{G}$, and $A[\varepsilon]$ (as a factor of the $B$-free module $A$) is free of finite rank over $B$. The $\varepsilon$-eigenspace $A[\varepsilon]$ is generated over $A^G$ by the lowest degree monomial $T_{2,j} := \bigoplus_{j \in \langle \varepsilon \rangle : \varepsilon = -1} T_j$, such that $b(T_{2,j}) = \varepsilon(b) T_{2,j}$ for any $b \in G$. Therefore $\Theta_i := \bigoplus_{j : \varepsilon = -1} \Theta_j$ generates $A[\varepsilon]$ over $A^G$. We have $A[\varepsilon] A[\varepsilon] \subset A[\varepsilon]$; in particular, $A[\varepsilon] = A^G \Theta_\varepsilon$ as $A^G$ contains the identity element. Since $I$ is stable under $G$, we have a decomposition $I = \bigoplus_{\varepsilon \neq 1} A[\varepsilon]$ with $I[\varepsilon] = I \cap A[\varepsilon]$ into the direct sum of $\varepsilon$-eigenspaces. Since $I$ is generated by $\Theta_j$, it contains $\Theta_\varepsilon$ for all $\varepsilon \neq 1$, and hence $I[\varepsilon] = A[\varepsilon]$ for $\varepsilon \neq 1$. Thus we conclude $I = I^G \oplus \bigoplus_{\varepsilon \neq 1} A[\varepsilon]$ and $A/I = A^G/I^G$, which we write $A^\text{cm}$ (so, $A^\text{cm}$ is $W[[H]]/(t - \gamma^k)$ or $W[[H]]$ according as $A = T/(t - \gamma^k)$ or $T$).

Since $p \nmid h_F$, we have $R_\infty/(T_1^-, \ldots, T_r^-) = W[[S_1^+, \ldots, S_r^+]]$ by Propositions 1.4 and 2.6, and we also conclude $R/(T_1^-, \ldots, T_r^-) = \Lambda[[S_1^+, \ldots, S_r^+]]$. This implies

$$B[[T_1^+, \ldots, T_r^+]] = \Lambda/(T_1^-, \ldots, T_r^-) = B[[S_1^+, \ldots, S_r^+]].$$

Therefore the graded algebra $\text{gr}(A)$ with respect to $(T_1^-, \ldots, T_r^-)$ of $A$ is isomorphic to the polynomial ring $B[[S_1^+, \ldots, S_r^+]]/[T_1^+, \ldots, T_r^-]$. This implies

$$A = \lim_{\rightarrow} B[[S_1^+, \ldots, S_r^+]]/(T_1^-, \ldots, T_r^-)^n = B[[S_1^+, \ldots, S_r^+, T_1^-, \ldots, T_r^-]]$$

and hence

$$A = \Lambda/[S_1^+, \ldots, S_r^+, S_1^-, \ldots, S_r^-]$$

$$= B[[S_1^+, \ldots, S_r^+, T_1^-, \ldots, T_r^-]]/(S_1^+, \ldots, S_r^+, S_1^-, \ldots, S_r^-) = B[[T_1^-, \ldots, T_r^-]]/(S_1^-, \ldots, S_r^-).$$
In other words, we may ignore the variables $S_i^+$ (even if they exist). Thus we now write $A := B[[T_1^-, \ldots, T_r^-]]$. Then we have
\[ A[ε] = B[[T_1^-, \ldots, T_r^-]][G]T_ε = B[[Y_1, \ldots, Y_r^-]][T_ε] \text{ and } (S_i^-, S_r^-)[ε] = A[ε] \cap (S_i^-, S_r^-). \]
We note a corollary of Lemma 7.2 which follows from the above argument and the definition of $S_j^-$ in Definition 4.3:

**Corollary 7.4.** For each $j = 1, 2, \ldots, r$, we have $S_j^- = f_j(Y_1, \ldots, Y_r^-)T_j^-$ with a power series $0 \neq f_j \in B[[Y_1, \ldots, Y_r^-]]$. Moreover $f_j$ (resp. $f_j(0)$) is not a unit in $B[[Y_1, \ldots, Y_r^-]]$ (resp. in $B$).

**Proof.** By Lemma 7.2 and the choice of $T_j^-$, $T_j$ and $S_j^-$ belong to the same eigenspace $A[ε_j]$ for all $b \in G$. Since $A[ε_j] = A^G[T_j^-]$ is $B[[Y_1, \ldots, Y_r^-]]$-free of rank 1, we find $S_j^- = f_jT_j^-$ for $0 \neq f_j \in B[[Y_1, \ldots, Y_r^-]] = A^G$. If $f_j$ is a unit, $S_j^-$ mod $m_A^2$ and $T_j^-$ mod $m_A^2$ generates the equal one dimension subspace of the cotangent space $t_b^*$ in which $S_j^- = 0$. This is a contradiction as $T_j^- \neq 0$ mod $m_A^2$ by Lemma 6.2. Thus $f_j$ is a non-unit. Hence $f_j(0)$ cannot be a unit in $B$.

There is another argument showing the non-unit property. Here it is: By Proposition 3.8, the $-1$ eigenspace dimension $d_Q^-$ of the tangent space of $T_Q$ and $T^Q$ is independent of $Q$. If one of $f_i$ is a unit, then the $d_Q^-$ of a contradiction. Thus $f_j$ ($j = 1, 2, \ldots, r$) are all non-unit. $\square$

Defining the $ε$-projection for $ε \in G$ by $π[ε](x) = \sum_{b \in G} ε(b)b(x)$ (so $X[ε] = π[ε](X)$ for any $p$-profinite module $X$ on which $G$ acts), if $b(x_i) = ε_i(b)x_i$ for all $b \in G$ (i.e., $x_i = y_iT_i^-$ with $y_i \in A^G$), we have
\[ π[ε](\sum_i a_i x_i) = \sum_i \sum_b a_i b(ε_i(b)) = \sum_i \sum_b b(ε_i(b)) x_i = \sum_i π[ε](a_i) x_i. \]
If $a_i$ runs over $A$, we have $π[ε](a_i) = b_i T_ε$ with $T_ε := 1$ for $b_i \in A^G$ (again $b_i$ runs over freely $A^G$). Thus we conclude, assuming that $b(x_i) = ε_i(b)x_i$ for all $b \in G$,
\[ (x_1, \ldots, x_r^-)[ε] = \sum_i x_i T_ε A^G = (x_1 T_ε, \ldots, x_r^- T_ε - ε) A^G. \]
Since $x_i = y_i T_i^-$ with $y_i \in A^G$, we have
\[ (x_1, \ldots, x_r^-)[ε] = (y_1 T_i^- T_ε, \ldots, y_r^- T_r^- T_ε - ε) A^G. \]
If $ε(b_i) = 1$ (i.e., $ε$ does not contain $ε_i$), we have $T_i^- T_ε = (T_i^-)^{-1}2 T_i = y_i T_i$, and if $ε(b_i) = -1$ (i.e., $ε$ does contain $ε_i$), we have $T_i^- T_ε = -T_i^-$. This shows
\[ I[ε_j] = (T_1^-, \ldots, T_r^-)[ε_j] = (T_ε - ε) A^G \quad \text{and hence } I[ε] = (T_ε - ε) A^G = A[ε] \quad \text{if } ε \neq 1, \quad I^G = (Y_1, \ldots, Y_r^-). \]
Similarly, for $A = (S_1^-, \ldots, S_r^-)$, its $ε$-eigenspace is given by
\[ A[ε] = T_ε a_ε \quad \text{for } a_ε := \left( \sum_{j : ε(b_j) = -1} f_j A^G + \sum_{i : ε(b_i) = 1} Y_i f_i A^G \right). \]
Note here $a_ε$ is an ideal of $A^G$. We have, if $ε \neq 1$,
\[ A[ε] = \frac{B[[Y_1, \ldots, Y_r^-]][T_ε]}{(S_i^-, S_r^-)[ε]} = \frac{I[ε]}{T_ε a_ε} \cong \frac{A_G}{a_ε}, \]
and
\[ A^G = \frac{B[[T_1^-, \ldots, T_r^-]]}{((T_1^-)^2 f_1, \ldots, (T_r^-)^2 f_r A^G)} \cong \frac{B[[Y_1, \ldots, Y_r^-]]}{(Y_1 f_1(Y), \ldots, Y_r^- f_r(Y)) A^G}, \]
where the last identity is under the identification of $B[[T_1^-, \ldots, T_r^-]]$ with $B[[Y_1, \ldots, Y_r^-]]$ under $(T_j^-)^2 \mapsto Y_j$. Thus $A^G$ is a complete intersection, and $f := \{f_j(Y)\}_{j : ε(b_j) = -1 \cup \{Y_i f_i(Y)\}_{i : ε(b_i) = 1}$ is a regular sequence of $B[[Y_1, \ldots, Y_r^-]]$ since $A[ε]$ is free of finite rank over $B$ (see [CRT, Theorem 17.4]). Since $A[ε] = A^G T_ε$, we have a surjective morphism of $A^G$-modules:
\[ ω_ε : \frac{B[[Y_1, \ldots, Y_r^-]]}{(Y_1 f_1, \ldots, Y_r^- f_r A^G)} \to \frac{A[ε]}{a_ε} \cong \frac{B[[Y_1, \ldots, Y_r^-]]}{\sum_{j : ε(b_j) = -1} f_j A^G + \sum_{i : ε(b_i) = 1} Y_i f_i A^G}. \]
sending \( g \mapsto g \Theta e \) whose kernel is given by

\[
\operatorname{Ker}(\varpi_e) = \frac{\sum_{j:z(b_j)=-1} f_j A G + \sum_{i:z(b_i)=1} Y_i f_i A G}{(Y_1 f_1, \ldots, Y_{r-} f_{r-}(Y))} = \operatorname{Ann}_{A G}(\Theta e) = \sum_{j:z(b_j)=-1} \operatorname{Ann}_{A G}(\Theta_j).
\]

Here \( \operatorname{Ann}_{A G}(\Theta) \) is the annihilator \( A G \)-ideal of \( \Theta \). Thus \( \operatorname{Ker}(\varpi_e) \) is isomorphic to

\[
\frac{\sum_{j:z(b_j)=-1} f_j Y_i A G + \sum_{i:z(b_i)=1} Y_i f_i A G}{(Y_1 f_1(Y), \ldots, Y_{r-} f_{r-}(Y))} \cong \frac{\sum_{j:z(b_j)=-1} f_j Y_i A G}{(Y_1 f_1(Y), \ldots, Y_{r-} f_{r-}(Y))} \cap \sum_{j:z(b_j)=-1} f_j Y_i A G.
\]

Let \( D_e := \sum_{i:z(b_i)=1} f_i A + \sum_{j:z(b_j)=1} T_j^− A \) and \( \alpha' := (\sum_{j:z(b_j)=-1} f_j A + \sum_{i:z(b_i)=1} T_i^− f_i A) \) as \( A \)-ideals. Then we have

\[
\mathfrak{A} := (S_1^−, \ldots, S_{r-}^−) = (T_1^− f_1(Y), \ldots, T_{r-} f_{r-}(Y)) \subset \bigcap_e D_e.
\]

Similarly put \( \mathfrak{B}_e := \sum_{i:z(b_i)=1} f_i A G + \sum_{j:z(b_j)=1} Y_j A G \). In the same way, we have

\[
\mathfrak{B} := (Y_1 f_1(Y), \ldots, Y_{r-} f_{r-}(Y)) A G \subset \bigcap_e \mathfrak{B}_e.
\]

**Lemma 7.5.** Let \( g_e := \{f_j(Y)\}_{j:z(b_j)=1} \cup \{1\}_{i:z(b_i)=1} \) (resp. \( g'_e := \{f_j(Y)\}_{j:z(b_j)=-1} \cup \{T_i^− \}_{i:z(b_i)=1} \)) which is a minimal set of generators of \( \mathfrak{B}_e \) (resp. \( \mathfrak{D}_e \)). Then \( g_e \) and \( g'_e \) is a regular sequence in \( A G \) and \( A \), respectively. Moreover \( \mathfrak{B} = \bigcap_e \mathfrak{B}_e \), \( \mathfrak{A} = \bigcap_e \mathfrak{D}_e \), \( \mathfrak{A}_e = \bigcap_{i:z(b_i)=1} \mathfrak{B}_e \), \( \alpha'_e = \bigcap_{i:z(b_i)=1} \mathfrak{D}_e \), and \( A G / \mathfrak{B} \) and \( \mathfrak{A} / \mathfrak{B} \) are \( B \)-free of finite rank for all \( e \).

Since the proof is the same for \( \mathfrak{B} \) and \( \mathfrak{A} \), we shall give a proof for \( \mathfrak{B} \).

**Proof.** Since \( \mathfrak{B}_e \subset \mathfrak{B} \) and \( A G = A / \mathfrak{B} \) is free of finite rank over \( B \), for any system \( \{a_1, \ldots, a_j\} \subset m_B \) of parameters of \( B \) (so \( j = 1 \) or 2 according as \( B = W \) or \( A \)), \( A / \mathfrak{B}_e + (a_1, \ldots, a_j) \) is a finite ring; so, \( \operatorname{height}(\mathfrak{B}_e + (a_1, \ldots, a_j)) = \dim A \), and by [CRT, Theorem 17.4 (3)], \( g_e \cup \{a_1, \ldots, a_j\} \) is a system of parameters of \( A \). In particular \( g_e \) is a regular sequence of \( A \) by [CRT, Theorem 17.4 (1)].

Pick a minimal prime ideal \( P \supset \mathfrak{B} = (Y_1 f_1(Y), \ldots, Y_{r-} f_{r-}(Y)) \). Then \( P \) contains either \( f_j \) or \( f_j \). Then choose \( e \in \hat{G} \) so that \( e(b_j) = 1 \iff Y_j \in P \). By definition \( P \supset \mathfrak{B}_e \). Thus any minimal prime of \( A G \) contains \( \mathfrak{B}_e \) contains \( \mathfrak{B}_e = (\bigcap \mathfrak{B}_e) / \mathfrak{B} \). In other words, \( \mathfrak{B}_e \) is nilpotent as \( \bigcap_{P \supset \mathfrak{B}_e \text{ minimal prime} P \supset \mathfrak{B}} \cap \mathfrak{B}_e \) is the nilradical of \( \mathfrak{B} \) (e.g., [CRT, §3]). Since \( A G = A G / \mathfrak{B} \) is reduced, we find \( \mathfrak{B} = \bigcap \mathfrak{B}_e \). We claim that \( A G / \alpha_e \) is reduced. Indeed, if \( a \in \alpha_e \alpha_e \), then \( a_n \in \alpha_e \), for \( n > 0 \). Thus \( (a \Theta_e) = a_\Theta_e \subset a_\Theta_e = (0) \). Hence \( a \Theta_e \) is nilpotent. Since \( A \) is reduced, we find \( a \Theta_e = 0 \); so, \( \alpha_e \subset \operatorname{Ann}_{A G}(\Theta_e) \). This shows reducedness of \( A G / \alpha_e \). Then, by the same argument as above, since \( \alpha_e = (\sum_{i:z(b_i)=1} f_i A G + \sum_{i:z(b_i)=1} Y_i f_i A G) \alpha_e = \bigcap_{i:z(b_i)=1} \mathfrak{B}_e \).

Since \( \{Y_1 f_1(Y), \ldots, Y_{r-} f_{r-}(Y)\} \) is a regular sequence, \( P \) is of height \( r_− \) by [CRT, Theorems 17.4 and 17.6] and [BCM, IV.1.4.2]. Since \( A G = A G / \mathfrak{B} \) is free of finite rank over \( B \), if \( p \in P \), \( P \) is of height \( r_− + 1 \), a contradiction. Thus the residue field of \( P \) is of characteristic 0. Taking \( B = W \), \( A G / \mathfrak{B}_e \subset \prod_{P \supset \mathfrak{B}_e, P \text{ minimal prime}} A G / P \) is free of finite rank; so, \( B \)-free. If \( B = \Lambda \), we have \( A G / \mathfrak{B}_e / (t^{-k}) \) is free of finite rank by the above argument. By Nakayama's lemma, \( A G / \mathfrak{B}_e \) is generated by \( m \) elements as a \( \Lambda \)-module. Therefore we have a surjective \( \Lambda \)-algebra homomorphism \( \pi : \Lambda^m \to A G / \mathfrak{B}_e \). We can vary \( k \geq 2 \), and rank\( \Lambda^m \) of \( (t^{-k}) \) is independent of \( k \). Then we conclude \( \operatorname{Ker}(\pi) \subset \prod_{k \geq 2} (t^{-k}) \) is a torsion-free \( B \)-module, \( A G / \mathfrak{B}_e \) is \( B \)-free of rank \( m \).

Since \( A G \) is \( B \)-free of finite rank, by the exact sequence \( \mathfrak{B}_e / \mathfrak{B} \to A G / \mathfrak{B} \to A G / \mathfrak{B}_e \), \( B \)-freeness of \( A G / \mathfrak{B}_e \) tells us that \( \mathfrak{B}_e / \mathfrak{B} \) is \( B \)-free of finite rank. \( \square \)

**Lemma 7.6.** Let the notation be as above, and put \( A_{Gm} := A / I \). The characteristic element in \( B \) of the \( B \)-torsion module \( \Omega_{A/B} \otimes_A A_{Gm} \), is given by \( \prod_{j=1}^r f_j(0) \in B \). In particular, \( f_j(0) \) is non-zero and non-unit in \( B \) (i.e., the sequences \( \{f_j, Y_1, \ldots, Y_{r-}\} \) and \( \{f_j, T_1^−, \ldots, T_{r-}^−\} \) for any single \( j \) are regular sequences in \( A \) of length \( r_− + 1 \). More generally, for \( 1 \neq \varepsilon \in \hat{G} \), with \( \varepsilon(b_j) = 1 \), \( \{f_j\}_{j:z(b_j)=1} \cup \{f_i\} \cup \{Y_k\}_{z(b_k)=1} \} \) and \( \{f_j\}_{j:z(b_j)=-1} \cup \{f_i\} \cup \{T_k^−\}_{z(b_k)=1} \) are regular sequence in \( A \) of length \( r_− + 1 \).
Proof. By Corollary 7.4, \( f_j(0) \) cannot be a unit. Write \( A_{\text{ncm}} = A/\mathfrak{a} \) (so, \( \mathfrak{a} = (A_{\text{cm}} \oplus 0) \cap A \)). Then the annihilator of \( I = A(\sigma - 1)A \) regarded as an ideal of \( A_{\text{ncm}} \) is the zero ideal (since \( A_{\text{ncm}} \oplus B \text{Frac}(B) = I \otimes_B \text{Frac}(B) \)). We can now decompose \( A_{\text{ncm}} = \bigoplus_{\varepsilon \in \hat{G}} A_{\text{ncm}}[\varepsilon] \) into the sum of the \( \varepsilon \)-eigenspaces \( A_{\text{ncm}}[\varepsilon] \) under the action of \( G \). Since \( I = (\Theta_1, \ldots, \Theta_{r-1}) \) and \( A_{\text{ncm}}[\varepsilon] = A[\varepsilon] = I[\varepsilon] \) as long as \( \varepsilon \neq 1 \) as already remarked, we have \( I^G = \sum_{\varepsilon} A_{\text{ncm}}[1][\Theta_{\varepsilon}] \) and \( I = (\sum_{\varepsilon} A_{\text{ncm}}[1][\Theta_{\varepsilon}] ) \oplus \bigoplus_{\varepsilon \neq 1} A[\varepsilon] \).

Since \( I = (\Theta_1, \ldots, \Theta_{r-1}) \), we have \( I^2 = (\Theta_1 \Theta_1^2) \cup \cdots \cup (\sum_{\varepsilon} A_{\text{ncm}}[1][\Theta_{\varepsilon}] )^2 \). If \( \#|\varepsilon| \geq 2 \) writing \( |\varepsilon| := (j \epsilon(b_j) = -1) \), decomposing \( \epsilon = \epsilon_1 \epsilon_2 \) with non-empty and disjoint \( |\varepsilon| \) (\( j = 1, 2 \)), the \( \varepsilon \)-eigenspace \( (I^2)[\varepsilon] \) contains \( \Theta_\varepsilon = \Theta_{s_1} \Theta_{s_2} \) and hence \( A[\varepsilon] = I[\varepsilon] = (I^2)[\varepsilon] \). If \( \#|\varepsilon| = 1 \), we have \( (I^2)[\varepsilon] = I^G A[\varepsilon] \).

Thus we conclude

\[
I/I^2 = \bigoplus_{\varepsilon: \#|\varepsilon| = 1} (I/I^2)[\varepsilon] = \bigoplus_{\varepsilon: \#|\varepsilon| = 1} A[\varepsilon] / I^G A[\varepsilon].
\]

Recall that \( \epsilon_i \in \hat{G} \) is the dual of \( b_i \) (i.e., \( \epsilon_i(b_j) = (-1)^{\delta_{ij}} \) for the Kronecker delta \( \delta_{ij} \)). From the exact sequence

\[
0 \to (S_j^2)/S_j^2 = \bigoplus_j A \cdot dS_j^2 \to \bigoplus_j A \cdot dT_j - \Omega_{A/B} \to 0,
\]
tensoring with \( B \cong A_{\text{cm}} \) over \( A \) and taking \( \varepsilon_j \)-eigenspace, we get another exact sequence

\[
0 \to B \cdot dS_j^2 \to B \cdot dT_j - \Omega_{A/B} \cong A_{\text{ncm}}[\varepsilon_j] = (I/I^2)[\varepsilon_j] \to 0.
\]

Note that \( A/(T_1^-, \ldots, T_{r-}^-) = A/A(\sigma - 1)A = A/A(\sigma - 1)A = B = A_{\text{ncm}} \). Since \( S_j^2 = f_j T_j^- \) for \( f_j \in A^G = B[Y_1, \ldots, Y_{r-}] \), for the constant term \( f_j(0) \) of \( f_j(Y) \), under the notation of (7.3), we have

\[
(I/I^2)[\varepsilon_j] = \frac{B \cdot dT_j}{B \cdot dS_j} = \frac{f_j(0)(dT_j^0 + T_j^- df_j(Y_1, \ldots, Y_{r-}))}{f_j(0)(dT_j^0)} = \frac{B \cdot dT_j^0}{f_j(0)dT_j^0}.
\]

Thus we have \( (I/I^2)[\varepsilon_j] \cong B/(f_j(0)) \). This shows that \( \text{char}_B(\Omega_{A/B} \otimes_A A_{\text{ncm}}) = \prod_j f_j(0) \) which is non-zero as \( A \) is reduced and free of finite rank over \( B \) (and \( A \neq A_{\text{ncm}} \) by our non-triviality assumption). In particular, \( f_j(0) \neq 0 \), and the principal ideal \( (f_j(Y)) \subset B[Y_1, \ldots, Y_{r-}] \) is prime to \( Y_j \), and hence \( \{f_j, Y_1, \ldots, Y_{r-}\} \) is a regular sequence in \( A^G = B[Y_1, \ldots, Y_{r-}] \). Since \( Y_j = (T_j^-)^2 \), we also conclude that \( \{f_j, T_1^-, \ldots, T_{r-}^-\} \) is a regular sequence in \( A \).

We now prove the general case. Pick \( \varepsilon \neq 1 \), and recall \( \mathfrak{a}'_i := (\bigoplus_{j \neq (b_j) = -1} f_j A + \sum_{k: \epsilon(b_k) = 1} T_k^- f_k A) \) and \( \mathfrak{D}_i := (\bigoplus_{j \neq (b_j) = -1} f_j A + \sum_{k: \epsilon(b_k) = 1} T_k^- A) \). Then \( G \) acts on \( A' = A/\mathfrak{a}'_i \) faithfully. Since \( A' \) is a \( B \)-module of finite type, \( \Omega_{A'/B} \) is a \( B \)-torsion module of finite type. We have the following exact sequence

\[
0 \to \mathfrak{a}'_i / \mathfrak{a}'_i^2 = \bigoplus_{j \neq (b_j) = -1} A' \cdot dT_j \oplus \bigoplus_{k: \epsilon(b_k) = 1} A' \cdot d(T_k^- f_k) \to \bigoplus_{i=1}^{r-} A' \cdot dT_i \to \Omega_{A'/B} \to 0.
\]

Tensoring with \( B' := A/\mathfrak{D}_i \) and taking \( \varepsilon_i \)-eigenspace for \( i \) with \( \varepsilon(b_i) = 1 \),

\[
(a'_i / a'_i^2)_{\varepsilon_i} = \frac{B' \cdot dT_i^-}{B' \cdot dS_i} = \frac{B'}{f_i dT_i^0 + T_i^- df_i} = \frac{B' \cdot dT_i^-}{f_i dT_i^0},
\]

where \( \overline{x} = (x \mod \mathfrak{D}_i) \). This is because \( \overline{T_i^-} = 0 \). Since \( \frac{B' \cdot dT_i^-}{f_i dT_i^0} \cong B' / f_i B' \) is \( B \)-torsion and \( B' \) is \( B \)-free of finite rank (by Lemma 7.5), the multiplication by \( f_i \) on \( B' \) is injective. Therefore \( \{f_j \}_{j \neq (b_j) = -1} \cup \{f_i \} \cup \{T_k^- \}_{\epsilon(b_k) = 1} \) is a regular sequence of length \( r_- + 1 \) in \( A \). This implies that \( \{f_j \}_{j \neq (b_j) = -1} \cup \{f_i \} \cup \{Y_k \}_{\epsilon(b_k) = 1} \) is a regular sequence of length \( r_- + 1 \) in \( A^G \). \( \square \)

Since \( \Omega_{A/B} \otimes_A A_{\text{cm}} \cong I/I^2 \), from (7.2) and (7.4), we get

**Corollary 7.7.** We have

\[
\Omega_{A/B} \otimes_A A_{\text{cm}} \cong \bigoplus_j B \Theta_j \cong \bigoplus_j B/(f_j(0)).
\]

Though we do not need the following result in the sequel, we just record it here:
Corollary 7.8. We have
\[ \text{Frac}(A^G) = \text{Frac}(A^G / B) = \bigoplus_{\varepsilon} \text{Frac}(A^G / B_{\varepsilon}) \] and \[ \text{Frac}(A) = \text{Frac}(A / A) = \bigoplus_{\varepsilon} \text{Frac}(A / D_{\varepsilon}). \]

Again we prove the assertion only for \( B \).

Proof. Let \( P \) be a minimal prime ideal containing \( B \). Then as in the proof of Lemma 7.5, \( P \) determines \( \varepsilon \in \hat{G} \) so that \( \varepsilon(b_k) = 1 \) if and only if \( Y_k \in P \), and hence \( P \) contains \( B_{\varepsilon} \). We first show that \( f_{\varepsilon} \notin P \) if \( \varepsilon(b_1) = 1 \). Indeed, if \( f_{\varepsilon} \in P \), \( P \) contains \( \{ f_j \}_{j \in \{ b_1 \}} \cup \{ f_i \} \cup \{ Y_k \}_{i \in \{ b_k \}} \) which is a regular sequence of length \( r_+ + 1 \) in \( A^G \). Thus \( P \) has height \( r_+ + 1 \) and hence cannot be a minimal prime containing \( B \) generated by a regular sequence of length \( r_- \) (see [CRT, Theorem 17.6] and [BCM, IV.1.4.2]). Write \( \text{min}(a) \) for the set of minimal primes containing an \( A^G \)-ideal \( a \). From this, for each minimal prime \( P \) of \( A^G \) containing \( B \), we have a unique \( B_{\varepsilon} \) such that \( P \supseteq B_{\varepsilon} \) (i.e., \( P \supseteq B_{\varepsilon'} \) for any other \( \varepsilon' \neq \varepsilon \)). Since \( \text{Frac}(A^G / B) = \bigoplus_{P \in \text{min}(B)} \text{Frac}(A^G / P) \) and \( \text{Frac}(A^G / B_{\varepsilon}) = \bigoplus_{P \in \text{min}(B_{\varepsilon})} \text{Frac}(A^G / P) \), we conclude \( \text{Frac}(A^G) = \text{Frac}(A^G / B) = \bigoplus_{\varepsilon} \text{Frac}(A^G / B_{\varepsilon}). \)

By definition, we have \( A^\text{cm}[1] = A^G / B_1 \cong A / D_1 \cong A^\text{cm} \cong B \). Write \( A^\text{cm}[1] = A^\text{cm,-}[1] = (A^G / B_1)^{\perp} \) for the image of \( A^G \) in \( \bigoplus_{\varepsilon \neq 1} \text{Frac}(A^G / B_{\varepsilon}) \). We have \( A^\text{cm}[1] = T_+ \) or \( T_+^\text{ac} \) according as \( B = \Lambda \) or \( B = W \). We want to compute the congruence module
\[ C_0 := A^\text{cm}[1] \otimes_A C^\text{cm}[1] = (A^G / B_1)^{\perp} \otimes_A (A^G / B_1) \cong (A^G / B_1)^{\perp} \otimes_A B \]
and its relation to \( C_\varepsilon := (A^G / B_{\varepsilon}) \otimes_A (A^G / B_1) \) for \( \varepsilon \neq 1 \). Here the identity at \( * \) is because we identify \( A^\text{cm} \) and \( B \) by the \( B \)-algebra structure of \( A^\text{cm} \).

Corollary 7.9. Let the notation be as above. Then we have \( C_0 \cong B / \bigcap_{j=1}^r (f_j(0)) \), \( C_{\varepsilon} \cong B / (f_j(0)) \) and \( C_{\varepsilon} = B / (f_j(0)_{j \in \{ b_j \} = -1}, \text{where } (f_j(0))_{j \in \{ b_j \} = -1} \) is the \( B \)-ideal generated by \( f_j(0) \) for all \( j \) with \( \varepsilon(b_j) = -1 \). In particular \( \text{Spec}(C_{\varepsilon}) \) contains \( \text{Spec}(C_0) \) as long as \( \varepsilon(b_j) = -1 \).

Proof. Recall \( B_{\varepsilon} := \bigcap_{j \in \{ b_j \} = -1} f_j A^G + \bigcap_{j \in \{ b_j \} = 1} Y_j A^G \). Since
\[ \text{Spec}(C_{\varepsilon}) = \text{Spec}(A^G / B_1) \cap \text{Spec}(A^G / B_{\varepsilon}) = \text{Spec}(A^G / B_1) \times_{\text{Spec}(A^G)} \text{Spec}(A^G / B_{\varepsilon}), \]
we have \( C_{\varepsilon} = B / (f_j(0)_{j \in \{ b_j \} = -1} \) as \( \text{Spec}(C_{\varepsilon}) \) is the locus of \( Y_i = 0 \) for all \( i \) and \( f_j = 0 \) for \( j \) with \( \varepsilon(b_j) = -1 \). Since \( B = \bigcap_{\varepsilon} B_{\varepsilon} \) by Lemma 7.5, we have an inclusion \( A^G = A^G / B \rightarrow \prod_{\varepsilon} A^G / B_{\varepsilon}, \) which implies \( \text{Spec}(A^G) = \bigcup_{\varepsilon} \text{Spec}(A^G / B_{\varepsilon}) \) and \( \text{Spec}(A^\text{cm}[1]) = \bigcup_{\varepsilon \neq 1} \text{Spec}(A^G / B_{\varepsilon}). \) Thus we have \( \text{Spec}(C_0) = \bigcup_{\varepsilon \neq 1} \text{Spec}(C_{\varepsilon}) \) and \( \text{Spec}(C_0) \rightarrow \text{Spec}(C_{\varepsilon}) \) is a closed immersion. On the other hand, \( C_{\varepsilon} = B / (f_j(0)) \) and hence \( \text{Spec}(C_{\varepsilon}) \) contains \( \text{Spec}(C_0) \) as long as \( \varepsilon(b_j) = -1 \). Thus we conclude \( C_0 = B / \bigcap_j (f_j(0)) \) as \( \text{Spec}(C_0) = \bigcup_{\varepsilon} \text{Spec}(C_\varepsilon) \) and \( \text{Spec}(C_\varepsilon) \subseteq \text{Spec}(C_0) \).

Corollary 7.10. If \( B = \Lambda \), then \( \Lambda / (f_i(0), f_j(0)) \) is a finite ring as long as \( i \neq j \).

Proof. We compute \( C_0 \) different way. Define \( \text{Spec}(A / I^{\perp}) = \pi^{-1}(\text{Spec}(A^G / I^{\perp}) \subseteq \text{Spec}(A) \) for the projection \( \pi : A \rightarrow \text{Spec}(A^G) \). Then \( \text{Frac}(A) = \text{Frac}(A / I) \oplus \text{Frac}(A / I^{\perp}) \). Since \( A^\text{cm} := A / I \cong A^G / I^{\perp} \) as \( I^G = (Y_1, \ldots, Y_r) \), the isomorphism \( \text{Spec}(A / I) \cong \text{Spec}(A^G / I^{\perp}) \) induced by \( \pi \) gives rise to \( C_0 \cong A / I \otimes_A (A / I^{\perp}) \) or equivalently \( \text{Spec}(C_0) \cong \text{Spec}(A / I) \times_{\text{Spec}(A)} \text{Spec}(A / I^{\perp}) \). We use this expression to compute \( C_0 \) differently.

Recall \( A^\text{cm} / a \). Thus we have \( C_0 \cong A^\text{cm} / a \cong A^\text{cm} \otimes_A A^\text{cm} \cong A^\text{cm} / I = A^\text{cm}[1] / I^{\perp} \) because of \( I^G \otimes \bigoplus_{\varepsilon \neq 1} A[\varepsilon] \). Note that \( B = A^\text{cm} \) as \( p \mid h_F \).

Since \( A \) is local complete intersection over \( B \cong A^\text{cm} \) by the \( R = \mathbb{T} \) theorem, we have
\[ \text{char}_B(C_0) = \text{char}_B(A^\text{cm}[1] / I^{\perp}) = \text{char}_B(A^\text{cm} / a) = \text{char}_B(A^\text{cm} \otimes_A A^\text{cm}) \]
\[ \cong \text{char}_B \Omega_{A / B} \otimes_A A^\text{cm} \text{Corollary 7.7} \prod_j f_j(0). \]

Here the identity \( * \) follows from a theorem of Tate [MFG, Proposition 5.25] which assumes the complete intersection property of \( A \) over \( B \). This implies \( a = (\prod_j f_j(0)) \subset \Lambda = A^\text{cm} \).
By Corollary 7.9, we conclude $\bigcap_i (f_i(0)) = \text{char}_B(C_0) = \prod_j (f_j(0))$. This implies, if $B = \Lambda$, the principal ideals $(f_i(0))$ and $(f_j(0))$ are co-prime in $\Lambda$ if $i \neq j$, and hence $\Lambda/(f_i(0), f_j(0))$ is finite. 

The following corollary concludes the proof of Theorem A:

**Corollary 7.11.** Assume (h0–4). Let the notation be as above. Then, if $B = \Lambda$, $\Omega_{A/B} \otimes_A A^{cm}$ is pseudo isomorphic to $\Lambda/\prod_{j=1}^{\infty}(f_j(0))$.

**Proof.** From Corollary 7.7, we conclude that $\Omega_{A/B} \otimes_A A^{cm} = \bigoplus_j B \Theta_j \cong \bigoplus_j B/(f_j(0))$. Then by Corollary 7.10, $f_j(0)$ are mutually prime each other if $B = \Lambda$, and by Chinese reminder theorem, $\Omega_{A/B} \otimes_A A^{cm}$ is pseudo isomorphic to $\Lambda/\prod_{j=1}^{\infty}(f_j(0))$. 

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**8. CYCLICITY FOR A $\mathbb{Z}_p$-extension $K/F$**

Let $F_\infty^+ \subset F[\mu_{\infty}]$ be the cyclotomic $\mathbb{Z}_p$-extension of $F$. Then $F_\infty := F_\infty^+ F_\infty^-$ is the unique $\mathbb{Z}_p^2$-extension of $F$. Take a $\mathbb{Z}_p$-extension $K/F$ inside $F_\infty$; so, $F_\infty/K$ is also a $\mathbb{Z}_p$-extension. Let

$$L_\infty/F_\infty F(\phi), \ L_\infty^+/F_\infty F(\phi), \ L_\infty^-/F_\infty F(\phi), \ L^K_\infty/K$$

be the maximal $p$-abelian extension unramified outside $p$. Define

(8.1) \hspace{1cm} Y = \text{Gal}(L_\infty/F_\infty F(\phi)), \ Y^\pm = \text{Gal}(L_\infty^+/F_\infty F(\phi)), \ Y_K = \text{Gal}(L^K_\infty/K F(\phi)).

Via canonical splitting

$$\text{Gal}(F_\infty F(\phi)/F) = \Gamma_F \times \text{Im}(\phi),$$

$$\text{Gal}(F_\infty^+ F(\phi)/F) = \Gamma_\pm \times \text{Im}(\phi) \text{ and } \text{Gal}(K F(\phi)/F) = \Gamma_K \times \text{Im}(\phi)$$

for $\Gamma_\pm = \text{Gal}(F_\infty^+/F)$ and $\Gamma_K = \text{Gal}(K/F)$, we define

$$Y(\phi) = Y \otimes_{\mathbb{Z}_p[\text{Im}(\phi)]} W, \ Y^\pm(\phi) = Y^\pm \otimes_{\mathbb{Z}_p[\text{Im}(\phi)]} W, \ Y_K(\phi) = Y_K \otimes_{\mathbb{Z}_p[\text{Im}(\phi)]} W.$$

Write $\mathbb{H}/F$ for the Hilbert class field over $F$, and put $\mathbb{H}(\phi) = \mathbb{H} F(\phi)$ (the composite of $\mathbb{H}$ and $F(\phi)$). Let $\mathcal{L}_\infty/F_\infty \mathbb{H}(\phi)$ (resp. $\mathcal{L}_\infty^+/F_\infty \mathbb{H}(\phi)$, $\mathcal{L}_\infty^-/F_\infty \mathbb{H}(\phi)$, $\mathcal{L}_\infty^K/K \mathbb{H}(\phi)$) be the maximal $p$-abelian extension unramified outside $p$. Put

$$\mathcal{Y} = \text{Gal}(\mathcal{L}_\infty/F_\infty \mathbb{H}(\phi)), \ \mathcal{Y}^\pm = \text{Gal}(\mathcal{L}_\infty^+/F_\infty \mathbb{H}(\phi)), \ \mathcal{Y}_K = \text{Gal}(\mathcal{L}_\infty^K/K \mathbb{H}(\phi)).$$

**Lemma 8.1.** Assume $p \nmid h_F$. Lifting the character $\varphi^-$ to $\text{Gal}(\mathbb{H}(\varphi^-)/F)$ for the composite $\mathbb{H}(\varphi^-) = \mathbb{H} F(\varphi^-)$, we have

$$Y^-(\varphi^-) \cong \mathcal{Y}^- \otimes_{\mathbb{Z}_p[\text{Gal}(\mathbb{H}(\varphi^-)/F)]} W, \ Y(\varphi^-) \cong \mathcal{Y} \otimes_{\mathbb{Z}_p[\text{Gal}(\mathbb{H}(\varphi^-)/F)]} W, \ Y_K(\varphi^-) \cong \mathcal{Y}_K \otimes_{\mathbb{Z}_p[\text{Gal}(\mathbb{H}(\varphi^-)/F)]} W$$

and

$$Y^-(\varphi^-) \cong \mathcal{Y}^- \otimes_{\mathbb{Z}_p[\text{Gal}(\mathbb{H}(\varphi^-)/F)]} W, \ Y(\varphi^-) \cong \mathcal{Y} \otimes_{\mathbb{Z}_p[\text{Gal}(\mathbb{H}(\varphi^-)/F)]} W, \ Y_K(\varphi^-) \cong \mathcal{Y}_K \otimes_{\mathbb{Z}_p[\text{Gal}(\mathbb{H}(\varphi^-)/F)]} W.$$

**Proof.** This follows from that fact that $\mathbb{H}(\varphi^-)$ is linearly disjoint from $L_\infty$ over $F(\varphi^-)$, since $[\mathbb{H}(\varphi^-) : F]$ is prime to $p$ by (h0) and $p \nmid h_F$. Indeed, as $[\mathbb{H}(\varphi^-) : F(\varphi^-)]$ is prime to $p$, we have $\text{Gal}(\mathcal{L}_\infty/F(\varphi^-)) = \text{Gal}(\mathbb{H}(\varphi^-)/F(\varphi^-)) \times \text{Gal}(\mathcal{L}_\infty/\mathbb{H}(\varphi^-))$, and hence $L_\infty = \mathcal{L}_\infty^{\text{Gal}(\mathbb{H}(\varphi^-)/F(\varphi^-))}$, which implies the identity

$$Y(\varphi^-) \cong \mathcal{Y} \otimes_{\mathbb{Z}_p[\text{Gal}(\mathbb{H}(\varphi^-)/F)]} W.$$

Replacing $(\mathcal{L}_\infty, L_\infty)$ by $(\mathcal{L}_\infty^K, L^K_\infty)$, respectively, we get

$$Y_K(\varphi^-) \cong \mathcal{Y}_K \otimes_{\mathbb{Z}_p[\text{Gal}(\mathbb{H}(\varphi^-)/F)]} W$$

by the same argument. This implies the other two identities (as the first and the third is special cases of the identity for $K$ taking $K := F_\infty^+$).

Let $a_{-} = \text{Ker}(W[[\Gamma_F]] \to W[[\Gamma_\pm]])$ and $a_K = \text{Ker}(W[[\Gamma_F]] \to W[[\Gamma_K]])$. Then we have a natural $W[[\Gamma_F]]$-linear maps

$$\pi_+: Y/a_{+} Y \to Y^+, \ \pi_K: Y/a_K Y \to Y_K \text{ and } \pi_-: Y/a_{-} Y \to Y^-.$$

If either $F_\infty/K$ is unramified outside $p$ ($\Leftrightarrow$ $p^r$ fully ramifies in $K/F$) or $\varphi^- \neq 1$, by Rubin [Ru91, Theorem 5.3 (i)-(ii)], we have $\text{Ker}(\pi_K) = \text{Ker}(\pi_\pm) = 0$ and $\text{Coker}(\pi_\pm) \cong \mathbb{Z}_p \cong \text{Coker}(\pi_K)$. Thus we get
Theorem 8.2. Suppose $p \nmid h_F$ and $\varphi^\circ \neq 1$. Let $K/F$ be a $\mathbb{Z}_p$-extension. Then $\pi_\pm$ and $\pi_K$ are all surjective. If either $F_\infty/K$ is unramified outside $p$ or $\varphi^\circ$ is non-trivial on $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, we have

$$Y(\varphi^\circ)/a_*Y(\varphi^\circ) \cong Y^-(\varphi^\circ) \text{ and } Y(\varphi^\circ)/a_KY(\varphi^\circ) \cong Y_K(\varphi^\circ)$$

as $W[\Gamma_F]$-modules.

Proof. Under the assumption of the theorem, by Rubin [Ru91, Theorem 5.3 (i)-(ii)], the corresponding assertions hold between $Y(\varphi^\circ)$ and $Y_K(\varphi^\circ)$. This is equivalent to the assertion of the theorem by Lemma 8.1.

Corollary 8.3. Assume $p \nmid h_F$ and $(h\circ-4)$. If either $F_\infty/K$ is unramified outside $p$ or $\varphi^\circ$ is non-trivial on $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, then cyclicity for $Y_K(\varphi^\circ)$ over $W[\Gamma_K]$, cyclicity of $Y^-(\varphi^\circ)$ over $W[\Gamma_-]$ and cyclicity of $Y(\varphi^\circ)$ over $W[\Gamma_F]$ are all equivalent. If $\varphi^\circ \neq 1$, then cyclicity of $Y(\varphi^\circ)$ over $W[\Gamma_F]$ implies cyclicity $Y_K(\varphi^\circ)$ of $W[\Gamma_K]$.

Proof. By Nakayama’s lemma applied to $W[\Gamma_F]$, as long as $Y(\varphi^\circ)/a_KY(\varphi^\circ) \cong Y_K(\varphi^\circ)$, cyclicity of $Y(\varphi^\circ)$ over $W[\Gamma_F]$ is equivalent to that of $Y_K(\varphi^\circ)$. This holds in particular for $K = F_\infty$ as $F_\infty/F_\infty^\circ$ is unramified everywhere. Then the first assertion follows from the above theorem.

If $Y(\varphi^\circ)$ is cyclic over $W[\Gamma_F]$, cyclicity $Y_K(\varphi^\circ)$ over $W[\Gamma_K]$, $Y(\varphi^\circ)/a_KY(\varphi^\circ) = W[\Gamma_K]$ follows from the surjectivity the projection: $Y(\varphi^\circ)/a_KY(\varphi^\circ) = Y_K(\varphi^\circ)$. Thus again the second assertion follows from the above theorem.

9. Degree of CM components over the Iwasawa algebra

We continue to assume that $F$ is imaginary. Let $\text{Spec}(\mathbb{T})$ be the connected component containing a CM component coming from $F$. As seen in [H15, Section 5] (and Corollary 2.5), under $(h\circ-4)$, any CM component of $\text{Spec}(\mathbb{T})$ is contained in $\text{Spec}(W[[H]])$ (and $\mathbb{T} = W[[\mathcal{H}]] \hookrightarrow L_p \subset W[[\mathcal{H}]]^\times$). Since $\iota : Z \cong H$ by (1.6), $H$ canonically contains $\iota(\Gamma)$ for $\Gamma = 1 + p\mathbb{Z}_p$ embedded into $O^K_{\mathbb{T}}$. We identify $\Gamma$ and $\iota(\Gamma) \subset H$. Denote $H = \Gamma_- \times \Delta$ for the torsion-free subgroup $\Gamma_- \subset \Gamma$ and a finite group $\Delta$, each irreducible CM component is isomorphic to $\text{Spec}(W[[\Gamma_-]])$. Since $\Gamma_- \cong \mathbb{Z}_p$, we find that $\dim_{\mathbb{Q}} \text{Frac}(W[[\Gamma_-]]) = \Gamma_-: \Gamma = p^m$ for some $m \geq 0$. Recall $C := \text{Gal}(F_{\mathbb{T}}/F)$ for the maximal $p$-abelian extension $F_{\mathbb{T}}/F$ of conductor dividing $p$. Since the image $C \hookrightarrow \text{cyclicity } Y(\varphi^\circ) \hookrightarrow C$ and $C \hookrightarrow \text{cyclicity } Y_K(\varphi^\circ) = \text{cyclicity } \Gamma_K(\varphi^\circ)$ under $(h\circ0)$ for the class group $\text{Cl}_F$ of $F$, if $h_F = \rho^a(\eta) (\eta, \in \mathbb{Z})$ with $p \nmid \eta$ for the class number $h_F = |\text{Cl}_F|$, we have $0 \leq m \leq h$. If we find an $O$-ideal $a$ prime to $p$ such that $a^p = (a)$ with $a \in O$ (for $0 < n \leq h$) with $a^{p-1} \equiv 1 \mod p$, we find that $m \geq n$.

By the last assertion of the above proposition, we can easily create many examples of CM components with $\text{Frac}(\mathbb{T}) \neq \mathbb{K}$. An interesting point is that the dimension $\dim_{\mathbb{Q}} \text{Frac}(\mathbb{T})$ is a $p$-power, while non CM component we studied earlier often satisfies $\dim_{\mathbb{Q}} \text{Frac}(\mathbb{T}) = 2$. As shown in [KhR15], there are also examples of non CM component with arbitrary large degree over $\Lambda$.

Take an irreducible component $\text{Spec}(\mathbb{T})$ of $\text{Spec}(\mathbb{T})$, and write its complementary (reduced) component as $\text{Spec}(\mathbb{T}^\perp)$. Thus we have $\text{Spec}(\mathbb{T}) = \text{Spec}(\mathbb{T}^\perp) \cup \text{Spec}(\mathbb{T})$, and $\text{Spec}(\mathbb{T}^\perp) \cup \text{Spec}(\mathbb{T}^\perp)$ has codimension $\geq 1$ in $\text{Spec}(\mathbb{T})$. Suppose that $\mathbb{I}$ is Gorenstein. This is true for CM components as it is isomorphic to the regular ring $W[[\Gamma_-]]$. If $\mathbb{T}^{\text{ncm}}$ is non-trivial and integral, $\mathbb{I} = \mathbb{T}^{\text{ncm}}$ is Gorenstein (as we proved that $\mathbb{T}^{\text{ncm}}$ is Gorenstein in Theorem 5.4 (1)); so, again this property is satisfied for many non CM components. Then as indicated in [EAI, Section 3.1, page 88], $\mathbb{I} \otimes \mathbb{T}^{\perp} = \mathbb{I}/(L_p(\text{Ad}(\rho_\mathbb{P})))$ for a $p$-adic L-function $L_p(\text{Ad}(\rho_\mathbb{P})) \in \mathbb{I}$ interpolating $L(1, \text{Ad}(\rho_\mathbb{P}))$ divided the canonical period for $P$ running through arithmetic points of $\text{Spec}(\mathbb{T}(\mathbb{Q}_p))$.

Suppose that $\mathbb{I}$ is a CM component. Since $\Omega [\mathbb{T}[[\mathcal{H}]][\Lambda]$ is a $p$-torsion module, we expect to have $p|L_p(\text{Ad}(\rho_\mathbb{P}))$ if $\mathbb{H} \neq \Gamma_-$. (see [MFG, §5.3.4]). The decomposition $\text{Ad}(\text{Ind}_{\mathbb{Q}}^F \varphi) \cong \chi \oplus \text{Ind}_{\mathbb{Q}}^F \varphi^\circ$ for $\varphi^\circ = \varphi^\circ \varphi(\sigma = 1)$ for a complex conjugation, we have $L_p(\text{Ad}(\rho_\mathbb{P})) = h_F \cdot L_p(\mathbb{I})$ for the projection $L_p(\mathbb{I})$ of the Katz $p$-adic L-function $L_p$ under $W[[\mathcal{H}]] \rightarrow \mathbb{I}$ (see [H15, Section 5]). Thus we get $h_F L_p(\text{Ad}(\rho_\mathbb{P}))$ in $\mathbb{I}$. This gives a plenty of examples of positivity of the $\mu$-invariant of $L_p(\text{Ad}(\rho_\mathbb{P}))$. One can then ask if the $\mu$-invariant of $L_p(\text{Ad}(\rho_\mathbb{P}))$ vanishes for non CM components $\mathbb{I}$. One can
produce some non CM component with $L_p(\Ad(\rho))$ having positive $\mu$ if $p = 2$. Thus for this question to be affirmative, we need to assume $p > 2$.

10. Divisibility of the adjoint $p$-adic $L$-function

We continue to assume that $F$ is imaginary. Recall our assumption $p \geq 5$. Picking an irreducible component $\Spec(\mathcal{I})$ of $\Spec(T)$ and writing $\mathcal{I}$ for the normalization of $\mathcal{I}$ (i.e., $\mathcal{I}$ is the integral closure of $\mathcal{I}$ in $\Frac(\mathcal{I})$), we put $\mathcal{I} = T \otimes_\Lambda \mathcal{I}$. Write $\pi : T \to \mathcal{I}$ for the projection inducing the inclusion $\Spec(\mathcal{I}) \hookrightarrow \Spec(T)$. Since $\Hom(\mathcal{T}, \Lambda) \cong T$, we have

$$\mathcal{I} \cong \Hom(\mathcal{T}, \Lambda) \otimes_\Lambda \mathcal{I} \cong \Hom(T \otimes_\Lambda \mathcal{I}, \mathcal{I}).$$

This follows from the fact that $\mathcal{I}$ is $\Lambda$-free of finite rank (as any reflexive module of finite type over a regular local domain of dimension 2 is free; see [H88a, Lemma 3.1] and [H88b, (5.5b)]). We fix the identification (10.1). Decompose $\Frac(\mathcal{I}) = \Frac(\mathcal{I}) \otimes S$ as a $K$-algebra direct sum, and define $\mathcal{I}^+$ for the image of $\mathcal{I}$ in $S$, where $\lambda : \mathcal{I} \to \Frac(\mathcal{I})$ is induced by the projection $\mathcal{I} = T \otimes_\Lambda \mathcal{I}$ given by $t \otimes i = \pi(t)i \in \mathcal{I}$. Regarding $\lambda : \mathcal{T} \to \mathcal{I}$, we take adjoint $\lambda^* : \mathcal{I} = \Hom(\mathcal{I}, \mathcal{I}) \to \Hom(\mathcal{T}, \mathcal{I}) = T$. Then define $L_p(\Ad(\rho)) := \lambda \circ \lambda^* \in \Hom(\mathcal{I}, \mathcal{I})$. As shown in [H86c, Lemma 1.6] (or [MFG, §5.3.3]), we have $C_0(\lambda, \mathcal{I}) := \mathcal{I}^+ \otimes_\mathcal{I} \mathcal{I} \cong \mathcal{I}/(L_p(\Ad(\rho)))$ as $\mathcal{I}$-modules. This $L_p(\Ad(\rho))$ interpolates the adjoint $L$-values $L(1, \Ad(\rho))/\Omega_p$ for arithmetic points $P$ for the canonical period $\Omega_p$, written as $U_\infty(f_P)U_P(f_P)$ in [H88b, Theorem 0.1] and coincides with the one introduced in the previous section if $\mathcal{I}$ is Gorenstein (i.e., $L_p(\Ad(\rho))$ is contained in $\mathcal{I}$ if $\mathcal{I}$ is Gorenstein).

In [H88b, Theorem 0.1], some restrictive assumptions [H88b, (0.8a,b,c)] are made. However, these assumptions are not necessary as long as $h$ is reduced (for example, $N$ is cube-free; see [H13, Section 1]). To see this, we consider the jacobian variety $J_1(Xp^r)$ of the modular curve $X_1(Np^r)$ over $\mathbb{Q}$. Then by the Albanese functoriality applied to the tower of modular curves:

$$\ldots \to X_1(Np^{r+1}) \to X_1(Np^r) \to \ldots \to X_1(Np),$$

we have the projective system of the ordinary parts of the $p$-adic Tate modules $\{T_pJ_1(Np^r)_{\text{ord}}\}_r$. Put $L := \lim_{\to} T_pJ_1(Np^r)_{\text{ord}}$. Then as shown in [H86b] (see also [H14, Sections 4–5]), $L$ is naturally an $h$-module and is also $\Lambda$-free of finite rank. As explained in [H13, Section 4] from the work of Ohta (and an earlier work by the author [H86b]), we have the following canonical exact sequence of $h$-modules:

$$0 \to h \to L \to h^\vee \to 0.$$  

When [H88b] was written, this sequence is only known under the one of the three conditions [H88b, (0.8a,b,c)]. This is the only point we used to prove [H88b, Theorem 0.1]; so, the result is valid without assuming these conditions (i.e., [H88b, Conjecture 0.2] is now known to be true; see [H16, §5.5] for more details of this).

We want to prove

**Theorem 10.1.** Suppose $p \geq 5$, let $\Spec(\mathcal{I})$ be an irreducible non CM component of $\Spec(T)$ satisfying (h0–4), and write $\mathcal{I}$ be the normalization of $\mathcal{I}$ in $\Frac(\mathcal{I})$. Then, under the equivalent conditions of Theorem 5.4, we have

1. If $\Spec(\mathcal{J})$ is a CM component of $\Spec(T)$ and $\varphi$ ramifies at $p$, then the ideal $L_p(\Ad(\rho))$ of $\mathcal{I}$ is generated by the $\varphi^r$-branch of the anticyclotomic Katz $p$-adic $L$-function times the $p$-part $h_F$ of the class number of $F$.

2. Suppose $p \nmid h_F$ and Conjecture 5.6. Then we have $\sqrt{L_p(\varphi^-)} \in \mathcal{I}$, $\text{rank}_\Lambda \mathcal{I} \geq 2$, the $p$-adic $L$-function $L_p(\Ad(\rho))$ is a non-unit in $\mathcal{I}$, and $\sqrt{L_p(\varphi^-)}$ divides $(L_p(\Ad(\rho)))$ in $\mathcal{I}$. If further $\text{rank}_\Lambda \mathcal{T}^\text{ncm} = 2$, then $\mathcal{I} = \mathcal{I} = \Lambda[\sqrt{L_p(\varphi^-)}]$ and $(L_p(\Ad(\rho))) = (\sqrt{L_p(\varphi^-)})$.

The example given in [H85, (10.0b)] shows the case (2) in the above theorem actually occurs, and indeed, in this case, $T = \Lambda[\sqrt{L_p}]$ and $(L_p)$ has a unique zero of multiplicity one in the unit disk $p\mathbb{Z}_p$.

**Proof.** The assertion (1) is a restatement of [H15, Proposition 7.10]. So we prove the other two assertions. We deal with (2). Write the composite map $\mathcal{I} = T \otimes_\Lambda \mathcal{I} \to \mathcal{I} \otimes_\Lambda \mathcal{I} \xrightarrow{m} \mathcal{I}$ as $\lambda$, where the right most arrow is the multiplication $(a \otimes b \mapsto ab)$. Since $\mathcal{I} = T \otimes_\Lambda \mathcal{I}$ surjects down to $\mathcal{I} \otimes_\Lambda \mathcal{I}$, we
have \( \text{Spec}(\overline{I}) \subseteq \text{Spec}(I \otimes \overline{A}) \subseteq \text{Spec}(T) \). Consider the congruence modules (see [MFG, §5.3.3] for congruence modules)

\[
C_0(\lambda; \overline{I}) := I^+ \otimes_{\mathbb{T},A} \overline{I} \quad \text{and} \quad C_0(m; \overline{I}) = I' \otimes_{\mathbb{T},m} \overline{I}
\]

for \( I' \) given by \( \text{Spec}(\overline{I'}) = \text{Spec}(I^+) \cap \text{Spec}(I \otimes \overline{A}) \) (i.e., \( \text{Spec}(\overline{I'}) \) is the complementary component of \( \text{Spec}(\overline{I}) \) in \( \text{Spec}(I \otimes \overline{A}) \)). Note that \( C_0(\lambda; \overline{I}) = I^+ \otimes_{\mathbb{T}} \overline{I} \cong \overline{I}/(L_p(\text{Ad}(p_1))) \) by definition. Thus we have a surjective \( I \)-linear map \( C_0(\lambda; \overline{I}) = \overline{I}/(L_p(\text{Ad}(p_1))) \rightarrow C_0(m; \overline{I}) \) as \( \text{Spec}(I^+) \cap \text{Spec}(\overline{I}) \subseteq \text{Spec}(\overline{I'}) \cap \text{Spec}(\overline{I}) \).

Note that the projection: \( T \twoheadrightarrow I \) factors through \( \mathbb{T}^{ncm} \). Write \( \lambda' \) for the composite \( \mathbb{T}^{ncm} \otimes A \rightarrow \mathbb{I} \rightarrow \mathbb{I} \) and define an \( I \)-ideal \( a \) by \( C_0(\lambda', \overline{I}) = \overline{I}/a \). By Theorem 5.4 (2), \( \mathbb{T}^{ncm} = \mathbb{T}^{ncm} \otimes \mathbb{T}^{ncm} \) with \( \theta^2 \in \mathbb{T}^{ncm} \), and by (5.2), \( \overline{I}/a = W[[H]]/(L_p) \) with \( (L_p(\text{Ad}(p_1))) = (h_p L_p(\varphi^-)) = (L_p(\varphi^-)) \) as \( p \nmid h_p \). By projecting \( \theta \) down to \( d \in I \), we find \((d^2) \cap A = (L_p(\varphi^-)); \) so, \( \sqrt{L_p(\varphi^-)} \in I \) (no need to extend \( W \) as \( W \supset W(\overline{F}_p) \)). Since divisibility just follows from localization, we may localize at height one primes \( P \mid (L_p(\varphi^-)) \) of \( A \). Thus \( \overline{I}/a \) is a semi-local normal ring finite flat over the valuation ring \( \Lambda_P \). Therefore, it is a regular ring (in particular, it is complete intersection); so, writing \( C_0(m, \overline{I}/a) = \overline{I}/d_P \), then \( d_P \) is the differnt of \( \overline{I}/\Lambda_P \) (cf. [MFG, Lemma 5.21]). Since \( \overline{I}/a \supset \Lambda_P[\sqrt{L_p(\varphi^-)}] \), its different \((\sqrt{L_p(\varphi^-)})\) is a factor of the different \( d_P \) of \( I \otimes \Lambda_P \), which is in turn a factor of \((L_p(\varphi^-)) \) (as \( C_0(m, \overline{I}) \) surjects down to \( C_0(m, \overline{A}) \)).

If further \( \mathbb{T}^{ncm} = \mathbb{I} \) and rank\( A \mathbb{T}^{ncm} = 2 \), then \( I = \Lambda[\sqrt{L_p(\varphi^-)}] \), and by the semi-simplicity conjecture, \( I \) is integrally closed; so, \( \overline{I} = \mathbb{I} \). Then, from \( W[[H]]/(L_p(\varphi^-)) \cong I/(\sqrt{L_p(\varphi^-)}) \), we find that

\[
\mathbb{T} = \{(x, y) \in W[[H]] \oplus I| (x \mod (L_p(\varphi^-))W[[H]]) = (y \mod \sqrt{L_p(\varphi^-)}I)\},
\]

where on the right-hand-side, we regard \( L_p(\varphi^-) \in I \subseteq \mathbb{I} \). From this, we can easily compute \( C_0(\lambda, I) = I/(\sqrt{L_p(\varphi^-)}) = I/(L_p(\text{Ad}(p_1))) \), which finishes the proof.

\[\square\]

11. DUALIZING MODULES

We describe purely ring theoretic results we have used in the paper. The theory of dualizing modules is initiated by Grothendieck [SGA 2.IV–V] and is developed by Hartshorne [RDD] and Kleiman [Kl80]. Let \( S \) be a base local ring. For any \( S \)-module \( M \), we define \( M^\dagger := \text{Hom}_S(M, S) \).

**Lemma 11.1.** Let \( S \) be a \( p \)-profinite Gorenstein local ring and \( A \) be a local \( S \)-algebra. Suppose that \( A \) is a local Cohen–Macaulay ring with \( \dim A = \dim S \). If \( A \) is an \( S \)-module of finite type, the following conditions are equivalent:

1. The local ring \( A \) is Gorenstein;
2. \( A^\dagger \cong A \) as \( A \)-modules.

**Proof.** Since \( S \) is Gorenstein, it has canonical module \( \omega_S \cong S \) (as \( S \)-modules; see [CMA, §21.3]). Then by [CMA, Theorem 21.15], \( A \) itself has its dualizing module \( \omega_A \) given by \( \text{Hom}_S(A, \omega_S) \). By [CMA, §21.3], a local ring \( R \) is Gorenstein if and only if \( \omega_R \cong R \) as \( R \)-modules for the dualizing module \( \omega_R \) of \( R \). Since \( \omega_S \cong S \), we find \( \omega_A \cong A^\dagger \), and hence \( A \) is Gorenstein if and only if \( A^\dagger \cong A \).

\[\square\]

Let \( A \) be a Gorenstein local \( S \)-algebra for a Gorenstein local ring \( S \). Suppose that \( A \) is reduced and free of finite rank over \( S \) and \( S \) is \( W \)-free of finite rank. Let \( \sigma \in \text{Aut}(A) \) be an \( S \)-algebra involution. We allow the case where \( \sigma \) acts non-trivially on \( S \). Put \( A_{\pm} := \{x \in A|\sigma(x) = \pm x\} \). Then by Lemma 11.1, we get \( A^\dagger \cong A \) as \( A \)-modules. Since \( \sigma \) acts by duality on \( A^\dagger \), we have \( A^\dagger = (A^\dagger)^\dagger := \{x \in A^\dagger|\sigma(x) = \pm x\} \). Note that \( A^\dagger = \text{Hom}_S(A_{\pm}, S) \). Thus \( \sigma \) acts on \( \text{Hom}_S(A_{\pm}, A) \) and \( \text{Isom}_A(A_{\pm}, A) \) just by \( \phi \mapsto \sigma \circ \phi \circ \sigma \). Indeed, by a computation: \( \sigma(\phi(ax)) = \sigma(\phi(\sigma(ax))) = \sigma(\sigma(a)\phi(\sigma(x))) = a\phi(\sigma(x)) = a\sigma(x) = \sigma(\phi(x)) \) for \( a \in A \), we conclude \( \sigma \circ \phi \circ \sigma \) is \( A \)-linear. We then consider the \( \pm \)-eigenspace \( \text{Hom}_A(A^\dagger, A)^\pm \) for \( a \in A \) and \( \text{Isom}_A(A_{\pm}, A)^\pm := \text{Hom}_A(A^\dagger, A)^\pm \cap \text{Isom}_A(A_{\pm}, A) \). Here \( \text{Isom}_A \subset \text{Hom}_A \) is made up of \( A \)-linear isomorphisms. The set \( \text{Isom}_A(A^\dagger, A)^\pm \) could be empty.

If \( \sigma \) fixes \( S \) point by point, we have \( (A^\dagger)^\dagger = (A_{\pm})^\dagger \), which we just write \( A^\dagger \).
Lemma 11.2. Let $A$ be a noetherian Gorenstein local $S$-algebra for a $p$-profinite Gorenstein local ring $S$ (for a prime $p > 2$). Suppose that $A$ is reduced and free of finite rank over $S$. Let $\sigma \in \text{Aut}_S(A)$ be an algebra involution fixing $S$ point by point.

1. At least for one sign $\varepsilon = \pm$, the set $\text{Isom}_A(A^\dagger, A^\varepsilon)$ is non-empty.
2. If either $\text{rank}_SA_+ > \text{rank}_SA_-$ or $\text{Isom}_A(A^\dagger, A^+ \neq \emptyset$, we have $A_+ \cong (A_+)^\dagger$ (i.e., $A_+$ is Gorenstein). Moreover we have $\text{Isom}_A(A^\dagger, A)^- = \emptyset$ if $\text{rank}_AA_+ > \text{rank}_AA_-.$
3. If $\text{rank}_SA_+ = \text{rank}_SA_-$ and $\text{Isom}_A(A^\dagger, A^\varepsilon \neq \emptyset$, we have $A_+ \cong A_1^{\varepsilon}[\varepsilon]$.
4. Suppose that $S$ is a domain. Then we have $\text{rank}_SA_+ \geq \text{rank}_SA_-.$

Proof. Since $A$ is Gorenstein, we have $A^\dagger \cong A$ as $A$-modules by Lemma 11.1. Thus we conclude $\text{Isom}_A(A^\dagger, A) \neq \emptyset$. Pick $\phi \in \text{Isom}_A(A^\dagger, A)$. Let $\phi^\dagger = \phi \pm \phi^\varepsilon$. Then for $a \in A$, we have

$$\phi^\dagger(ax) = \phi(ax) \pm \sigma(\phi(\sigma(ax))) = a\phi(x) \pm \sigma(a\phi(\sigma(x))) = a\phi(x) \pm a\sigma(\phi(\sigma(x))) = a\phi^\dagger(x).$$

Then $\phi^\dagger + \phi^\varepsilon = 2\phi$. If one $\phi^\varepsilon$ of $\phi^\dagger$ is not onto, we conclude $\text{Im}(\phi^\varepsilon) \subset A$ is a proper $A$-submodule of $A$; so, $\text{Im}(\phi^\varepsilon) \subset \mathfrak{m}_A$. This shows $\phi^\varepsilon = 2\phi - \phi^\varepsilon \equiv 2\phi \mod \mathfrak{m}_A$, which implies $\phi^\varepsilon$ is onto (as $p > 2$). Identifying $A^\dagger$ with $A$, we can iterate $\Phi := \phi^\varepsilon$, and $\text{Ker}(\Phi^n)$ is an ascending sequence of $A$-ideals. Since $A$ is noetherian, for some $n > 0$, we have $\text{Ker}(\Phi^n) = \text{Ker}(\Phi^{n+1})$. Thus we conclude

$$\Phi : A = \text{Im}(\Phi^n) = A/\text{Ker}(\Phi^n) \xrightarrow{\Phi} A/\text{Ker}(\Phi^{n+1}) = \text{Im}(\Phi^{n+1}) = A$$

and hence $\phi^{-\varepsilon}$ is an isomorphism.

If $\text{rank}_SA_+ > \text{rank}_SA_-$, by $A_+$-indecomposability of $A_+$ as $A_+$-modules, the Krull-Schmidt theorem tells us $A_\dagger_+ \cong A_+$ and hence $A_\dagger_- \cong A_-$. Moreover the decomposition $A = A_+ \oplus A_-$ is a unique decomposition of the $A_+$-module $A$ into the sum of the indecomposable $A_+$ of the largest $S$-rank and an $A_-$-submodule $A_-$ of less $S$-rank. Therefore, any $\phi \in \text{Isom}_A(A^\dagger, A)$ is forced to preserve $A_+$ and $A_-$; so, we have $\text{Isom}_A(A^\dagger, A^\varepsilon \neq \emptyset$ and $\text{Isom}_A(A^\dagger, A)^- = \emptyset$. Thus we get $A^\dagger_+ \cong A_+$ as $A_+$-modules (i.e., $A_+$ is Gorenstein).

Now suppose $\text{rank}_SA_+ = \text{rank}_SA_-$ and $\text{Isom}_A(A^\dagger, A^\varepsilon \neq \emptyset$. Thus $A_\dagger_\pm \cong A_\dagger_\pm$ as $A_\dagger_\pm$-modules, and $\text{Isom}_A(A^\dagger_+, A^\varepsilon \neq \emptyset$ implies $A^\dagger_+ \cong A_+$ as $A_+$-modules (i.e., $A$ is Gorenstein). Similarly $\text{Isom}_A(A^\dagger, A)^- = \emptyset$ implies $\text{Isom}_A(A^\dagger, A)^- \neq \emptyset$ by (1), and $A^\dagger_\pm \cong A_\dagger_\pm$ as $A_\dagger_\pm$-modules.

Since $\text{Frac}(A)$ is a product of fields, for each simple component $K$ of $\text{Frac}(A)$, either $\sigma$ acts non-trivially or $\sigma$ fixes $K$ element by element. Since $A^\dagger_\pm$ is a direct summand of the $S$-free module $A$ of finite rank, $A^\dagger_\pm$ is $S$-free of finite rank as $S$ is a local ring. Thus we get

$$\text{rank}_SA_+ = \text{dim}_{\text{Frac}(S)}A_+ \otimes_S \text{Frac}(S) \geq \text{dim}_{\text{Frac}(S)}A_- \otimes_S \text{Frac}(S) = \text{rank}_SA_-,$$

proving (4). □

We now study relative dualizing modules and show that a Gorenstein local domain quadratic over a Gorenstein subalgebra is generated by a single element over the subalgebra. Let $B$ be a commutative $p$-profinite local ring for a prime $p > 2$. Consider a local $B$-algebra $A$ finite over $B$ with $B \hookrightarrow A$. Write $\omega_{A/B}$ for the dualizing module for the finite (hence proper) morphism $X \leftarrow \text{Spec}(A) \xrightarrow{f} \text{Spec}(B) \leftarrow Y$ if it exists (in the sense of [Kl80, (6)]). For the dualizing functor $f^!$ from quasi coherent $Y$-sheaves into quasi coherent $X$-sheaves defined in [Kl80, (2)], we have $\text{Hom}_A(F, f^!N) = \text{Hom}_B(f_*F, N)$ for any quasi-coherent sheaves $F$ over $X$ and $N$ over $Y$; so, if $\omega_{A/B}$ exists (i.e., $f^!(N) = N \otimes_B \omega_{A/B}$), taking $F = A$ and $N = B$, we have $\omega_{A/B} = f^!(O_Y) = \text{Hom}_B(A, B)$ as $A$-modules. As shown in [Kl80, (21)], $\text{Spec}(A) \xrightarrow{f} \text{Spec}(B)$ has dualizing module if and only if $f$ is Cohen Macaulay (e.g., if $B$ is regular and $A$ is free of finite rank over $B$). Even if we do not have dualizing module $\omega_{A/B}$, we just define $\omega_{A/B} := \text{Hom}_B(A, B)$ generally.

Suppose that we have an involution $\sigma \in \text{Aut}(A/B)$. Let $A_{\pm} = A^\sigma$ for the order 2 subgroup $G$ of $\text{Aut}(A/B)$ generated by $\sigma$. Under the following four conditions:

1. $B$ is a regular local ring.
2. $A$ is free of finite rank over $B$.
3. $A$ and $A_+$ are Gorenstein ring.
4. $A/B$ is generically étale (i.e., $\text{Frac}(A)$ is reduced separable over $\text{Frac}(B)$),
in [RDF, §3.5.a], the module of regular differentials \( \omega_{\mathcal{C}/A} \) for \((\mathcal{C}, \Delta) = (A, B), (A, A_+), (A_+, B) \) is defined as fractional ideals in Frac(\( \mathcal{C} \)). By (1) and (2), \( A/B \) and \( A_+/B \) are Cohen Macaulay; so, \( \omega_{A/B} \) and \( \omega_{A_+/B} \) as above are the dualizing modules.

We now identify the dualizing module with more classical “inverse different” (realized as a fractional ideal). Let \( C \subset B \) be reduced algebras. By abusing notation, write \( \omega_{C/B} := \text{Hom}_B(C, B) \) in general. Suppose that Frac(\( C \)) is étale; so, we have a well defined trace map Tr: Frac(\( C \)) → Frac(\( B \)), and \( \omega_{\text{Frac}(C)/\text{Frac}(B)} = \text{Frac}(C) \text{Tr} \) by the trace pairing \((x, y) \mapsto \text{Tr}(xy)\). We define an \( C \)-fractional ideal by

\[
\mathcal{C}^{-1}_{C/B} := \{ x \in C | \text{Tr}(xC) \subset B \}.
\]

In other words, \( \omega_{C/B} = \text{Hom}_B(C, B) \leftarrow \text{Hom}_{\text{Frac}(C)}(\text{Frac}(C), \text{Frac}(B)) = \text{Frac}(C) \text{Tr} \) has image \( \mathcal{C}^{-1}_{C/B} \text{Tr} \). Thus we have \( \mathcal{C}^{-1}_{C/B} \cong \omega_{C/B} \). If \( C = B[\delta] \) is free of rank 2 over \( B \) with an \( B \)-basis \( 1, \delta \) with \( \delta^2 \in B \), we have \( \mathcal{C}^{-1}_{C/B} = \delta^{-1} C \) for \( \delta^{-1} \in \text{Frac}(C) \). Here is a version of Dedekind’s formula of transitivity of inverse differentes proven in [KDF, Proposition G.13] (see also [RDP, Theorem 8.6], [Ki80, (26) (vii)] and [Hu89]):

**Proposition 11.13.** Let \( B \) be a regular \( p \)-profinite local ring. Suppose that \( D/C/B \) is generically étale finite extensions of reduced algebras such that \( D \) and \( C \) are \( B \)-flat, \( \omega_{D/C/B} \cong B \) is \( B \)-modules (i.e., \( B \) is Gorenstein) and that Frac(\( D \)) is Frac(\( C \))-free. Then we have \( \mathcal{C}^{-1}_{D/C} \mathcal{C}^{-1}_{C/B} = \mathcal{C}^{-1}_{D/B} \) and \( \omega_{D/C} \otimes_C \omega_{C/B} \cong \omega_{D/B} \).

Let \( A \) be a reduced noetherian algebra with an involution \( \sigma \). Put \( A^\pm = A_+ := \{ x \in A | \sigma(x) = \pm x \} \) and write \( \mathcal{G} \) for the subgroup of \( \text{Aut}(A) \) of order 2 generated by \( \sigma \); so, \( A^+ = A^\delta = H^0(\mathcal{G}, A) \).

**Lemma 11.4.** Let \( S \) be a \( p \)-profinite Gorenstein integral domain for a prime \( p > 2 \) and \( A \) be a reduced local \( S \)-algebra free of finite rank over \( S \). Suppose

1. \( A \) and \( A_+ \) are Gorenstein,
2. Frac(\( A \)) is an étale extension of \( \text{Frac}(A_+) \)
3. Frac(\( A \)) is free of rank 2 over Frac(\( A_+ \)),
4. \( \mathcal{O}_{A_+, A} \subset M_A \) or \( A \) is flat over \( A_+ \) or \( A_- \) is generated by one element over \( A_+ \).

Then \( A \) is free of rank 2 over \( A_+ \) and \( A = A_+ \oplus A_+ \delta \) for an element \( \delta \in A \) with \( \sigma(\delta) = -\delta \).

For \( A_+ \)-module \( M \), we write \( M^* \) for the \( A_+ \)-dual \( \text{Hom}_{A_+}(M, A_+) \).

**Proof.** From Lemma 11.1, we conclude \( A^* \cong \omega_{A/A_+} \cong A \). Thus we conclude

\[
\omega_{A/A_+} \cong \mathcal{C}^{-1}_{A/A_+} = A\theta^{-1}
\]

with a non-zero divisor \( \theta \in A \). Similarly \( \mathcal{C}^{-1}_{A/S} = A\theta^{-1}/S \) and \( \mathcal{C}^{-1}_{A+/S} = \theta_{A+/S}^{-1} A^+ \). We may assume that \( \theta \theta_{A+/S} = \theta_{A/S} \) by Proposition 11.13. Define \([x, y] := \text{Tr}_{A/A_+}(\theta^{-1}xy)\), which induces the self \( A_+ \)-duality on \( A \). If \( \theta \in A_+ \), we have \( \text{Tr}_{A/A_+}(\theta^{-1}xy) = \theta^{-1} \text{Tr}_{A_+/A_+}(xy) \); so,

\[
A_+ = [A, A] = \text{Tr}_{A/A_+}(\theta^{-1}A) = \theta^{-1} \text{Tr}_{A/A_+}(A) = \theta^{-1} A_+.
\]

Thus \( \theta \) is a unit. The multiplication of \( \theta \) gives rise to \( \text{Isom}_A(A^*, A) \cong \text{Isom}_A(\mathcal{C}^{-1}_{A/A_+}, A) \).

Suppose \( \mathcal{C}^{-1}_{A/A_+} \subset M_A \). Then \( \theta \) cannot be a unit. We conclude \( \theta \notin A_+ \); so, \( \text{Isom}_A(A^*, A^+ = \emptyset \).

Thus by Lemma 11.2, \( \text{Isom}_A(A^*, A) = \emptyset \). In other words, writing \( f(x) \) for the minimal monic quadratic polynomial of \( \theta \) in \( A_+[x] \), we have \( \mathcal{C}^{-1}_{A/A_+} = A\delta \) with \( \delta = f'(\theta) = \theta - \sigma(\theta) \) (i.e., the multiplication of \( \delta \) gives rise to an element in \( \text{Isom}_A(A^*, A) \)). Indeed, by the trace pairing \([x, y] = \text{Tr}_{A/A_+}(xy)\), we have the identity \( \mathcal{C}^{-1}_{A/A_+} \cong A^* = A_+^* \oplus A^*_{\delta} \) and \( A^* = A_+^* \oplus A_+^* \delta \) under this isomorphism. Taking the dual under the trace pairing, we get \( A_- = (A^*_{\delta})^* = A_+ \delta \) and \( A = A_+ \oplus A_- \); so, \( A_+ = A_+ \delta \) and \( A = A_+ \oplus A_+ \delta \), as desired.

Under flatness of \( A \) over \( A_+ \), plainly by (3), \( A_- \) is generated by a single element \( \delta \). The assertion is plain in the case where \( A_- = A_\delta \).

\( \square \)
Books

ANTICYCLOTOMIC CYCLICITY CONJECTURE


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