SYMMETRIC POWER CONGRUENCE IDEALS AND SELMER GROUPS

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Contents

1. Introduction

In this paper we define relative congruence ideals for various automorphic symmetric powers $\mathrm{Symm}^m\,\mathbf{f}$ of a Hida family $\mathbf{f}$ over $\mathbb{Q}$ in big ordinary Hecke algebras for symplectic and unitary groups (these powers are now known to be automorphic for $m \leq 8$) and we prove, under some assumptions, that they coincide with the characteristic power series of (the Pontryagin duals of) Greenberg Selmer groups over $\mathbb{Q}$ for related symmetric powers $\mathcal{A}^n_{\mathbf{f}} = \mathrm{Symm}^{2n} \otimes \det^{-n} \rho_{\mathbf{f}}$ of the Galois representation of the family $\mathbf{f}$. Note that these Selmer groups over $\mathbb{Q}$ are modules over the weight variable Iwasawa algebra which are finitely generated but *a priori* not known to be torsion except for the symmetric square. It follows from our result that they are. Similar results when one includes the cyclotomic variable (that is, for Selmer groups over the $\mathbb{Z}_p$-extension $\mathbb{Q}_\infty$ of $\mathbb{Q}$) could probably also be studied but are not dealt with in this paper. Let us be a little more precise.

Let $N \geq 1$ and let p be an odd prime not dividing N. Let $E/\mathbb{Q}_p$ be a finite extension of $\mathbb{Q}_p$, $\mathcal{O}$ its valuation ring, ϖ a uniformizing parameter and $k = \mathcal{O}/(\varpi)$ its residue field. Let $\Lambda_1 = \mathcal{O}[[X]]$

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be the one variable Iwasawa algebra identified to the completed group algebra of $1+p\mathbb{Z}_p$ over $\mathcal{O}$ by the choice of a topological generator u of the group $1+p\mathbb{Z}_p$. Let h_1 be the cuspidal Hida Hecke algebra (generated over $\mathcal{O}$ by the T_ℓ's for ℓ prime to Np, by the diamond operators $< a >_0$ for $a \in \mathbb{Z}_p^\times$ for the cohomological weight 0, and by U_p. The ring h_1 is reduced; it is endowed with a structure of Λ_1-algebra by the homomorphism sending X to $\langle u\rangle_0 - 1$ for which it is finite and flat. Let $\mu\colon h_1 \to A_1$ be a surjective Λ_1-algebra homomorphism onto a local domain which is finite and torsion free over Λ_1. Geometrically, it amounts to considering an irreducible component of $\mathrm{Spec}\, h_1$. We put $\mu(U_p) = \alpha$. We assume that $\mathcal{O}$ is sufficiently big so that the residue field $A_1/\mathfrak{m}_{A_1}$ of A_1 coincides with k. We denote by $\overline{\gamma}$ the reduction of an element $\gamma \in A_1$. Let $\Gamma_{\mathbb{Q}}$ be the absolute Galois group. We assume that the residual Galois representation $\overline{\rho}_\mu\colon \Gamma_{\mathbb{Q}} \to \mathrm{GL}_2(k)$ associated to μ is irreducible. It is well-known that under this assumption there exists a continuous Galois representation $\rho_\mu\colon \Gamma_{\mathbb{Q}} \to \mathrm{GL}_2(A_1)$ associated to μ. Let $D_p \subset \Gamma_{\mathbb{Q}}$ be a decomposition group at p. Let ω be the cyclotomic character modulo p (we shall also denote by ω its Teichmüller lift). The ordinarity condition for μ implies that there exists a unique integer $a \in [0, p-2]$ and a (not necessarily unique) D_p-stable two-step filtration $0 \subset F^0 \subset F^{-a-1} = \overline{\rho}_\mu$ with D_p acting by $\mathrm{unr}(\overline{\alpha})$ on gr^0 and by $\mathrm{unr}(\overline{\alpha}^{-1})\omega^{-a-1}$ on gr^{-a-1}. In other words, the restriction of the representation $\overline{\rho}_\mu$ to a decomposition group D_p at p is conjugate to

$$\begin{pmatrix} \mathrm{unr}(\overline{\alpha}) & * \\ 0 & \mathrm{unr}(\overline{\alpha}^{-1})\omega^{-a-1} \end{pmatrix}$$

If $a+1 < p-1$, the filtration is uniquely determined, but we want to allow the possibility $a+1 \equiv 0 \pmod{p-1}$. This is why we assume throughout the paper the condition (RFR) of residual Frobenius regularity (to be reinforced later)

$$(RFR) \quad \overline{\alpha}^2 \neq 1$$

Under this condition, the filtration defined above is unique, even if $a+1 \equiv 0 \pmod{p-1}$. Let St_2 be the standard representation of GL_2. For any $j \geq 1$, let $\mathcal{A}^j = \mathrm{Symm}^{2j} \otimes \det^{-j} \mathrm{St}_2$; we assume from now on that $p > 2j+1$; in particular, viewed as a $\mathbb{Z}_p$-schematic representation, the $(2j+1)$-dimensional representation of GL_2, $\mathcal{A}^j$ has irreducible geometric fibers. Let $\mathcal{A}^j_\mu = \mathcal{A}^j \circ \rho_\mu\colon \Gamma_{\mathbb{Q}} \to \mathrm{GL}_{2j+1}(A_1)$. Let $\mathbb{X}\colon \Gamma_{\mathbb{Q}} \to \Lambda_1^\times$ denote the restriction (to D_p) of the universal deformation of the p-adic cyclotomic character χ unramified outside $p\infty$: if $\chi(\sigma) = \omega(\sigma)u^{\ell(\sigma)}$ (that is, $\ell([z, \mathbb{Q}_p]) = \log_p(z)/\log_p(u)$ for $z \in \mathbb{Z}_p^\times$). then

$$\mathbb{X}(\sigma) = \chi(\sigma)(1+X)^{\ell(\sigma)}$$

Let $\Phi_a = \omega^a\mathbb{X}$. Then the restriction of $\mathcal{A}^j_\mu$ to the local Galois group D_p leaves stable a filtration $(F^{k(a+1)}\mathcal{A}^j_\mu)_k$ with A_1-free graded pieces $\mathrm{gr}^{k(a+1)}\mathcal{A}^j_\mu$ on which D_p acts by $\mathrm{unr}(\alpha^{2k})\Phi_a^k$. In order to define the ordinary Selmer group of $\mathcal{A}^j_\mu$ even if the residual action of the inertia is trivial, it is crucial for us that this filtration be well defined modulo p. To achieve this, we assume throughout the paper the following reinforcement of (RFR):

$$(RFR_n) \quad \overline{\alpha}^{2c_n} \neq 1$$

where c_n denotes the least common multiple of the integers $1, 2, \ldots, 2n$. Note that (RFR_n) will be enough to guarantee $R = T$ theorems for $U(n+1)$ but for actually deducing from these results a relation between congruence modules and Selmer groups, the stronger assumption $n(a+1) < p-1$ will be ultimately needed.

Let $\widetilde{A}_1$ be the normal closure of A_1. It is a two dimensional normal local ring, hence it is Cohen-Macaulay so that $\widetilde{A}_1$ is free over Λ_1. For any $\mathcal{O}$-module M, we denote by $M^* = \mathrm{Hom}_{\mathcal{O}}(M, E/\mathcal{O})$ its Pontryagin dual. Let us consider the minimal p-ordinary Selmer group associated to $\mathcal{A}^j_\mu$:

$$\mathrm{Sel}(\mathcal{A}^j_\mu) = \mathrm{Ker}\left(\mathrm{H}^1(\mathbb{Q}, \mathcal{A}^j_\mu \otimes_{A_1} \widetilde{A}_1^*) \to \prod_{\ell \neq p} \mathrm{H}^1(I_\ell, \mathcal{A}^j_\mu \otimes_{A_1} \widetilde{A}_1^*) \times \mathrm{H}^1(D_p, (\mathcal{A}^j_\mu/F^1\mathcal{A}^j_\mu) \otimes_{A_1} \widetilde{A}_1^*)/L_p\right)$$

where L_p denotes the image in $\mathrm{H}^1(D_p, \mathcal{A}^j_\mu \otimes_{A_1} \widetilde{A}_1^*)$ of

$$L'_p = \mathrm{Ker}\left(\mathrm{H}^1(D_p, F^0\mathcal{A}^j_\mu \otimes_{A_1} \widetilde{A}_1^*) \to \mathrm{H}^1(I_p, (F^0\mathcal{A}^j_\mu/F^1\mathcal{A}^j_\mu) \otimes_{A_1} \widetilde{A}_1^*)\right)$$

Its Pontryagin dual $\mathrm{Sel}(\mathcal{A}^j_\mu)^*$ is finitely generated over $\widetilde{A}_1$. Recall that the Greenberg-Iwasawa main conjecture implies (by taking the cyclotomic variable s to be 1) that

1) there is a p-adic L function $L_p(\mathcal{A}^j_\mu)$ in $\widetilde{A}_1$ interpolating normalized special values $L^*(\mathcal{A}^j_{\mathbf{f}_k}, 1)$ where $\mathbf{f}_k$ runs over the eigenforms of classical weights $k \geq 2$ occuring in μ.

2) $\mathrm{Sel}(\mathcal{A}^j_\mu)^*$ is torsion and a characteristic power series is equal to $L_p(\mathcal{A}^j_\mu)$ up to a unit in $\widetilde{A}_1$; this means, more precisely, that the localization at each height one prime of $\widetilde{A}_1$ of its first Fitting ideal $\mathrm{Fitt}_0\left(\mathrm{Sel}(\mathcal{A}^j_\mu)^*\right)$ is generated by $L_p(\mathcal{A}^j_\mu)$.

Let $\mathbf{T}_1$ be the localization of h_1 at the maximal ideal corresponding to the residual representation $\overline{\rho}_\mu$. The homomorphism μ factors through $\mathbf{T}_1$. We still denote $\mu\colon \mathbf{T}_1 \to A_1$ the resulting homomorphism. Extending the scalars, it gives rise to a surjective homomorphism $\widetilde{\mathbf{T}}_1 = \mathbf{T}_1 \otimes_{\Lambda_1} \widetilde{A}_1 \to \widetilde{A}_1$ which we again denote by μ. The context should make clear the meaning of this notation. Since $\mathbf{T}_1$ is reduced and $\widetilde{A}_1$ is a domain which is flat over Λ_1, we see that $\widetilde{\mathbf{T}}_1$ is reduced too. By tensoring with $\mathcal{K}_1 = \mathrm{Frac}(A_1)$, we have a splitting

$$\widetilde{\mathbf{T}}_1 \otimes_{\widetilde{A}_1} \mathcal{K}_1 \cong \mathcal{K}_1 \times \widetilde{\mathbf{T}}'_{1,\mathcal{K}_1}$$

where the first projection is given by $\mu \otimes \mathrm{Id}_{\mathcal{K}_1}$. Let $\widetilde{\mathbf{T}}'_1$ be the image of the second projection $\widetilde{\mathbf{T}}_1 \to \widetilde{\mathbf{T}}'_{\mu,\mathcal{K}_1}$. One defines the congruence ideal of μ by $\mathfrak{c}_\mu = \widetilde{\mathbf{T}}_1 \cap (\widetilde{A}_1 \times \{0_{\widetilde{\mathbf{T}}'_\mu}\})$. We view this ideal as an ideal of $\widetilde{A}_1$. Recall that for a finitely generated $\widetilde{A}_1$-module M, any generator of the smallest principal ideal $\mathcal{X}_M$ containing the first Fitting ideal $\mathrm{Fitt}_0(M)$ of M is called a characteristic power series : $\mathcal{X}_M = (\mathrm{Char}(M))$. It is non zero if and only if M is torsion. Consider the assumption

$(*)$ N is squarefree, there exists a subfield $k' \subset k$ such that $\mathrm{SL}_2(k') \subset \mathrm{Im}\ \overline{\rho}_\mu \subset \mathrm{GL}_2(k')$, moreover for any prime ℓ dividing N, the restriction to I_ℓ of $\overline{\rho}_\mu$ is non trivial (hence unipotent).

Let R_1 be the universal deformation ring for N-minimal p-ordinary deformations of $\overline{\rho}_\mu$. It can be viewed as a Λ_1-algebra in two ways, which are equivalent because our ground field is $\mathbb{Q}$: one way is using the determinant of ρ^{univ}: it is a deformation of the global character $\omega^{-(a+1)}\colon \Gamma_\mathbb{Q} \to k^\times$ Since the pair (Λ_1, Φ_a^{-1}) can be viewed as the universal deformation of the character $\omega^{-(a+1)}$, the global character ρ^{univ} defines a structural morphism $\Lambda_1 \to R_1$. Another is to consider the restriction of ρ^{univ} to I_p:

$$\rho^{univ}|_{I_p} \sim \begin{pmatrix} 1 & * \\ 0 & \Psi_a^{univ} \end{pmatrix}$$

Since the local character $\Psi_a^{univ}\colon I_p \to k^\times$ is a deformation of $\omega^{-(a+1)}$, it also gives rise to a structural morphism $\Lambda_1 \to R_1$. These two structures coincide. Recall a special case of theorems by Wiles and Hida

Theorem 1.1. *Assume $(*)$ and either (RFR) or $a+1 < p-1$, then the natural surjection $R_1 \to \mathbf{T}_1$ is an isomorphism of Λ_1-algebras and $\mathbf{T}_1$ is local complete intersection over Λ_1. Moreover, $\mathfrak{c}_\mu$ is a principal ideal of $\widetilde{A}_1$.*

The precise definition and some properties of local complete intersection algebras are given Section 8.4 below. From the theorem, one can deduce equalities

Corollary 1.2. (a) *Assume $(*)$ and $a+1 < p-1$, then $\mathfrak{c}_\mu = (\mathrm{Char}(\mathrm{Sel}(\mathcal{A}^1_\mu)^*))$,*
(b) *Assume $(*)$ and (RFR) or $a+1 < p-1$, then $\mathfrak{c}_\mu = (L_p(\mathcal{A}^1_\mu))$.*

Note that the p-adic L function $L_p(\mathcal{A}^1_\mu)$ interpolating the special values $L^*(\mathcal{A}^1_{\mathbf{f}_k}))$ has been constructed by one of the authors [H88] (see also his notes of the Pune course [H16]).

The goal of this paper is to establish analogues of part (a) of the theorem above for higher j's, provided the automorphic base change is established for Symm^m for certain values of m less than $2j$. For any imaginary quadratic field K of discriminant D prime to N in which p splits, let $U(n)$ be a definite unitary group for K

Theorem 1.3. *Assume $(*)$ and $3(a+1) < p-1$. Then,*

- *for $j = 3$, the characteristic power series of $\mathrm{Sel}(\mathcal{A}^j_\mu)^*$ is a generator of the ideal of congruences between the family $\mathrm{Symm}^3\mu$ and Siegel families which are not of the form $\mathrm{Symm}^3\mu'$ for other GL_2-families μ',*

- *for $j = 2$, for $U(4)$ a unitary group as above, the characteristic power series of* $\mathrm{Sel}(\mathcal{A}_\mu^j)^*$ *is a generator of the congruence ideal between the $U(4)$ Hida family associated to* $\mathrm{Symm}^3\mu$ *and families on $U(4)$ which don't come from Siegel families.*

Similarly, we have

Theorem 1.4. *If one assumes $(*)$ and $4(a+1) < p-1$, then for $j = 4$, for $U(5)$ a unitary group as above, the characteristic power series of* $\mathrm{Sel}(\mathcal{A}_\mu^j)^*$ *is a generator of the congruence ideal between the $U(5)$ Hida family* $\mathrm{Symm}^4\mu$ *and families of unitary forms on $U(5)$ which don't come from congruences between* $\mathrm{Symm}^3\mu$ *and families on* GSp_4 *by the integral transfer from* GSp_4 *to $U(5)$.*

See Sections 3 and 4 for a more precise form of the statement and its proof. To put these results in perspective, let us mention a more elementary result.

Let $\mathfrak{p}$ be a prime of $\widetilde{A}_1$. For $j = 3, 2, 4$, consider the condition

$$(S_j) \quad \mathrm{Fitt}_0(\mathrm{Sel}(\mathcal{A}_\mu^j)^*) \subset \mathfrak{p}$$

and the conditions

(C_3) there exists a Hida family G of Iwahori level N on GSp_4 which is not the Symm^3 of a Hida family on GL_2 and such that $\mathrm{Symm}^3\mu \equiv G \pmod{\mathfrak{p}}$

(C_2) there exists a Hida family G of Iwahori level N on $U(4)$ which does not come from GSp_4 by base change and such that $\mathrm{Symm}^3\mu \equiv G \pmod{\mathfrak{p}}$

(C_4) there exists a Hida family G of Iwahori level N on $U(5)$ which does not come from GSp_4 by base change and such that $\mathrm{Symm}^4\mu \equiv G \pmod{\mathfrak{p}}$,

Theorem 1.5. *Assume $(*)$, then (C_3) implies (S_3) or (S_2) or (S_4).*

See Section 5. It requires a theorem of big image of Galois established by [HT15] when $A_1 = \Lambda_1$ and by A. Conti in his thesis [Con16b] in general. Note that the conditions (C_j) are not mutually exclusive so that the difficulty of separating *a priori* the possible conclusions (S_j) is not so surprising.

Our theorems do separate the conclusions and imply in particular for $j = 3, 2, 4$ that (C_j)implies (S_j). Their proof require using more advanced tools, namely $R = T$ type theorems in the minimal level case and Hida–Tate theory of conguence ideals for Gorenstein rings. Actually our method applies to more cases:

Theorem 1.6. *Assume $(*)$ and $p-1 > max(n(a+1), (n-1))$ hold. Assume also that N has at least two prime factors q_1 and q_2. Assume that the transfers* Symm^{n-1} *and* Symm^n *from $GL_2(\mathbb{Q})$ to* GL_n *resp. GL_{n+1} are established. Then, for any imaginary quadratic field K of discriminant prime to Np in which p and q_1 split and q_2 is inert and for $U(n)$ and $U(n+1)$ corresponding unitary groups for K, the characteristic power series of* $\mathrm{Sel}(\mathcal{A}_\mu^n)^*$ *is a generator of the quotient of the congruence ideal between the family* $\mathrm{Symm}^n\mu$ *and families of unitary forms on $U(n+1)$ by the congruence ideal between the family* $\mathrm{Symm}^{n-1}\mu$ *and families of unitary forms on $U(n)$. In particular, the quotient of these ideals is integral and principal.*

This theorem applies for $n = 5, 6, 7, 8$ by [CT15] where the Symm^m transfer is established for $m \leq 8$. See Section 6 for a more precise statement and the proof.

We also give in Section 7 an analogue result starting from a Hida family σ on $\mathrm{GSp}_4(\mathbb{Q})$ instead of a Hida family μ on $\mathrm{GL}_2(\mathbb{Q})$. We still need to assume that it has Iwahori auxiliary level $\Gamma^{(2)}(N)$ for a squarefree integer N and that its Galois representation satisfies N-minimality, p-distinguishability and residual bigness (see Section 7). The method and result are similar although the Hida family σ is two variable so that the commutative algebra results involve three–dimensional local rings, so that we can only compare localizations at height one primes of the congruence ideal and the characteristic power series of the standard (degree 5) Selmer group. The tool this time is the base change from $\mathrm{GSp}_4(\mathbb{Q})$ to $U(4)$ (for an imaginary quadratic field) established by C.-P. Mok [Mok14] and [Clo91] and the conclusion is that the two variable characteristic power series of the degree 5 Galois representation associated to the family σ generates the height one part of the ideal of congruences between the base change of σ to $U(4)$ and families on $U(4)$ which don't come from $\mathrm{GSp}_4(\mathbb{Q})$.

For such a family σ, we formulate in Section 7 assumptions

- $(*)^{(2)}$ the Galois representation ρ_σ has residual large image and is N-minimal
- $(RFR)^{(2)}$ $\overline{\rho}_\sigma$ is p-distinguished,

which are analogue of $(*)$ and (RFR) for GL_2. The transfer homomorphism between Hecke algebras $\mathbf{T}_3^u \to \mathbf{T}_2^s$ induces by specialization to the automrphic weight $(1,2,2)$, resp. $(2,2)$, a transfer homomorphism $\mathcal{T}_{1,2,2}^u(N,\mathcal{O}) \to \mathcal{T}_{2,2}^s(N,\mathcal{O})$ from the Hida Hecke algebra for $U(3,1)$ of automorphic weight $(1,2,2)$ to the Hida Hecke algebra for $\mathrm{GSp}(4)$ of weight $(2,2)$. We can consider this homomorphism as being associated to a p-adic transfer map from p-adic Siegel cusp forms to "stable" p-adic automorphic forms on $U(3,1)$ (stable means here p-adic limits of stable forms). Let us mention two corollaries (see Corollaries 7.5 and 7.6).

Corollary 1.7. *Let $f \in S_{2,2}(\Gamma^{(2)}(N))$ be a Siegel cusp eigenform of automorphic weight $(2,2)$ such that $(*)^{(2)}$ and $(RFR)^{(2)}$ hold. Assume that the specialization $\mathcal{T}_{2,2}^s(N,\mathcal{O})$ in automorphic weight $(2,2)$ of the Hida Hecke algebra for* $\mathrm{GSp}(4)$ *and the specialization $\mathcal{T}_{1,2,2}^u(N,\mathcal{O})$ in automorphic weight $(1,2,2)$ of the Hida Hecke algebra for $U(3,1)$, are reduced. Then the $\mathcal{O}$-length of the Selmer group* $\mathrm{Sel}(\mathrm{St}_f)$ *of the 5-dimensional representation associated to f is equal to the valuation of the ideal of congruence between a transfer $f_{U(4)}$ of f to $U(4)$ and p-adic automorphic forms on $U(4)$ of automorphic weight $(k_1,k_2,k_3) = (1,2,2)$ and Iwahori level N which don't come by transfer from* $\mathrm{GSp}(4)$.

Note that there are no algebraic automorphic forms on $U(4)$ of automorphic weight $(k_1,k_2,k_3) = (1,2,2)$ because the weight is not cohomological. But even if one transfers from the definite unitary group $U(4)$ to a quasi-split group $U(3,1)$, there are still no classical holomorphic automorphic forms on $U(3,1)$ of automorphic weight $(k_1,k_2,k_3) = (1,2,2)$ because the weight is even not in the non-degenerate limit of discrete series. However, by using Hida theory [H02, Theorem 6.8] for $U(3,1)$, we do have an action of the Hecke algebra h_3^u on ordinary p-adic holomorphic automorphic forms of this weight.

The second corollary applies to abelian surfaces:

Corollary 1.8. *Let A be a modular abelian surface defined over $\mathbb{Q}$, ordinary at p of squarefree conductor N (p prime to N). Assume that the rings $\mathcal{T}_{2,2}^s(N,\mathbb{Z}_p)$ and $\mathcal{T}_{1,2,2}^u(N,\mathbb{Z}_p)$ are reduced. Let T_pA be the Tate module and $S_pA \subset \bigwedge^2 T_pA$ the associated rank 5 Galois representation. Assume that the residual representation $A[p]$ is modular for a Siegel modular form of weight $(2,2)$ for which $(*)^{(2)}$ and $(RFR)^{(2)}$ hold. Then the cardinality of* $\mathrm{Sel}(S_pA)$ *spans the $\mathbb{Z}_p$-ideal of congruence between a transfer $f_{U(4)}$ of f to $U(4)$ and p-adic automorphic forms on $U(4)$ of automorphic weight $(k_1,k_2,k_3) = (1,2,2)$ and Iwahori level N which don't come by transfer from* $\mathrm{GSp}(4)$.

See the end of Section 7 for a more precise statement and the proof.

The last Section presents the formalism of congruence modules, in particular the transfer formula (Corollary Corollary 8.6) which is used throughout the paper.

2. Theorems $R_{n-1} = \mathbf{T}_{n-1}^u$ for $n \geq 4$

2.1. **Big ordinary Hecke algebra for unitary groups.** Recall that we fixed a squarefree integer $N = q_1 \cdot \ldots \cdot q_k$ prime to p. As stated in Theorem 1.6, we will need to assume in some cases that $k \geq 2$. Hida theory for unitary groups [PAF, Chapt.8] is developed using coherent cohomology but hereafter we follow the presentation of [Ge10, Section 2] (see also [Ge16]) using definite forms of unitary groups. We fix an auxiliary imaginary quadratic field $K = \mathbb{Q}(\sqrt{-\Delta})$ of negative discriminant $-\Delta$ relatively prime to Np such that $p = \mathfrak{p}\mathfrak{p}^c$ and $q_1 = \mathfrak{q}_1\mathfrak{q}_1^c$ split and q_2 remains inert in K. Let D be a central division algebra over K of rank n^2 whose ramification set S_D consists in the primes above q_1. From the calculations of [Clo91, (2.3) and Lemma 2.2], we see that

(Case 1) If n is odd or is divisible by 4, then for any $k \geq 1$, there exists an involution of second kind $*$ on D which is positive definite at ∞ and such that the unitary group $U(D,*)$ is quasisplit at all inert places.

(Case 2) If $n = 2m$ with m odd; for $k \geq 2$ there exists an involution of second kind $*$ on D which is positive definite at ∞ and such that the unitary group $U(D,*)$ is quasisplit at all inert places except q_2.

We fix $G = U(D,*)$ as above.

Definition 2.1. *We fix an auxiliary level group $U^p = \prod_{i=1}^k U_{q_i} \times U^{Np}$ of Iwahori type of squarefree level N; this means that for each prime q dividing N, U_q is*

-equal to the standard Iwahori subgroup of G_q if G is quasi-split at q (that is, either $G_q = \mathrm{GL}_n(K_\mathfrak{q})$ if q splits in K, or if G_q is the quasi-split unitary group),

-is a minimal parahoric subgroup if G_q is not quasisplit.

Remark 2.2. *Let Π_G be any cuspidal automorphic representation on G with cohomological weight and level $U = U^p \times U_p$. Let q be a prime dividing N which is inert in K; the condition $\Pi_{G,q}^{U_q} \neq 0$ implies that the base change of $\Pi_{G,q}$ to K_q has fixed vectors by the Iwahori subgroup of $GL_n(K_q)$. Let $\sigma_{\Pi_{G,q}}$ be the p-adic Weil-Deligne representation of $\Pi_{G,q}$. Let $\widetilde{\sigma}_{\Pi_{G,q}}$ be its restriction to the inertia subgroup I_q. If the reduction modulo p of $\widetilde{\sigma}_{\Pi_{G,q}}$ is regular unipotent, the same holds for $\widetilde{\sigma}_{\Pi_{G,q}}$ and $\Pi_{G,q}$ is the twist of the Steinberg representation by an unramified -at most quadratic, character.*

This remark will be useful later.

We fix an isomorphism $i_p \colon G_p = G(\mathbb{Q}_p) \cong \mathrm{GL}_n(\mathbb{Q}_p)$, which we use to identify these groups. Thus, we can view $U_p = i_p^{-1}(\mathrm{GL}_n(\mathbb{Z}_p))$ as a hyperspecial maximal compact subgroup of G_p. From now on, we omit the mention of i_p and we simply write $U_p = \mathrm{GL}_n(\mathbb{Z}_p)$. Let $I_p \subset U_p$ be the Iwahori subgroup and for $0 \leq b \leq c$, $I_p^{b,c} \subset U_p$ be the subgroup of matrices whose reduction modulo p^c, resp. p^b, belong to the group of $\mathbb{Z}/p^c\mathbb{Z}$-points of the subgroup B of upper triangular matrices of GL_n, resp. to the group of $\mathbb{Z}/p^b\mathbb{Z}$-points of the group $N' = \mathrm{diag}(1, \ldots, 1, *) \cdot N^+$ where N^+ is the group of upper unipotent matrices. Note the difference with [Ge10, Def.2.1] where the condition modulo p^b is that $u \in N^+(\mathbb{Z}/p^b\mathbb{Z})$. Here we enlarge the group N^+ to N'. This is because we want to define a big ordinary Hecke algebra depending only on the semisimple variables of the diagonal torus T, not on the whole of T. Let $T^{ss} = \mathrm{Ker}(\det \colon T \to \mathbb{G}_m)$. We have a decomposition $T \cong T^{ss} \times \mathbb{G}_m$ given by $u \mapsto (u^{ss}, \det u)$ where $u = \mathrm{diag}(u_1, \ldots, u_n)$ and $u^{ss} = \mathrm{diag}(u_1, \ldots, u_{n-1}, (u_1 \cdot \ldots \cdot u_{n-1})^{-1})$.

Let G_f be the locally compact group of finite adèles of G and $G_\mathbb{Q}$ be the subgroup of principal adèles. By compactness of G_∞, $G_\mathbb{Q}$ is discrete in G_f and for any compact open subgroup U of G_f, the quotient $G_\mathbb{Q}\backslash G_f/U$ is finite (see [PR94, Chap.5, Section 3, Th.5.5]). We fix from now on the auxiliary level group $U = U^p \times U_p$ of Iwahori type of squarefree level N in the sense of Definition 2.1. As usual, one can add another auxiliary prime r (in the sense of Taylor-Wiles) prime to Np to assure that U is sufficiently small: $G_\mathbb{Q} \cap U = 1$. Note that at the prime r, U is no longer of Iwahori type but of strict Iwahori type. After localization at a suitable maximal ideal, it will not introduce extra ramification at r for the automorphic forms occuring in the Hida-Geraghty Hecke algebra of auxilary level U defined below.

For $c \geq b \geq 0$, let $U^{b,c} = U^p \times I_p^{b,c}$. Let E be a sufficiently large p-adic field; let $\mathcal{O}$ be its valuation ring. Any $(n-1)$-tuple $\lambda = (\lambda_1, \ldots, \lambda_{n-1}) \in \mathbb{Z}^{n-1}$ defines a character of the diagonal torus T of GL_n (and of $T^{ss} = T \cap \mathrm{SL}_n$) by

$$\mathrm{diag}(t_1, \ldots, t_n) \mapsto t_1^{\lambda_1} \cdot \ldots \cdot t_{n-1}^{\lambda_{n-1}}$$

Let us assume that $\lambda_1 \geq \ldots \lambda_{n-1} \geq 0$, let $L_\lambda(\mathcal{O})$ be the "maximal" $\mathcal{O}$-representation of GL_n of highest weight λ (see [PT02]). Let w_0 be the longest element of the Weyl group of GL_n. Then $L_\lambda(\mathcal{O})$ is defined as the algebraic induction of $w_0\lambda$ from B to GL_n, that is, the $\mathcal{O}$-module of rational functions $\phi \in \mathcal{O}[\mathrm{GL}_n]$ such that $\phi(tn^+g) = (w_0\lambda)(t)\phi(g)$ for any $b = tn^+ \in B$. We define the $\mathcal{O}$-module $S_\lambda(U^{b,c}; \mathcal{O})$ of cuspidal forms of level $U^{b,c}$ for G by

$$S_\lambda(U^{b,c}; \mathcal{O}) = \{s \colon G_\mathbb{Q}\backslash G_f \to L_\lambda(\mathcal{O}); s(xu) = u_p^{-1} \cdot s(x) \quad \text{for any}\, u \in U^{b,c}\}$$

For $c > 0$, let $h_\lambda(U^{b,c}; \mathcal{O})$ be the $\mathcal{O}$-algebra of endomorphisms of $S_\lambda(U^{b,c}; \mathcal{O})$ generated by the Hecke operators

- $T_{\xi,i} = [U^{b,c}\alpha_{\varpi_\xi}^{(i)}U^{b,c}]_\lambda$, where $i = 1, \ldots, n$, $\alpha_\varpi^{(i)} = \begin{pmatrix} \varpi_\xi 1_i & 0 \\ 0 & 1_{n-i} \end{pmatrix}$, and ξ runs over the degree one primes of $\mathcal{O}_K$, relatively prime to $M\Delta p$.
- $U_{\varpi,i} = (w_0\lambda)(\alpha_\varpi^{(i)})^{-1}[U^{b,c}\alpha_\varpi^{(i)}U^{b,c}]_\lambda$, $i = 1, \ldots, n-1$, where ϖ is a uniformizing parameter of $\mathfrak{p}$ and $\alpha_\varpi^{(i)} = \begin{pmatrix} \varpi 1_i & 0 \\ 0 & 1_{n-i} \end{pmatrix}$ as before,

- $\langle u\rangle_\lambda = [U^{b,c}uU^{b,c}]_\lambda$ where $u \in T(\mathbb{Z})$ (actually, it depends only on the image of u in $T^{ss}(\mathbb{Z}/p^b\mathbb{Z})$).

Recall that $[U^{b,c}\alpha U^{b,c}]_\lambda$ acts by $([U^{b,c}\alpha U^{b,c}]_\lambda\cdot s)(x) = \sum_i \alpha_{i,p}\cdot s(x\alpha_i)$ where $U^{b,c}\alpha U^{b,c} = \bigsqcup_i \alpha_i U^{b,c}$ (see beginning of [Ge10, Sect. 2.3]). The operators $T_{\xi,i}$, $U_{\varpi,i}$ and $\langle u\rangle_\lambda$ preserve integrality [Ge10, Def. 2.3.1 and 2.3.2].

Let e be the ordinary idempotent associated to $U_\varpi = \prod_{i=1}^{n-1} U_{\varpi,i}$. We define

$$h_{n-1} = \varprojlim_c e \cdot h_\lambda(U^{c,c};\mathcal{O})$$

It does not depend on the dominant weight λ [Ge10, Prop.2.6.1]. It is reduced [Ge10, Lemma 2.4.4]. Let $T_0 = T(\mathbb{Z}_p)$, $T_0^{ss} = T^{ss}(\mathbb{Z}_p)$; and similarly let $T_b = \mathrm{Ker}(T(\mathbb{Z}_p) \to T(\mathbb{Z}/p^b\mathbb{Z}))$, $T_b^{ss} = \mathrm{Ker}(T^{ss}(\mathbb{Z}_p) \to T^{ss}(\mathbb{Z}/p^b\mathbb{Z}))$. We can decompose $T_0 = T(\mathbb{Z}/p\mathbb{Z}) \times T_1$ and $T_1 = T_1^{ss} \times (1+p\mathbb{Z}_p)$. For $p > 2$, let $u = 1+p$. We can identify the $\mathcal{O}$-algebra Λ_{n-1} of power series in $n-1$ variables to the completed group algebra $\mathcal{O}[[T_1^{ss}]]$ by sending $1+X_i$ to $\mathrm{diag}(1_{i-1}, u, 1_{n-1-i}, u^{-1})$. We view h_{n-1} as a Λ_{n-1}-algebra via the weight 0 diamond action $T_1 \to h_{n-1}^\times$, $u \to \langle u\rangle_0$. As a Λ_{n-1}-algebra, h_{n-1} is finite torsion-free. Indeed, the proof of [Ge10, Prop.2.5.3] goes through when one replaces the group T_b by the group T_b^{ss}, because with our modified definition of the groups $U^{b,c}$, we do have

$$S_\lambda(U^{c,c};\mathcal{O})^{T_b^{ss}} = S_\lambda(U^{b,c};\mathcal{O})$$

hence, by Hida's lemma (see [Ge10, Lemma 2.5.2], we see that

$$e \cdot S_\lambda(U(p^\infty); E/\mathcal{O})^{T_b^{ss}} = e \cdot S_\lambda(U^{b,b}; E/\mathcal{O})$$

which is the key step for the vertical control theorem and its corollary [Ge10, Coroll.2.5.4]. From this fact, the finiteness and torsion-freeness of our Hecke algebra over Λ_{n-1} follow as in [Ge10, Coroll.2.5.4].

2.2. Symm^{n-1} **Langlands functoriality.** We assume that the Symm^{n-1} Langlands functoriality from GL_2 to GL_n is established (sending non CM classical cusp eigensystems to cuspidal eigensystems on GL_n). It is known for $n-1 \leq 8$ thanks to the works of Kim-Shahidi [KS02b], Kim [Kim03] and Clozel-Thorne [CT14], [CT15]. Let π be a non CM holomorphic cuspidal representation of $\mathrm{GL}_2(\mathbf{A}_\mathbb{Q})$ cohomological for a local system of highest weight $a' > 0$ (later, a' will vary in the arithmetic progression $a + (p-1)\mathbb{Z}$ for a as in the introduction), with conductor N and level group $U_0^{(1)}(N) = \{u \in \mathrm{GL}_2(\widehat{\mathbb{Z}}); u \pmod N \text{ upper triangular}\}$, (that is, $\dim \pi^{U_0^{(1)}(N)} = 1$). The Langlands parameter $r_\infty\colon W_\mathbb{R} \to GL_2(\mathbb{C})$ of π_∞ is given by $r_\infty(z) = \mathrm{diag}((z/\overline{z})^{(a'+1)/2}, (\overline{z}/z)^{a'+1)/2})$ for $z \in W_\mathbb{C}$ and $r_\infty(j) = \begin{pmatrix} 0 & 1 \\ (-1)^{a'+1} & 0 \end{pmatrix}$.

By assumption, there is an automorphic cuspidal representation $\Pi = \mathrm{Symm}^{n-1}\pi$ on GL_n. The Langlands parameter $R_\infty\colon W_\mathbb{R} \to GL_n(\mathbb{C})$ of Π_∞ is given by its restriction to $W_\mathbb{C}$ by

$$R_\infty(z) = \mathrm{diag}((z/\overline{z})^{(n-1)(a'+1)/2}, (z/\overline{z})^{(n-3)(a'+1)/2}\ldots, (z/\overline{z})^{-(n-1)(a'+1)/2}).$$

It follows from the local Langlands correspondence for $\mathrm{GL}_n(\mathbf{A}_\mathbb{Q})$ that Π is Steinberg at all primes dividing N. Let Π_K the base change of Π to $\mathrm{GL}_n(\mathbf{A}_K)$ (see [AC89, III,5]); the Langlands parameter of $\Pi_{K,\infty}$ is $R_\infty|_{W_\mathbb{C}}$. It is cohomological. Moreover, $\Pi_{K,\mathfrak{q}}$ is Steinberg at all primes $\mathfrak{q}$ of K dividing N. In particular, Π_K is square-integrable at both places of S_D; therefore, by the Jacquet-Langlands correspondence for GL_n (see [Vi84] and [AC89]), it descends to a cuspidal representation Π_D on $D^\times(\mathbf{A}_K)$. Note that $\Pi_{D,\infty} = \Pi_{K,\infty}$ is cohomological. By [Clo91, Lemma 3.8 and Prop.4.11], Π_D descends as a cuspidal representation Π_G on G (for more general results of descent from $D^\times$ to G, see Labesse [Lab09, Th.5.4] and C.-P. Mok [Mok14]). The difference with [Clo91, Prop.4.11] is that here $\Pi_{G,\infty}$ is the irreducible representation of highest weight $((n-1)a', (n-2)a', \ldots, a', 0)$ of the compact group $U(n)$ (instead of being a cohomological representation of $U(n-1,1)$); moreover, $\Pi_{G,\mathfrak{q}}$ is Steinberg at all places $\mathfrak{q}$ of K dividing N. Note that

- For any rational prime q prime to Np which splits in K, say, $q = \xi\xi^c$, the Hecke eigenvalues $t_{\xi,i}$ on the 1-dimensional space $\Pi_\xi^{U_\xi}$ of the Hecke operators $T_{\xi,i}$, $i = 1,\ldots,n-1$, are determined by the relation between Hecke polynomials:

$$P^{(n-1)}_{\Pi_\xi}(T) = \mathrm{Sym}^{n-1} P^{(1)}_{\pi_q}(T) \in E[T]$$

where

$$P^{(1)}_{\pi_q}(T) = T^2 - a_q T + q^{a'+1} = (T-\alpha_q)(T-\beta_q),$$

$$\mathrm{Sym}^{n-1} P^{(1)}_{\pi_q}(T) = (T-\alpha_q^{n-1})(T-\alpha_q^{n-2}\beta_q)\dots(T-\beta_q^{n-1}),$$

and

$$P^{(n-1)}_{\Pi_\xi}(T) = T^n - t_{\xi,1}T^{n-1} + \dots + (-1)^j q^{j(j+1)/2} t_{\xi,j} T^{n-j} + \dots + (-1)^n q^{n(n+1)/2} t_{\xi,n}$$

- since $a' \neq 0$, the local component $\Pi_{\mathfrak{p}}$ is unramified and the eigenvalues $u_{\varpi,i}$ of the normalized Atkin-Lehner operators $U_{\varpi,i}$ $(i = 1,\dots,n-1)$, on the finite dimensional vector space $\Pi_{\mathfrak{p}}^{I_p}$ are given by

$$\prod_{i=1}^{n}(T - p^{i-1}\frac{u_{\varpi,i}}{u_{\varpi,i-1}}\varpi^{(i-1)a'}) = \mathrm{Sym}^{n-1} P^{(1)}_{\pi_p}(T)$$

  where one has put $U_{\varpi,0} = U_{\varpi,n} = \mathrm{Id}$. Explicitely, one has $u_{\varpi,1} = \alpha_p^{n-1}$, $u_{\varpi,2} = \alpha_p^{n-2}\frac{\beta_p}{p\varpi^{a'}}$,..., $u_{\varpi,n-1} = \alpha_p(\frac{\beta_p}{p\varpi^{a'}})^{n-2}$, where α_p is the unit root of $P^{(1)}_{\pi_p}(T)$. Note that the eigenvalues $u_{\varpi,i}$ are p-adic units since $\frac{\varpi}{p}$ is. This follows from Lemma 2.7.5 of [Ge10] because the weight $\lambda = (\lambda_1,\dots,\lambda_n)$ is given by $((n-1)a', (n-2)a', \dots, a', 0)$, hence it is regular if $a' \neq 0$, hence the lemma applies.

Let h_1^{N-new} be the N-new quotient of h_1. For any prime q prime to Np splitting in K as $\xi\xi^c$, let

$$P^{(n-1)}_{\xi}(T) = T^n - T_{\xi,1}T^{n-1} + \dots + (-1)^j q^{j(j+1)/2} T_{\xi,j} T^{n-j} + \dots + (-1)^n q^{n(n+1)/2} T_{\xi,n}$$

be the universal Hecke polynomial of the spherical Hecke algebra of GL_n at ξ and $P^{(1)}_q(T) = T^2 - T_qT + qS_q$ the universal Hecke polynomial of the spherical Hecke algebra for GL_2 at q. Recall that $\ell\colon \mathbb{Z}_p^\times \to p\mathbb{Z}_p$ is defined by $x = \omega(x)u^{\ell(x)})$. We can interpolate the formulas above:

Proposition 2.3. *There exists a ring homomorphism $\theta\colon h_{n-1} \to h_1$ above the algebra homomorphism $\Lambda_{n-1} \to \Lambda_1$ given by $1 + X_i \to (1+X)^{n-i}$ for $i = 1,\dots,n-1$. The homomorphism θ is characterized by the fact that for any prime q prime to Np splitting in K as $\xi\xi^c$, the image by θ of the universal Hecke polynomial $P^{(n-1)}_\xi(T)$ is $\mathrm{Sym}^{n-1} P^{(1)}_q(T)$, while the images $\theta(U_{\varpi,i})$ are given by $(U_p^{(1)})^{n-2i+1}\cdot(\omega(\frac{p}{\varpi})(1+X)^{\ell(\frac{p}{\varpi})})^{i-1}$. Let π be an N-new p-ordinary holomorphic cuspidal automorphic form π on $GL_2(\mathbb{Q})$ of highest weight $a > 0$. Let $\mu_\pi\colon h_1^{N-new} \to \mathcal{O}$ be the associated eigensystem. Let $\theta_{a'} = \theta \pmod{X - u^{a'} + 1}$. Then for any rational prime q split in K as $\xi\xi^c$, $\mu_\pi \circ \theta_{a'}$ sends the universal polynomial $P^{(n-1)}_\xi(T)$ to $\mathrm{Sym}^{n-1} P_{\pi_q}(T)$ and if we put $P_{\pi_p}(T) = (T-\alpha_p)(T-\beta_p)$, $\mathrm{ord}_p(\alpha_p) = 0$, we have $\mu_\pi\circ\theta_{a'}(U_{\varpi,i}) = \alpha_p^{n-i}(\frac{\beta_p}{p\varpi^{a'}})^{i-1}$ for $i = 1,\dots,n-1$.*

Proof. For primes $q \neq p$, the statement is obvious. For the prime p, for any $a' > 0$, one gets

$$\theta(U_{\varpi,i}) \equiv (U_p^{(1)})^{n-2i+1}\cdot(\frac{p}{\varpi})^{a'(i-1)} \pmod{X - u^{a'} + 1}.$$

But we have for $a' > 0$

$$\alpha_p^{n-i}(\frac{\beta_p}{p\varpi^{a'}})^{i-1} = \alpha_p^{n-i}(\frac{\beta_p}{p^{a'+1}})^{i-1}(\frac{p}{\varpi})^{a'(i-1)} = \alpha_p^{n-2i+1}(\frac{p}{\varpi})^{a'(i-1)}$$

as desired. ∎

2.3. **Galois representations.** Let $\Gamma_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and $\Gamma_K = \mathrm{Gal}(\overline{K}/K)$. In this section we use notations and results of [CHT08, Sect.2.1]. Let $\mathcal{G}_n = (GL_n \times \mathrm{GL}_1) \rtimes \{1, j\}$ where $j^2 = 1$ and $j(g,\mu)j^{-1} = ({}^tg^{-1}\mu, \mu)$. It is a non connected group scheme over $\mathbb{Z}$. Let $\nu\colon \mathcal{G} \to \mathrm{GL}_1$ be the homomorphism given by $(g,\mu) \mapsto \mu$ and $\nu(j) = -1$. We have the inclusions of Lie algebras $\mathfrak{sl}_n \subset \mathfrak{gl}_n \subset \mathrm{Lie}\,\mathcal{G}_n$. Note that $ad(g,\mu)(X) = gXg^{-1}$ and $ad(j)(X) = -{}^tX$. We fix a sufficiently large p-adic field E with valuation ring $\mathcal{O}$. In this section, we consider representations $\rho\colon \Gamma_K \to \mathrm{GL}_n(R)$ and homomorphisms $r\colon \Gamma_{\mathbb{Q}} \to \mathrm{GL}_n(R)$ for various $\mathcal{O}$-algebras R. The theorem below follows from [Ge10, Proposition 2.7.2] (see also [CHT08, Proposition 3.3.4]) and [Ge10, Corollary 2.7.8].

Theorem 2.4. *Let λ' be a dominant weight for GL_n ; for any cuspidal automorphic representation Π'_G of $G(\mathbb{A}_{\mathbb{Q}})$ occuring in $e \cdot S_{\lambda'}(U^{0,1}, E)$, there exists a continuous semisimple representation*

$$\rho_{\Pi'_G}\colon \Gamma_K \to \mathrm{GL}_n(E)$$

such that

(i) for q prime to Np splitting in K as $\xi\xi^c$, $\rho_{\Pi'_G}$ is unramified at q and the characteristic polynomial of $Frob_\xi$ is $P_{\Pi'_{G,\xi}}(T)$,

(ii) $\rho^c_{\Pi'_G} \cong \rho^\vee_{\Pi'_G}\chi^{1-n}$

(iii) for any prime q inert in K not dividing Np, $\rho_{\Pi'_G}$ is unramified at q

(iv) if moreover λ' is regular, $\rho_{\Pi'_G}$ is crystalline and ordinary at $\mathfrak{p}$ and $\mathfrak{p}^c$. For instance at $\mathfrak{p}$:

$$\rho_{\Pi'_G}|_{\Gamma_{K_\mathfrak{p}}} \sim \begin{pmatrix} \psi_{\mathfrak{p},1} & * & \cdots & * \\ & \chi^{-1}\psi_{\mathfrak{p},2} & \cdots & * \\ & & \ddots & \\ & & & \chi^{-n+1}\psi_{\mathfrak{p},n} \end{pmatrix}$$

where $\psi_{\mathfrak{p},i} \circ \mathrm{Art}_\mathfrak{p}\colon K_\mathfrak{p}^\times \to E^\times$ is given on $\mathcal{O}_\mathfrak{p}^\times$ by $x \mapsto x^{-\lambda'_{n-i+1}}$ and by $\psi_{\mathfrak{p},i} \circ \mathrm{Art}_\mathfrak{p}(\varpi) = u_{\Pi',i}/u_{\Pi',i-1}$, $i = 1, \ldots, n-1$ where $u_{\Pi',i}$ is the unique unit eigenvalue of $U_{\varpi,i}$ on $(\Pi'_{G,\mathfrak{p}})^{I_p}$ for $i = 1, \ldots, n-1$.

In particular, denoting by Art *the global Artin symbol of K, we have for any $x \in \mathcal{O}_{K,\mathfrak{p}}^\times$*

$$\det\,\rho_{\Pi'_G} \circ \mathrm{Art}(x) = x^{-\sum_{i=1}^n(\lambda'_{n-i+1}+i-1)}.$$

As explained in [Ge10, Proposition 2.7.2], the key ingredient in order to apply the main result of [HaT01] is Coroll.5.3 of [Lab09]. The uniqueness of the unit eigenvalue of $U_{\varpi,i}$ on $(\Pi')^{I_p}_{G,\mathfrak{p}}$ for $i = 1, \ldots, n-1$ is proven in [Ge10, Lemma 2.7.5 (2)].

By Theorem 2.4, we have a perfect pairing $\langle\,,\,\rangle\colon E^n \times E^n \to E$ and a character $\mu = \chi^{1-n}\colon \Gamma_{\mathbb{Q}}\colon \Gamma_{\mathbb{Q}} \to E^\times$ such that

- $\langle y, x\rangle = -\mu(c)\langle x, y\rangle$
- $\langle \rho_{\Pi'_G}(\delta)x, \rho_{\Pi'_G}(c\delta c)y\rangle = \mu(\delta)\langle x, y\rangle$.

By [CHT08, Lemma 2.1.1], there is a bijection between "polarized representations" $(\rho, \mu, \langle\,,\,\rangle)$ where $\rho\colon \Gamma_K \to \mathrm{GL}_n(E)$, $\mu\colon \Gamma_{\mathbb{Q}}\colon \Gamma_{\mathbb{Q}} \to E^\times$ are homomorphisms and $\langle\,,\,\rangle\colon E^n \times E^n \to E$ is a perfect pairing, and homomorphisms $r\colon \Gamma_{\mathbb{Q}} \to \mathcal{G}_n(E)$. We have $-\mu(c) = (-1)^n$. Therefore, Theorem 2.4 yields

Corollary 2.5. *For Π'_G as above, there exists a continuous homomorphism*

$$R_{\Pi'_G}\colon \Gamma_{\mathbb{Q}} \to \mathcal{G}_n(E)$$

such that

- *for $\delta \in \Gamma_K$, one has $R_{\Pi'_G}(\delta) = (\rho_{\Pi'_G}(\delta), \chi^{1-n}(\delta))$,*
- *$R_{\Pi'_G}(c) = (J_n^{-1}, (-1)^n)j$.*

Let π be an N-new p-ordinary cuspidal holomorphic representation of level N cohomological of highest weight $a \geq 0$ occuring in the Hida family μ. Let $\rho_\pi\colon \Gamma_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{O})$ its p-adic Galois representation. Assume that the residual representation $\overline{\rho} = \overline{\rho}_\pi\colon \Gamma_{\mathbb{Q}} \to \mathrm{GL}_2(k)$ has big image, in the sense that there exists a subfield $k' \subset k$ such that

$$\mathrm{SL}_2(k') \subset \mathrm{Im}\,\overline{\rho}_\pi \subset \mathrm{GL}_2(k').$$

Note that up to conjugation the restriction of $\overline{\rho}_\pi$ to a decomposition group D_p at p is given by

$$\begin{pmatrix} \mathrm{unr}(\overline{\alpha}) & * \\ \mathbf{0} & \mathrm{unr}(\overline{\alpha}^{-1})\omega^{-a-1} \end{pmatrix}$$

where $\alpha = \mu(U_p)$.

Let $\Pi = \mathrm{Symm}^{n-1}\pi$ be the $n-1$-symmetric power cuspidal representation of π on $\mathrm{GL}_n(\mathbb{Q})$ and Π_G its base change to G. Let

$$R_\Pi = \mathrm{Symm}^{n-1}\rho_\pi \colon \Gamma_\mathbb{Q} \to \mathrm{GL}_n(\mathcal{O})$$

be the Galois representation associated to Π.

By [CHT08, Lemma 2.1.2], the continuous homomorphism

$$R_{\Pi_G} \colon \Gamma_\mathbb{Q} \to \mathcal{G}_n(\mathcal{O})$$

associated to Π_G is given as follows. Let c be a complex conjugation in $\Gamma_\mathbb{Q}$. For $\sigma \in \Gamma_K$, we put

$$R_{\Pi_G}(\sigma) = (R_\Pi(\sigma), (\det\, \rho_\pi(\sigma))^{n-1})$$

and for $\sigma \in \Gamma_\mathbb{Q}\backslash\Gamma_K$, and $J_n = \mathrm{Symm}^{n-1}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (so that $J_n = \mathrm{antidiag}(1, -1, 1, \ldots, (-1)^{n-1})$). Then we put

$$R_{\Pi_G}(\sigma) = (R_\Pi(\sigma)J^{-1}, (-1)^{n-1}(\det\, \rho_\pi(\sigma))^{n-1})j.$$

Moreover, we have

$$\nu \circ R_{\Pi_G} = \delta^n \cdot (\det\, R_\Pi)^{n-1}.$$

where $\delta \colon \Gamma_\mathbb{Q}/\Gamma_K \cong \{\pm 1\}$. It is ordinary at $\mathfrak{p}$ and each prime $\mathfrak{q}$ of K dividing N, its restriction to the inertia subgroup $I_\mathfrak{q}$ is regular unipotent.

Note that by our assumption, there exists a subfield $k' \subset$ such that

$$\mathrm{Symm}^{n-1}\mathrm{SL}_2(k') \subset \mathrm{Im}\ \overline{R}_\Pi \subset k'^\times \cdot \mathrm{Symm}^{n-1}\mathrm{GL}_2(k')$$

This implies that the residual image of R_Π is big in the sense of [CHT08, Definition 2.5.1] (see [CHT08, Lemma 2.5.4] for details). Let $\mathfrak{m}$ be the maximal ideal of h_{n-1} associated to the residual representation

$$\overline{R} = \overline{R}_{\Pi_G} \colon \Gamma_\mathbb{Q} \to \mathcal{G}_n(k).$$

We fix a decomposition group $D_p = \Gamma_{K_\mathfrak{p}}$ at p in $\Gamma_\mathbb{Q}$. Note that after a given conjugation, one can assume that the restriction to D_p is upper triangular, with diagonal

$$\mathrm{diag}\left(\mathrm{unr}(\overline{\alpha})^{n-1}, \mathrm{unr}(\overline{\alpha})^{n-2}\omega^{-a-1}, \ldots, \omega^{-(n-1)(a+1)}\right)$$

which we rewrite as

$$\mathrm{diag}\left(\overline{\psi}_1, \overline{\psi}_2\omega^{-1}, \ldots, \overline{\psi}_n\omega^{-(n-1)}\right).$$

Let $\mathbf{T}_{n-1}$ be the localization of h_{n-1} at $\mathfrak{m}$ and, for any weight $\lambda' \in \mathbb{Z}^n_+$ congruent to $((n-1)a, (n-2)a, \ldots, a, 0)$ modulo $p-1$, let $\mathbf{T}_{\lambda'}(U^{0,1}, \mathcal{O})$ be the image of $\mathbf{T}_{n-1}$ in $\mathrm{End}_\mathcal{O}(e \cdot S_{\lambda'}(U^{0,1}, E))$ We shall compare, under certain assumptions, $\mathbf{T}_{n-1}$, resp. $\mathbf{T}_{\lambda'}(U^{0,1}, \mathcal{O})$, with the universal ordinary deformation ring R_{n-1}, resp. and $R_{\lambda'}$ of the representation $\overline{R}$ defined as follows. Let $\mathrm{CNL}_\mathcal{O}$ be the category of complete noetherian local $\mathcal{O}$-algebras A with residue field $k = \mathcal{O}/\varpi_E\mathcal{O}$. For an object A of $\mathrm{CNL}_\mathcal{O}$, a lifting $r \colon \Gamma_\mathbb{Q} \to \mathcal{G}_n(A)$ of $\overline{R}$ is a continuous homomorphism such that $r \pmod{\mathfrak{m}_A} = \overline{R}$. Two liftings $r, r' \colon \Gamma_\mathbb{Q} \to \mathcal{G}_n(A)$ of $\overline{R}$ are equivalent if there exists $g \in 1 + \mathfrak{m}_A M_n(A)$ such that $r' = g \cdot r \cdot g^{-1}$. We consider the functors of liftings $\mathcal{D}$ and $\mathcal{D}_{\lambda'}$ from $\mathrm{CNL}_\mathcal{O}$ to Sets defined as follows:

- The functor $\mathcal{D}$ sends an object A to the set of equivalence classes of liftings $r \colon \Gamma_\mathbb{Q} \to \mathcal{G}_n(A)$ of $\overline{R}$ which satisfy the two following conditions

  1) r is N-minimal: for each prime q dividing N there exists $g_q \in \mathbf{1}_n + \mathfrak{m}_A \cdot M_n(A)$ such that for any $\sigma \in I_q$,

$$\mathrm{pr}_1 \circ r(\sigma) = g_q \cdot \exp(t_p(\sigma)N_n) \cdot g_q^{-1}$$

where

$$N_n = \begin{pmatrix} 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ 0 & 0 & \ddots & \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

2) $r|_{D_p}$ is ordinary, that is, there exist characters $\psi_{r,1}, \dots, \psi_{r,n} \colon \Gamma_{K_\mathfrak{p}} \to A^\times$ and $g \in 1_n + \mathfrak{m}_A M_n(A)$ such that for any $\sigma \in \Gamma_{K_\mathfrak{p}}$,

$$\mathrm{pr}_1 \circ r(\sigma) = g \cdot \begin{pmatrix} \psi_{r,1}(\sigma) & * & \dots & * \\ & \chi^{-1}\psi_{r,2}(\sigma) & \dots & \\ & & \ddots & \\ & & & \chi^{-n+1}\psi_{r,n}(\sigma) \end{pmatrix} \cdot g^{-1}$$

with the condition that for any $j = 1, \dots n$, $\psi_{r,j}$ is a lifting of $\overline{\psi}_{r,j}$.

- The functor $\mathcal{D}_{\lambda'}$ is defined similarly, replacing condition 2) by the stronger condition $2)_{\lambda'}$ $r|_{D_p}$ is ordinary with characters $\psi_{r,j}$ lifting of $\overline{\psi}_j$ $(j = 1, \dots, n)$ and for any $j = 1, \dots n$ and for any $x \in \mathcal{O}_\mathfrak{p}^\times$,

$$\psi_{r,j}(\mathrm{Art}_\mathfrak{p}(x)) = x^{-\lambda'_{n-j+1}}.$$

The functor $\mathcal{D}$, resp. $\mathcal{D}_{\lambda'}$ is the functor of N-minimal ordinary, resp. N-minimal ordinary of weight λ' deformations of $\overline{R}$. Note that condition that the characters $\psi_{r,j}$ are liftings of $\overline{\psi}_{r,j}$ implies that $\psi_{r,j}|_{I_p}$ is a lifting of $\omega^{-(j-1)(a+1)}$ $(j = 1, \dots, n)$.

Let d_n be the least common multiple of all integers k less than n.

Let us consider the following conditions

1a) $\alpha^{2d_{n-1}} \not\equiv 1 \pmod{\varpi_E}$ holds,

1b) $(n-1)(a+1) < p-1$ holds.

Condition 1a) implies that the characters $\overline{\psi}_j \colon D_p \to k^\times$ associated to $\overline{R}|_{D_p}$ are mutually distinct on the Frobenius element $[p, \mathbb{Q}_p]$, while condition 1b) implies that the restrictions to the inertia subgroup of the characters $\overline{\psi}_j$ are mutually distinct. In the following subsections devoted to the proof that $R_{n-1} \cong \mathbf{T}^u_{n-1}$, assumption 1a) is assumed (although we could have assumed 1b) instead).

Lemma 2.6. *Assuming condition 1a) or 1b) , the functors $\mathcal{D}$ and $\mathcal{D}_{\lambda'}$ are representable by universal couples (R_{n-1}, r^{univ}) and $(R_{\lambda'}, r^{univ}_{\lambda'})$ where R_{n-1} and $R_{\lambda'}$ are objects of $\mathrm{CNL}_\mathcal{O}$ and $r^{univ} \colon \Gamma_\mathbb{Q} \to \mathcal{G}_n(R_{n-1})$ and $r^{univ}_{\lambda'} \colon \Gamma_\mathbb{Q} \to \mathcal{G}_n(R_{\lambda'})$ are continuous homomorphisms such that $\mathrm{pr}_1 \circ r^{univ}$, resp. $r^{univ}_{\lambda'}$ is conjugate in $1 + \mathfrak{m}_{R_{n-1}} M_n(R_{n-1})$, resp. in $1 + \mathfrak{m}_{R_{\lambda'}} M_n(R_{\lambda'})$, to*

$$\begin{pmatrix} \psi_1^{univ}(\sigma) & * & \dots & * \\ & \chi^{-1}\psi_2^{univ}(\sigma) & \dots & \\ & & \ddots & \\ & & & \chi^{-n+1}\psi_n^{univ}(\sigma) \end{pmatrix}$$

with ψ_j^{univ} lifting $\overline{\psi}_j$ $(j = 1, \dots, n)$, resp.

$$\begin{pmatrix} \psi_{\lambda',1}^{univ}(\sigma) & * & \dots & * \\ & \chi^{-1}\psi_{\lambda',2}^{univ}(\sigma) & \dots & \\ & & \ddots & \\ & & & \chi^{-n+1}\psi_{\lambda',n}^{univ}(\sigma) \end{pmatrix}$$

with the same lifting condition, and such that the restriction of $\psi^{univ}_{\lambda',j} \circ \mathrm{Art}_\mathfrak{p}$ to $\mathcal{O}_\mathfrak{p}^\times$ is given by $x \mapsto x^{-\lambda'_{n-j+1}}$ $(j = 1, \dots, n)$.

Proof. As noted above, the restriction of $\mathrm{pr}_1 \circ \overline{R}$ to the decomposition group D_p at p is upper triangular and its diagonal is given by $\mathrm{diag}\left(\mathrm{unr}(\overline{\alpha})^{n-1}, \mathrm{unr}(\overline{\alpha})^{n-3}\omega^{-a-1}, \dots, \mathrm{unr}(\overline{\alpha})^{-n+1}\omega^{-(n-1)(a+1)}\right)$.

Hence either assumption 1a), resp. 1b), assures that the characters on the diagonal are mutually distinct on D_p, resp. I_p. This is well known to assure that the functors $\mathcal{D}$ and $\mathcal{D}_{\lambda'}$ satisfy Schlessinger's criterion for representability (see for instance [Ti02]). □

The ring R_{n-1} has a natural structure of Λ_{n-1}-algebra given by the characters $\psi_i^{univ}\colon K_{\mathfrak{p}}^\times \to R_{n-1}^\times$. More precisely, by identifying $\mathbb{Z}_p = \mathcal{O}_{\mathfrak{p}}$, we view the topological generator u of $1+p\mathbb{Z}_p$ as topological generator of $1+\mathfrak{p}$. we then define the structural morphism $\Lambda_{n-1} \to R_{n-1}$ by sending $1+X_i$ to $\psi_{n-i+1}^{univ} \circ \mathrm{Art}_{\mathfrak{p}}(u)^{-1}$ for $i = 1,\ldots,n-1$. Note that ψ_1 is determined by the determinant relation $\prod_{i=1}^n \chi^{-i+1}\psi_i = \chi^{-n(n+1)/2}$.

Let $P_{\lambda'}$ be the prime ideal of Λ_{n-1} defined as the kernel of the morphism

$$T_1^{ss} \to \mathcal{O}^\times, \quad \mathrm{diag}(t_1,\ldots,t_n) \mapsto t_1^{\lambda'_1}\cdot\ldots\cdot t_n^{\lambda'_n}.$$

Lemma 2.7. *For any λ' congruent to $((n-1)a,(n-2)a,\ldots,a,0)$ modulo $p-1$, the natural ring homomorphism*

$$R_{n-1} \to R_{\lambda'}$$

induces an isomorphism

$$R_{n-1}/P_{\lambda'}R_{n-1} \cong R_{\lambda'}.$$

Proof. It suffices to check for each $j = 1,\ldots,n$ that $\psi_j^{univ} \circ \mathrm{Art}_{\mathfrak{p}}$ modulo $P_{\lambda'}$ is given on $\mathcal{O}_{\mathfrak{p}}^\times$ by $x \mapsto x^{-\lambda'_{n-j+1}}$. This is the case on $1+\mathfrak{p}$ by definition of $P_{\lambda'}$. This is also the case on $\mathcal{O}_{\mathfrak{p},tors}^\times = \mu_{p-1}$ since ψ_j^{univ} is a lifting of $\overline{\psi}_j$. □

Proposition 2.8. *There is a unique lifting $R^h\colon \Gamma_{\mathbb{Q}} \to \mathcal{G}(\mathbf{T}_{n-1}^u)$, resp. $R_{\lambda'}^h\colon \Gamma_{\mathbb{Q}} \to \mathcal{G}_n(\mathbf{T}_{\lambda'}(U^{0,1},\mathcal{O}))$ of $\overline{R}$ such that for any Hecke eigensystem $\theta_{\Pi'_G}\colon \mathbf{T}_{n-1}^u \to \mathcal{O}'$, resp. $\theta_{\Pi'_G}\colon \mathbf{T}_{\lambda'}(U^{0,1},\mathcal{O}) \to \mathcal{O}'$ associated to a cuspidal representation Π'_G on G, one has $\theta_{\Pi'_G}\circ R^h = R_{\Pi'_G}$. By universal property, the homomorphism R^h gives rise to a surjective Λ_{n-1}-algebra homomorphism $\phi_{R^h}\colon R_{n-1} \to \mathbf{T}_{n-1}^u$. Similarly, the homomorphism $R_{\lambda'}^h$ gives rise to a surjective $\mathcal{O}$-algebra homomorphism $\phi_{R_{\lambda'}^h}\colon R_{\lambda'} \to \mathbf{T}_{\lambda'}(U^{0,1},\mathcal{O})$*

Proof. The existence of R^h resp. $R_{\lambda'}^h$ follows from [Ge10, Prop.2.74]. Its restriction to the inertia group I_q at any prime divisor q of N is regular unipotent by Remark 2.2. Its ordinarity at p follows from [Ge10, Cor.3.1.4]. This gives rise to ordered characters ψ_i^h, $i = 1,\ldots,n$ lifting the ordered characters $\overline{\psi}_i^h$, $i = 1,\ldots,n$ and such that

$$pr_1 \circ R^h|_{\Gamma_{K_{\mathfrak{p}}}} \sim \begin{pmatrix} \psi_1^h & * & \cdots & * \\ & \chi^{-1}\psi_2^h(\sigma) & \cdots & \\ & & \ddots & \\ & & & \chi^{-n+1}\psi_n^h(\sigma) \end{pmatrix}.$$

Actually, viewing the topological generator u of $1+p\mathbb{Z}_p$ as topological generator of $1+\mathfrak{p}$, we have $\psi_{n-i+1}^h \circ \mathrm{Art}_{\mathfrak{p}}(u) = (1+X_i)^{-1}$ as the series $(1+X_i)^{-1}$ interpolates the values $u^{-\lambda'_i}$ at $X_i = u^{\lambda'_i}$. The N-minimality of R^h follows from that of $R_{\Pi'_G}$ for all Π'_G's occuring in $\mathbf{T}_{n-1}^u$. By universal property, this yields the existence of a unique ring homomorphism $\phi_{R^h}\colon R_{n-1} \to \mathbf{T}_{n-1}^u$ such that $\phi_{R^h}\circ R^{univ} \sim R^h$. The relation $\phi_{R^h}\circ\psi_i^{univ} = \psi_i^h$ implies that ϕ_{R^h} is Λ_{n-1}-linear.

The surjectivity of ϕ_{R^h} and $\phi_{R_{\lambda'}^h}$ follows from the absolute irreducibility of $\overline{R}$ and Carayol's theorem : $\mathbf{T}_{n-1}^u = \Lambda_{n-1}[\mathrm{Tr}\,R^h|_{\Gamma_K}]$ and $R_{n-1} = \Lambda_{n-1}[\mathrm{Tr}\,R^{univ}|_{\Gamma_K}]$ hence $\mathbf{T}_{n-1}^u = \phi_{R^h}(R_{n-1})$. Similarly for $\phi_{{R^h}_{\lambda'}}$ □

Let $\lambda' \in \mathbb{Z}_+^n$ be an arbitrary regular dominant weight congruent to $((n-1)a,(n-2)a,\ldots,a,0)$ modulo $p-1$. We shall use the technique of classical Taylor-Wiles systems to prove that $\phi_{R_{\lambda'}^h}$ is an isomorphism and that the rings $R_{\lambda'}$ and $\mathbf{T}_{\lambda'}$ are local complete intersection over $\mathcal{O}$. From this it will be easy by varying λ' to deduce that ϕ_{R^h} is an isomorphism and that R_{n-1} and $\mathbf{T}_{n-1}^u$ are local complete intersection over Λ_{n-1}.

We follow (in an easier situation) the proof of [Ge10, Section 3] which itself relies on calculations of [CHT08, Section 3.5].

2.4. **Galois cohomology.** Let $p > 2$ and $\overline{R} = \mathrm{Symm}^{n-1}\overline{\rho}_\pi$ where π is holomorphic cuspidal on $GL_2(\mathbb{Q})$, of square free conductor N and cohomological for a local system of highest weight $a \geq 0$. We assume it is p-ordinary. Therefore it occurs in a (unique) Hida family μ. We assume that $\overline{\rho}_\pi$ has big image and that it is N-minimal as above. We also assume 1a).

Note that the image $\overline{R}(c)$ of the complex conjugation c is conjugate in $\mathcal{G}_n(k)$ to $(J_n^{-1}, (-1)^n)j$ where $J_n = \mathrm{antidiag}(1, -1, \ldots, 1, (-1)^{n-1})$ From this we have as in [CHT08, Lemma 2.1.3] :

Lemma 2.9. $\dim_k(\mathfrak{gl}_n)^{c=1} = n(n-1)/2$.

Proof. For $X \in M_n(k)$, we have $\mathrm{Ad}\,\overline{R}(c)(X) = J_n^{-1} j X j^{-1} J_n$. We have $J_n^{-1} = (-1)^{n-1} J_n = {}^t J_n$ and $jXj^{-1} = -{}^tX$, hence

$\mathrm{Ad}\,\overline{R}(c)(X) = X$ if and only if $J_n^{-1}X$ is antisymmetric. The subspace of these matrices has dimension $n(n-1)/2$. $\square$

Let $M = \mathfrak{gl}_n(k) = \mathrm{Ad}_{\mathfrak{gl}_n}\overline{R}$ and $M^* = \mathrm{Hom}_k(M, k(1))$ its k-Cartier dual. For $N = M$ or M^*, let $h^0(N) = \dim_k \mathrm{H}^0(\Gamma_{\mathbb{Q}}, N)$; for any place v of $\mathbb{Q}$, and any fixed decomposition group $D_v \subset \Gamma_{\mathbb{Q}}$, let $h^i_v(M) = \dim_k \mathrm{H}^i(D_v, M)$ $(i = 0, 1, 2)$. Let $Q = \{q_1, \ldots, q_r\}$ be a finite set of primes disjoint of those dividing Np such that for any $i = 1, \ldots, r$, $q_i = v_i v_i^c$ splits in K. For any finite place $v \neq p$ of $\mathbb{Q}$ with $v \notin Q$, let

$$L_{Q,v} = L_v(M) = \mathrm{H}^1_{\mathrm{unr}}(\Gamma_v, M) = \mathrm{Ker}(\mathrm{H}^1(\Gamma_v, M) \to \mathrm{H}^1(I_v, M)) = \mathrm{Ker}(\mathrm{H}^1(D_v/I_v, M^{I_v})$$

for each $v \in Q$, let $L_{Q,v} \subset \mathrm{H}^1(D_v, M)$ to be specified later in such a way that $\dim_k L_{Q,v} - h^0_v(M) = 1$. We also put $L_\infty = 0$ and

$$L_{Q,p} = L_p(M) = \mathrm{Im}(L'_p(M) \to \mathrm{H}^1(\Gamma_p, M)$$

where $L'_p(M) = \mathrm{Ker}(\mathrm{H}^1\Gamma_p, F^0M) \to \mathrm{H}^1\Gamma_p, F^0(M)/F^1(M)))$.

For any place v of $\mathbb{Q}$, let $L^\perp_{Q,v}$ be the orthogonal in $\mathrm{H}^1(D_v, M^*)$ of $L_{Q,v} \subset \mathrm{H}^1(D_v, M)$ for the local Tate duality $\mathrm{H}^1(D_v, M) \times \mathrm{H}^1(D_v, M^*) \to k$. When $L_{Q,v} = \mathrm{H}^1_{\mathrm{unr}}(\Gamma_v, M)$, one has $L^\perp_{Q,v} = \mathrm{H}^1_{\mathrm{unr}}(\Gamma_v, M^*)$. Moreover it is easy to check that $L_p(M)^\perp = L_p(M^*)$ associated to the p-ordinarity filtration of M^* given by $F^i(M^*) = (F^{b-a+1-i}(M))^\perp$ where a, resp. b is the smallest, resp. largest weight of $F^\bullet(M)$ (that is, $F^a(M) = M \neq F^{a+1}(M)$, and $F^b(M) \neq 0$ but $F^{b+1}(M) = 0$). Let $\mathcal{L}_Q = (L_{Q,v})_v$ and $\mathcal{L}^\perp_Q = (L^\perp_{Q,v})_v$. We define the Selmer groups

$$\mathrm{H}^1_{\mathcal{L}_Q}(M) = \mathrm{Ker}\left(\mathrm{H}^1(\Gamma, M) \to \bigoplus_v \mathrm{H}^1(D_v, M)/L_{Q,v}\right)$$

and

$$\mathrm{H}^1_{\mathcal{L}^\perp_Q}(M^*) = \mathrm{Ker}\left(\mathrm{H}^1(\Gamma, M^*) \to \bigoplus_v \mathrm{H}^1(D_v, M^*)/L^\perp_{Q,v}\right)$$

They are finite and their cardinalities are denoted by $h^1_{\mathcal{L}_Q}(M)$ resp. $h^1_{\mathcal{L}^\perp_Q}(M^*)$. The Poitou-Tate Euler characteristic formula, as formulated for instance in [DDT94, Theorem 2.18], yields

$$h^1_{\mathcal{L}_Q}(M) - h^1_{\mathcal{L}^\perp_Q}(M^*) = h^0(M) - h^0(M^*) + \sum_v (\dim_k L_{Q,v} - h^0_v(M)).$$

Proposition 2.10. *We have:*
(i) $\dim_k L_{Q,\ell} - h^0_\ell(M) = 0$ *for any* $\ell \notin Q$ *and* $\ell \neq p$,
(ii)$\dim_k L_{Q,q} - h^0_q(M) = 1$ *for* $q \in Q$,
(iii) $\dim_k L_{Q,p} - h^0_p(M) \leq n(n-1)/2$, *(this uses 1a) and 1b)).*
(iv) $h^0_\infty = h^0(D_\infty, M) = n(n-1)/2$
(v) $h^0(M) = 0 = h^0(M^*) = 0$.
It follows that

$$h^1_{\mathcal{L}_Q}(M) - h^1_{\mathcal{L}^\perp_Q}(M^*) \leq \sharp Q = r.$$

Proof. From the Poitou-Tate Euler characteristic formula, the last inequality follows from the four first formulas. The first equality is clear from the exact sequence

$$0 \to \mathrm{H}^0(D_\ell, M) \to M^{I_\ell} \overset{F_\ell - 1}{\to} M^{I_\ell} \to \mathrm{H}^1(D_\ell/I_\ell, M^{I_\ell}) \to 0.$$

Let us check the inequality at p. We proceed as in [GeTi05, Lemma 10.4.4]. Let $\mathfrak{b}_n$, $\mathfrak{n}_n$, $\mathfrak{t}_n$ be the Lie algebras of the upper triangular, upper unipotent subgroup, resp. of their quotient. We have an exact sequence

$$\mathrm{H}^0(D_p, \mathfrak{n}_n) \to \mathrm{H}^0(D_p, \mathfrak{b}_n) \to \mathrm{H}^0(D_p, \mathfrak{b}_n/\mathfrak{n}_n)) \to \mathrm{H}^1(D_p, \mathfrak{n}_n) \to \mathrm{H}^1(D_p, \mathfrak{b}_n) \to \mathrm{H}^1(D_p, \mathfrak{b}_n/\mathfrak{n}_n))$$

Moreover, we also have an exact sequence

$$0 \to \mathrm{H}^1(D_p/I_p, \mathfrak{b}_n/\mathfrak{n}_n) \to \mathrm{H}^1(D_p, \mathfrak{b}_n/\mathfrak{n}_n) \to \mathrm{H}^1(I_p, \mathfrak{b}_n/\mathfrak{n}_n)$$

By assumptions 1a) and 1b), we have $h^0(D_p, \mathfrak{n}_n) = 0$ and $h^2(D_p, \mathfrak{n}_n) = h^0(D_p, \mathfrak{n}_n^\vee(1)) = 0$. Therefore, we have $L_p(M)' = L_p(M)$ and

$$h^0(D_p, \mathfrak{b}_n) - h^0(D_p, \mathfrak{b}_n/\mathfrak{n}_n) + h^1(D_p, \mathfrak{n}) - \dim_k\, L_{Q,p} + h^1(D_p/I_p, \mathfrak{b}_n/\mathfrak{n}_n) = 0$$

By cyclicity of D_p/I_p, we have $h^0(D_p, \mathfrak{b}_n/\mathfrak{n}_n) = h^1(D_p/I_p, \mathfrak{b}_n/\mathfrak{n}_n)$. Moreover By Tate local duality, we have

$$h^0(D_p, \mathfrak{n}_n) - h^1(D_p, \mathfrak{n}_n) + h^2(D_p, \mathfrak{n}_n) = -\dim_k \mathfrak{n}_n$$

Hence $h^1(D_p, \mathfrak{n}_n) = \dim_k \mathfrak{n}_n$. We conclude that $\dim_k\, L_{Q,p} - h^0(D_p, \mathfrak{b}_n) = \dim_k \mathfrak{n}_n = n(n-1)/2$. It implies $\dim\, L_{Q,p} - h^0(D_p, \mathfrak{gl}_n) \le n(n-1)/2$ as desired. Satement (iv) follows from 2.9. Statement (v) follows from the fact that M (and M^*) is the sum of the irreducible Γ-modules $\overline{\mathcal{A}}_\mu^j = \mathcal{A}_\mu^j(k)$ ($j = 1, \ldots, n-1$) which satisfy $\mathrm{H}^0(\Gamma, \overline{\mathcal{A}}_\mu^j) = 0$ since $2j < p$ and $\mathrm{SL}_2(k') \subset \mathrm{Im}\ \overline{\rho}_\pi \subset \mathrm{GL}_2(k')$. □

2.5. **Application of Chebotarev density theorem.** Let $M = \mathfrak{gl}_n(k)$. Let $r = \dim_k \mathrm{H}^1_{\mathcal{L}_\emptyset^\perp}(\Gamma_\mathbb{Q}, M^*)$. For q prime to Np splitting in K, let $X^2 - \overline{a}_{\pi,q}X + q^{a'+1} = (X - \overline{\alpha}_q)(X - \overline{\beta}_q)$, where $\overline{\alpha}_q, \overline{\beta}_q \in k$. We write $\overline{R}|_{D_q} = \overline{\psi}_q \oplus \overline{s}_q$ where $\overline{\psi}_q = (\overline{\alpha}_q)^{n-1}$ and $\overline{s}_q$ is the unramified D_q-module given by the sum of all eigenspaces corresponding to the other eigenvalues $\overline{\alpha}_q^{n-i}\overline{\beta}_q^i$, $i \neq 0$. We assume that $(\overline{\alpha}_q/\overline{\beta}_q)^{c_{n-1}} \not\equiv 1 \pmod{\mathfrak{m}_E}$ so that the eigenvalues $\overline{\alpha}_q^{n-i}\overline{\beta}_q^i$, $i = 0, \ldots, n-1$ are mutually distinct. Therefore $\overline{s}_q$ does not contain $\overline{\psi}_q$ as D_q-submodule. Given a finite set of primes q prime to Np, split in K as above and such that $q \equiv 1 \pmod{p}$, let us define

$$L_{Q,q} = \mathrm{H}^1(D_q/I_q, \mathrm{Ad}\,\overline{s}_q) \oplus \mathrm{H}^1(D_q, \mathrm{Ad}\,\overline{\psi}_q)$$

We notice as in [CHT08, Section 2.4.6] the obvious

Lemma 2.11. *We have* $\dim_k L_{Q,q} - h^0_q(M) = \dim_k \mathrm{H}^1(I_q, \mathrm{Ad}\,\overline{\psi}_q)^{D_q} = 1$.

Using $\mathcal{L}_Q = (L_{Q,v})_v$ with $L_{Q,v}$ as before for $v \notin Q$, and the definition above for $v \in Q$, we define $\mathrm{H}^1_{\mathcal{L}_Q}(\Gamma_\mathbb{Q}, M)$ and $\mathrm{H}^1_{\mathcal{L}_Q^\perp}(\Gamma_\mathbb{Q}, M^*)$. Note that, as in [CHT08, Prop.2.4.9], we have a short exact sequence

$$0 \to \mathrm{H}^1_{\mathcal{L}_Q^\perp}(\Gamma_Q, M^*) \to \mathrm{H}^1_{\mathcal{L}^\perp}(\Gamma_Q, M^*) \to \bigoplus_{q \in Q} \mathrm{H}^1(D_q/I_q, \mathrm{Ad}\,\overline{\psi}_q(1))$$

given by the maps $\omega_q \colon [c] \mapsto [c_q]$ for $q \in Q$, where, for any $\sigma \in D_q$

$$c_q(\sigma) = pr_{\overline{\psi}_q} \circ c(\sigma) \circ i_{\overline{\psi}_q}$$

where $i_{\overline{\psi}_q}$ is the inclusion of the $\overline{\psi}_q$-line and $pr_{\overline{\psi}_q}$ is the projection onto this line parallely to $\overline{s}_q$.

Moreover, each term of the right hand side sum is one-dimensional.

Theorem 2.12. *For any $m \ge 1$ there exists a set Q_m of primes q splitting in K, say, $(q) = \mathfrak{q}\mathfrak{q}^c$, and relatively prime to Np, such that*

- $\sharp Q_m = r$
- *for any $q \in Q_m$, one has $q \equiv 1 \pmod{p^m}$*
- $\mathrm{H}^1_{\mathcal{L}_{Q_m}^\perp}(\Gamma_\mathbb{Q}, M^*) = 0$
- *for any $q \in Q_m$, $\overline{R}(\mathrm{Frob}_q)$ has distinct eigenvalues in k.*

Proof. We follow the proof of [CHT08, Proposition 2.5.9]. Let us first assume that we chose primes q which split totally in $K(\zeta_{p^m})$ and such that $\overline{R}(\mathrm{Frob}_q)$ has distinct eigenvalues in k. The condition $\mathrm{H}^1_{\mathcal{L}^{\perp}_{Q_m}}(\Gamma_{\mathbb{Q}}, M^*) = 0$ is implied by the isomorphism

$$\mathrm{H}^1_{\mathcal{L}^{\perp}_{\emptyset}}(\Gamma_{\mathbb{Q}}, M^*) \cong \bigoplus_{q \in Q_m} \mathrm{H}^1(D_q/I_q, \mathrm{Ad}\,\overline{\psi}_q)$$

of the sum of the maps ω_q defined above Since $M \cong M^\vee$, we have $M \cong M^*$ as $\Gamma_{\mathbb{Q}(\zeta_p)}$-modules. For this, it is enough to show that for each non-zero class $[c] \in \mathrm{H}^1_{\mathcal{L}_\emptyset}(\Gamma_{\mathbb{Q}}, M^*)$ there is a prime q such that $\omega_q([c]) \in \mathrm{H}^1(D_q/I_q, \mathrm{Ad}\,\overline{\psi}_q)$ is non-zero.

By Chebotarev density theorem, it is enough to find for each non zero class $[c] \in \mathrm{H}^1_{\mathcal{L}_\emptyset}(\Gamma_{\mathbb{Q}}, M^*)$ an element $\sigma \in \Gamma_{\mathbb{Q}}$ such that $\sigma|_{K(\zeta_{p^m})} = 1$, $\overline{R}(\sigma)$ admits an eigenvalue γ with multiplicity 1, and $\mathrm{pr}_{\overline{\gamma}} \circ c(\sigma) \circ i_{\overline{\gamma}} \neq 0$ where $i_{\overline{\gamma}}$ is the injection of the $\overline{\gamma}$-eigenspace of $\overline{R}(\sigma)$ into the space of $\overline{R}$ and $\mathrm{pr}_{\overline{\gamma}}$ the projection to this eigenspace.

Let F_m be the extension of $K(\zeta_{p^m})$ cut out by $\mathrm{Ad}\,\overline{R}$, that is, the field fixed by the kernel of $\mathrm{Ad}\,\overline{R}|_{\Gamma_{F_m}}$. Let us show that $c(\Gamma_{F_m}) \neq 0$. By the inflation-restriction exact sequence

$$\mathrm{H}^1(\mathrm{Gal}(F_m/\mathbb{Q}), \mathrm{Ad}\,\overline{R}) \to \mathrm{H}^1(\Gamma_{\mathbb{Q}}, \mathrm{Ad}\,\overline{R}) \to \mathrm{Hom}(\Gamma_{F_m}, \mathrm{Ad}\,\overline{R})$$

it suffices to see that $\mathrm{H}^1(\mathrm{Gal}(F_m/\mathbb{Q}), \mathrm{Ad}\,\overline{R}) = 0$. Consider the inflation-restriction exact sequence

$$0 \to \mathrm{H}^1(\mathrm{Gal}(F_0/\mathbb{Q}), \mathrm{Ad}\,\overline{R}^{\mathrm{Gal}(F_m/F_0)}) \to \mathrm{H}^1(\mathrm{Gal}(F_m/\mathbb{Q}), \mathrm{Ad}\,\overline{R}) \to \mathrm{H}^1(\mathrm{Gal}(F_m/F_0), \mathrm{Ad}\,\overline{R})^{\Gamma_{\mathbb{Q}}}$$

Since F_1/F_0 is of degree prime to p, we have

$$\mathrm{H}^1(\mathrm{Gal}(F_m/F_0), \mathrm{Ad}\,\overline{R})^{\Gamma_{\mathbb{Q}}} = \mathrm{Hom}(\mathrm{Gal}(F_m/F_1), \mathrm{Ad}\,\overline{R}^{\Gamma_{\mathbb{Q}}})$$

but $\mathrm{Ad}\,\overline{R}^{\Gamma_{\mathbb{Q}}} = 0$ (by bigness), hence the right hand side vanishes. Similarly, $\mathrm{Gal}(F_m/F_0)$ acts trivially on V and $\mathrm{H}^1(\mathrm{Gal}(F_0/\mathbb{Q}), \mathrm{Ad}\,\overline{R})$ vanishes by bigness (see [CHT08, Cor. 2.5.4]).

Now, we consider $c(\Gamma_{\mathbb{Q}(\zeta_{p^m})})$ as a $\mathrm{Gal}(F_m/\mathbb{Q}(\zeta_{p^m}))$-submodule of $\mathrm{Ad}\,\overline{R}$. Note that $\mathrm{Gal}(F_m/\mathbb{Q}(\zeta_{p^m}))$ contains $\mathrm{PSL}_2(k')$ hence there exists $g \in \mathrm{Gal}(F_m/\mathbb{Q}(\zeta_{p^m}))$ of order not dividing p fixing a non zero element of $c(\Gamma_{\mathbb{Q}(\zeta_{p^m})})$; we can even assume that g acting on $\mathrm{Ad}\,\overline{R}$ has distinct eigenvalues in k, again by bigness of $\overline{\rho}_\pi$, provided that $p-1$ does not divide c_{n-1}. Let $\sigma_0 \in \Gamma_{\mathbb{Q}(\zeta_{p^m})}$ lifting g and let $\sigma = \tau\sigma_0$ where $\tau \in \Gamma_{F_m}$ is such that $c(\sigma) = c(\tau) + c(\sigma_0) \notin (\sigma_0 - 1)V$. Such a τ exists because $c(\Gamma_{\mathbb{Q}(\zeta_{p^m})}) \not\subset (g-1)\mathrm{Ad}\,\overline{R}$. The corresponding element σ satisfies the desired conditions for some $\overline{\gamma} \in k^\times$. □

For $q \in Q_m$, let $X^2 - \overline{a}_{\pi,q}X + q^{a+1} = (X - \overline{\alpha}_q)(X - \overline{\beta}_q)$, where $\overline{\alpha}_q, \overline{\beta}_q \in k$. We write $\overline{R}|_{D_q} = \overline{\psi}_q \oplus \overline{s}_q$ where $\overline{\psi}_q = \mathrm{unr}(\overline{\alpha}_q)^{n-1}$ and $\overline{s}_q$ is the unramified D_q-module given by the sum of all eigenspaces corresponding to the other eigenvalues $\overline{\alpha}_q^{n-i}\overline{\beta}_q^{i}$, $i \neq 0$. Note that $\overline{s}_q$ does not contain $\overline{\psi}_q$ as D_q-submodule. We define

$$L_{Q_m,q} = \mathrm{H}^1(D_q/I_q, \mathrm{Ad}\,\overline{s}_q) \oplus \mathrm{H}^1(D_q, \mathrm{Ad}\,\overline{\psi}_q)$$

We notice as in [CHT08, Section 2.4.6] the obvious

Lemma 2.13. *We have* $\dim_k L_{Q_m,q} - h^0_q(M) = \dim_k \mathrm{H}^1(I_q, \mathrm{Ad}\,\overline{\psi}_q)^{D_q} = 1$.

Using $\mathcal{L}_{Q_m} = (L_{Q_m,v})_v$ with $L_{Q_m,v}$ as before for $v \notin Q_m$, and the definition above for $v \in Q_m$, we define $\mathrm{H}^1_{\mathcal{L}_{Q_m}}(\Gamma_Q, M)$ and $\mathrm{H}^1_{\mathcal{L}^{\perp}_{Q_m}}(\Gamma_Q, M^*)$.

Corollary 2.14. *For any set* Q_m *as above, one has* $\dim_k \mathrm{H}^1_{\mathcal{L}_{Q_m}}(\Gamma_Q, M) \leq \sharp Q_m = r$.

2.6. **Construction of a Taylor-Wiles system.** Recall we fixed in Definition 2.1 a level subgroup $U \subset G(\widehat{\mathbb{Z}})$ of level N. Let Q be a finite set of primes q splitting in K such that $q \equiv 1 \pmod p$ and $(\alpha_q/\beta_q)^{c_{n-1}} \not\equiv 1 \pmod{\mathfrak{m}_E}$. Let Δ_q be the p-Sylow of $(\mathbb{Z}/q\mathbb{Z})^\times$. We write $(\mathbb{Z}/q\mathbb{Z})^\times = \Delta_q \times \Delta_q^p$. For each $q \in Q$, $q = \mathfrak{q}\mathfrak{q}^c$, we fix an isomorphism $i_{\mathfrak{q}}\colon U_q \cong \mathrm{GL}_n(\mathcal{O}_{\mathfrak{q}})$; we identify $\mathcal{O}_{\mathfrak{q}}/\mathfrak{q} = \mathbb{Z}/q\mathbb{Z}$. Let

$$U'_q = \{g \in U_q; i_{\mathfrak{q}}(g) \equiv \begin{pmatrix} g_{n-1} & * \\ 0 & \delta \end{pmatrix} \pmod q \quad \delta \in \Delta_q^p, g_{n-1} \in \mathrm{GL}_{n-1}(\mathcal{O}_{\mathfrak{q}})\}$$

We also write $U'_{\mathfrak q}=i_{\mathfrak q}(U'_q)$. We also consider the parahoric group

$$U_{q,0}=\{g\in U_q; i_q(g)\equiv\begin{pmatrix} g_{n-1} & * \\ 0 & \delta\end{pmatrix}\pmod q\quad \delta\in(\mathbb{Z}/q\mathbb{Z})^\times, g_{n-1}\in \mathrm{GL}_{n-1}(\mathcal{O}_{\mathfrak q})\}$$

associated to the maximal parabolic subgroup $P\subset \mathrm{GL}_n$ fixing the line $< e_n >$. Note that $U_{q,0}/U'_q\cong\Delta_q$. Let $U'_Q=\prod_{q\in Q}U'_q\times U^Q$, $U_{Q,0}=\prod_{q\in Q}U_{q,0}\times U^Q$ and $\Delta_Q=\prod_{q\in Q}\Delta_q$. Note that $U_{Q,0}/U'_Q=\Delta_Q$. Let $h_{n-1,Q}$, resp. $\widetilde{h}_{n-1,Q}$, be the (cuspidal) Hida Hecke algebra of auxiliary level group U'_Q excluding the Hecke operators at NQ, resp.including the Atkin-Lehner Hecke operators $U_{q,i}$ at $q\in Q$. This is naturally an $\mathcal{O}[\Delta_Q]$-algebra. We denote by $\mathfrak{a}_Q=([\delta]-1,\delta\in\Delta_Q)$ the augmentation ideal of $\mathcal{O}[\Delta_Q]$.

If we put $U_Q^{b,c}=U'_Q\cap U^{b,c}$, we note that these algebras both act faithfully on $e\cdot\mathbb{S}_Q(E/\mathcal{O})=\varinjlim_c e\cdot S_\lambda(U_Q^{c,c};E/\mathcal{O})$ (where λ is an arbitrary dominant weight, for instance $\lambda=0$). By using the diamonds of weight 0, one endows these $\mathcal{O}$-algebras with a structure of $\Lambda_{n-1}=\mathcal{O}[[T_1^{ss}]]$-algebra. We have a morphism $h_{n-1,Q}\to h_{n-1}$ which factors through $h_{n-1,Q}\to h_{n-1,Q}/\mathfrak{a}_Q h_{n-1,Q}$.

For any fixed dominant weight λ' congruent to $((n-1)a,(n-2)a,\dots,a,0)$, we also consider the Hecke algebras $e\cdot h_{\lambda'}(U_Q^{0,1},\mathcal{O})$, resp. $e\cdot\widetilde{h}_{\lambda'}(U_Q^{0,1},\mathcal{O})$ of weight λ'. Recall that

$$h_{n-1,Q}/P_{\lambda'}h_{n-1,Q}\to e\cdot h_{\lambda'}(U_Q^{0,1},\mathcal{O})$$

is a surjection with nilpotent kernel. Let $\mathfrak{m}_Q$ be the maximal ideal of $h_{n-1,Q}$ associated to the residual representation $\overline{R}$. Let $\mathbf{T}_{n-1,Q}=(h_{n-1,Q})_{\mathfrak{m}_Q}$ resp. $\mathbf{T}_{\lambda',Q}=e\cdot h_{\lambda'}(U_Q^{0,1},\mathcal{O})_{\mathfrak{m}_Q}$ be the corresponding localization-completion. We denote by $R_Q^h\colon\Gamma_{\mathbb{Q}}\to\mathcal{G}_n(\mathbf{T}_{n-1,Q})$ the lifting of $\overline{R}$ constructed as $R^h=R^h_\emptyset$ over $\mathbf{T}^u_{n-1}=\mathbf{T}_{n-1,\emptyset}$. Similarly for $R^h_{\lambda',Q}\colon\Gamma_{\mathbb{Q}}\to\mathcal{G}_n(\mathbf{T}_{\lambda',Q})$.

For $q\in Q$ and for $\beta\in K_{\mathfrak q}^\times$, we define compatible Frobenius Hecke operators

$${}_qV_\beta^{c,c}=i_{\mathfrak q}^{-1}\left(U'_{\mathfrak q}\begin{pmatrix}1_{n-1} & 0\\ 0 & \beta\end{pmatrix}U'_{\mathfrak q}\right)\times(U_Q^{c,c})^q$$

It defines an endomorphism of $S_0(U_Q^{c,c},\mathcal{O})_{\mathfrak{m}_Q}$. We fix a lifting ϕ_q of the geometric Frobenius to $\overline{\mathbb{Q}}_q$ given by the Artin symbol $[q,\mathbb{Q}_q]$ on $\mathbb{Q}_q^{ab}$. Let $\mathbf{T}^{c,c}_{n-1,Q}=e\cdot h_0(U_Q^{c,c};\mathcal{O})_{\mathfrak{m}_Q}$ and Let $R_Q^{c,c}\colon\Gamma_{\mathbb{Q}}\to\mathcal{G}_n(\mathbf{T}^{c,c}_{n-1,Q})$ be the push-forward of R_Q^h via the surjective homomorphism $\mathbf{T}_{n-1,Q}\to\mathbf{T}^{c,c}_{n-1,Q}$. Let $A_q^{c,c}$ be the unique root of Char $R_Q^{c,c}(\phi_q)$ in $\mathbf{T}_{n-1,Q}$ lifting the root $\alpha_q^{n-1}\in k^\times$ of Char $\overline{R}(\phi_q)$. By Hensel's lemma, for any $c\geq 1$ we have factorizations in $\mathbf{T}^{c,c}_{n-1}$:

$$\mathrm{Char}\ R^{c,c}(X)=(X-A_q^{c,c})Q_q^{c,c}(X).$$

Let $Y_Q^{c,c}=\prod_{q\in Q}Q_q^{c,c}({}_qV_q^{c,c})eS_0(U_Q^{c,c},\mathcal{O})_{\mathfrak{m}_Q}$; it is the largest $e\cdot h_0(U_Q^{c,c};\mathcal{O})_{\mathfrak{m}_Q}[({}_qV_q)_{q\in Q}]$-submodule of $S_0(U_Q^{c,c},\mathcal{O})_{\mathfrak{m}_Q}$ on which for any $q\in Q$, ${}_qV_q^{c,c}-A_q^{c,c}$ is topologically nilpotent. Let $\mathbf{T}_{n-1}(Y_Q^{c,c})$ be the image of $e\cdot h_0(U_Q^{c,c};\mathcal{O})_{\mathfrak{m}_Q}$ in $\mathrm{End}_{\mathcal{O}}(Y_Q^{c,c})$. We denote by $R_{Y_Q^{c,c}}$ be the image of the representation R^h by the projection $\mathbf{T}^u_{n-1}\to\mathbf{T}_{n-1}(Y_Q^{c,c})$. Recall that by [CHT08, Prop.3.4.4, 8] (here our level groups $U_Q^{c,c}$ play the role of the group U there).

Proposition 2.15. *For any $c\geq 1$,*

- *for any $\beta\in K_{\mathfrak q}^\times\cap\mathcal{O}_{\mathfrak q}$, we have ${}_qV_\beta^{c,c}\in\mathbf{T}_{n-1}(Y_Q^{c,c})$, and $X-{}_qV_q^{c,c}$ divides* Char $R^{c,c}(\phi_q)$ *in $\mathbf{T}_{n-1}(Y_Q^{c,c})[X]$.*
- *The map given by ${}_qV^{c,c}([\beta,\mathbb{Q}_q])={}_qV_\beta^{c,c}$ for $\beta\in K_{\mathfrak q}^\times\cap\mathcal{O}_{\mathfrak q}$ extends into a continuous character ${}_qV^{c,c}\colon D_q\to(\mathbf{T}_{n-1}(Y_Q^{c,c}))^\times$ and we have*

$$R_Y^{c,c}|_{D_q}=s_q\oplus{}_qV^{c,c}$$

where s_q in unramified of rank $n-1$.

Proof. We refer to [CHT08, Prop.3.4.4, 8] for the details; we simply mention that the proof relies on [CHT08, Lemma 3.1.5] which analyzes the q-component of a cuspidal representation occuring in $\mathbf{T}_{n-1}(Y_Q^{c,c})$; the possibility of a partial Steinberg component is excluded by the condition $q\equiv 1\pmod p$. ☐

Actually, by Hensel's lemma one can even define a unique root $A_q \in \mathbf{T}_{n-1,Q}$ of Char $R^h(\phi_q)$ congruent to α_q^{n-1} modulo the maximal ideal such that

$$\text{Char } R^h(X) = (X - A_q^{c,c})Q_q(X)$$

with $Q_q(X) \in \mathbf{T}_{n-1,Q}[X]$ and $Q_q(A_q) \in \mathbf{T}_{n-1,Q}^\times$. For any $c \geq 1$, A_q interpolates the $A_q^{c,c}$ via the morphisms $\mathbf{T}_{n-1,Q} \to \mathbf{T}_{n-1,Q}^{c,c}$. For any $\beta \in K_\mathfrak{q}^\times \cap \mathcal{O}_\mathfrak{q}$, the operators ${}_qV_\beta^{c,c}$ are compatible when $c \geq 1$ varies; they give rise to an element ${}_qV_\beta \in \mathbf{T}_{n-1,Q}$ and to a continuous homomorphism

$$D_q \to \mathbf{T}_{n-1,Q}^\times.$$

One can then define a subspace $\mathbb{Y}_Q \subset \mathbb{S}_Q(E/\mathcal{O})_{\mathfrak{m}_Q}$ by $\mathbb{Y}_Q = \prod_{q\in Q} Q_q(V_q)\mathbb{S}_Q(E/\mathcal{O})_{\mathfrak{m}_Q}$ such that for any $c \geq 1$, one has $\mathbb{Y}_Q^{T_c^{ss}} = Y_Q^{c,c} \otimes \mathbb{Q}_p/\mathbb{Z}_p$ via the identification

$$\mathbb{S}_Q^{T_c^{ss}} = e \cdot S_0(U^{c,c}; E/\mathcal{O}).$$

We conclude from 2.15 that for any $q \in Q$,

$$R_Q^h|_{D_q} = s_q \oplus {}_qV.$$

where s_q is unramified and ${}_qV([q, \mathbb{Q}_q]) = A_q$.

Let $\mathcal{D}_Q$ be the deformation subfunctor of $\mathcal{D}$ imposing that for any $q \in Q$, the liftings $r \in \mathcal{D}_Q(A)$ of $\overline{R}$ when restricted to D_q are of the form

$$s_q \oplus A(\psi_q)$$

where s_q is unramified of rank $n-1$ and lifts $\overline{s}_q$, and ${}_qV_q \colon D_q \to A^\times$ is such that $\psi_q([q, \mathbb{Q}_q]) \equiv \alpha_q^{n-1}$ (mod $\mathfrak{m}_A$). Let $R_{n-1,Q}$ be the universal deformation ring of $\mathcal{D}_Q$. It is endowed with a universal lifting

$$R_Q^{univ} \colon \Gamma_Q \to \mathcal{G}_n(R_Q^{univ})$$

and characters $\psi_q^{univ} \colon D_q \to (R_Q^{univ})^\times$ such that

$$R_Q^{univ}|_{D_q} = s_q^{univ} \oplus \psi_q^{univ}.$$

where s_q^{univ} is unramified and lifts $\overline{s}_q$,and $\psi_q^{univ}([q, \mathbb{Q}_q])$ lifts the root $\overline{\alpha}_q^{n-1} \in k^\times$ of $\overline{R}(\text{Frob}_q)$. Similarly, we have

$$R_{\lambda',Q}^h \colon \Gamma_\mathbb{Q} \to \mathcal{G}_n(\mathbf{T}_{\lambda',Q})$$

such that for any $q \in Q$,

$$R_{\lambda',Q}^h|_{D_q} = s_{\lambda',q}^h \oplus \psi_{\lambda',q}^h.$$

where $\psi_{\lambda',q}^h = {}_qV_q^{0,1}$. By Proposition 2.15, these automorphic liftings give rise to surjective ring homomorphisms

$$R_{n-1,Q} \to \mathbf{T}_{n-1,Q}(\mathbb{Y}_Q) \quad \text{and}\, R_{\lambda',Q} \to \mathbf{T}_{\lambda',Q}(\mathbb{Y}_Q)$$

sending R_Q^{univ} to R_Q^h and ψ_q^{univ} to ${}_qV_q$, resp. $R_{\lambda',Q}^{univ}$ to $R_{\lambda',Q}^h$ and $\psi_{\lambda',q}^{univ}$ to ${}_qV_q^{0,1}$. Let $\mathfrak{m}_{R_{n-1,Q}}$ resp. $\mathfrak{m}_{R_{\lambda',Q}}$ be the maximal ideal of $R_{\lambda',Q}$. Note that we have canonically

$$\mathfrak{m}_{R_{n-1,Q}}/(\mathfrak{m}_\mathcal{O} + \mathfrak{m}_{R_{n-1,Q}}^2) = \mathfrak{m}_{R_{\lambda',Q}}/(\mathfrak{m}_\mathcal{O} + \mathfrak{m}_{R_{\lambda',Q}}^2)$$

and that the k-dual of this space is canonically isomorphic to $\mathrm{H}^1_{\mathcal{L}_Q}(\Gamma_\mathbb{Q}, M)$. Moreover, it follows from 2.14 that

Corollary 2.16. *For any set Q_m as above and for $r = \dim_k \mathrm{H}^1_{\mathcal{L}_\emptyset}(\Gamma_\mathbb{Q}, M)$, one has*

$$\dim_k \mathfrak{m}_{R_{n-1,Q_m}}/(\mathfrak{m}_\mathcal{O} + \mathfrak{m}_{R_{n-1,Q_m}}^2) = \dim_k \mathrm{H}^1_{\mathcal{L}_{Q_m}}(\Gamma_\mathbb{Q}, M) \leq \sharp Q_m = r$$

2.7. **End of the proof.** We first fix a regular dominant weight λ' congruent modulo $p-1$ to $((n-1)a,(n-2)a,\dots,a,0)$. We assume either 1a) or 1b), so that the characters on the diagonal of $\mathrm{Symm}^{n-1}\overline{\rho}_\mu|_{D_d}$ are mutually distinct. We consider the diagram of morphisms

$$\begin{array}{ccc} R_{\lambda',Q_m} & \to & \mathbf{T}_{\lambda',Q_m}(\mathbb{Y}_{Q_m}) \\ \downarrow & & \downarrow \\ R_{\lambda'} & \to & \mathbf{T}_{\lambda'}(U^{0,1},\mathcal{O}) \end{array}$$

The first line is $\mathcal{O}[\Delta_{Q_m}]$-linear. Let $\mathbf{M}_{\lambda',Q_m}$ be the Pontryagin dual of $\mathbb{Y}_{Q_m}$ and $\mathbf{M}_{\lambda',0,Q_m}$ be the Pontryagin dual of the analogue $\mathbb{Y}_{Q_m,0}$ of $\mathbb{Y}_{Q_m}$ obtained by replacing the level group U_{Q_m} by $U_{Q_m,0}$. Similarly, let $\mathbf{M}_{\lambda',\emptyset}$ be the Pontryagin dual of $e\cdot S_{\lambda'}(U^{0,1},E/\mathcal{O})_{\mathfrak{m}}$. We know that $\mathbf{M}_{\lambda',Q_m}$ is free of finite rank over $\mathcal{O}[\Delta_{Q_m}]$ and admits a faithful action of $\mathbf{T}_{\lambda',Q_m}(\mathbb{Y}_{Q_m})$. By [Ge10, Lemma 2.2.6], we have

$$\mathbf{M}_{\lambda',Q_m}/\mathfrak{a}_{Q_m}\mathbf{M}_{\lambda',Q_m}\cong\mathbf{M}_{\lambda',0,Q_m}$$

One also knows that

$$R_{\lambda',Q_m}/\mathfrak{a}_{Q_m}R_{\lambda',Q_m}\cong R_{\lambda'}$$

By Cor.2.14, there are surjections in $CNL_{\mathcal{O}}$:

$$\mathcal{O}[[Y_1,\dots,Y_r]]\to R_{\lambda',Q_m}$$

Let Ψ_m: be the composition

$$\mathcal{O}[[Y_1,\dots,Y_r]]\to R_{\lambda',Q_m}\to R_{\lambda'}$$

We also have surjections

$$\mathcal{O}[[Z_1,\dots,Z_r]]\to\mathcal{O}[\Delta_{Q_m}]$$

whose kernels $\mathfrak{n}_m$ satisfy $\bigcap_m\mathfrak{n}_m=(0)$. We can lift the map

$$\mathcal{O}[[Z_1,\dots,Z_r]]\to\mathcal{O}[\Delta_{Q_m}]\to R_{\lambda',Q_m}$$

to a map

$$\Phi_m\colon\mathcal{O}[[Z_1,\dots,Z_r]]\to\mathcal{O}[[Y_1,\dots,Y_r]]$$

The composition

$$\Psi_m\circ\Phi_m\colon\mathcal{O}[[Z_1,\dots,Z_r]]\to R_{\lambda'}/\mathfrak{m}_{\mathcal{O}}R_{\lambda'}$$

has kernel $(Z_1,\dots,Z_r)+\mathfrak{m}_{\mathcal{O}}$.

On the other hand, it follows from [CHT08, Cor.3.1.5] that we have a Hecke linear isomorphism

$$\mathbf{M}_{\lambda',Q_m,0}\cong\mathbf{M}_{\lambda',\emptyset}$$

so that

$$\mathbf{M}_{\lambda',Q_m}/\mathfrak{a}_{Q_m}\mathbf{M}_{\lambda',Q_m}\cong\mathbf{M}_{\lambda',\emptyset}$$

One can now apply Diamond-Fujiwara's version of the Taylor-Wiles machine (see Th.2.1 of [Dia97] as at the end of the proof of [CHT08, Theorem 3.5.1] to conclude that the morphism

$$R_{\lambda'}\to\mathbf{T}_{\lambda'}(U^{0,1},\mathcal{O})$$

is an isomorphism in $CNL_{\mathcal{O}}$, that $M_{\lambda',\emptyset}$ is free over $\mathbf{T}_{\lambda'}(U^{0,1},\mathcal{O})$ and that these algebras are local complete intersection.

In order to deduce that $R_{n-1}\to\mathbf{T}^u_{n-1}$ is an isomorphism of Λ_{n-1}-algebras and that they are local complete intersection, we proceed as in [Ti06, Sect.3.2]. We choose a regular dominant weight λ' congruent to $((n-1)a,\dots,a,0)$ and we consider the diagram

$$\begin{array}{ccc} R_{n-1}/PR_{n-1} & \to & \mathbf{T}^u_{n-1}/P\mathbf{T}^u_{n-1} \\ \downarrow & & \downarrow \\ R_{\lambda'} & \to & \mathbf{T}_{\lambda'}(U^{0,1},\mathcal{O}) \end{array}$$

where $P=P_{\lambda'}$. We know that the bottom line is an isomorphism and that the first column is an isomorphism. It follows that the first line is an isomorphism. and that $\mathbf{T}^u_{n-1}/P\mathbf{T}^u_{n-1}\cong\mathbf{T}^u_{\lambda'}$. Moreover, by Hida's control theorem $\mathbf{M}_{n-1}/P\mathbf{M}_{n-1}=\mathbf{M}_{\lambda',\emptyset}$; hence $\mathbf{M}_{n-1}/P\mathbf{M}_{n-1}$ is free over $\mathbf{T}^u_{n-1}/P\mathbf{T}^u_{n-1}$. This implies by Nakayama's lemma that $\mathbf{M}_{n-1}$ is free over T^u_{n-1} (use that $\mathbf{M}_{n-1}$ is free over Λ_{n-1} by Hida theory). In particular, $\mathbf{T}^u_{n-1}$ is free over Λ_{n-1}. From this we can deduce by a similar argument that the injectivity of $R_{n-1}/PR_{n-1}\to\mathbf{T}^u_{n-1}/P\mathbf{T}^u_{n-1}$ implies the injectivity of

$R_{n-1} \to \mathbf{T}^u_{n-1}$. Since P is generated by a regular sequence in Λ_{n-1} and since $\mathbf{T}^u_{n-1}/P\mathbf{T}^u_{n-1}$ is local complete intersection over Λ_{n-1}/P, the same holds for $\mathbf{T}^u_{n-1}$ over Λ_{n-1}.

3. Proof of Theorem 1.3

3.1. **The case $j = 3$.** The proof of Theorem 1.3 makes use of the Symm^3 base change from $GL_2(\mathbb{Q})$ to $\mathrm{GSp}_4(\mathbb{Q})$ as established in [RS07]. The level of the Symm^3 of a newform of squarefree Iwahori level is still squarefree Iwahori [RS07]. Let h_2 be the Hida Hecke algebra constructed in [TU99] (see also [H02] or [Ti06]). It is a finite torsion-free algebra over $\Lambda_2 = \mathbb{Z}_p[[X_1, X_2]]$. Calculations detailed in [Con16a, Section 3.3] describe the only possible homomorphism

$$\mathcal{H}(\mathrm{GSp}_4)^{Np} \otimes \mathcal{H}_p(\mathrm{GSp}_4)^{Iw,-} \to \mathcal{H}(\mathrm{GL}_2)^{Np} \otimes \mathcal{H}_p(\mathrm{GL}_2)^{Iw,-}$$

between our abstract Hecke algebras, deduced from the base change map from GL_2 to GSp_4 and compatible with the ordinarity condition. In fact, in [Con16a, Proposition 3.3.5], A. Conti defines eight homomorphisms $\lambda^{Np} \otimes \lambda_{p,i}$, $i = 1, \ldots, 8$ in the context of finite slope case, but only the first is compatible with our ordinarity assumption. It provides a commutative diagram of algebra homomorphisms

$$\begin{array}{ccc} h_2 & \stackrel{\theta}{\to} & h_1 \\ \uparrow & & \uparrow \\ \Lambda_2 & \to & \Lambda_1 \end{array}$$

the bottom homomorphism is induced on the highest weights of local systems by $n \geq 0 \mapsto (a, b)$ where $a \geq b \geq 0$ are given by $a = 2n$ and $b = n$. Here, $n \geq 0$ corresponds to the irreducible representation $\mathrm{Symm}^n\mathrm{St}_2$ of highest weight n of $\mathrm{GL}_2(\mathbb{Q})$ and (a, b) corresponds to the similar Weyl representation of $\mathrm{GSp}_4(\mathbb{Q})$. For any prime ℓ not dividing Np, let the universal genus 2 Hecke polynomial at ℓ

$$P^{(2)}_\ell(X) = X^4 - T_\ell X^3 + \ell(R_\ell + (1 + \ell^2 S_\ell))X^2 - \ell^3 T_\ell S_\ell X + \ell^6 S^3_\ell$$

where the coefficients are given by the universal Hecke operators with the notations of Conjecture 2 Section 7 of [TU99]. Then,the homomorphism θ is defined as follows. For any prime ℓ not dividing Np, it sends the coefficients of $P^{(2)}_\ell(X)$ to those of the symmetric cube $\mathrm{Symm}^3 P^{(1)}_\ell = (X - U^3_\ell)(X - U^2_\ell V_\ell)(X - U_\ell V^2_\ell)(X - V^3_\ell)$ of the genus 1 universal Hecke polynomial $P^{(1)}_\ell = X^2 - T_\ell X + \ell S_\ell = (X - U_\ell)(X - V_\ell)$. For $\ell = p$, let $U_{p,1}$, resp. $U_{p,2}$, be the double class of $\mathrm{diag}(p, p, 1, 1)$, resp. $\mathrm{diag}(p, p^2, p, 1)$, for the Iwahori subbgroup of $\mathrm{GSp}_4(\mathbb{Z}_p)$. Then by standard calculations (see [Con16a, Proposition 3.3.5 and Corollary 3.3.9], we see that θ sends $U_{p,1}$ to U^3_p. and $U_{p,2}$ to U^4_p

Let $\mathbf{T}^s_2$ be the localization of h_2 at the maximal ideal associated to $\mathrm{Symm}^3\overline{\rho}_\mu$; the morphism θ factors through $\mathbf{T}^s_2 \to \mathbf{T}_1$. Let $\lambda = \mu \circ \theta$. Let $\widetilde{\mathbf{T}}^s_2 = \mathbf{T}^s_2 \otimes_{\Lambda_2} \widetilde{A}_1$ Let us still denote by the same letters the homomorphisms obtained by extensions of scalars to $\widetilde{A}_1$:

$$\widetilde{\mathbf{T}}^s_2 \stackrel{\theta}{\to} \widetilde{\mathbf{T}}_1 \stackrel{\mu}{\to} \widetilde{A}_1$$

and their composition $\lambda = \mu \circ \theta$. Let us recall Theorem 4.2 of [Pi12b] (especially, in the context of deformations of a residual representation $\mathrm{Symm}^3\overline{\rho}_\mu$, treated in section 5.8.2 of this paper). Let R_2 be the minimal p-ordinary universal deformation ring of $\mathrm{Symm}^3\overline{\rho}_\mu$.

Theorem 3.1. *Assuming the assumptions* $(*)$ *and that either* $\alpha^{12} \not\equiv 1 \pmod{\mathfrak{m}_{A_1}}$ *or* $3(a+1) < p-1$, *we have* $R_2 = \mathbf{T}^s_2$ *and* $\mathbf{T}^s_2$ *is local complete intersection over* Λ_2 *; moreover it is finite flat over* Λ_2.

Note that the auxiliary level of h_2 is Iwahori of the same squarefree level N as h_1.

Corollary 3.2. *Assuming the assumptions* $(*)$ *and* $3(a+1) < p - 1$, *we have*

1) the homomorphism $\theta \colon \mathbf{T}^s_2 \to \mathbf{T}_1$ *is surjective, moreover*

2) the Λ_1*-algebra* $\widetilde{\mathbf{T}}^s_2$ *is reduced.*

Proof. By absolute irreducibility of $\overline{\rho}$, and of its symmetric cube and by $R_1 = \mathbf{T}_1$ and $R^s_2 = \mathbf{T}^s_2$, we see that the rings $\mathbf{T}^s_2$ and $\mathbf{T}_1$ are generated by the traces. Let $\mathbf{T}'_1$ be the image of $\mathbf{T}^s_2$ by θ; it is a Λ_1-subalgebra of $\mathbf{T}_1$. It contains $U_p = \theta(U_{p,2} \cdot U^{-1}_{p,1})$. For any prime ℓ relatively prime to Np, we have $\mathrm{Tr}\,\mathrm{Symm}^3\rho(\mathrm{Fr}_\ell) = T^3_\ell - 2\ell < \ell >_0 \cdot T_\ell$. Let $\xi_\ell \in \mathbf{T}'_1$ be this quantity. The polynomial $X^3 - 2\ell < \ell >_0 \cdot X - \xi_\ell$ admits a root $T_\ell \in \mathbf{T}_1$. Hence in the residue field $k = \mathbf{T}_1/\mathfrak{m}_{\mathbf{T}_1} = \mathbf{T}'_1/\mathfrak{m}_{\mathbf{T}'_1}$, it has a root t_ℓ. If this root is simple, we can conclude by Hensel's lemma that $T_\ell \in \mathbf{T}_1'$. Let us show that we can find a finite set Σ of primes ℓ's prime to Np such that

$$(*) \quad \mathbf{T}_1 \subset \Lambda_1[U_p, T_\ell, \ell \in \Sigma] + \mathfrak{m}_{\mathbf{T}_1}$$

and such that for each $\ell \in \Sigma$, t_ℓ is simple.

Recall that the subgroup of elements B of $\mathrm{GL}_2(k')$ such that $\det B \in (\mathbb{F}_p)^{a+1}$ is precisely equal by [Ri76] to $\mathrm{Im}\,\overline{\rho}$. Let $\ell \in \Sigma$. Let α and β be the roots in $\overline{k}$ of $X^2 - t_\ell X + \ell^{a+1} = 0$. We have $\overline{\rho}(\mathrm{Fr}_\ell) \sim A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$. The condition $3(\alpha+\beta)^3 - 2\alpha\beta = 0$ implies that $x = \frac{\alpha}{\beta}$ is a root of $3x^2 + 4x + 3 = 0$. If $p = 5$, $x = 1$ is double root; if $p > 5$, the roots are inverse one of the other and are distinct. If $p = 5$ and we do have $\alpha = \beta$, we have $T_\ell \in \Lambda_1 + \mathfrak{m}_{\mathbf{T}_1}$ hence we can omit T_ℓ from our set of generators.

If $p > 5$ and if $x = \frac{\alpha}{\beta}$ is a root of $3x^2 + 4x + 3 = 0$, Let us find two primes ℓ' and ℓ'' such that $T_\ell \in \Lambda_1[T_{\ell'}, T_{\ell''}] \subset \mathbf{T}_1$ and

$$3(T_{\ell'})^2 - 2\ell' < \ell' > \not\equiv 0 \pmod{\mathfrak{m}_{\mathbf{T}_1}} \quad \text{and} \quad 3(T_{\ell''})^2 - 2\ell'' < \ell'' > \not\equiv 0 \pmod{\mathfrak{m}_{\mathbf{T}_1}}.$$

Let $c \in (\mathbb{F}_p^\times)^{a+1}$. In particular, there exists σ' and $\sigma'' \in \Gamma_{\mathbb{Q}}$ such that

$$\overline{\rho}(\sigma') \sim A' = \begin{pmatrix} \alpha & 0 \\ 0 & \beta c \end{pmatrix} \text{and } \overline{\rho}(\sigma'') \sim A'' = \begin{pmatrix} \beta & 0 \\ 0 & \alpha c \end{pmatrix}$$

where for both relations, the conjugation is by the same matrix as for A. Now we choose $c \neq \pm 1$ such that $3(\alpha + \beta c)^2 - 2\alpha\beta c \neq 0$ and $3(\alpha c + \beta)^2 - 2\alpha\beta c \neq 0$. This rules out at most four values. If $p \neq 5$, it is easy to see that $(\mathbb{F}_p^\times)^{a+1}$ has order at least 5 under the assumption $p - 1 > 3(a+1)$. Hence such a c exists. Choose ℓ' and ℓ'' by Chebotarev density such that $\overline{\rho}(\mathrm{Fr}_{\ell'})$, resp. $\overline{\rho}(\mathrm{Fr}_{\ell''})$, belongs to the conjugacy class of σ', resp. σ''. Note that $c + 1 \neq 0$ and

$$T_\ell \equiv (c+1)^{-1} \cdot (T_{\ell'} + T_{\ell''}) \pmod{\mathfrak{m}_{\mathbf{T}_1}}.$$

This shows that we can achieve the inclusion $(*)$ by replacing the set of indexes Σ for our generators T_q by $(\Sigma - \{\ell\}) \cup \{\ell', \ell''\}$. Then, an obvious induction shows that we can assume $\mathbf{T}_1 \subset \Lambda_1[U_p, T_\ell, \ell \in \Sigma]$ with t_ℓ simple for any $\ell \in \Sigma$. We have therefore proved that $\mathbf{T}_1' = \mathbf{T}_1$.

2) The proof of Theorem 3.1 also shows that $\overline{R}_2^s = R_2 \otimes_{\lambda_2} \Lambda_1$ is isomorphic to the Λ_1-algebra $\overline{\mathbf{T}}_2^s$ image of $\mathbf{T}_2^s$ in $\prod_{\lambda'} \mathbf{T}_{\lambda'}(U^{0,1}, \mathcal{O})$ where λ' runs over the set of regulars weights of the form $(2n', n')$ for all integers $n' \geq 0$. This algebra is clearly reduced. By Theorem 3.1, it coincides with $\mathbf{T}_2^s \otimes_{\Lambda_2} \Lambda_1$. By tensoring by $\widetilde{A}_1$, this implies that $\widetilde{\mathbf{T}}_2^s$ is reduced. □

Using Corollary 3.2, we have quasi-splittings of μ, θ and λ:

$$(1) \quad \widetilde{\mathbf{T}}_1 \otimes_{\widetilde{A}_1} \mathcal{K}_1 \cong \mathcal{K}_1 \times \widetilde{\mathbf{T}}'_{\mu, \mathcal{K}_1}$$

$$(2) \quad \widetilde{\mathbf{T}}_2^s \otimes_{\widetilde{A}_1} \mathcal{K}_1 \cong (\widetilde{\mathbf{T}}_1 \otimes_{\widetilde{A}_1} \mathcal{K}_1) \times \widetilde{\mathbf{T}}'_{\theta, \mathcal{K}_1}$$

$$(3) \quad \widetilde{\mathbf{T}}_2^s \otimes_{\widetilde{A}_1} \mathcal{K}_1 \cong \mathcal{K}_1 \times \widetilde{\mathbf{T}}'_{\lambda, \mathcal{K}_1}.$$

Let $\widetilde{\mathbf{T}}'_\theta$, resp. $\widetilde{\mathbf{T}}'_\lambda$, be the image of $\widetilde{\mathbf{T}}_2^s$ by the second projection in (2), resp. in (3). Besides the ideal $\mathfrak{c}_\mu$ already defined, one can define two other congruence ideals :

$$\mathfrak{c}_\theta = \widetilde{\mathbf{T}}_2^s \cap (\widetilde{\mathbf{T}}_1 \times \{0_{\widetilde{\mathbf{T}}'_\theta}\})$$

and

$$\mathfrak{c}_\lambda = \widetilde{\mathbf{T}}_2^s \cap (\widetilde{A}_1 \times \{0_{\widetilde{\mathbf{T}}'_\lambda}\}).$$

Corollary 3.3. *Assume $(*)$ and $3(a+1) < p-1$, then the ideals $\mathfrak{c}_\lambda$, $\mathfrak{c}_\mu$ and $\lambda(\mathfrak{c}_\theta)$ are principal and we have the relation*

$$\mathfrak{c}_\lambda = \mathfrak{c}_\mu \lambda(\mathfrak{c}_\theta)$$

Proof. We know that $\mathbf{T}_1$ and $\mathbf{T}_2^s$ are local complete intersection over Λ_i $(i=1,2)$. By flatness of $\widetilde{A}_1$ over Λ_1, it follows that $\widetilde{\mathbf{T}}_1$ and $\widetilde{\mathbf{T}}_2^s$ are local complete intersection over $\widetilde{A}_1$. Thus, the statement follows from Lemma 8.5 and Lemma 8.8. □

Proposition 3.4. *Under the same assumptions as above, the ideal* $\mathfrak{c}_\lambda$ *is generated by* $\mathrm{Char}(\mathrm{Sel}(\mathrm{Ad}_{\mathfrak{sp}_4}\rho_\mu))$.

Proof. Let $\mathcal{L}$ be the composition of the isomorphism $R_2 \to \mathbf{T}_2^s$ with $\lambda\colon \mathbf{T}_2^s \to A_1$,$\widetilde{R}_2 = R_2 \otimes_{\Lambda_2} \widetilde{A}_1$ and $\widetilde{\mathcal{L}}\colon \widetilde{R}_2 \to \widetilde{A}_1$ the composition of $\mathcal{L} \otimes \mathrm{Id}_{\widetilde{A}_1}$ with the multiplication $A_1 \otimes \widetilde{A}_1 \to \widetilde{A}_1$. By flatness of $\widetilde{A}_1$ over Λ_1, we see that $\widetilde{R}_2$ is local complete intersection over $\widetilde{A}_1$. We first apply 8.7 to see that the principal ideal $\mathfrak{c}_\lambda$ coincides with the reflexive envelope of $\mathrm{Fitt}_0(C_1(\widetilde{\mathcal{L}}, \widetilde{A}_1))$. It remains to see that

$$C_1(\widetilde{\mathcal{L}}, \widetilde{A}_1) \cong \mathrm{Sel}(\mathrm{Ad}_{\mathfrak{sp}_4}\rho_\mu)$$

Let $I = \mathrm{Ker}(\Lambda_2 \to \Lambda_1)$. It is a principal ideal, say $I = (\xi)$. The quotient $R_2' = R_2/\xi R_2$ is local complete intersection over Λ_1 and is the deformation ring of symplectic N-minimal ordinary deformations whose Hodge-Tate weights are of the form $(3h, 2h, h, 0)$. We have $\Omega_{R_2/\Lambda_2} \otimes_{\Lambda_2} \widetilde{A}_1 = \Omega_{R_2'/\Lambda_1} \otimes_{\Lambda_1} \widetilde{A}_1$. By flatness of $\widetilde{A}_1$ over Λ_1, we conclude

$$\Omega_{R_2/\Lambda_2} \otimes_{\Lambda_2} \widetilde{A}_1 = \Omega_{\widetilde{R}_2/\widetilde{A}_1}.$$

Let us now compute the Pontryagin dual of $C_1(\widetilde{\mathcal{L}}, \widetilde{A}_1)$:

$$C_1(\widetilde{\mathcal{L}}, \widetilde{A}_1)^* = \mathrm{Hom}_{R_2}(\Omega_{\widetilde{R}_2/\widetilde{A}_1}, \widetilde{A}_1^*) = \mathrm{Hom}_{R_2}(\Omega_{R_2/\Lambda_2} \otimes_{\Lambda_2} \widetilde{A}_1, \widetilde{A}_1^*)$$

$$= \mathrm{Hom}_{R_2}(\Omega_{R_2/\Lambda_2}, \widetilde{A}_1^*) = \mathrm{Der}_{\Lambda_2}(R_2, \widetilde{A}_1^*)$$

Let $R_2(\widetilde{A}_1^*) = R_2 \oplus \epsilon \cdot \widetilde{A}_1^*$ with $\epsilon^2 = 0$, then

$$\mathrm{Der}_{\Lambda_2}(R_2, \widetilde{A}_1^*) = \mathrm{Hom}_{\mathcal{L}}(R_2(\widetilde{A}_1^*), \widetilde{A}_1^*) = \{\rho \in \mathcal{D}(R_2(\widetilde{A}_1^*)); \rho(\sigma) = (\mathbf{1} + \epsilon \cdot c(\sigma))\rho^{univ}(\sigma)\}/\sim.$$

The map $\rho \mapsto c$ induces an injective map Φ from this set (which happens to be a group) to the group of cohomology classes $[c] \in \mathrm{H}^1(\Gamma_{\mathbb{Q}}, \mathfrak{gl}_n(A_1) \otimes_{A_1} \widetilde{A}_1^*)$.

Recall that by the assumption $\alpha^{12} \not\equiv 1 \pmod{\mathfrak{m}_{A_2}}$ or $3(a+1) < p-1$, the p-ordinarity filtrations are well defined on ρ^{univ}, on $\mathrm{Ad}_{\mathfrak{sp}_4}\mathrm{Symm}^3\,\rho_\mu$ and on $\mathcal{A}_\mu^3$. Moreover, the image of Φ is contained in the subgroup of the cohomology classes such that for ℓ dividing N, $c(I_\ell) = 0$ (this is the N-minimality condition) and such that $c|_{I_p}$ takes values, up to conjugation, in $\mathfrak{n}_n^+(A_1) \otimes_{A_1} \widetilde{A}_1$ where $\mathfrak{n}_n^+(A_1) = Fil^1\mathrm{Ad}\,(\mathrm{Symm}^3\rho_\mu)$ is the upper nilpotent subgroup of $\mathfrak{gl}_n(A_1) = \mathrm{Ad}\,(\mathrm{Symm}^3\rho_\mu)$. This is the min-ord condition. Note that if ρ is upper triangular on I_p it is still so on D_p.

Let us show the surjectivity of Φ. Indeed, any cocycle c define a unique conjugacy class $[\rho]$ of liftings of $\overline{\rho}$We need to check that $[\rho]$ defines a deformation in $\mathcal{D}(R_2(\widetilde{A}_1^*))$; it amounts to verifying that the characters defined by $\rho|_{D_p}$ are in the right order on the full decomposition group. This is imposed by the uniqueness of the p-ordinarity filtration.

□

On the other hand, for $p > 3$, we have a decomposition of $\mathbb{Z}_p$-representations of $\mathrm{GL}_2 : \mathrm{Ad}_{\mathfrak{sp}_4} = \mathcal{A}^1 \oplus \mathcal{A}^3$. This implies a decomposition of minimal p-ordinary Selmer groups

$$\mathrm{Sel}(\mathrm{Ad}_{\mathfrak{sp}_4}\rho_\mu) = \mathrm{Sel}(\mathcal{A}_\mu^1) \oplus \mathrm{Sel}(\mathcal{A}_\mu^3).$$

Since we know that $\mathfrak{c}_\mu = (\mathrm{Char}\,\mathrm{Sel}(\mathcal{A}_\mu^1))$ and $\mathfrak{c}_\lambda = \mathrm{Char}(\mathrm{Sel}(\mathrm{Ad}_{\mathfrak{sp}_4}\rho_\mu))$, we conclude by division

Corollary 3.5. *The ideal* $\lambda(\mathfrak{c}_\theta)$ *is principal generated by* $\mathrm{Char}(\mathrm{Sel}(\mathcal{A}_\mu^3))$.

By definition, the associated primes of $\lambda(\mathfrak{c}_\theta)$ in in $\widetilde{A}_1$ are congruence primes between $\mathrm{Symm}^3(\mu)$ and Siegel families which are not Symm^3 of $GL_2(\mathbb{Q})$ families. Because of the Greenberg-Iwasawa conjecture, it is natural to conjecture that the (not yet constructed) p-adic L function of $\mathcal{A}_\mu^3$ generates the ideal $\lambda(\mathfrak{c}_\theta)$, hence controls the congruences of the above type. This would require a decomposition, not only up to algebraic numbers but up to p-adic units in a number field, of normalized special values at 1 of the specializations f_k in the Hida family, according to the decomposition of the complex L functions

$$L(\mathrm{Ad}_{\mathfrak{sp}_4}(f_k), s) = L(\mathcal{A}^1_{f_k}, s)L(\mathcal{A}^3_{f_k}, s).$$

3.2. **The case $j = 2$.** In order to treat the case $j = 2$, we assume that the integer a associated to $\mathbf{T}_1$ (hence to μ) satisfies $3(a+1) < p-1$. We also use another decomposition of $\mathbb{Z}_p$-representations of GL_2 (valid if $p > 3$) :

$$\mathrm{Ad}_{\mathfrak{sl}_4} = \mathcal{A}^1 \oplus \mathcal{A}^2 \oplus \mathcal{A}^3$$

The general formulas over $\mathbb{Z}_p$ are actually

$$\mathrm{Ad}_{\mathfrak{sl}_m} = \bigoplus_{i=1}^{m-1} \mathcal{A}^i$$

for any $m \geq 2$, provided $p > m-1$, and

$$\mathrm{Ad}_{\mathfrak{sp}_{2m}} = \bigoplus_{j=1}^{m} \mathcal{A}^{2j-1}$$

for any $m \geq 1$ provided $p > 2m-1$.

As already mentioned, the Symm^3 base change to $GL_4(\mathbb{Q})$ is established by Kim. Recall that we fixed a squarefree level N and a prime factor q_1 thereof. As above, we choose an auxiliary imaginary quadratic field in which p and q_1 split. We then choose a degree 16 skew field of center K with second kind involution, which ramifies exactly at those two primes. There exists a unitary group $U(4)$ compact at infinity, quasi split at all inert primes. By Arthur and Clozel, automorphic forms can be transfered from $GL_2(\mathbb{Q})$ to $U(4)$. In his thesis [Ge10], D. Geraghty defined a Hida Hecke algebra h_3^u associated to $U(4)$ which is finite torsion free over the Iwasawa algebra $\Lambda_3 = \mathbb{Z}_p[[X_1, X_2, X_3]]$. At this stage, it is better to write h_2^s for the Hecke algebra for symplectic forms previously denoted h_2, in order to distinguish unitary and symplectic group Hecke algebras. The Symm^3 base change provides a ring homomorphism $h_3^u \to h_1$ with a commutative diagram

$$\begin{array}{ccc} h_3^u & \to & h_1 \\ \uparrow & & \uparrow \\ \Lambda_3 & \to & \Lambda_1 \end{array}$$

The bottom map is induced by $n \mapsto (\lambda_1, \lambda_2, \lambda_3)$ where $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \mathbf{0}$ are given by $\lambda_1 = 3n$, $\lambda_2 = 2n$ and $\lambda_3 = n$. But we need a more precise information about this diagram. For this, we note that the base change from GSp_4 to GL_4 has also been established [Mok14], so that there is also a commutative diagram of ring homomorphisms

$$\begin{array}{ccccccc} h_3^u & \xrightarrow{\theta'} & h_2^s & \xrightarrow{\theta} & h_1 & \xrightarrow{\mu} & A_1 \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \Lambda_3 & \to & \Lambda_2 & \to & \Lambda_1 & = & \Lambda_1 \end{array}$$

where the first bottom arrow is given by $(a, b) \mapsto (\lambda_1, \lambda_2, \lambda_3)$ where $a \geq b \geq 0$ and $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \mathbf{0}$ are given by $\lambda_1 = a + b$, $\lambda_2 = a$ and $\lambda_3 = b$. Let $\mathbf{T}_3^u$ be the localization of h_3^u at the maximal prime corresponding to $\mathrm{Symm}^3\overline{\rho}_\mu$. The morphism θ' factors through $\mathbf{T}_3^u$ and is still denoted as $\theta'\colon \mathbf{T}_3^u \to \mathbf{T}_2^s$. Let $\lambda' = \lambda \circ \theta'$. Let $\widetilde{\mathbf{T}}_3^u = \mathbf{T}_3^u \otimes_{\Lambda_3} \widetilde{A}_1$. We tensorize the morphisms by $\widetilde{A}_1$ (without changing the notation) and we get $\widetilde{A}_1$-algebra homomorphisms

$$\widetilde{\mathbf{T}}_3^u \xrightarrow{\theta'} \widetilde{\mathbf{T}}_2^s \xrightarrow{\lambda} \widetilde{A}_1$$

Let $\mathcal{G}_4 = (\mathrm{GL}_4 \times \mathrm{GL}_1) \rtimes \{1, j\}$ where $j(g, \nu)j^{-1} = (\nu {}^t g^{-1}, \nu)$. Let R_3 be the minimal p-ordinary universal ring of deformations $\rho\colon G_\mathbb{Q} \to \mathcal{G}_4(B)$ of $\mathrm{Symm}^3\overline{\rho}_\mu\colon G_\mathbb{Q} \to GL_4(k)$. By treating a simpler case than in [Ge10], we prove

Theorem 3.6. *Assuming $(*)$ and either $\alpha^{12} \not\equiv 1 \pmod{A_1}$ or $3(a+1) < p-1$, we have $R_3 = \mathbf{T}_3^u$, and this ring is local complete intersection over Λ_3 ; in particular it is finite flat over Λ_3.*

Corollary 3.7. *Assuming the assumptions* $(*)$ *and* $3(a+1)<p-1$*, we have*

1) the homomorphism $\theta'\colon \mathbf{T}_3^u\to\mathbf{T}_2^s$ *is surjective, moreover*

2) the Λ_1*-algebra* $\widetilde{\mathbf{T}}_3^u$ *is reduced.*

Proof. 1) Over the Iwasawa algebras, both rings are generated by the traces of the universal modular representations $\rho_{\mathbf{T}_3^u}$ and $\rho_{\mathbf{T}_2^s}$. As in Section 3.1, we denote by T_ℓ, R_ℓ, S_ℓ the Hecke operators at ℓ in $\mathbf{T}_2^s$. By Chebotarev density theorem, $\mathbf{T}_2^s$ is generated by the T_ℓ's for primes ℓ relatively prime to Np and split in the imaginary quadratic field K. For such an ℓ, we have $\mathrm{Tr}\,\rho_{\mathbf{T}_3^u}(\mathrm{Fr}_\ell)=T_{\ell,1}$ and $\mathrm{Tr}\,\rho_{\mathbf{T}_2^s}(\mathrm{Fr}_\ell)=T_\ell$, hence $\theta'(T_{\ell,1})=T_\ell$ and these elements generate $\mathbf{T}_2^s$ over Λ_2.

2) As in the symplectic case (see Corollary 3.2), the proof given at the end of Section 2.7 shows also that $\overline{R}_3^u=R_3^u\otimes_{\Lambda_3}\Lambda_1$ is isomorphic to the image $\overline{\mathbf{T}}_3^u$ of $\mathbf{T}_3^u$ in $\prod_{\lambda'}\mathbf{T}_{\lambda'}^u(U^{0,1},\mathcal{O})$ where λ' runs over the set of regular weights of the form $(3n',2n',n')$, $n'\geq 0$. Thus $\overline{\mathbf{T}}_3^u$ is clearly reduced. It is isomorphic to $\mathbf{T}_3^u\otimes_{\Lambda_3}\Lambda_1$ by Theorem 3.6.

□

Thus Corollary 3.7 and Corollary 3.2 imply that, assuming $3(a+1)<p-1$, the Λ_1-algebras $\widetilde{\mathbf{T}}_3^u$ and $\widetilde{\mathbf{T}}_2^s$ are reduced and we have quasi-splittings of (λ and) θ' and λ' :

$$(2')\quad \widetilde{\mathbf{T}}_3^u\otimes_{\widetilde{A}_1}\mathcal{K}_1\cong(\widetilde{\mathbf{T}}_2^s\otimes_{\widetilde{A}_1}\mathcal{K}_1)\times\widetilde{\mathbf{T}}'_{\theta',\mathcal{K}_1}$$

$$(3')\quad \widetilde{\mathbf{T}}_3^u\otimes_{\widetilde{A}_1}\mathcal{K}_1\cong\mathcal{K}_1\times\widetilde{\mathbf{T}}'_{\lambda',\mathcal{K}_1}.$$

Let $\widetilde{\mathbf{T}}'_{\theta'}$, resp. $\widetilde{\mathbf{T}}'_{\lambda'}$, be the image of $\widetilde{\mathbf{T}}_3^u$ by the second projection in $(2')$, resp. in $(3')$. Besides the ideal $\mathfrak{c}_\mu$ already defined, one can define two other congruence ideals :

$$\mathfrak{c}_{\theta'}=\widetilde{\mathbf{T}}_3^u\cap(\widetilde{\mathbf{T}}_2^s\times\{0_{\widetilde{\mathbf{T}}'_{\theta'}}\})$$

and

$$\mathfrak{c}_{\lambda'}=\widetilde{\mathbf{T}}_3^u\cap(\widetilde{A}_1\times\{0_{\widetilde{\mathbf{T}}'_{\lambda'}}\}).$$

The formalism of Sections 8.3-8.5 yields the following

Corollary 3.8. *Assume* $(*)$ *and* $3(a+1)<p-1$*, then*

1) the ideals $\mathfrak{c}_{\lambda'}$, $\mathfrak{c}_\lambda\lambda(\mathfrak{c}_{\theta'})$ *and* $\mathfrak{c}_{\theta'}$ *are principal and we have the relation*

$$\mathfrak{c}_{\lambda'}=\mathfrak{c}_\lambda\lambda(\mathfrak{c}_{\theta'}).$$

2) the ideal $\mathfrak{c}_{\lambda'}$ *is generated by* $\mathrm{Char}(\mathrm{Sel}(\mathrm{Ad}_{\mathfrak{sp}_4}\rho_\mu))$,

Proof. Same proof as in Corollary 3.3. □

Note that the associated primes of $\lambda(\mathfrak{c}_{\theta'})$ in $\widetilde{A}_1$ are congruence primes between $\mathrm{Symm}^3(\mu)$ and unitary families which don't come from Siegel families.

Moreover, according to the Greenberg-Iwasawa main conjecture, the ideal $\lambda(\mathfrak{c}_{\theta'})$ should be generated by the (still conjectural) p-adic L function $L_p(\mathcal{A}^2_\mu)$. On the other hand, for $p>3$, we have a decomposition of $\mathbb{Z}_p$-representations of GL_2 : $\mathrm{Ad}_{\mathfrak{sl}_4}=\mathrm{Ad}_{\mathfrak{sp}_4}\oplus\mathcal{A}^2$. This implies a decomposition of minimal p-ordinary Selmer groups

$$\mathrm{Sel}(\mathrm{Ad}_{\mathfrak{sl}_4}\rho_\mu)=\mathrm{Sel}(\mathrm{Ad}_{\mathfrak{sp}_4}\rho_\mu)\oplus\mathrm{Sel}(\mathcal{A}^2_\mu)$$

Since we know that $\mathfrak{c}_\lambda=\mathrm{Char}(\mathrm{Sel}(\mathrm{Ad}_{\mathfrak{sp}_4}\rho_\mu)^*)$, we conclude by division :

Corollary 3.9. *The ideal* $\lambda(\mathfrak{c}_{\theta'})$ *is principal generated by* $\mathrm{Char}(\mathrm{Sel}(\mathcal{A}^2_\mu))^*$.

Proof. Same proof as in Corollary 3.3. □

4. The case $j = 4$

To treat this case, we fix an auxiliary imaginary quadratic field as above and we choose unitary groups $U(4)$ and $U(5)$ which are compact at infinity and with the same local conditions at finite places.

Besides the Symm^3 base change, we also consider the Symm^4 base change from GL_2 (established by H. Kim[Kim03] to GL_5, and by Clozel to $U(5)$). We note the commutative diagram of group schemes over $\mathbb{Z}_p$:

$$\begin{array}{ccc} & \mathrm{GL}_2 & \\ \mathrm{Symm}^3 \swarrow & & \searrow \mathrm{Symm}^4 \\ \mathrm{GSp}_4 & \longrightarrow & \mathrm{GSO}_5 \end{array} \tag{4}$$

where the bottom arrow is the standard $(2:1)$-covering coming from the exceptional isomorphism $\mathrm{GSp}(4) \cong \mathrm{GSpin}_5$. Recall that by definition $\mathrm{GSO}_5 = \mathbb{G}_m \times \mathrm{SO}_5$. Therefore, the adjoint action of $\mathrm{Symm}^4\mathrm{GL}_2$ on $\mathfrak{so}_5$ coincides with the adjoint action of $\mathrm{Symm}^3\mathrm{GL}_2$ on $\mathfrak{sp}_4$. We therefore have the following $\mathbb{Z}_p$-decompositions for the action of $\mathrm{Symm}^4\mathrm{GL}_2$:

$$\mathfrak{so}_5 = \mathcal{A}^1 \oplus \mathcal{A}^3$$

and

$$\mathfrak{sl}_5 = \mathcal{A}^1 \oplus \mathcal{A}^2 \oplus \mathcal{A}^3 \oplus \mathcal{A}^4.$$

Let R_4 be the minimal p-ordinary universal ring of deformations $\rho\colon G_{\mathbb{Q}} \to \mathcal{G}_5(B)$ of $\mathrm{Symm}^4\overline{\rho}_\mu$. Since 5 is odd, one can choose an imaginary quadratic field in which p and q_1 split, a skewfield D split outside $\mathfrak{q}_1$ and qG_1^c and a second kind involution $*$ on D such that $G = U(D, *)$ is definite at ∞ and quasisplit at all inert places. Again, by Geraghty's thesis, the Hida Hecke algebra h_4^u associated to $U(5)$ is finite torsion free Λ_4. Its localization $\mathbf{T}_4^u$ at the maximal ideal associated to $\mathrm{Symm}^4\overline{\rho}_\mu$ satisfies

Theorem 4.1. *Assuming $\alpha^{12} \not\equiv 1 \pmod{\mathfrak{m}_{A_1}}$ or $4(a+1) < p-1$, we have $R_4 = \mathbf{T}_4^u$; this ring is local complete intersection over Λ_4. In particular it is finite flat over Λ_4.*

Note that the universal morphism $R_4 \to R_2$ induced by the diagram (4) gives rise by identification to a non obvious base change morphism $\theta''\colon \mathbf{T}_4^u \to \mathbf{T}_2^s$ above the morphism $\Lambda_4 \to \Lambda_2$ (also induced by the diagram (4)). We prove as in Corollary 3.7 that θ'' is surjective. Let $\widetilde{\mathbf{T}}_4^u = \mathbf{T}_4^u \otimes_{\Lambda_4} \widetilde{A}_1$. We consider the morphisms

$$\widetilde{\mathbf{T}}_4^u \stackrel{\theta''}{\to} \widetilde{\mathbf{T}}_2^s \stackrel{\lambda}{\to} \widetilde{A}_1$$

Let $\lambda'' = \lambda \circ \theta''$. We can define the congruence ideal $\mathfrak{c}_{\lambda''}$ as before by

$$\mathfrak{c}_{\lambda''} = \widetilde{\mathbf{T}}_4^u \cap \left(\widetilde{A}_1 \times \{0_{\widetilde{\mathbf{T}}'_{\lambda''}}\}\right).$$

Corollary 4.2. *Under the assumptions $(*)$ and $4(a+1) < p-1$, the ideal $\mathfrak{c}_{\lambda''}$ is principal, generated by a characteristic power series of* $\mathrm{Sel}(\mathrm{Ad}_{\mathfrak{sl}_5}\mathrm{Symm}^4\rho_\mu)^*$.

Proof. Same proof as in Corollary 3.3. □

We define a new congruence ideal $\mathfrak{c}_{\theta''}$ by

$$\mathfrak{c}_{\theta''} = \widetilde{\mathbf{T}}_4^u \cap \left(\widetilde{\mathbf{T}}_2^s \times \{0_{\widetilde{\mathbf{T}}'_{\theta''}}\}\right).$$

Then,

Corollary 4.3. *We have*

$$\mathfrak{c}_{\lambda''} = \mathfrak{c}_\lambda \lambda(\mathfrak{c}_{\theta''}).$$

Moreover, the congruence ideal $\lambda(\mathfrak{c}_{\theta'})$ divides $\lambda(\mathfrak{c}_{\theta''})$ and

$$\frac{\lambda(\mathfrak{c}_{\theta''})}{\lambda(\mathfrak{c}_{\theta'})} = \mathrm{Char}(\mathrm{Sel}(\mathcal{A}^4_\mu)^*).$$

Proof. We have a decomposition of Selmer groups

$$\mathrm{Sel}(\mathrm{Ad}_{\mathfrak{sl}_5}\mathrm{Symm}^4\rho_\mu) = \mathrm{Sel}(\mathrm{Ad}_{\mathfrak{so}_5}\mathrm{Symm}^4\rho_\mu) \oplus \mathrm{Sel}(\mathcal{A}^2_\mu) \oplus \mathrm{Sel}(\mathcal{A}^4_\mu)$$

□

Of course this suggests that the prime factors in $\widetilde{A}_1$ of the (not yet constructed) p-adic L function $L_p(\mathcal{A}_\mu^4)$ are congruence primes between $\mathrm{Symm}^4\mu$ and forms on $U(5)$ which don't come from $\mathrm{GSp}(4)$ (by the base change given by $\mathrm{GSpin}_5 \to \mathrm{GSO}_5$).

5. Digression: a Kummer type criterion for the non triviality of certain Selmer groups

We keep the notations of the introduction and we assume $(*)$. Let $\mathfrak{p}$ be a prime of $\widetilde{A}_1$. For $j = 3, 2, 4$, consider the condition

$$(S_j) \quad \mathrm{Fitt}_0(\mathrm{Sel}(\mathcal{A}_\mu^j)^*) \subset \mathfrak{p}$$

and the condition

(C_3) there exists a Hida family G of Iwahori level N on GSp_4 which is not the Symm^3 of a Hida family on GL_2 and such that $\mathrm{Symm}^3\mu \equiv G \pmod{\mathfrak{p}}$.

We prove

Theorem 5.1. *Assume $(*)$, then (C_3) implies (S_3) or (S_2) or (S_4).*

Proof. Let $\nu_G\colon \mathbf{T}_2^s \to A_2$ be the Hida family associated to G and $\rho_G\colon \Gamma_\mathbb{Q} \to \mathrm{GSp}_4(A_2)$ be the Galois representation associated to this Hida family. It is well defined because $\overline{\rho}_G = \mathrm{Symm}^3\overline{\rho}_\mu$ is absolutely irreducible by $(*)$. We define $\widetilde{\mathbf{T}}_1 = \mathbf{T}_1 \otimes_{\Lambda_1} \widetilde{A}_1$ and $\widetilde{\mathbf{T}}_2^s = \mathbf{T}_2^s \otimes_{\Lambda_2} \widetilde{A}_1$ and we consider

$$\widetilde{\mathbf{T}}_2^s \xrightarrow{\widetilde{\sigma}_3} \widetilde{\mathbf{T}}_1 \xrightarrow{\widetilde{\mu}} \widetilde{A}_1$$

We can decompose $A_2 \otimes_{\Lambda_2} \widetilde{A}_1$ as a product of domains B_i which are finite extensions of $\widetilde{A}_1$. Let C be the normalisation of B_1. By assumption, $\rho_{G,C} = \rho_G \otimes_{A_2} C$ is not a Symm^3 but there exists a prime ideal $\mathfrak{p}_C$ of C above $\mathfrak{p}$ such that, denoting by ϕ, resp. ι, the homomorphism $\phi\colon C \to C/\mathfrak{p}_C$, resp. $\iota\colon \widetilde{A}_1/\mathfrak{p} \hookrightarrow C/\mathfrak{p}_C$, we have $\phi_*\rho_{G,C} = \iota_*\mathrm{Symm}^3\,\rho_\mu \pmod{\mathfrak{p}}$ up to conjugation in $C/\mathfrak{p}_C$. Note that $\mathfrak{p}_C = \mathrm{Ker}\,\phi$ is a height one prime of the normal ring C. Since $\rho_{G,C}$ is not the Symm^3 of a Hida family on GL_2, it follows by a theorem of Conti [Con16b] that $\mathrm{Im}\,\rho_{G,C}$ contains a congruence subgroup of $\mathrm{GSp}_4(\Lambda_1)$ up to conjugation by an element of $\mathrm{GSp}_4(C)$. In particular, the adjoint action $\mathrm{Ad}^0\,\rho_{G,C}$ on $\mathfrak{sp}_4(C)$ is irreducible while it becomes reducible when one applies ϕ. More precisely:

$$\phi_*(\mathrm{Ad}^0\,\rho_{G,C}) \cong \mathcal{A}_\mu^1(C/\mathfrak{p}_C) \oplus \mathcal{A}_\mu^3(C/\mathfrak{p}_C)$$

Note that $C_{\mathfrak{p}_C}$ is a dvr. Let $K = \mathrm{Frac}(C/\mathfrak{p}_C)$. By [Ri76, Proposition 4.2], there exists a $C_{\mathfrak{p}_C}$-lattice L in $\mathfrak{sp}_4(\mathrm{Frac}(C))$ with Galois action such that on the quotient $L/\mathfrak{p}_C L$, the action of Galois is a non trivial extension of $\mathcal{A}_\mu^1(K)$ by $\mathcal{A}_\mu^3(K)$. But $\mathrm{Hom}(\mathcal{A}_\mu^1(K), \mathcal{A}_\mu^3(K)) = \mathcal{A}_\mu^3(K) \oplus \mathcal{A}_\mu^2(K) \oplus \mathcal{A}_\mu^4(K)$ as $\Gamma_\mathbb{Q}$-modules. Since we are dealing with N-minimal p-ordinary Selmer groups, the non triviality of one of the three Selmer groups over K follows. □

6. The case $j = n$

It follows from [PT02] that for any n with $1 \leq n < p-1$, we have a decomposition $\mathfrak{sl}_{n+1} = \bigoplus_{j=1}^n \mathcal{A}^j$ over $\mathbb{Z}_p$ which is GL_2-equivariant for the action on the left hand side by $\mathrm{Ad}\,\mathrm{Symm}^n$. It follows that for $n \in [1, p-1[$, we have

$$\mathfrak{sl}_{n+1} = \mathfrak{sl}_n \oplus \mathcal{A}^n$$

where GL_2 acts by $\mathrm{Ad}\,\mathrm{Symm}^n$ on the left hand side and by $\mathrm{Ad}\,\mathrm{Symm}^{n-1}$ on the first factor or the right-hand side.

From now on, we take $j = n-1$ or $j = n$ Let h_j^u be the Hida Hecke algebra associated to the unitary group $U(j+1)$ chosen to be compact at infinity and with local conditions af finite primes as before. This algebra is finite torsion free over the Iwasawa algebra Λ_j in j variables. Let us assume the automorphic base change is established for Symm^j for $j = n-1$ and $j = n$. This gives rise to algebra homomorphisms $\theta_j\colon h_j^u \to h_1$ above the homomorphism $\Lambda_j \to \Lambda_1$ induced by $m \geq 0 \mapsto (x_1, \cdots, x_j)$ with $x_1 = m$, $x_2 = 2m$,...,$x_j = jm$. Let $\mu\colon h_1 \to A_1$ be a Hida family and let $\lambda_j = \mu \circ \theta_j$, for $j = n-1, n$. We assume that the image of the residual representation $\overline{\rho}_\mu$ contains $\mathrm{SL}_2(\mathbb{F}_p)$. Let $\mathbf{T}_j^u$ be the localization of h_j^u at the maximal ideal associated to $\mathrm{Symm}^j\overline{\rho}_\mu$. Let R_j be the universal deformation ring for minimal p-ordinary lifts $\rho\colon G_{\mathbb{Q}} \to \mathcal{G}_{j+1}(B)$ of $\mathrm{Symm}^j\overline{\rho}_\mu$. We have proven in Section 2

Theorem 6.1. *Assuming* $(*)$*, and* $\alpha^{2d_n} \not\equiv 1 \pmod{\mathfrak{m}}_{A_1}$ *or* $n(a+1) < p-1$*, and that the transfer* Symm^j *is established for* $j = n-1, n$*, then for* $j = n-1, n$*, we have* $R_j = \mathbf{T}_j^u$ *and these rings are local complete intersection over* Λ_j*.*

One shows exactly as in Corollary 3.2 the

Corollary 6.2. *Assuming* $(*)$ *and* $\max(n(a+1), 2(n-1)) < p-1$*,* $\mathbf{T}_j^u \otimes_{\Lambda_j} \Lambda_1$ *is reduced and the morphisms* $\mathbf{T}_j^u \to \mathbf{T}_1$ *for* $j = n-1, n$ *are sujective.*

From this it follows that for $j = n-1, n$, the congruence modules $\mathfrak{c}_{\lambda_j}$ and $\mathfrak{c}_{\theta_j}$ are principal and related by the relation

$$\mathfrak{c}_{\lambda_j} = \lambda_j(\mathfrak{c}_{\theta_j})\mathfrak{c}_\mu$$

Moreover, if one assumes moreover that $\alpha^{2c_n} \not\equiv 1 \pmod{\mathfrak{m}}_{A_1}$, it also follows from the theorem that for $j = n-1, n$, the ideal $\mathfrak{c}_{\lambda_j}$ is generated by $\mathrm{Char}\left((\mathrm{Sel}(\mathrm{Ad}\,\mathrm{Symm}^j\rho_\mu))^*\right)$. Since we have

$$\mathrm{Sel}(\mathrm{Ad}\,\mathrm{Symm}^j\rho_\mu)) = \mathrm{Sel}(\mathrm{Ad}\,\mathrm{Symm}^{j-1}\rho_\mu)) \oplus \mathrm{Sel}(\mathcal{A}_\mu^j),$$

we deduce by passing to the characteristic power series of the Pontryagin duals that

$$\mathfrak{c}_{\lambda_n} = \mathfrak{c}_{\lambda_{n-1}}\left(\mathrm{Char}(\mathrm{Sel}\,\mathcal{A}_\mu^n)\right)$$

and dividing by the invertible ideal $\mathfrak{c}_\mu$, we conclude that $\lambda_{n-1}(\mathfrak{c}_{\theta_{n-1}})$ divides $\lambda_n(\mathfrak{c}_{\theta_n})$ and that the quotient is the principal ideal $\left(\mathrm{Char}(\mathrm{Sel}\,\mathcal{A}_\mu^n)\right)$.

This theorem applies to $n = 4, 5, 6, 7, 8$ since the transfers Symm^j, $j = 3, 4, 5, 6, 7, 8$ of a classical form of weight ≥ 2 have been established in [CT15]. For $n = 4$, we obtain a different proof of the Theorem 4.3 given in 4 relating congruences on $U(4)$ and $U(5)$. In that case, the congruence ideals refer to transfers from $\mathrm{GSp}(4)$ to $U(4)$ and $U(5)$, while here they refer to the congruences between Symm^3, resp. Symm^4 transfers and families on $U(4)$ resp. $U(5)$. For $n \geq 5$, there is no alternative proof because there is no known transfer from GSp_4 to $U(n+1)$ compatible to Symm^3 and Symm^n.

The meaning of this is that any congruence prime between $\mathrm{Symm}^{n-1}\mu$ and a family of $U(n)$-forms which are not Symm^{n-1} from GL_2 is also a congruence prime between $\mathrm{Symm}^n\mu$ and a family of $U(n+1)$-forms which are not Symm^n from GL_2. However, it doesn't seem that one can define a cuspidal base change from $U(n)$ to $U(n+1)$ which would explain this phenomenon by Tate–Hida formalism. It appears for the moment only as a consequence of our congruence ideal main conjecture theorem.

7. The case of the standard representation of GSp(4)

Let N be squarefree, prime to p. We consider the cuspidal Hida-Hecke algebra h_2^s of auxiliary level $\Gamma_{Iw}(N)$ Let us consider a Hida family σ of Siegel cusp forms that is, a Λ_2-algebra homomorphism $\sigma\colon h_2^s \to A_2$ onto a domain A_2 which is finite and torsion free over Λ_2. We assume that it is N-minimal :

(N-Min) for any prime ℓ dividing N, $\overline{\rho}_\sigma(I_\ell)$ contains a regular unipotent element.

Let $\widetilde{A}_2$ be the normal closure of A_2. It is not necessarily flat over Λ_2. Let K an imaginary quadratic field in which the prime p and all the primes dividing N split. As we noted above, using the base change from $\mathrm{GSp}(4)$ to $GL(4)$ established in [Mok14] and Clozel's descent to $U(4)$, we constructed a morphism $\theta'\colon h_3^u \to h_2^s$. Let $\mathbf{T}_3^u$, resp. $\mathbf{T}_2^s$ be the localization of the rings h_3^u resp. h_2^s at the maximal ideals associated to $\overline{\rho}_\sigma$. We localize the morphism θ' at these maximal ideals. We write $\theta'\colon \mathbf{T}_3^u \to \mathbf{T}_2^s$ for its localization. It is over the homomorphism from $\Lambda_3 = \mathcal{O}[[Y_1, Y_2, Y_3]]$ to $\Lambda_2 = \mathcal{O}[[X_1, X_2]]$ which sends $1 + Y_1 \mapsto (1 + X_1)(1 + X_2)$, $Y_2 \mapsto X_1$, $Y_3 \mapsto X_2$; for a further application, let us note that in particular, the inverse image of the prime ideal $P_{-1,-1} = (1 + X_1 - u^{-1}, 1 + X_2 - u^{-1})$ is $P_{-2,-1,-1} = (1 + Y_1 - u^{-2}, 1 + Y_2 - u^{-1}, 1 + Y_3 - u^{-1})$. Let $\widetilde{\mathbf{T}}_3^u = \mathbf{T}_3^u \otimes_{\Lambda_3} \widetilde{A}_2$ and $\widetilde{\mathbf{T}}_2^s = \mathbf{T}_2^s \otimes_{\Lambda_2} \widetilde{A}_2$. we set $\lambda = \sigma \circ \theta'$. Let $\widetilde{\mathbf{T}}_3^u = \mathbf{T}_3^u \otimes_{\Lambda_3} \widetilde{A}_2$ and $\widetilde{\mathbf{T}}_2^s = \mathbf{T}_2^s \otimes_{\Lambda_2} \widetilde{A}_2$. Let $\widetilde{\theta}'\colon \widetilde{\mathbf{T}}_3^u \to \widetilde{\mathbf{T}}_2^s$ and $\widetilde{\sigma}\colon \widetilde{\mathbf{T}}_2^s \to \widetilde{A}_2$ the $\widetilde{A}_2$-algebra homomorphisms obtained by extension of scalars. We put $\widetilde{\lambda} = \widetilde{\sigma} \circ \widetilde{\theta}'$.

Let $\alpha = \sigma(U_{p,1})$, β' such that $\sigma(U_{p,2}) = \alpha\beta'$ and $\gamma' = (\beta')^{-1}$, and $\delta' = \alpha^{-1}$. These four elements belong to $A_2^\times$.

We assume that the residual Galois representation $\overline{\rho}_\sigma$ is absolutely irreducible and that $(RFR^{(2)})$ the four elements α, β', γ' and δ' of A_2 are mutually distinct modulo $\mathfrak{m}_{A_2}$.

By ordinarity the restriction of $\overline{\rho}_\sigma$ to a decomposition group D_p at p is conjugate to

$$\begin{pmatrix} \mathrm{unr}(\overline{\alpha}) & * & * & * \\ & \mathrm{unr}(\overline{\beta'})\omega^{-(a_2+1)} & * & * \\ & & \mathrm{unr}(\overline{\gamma'})\omega^{-(a_1+2)} & * \\ & & & \mathrm{unr}(\overline{\delta'})\omega^{-()} \end{pmatrix}$$

for a pair of integers $a_1 \geq a_2 \geq 0$. Assume the residual Galois image is big : either Im $\overline{\rho}_\sigma$ contains $\mathrm{Sp}_4(k')$ or $\mathrm{Symm}^3\mathrm{SL}_2(k') \subset \mathrm{Im}\ \overline{\rho}_\sigma \subset k^\times \mathrm{Symm}^3\mathrm{GL}_2(k')$.

If the residual image is big (in the sense of [Pi12b] Section 5.8), and that the four Hecke eigenvalues at p are distinct modulo $\mathfrak{m}_{A_1}$. Then, let R_i for $i = 2, 3$ be the minimal p-ordinary universal deformation rings of $\overline{\rho}_\sigma$ (for deformations into $\mathrm{GSp}_4(B)$ resp. $\mathcal{G}_4(B)$), we can prove

Theorem 7.1. *Assume that ρ_σ has residual big image, is N-minimal and that $(RFR^{(2)})$ holds; then we have $R_2 = \mathbf{T}_2^s$ and $R_3 = \mathbf{T}_3^u$, and the rings $\mathbf{T}_2^s$ resp. $\mathbf{T}_3^u$ is local complete intersection over Λ_2 resp. Λ_3.*

As in Corollary 3.7 and its proof, this implies that

Corollary 7.2. *The homomorphism $\mathbf{T}_3^u \to \mathbf{T}_2^s$ is surjective.*

We can define three congruence ideals $\mathfrak{c}_\lambda$, $\mathfrak{c}_\sigma$ $\mathfrak{c}'_\theta$. Because of the assumption $(RFR^{(2)})$, we see as in the proof of Proposition 3.4 that the differential module $\Omega_{\mathbf{T}_3^u/\Lambda_2} \otimes_{\mathbf{T}_3^u} \widetilde{A}_2$ is isomorphic to $\mathrm{Sel}(\mathrm{Ad}_{\mathfrak{sl}_4}\rho_\sigma))^*$ and that similarly $\Omega_{\mathbf{T}_2^s/\Lambda_2} \otimes_{\mathbf{T}_2^s} \widetilde{A}_2$ is isomorphic to $\mathrm{Sel}(\mathrm{Ad}_{\mathfrak{sp}_4}\rho_\sigma))^*$. Hence by Theorem 8.7, we conclude that $\widetilde{\mathfrak{c}_\lambda} = \mathrm{Char}\left((\mathrm{Sel}(\mathrm{Ad}_{\mathfrak{sl}_4}\rho_\sigma))^*\right)$ and $\widetilde{\mathfrak{c}_\sigma} = \mathrm{Char}\left((\mathrm{Sel}(\mathrm{Ad}_{\mathfrak{sp}_4}\rho_\sigma))^*\right)$. We also have the transfer formula of Proposition 8.14 :

$$\widetilde{\mathfrak{c}_\lambda} = \widetilde{\mathfrak{c}_\sigma}\widetilde{\sigma(\mathfrak{c}_{\theta'})}.$$

On the other hand, we have

$$\mathrm{Ad}_{\mathfrak{sl}_4}(\rho_\sigma) = \mathrm{Ad}_{\mathfrak{sp}_4}(\rho_\sigma) \oplus \mathrm{St}_\sigma$$

where St_σ is the composition of σ with $\mathrm{St}\colon \mathrm{GSp}_4 \to \mathrm{GSO}_5$. From this and Proposition 8.14 (for $\nu = 2$) and Theorem 8.15, we conclude

Theorem 7.3. *The reflexive envelope $\widetilde{\sigma(\mathfrak{c}_{\theta'})}$ of the "non base change" congruence ideal $\sigma(\mathfrak{c}_{\theta'})$ of $\widetilde{A}_2$ is principal and is generated by* $\mathrm{Char}((\mathrm{Sel}(\mathrm{St}_\sigma))^*)$.

Remark 7.4. *1) A p-adic standard L function $L_p(\mathrm{St}_\sigma)$ associated to the Hida family σ has been constructed by Z. Liu* [Liu18] *(her work includes the case of an arbitrary genus g). The main conjecture implies that $\sigma(\mathfrak{c}_{\theta'})$ is generated by $L_p(\mathrm{St}_\sigma)$. It is natural to ask whether the height one prime factors of $L_p(\mathrm{St}_\sigma)$ in $\widetilde{A}_2$ are congruence primes between σ and families on $U(4)$ which don't come from* $\mathrm{GSp}(4)$.

2) Let ξ be the quadratic character associated to the imaginary quadratic field K defining the unitary group $U(4)$. Following the method of [Liu18]*, X. Zhang* [Zh18] *proved that for a cusp Siegel Hecke eigensystem σ, the normalized special value $L^{norm}(\mathrm{St}(\sigma) \otimes \xi, 1)$ is in and generates, under certain assumptions, another non base change congruence ideal between the Theta lift of σ to $U(4)$ and "non Theta lift" Hecke eigensystems on $U(4)$. Note however that this Theta lift is not the same functoriality as the one used in Theorem 7.3 above. Indeed, the Theta lift involves a twist by ξ while no such twist occurs in our functoriality. The Bloch-Kato conjecture suggests that in the case of* [Zh18]*, one should have* $(L^{norm}(\mathrm{St}_\sigma \otimes \xi, 1)) = \mathrm{Char}\,\mathrm{Sel}(\mathrm{St}(\rho_\sigma) \otimes \xi)$.

We give below an application of this theorem. Let us take $a_1 = a_2 = p-2$. Let $f \in S_{2,2}(\Gamma_{Iw}(N))$ be a Siegel cusp form of weight $(2,2)$. Let $\rho_f \colon \Gamma_\mathbb{Q} \to \mathrm{GSp}_4(\mathcal{O})$ be the associated Galois representation. Assume the residual Galois image is big : either Im $\overline{\rho}_f$ contains $\mathrm{Sp}_4(k')$ or $\mathrm{Symm}^3\mathrm{SL}_2(k') \subset \mathrm{Im}\ \overline{\rho}_f \subset k^\times\mathrm{Symm}^3\mathrm{GL}_2(k')$ for a subfield k' of k, and that $\overline{\rho}$ is N-minimal (the image of the inertia at any prime dividing N contains a regular unipotent element). Let α, β, γ,δ be the Satake parameters at p ordered in such a way that $ord_p(\alpha)=\mathrm{ord}_p(\beta)=0$ and $\mathrm{ord}_p(\gamma)=\mathrm{ord}_p(\delta)=1$, with $\alpha\delta=\beta\gamma$. Assume that α, β, γ/p,δ/p belong to $\mathcal{O}$ and are mutually distinct modulo ϖ_E. There are two such orderings. We fix one. This choice fixes a p-ordinarity filtration of $\rho_f|_{D_p}$ and of $\overline{\rho}_f|_{D_p}$ so that the the minimal p-ordinary Selmer group of $\mathrm{St}_f=\mathrm{St}\circ\rho_f$ is well defined. Let f^{st} be the p-stabilization of f such that $f^{st}|U_{p,1}=\alpha f^{st}$ and $f^{st}|U_{p_2}=\alpha\beta f^{st}$. There exists a family $\sigma\colon h_2^s\to A_2$ whose specialization in weight $(a_1,a_2)=(-1,-1)$ gives the Hecke eigenvalues of f^{st}. More precisely, there exists an arithmetic prime $\mathfrak{P}_{-1,-1}$ of A_2 above $P_{-1,-1}$. After extending the coefficient ring $\mathcal{O}$, we may and do assume that $A_2/\mathfrak{P}_{-1,-1}=\mathcal{O}$. Let $\mathcal{T}_3^u=\mathbf{T}_3^u\otimes_{\Lambda_3}A_2$ and $\mathcal{T}_2^s=\mathbf{T}_2^s\otimes_{\Lambda_2}A_2$ We still write $\sigma\colon\mathcal{T}_2^s\to A_2$, $\theta'\colon\mathcal{T}_3^u\to\mathcal{T}_2^s$ and $\lambda=\sigma\circ\theta'$. Let $\mathcal{T}_{1,2,2}^u=\mathcal{T}_3^u/\mathfrak{P}_{-1,-1}\mathcal{T}_3^u$ and $\mathcal{T}_{2,2}^s=\mathcal{T}_2^s/\mathfrak{P}_{-1,-1}\mathcal{T}_2^s$. Let us explain the notations. We fix a compact Harris-Taylor Shimura variety for $U(3,1)$ [HaT01] and we let $eV_3^u(N,\mathcal{O})$ be the $\mathcal{O}$-module of ordinary p-adic automorphic forms on $U(3,1)$ of auxiliary Iwahori level N. Similarly, we let $eV_2^s(N,\mathcal{O})$ be the $\mathcal{O}$-module of ordinary p-adic automorphic forms on GSp(4) of auxiliary Iwahori level N. Let similarly $eV_{1,2,2}^u(N,\mathcal{O})$ be the space of ordinary p-adic automorphic forms on $U(3,1)$ of Iwahori level N and automorphic weight $(1,2,2)$, resp. $eV_{2,2}^s(N,\mathcal{O})$ the space of ordinary p-adic automorphic forms on GSp(4) of Iwahori level N and automorphic weight $(2,2)$. Note that $eS_{2,2}^s(N,\mathcal{O})\subset eV_{2,2}^s(N,\mathcal{O})$ is not an equality in general. By [H02, Th.1.1 (6) and Th.6.7 (6)] , we have exact control:

$$eV_2^s(N,\mathcal{O})[P_{-1,-1}]=eV_{2,2}^s(N,\mathcal{O}),\quad eV_3^u(N,\mathcal{O})[P_{-2,-1,-1}]=eV_{1,2,2}^u(N,\mathcal{O})$$

On the other hand, it follows from [Pi12b, Th.7.1] that the algebra $\mathcal{T}_{2,2}^s(N,\mathcal{O})$ is local complete intersection over $\mathcal{O}$. Indeed, it is a quotient of the local complete intersection Λ_2-algebra $\mathbf{T}_2^s(N,\mathcal{O})$ by the regular sequence $((1+X_1)-u^{-1},(1+X_2)-u^{-1})$ and acts faithfully on $eV_{2,2}^s(N,\mathcal{O})$. The same method proves similarly that the algebra $\mathcal{T}_{1,2,2}^u(N,\mathcal{O})$ is local complete intersection over $\mathcal{O}$ and acts faithfully on $eV_{1,2,2}^u(N,\mathcal{O})$. However, it is not known whether these algebras are reduced because they may not act by normal operators on the spaces of p-adic modular forms.

We consider $\sigma_{2,2}=\sigma \pmod{\mathfrak{P}_{-1,-1}}$, $\theta'_{2,2}=\theta' \pmod{\mathfrak{P}_{-1,-1}}$ and $\lambda_{1,2,2}=\theta'_{1,2,2}\circ\sigma_{2,2}$. These morphisms give rise to the diagram

$$\mathcal{T}_{1,2,2}^u(N,\mathcal{O})\to\mathcal{T}_{2,2}^s(N,\mathcal{O})\to\mathcal{O}.$$

Let $\mathbf{T}_{2,2}^s(Np,\mathcal{O})$ denotes the Iwahori level Hecke algebra acting faithfully on the space of classical Siegel cusp forms of Iwahori level Np and automorphic weight $(2,2)$ and let

$$\sigma_{f^{st}}\colon\mathbf{T}_{2,2}^s(Np,\mathcal{O})\to\mathcal{O}$$

be the eigensystem associated to f^{st}. Composing with the surjection $\pi_{2,2}\colon\mathcal{T}_2^s(N,\mathcal{O})\to\mathbf{T}_{2,2}^s(Np,\mathcal{O})$, we have $\sigma_{2,2}=\sigma_{f^{st}}\circ\widetilde{\pi}_{2,2}$. Note that $\pi_{2,2}$ might have a non zero kernel due to non classical holomorphic Siegel cusp forms of cohomological weight $(2,2)$. We can nevertheless define the congruence ideal $\sigma_{2,2}(\mathfrak{c}_{\theta'_{1,2,2}})$.

Corollary 7.5. *Assume that $\overline{\rho}_f$ satisfies $(*)^{(2)}$ and $(RFR^{(2)})$ as above. Assume that the rings $\mathcal{T}_{2,2}^s(N,\mathcal{O})$ and $\mathcal{T}_{1,2,2}^u(N,\mathcal{O})$ are reduced, then the $\mathcal{O}$-length of* $\mathrm{Sel}(\mathrm{St}_f)$ *is equal to the ϖ-adic valuation of the congruence ideal $\sigma_{2,2}(\mathfrak{c}_{\theta'_{1,2,2}})$.*

This ideal $\sigma_{2,2}(\mathfrak{c}_{\theta'_{1,2,2}})$ measures congruences between the transfer of f to $U(3,1)$ and p-adic automorphic forms of automorphic weight $(1,2,2)$ and Iwahori level N on $U(3,1)$. Note that the space of classical automorphic forms of weight $(1,2,2)$ and Iwahori level Np is null, since $1<2$.

Proof. By [Pi12b, Th.7.1], the algebra $\mathcal{T}_{2,2}^s(N,\mathcal{O})$ is isomorphic to the universal deformation ring $R_{2,2}^s$ of GSp(4)-valued minimal ordinary deformations of $\overline{\rho}_f$ of Hodge-Tate weights $0,0,1,1$. One can prove similarly that the algebra $\mathcal{T}_{1,2,2}^u(N,\mathcal{O})$ is isomorphic to the universal deformation ring $R_{1,2,2}^s$ of $\mathcal{G}_4$-valued minimal ordinary deformations of $\overline{\rho}_f$ viewed in $\mathcal{G}_4(k)$, of Hodge-Tate weights $0,0,1,1$. In particular, $\mathcal{O}/\mathfrak{c}_{\lambda_{1,2,2}}$ has same length as $\mathrm{Sel}(\mathrm{Ad}_{\mathfrak{sl}_4}\rho_f)$. and $\mathcal{O}/\mathfrak{c}_{\sigma_{2,2}}$ has same length as $\mathrm{Sel}(\mathrm{Ad}_{\mathfrak{sp}_4}\rho_f)$. Recall that $\mathrm{Ad}_{\mathfrak{sl}_4}\rho_f=\mathrm{Ad}_{\mathfrak{sp}_4}\rho_f\oplus\mathrm{St}_f$. On the other hand, the rings $\mathcal{T}_{1,2,2}^u(N,\mathcal{O})$ and $\mathcal{T}_{2,2}^s(N,\mathcal{O})$ being local complete intersection over $\mathcal{O}$, we can apply Corollary 8.6 and conclude that $\mathfrak{c}_{\lambda_{1,2,2}}=\mathfrak{c}_{\sigma_{2,2}}\cdot\sigma_{2,2}(\mathfrak{c}_{\theta'_{1,2,2}})$. We then obtain the result by division as before. $\square$

This corollary applies in particular to modular abelian surfaces

Corollary 7.6. *Let A be a modular abelian surface defined over $\mathbb{Q}$, ordinary at p of squarefree conductor N (p prime to N). Assume the Galois representation on $A[p]$ has big image and is N-minimal. Assume that the rings $\mathcal{T}^s_{2,2}(N,\mathbb{Z}_p)$ and $\mathcal{T}^u_{1,2,2}(N,\mathbb{Z}_p)$ are reduced. Let T_pA be the Tate module and $S_pA \subset \bigwedge^2 T_pA$ the associated rank 5 Galois representation. Then the cardinality of* $\mathrm{Sel}(S_pA)$ *spans the $\mathbb{Z}_p$-ideal $\sigma_{2,2}(\mathfrak{c}_{\theta'_{1,2,2}})$.*

8. Congruence ideal formalism

We recall a formalism developed by Hida based on Tate's appendix to [MR70], alongside we introduce the notion of congruence modules and differential modules for general rings and basic facts about it. We apply the theory to Hecke algebras and deformation rings to show that these two torsion modules have the same size (that is, the equal characteristic ideals and Fitting ideals).

8.1. **Differentials.** We recall here the definition of 1-differentials and some of their properties for our later use. Let R be a A-algebra, and suppose that R and A are objects in CNL_W, where W is a finite flat extenion of $\mathbb{Z}_p$ The module of 1-differentials $\Omega_{R/A}$ for a A-algebra R ($R, A \in CNL_W$) indicates the module of **continuous** 1-differentials with respect to the profinite topology.

For a module M with continuous R-action (in short, a continuous R-module), let us define the module of A-*derivations* by

$$\mathrm{Der}_A(R,M) = \left\{ \delta : R \to M \in \mathrm{Hom}_A(R,M) \,\middle|\, \begin{array}{c} \delta\text{: continuous} \\ \delta(ab) = a\delta(b) + b\delta(a) \\ \text{for all } a,b \in R \end{array} \right\}.$$

Here the A-linearity of a derivation δ is equivalent to $\delta(A) = 0$, because

$$\delta(1) = \delta(1 \cdot 1) = 2\delta(1) \Rightarrow \delta(1) = 0.$$

Then $\Omega_{R/A}$ represents the covariant functor $M \mapsto \mathrm{Der}_A(R,M)$ from the category of **continuous** R-modules into MOD.

The construction of $\Omega_{R/A}$ is easy. Let $R\widehat{\otimes}_A R$ be the completion of $R \otimes_A R$ with respect to the $(\mathfrak{m}_R \otimes_A R + R \otimes_A \mathfrak{m}_R)$-adic topology. The multiplication $a \otimes b \mapsto ab$ induces a A-algebra homomorphism $m : R\widehat{\otimes}_A R \to R$ taking $a \otimes b$ to ab. We put $I = \mathrm{Ker}(m)$, which is an ideal of $R\widehat{\otimes}_A R$. Then we define $\Omega_{R/A} = I/I^2$. We endow it with a structure of R-module by action of $R\widehat{\otimes}1$. It is a complete module for the $\mathfrak{m}_R$-topology. One checks that the map $d : R \to \Omega_{R/A}$ given by $d(a) = a \otimes 1 - 1 \otimes a \mod I^2$ is a continuous A-derivation. Thus we have a morphism of functors: $\mathrm{Hom}_R(\Omega_{R/A}, ?) \to \mathrm{Der}_A(R, ?)$ given by $\phi \mapsto \phi \circ d$. Since $\Omega_{R/A}$ is generated by $d(R)$ as R-modules (left to the reader as an exercise), the above map is injective. To show that $\Omega_{R/A}$ represents the functor, we need to show the surjectivity of the above map, which is well known (see [CRT,]).

Proposition 8.1. *The above morphism of two functors $M \mapsto \mathrm{Hom}_R(\Omega_{R/A}, M)$ and $M \mapsto \mathrm{Der}_A(R,M)$ is an isomorphism, where M runs over the category of complete R-modules. In other words, for each A-derivation $\delta : R \to M$, there exists a unique R-linear homomorphism $\phi : \Omega_{R/A} \to M$ such that $\delta = \phi \circ d$.*

We have the following fundamental exact sequences:

Corollary 8.2. *Let the notation be as in the proposition.*

(i) *Suppose that A is a C-algebra for an object $C \in CL_W$. Then we have the following natural exact sequence:*

$$\Omega_{A/C}\widehat{\otimes}_A R \longrightarrow \Omega_{R/C} \longrightarrow \Omega_{R/A} \to 0.$$

(ii) *Let $\pi : R \twoheadrightarrow C$ be a surjective morphism in CL_W, and write $J = Ker(\pi)$. Then we have the following natural exact sequence:*

$$J/J^2 \xrightarrow{\beta^*} \Omega_{R/A}\widehat{\otimes}_R C \longrightarrow \Omega_{C/A} \to 0.$$

Moreover if $A = C$, then $J/J^2 \cong \Omega_{R/A}\widehat{\otimes}_R C$.

For any continuous R-module M, we write $R[M]$ for the R-algebra with square zero ideal M. Thus $R[M] = R \oplus M$ with the multiplication given by

$$(r \oplus x)(r' \oplus x') = rr' \oplus (rx' + r'x).$$

It is easy to see that $R[M] \in CNL_W$, if M is of finite type, and $R[M] \in CL_W$ if M is a p-profinite R-module. By definition,

$$\mathrm{Der}_A(R, M) \cong \left\{\phi \in \mathrm{Hom}_{A-alg}(R, R[M]) \middle| \phi \mod M = \mathrm{id}\right\}, \tag{8.1}$$

where the map is given by $\delta \mapsto (a \mapsto (a \oplus \delta(a))$. Note that $i : R \to R\widehat{\otimes}_A R$ given by $i(a) = a \otimes 1$ is a section of $m : R\widehat{\otimes}_A R \to R$. We see easily that $R\widehat{\otimes}_A R/I^2 \cong R[\Omega_{R/A}]$ by $x \mapsto m(x) \oplus (x - i(m(x)))$. Note that $d(a) = 1 \otimes a - i(a)$ for $a \in R$.

8.2. **Congruence and differential modules.** Let R be an algebra over a normal noetherian domain A. We assume that R is an A-flat module of finite type. Let $\phi : R \to A$ be an A-algebra homomorphism. We define

$$C_1(\phi; A) = \Omega_{R/A} \otimes_{R,\phi} \mathrm{Im}(\phi)$$

which we call the *differential* module of ϕ . We have seen (for instance Corollary 3.3, (2)) that if R is a deformation ring, this module is the dual of the associated adjoint Selmer group. If ϕ is surjective, we just have

$$C_1(\phi; A) = \Omega_{R/A} \otimes_{R,\phi} A.$$

We suppose that R is reduced (having zero nilradical of R). Then the total quotient ring $\mathrm{Frac}(R)$ can be decomposed uniquely into $\mathrm{Frac}(R) = \mathrm{Frac}(\mathrm{Im}(\phi)) \oplus X$ as an algebra direct product. Write 1_ϕ for the idempotent of $\mathrm{Frac}(\mathrm{Im}(\phi))$ in $\mathrm{Frac}(R)$. Let $\mathfrak{a} = \mathrm{Ker}(R \to X) = (1_\phi R \cap R)$, $S = \mathrm{Im}(R \to X)$ and $\mathfrak{b} = \mathrm{Ker}(\phi)$. Here the intersection $1_\phi R \cap R$ is taken in $\mathrm{Frac}(R) = \mathrm{Frac}(\mathrm{Im}(\phi)) \oplus X$. Then we put

$$C_0(\phi; A) = (R/\mathfrak{a}) \otimes_{R,\phi} \mathrm{Im}(\phi) \cong \mathrm{Im}(\phi)/(\phi(\mathfrak{a})) \cong 1_\phi R/\mathfrak{a} \cong S/\mathfrak{b} \cong R/(\mathfrak{a} \oplus \mathfrak{b}),$$

which is called the *congruence* module of ϕ but is actually a ring (cf. [H88] Section 6). We can split the isomorphism $1_\phi R/\mathfrak{a} \cong S/\mathfrak{b}$ as follows: First note that $\mathfrak{a} = (R \cap (1_\phi R \oplus 0))$ in $\mathrm{Frac}(\mathrm{Im}(\phi)) \oplus X$. Then $\mathfrak{b} = (0 \oplus X) \cap R$, and we have

$$1_\phi R/\mathfrak{a} \cong R/(\mathfrak{a} \oplus \mathfrak{b}) \cong S/\mathfrak{b},$$

where the maps $R/(\mathfrak{a} \oplus \mathfrak{b}) \to 1_\phi R/\mathfrak{a}$ and $R/(\mathfrak{a} \oplus \mathfrak{b}) \to S/\mathfrak{b}$ are induced by two projections from R to $1_\phi R$ and S.

Write $K = \mathrm{Frac}(A)$. Fix an algebraic closure $\overline{K}$ of K. Since the spectrum $\mathrm{Spec}(C_0(\phi; A))$ of the congruence ring $C_0(\phi; A)$ is the scheme theoretic intersection of $\mathrm{Spec}(\mathrm{Im}(\phi))$ and $\mathrm{Spec}(R/\mathfrak{a})$ in $\mathrm{Spec}(R)$:

$$\mathrm{Spec}(C_0(\lambda; A)) = \mathrm{Spec}(\mathrm{Im}(\phi)) \cap \mathrm{Spec}(R/\mathfrak{a}) := \mathrm{Spec}(\mathrm{Im}(\phi)) \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R/\mathfrak{a}),$$

we conclude that

Proposition 8.3. *Let the notation be as above. Then a prime $\mathfrak{p}$ is in the support of $C_0(\phi; A)$ if and only if there exists an A-algebra homomorphism $\phi' : R \to \overline{K}$ factoring through $R/\mathfrak{a}$ such that $\phi(a) \equiv \phi'(a) \mod \mathfrak{p}$ for all $a \in R$.*

In other words, ϕ mod $\mathfrak{p}$ factors through $R/\mathfrak{a}$ and can be lifted to ϕ'. Therefore, if A is the integer ring of a sufficiently large number field in $\overline{\mathbb{Q}}$, $\bigcup_\phi \mathrm{Supp}(C_0(\phi; A))$ is made of primes dividing the absolute different $\mathfrak{d}(R/\mathbb{Z})$ of R over $\mathbb{Z}$, and each prime appearing in the absolute discriminant of $R/\mathbb{Z}$ divides the order of the congruence module for some ϕ.

By Corollary 8.2 applied to the exact sequence: $0 \to \mathfrak{b} \to R \xrightarrow{\phi} A \to 0$, we know that

$$C_1(\phi; A) \cong \mathfrak{b}/\mathfrak{b}^2. \tag{8.2}$$

Since $C_0(\phi; A) \cong S/\mathfrak{b}$, we may further define *higher congruence modules* by $C_n(\phi; A) = \mathfrak{b}^n/\mathfrak{b}^{n+1}$.

8.3. **Transfer property of congruence modules.** Let B be a normal profinite local domain of characteristic p residue field. We suppose to have a sequence of surjective B-algebra homomorphisms: $R \xrightarrow{\theta} S \xrightarrow{\mu} A$ of reduced local rings finite flat over B. We put $\lambda = \mu \circ \theta : R \to A$. We assume that R, S, A are all Gorenstein rings over B. This means that

$$\mathrm{Hom}_B(R, B) \cong R,\ \mathrm{Hom}_B(S, B) \cong S \text{ and } \mathrm{Hom}_B(A, B) \cong A \text{ as } R\text{-modules.} \tag{8.3}$$

We write $B = \Lambda$. Since R is reduced, the total quotient ring $Q(R)$ of R is a product of fields, and we have $Q(R) = Q_S \oplus Q(S)$ for the complementary semi-simple algebra Q_S. Let R_S be the projection of R in Q_S. We have the following (unique) decomposition

(1) $\mathrm{Spec}(R) = \mathrm{Spec}(R_S) \cup \mathrm{Spec}(S)$, union of closed subschemes inducing $R \hookrightarrow (R_S \oplus S)$ with Λ-torsion module $C_0(\theta, S) := (R_S \oplus S)/R$.

Similarly, we have $Q(S) = Q_A \oplus Q(A)$ and $Q(R) = Q'_A \oplus Q(A)$ as algebra direct sums. Write S_A (resp. R_A) for the projected image of S (resp. R) in Q_A (resp. Q'_A). Then we have

(2) $\mathrm{Spec}(S) = \mathrm{Spec}(S_A) \cup \mathrm{Spec}(A)$, union of closed subschemes inducing $S \hookrightarrow (S_A \oplus A)$ with Λ-torsion module $C_0(\mu, A) := (S_A \oplus A)/S$.

(3) $\mathrm{Spec}(R) = \mathrm{Spec}(R_A) \cup \mathrm{Spec}(A)$, union of closed subschemes inducing $R \hookrightarrow (R_A \oplus A)$ with Λ-torsion module $C_0(\lambda, A) := (R_A \oplus A)/R$.

By [H88, Lemma 6.3] (or [MFG, §5.3.3]), we get the following isomorphisms of R-modules:

$$C_0(\lambda; A) \cong R_A \otimes_R A,\ C_0(\theta; S) \cong R_S \otimes_R S \text{ and } C_0(\mu; A) \cong S_A \otimes_S A. \tag{8.4}$$

Write $\pi_S : R \twoheadrightarrow R_S$ and $\pi : R \to S$ for the two projections and $(\cdot,\cdot)_R : R \times R \to B$ and $(\cdot,\cdot)_S : S \times S \to B$ for the pairing giving the self-duality (8.3). We recall [H86, Lemma 1.6]:

Lemma 8.4. *The S-ideal $\mathrm{Ker}(\pi_S : R \to R_S)$ is principal and S-free of rank 1.*

Proof. Let $\mathfrak{b} = \mathrm{Ker}(\theta : R \to S)$ and $\mathfrak{a} = \mathrm{Ker}(\pi_S : R \to R_S)$. By assumption, R and S are B-free of finite rank; so, $\mathfrak{b}$ is B-free, and by duality, we have an exact sequence $0 \to S^* \xrightarrow{\theta^*} R^* \to \mathfrak{b}^* \to 0$. Note that $\mathfrak{b}^*$ is naturally an R_S-module which is free of finite rank over B. Thus identifying $S = S^*$ and $R^* = R$ by (8.3), we have $\theta^*(S^*) = \{r \in R; r \cdot \mathfrak{b} = 0\} = (Q(S) \oplus 0) \cap R = \mathfrak{a}$; hence θ^* induces $S = S^* \cong \mathfrak{a}$. □

Recall the following fact first proved in [H88, Theorem 6.6]:

Lemma 8.5. *We have the following exact sequence of R-modules:*

$$0 \to C_0(\mu; A) \to C_0(\lambda; A) \to C_0(\theta; S) \otimes_S A \to 0.$$

Proof. Write $M^* = \mathrm{Hom}_B(M, B)$ as an R-module for a R-module M. Note that

$\mathrm{Ker}(\theta) = R \cap (R_S \oplus 0) \subset R_S \oplus S$, $\mathrm{Ker}(\lambda) = R \cap (R_A \oplus 0) \subset R_A \oplus A$ and $\mathrm{Ker}(\mu) = S \cap (S_A \oplus 0) \subset S_A \oplus A$.

From an exact sequence $0 \to \mathrm{Ker}(\theta) \to R \to S \to 0$, we have the following commutative diagram with exact rows (for $\mathfrak{a} = \mathrm{Ker}(\pi_S : R \to R_S)$):

$$\begin{tikzcd}
S^* \arrow[r, hook] \arrow[d, "\wr"'] & R^* \arrow[r, two heads] \arrow[d, "\wr"'] & \mathrm{Ker}(\theta)^* \arrow[d] \\
S \cong \mathfrak{a} \arrow[r, hook] & R \arrow[r, two heads] & R_S,
\end{tikzcd}$$

which shows $R_S \cong \mathrm{Ker}(\theta)^* = ((R_S \oplus 0) \cap R)^*$ as R-modules. Similarly, we get $\mathrm{Ker}(\lambda)^* \cong R_A$, $\mathrm{Ker}(\mu)^* \cong S_A$ We have a commutative diagram with exact rows:

$$\begin{tikzcd}
\mathrm{Ker}(\theta) \arrow[r] \arrow[d] & R \arrow[r, "\theta"] \arrow[d, "\lambda"'] & S \arrow[r] \arrow[d, "\mu"'] & 0 \\
0 \arrow[r] & A \arrow[r, equal] & A \arrow[r] & 0
\end{tikzcd}$$

Applying the snake lemma, we get an exact sequence of R-modules:

$$0 \to \mathrm{Ker}(\theta) \to \mathrm{Ker}(\lambda) \to \mathrm{Ker}(\mu) \to 0.$$

By B-freeness of A and S, all the terms of the above exact sequence are B-free. Thus the above sequence is split as a sequence of B-modules, and we have the dual exact sequence:

$$\begin{array}{ccccc} \mathrm{Ker}(\mu)^* & \xrightarrow{\hookrightarrow} & \mathrm{Ker}(\lambda)^* & \xrightarrow{\twoheadrightarrow} & \mathrm{Ker}(\theta)^* \\ \wr\downarrow & & \wr\downarrow & & \wr\downarrow \\ S_A & \xrightarrow[\hookrightarrow]{} & R_A & \xrightarrow[\twoheadrightarrow]{} & R_S. \end{array}$$

Tensoring with A over R, from (8.4), we get an exact sequence:

$$\mathrm{Tor}^1_R(R_S, A) \to C_0(\mu; A) \to C_0(\lambda; A) \to C_0(\theta; S) \otimes_R A \to 0.$$

Thus we need to show the vanishing: $\mathrm{Tor}^1_R(R_S, A) = 0$. To see this, we recall $\mathfrak{a} \cong S$. Thus the exact sequence $\mathfrak{a} \hookrightarrow R \twoheadrightarrow R_S$ can be rewritten as $S \hookrightarrow R \twoheadrightarrow R_S$. Tensoring with A over R, we get an exact sequence

$$0 = \mathrm{Tor}^1_R(R, A) \to \mathrm{Tor}^1_R(R_S, A) \to S \otimes_R A \xrightarrow{\alpha} R \otimes_R A \to R_S \otimes_R A \to 0.$$

Since we have a commutative diagram:

$$\begin{array}{ccc} S \otimes_R A & = & A \\ \alpha\downarrow & & \downarrow \\ R \otimes_R A & = & A \end{array}$$

and $\mathrm{Coker}(\alpha)$ is a torsion A-module, α is a nontrivial A-linear map of the integral domain A into itself; so, α is injective, and we conclude $\mathrm{Tor}^1_R(R_S, A) = 0$ as desired. □

By (8.4), the three congruence modules $C_0(\mu; A), C_0(\lambda; A), C_0(\theta; S) \otimes_R A$ are residue rings of R; so, cyclic A-modules. Moreover, by Lemma 8.4, they are the ring A modulo principal ideals. Write their generators as $Ac_\lambda = A \cap R \subset (R_A \oplus A)$, $Ac_\mu = A \cap S \subset (S_A \oplus A)$ and $Sc_\theta = S \cap R \subset (R_S \oplus S)$. Thus we have $C_0(\lambda; A) = A/c_\lambda A$, $C_0(\mu; A) = A/c_\mu A$ and $C_0(\theta; S) \otimes_S A = A/\lambda(c_\theta)A$ for the image $\lambda(c_\theta) \in A$ of $c_\theta \in S$. By the above lemma, we conclude the following result:

Corollary 8.6. *We have $\lambda(c_\theta) \cdot c_\mu = c_\lambda$ up to units in A; so, for the ideals $\mathfrak{c}_?$ generated by $c_?$, we have $\lambda(\mathfrak{c}_\theta) \cdot \mathfrak{c}_\mu = \mathfrak{c}_\lambda$.*

Note here $\mathfrak{c}_?$ (resp. $\lambda(\mathfrak{c}_\theta)$) is the annihilator $\mathrm{Ann}_A(C_0(?; A))$ (resp. $\mathrm{Ann}_A(C_0(?; S) \otimes_S A)$ of $C_0(?; A)$ in A, and $\mathfrak{c}_\theta$ is the annihilator $\mathrm{Ann}_S(C_0(\theta; A))$ of $C_0(\theta; S)$ in S.

8.4. **Local complete intersections.** Let A be a complete normal local domain (for example, a complete regular local rings like $A = W$ or $A = W[[T]]$ or $A = W[[T_1, \dots, T_r]]$ (power series ring)). Any local A-algebra R free of finite rank over A has a presentation $R \cong A[[X_1, \dots, X_n]]/(f_1, \dots, f_m)$ for $f_i \in A[[X_1, \dots, X_n]]$ with $m \geq n$. If $m = n$, then R is called a *local complete intersection* over A. Note that if B is a complete normal local domain which is finite flat over A, the extension $R \otimes_A B$ of an A-algebra R which is local complete intersection over A is local complete intersection over B. There is a theorem of Tate giving the identity of the Fitting ideals of the differential module and the congruence module for local complete intersection rings. To introduce this, let us explain the notion of pseudo-isomorphisms between torsion A-modules (see [BCM, VII.4.4] for a more detailed treatment). For two A-modules M, N of finite type, a morphism $\phi : M \to N$ is called a pseudo isomorphism if the annihilator of $\mathrm{Ker}(\phi)$ and $\mathrm{Coker}(\phi)$ each has height at least 2 (i.e., the corresponding closed subscheme of $\mathrm{Spec}(A)$ has co-dimension at least 2). If $A = W$, a pseudo-isomorphism is an isomorphism, and if $A = W[[T]]$, it is an isogeny (having finite kernel and cokernel). The classification theorem of torsion A-modules M of finite type tells us that we have a pseudo isomorphism $M \to \bigoplus_i A/\mathfrak{f}_i$ for finitely many reflexive ideal $0 \neq \mathfrak{f}_i \in A$. An ideal $\mathfrak{f}$ is *reflexive* if $\mathrm{Hom}_A(\mathrm{Hom}_A(\mathfrak{f}, A), A) \cong \mathfrak{f}$ canonically as A-modules (and equivalently $\mathfrak{f} = \bigcap_{\lambda \in A, (\lambda) \supset \mathfrak{f}} (\lambda)$; i.e., close to be principal). Then the *characteristic ideal* $\mathrm{Char}(M)$ of M is defined by $\mathrm{Char}(M) := \prod_i \mathfrak{f}_i \subset A$. If A is a unique factorization domain (for example, if A is regular; a theorem of Auslander-Buchsbaum [CRT, Theorem 20.3]), any reflexive ideal is principal. If $A = W$, then $|W/\mathrm{Char}(M)|_p = \big||M|\big|_p$, and if further $A = \mathbb{Z}_p$, we have $\mathrm{Char}(M) = (|M|)$.

Theorem 8.7 (J. Tate). *Assume that R is a local complete intersection over a complete normal noetherian local domain A with an algebra homomorphism $\lambda : R \to A$. If after tensoring the quotient field Q or A, $R \otimes_A Q = (\mathrm{Im}(\lambda) \otimes_A Q) \oplus S$ as algebra direct sum for some Q-algebra S, then $C_j(\lambda; A)$ is a torsion A-module of finite type, and we have*

$$\mathrm{Ann}_A(C_0(\lambda; A)) = \mathrm{Char}(C_0(\lambda; A)) = \mathrm{Char}(C_1(\lambda; A)).$$

For the reader's convenience, we shall give a proof of this theorem in the following subsection. Actually we prove

$$\mathrm{length}_A(C_0(\lambda; A)) = \mathrm{length}_A(C_1(\lambda; A)), \tag{8.5}$$

assuming that A is a discrete valuation ring (see Proposition 8.12). If A is a normal noetherian domain, $\mathrm{Char}_A(M) = \prod_P P^{\mathrm{length}_{A_P} M_P}$ for the localization M_P at height 1-primes P for a given A-torsion module M. Since A_P is a discrete valuation ring if and only if P has height 1, this implies the above theorem.

8.5. **Proof of Tate's theorem.** We reproduce the proof from [MR70, Appendix] (which actually determines the Fitting ideal of M more accurate than $\mathrm{Char}(M)$). We prepare some preliminary results; so, we do not assume yet that R is a local complete intersection over A. Let A be a normal noetherian integral domain of characteristic 0 and R be a reduced A–algebra free of finite rank r over A. The algebra R is called a *Gorenstein* algebra over A if $\mathrm{Hom}_A(R, A) \cong R$ as R–modules. Since R is free of rank r over A, we choose a base $(x_1, \ldots, x_r)$ of R over A. Then for each $y \in R$, we have $r \times r$–matrix $\rho(y)$ with entries in A defined by $(yx_1, \ldots, yx_r) = (x_1, \ldots, x_r)\rho(y)$. Define $\mathrm{Tr}(y) = \mathrm{Tr}(\rho(y))$. Then $\mathrm{Tr} : R \to A$ is an A–linear map, well defined independently of the choice of the base. Suppose that $\mathrm{Tr}(xR) = 0$. Then in particular, $\mathrm{Tr}(x^n) = 0$ for all n. Therefore all eigenvalues of $\rho(x)$ are 0, and hence $\rho(x)$ and x is nilpotent. By the reducedness of R, $x = 0$ and hence the pairing $(x, y) = \mathrm{Tr}(xy)$ on R is non-degenerate.

Lemma 8.8. *Let A be a normal noetherian integral domain of characteristic* 0 *and R be an A–algebra. Suppose the following three conditions:*

(1) *R is free of finite rank over A;*
(2) *R is Gorenstein; i.e., we have $i : \mathrm{Hom}_A(R, A) \cong R$ as R–modules;*
(3) *R is reduced.*

Then for an A–algebra homomorphism $\lambda : R \to A$, we have

$$C_0(\lambda; A) \cong A/\lambda(i(\mathrm{Tr}_{R/A}))A.$$

In particular, $\mathrm{length}_A C_0(\lambda; A)$ *is equal to the valuation of $d = \lambda(i(\mathrm{Tr}_{R/A}))$ if A is a discrete valuation ring.*

Proof. Let $\phi = i^{-1}(1)$. Then $\mathrm{Tr}_{R/A} = \delta\phi$. The element $\delta = \delta_{R/A}$ is called the different of R/A. Then the pairing $(x, y) \mapsto \mathrm{Tr}_{R/A}(\delta^{-1}xy) \in A$ is a perfect pairing over A, where $\delta^{-1} \in S = Frac(R)$ and we have extended $\mathrm{Tr}_{R/A}$ to $S \to K = Frac(A)$. Since R is commutative, $(xy, z) = (y, xz)$. Decomposing $S = K \oplus X$, we have

$$C_0(\lambda; A) = \mathrm{Im}(\lambda)/\lambda(\mathfrak{a}) \cong A/R \cap (K \oplus 0).$$

Then it is easy to conclude that the pairing (,) induces a perfect A–duality between $R \cap (K \oplus 0)$ and $A \oplus 0$. Thus $R \cap (K \oplus 0)$ is generated by $\lambda(\delta) = \lambda(i(\mathrm{Tr}_{R/A}))$. □

Next we introduce two A–free resolutions of R, in order to compute $\delta_{R/A}$. We start slightly more generally. Let X be an algebra. A sequence $f = (f_1, \ldots, f_n) \in X^n$ is called *regular* if $x \mapsto f_j x$ is injective on $X/(f_1, \ldots, f_{j-1})$ for all $j = 1, \ldots, n$. We now define a complex $K^\bullet_X(f)$ (called *the Koszul complex*) out of a regular sequence f (see [CRT, Section 16]). Let $V = X^n$ with a standard base $e_1, \ldots, e_n$. Then we consider the exterior algebra

$$\bigwedge{}^{\bullet} V = \bigoplus_{j=0}^{n} (\wedge^j V).$$

The graded piece $\wedge^j V$ has a base $e_{i_1,\dots,i_j} = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_j}$ indexed by sequences $(i_1, \dots, i_j)$ satisfying $0 < i_1 < i_2 < \cdots < i_j \le n$. We agree to put $\bigwedge^0 V = X$ and $\bigwedge^j V = 0$ if $j > n$. Then we define X–linear differential $d : \bigwedge^j X \to \bigwedge^{j-1} X$ by

$$d(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_j}) = \sum_{r=1}^{j} (-1)^{r-1} f_{i_r} e_{i_1} \wedge \cdots \wedge e_{i_{r-1}} \wedge e_{i_{r+1}} \wedge \cdots \wedge e_{i_j}.$$

In particular, $d(e_j) = f_j$ and hence,

$$\bigwedge^0 V / d(\bigwedge^1 V) = X/(f).$$

Thus, $(K_X^\bullet(f), d)$ is a complex and X–free resolusion of $X/(f_1, \dots, f_n)$. We also have

$$d_n(e_1 \wedge e_2 \wedge \cdots \wedge e_n) = \sum_{j=1}^{n} (-1)^{j-1} f_j e_1 \wedge \cdots \wedge e_{j-1} \wedge e_{j+1} \wedge \cdots \wedge e_n.$$

Suppose now that X is a B–algebra. Identifying $\bigwedge^{n-1} V$ with V by

$$e_1 \wedge \cdots \wedge e_{j-1} \wedge e_{j+1} \wedge \cdots \wedge e_n \mapsto e_j$$

and $\bigwedge^n V$ with X by $e_1 \wedge e_2 \wedge \cdots \wedge e_n \mapsto 1$, we have

$$\mathrm{Im}(d_n^* : \mathrm{Hom}_B(\bigwedge^{n-1} V, Y) \to \mathrm{Hom}_B(\bigwedge^n V, Y)) \cong (f)\mathrm{Hom}_B(X, Y),$$

where $(f)\mathrm{Hom}_B(X, Y) = \sum_j f_j \mathrm{Hom}_B(X, Y)$, regarding $\mathrm{Hom}_B(X, Y)$ as an X–module by $y\phi(x) = \phi(xy)$. This shows that if X is an B–algebra free of finite rank over B, $K_X^\bullet(f)$ is a B–free resolution of $X/(f)$, and

$$\mathrm{Ext}_B^n(X/(f), Y) = H^n(\mathrm{Hom}_B(K_X^\bullet(f), Y)) \cong \frac{\mathrm{Hom}_B(X, Y)}{(f)\mathrm{Hom}_B(X, Y)} \tag{8.6}$$

for any B–module Y.

We now suppose that R is a local complete intersection over A. Thus R is free of finite rank over A and $R \cong B/(f_1, \dots, f_n)$ for $B = A[[T_1, \dots, T_n]]$. Write t_j for $T_j \mod (f_1, \dots, f_n)$ in R. Since R is local, t_j are contained in the maximal ideal $\mathfrak{m}_R$ of R. We consider $C = B \otimes_A R \cong R[[T_1, \dots, T_n]]$. Then

$$R = R[[T_1, \dots, T_n]]/(T_1 - t_1, \dots, T_n - t_n),$$

and $g = (T_1 - t_1, \dots, T_n - t_n)$ is a regular sequence in $C = R[[T_1 \dots, T_n]]$. Since C is B–free of finite rank, the two complexes $K_B^\bullet(f) \twoheadrightarrow R$ and $K_C^\bullet(g) \twoheadrightarrow R$ are B–free resolutions of R.

We have a Λ–algebra homomorphism $\Phi : B \hookrightarrow C$ given by $\Phi(x) = x \otimes 1$. We extend Φ to $\Phi^\bullet : K_B^\bullet(f) \to K_C^\bullet(g)$ in the following way. Write $f_i = \sum_{j=1}^n b_{ij} g_j$. Then we define $\Phi^1 : K_B^1(f) \to K_C^1(g)$ by $\Phi^1(e_i) = \sum_{j=1}^n b_{ij} e_j$. Then $\Phi^j = \bigwedge^j \Phi^1$. One can check that this map $\Phi^\bullet$ is a morphism of complexes. In particular,

$$\Phi_n(e_1 \wedge \cdots \wedge e_n) = \det(b_{ij}) e_1 \wedge \cdots \wedge e_n. \tag{8.7}$$

Since $\Phi^\bullet$ is the lift of the identity map of R to the B–projective resolutions $K_B^\bullet(f)$ and $K_C^\bullet(g)$, it induces an isomorphism of extension groups computed by $K_C^\bullet(g)$ and $K_B^\bullet(f)$:

$$\Phi^* : H^\bullet(\mathrm{Hom}_B(K_C^\bullet(g), B)) \cong \mathrm{Ext}_B^j(R, B) \cong H^\bullet(\mathrm{Hom}_B(K_B^\bullet(f), B)).$$

In particular, identifying $\bigwedge^n B^n = B$, we have from (8.6) that

$$H^n(\mathrm{Hom}_B(K_B^\bullet(f), B)) = \mathrm{Hom}_B(B, B)/(f)\mathrm{Hom}_B(B, B) = B/(f) = R$$

and similarly

$$H^n(\mathrm{Hom}_B(K_C^\bullet(g), B)) = \frac{\mathrm{Hom}_B(C, B)}{(g)\mathrm{Hom}_B(C, B)}.$$

The isomorphism between R and $\frac{\mathrm{Hom}_B(C,B)}{(g)\mathrm{Hom}_B(C,B)}$ is induced by Φ_n which is a multiplication by $d = \det(b_{ij})$ (see (8.7)). Thus we have

Lemma 8.9. *Suppose that R is a local complete intersection over A. Let $\pi : B = A[[T_1, \dots, T_n]] \twoheadrightarrow R$ be the projection as above. We have an isomorphism:*

$$h : \frac{\mathrm{Hom}_B(C, B)}{(T_1 - t_1, \dots, T_n - t_n)\mathrm{Hom}_B(C, B)} \cong R$$

given by $h(\phi) = \pi(\phi(d))$ for $d = \det(b_{ij}) \in C$.

We have a base change map:

$$\iota : \mathrm{Hom}_A(R, A) \longrightarrow \mathrm{Hom}_B(C, B) = \mathrm{Hom}_B(B \otimes_A R, B \otimes_A A),$$

taking ϕ to $\mathrm{id} \otimes \phi$. Identifying C and B with power series rings, $\iota(\phi)$ is just applying the original ϕ to coefficients of power series in $R[[T_1, \dots, T_n]]$. We define $I = h \circ \iota : \mathrm{Hom}_A(R, A) \to R$.

Lemma 8.10. *Suppose that R is a local complete intersection over A. Then the above map I is an R–linear isomorphism, satisfying $I(\phi) = \pi(\iota(\phi(d))$. Thus the ring R is Gorenstein.*

Proof. We first check that I is an R–linear map. Since $I(\phi) = \pi(\iota(\phi(d))$, we compute $I(\phi \circ b))$ and $rI(\phi)$ for $b \in B$ and $r = \pi(b)$. By definition, we see

$$I(\pi(bx)) = \pi(\iota(\phi(r \otimes 1)d)) \text{ and } rI(\phi) = \pi(b\iota(\phi(d)).$$

Thus we need to check $\pi(\iota(\phi)((r \otimes 1 - 1 \otimes b)d)) = 0$. This follows from:

$$r \otimes 1 - 1 \otimes b \in (g) \text{ and } \det(b_{ij})g_i = \sum_i b'_{ij} f_i,$$

where b'_{ij} are the (i, j)–cofactors of the matrix (b_{ij}). Thus I is R–linear. Since $\iota \mod \mathfrak{m}_B$ for the maximal ideal $\mathfrak{m}_B$ of B is a surjective isomorphism from

$$\mathrm{Hom}_A((A/\mathfrak{m}_A)^r, A/\mathfrak{m}_A) = \mathrm{Hom}_A(R, A) \otimes_A A/\mathfrak{m}_A$$

onto

$$\mathrm{Hom}_B((B/\mathfrak{m}_B)^r, B/\mathfrak{m}_B) = \mathrm{Hom}_B(C, B) \otimes_B B/\mathfrak{m}_B,$$

the map ι is non-trivial modulo $\mathfrak{m}_C$. Thus $I \mod \mathfrak{m}_R$ is non-trivial. Since h is an isomorphism, $\mathrm{Hom}_B(C, B) \otimes_C C/\mathfrak{m}_C$ is 1–dimensional, and hence $I \mod \mathfrak{m}_R$ is surjective. By Nakayama's lemma, I itself is surjective. Since the target and the source of I are A–free of equal rank, the surjectivity of I tells us its injectivity. This finishes the proof. □

Corollary 8.11. *Suppose that R is a local complete intersection over A. We have $I(\mathrm{Tr}_{R/A}) = \pi(d)$ for $d = \det(b_{ij})$, and hence the different $\delta_{R/A}$ is equal to $\pi(d)$.*

Proof. The last assertion follows from the first by $I(\phi) = \pi(\iota(\phi(d)))$. To show the first, we choose dual basis $x_1, \dots, x_r$ of R/A and $\phi_1, \dots, \phi_r$ of $\mathrm{Hom}_A(R, A)$. Thus for $x \in R$, writing $xx_i = \sum_i a_{ij}x_j$, we have $\mathrm{Tr}(x) = \sum_i a_{ii} = \sum_i \phi_i(xx_i) = \sum_i x_i\phi_i(x)$. Thus $\mathrm{Tr} = \sum_i x_i\phi_i$.

Since x_i is also a base of C over B, we can write $d = \sum_j b_j x_i$ with $\iota(\phi_i)(d) = b_i$. Then we have

$$I(\mathrm{Tr}_{R/A}) = \sum_i x_i I(\phi_i) = \sum_i x_i \pi(\iota(\phi_i)(d)) = \sum_i x_i \pi(b_i) = \pi(\sum_i b_i x_i) = \pi(d).$$

This shows the desired assertion. □

We now finish the proof of (8.5):

Proposition 8.12. *Let A be a discrete valuation ring, and let R be a reduced local complete intersection over A. Then for an A–algebra homomorphism $R \to A$, we have*

$$\mathrm{length}_A C_0(\lambda, A) = \mathrm{length}_A C_1(\lambda, A).$$

Proof. Let X be a torsion A–module, and suppose that we have an exact sequence:

$$A^r \xrightarrow{L} A^r \to X \to 0$$

of A–modules. Then we claim $\mathrm{length}_A X = \mathrm{length}_A A/\det(L)A$. By elementary divisor theory applied to L, we may assume that L is a diagonal matrix with diagonal entry $d_1, \dots, d_r$. Then the assertion is clear, because $X = \bigoplus_j A/d_jA$ and length A/dA is equal to the valuation of d.

Since R is reduced, $\Omega_{R/A}$ is a torsion R–module, and hence $\Omega_{R/A} \otimes_R A = C_1(\lambda; A)$ is a torsion A–module. Since R is a local complete intersection over A, we can write

$$R \cong A[[T_1, \ldots, T_r]]/(f_1, \ldots, f_r).$$

Then by Corollary 8.2 (ii), we have the following exact sequence for $J = (f_1, \ldots, f_r)$:

$$J/J^2 \otimes_{A[[T_1,\ldots,T_r]]} A \longrightarrow \Omega_{A[[T_1,\ldots,T_r]]/A} \otimes_{A[[T_1,\ldots,T_r]]} A \longrightarrow \Omega_{R/A} \otimes_R A \to 0.$$

This gives rise to the following exact sequence:

$$\bigoplus_j A df_j \xrightarrow{L} \bigoplus_j A dT_j \longrightarrow C_1(\lambda; A) \to 0,$$

where $df_j = f_j \mod J^2$. Since $C_1(\lambda; A)$ is a torsion A–module, we see that $\mathrm{length}_A(A/\det(L)A) = \mathrm{length}_A\, C_1(\lambda; A)$. Since $g = (T_1 - t_1, \ldots, T_n - t_n)$, we see easily that $\det(L) = \pi(\lambda(d))$. This combined with Corollary 8.11 and Lemma 8.8 shows the desired assertion. □

8.6. **A more general setting.** Let $\mathbf{\Lambda}_j$ be the power series ring $W[[T_1, \ldots, T_j]]$. We consider the following commutative diagram of local profinite W-algebras sharing the same residue field $\mathbb{F}$ with W:

$$\begin{array}{ccccc} R_m & \xrightarrow[\twoheadrightarrow]{\theta'} & R_n & \xrightarrow[\twoheadrightarrow]{\mu'} & A \\ \cup\uparrow & & \cup\uparrow & & \cup\uparrow \\ \mathbf{\Lambda}_m & \xrightarrow[\twoheadrightarrow]{\pi_{m,n}} & \mathbf{\Lambda}_n & \xrightarrow[\twoheadrightarrow]{\pi_{n,\nu}} & \mathbf{\Lambda}_\nu. \end{array} \tag{8.8}$$

We put $\lambda' := \mu' \circ \theta'$. Consider the following conditions:

(A0) A is an integral domain.
(A1) $\pi_{j,k}$ indices the identity $\mathbf{\Lambda}_j \otimes_{\mathbf{\Lambda}_j, \pi_{j,k}} \mathbf{\Lambda}_k = \mathbf{\Lambda}_k$ for $(j,k) = (m,n)$ and $(j,k) = (n,\nu)$; so, $m \geq n \geq \nu$.
(A2) R_j is free of finite rank over $\mathbf{\Lambda}_j$ for $j = m, n$, and A is a torsion-free $\mathbf{\Lambda}_\nu$-modules of finite type.
(A3) $\mathrm{Hom}_{\mathbf{\Lambda}_j}(R_j, \mathbf{\Lambda}_j) \cong R_j$ as R_j-modules for $j = m, n$.
(A4) $R_j \otimes_{\mathbf{\Lambda}_j} \mathbf{\Lambda}_\nu$ is a reduced algebra for $j = m, n$.
(A5) R_j is a local complete intersection over $\mathbf{\Lambda}_j$ for $j = m, n$.

Note that (A5) implies (A4) (e.g., [CRT, Theorem 21.3]).

Lemma 8.13. *Suppose* (A0–3). *Let $\widetilde{A}$ be the normalization of A and put $\widetilde{R}_j = R_j \otimes_{\mathbf{\Lambda}_j} \widetilde{A}$ for $j = m, n$.*

(1) *Suppose that $\nu = 1$. Then $\widetilde{R}_j$ is free of finite rank over $\widetilde{A}$ and satisfies $\mathrm{Hom}_{\widetilde{A}}(\widetilde{R}_j, \widetilde{A}) \cong \widetilde{R}_j$ as $\widetilde{R}_j$-modules;*
(2) *Suppose $\nu > 1$. Then for each height 1 prime P of $\widetilde{A}$, $\widetilde{R}_{j,P} = \widetilde{R}_j \otimes_{\widetilde{A}} \widetilde{A}_P$ for the localization $\widetilde{A}_P$ of $\widetilde{A}$ at P is free of finite rank over $\widetilde{A}_P$ and satisfies $\mathrm{Hom}_{\widetilde{A}_P}(\widetilde{R}_{j,P}, \widetilde{A}_P) \cong \widetilde{R}_{j,P}$ as $\widetilde{R}_{j,P}$-modules for $j = m, n$.*

Proof. We only prove the assertion (1), since the assertion (2) is easier to prove after localization at P as $\widetilde{A}_P$ is a discrete valuation ring. If $\nu = 1$, $\widetilde{A}$ is reflexive and hence flat over $\mathbf{\Lambda}_1$. Thus we get

$$\mathrm{Hom}_{\mathbf{\Lambda}_j}(R_j, \mathbf{\Lambda}_j) \otimes_{\mathbf{\Lambda}_j} \widetilde{A} \cong \mathrm{Hom}_{\widetilde{A}}(\widetilde{R}_j, \widetilde{A})$$

from [BAL, II.5.4]. Since $\mathrm{Hom}_{\mathbf{\Lambda}_j}(R_j, \mathbf{\Lambda}_j) \cong R_j$, we get from the above identity

$$\widetilde{R}_j = R_j \otimes_{\mathbf{\Lambda}_j} \widetilde{A} \cong \mathrm{Hom}_{\mathbf{\Lambda}_j}(R_j, \mathbf{\Lambda}_j) \otimes_{\mathbf{\Lambda}_j} \widetilde{A} \cong \mathrm{Hom}_{\widetilde{A}}(\widetilde{R}_j, \widetilde{A})$$

as $\widetilde{R}_j$-modules. □

By Lemma 8.13, under (A0–4), the sequences $\widetilde{R}_m \xrightarrow{\theta} \widetilde{R}_n \xrightarrow{\mu} \widetilde{A}$ if $\nu = 1$ and $\widetilde{R}_{m,P} \xrightarrow{\theta} \widetilde{R}_{n,P} \xrightarrow{\mu} \widetilde{A}_P$ if $\nu = 1$ for $\theta = \theta' \otimes 1$ and $\mu = \mu' \otimes 1$ satisfies the requirement of $R \xrightarrow{\theta} S \xrightarrow{\mu} A$ for the transfer property of congruence modules in 8.3. Thus we get for the annihilators $\mathfrak{c}_? := \mathrm{Ann}_{\widetilde{A}}(C_0(?;\widetilde{A}))$ with $? = \lambda, \mu$ and $\mathfrak{c}_\theta := \mathrm{Ann}_{\widetilde{R}_n}(C_0(\theta;\widetilde{R}_n))$ the following transfer formula:

Proposition 8.14. *Assume* (A0–4).

(1) *If $\nu = 1$, the ideals $\mathfrak{c}_?$ ($? = \lambda, \mu, \theta$) are all principal, and satisfies $\lambda(\mathfrak{c}_\theta) \cdot \mathfrak{c}_\mu = \mathfrak{c}_\lambda$.*

(2) *If $\nu > 1$, writing $\widetilde{M}$ for the reflexive closure of a torsion-free $\mathbf{\Lambda}_\nu$-module M of finite type, we have the following identity $\widetilde{\lambda(\mathfrak{c}_\theta) \cdot \mathfrak{c}_\mu} = \widetilde{\mathfrak{c}_\lambda}$.*

Proof. By Corollary 8.6, we get the assertion (1) and also the localized identity: $\lambda(\mathfrak{c}_\theta)_P \cdot \mathfrak{c}_{\mu,P} = \mathfrak{c}_{\lambda,P}$ for each height 1 prime P in the setting of (2), since $\lambda(\mathfrak{c}_{\theta,P}) = \lambda(\mathfrak{c}_\theta)_P$ and $\mathfrak{c}_{?,P}$ is the annihilator of the corresponding P-localized congruence module by the definition of the congruence module. Since $\widetilde{M} = \bigcap_P M_P$ inside $M \otimes_{\widetilde{A}} \mathrm{Frac}(\widetilde{A})$, we get the assertion (2). □

Now suppose (A5); so, $R_j \cong \mathbf{\Lambda}_j[[X_1, \dots, X_k]]/(f_1, \dots, f_k)$ for a regular sequence $(f_1, \dots, f_k)$ in $\mathfrak{m}_{\mathbf{\Lambda}_j[[X_1,\dots,X_k]]}$. Tensoring $\widetilde{A}$ over $\mathbf{\Lambda}_j$ with the exact sequence:

$$0 \to (f_1, \dots, f_k) \to \mathbf{\Lambda}_j[[X_1, \dots, X_k]] \to R_j \to 0,$$

we get a sequence,

$$0 \to (f_1, \dots, f_k) \to \widetilde{A}[[X_1, \dots, X_k]] \to \widetilde{R}_j \to 0,$$

which is exact. Since R_j is $\mathbf{\Lambda}_j$-free of finite rank, the first sequence of $\mathbf{\Lambda}_j$-modules is split exact; so, the exactness is kept after tensoring $\widetilde{A}$. Thus (A5) implies that

(A′5) $\widetilde{R}_j$ is a local complete intersection over $\widetilde{A}$ for $j = m, n$.

Thus we may apply Tate's formula Theorem 8.7 to our setting $\widetilde{R}_{m,P} \twoheadrightarrow \widetilde{R}_{n,P} \twoheadrightarrow \widetilde{A}_P$ for each height 1 primes and get the following fact:

Theorem 8.15. *Assume* (A0–5). *Then we have*

$$\widetilde{\lambda(\mathfrak{c}_\theta)} \cdot \mathrm{Char}(C_1(\mu, \widetilde{A})) = \mathrm{Char}(C_1(\lambda, \widetilde{A})).$$

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