Iwasawa theory for deformation of ordinary elliptic curves

1 Special values of L-functions of ordinary elliptic curves

1.1 Review of elliptic curves

Let E be an elliptic curve over $\mathbb{Q}$ of conductor N. Let K be an imaginary quadratic extension of $\mathbb{Q}$ of discriminant d_K lesser than 4 and prime to N and let η_K be the quadratic character of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ attached to K. Let $p \geq 5$ be a rational prime which does not divide Nd_K

1.1.1 Geometric data

Let G_K be $\mathrm{Gal}(\bar{K}/K)$. The group $E(\bar{K})$ being divisible, there is an exact sequence

$$0 \longrightarrow E(\bar{K})[p^n] \longrightarrow E(\bar{K}) \xrightarrow{p^n} E(\bar{K}) \longrightarrow 0$$

and thus an exact sequence in G_K-cohomology:

$$0 \longrightarrow E(K)/p^nE(K) \longrightarrow H^1(G_K, E[p^n]) \longrightarrow H^1(G_K, E)[p^n] \longrightarrow 0$$

Doing the same with all the completions of K at finite places, we obtain a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & E(K)/p^nE(K) \arrow[r] \arrow[d] & H^1(G_K, E[p^n]) \arrow[r] \arrow[d] & H^1(G_K, E)[p^n] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \prod_v E(K_v)/p^nE(K_v) \arrow[r] & \prod_v H^1(G_{K_v}, E[p^n]) \arrow[r] & \prod_v H^1(G_{K_v}, E)[p^n] \arrow[r] & 0
\end{tikzcd}$$

Let $\mathrm{Sel}^n(E/K)$ and $\text{Ш}(E/K)$ be:

$$\mathrm{Sel}^n(E/K) = \ker\left(H^1(G_K, E[p^n]) \longrightarrow \prod_v H^1(G_{K_v}, E)\right)$$

$$\text{Ш}(E/K) = \ker\left(H^1(G_K, E) \longrightarrow \prod_v H^1(G_{K_v}, E)\right)$$

Using the snake lemma, we see that there is a short exact sequence

$$0 \longrightarrow E(K) \otimes \mathbb{Z}/p^n\mathbb{Z} \longrightarrow \mathrm{Sel}^n(E/K) \longrightarrow \text{Ш}(E/K)[p^n] \longrightarrow 0$$

which is also interesting after taking direct and inverse limit on n:

$$0 \longrightarrow E(K) \otimes \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow \mathrm{Sel}^\infty(E/K) \longrightarrow \text{Ш}(E/K)[p^\infty] \longrightarrow 0$$

$$0 \longrightarrow E(K) \otimes \mathbb{Z}_p \longrightarrow \mathrm{Sel}_\infty(E/K) \longrightarrow T_p\text{Ш}(E/K) \longrightarrow 0$$

1.1.2 Automorphic data

The reduction modulo a prime ℓ of $E/\mathbb{Q}$ defines a curve $E/\mathbb{F}_\ell$, which is elliptic for primes ℓ not dividing N. Let a_ℓ be $1+\ell-\#E/\mathbb{F}_\ell$ and $P_\ell(X)$ be $1-a_\ell X+\ell X^2$ if $\ell\nmid N$ and or $1-a_\ell X$ if $\ell|N$. The local L-function L_ℓ is then defined to be:

$$L_\ell(E/\mathbb{Q},s)=\frac{1}{P_\ell(\ell^{-s})}$$

The product of local L-functions defines the complex L-function:

$$L(E/\mathbb{Q},s)=\prod_\ell L_\ell(E/\mathbb{Q},s)=\prod_{\ell\nmid N}\frac{1}{1-a_\ell\ell^{-s}+\ell^{1-2s}}\prod_{\ell|N}\frac{1}{1-a_\ell\ell^{-s}}$$

By [Wil95, BCDT01], the complex L-function $L(E/\mathbb{Q},s)$ coincides with the Mellin transform of a modular form $f\in S_2(\Gamma_0(N))$ which is an eigenvector under the action of the Hecke operators T_ℓ for $\ell\nmid N$. The defining property of invariance under $\Gamma_0(N)$ of modular forms implies that their Mellin transforms satisfy functional equations. Hence, $L(E/\mathbb{Q},s)$ satisfies the functional equation:

$$L(E/\mathbb{Q},s)=\varepsilon C(s)L(E/\mathbb{Q},2-s)$$

The factor $C(s)$ comes from the local factors at ∞ of $L(E/\mathbb{Q},s)$ and $L(E/\mathbb{Q},2-s)$. Because $C(1)$ is equal to 1 and because we are only interested in the order of vanishing of $L(E/\mathbb{Q},1)$ at 1, we ignore it in the following. The ε-factor ε is equal to ±1 and can be defined and computed locally, though computations are actually quite involved when $\pi(f)_\ell$ is supercuspidal. The important property for what follows is that ε_ℓ is equal to 1 for $\ell\nmid N\infty$ and that ε_ℓ only depends on $E/\mathbb{Q}_\ell$.

We will also consider the twisted L-function $L(E,\eta_K,s)$ defined by:

$$L(E,\eta_K,s)=\prod_{\ell\nmid N}\frac{1}{1-a_\ell\eta_K(\ell)\ell^{-s}+\ell^{1-2s}}\prod_{\ell|N}\frac{1}{1-a_\ell\eta_K(\ell)\ell^{-s}}$$

The L-function of the elliptic curve E/K is defined to be:

$$L(E/K,s)=L(E/\mathbb{Q},s)L(E/\mathbb{Q},\eta_K,s)$$

This L-function is the automorphic L-function of the base change $\pi_K(f)$ of $\pi(f)$ to $\mathbb{A}_K$ as well as the Rankin-Selberg L-function $\pi(f)\times\theta(\chi_{triv})$. It satisfies the functional equation:

$$L(E/K,s)=\varepsilon_K C_K(s)L(E/K,2-s)$$

We ignore the contribution of $C_K(s)$ for the same reasons as above. The ε factor ε_K is equal to:

$$\varepsilon_K=(-1)^{\#S},\ S=\{\ell|\varepsilon_{K,\ell}\neq\eta_{K,\ell}(-1)\}$$

The values of the local $\varepsilon_{K,\ell}$ factors is summed up below:

1. Assume $\pi_K(f)_\ell$ is a principal series. Then $\varepsilon_{K,\ell}=\eta_{K,\ell}(-1)$ (hence $\ell\notin S$).
2. Assume $\pi_K(f)_\ell=St(\mu)$ is Steinberg. Then $\varepsilon_{K,\ell}=-\eta_{K,\ell}(-1)$ if ℓ is not split in K and $\mu=1$, else it is equal to $\eta_v(-1)$ (hence ℓ belongs to S if and only if ℓ is inert in K and $\operatorname{ord}_\ell N$ is odd).
3. Assume $\pi_K(f)_\ell$ is supercuspidal. Then $\varepsilon_{K,\ell}=\eta_\ell(\ell)^{\operatorname{ord}_\ell N}$ (Hence ℓ belongs to S if and only if ℓ is inert in K and $\operatorname{ord}_\ell N$ is odd).
4. The ε-factor at ∞ is equal to 1 (hence ∞ belongs to S).

We assume henceforth that all primes dividing N split in K. Then, ε_K is equal to -1 so the L-function $L(E/K,s)$ vanishes at odd, and hence non-zero, order at 1.

1.1.3 Galois theoretic data

The Tate module T_pE of E and its p-torsion $E[p]$ define $G_{\mathbb{Q}}$ and G_K-representations ρ_E and $\bar{\rho}_E$ of degree 2 with coefficients in $\mathbb{Z}_p$ and $\mathbb{F}_p$ respectively. At $\ell \nmid Np$, ρ_E is unramified and:

$$\det(1-\operatorname{Fr}(\ell)X) = 1 - a_\ell X + \ell X^2$$

If $\ell | N$, the ρ_E is ramified, but $\bar{\rho}_E$ need not be, and we define $\operatorname{Tam}_p(E,\ell)$ to be the greatest power p^n such that $E[p^n]$ is unramified at ℓ. We assume that $\bar{\rho}_E$ is an irreducible $G_{\mathbb{Q}}$-representation (in which case it is in fact absolutely irreducible).

Because of our assumptions on p, if ℓ is such that $\operatorname{Tam}_p(E,\ell)$ is not trivial, then $\pi(f)_\ell$ is Steinberg so $\rho_{E,G_{\mathbb{Q}_\ell}}$ is reducible.

We assume henceforth that E is ordinary at p, which means that $p \nmid a_p$. Then ρ_E is reducible at p and admits an unramified quotient.

1.2 The Birch and Swinnerton-Dyer conjecture and the Gross-Zagier formula

The order of vanishing part of the Birch and Swinnerton-Dyer conjecture states that:

$$\operatorname{ord}_{s=1} L(E/K,s) \overset{?}{=} \operatorname{rank} E(K)$$

Under our hypotheses, there thus should exist points of infinite order on E which are rational over K.

The fact that $E/\mathbb{Q}$ is modular implies that there exists an isogeny from the Jacobian variety $J_0(N)$ of $X_0(N)$ to E. Gross-Zagier proved the remarkable fact that $L'(E/K,1)$ is non-zero if and only if the height of a special point z on $X_0(N)$ is non-zero. More precisely, if c is the degree of the modular parametrization:

$$L'(E/K,1)/\Omega = h(z)\frac{1}{c^2\sqrt{|d_K|}}$$

When $\operatorname{ord}_{s=1} L(E/K,s)$ is exactly equal to 1, this point is thus non-torsion. On the other hand, the conjecture of Birch and Swinnerton-Dyer then also predicts the exact value of $L'(E/K,s)$.

$$L'(E/K,1)/\Omega \overset{?}{=} h(z)\frac{\#Ш(E/K)\prod \operatorname{Tam}(E/K,\lambda)}{[E(K):z]^2\sqrt{|d_K|}}$$

Putting these 2 equations together, we get:

$$\#Ш(E/K) \overset{?}{=} \left(\frac{[E(K):z]}{c\prod\limits_{\ell|N} \operatorname{Tam}(E/\mathbb{Q},\ell)}\right)^2$$

Specializing to the p-part of the previous equation, we get:

$$\#Ш(E/K)[p^\infty]\left(\prod_{\ell|N}\operatorname{Tam}_p(E/\mathbb{Q},\ell)\right)^2 = [E(K):z]^2 \tag{1.2.1}$$

Shortly after [GZ86], V.Kolyvagin proved in [Kol90] (under slightly stronger hypotheses that we are working here) that $Ш(E/K)[p^\infty]$ was indeed finite, that $E(K)$ was of rank 1 and that:

$$\#Ш(E/K)[p^\infty] | [E(K):z]^2 \tag{1.2.2}$$

2 Families of G_K-representations containing ρ_E

2.1 Iwasawa-theoretic families

The number field K admits a unique pro-finite extension K_∞ such that $\mathrm{Gal}(K_\infty/K)$ is isomorphic to $\mathbb{Z}_p^2$. Among the sub-extensions of K_∞ whose Galois group over K is isomorphic to $\mathbb{Z}_p$, 2 play a special roles: the cyclotomic $\mathbb{Z}_p$-extension, which is a sub-extension of $K(\zeta_{p^\infty})$ and which is thus abelian over $\mathbb{Q}$, and the anticyclotomic or dihedral $\mathbb{Z}_p$-extension D_∞, which is pro-dihedral over $\mathbb{Q}$, that is to say which verifies the relation $\tau\sigma\tau = \sigma^{-1}$ for all $\sigma \in \mathrm{Gal}(D_\infty/K)$ and for τ a generator of $\mathrm{Gal}(K/\mathbb{Q})$. The former is studied in [Gre91] but we will not say anything about it in the following.

Let Γ_a be $\mathrm{Gal}(D_\infty/K)$ and Λ_a be $\mathbb{Z}_p[[\Gamma_a]]$. The ring Λ_a is isomorphic to $\mathbb{Z}_p[[X]]$ and hence regular of Krull dimension 2. Its group of unit is endowed with a continuous action of G_K by the surjection from G_K onto Γ_a composed with the inclusion of Γ_a inside $\Lambda_a^\times$. We denote by χ_a this character.

Let $T_{\mathrm{Iw}} = T_pE \otimes_{\mathbb{Z}_p} \Lambda_a$ be the G_K-representation with G_K-action on T_pE via ρ_E and on Λ_a via χ_a. We can view $T_pE \otimes_{\mathbb{Z}_p} \Lambda_a$ as a family of p-adic representation by the following specialization process. Let S be the ring of integers of a finite extension of $\mathbb{Q}_p$ and let $\mathbf{Sp}$ be a $\mathbb{Z}_p$-algebra morphism from Λ_a to S. Then $T_{\mathbf{Sp}}$ is the $S[G_K]$-module $T_{\mathrm{Iw}} \otimes_{\mathbf{Sp}} S$ with trivial G_K-action on S.

2.2 Hida-theoretic families

Let $X(N,s)$ be the modular curve $(\Gamma_0(N)\cap\Gamma_1(p^s))\backslash\mathcal{H}^*$ and let $J(N,s)$ its Jacobian variety. There is a surjective isogeny from $J(N,0)$ onto E. The ring of diamond operators of level s is the ring $\mathbb{Z}_p[(\mathbb{Z}/p^s\mathbb{Z})^\times]$. Let $\mathbf{T}_k(N,s)$ be the image of the $\mathbb{Z}_p$-algebra generated by all Hecke operators $T(\ell)$ with $\ell \nmid N$ and all diamond operators $<a>$ of level s inside the endomorphism ring of $S_k(\Gamma_0(N)\cap\Gamma_1(p^s))$ the set of cuspforms of weight k for the congruence subgroup $\Gamma_0(N)\cap\Gamma_1(p^s)$. It is an algebra over the ring of diamond operators.

Let Γ be the torsion-free part of $\varprojlim_s (\mathbb{Z}/p^s\mathbb{Z})^\times$ and Λ be the regular local ring $\mathcal{O}[[\Gamma]]$, which is also the torsion-free part of the inverse limit on s of the ring of diamond operators. Let γ be a topological generator of Γ. For $k \geq 2$ and integer and ϵ a finite order character of Γ, an arithmetic point of weight k and character ϵ of Λ is a $\mathbb{Z}_p$-algebra morphism:

$$\begin{aligned} \phi: \quad \Lambda &\longrightarrow \bar{\mathbb{Q}}_p^\times \\ \gamma &\longmapsto \epsilon(\gamma)\gamma^{k-2} \end{aligned}$$

Arithmetic points of a Λ-algebra are $\mathbb{Z}_p$-algebra morphism which coincide with arithmetic points after restriction to Λ.

Let e^{ord} be Hida's projector, this is to say the idempotent:

$$e^{ord} = \lim_{n\to\infty} T(p)^{n!}$$

The ordinary Hecke algebra $\mathbf{T}_k^{\mathrm{ord}}(N)$ is the inverse limit on s:

$$\mathbf{T}_k^{\mathrm{ord}}(N) = \varprojlim_s e^{ord}\mathbf{T}_k(N,s)$$

It is a torsion-free Λ-algebra independent of k, finite and flat as Λ-module and independent of k in the sense that $\mathbf{T}^{\mathrm{ord}} = \mathbf{T}_2^{\mathrm{ord}}(N) = \mathbf{T}_k^{\mathrm{ord}}(N)$. The morphism λ_E given by E factors through the localization of $\mathbf{T}^{\mathrm{ord}}$ at a maximal ideal $\mathfrak{m}$ and in fact through $R = \mathbf{T}_{\mathfrak{m}}^{\mathrm{ord}}/\mathfrak{a}$ for a unique minimal prime $\mathfrak{a}$. Define:

$$J_\infty = \varinjlim_s J(N,s)[p^\infty]$$

And:

$$T_{\mathrm{Hi}} = \mathrm{Hom}_{\mathbb{Z}_p}(e^{ord}_{\mathfrak{m}} J_\infty, \mu_{p^\infty}) \otimes_{\mathbf{T}^{\mathrm{ord}}_{\mathfrak{m}}} R \ \ (= \varprojlim_{s} e^{ord}_{\mathfrak{m}} H^1_{et}(X(N,s) \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathbb{Z}_p) \otimes_{\mathbf{T}^{\mathrm{ord}}_{\mathfrak{m}}} R)$$

The R-module T_{Hi} is free of rank 2. We assume that R is a Gorenstein ring (this means for example that $\mathrm{Hom}_\Lambda(R,\Lambda)$ is a free R-module of rank 1). In the talk, I unfortunately claimed that R is a Gorenstein, but as Professor Hida remarked, this is not known to be true. Arithmetic specializations of T_{Hi} are the $G_{\mathbb{Q}}$-representations attached to ordinary eigenforms with same residual representation as E in the sense that for $\ell \nmid Np$, the $G_{\mathbb{Q}}$-representation T_{Hi} verifies

$$\det(1 - \mathrm{Fr}(\ell)X) = 1 - \lambda(T_\ell)X + \chi_\Gamma(\ell)\ell X^2$$

where λ is the natural morphism from $\mathbf{T}^{\mathrm{ord}}$ to R and χ_Γ is the inclusion of Γ inside R.

2.3 Combining the two

Let R_{Iw} be $R[[\Gamma_a]]$ and $\mathcal{T}$ be $T_{\mathrm{Hi}} \otimes_R R_{\mathrm{Iw}}$ with G_K-action on both side of the tensor product. The module $\mathcal{T}$ is free of rank 2 over a Gorenstein ring of Krull dimension 3. In fact, we will consider self-dual twists of T_{Hi} and $\mathcal{T}$. The determinant of T_{Hi} verifies:

$$\det{}_R T_{\mathrm{Hi}} = R(1) \otimes \chi_\Gamma$$

The character χ_Γ is a square because $1 + p\mathbb{Z}_p$ is 2-divisible. Choose χ such that $\chi^2 = \chi_\Gamma$ and define:

$$T^\dagger_{\mathrm{Hi}} = T_{\mathrm{Hi}} \otimes \chi^{-1}, \ \ \mathcal{T}^\dagger = \mathcal{T} \otimes \chi^{-1}$$

Then T_{Hi} and $\mathcal{T}^\dagger$ are self-dual. Note that an arithmetic specialization of T_{Hi} of even weight k is equal to the (cohomological) $G_{\mathbb{Q}}$-representation of f twisted by $k/2$.

3 A question

For T equal to T_{Iw} or to $T^\dagger_{\mathrm{Hi}}$ or to $\mathcal{T}^\dagger$, is it possible to formulate and prove equivalents of (1.2.1) and of (1.2.2) for the G_K-representation T. Stating a generalization of (1.2.1) would entail the following:

1. Define a generalization $\tilde{H}^1_f(K,T)$ of the group of rational points and $\tilde{H}^2_f(K,T)$ of Ш for T. Presumably, these should be subgroups of Galois cohomology with coefficients in the ring of coefficients of T and we would expect $\tilde{H}^1_f(K,T)$ to be of rank 1.
2. Define a generalization z of Heegner points in this setting. Presumably, z should belong to $\tilde{H}^1_f(K,T)$, this is to say to the generalization of the group of rational points.
3. The equation (1.2.2) involves the cardinal of Ш and of $E(K)/z$. It doesn't seem reasonable to hope for our more general $\tilde{H}^i_f(K,T)$ to be of finite cardinality. Find a suitable generalization.

In the setting described, everything in the above has in fact been achieved in [Gre91, Nek06, How07] when the ring of coefficients R is assumed to be regular, so automatically for T_{Iw} but under the supplementary hypothesis that R is a regular ring for $T^\dagger_{\mathrm{Hi}}$ and $\mathcal{T}^\dagger$. For general R, one can look up [Fou09]. As (1.2.1) stems for the Birch and Swinnerton-Dyer conjecture, it is moreover not unreasonable to make the vague requirement that $\tilde{H}^i_f(G_K,T)$ be linked with the special values of a suitable p-adic L-function.

4 Some progress towards the question

4.1 Selmer structures

4.1.1 Some motivation

Since [BK90], it is known that a good equivalent of the group of K-rational points of an elliptic curve or an abelian variety for a general p-adic G_K-representation can be constructed by looking at subgroups of Galois cohomology $H^1_f(K,T)\subset H^1(K,T)$ satisfying local conditions, in the sense that $c\in H^1(K,T)$ belongs to $H^1_f(K,T)$ if and only if for all v, the local class $c_v\in H^1(K_v,T)$ verifies some supplementary condition. Among the conditions we can impose on local cohomology, the following ones have proven useful:

1. The relaxed condition: c_v verifies no supplementary condition.
2. The strict condition: c_v is trivial.
3. The unramified condition: c_v is trivial in $H^1(I_v,T)$.
4. The ordinary or Greenberg condition at $v|p$: c_v is trivial in $H^1(K_v,T_v^-)$ for a given quotient T_v^- of T.
5. The crystalline or Bloch-Kato condition at $v|p$: c_v is trivial in $H^1(K_v,V\otimes B_{\mathrm{cris}})$ where $V=T\otimes\mathbb{Q}_p$.

In general, the best equivalent of $E(K)$ is given by the imposition of the unramified condition at $v\nmid p$ and of the Bloch-Kato condition at $v|p$ (indeed, in that case, $H^1_f(K,T_pE)$ coincides with the image of $E(K)$ inside $H^1(K,T_pE)$). However, the Bloch-Kato condition suffers from the defect that it is only defined at present for $\mathbb{Q}_p$-modules, so it cannot be used for our more general representations T_{Iw} or $\mathcal{T}^\dagger$.

4.1.2 Selmer complexes

Let T be T_pE, T_{Iw}, $T^\dagger_{\mathrm{Hi}}$ or $\mathcal{T}^\dagger_{\mathrm{Iw}}$ and let S be its ring of coefficients (hence always a Gorenstein domain). Let Σ be a finite set of places of K containing the places above Np and $G_{K,\Sigma}$ be the Galois group of the maximal extension of K unramified outside Σ. Let $C^\bullet_{\mathrm{cont}}(G_{K,\Sigma},T)$ be the complex of continuous cochains with values in T and for $v\in\Sigma$, let $C^\bullet_{\mathrm{cont}}(G_{K_v},T)$ be the complex of local continuous cochains with values in T. If $v|p$, let $C_f(G_{K_v},T)$ be the complex of continuous cochains $C^\bullet_{\mathrm{cont}}(G_{K_v},T_v^+)$. If v belongs to Σ but $v\nmid p$, let $C_f(G_{K_v},T)$ be the complex of continuous cochains $C^\bullet_{\mathrm{cont}}(G_{K_v}/I_v,T)$. Remark that for all $v\in\Sigma$, there is a morphism of complexes i_v from $C_f(G_{K_v},T)$ to $C^\bullet_{\mathrm{cont}}(G_{K_v},T)$ induced by the inclusion of T_v^+ inside T if $v|p$ and by inflation if $v\nmid p$. Let $\mathrm{R}\Gamma_f(G_{K,\Sigma},T)$ be the object in the derived category corresponding to:

$$\mathrm{Cone}\left(C^\bullet_{\mathrm{cont}}(G_{K,\Sigma},T)\oplus\bigoplus_{v\in\Sigma}C_f(G_{K_v},T)\xrightarrow{\mathrm{res}_v-i_v}\bigoplus_{v\in\Sigma}C^\bullet_{\mathrm{cont}}(G_{K,v},T)\right)[-1]$$

We write $\tilde{H}^i_f(K,T)$ for the i-th cohomology group of $\mathrm{R}\Gamma_f(G_{K,\Sigma},T)$. This notation introduces no ambiguity because $\mathrm{R}\Gamma_f(G_{K,\Sigma},T)$ is independent of the choice of Σ as long as it verifies what we required.

Our generalization of $E(K)\otimes\mathbb{Z}_p$ and $\text{Ш}(E/K)[p^\infty]$ is $\tilde{H}^1_f(K,T)$ and $\tilde{H}^2_f(K,T)_{\mathrm{tors}}$.

4.2 The Iwasawa-theoretic case

In this subsection, T is taken to be T_{Iw}. The ring of coefficients Λ_a being a 2-dimensional regular ring, for every Λ_a-module M, there is a pseudo-isomorphism (this is to say, in that case, a morphism with finite kernel and co-kernel)

$$M \overset{\cdot}{\xrightarrow{\sim}} \Lambda_a^r \oplus \bigoplus_{\mathfrak{p}} \Lambda_a/\mathfrak{p}^n$$

where the $\mathfrak{p}$ are prime ideal of height one. If M is a torsion Λ_a-module, let $\operatorname{char}_{\Lambda_a} M$ be the ideal generated by the products of the $\mathfrak{p}^n$. Under the inspiration of B.Mazur, B.Perrin-Riou proposed in [PR87] the following conjecture:

1. The Λ_a-module $\tilde{H}^1_f(K, T_{\mathrm{Iw}})$ is free of rank 1.
2. There exists a non-trivial class z_{Iw} in $\tilde{H}^1_f(K, T_{\mathrm{Iw}})$ coming from Heegner points in $\varprojlim_n E(D_n)$.
3. There is the following equality of characteristic ideals:

$$\operatorname{char}_{\Lambda_a} \tilde{H}^2_f(K, T_{\mathrm{Iw}})_{\mathrm{tors}} = \left(\operatorname{char}_{\Lambda_a} \tilde{H}^1_f(K, T_{\mathrm{Iw}})/\Lambda_a \cdot z_{\mathrm{Iw}}\right)^2$$

This generalizes neatly (1.2.1). Note that in this conjecture, it is not assumed that $L(E/K, s)$ vanishes at order 1. This reflects a principle called Mazur's conjecture that the generic order of vanishing of the L-function, hence the generic rank of $\tilde{H}^1_f(K, T)$ should be minimal.

Thanks to [Cor02, CV04], the first two numbers of this conjecture has known. Thanks to [How04], it is known (the result is stated under slightly stronger hypotheses but ours are enough to prove it) that:

$$\operatorname{char}_{\Lambda_a} \tilde{H}^2_f(K, T_{\mathrm{Iw}})_{\mathrm{tors}} | \left(\operatorname{char}_{\Lambda_a} \tilde{H}^1_f(K, T_{\mathrm{Iw}})/\Lambda_a \cdot z_{\mathrm{Iw}}\right)^2$$

4.3 The general regular case

In this subsection, T is taken to be $\mathcal{T}^\dagger$ and R is assumed to be a regular ring (hence the same is true of R_{Iw}). The conjecture of the previous subsection has been generalized by B.Howard in [How07] to:

1. The R_{Iw}-module $\tilde{H}^1_f(K, \mathcal{T}^\dagger)$ is torsion-free of rank 1.
2. There exists a non-trivial class z_∞ in $\tilde{H}^1_f(K, \mathcal{T}^\dagger)$ coming from Heegner points in the tower of modular curves $\varprojlim_n \varprojlim_s X_1(Np^s)(D_n)$.
3. There is the following equality of characteristic ideals:

$$\operatorname{char}_{R_{\mathrm{Iw}}} \tilde{H}^2_f(K, \mathcal{T}^\dagger)_{\mathrm{tors}} = \left(\operatorname{char}_{R_{\mathrm{Iw}}} \tilde{H}^1_f(K, \mathcal{T}^\dagger)/R_{\mathrm{Iw}} \cdot z_\infty\right)^2$$

Moreover, B.Howard proved the first two numbers. A proof of the fact that

$$\operatorname{char}_{R_{\mathrm{Iw}}} \tilde{H}^2_f(K, \mathcal{T}^\dagger)_{\mathrm{tors}} | \left(\operatorname{char}_{R_{\mathrm{Iw}}} \tilde{H}^1_f(K, \mathcal{T}^\dagger)/R_{\mathrm{Iw}} \cdot z_\infty\right)^2$$

can be found in [Fou08].

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